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# CONSTRUCTION OF MOLLER WAVE OPERATORS FOR VLASOV EQUATIONS LINEARIZED AROUND BOLTZMANNIAN STATES\*

BRUNO DESPRÉS†

**Abstract.** We detail a possible construction of the Moller wave operators for linear Vlasov-Poisson and Vlasov-Ampere equations. This is based on an explicit and detailed calculation of the eigenstructure. A simple non homogeneous case is detailed. We finally show that, for the homogeneous case, the Morrison transform is exactly the Moller wave operator.

**Key words.** Vlasov-Poisson equation, integro-differential operator, scattering theory, Moller wave operators, Landau damping.

**AMS subject classifications.** 35B35, 35L60, 82D10

**1. Introduction.** This work is an extension of a previous one [6] on the linearized Vlasov-Poisson or Vlasov-Ampere equation and is dedicated to new connections between Moller wave operators (in the context of the abstract theory of scattering [15, 19, 13, 32]) for general homogeneous or non homogeneous states and the Morrison integral transform [24, 25]. It yields an alternative to the standard Hamiltonian framework privileged in Morrison works. Indeed it seems that the abstract scattering theory *per se* has never been adapted to the study of linearized Vlasov-Poisson equation, or more precisely, the fact that Moller wave operators give a proof of linear Landau damping has never been explicitly formulated [27]. The associated algebra gives some tracks in the direction of a proof of linear Landau damping [18, 27] for non homogeneous states [28, 3]. In particular we rigorously derive an integral equation which appears to be fundamental for the definition of Moller operators for non homogeneous Boltzmannian states.

The model problem is a Vlasov-Ampère equation in dimension 1+1

$$\begin{cases} \partial_t f + v\partial_x f - E\partial_v f = 0, & t > 0, \quad (x, v) \in I \times \mathbb{R}, \\ \partial_t E = \int_{\mathbb{R}} v f dv, & t > 0, \quad x \in I, \end{cases} \quad (1.1)$$

where  $I = [0, 1]$  is a periodic domain. The system (1.1) is equivalent to the Vlasov-Poisson equation

$$\begin{cases} \partial_t f + v\partial_x f - E\partial_v f = 0, & t > 0, \quad (x, v) \in I \times \mathbb{R}, \\ \partial_x E = \rho_0(x) - \int_{\mathbb{R}} f dv, & t > 0, \quad x \in I. \end{cases} \quad (1.2)$$

considering that the following conditions are fulfilled at initial time

$$\begin{cases} \partial_x E = \rho_0(x) - \int f dv & \text{at } t = 0 \text{ and } \forall x \in I, \\ \int_I \int_{\mathbb{R}} f dv dx = \int_I \rho_0(x) dx, & \text{at } t = 0, \\ \int_I \int_{\mathbb{R}} f v dv dx = 0, & \text{at } t = 0. \end{cases} \quad (1.3)$$

These conditions are propagated by the equation. The solutions of the Vlasov-Poisson-Ampère equation satisfy the conservation of the physical energy  $\frac{d}{dt} \left( \int_I \int_{\mathbb{R}} f(t, x, v) v^2 dv dx + \int_I E(t, x)^2 dx \right) = 0$  and the boundedness of the density  $0 \leq f(t, x, v) \leq \|f_0\|_{L^\infty(I \times \mathbb{R})}$ ,  $f(0, x, v) = f_0(x, v)$ .

We will study the linear stability around general non homogeneous states

$$v\partial_x f_0 - E_0(x)\partial_v f_0 = 0. \quad (1.4)$$

A natural possibility to represent such states is  $f_0(x, v) = F\left(\frac{v^2}{2} - \varphi(x)\right)$  where  $\varphi$  is the 1-periodic electric potential such that

$$\varphi'(x) = -E_0(x) \quad (1.5)$$

and  $F$  is an arbitrary function which can be multivalued as well. For the simplicity of the expository, we will restrict a lot by considering that the stationary state is Boltzmannian, that is  $F(w) = \exp(-w)$  so that we can write

$$f_0(x, v) = n_0(x)G(v), \quad n_0(x) = \exp(\varphi(x)), \quad G(v) = \exp\left(-\frac{v^2}{2}\right).$$

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Notice that

$$-\varphi'' + \alpha^2 \exp \varphi = \rho_0 \text{ with } \alpha^2 = \int_{\mathbb{R}} \exp\left(-\frac{v^2}{2}\right) dv = \sqrt{2\pi}.$$

This assumption could probably be partially relaxed, see [6].

Starting from the Kruzkal-Obermann-Antonov identity [17, 2], the linearized Vlasov-Ampere equations are recast in the next section as

$$\partial_t U = iHU, \quad i^2 = -1, \quad (1.6)$$

where  $U(t)$  is in a convenient Hilbert space  $V$  and one has the formal identity  $H^* = H$ . The notation is formal since  $iH$  is a real valued operator. Looking at the structure of the equations, one realizes moreover that

$$H = H_0 + K$$

where  $H_0^* = H_0$  models the particles without interaction with the electric field and  $K$  is a compact operator that models the interaction between the particles and the electric field. More precisely, writing all quantities with moments against Hermite functions, the operator  $K$  is realized as an infinite matrix but with finite rank, so the compactness. This is the framework of scattering theory. So, in this context, it is natural to embark in the construction the Moller wave operator  $L$  such that

$$L^*H = H_0L^* \iff HL = LH_0 \quad (\text{formally}). \quad (1.7)$$

Assuming invertibility of  $L^*$ , it yields a representation formula for the solution of (1.6)

$$U(t) = \exp(iHt)U_0 \iff U(t) = L^{-*} \exp(iH_0t)L^*U_0. \quad (1.8)$$

The Moller wave operator is a fundamental object in scattering theory, we just quote seminal references [15, 19], see also [13, 32].

In this work, we concentrate on an explicit representation for  $L$  and notice that (1.7) implies the formal identity

$$V_\lambda = LU_\lambda$$

where  $V_\lambda$  is a (generalized) eigenvector for the full Hamiltonian, that is  $HV_\lambda = \lambda V_\lambda$ , and  $U_\lambda$  is a (generalized) eigenvector for the reduced or free Hamiltonian, that is  $H_0U_\lambda = \lambda U_\lambda$ . It appears that  $U_\lambda$  and  $V_\lambda$  can be constructed almost explicitly, which ultimately is a method to construct the wave operator  $L$ . In this work, we try to make everything as discrete as possible by using a Hilbert basis conveniently defined with moment against the Hermite functions: this construction is done with the help of the Koopman-von Neumann (KvN) method [16, 30, 31] which yields a natural bridge between the method of characteristic for the transport equation and the Hilbertian formulation. These ideas go back to the very early stage of quantum mechanics.

The results of this general strategy are pushed in two directions. Firstly we construct, for a simple non homogeneous state, original series formulas and a new Lipmann-Schwinger integral equation. A necessary structural condition shows up for the principal values to make sense. One needs that the time needed to travel along characteristics must be monotone function. When applied to the case of study, one obtains the structural condition which writes:

the function  $h$  is strictly concave

$$h(x) = \sqrt{\varphi_+ - \varphi(x)}, \quad \varphi_+ = \max_I \varphi(x). \quad (1.9)$$

Additional technical difficulties associated to the integral equation in the general case are left for further research. Secondly we show that, in the homogeneous case, the Moller wave operator is exactly the integral transform of Morrison, which can be used to obtain an elegant proof of linear Landau damping [24, 25, 6].

The plan of this work is as follows. Firstly we linearize the Vlasov equation around a general family of Boltzmannian stationary states. It gives the definition of  $iH$ ,  $iH_0$  and  $iK$ . The KvN setting is used to construct the generalized eigenvectors of  $iH_0$  and we prove that the completeness of the construction. The next section is devoted to the construction of the eigenvectors  $V_\lambda$ . It yields an new equation of Lipmann-Schwinger type. For homogeneous states the solution is trivial and it gives the Moller wave operator which is exactly the Morrison transform. We indicate some open problems related to the solution of this integral equation.

**2. Vlasov equation.** We provide elementary considerations on linearized Vlasov equations which justify (1.6).

**2.1. Linearization.** The decomposition

$$f(t, x, v) = f_0(x, v) + g(t, x, v) \text{ and } E(t, x) = E_0(x) + F(t, x)$$

injected in (1.3) yields, after dropping the quadratic terms, the linear system

$$\begin{cases} \partial_t g + v\partial_x g - E_0(x)\partial_v g + Fvn_0(x)G(v) = 0, & t > 0, \quad (x, v) \in I \times \mathbb{R}, \\ \partial_t F = \int_{\mathbb{R}} vgdv, & t > 0, \quad x \in I. \end{cases} \quad (2.1)$$

The Gauss relation reads  $\partial_x F = -\int_{\mathbb{R}} gdv$ . This equation is endowed with an important weighted  $L^2$  conservation property

$$\frac{d}{dt} \left( \int_I \int_{\mathbb{R}} \frac{g^2}{n_0 G} dv dx + \int_I E^2 dx \right) = 0 \quad (2.2)$$

which can be traced back to the early works of Kruzkal-Obermann and Antonov. Let us define

$$M(x, v) = \sqrt{n_0(x)G(v)} = \exp \left( -\frac{v^2}{4} + \frac{\varphi(x)}{2} \right). \quad (2.3)$$

and the function  $u = \frac{g}{M}$ . Using that  $(v\partial_x - E_0\partial_v)M = 0$ , one gets the linear system

$$\begin{cases} \partial_t u + v\partial_x u - E_0\partial_v u = -vMF, & t > 0, \quad (x, v) \in I \times \mathbb{R}, \\ \partial_t F = \int_{\mathbb{R}} uvMdv, & t > 0, \quad x \in I, \end{cases} \quad (2.4)$$

with the energy identity

$$\frac{d}{dt} \left( \int_I \int_{\mathbb{R}} u^2 dv dx + \int_I F^2 dx \right) = 0. \quad (2.5)$$

The Gauss law

$$\partial_x F = -\int_{\mathbb{R}} uMdv \quad (2.6)$$

is propagated by the equations. The system (2.4) shows that one can recast the linear Vlasov-Ampere equation (2.4) as  $\partial_t U(t) = iHU(t)$  where the unknown is

$$U = \begin{pmatrix} u \\ F \end{pmatrix} \in V = L^2(I \times \mathbb{R}) \times L^2(I)$$

and the operator writes

$$iH = \left( \begin{array}{c|c} -v\partial_x + E_0\partial_v & -vM \\ \int_{\mathbb{R}} \cdot vMdv & 0 \end{array} \right).$$

Since this operator is (formally) anti-hermitian, it shows that  $H$  is (formally) hermitian. The space  $V$  is endowed with the natural real scalar product

$$(U_1, U_2) = \int_I \int_{\mathbb{R}} u_1 u_2 dx dv + \int_I F_1 F_2 dx, \quad U_1 = \begin{pmatrix} u_1 \\ F_1 \end{pmatrix}, \quad U_2 = \begin{pmatrix} u_2 \\ F_2 \end{pmatrix} \in V.$$

The quadratic norm is

$$\|U\| = \left( \int_I \int_{\mathbb{R}} u^2 dx dv + \int_I F^2 dx \right)^{\frac{1}{2}}.$$

**2.2. Reduction.** We show that a reduction method is possible which allows to work in the reduced space  $L^2(I \times \mathbb{R})$  instead of  $V = L^2(I \times \mathbb{R}) \times L^2(I)$ . This is perhaps more a curiosity but we indicate it since the algebra is interesting.

The idea is to find a weight  $\gamma(x)$  and a new function

$$w = u + \gamma(x)M(x, v)F \quad (2.7)$$

such that

$$\int_I \int_{\mathbb{R}} w^2 dv dx = \int_I \int_{\mathbb{R}} u^2 dv dx + \int_I F^2. \quad (2.8)$$

The electric potential  $\varphi$  in the next formula is the periodic function defined in (1.5).

PROPOSITION 1. *Assume the 1-periodic function  $\gamma$  is solution to the equation*

$$\partial_x \gamma + \alpha^2 \gamma^2 \exp \varphi = 1.$$

*Then the identity (2.8) holds.*

*Proof.* By definition (2.3), one has that  $M^2 = \exp\left(-\frac{v^2}{2} + \varphi(x)\right)$ . So  $\int_{\mathbb{R}} M^2(x, v) dv = \alpha^2 \exp(\varphi(x))$ . Therefore

$$\begin{aligned} \int_I \int_{\mathbb{R}} w^2 dv dx &= \int_I \int_{\mathbb{R}} u^2 dv dx + 2 \int_I \gamma F \int u M dv dx + \alpha^2 \int_I \gamma^2 F^2 \exp \varphi dx \\ &= \int_I \int_{\mathbb{R}} u^2 dv dx + 2 \int_I \gamma F (-\partial_x F) dx + \alpha^2 \int_I \gamma^2 F^2 \exp \varphi dx \\ &= \int_I \int_{\mathbb{R}} u^2 dv dx + \int_I \partial_x \gamma F^2 dx + \alpha^2 \int_I \gamma^2 F^2 \exp \varphi dx \end{aligned}$$

from which the result proceeds.  $\square$

Set  $g = \frac{1}{\gamma}$  which a formal solution of the Riccati equation

$$g'(x) + g(x)^2 = \alpha^2 \exp(\varphi(x)).$$

PROPOSITION 2. *There exists one unique 1-periodic solution  $g = \frac{1}{\gamma}$  of the Riccati equation. It is a positive function.*

*Proof.* The proof of the existence is by a shooting method. We consider the solution given by the Cauchy-Lipshitz theorem of the equation in the interval  $(0, 1)$

$$\begin{cases} g'_a(x) = \alpha^2 \exp(\varphi(x)) - g_a(x)^2, & 0 < x \leq 1, \\ g_a(0) = a \end{cases}$$

and define the function  $H(a) = g_a(1)$ .

Define  $K = \alpha \max_{x \in I} \exp(\varphi(x)/2) + 1$ . We claim that the trajectories are well defined for  $a \in [0, K]$  and that  $g(x) \in [0, K]$  for all  $x$ : this is actually trivial since if  $g(x) = 0$  then  $g'(x) > 0$ , and if  $g(x) = K$  then  $g'(x) < 0$ . Therefore  $H[0, K] \subset [0, K]$ . Since  $H$  is a continuous function, there exists  $b \in [0, K]$  such that  $H(b) = b$ . Therefore the trajectory such that  $g(0) = b$  is periodic. Of course  $b$  cannot be equal to zero, nor to  $K$ . So this trajectory is globally positive.

Assume there exists another solution  $\tilde{g}$  (without any sign condition). Note  $z = g - \tilde{g}$ . As usual for Riccati equations, one gets a simpler equation for the difference

$$z' + 2gz + z^2 = 0.$$

$$\left( \exp\left(2 \int_0^x g(y) dy\right) z(x) \right)' = -z^2 \exp\left(2 \int_0^x g(y) dy\right).$$

The derivative on the left hand side is 1-periodic function. So after integration

$$- \int_0^1 z^2 \exp\left(2 \int_0^x g(y) dy\right) dx = 0.$$

Therefore  $z = 0$  which yields the uniqueness and ends the proof.  $\square$

Now that  $w$  is defined from  $u$  and  $F$ , we desire to show that the sole knowledge of  $w$  is enough to reconstruct  $u$  and  $F$ . Indeed the integration of the identity (2.7) (multiplied by  $M$ ) in the velocity direction yields

$$-\int_{\mathbb{R}} wM(x, v)dv = -\int_{\mathbb{R}} uM(x, v)dv - \gamma\alpha^2 \exp(\varphi(x)) F = \partial_x F - \alpha^2 \gamma(x) \exp(\varphi(x)) F$$

where we use the Gauss law (2.6). One obtains an equation for  $F$

$$\partial_x F + \left( \frac{\partial_x \gamma}{\gamma} - \frac{1}{\gamma} \right) F = -\int_{\mathbb{R}} wM(x, v)dv$$

or

$$\partial_x(\gamma F) - \partial_x F = -\gamma(x) \int_{\mathbb{R}} wM(x, v)dv.$$

Set  $G = \gamma F$  so that

$$\partial_x G - \frac{1}{\gamma} \partial_x G = -\gamma(x) \int_{\mathbb{R}} wM(x, v)dv$$

and

$$\partial_x \left( \exp \left( -\int_0^x \frac{dy}{\gamma(y)} \right) G \right) = -\exp \left( -\int_0^x \frac{dy}{\gamma(y)} \right) \gamma(x) \int_{\mathbb{R}} wM(x, v)dv. \quad (2.9)$$

**PROPOSITION 3.**  *$F$  and  $u$  can be reconstructed from  $w$ .*

*Proof.* It is sufficient to show that the equation (2.9) is solvable in the space of periodic functions. By integration in  $I$  one gets

$$\left( \exp \left( -\int_0^1 \frac{dy}{\gamma(y)} \right) - 1 \right) G(0) = -\int_0^1 \left( \exp \left( -\int_0^x \frac{dy}{\gamma(y)} \right) \gamma(x) \int_{\mathbb{R}} wM(x, v)dv \right) dx.$$

Since  $\exp \left( -\int_0^1 \frac{dy}{\gamma(y)} \right) < 1$  as a consequence of  $\gamma > 0$ , the Cauchy data  $G(0)$  is uniquely determined, which is enough to construct the function in the entire interval with the Cauchy-Lipshitz theorem. Therefore  $F$  is known from  $w$ . After that  $u = w - \gamma(x)M(x, v)F$ . The proof is ended.  $\square$

Since the system (2.4) preserves the quadratic norm of the pair  $(u, F)$  which is equal to the quadratic norm of  $w$ , it is not surprising  $w$  is the solution of an homogeneous equation. Define  $\mathcal{H} = \mathcal{H}^*$  by

$$-i\mathcal{H}w = Dw + \gamma(x) \left( vM(x, v) \int wM(x, v)dv - M(x, v) \int wvM(x, v)dv \right)$$

with  $D = \partial_x v - E_0(x)\partial_v$ .

**PROPOSITION 4.** *The function  $w$  is solution of  $\partial_t w = i\mathcal{H}w$ .*

*Proof.* Indeed

$$\begin{aligned} \partial_t w + Dw &= \partial_t u + Du + \gamma M(x, v) \int wMdv + D(\gamma(x)FM(x, v)) \\ &= -vMF + \gamma(x)M(x, v) \int wvMdv + MD(\gamma(x)F) \\ &= -vMF + \gamma M(x, v) \int wvMdv - \gamma M(x, v) \int (\gamma(x)FM)vMdv + Mv\partial_x(\gamma(x)F). \end{aligned}$$

One has that  $\int_{\mathbb{R}} vM^2(x, v)dv = 0$ . So

$$\partial_t w + Dw = \gamma M(x, v) \int wvMdv - vMF + Mv\gamma(x)\partial_x F + MvF\partial_x \gamma$$

$$\begin{aligned}
&= \gamma M(x, v) \int wvMdv - vMF - Mv\gamma \int uMdv + MvF(1 - \alpha^2\gamma^2 \exp(\varphi)) \\
&= \gamma M(x, v) \int wvMdv - Mv\gamma \int uMdv - MvF\alpha^2\gamma^2 \exp(\varphi) \\
&= \gamma M(x, v) \int wvMdv - \gamma vM \int wMdv + \gamma vM \int (\gamma MF)Mdv - MvF\alpha^2\gamma^2 \exp(\varphi) \\
&= \gamma M(x, v) \int wvMdv - \gamma vM \int wMdv.
\end{aligned}$$

The proof is ended.  $\square$

**3. The KvN setting.** The scattering method exposed in the introduction is based on the construction of the eigenstructure of  $H_0$ . Since we desire to obtain a rigorous decomposition, we decide to construct a convenient basis of the space  $L^2(I \times \mathbb{R})$  with moments against Hermite functions, as proposed in [12, 6]. Moreover it appears that the equation which defines the generalized eigenvectors of  $v\partial_x - E_0(x)\partial_v$  can more easily be defined if one has in mind the structure of the characteristic lines of the transport operator because the generalized eigenvectors can be interpreted as some Dirac masses which move along the characteristic lines. This approach, which makes an explicit connection between a transport/Liouville equation and the underlying Hilbert structure, is called the KvN approach, giving full credit to a formalization of quantum mechanics achieved by Koopman [16] and von Neuman [30, 31]. This approach is sometimes referred to as Koopmanism.

**3.1. Notations.** We introduce the Hermite polynomials  $H_n(v)$  which are orthonormal with respect to the Maxwellian weight  $G(v)$ , see [1, 12]. The Hermite polynomials are  $H_n(v) = (-1)^n G(v)^{-1} \frac{d^n}{dv^n} G(v)$ . The degree of  $H_n$  is  $n$ . The parity of  $H_n$  is the parity of  $n$ . Hermite polynomials are orthogonal with respect to the Maxwellian weight

$$\int H_n(v)H_m(v)G(v)dv = (2\pi)^{\frac{1}{2}}n! \delta_{nm}, \quad n, m \in \mathbb{N}.$$

One has the recursion formula  $H_{n+1}(v) = vH_n(v) - H_{n-1}(v)$ . The family of Hermite polynomials is a Hilbert basis of the space of functions such that  $\int_{\mathbb{R}} f^2(v)G(v)dv < \infty$ . Define for convenience  $I_n(v) = (2\pi)^{-\frac{1}{4}}n!^{-\frac{1}{2}}H_n(v)$  and the Hermite functions which constitute a Hilbert basis of  $L^2(\mathbb{R})$

$$\psi_n(v) = I_n(v)G^{\frac{1}{2}}(v). \quad (3.1)$$

The family  $(\psi_n)_{n \in \mathbb{N}}$  is by construction orthonormal:  $\int_{\mathbb{R}} \psi_p(v)\psi_q(v)dv = \delta_{pq}$ . The first terms of the series are

$$\psi_0(v) = \frac{G^{\frac{1}{2}}(v)}{\alpha}, \quad \psi_1(v) = \frac{vG^{\frac{1}{2}}(v)}{\alpha}, \quad \psi_2(v) = \frac{(v^2 - 1)G^{\frac{1}{2}}(v)}{\alpha\sqrt{2}}, \quad \alpha = (2\pi)^{\frac{1}{4}}. \quad (3.2)$$

The recursion formula becomes after rescaling

$$v\psi_n(v) = \sqrt{n+1}\psi_{n+1}(v) + \sqrt{n}\psi_{n-1}(v), \quad n \in \mathbb{N}. \quad (3.3)$$

Another fundamental relation writes

$$\psi'_n(v) = \frac{1}{2}(-\sqrt{n+1}\psi_{n+1}(v) + \sqrt{n}\psi_{n-1}(v)), \quad n \in \mathbb{N}. \quad (3.4)$$

The system (2.4) is therefore rewritten as

$$\begin{cases} \partial_t u + v\partial_x u - E_0\partial_v u &= -\alpha n_0(x)\psi_1(v)F, & t > 0, & (x, v) \in I \times \mathbb{R}, \\ \partial_t F &= \alpha n_0(x) \int_{\mathbb{R}} \psi_1(v)u dv, & t > 0, & x \in I. \end{cases} \quad (3.5)$$

Assuming in view of the energy identity that  $u(t) \in L^2(I \times \mathbb{R})$ , we define the moments  $\alpha_n(t) \in L^2(I)$  by

$$u(t, x, v) = \sum_n \alpha_n(t, x)\psi_n(v), \quad \alpha_n = \int_{\mathbb{R}} u\psi_n dv.$$

By construction  $\|u\|_{L^2(I \times \mathbb{R})}^2 = \sum_{n \in \mathbb{N}} \|u_n\|_{L^2(I)}^2$ . We construct the vector

$$U(t, \cdot) = (F(t, \cdot), u_0(t, \cdot), u_1(t, \cdot), u_2(t, \cdot), \dots)^t \quad (3.6)$$

and the matrices

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & 2^{\frac{1}{2}} & 0 & \dots \\ 0 & 0 & 2^{\frac{1}{2}} & 0 & 3^{\frac{1}{2}} & 0 \\ 0 & 0 & 0 & 3^{\frac{1}{2}} & 0 & 4^{\frac{1}{2}} \\ \dots & \dots & \dots & \dots & 4^{\frac{1}{2}} & \dots \end{pmatrix} = A^t, \quad (3.7)$$

$$B = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & -1 & 0 & 2^{\frac{1}{2}} & 0 & \dots \\ 0 & 0 & -2^{\frac{1}{2}} & 0 & 3^{\frac{1}{2}} & 0 \\ 0 & 0 & 0 & -3^{\frac{1}{2}} & 0 & 4^{\frac{1}{2}} \\ \dots & \dots & \dots & \dots & -4^{\frac{1}{2}} & \dots \end{pmatrix} = -B^t, \quad (3.8)$$

and

$$D = \begin{pmatrix} 0 & 0 & -\alpha & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ \alpha & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} = -D^t. \quad (3.9)$$

With these notations (3.5) is recast as

$$\partial_t U = iHU, \quad iH = i(H_0 + K) \quad (3.10)$$

where  $iH_0 = -A\partial_x U + E_0(x)BU$  and  $iKU = \sqrt{n_0(x)}DU$ . The matrix  $K$  has a finite rank, and so is a compact perturbation of  $H_0$ . The Hilbert space that will be used from now on is

$$V = \left\{ U = (\alpha_n)_{n \in \mathbb{N}} \in L^2(I)^{\mathbb{N}}, \sum_{n \in \mathbb{N}} \|\alpha_n\|_{L^2(I)}^2 < \infty \right\}. \quad (3.11)$$

The norm in  $V$  is

$$\|U\|^2 = \sum_n \int_I |\alpha_n|^2(x) dx.$$

We will consider that  $\alpha_n(x) \in \mathbb{C}$  as well, since it is convenient for Fourier decompositions. The hermitian product is

$$(U^1, U^2) = \sum_n \int_I \alpha_n^1(x) \overline{\alpha_n^2(x)} dx, \quad U^1, U^2 \in V.$$

A larger space is

$$V_{\text{loc}} = L^2(I)^{\mathbb{N}}. \quad (3.12)$$

These two spaces are functional extensions in  $L^2(I)$  of the classical Hilbert space

$$l^2 = \left\{ U = (\alpha_n)_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}, \sum_{n \in \mathbb{N}} \|\alpha_n\|^2 < \infty \right\}.$$

The notation for the hermitian product in  $l^2$  will be

$$U^1 \cdot U^2 = \sum_n \alpha_n \overline{\alpha_n^2}, \quad U^1, U^2 \in l^2.$$

We also define

$$W_0 = \{U = (\alpha_n)_{n \in \mathbb{N}} \in V, \text{ there exists } N > 0 \text{ such that} \\ \alpha_n \in H^1(I) \text{ for } n \leq N \text{ and } \alpha_n = 0 \text{ for } n > N\}.$$

The space  $W_0$  is dense in  $V$ , and its elements have compact support with respect to the index  $n$ . Note that

$$W_0 \subset V \subset V_{\text{loc}}.$$

We begin the construction of the eigenstructure by simple considerations on the operator/matrix  $A$  which can be viewed as an unbounded hermitian operator in  $l^2$ . So it admits a canonical eigenstructure decomposition. It can be detailed with the help of the Hermite functions. We will use the notation that

$$e_p = (0, \dots, 0, 1, 0, \dots)^t, \quad p = 0, 1, \dots,$$

where the coefficient 1 is in the  $p$ th position. We say that  $W \in \mathbb{R}^{\mathbb{N}}$  is a generalized eigenvector of  $A$  if it satisfies the equation

$$AW = \mu W, \quad \mu \in \mathbb{R}.$$

If  $W \in l_2$ , then one says that  $W$  is a classical eigenvector: in this case  $\mu$  is in the discrete spectrum.

REMARK 5. *To make no confusion, the standard notation  $\lambda$  is reserved for the eigenvalues of  $iH$  and  $iH_0$ .*

PROPOSITION 6. *There is one eigenvector  $e_0$  associated to the eigenvalue 0 ( $Ae_0 = 0$ ). For all  $\mu \in \mathbb{R}$ ,  $U_\mu$  is in the continuous spectrum*

$$U_\mu = (0, \psi_0(\mu), \psi_1(\mu), \psi_2(\mu), \dots)^t, \quad AU_\mu = \mu U_\mu, \quad (3.13)$$

*Proof.* Evident from the recurrence relation (3.3) for Hermite functions.  $\square$

PROPOSITION 7. *These eigenvectors are complete: for all  $U \in l^2$ , one has the representation formula*

$$U = U \cdot e_0 e_0 + \int_{\mathbb{R}} U \cdot U_\mu U_\mu d\mu$$

with

$$U \cdot U = |U \cdot e_0|^2 + \int_{\mathbb{R}} |U \cdot U_\mu|^2 d\mu. \quad (3.14)$$

*Proof.* The formulas come from the fact that the family of Hermite functions  $(\psi_n(\mu))_{n \in \mathbb{N}}$  is an Hilbert basis of  $L^2(\mathbb{R})$ .  $\square$

So the discrete spectrum is  $S_d(A) = \{0\}$  and the continuous spectrum is  $S_c(A) = \mathbb{R}$ . The goal is now to extended the results to the operator  $v\partial_x + \varphi'(x)\partial_v$  viewed as an unbounded Hermitian operator in  $V$ .

**3.2. Case of study.** To more specific, we consider the case of study described in figure 3.1 where the characteristic lines of the operator  $v\partial_x + \varphi'(x)\partial_v = v\partial_x - E_0(x)\partial_v$  are plotted. We make an additional assumption which will be reinforced later.

ASSUMPTION 8. *The electric potential is a 1-periodic function, is smooth  $\varphi \in C^2(I)_{\text{per}}$ , and is monotone*

$$\begin{cases} \varphi'(x) \geq 0 \text{ for } 0 < x < x_0, \\ \varphi'(x) \leq 0 \text{ for } x_0 < x < 1. \end{cases} \quad (3.15)$$

Moreover we normalize by translation in the  $x$  direction, so that

$$\varphi_- = \min_x \varphi = \varphi(0).$$

The structure of the characteristic lines is depicted in figure 3.1 where the separatrix between these zones are the curves  $\frac{v^2}{2} - \varphi(x) = -\varphi_-$ . Fo an electric potential with a more complex structure, the internal zone might be decomposed into any finite number of connected subdomains, such that a set of closed characteristic is defined in each of these subdomains. But, for the simplicity of the presentation, we focus only on the situation depicted in the graphics.

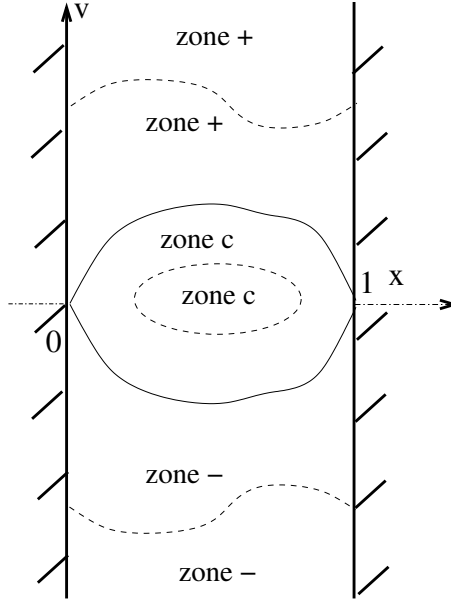


FIG. 3.1. Case of study: the characteristics in dashed split the domain in three zones. Zone 'z = +' corresponds to  $\{\frac{v^2}{2} - \varphi(x) > -\varphi_-\} \cap \{v > 0\}$ . Zone 'z = -' is the symmetric  $\{\frac{v^2}{2} - \varphi(x) > -\varphi_-\} \cap \{v < 0\}$ . Finally central zone 'z = c' corresponds to the closed loops that cross the horizontal axis. The separatrix between these zones are the curves  $\frac{v^2}{2} - \varphi(x) = -\varphi_-$ .

**3.3. Eigenstructure of  $iH_0 = -A\partial_x + E_0(x)B$ .** This is the next step in the construction. We will show that the KvN philosophy is exactly what is needed to obtain an explicit representation of the generalized eigenstructure of the operator

$$iH_0 = -A\partial_x + E_0(x)B$$

in relation with a convenient characteristic equation.

- The first line of the infinite matrix  $iH_0$  vanishes. So any function of the form

$$u_0(\cdot) = w(\cdot)e_0 \in V, \quad w \in L^2(I), \quad (3.16)$$

is an eigenvector with eigenvalue 0 which is therefore in the discrete spectrum.

- To construct generalized eigenvectors of  $iH_0$ , we start from the formula

$$U(x) = \tau(x)U_{\mu(x)} = \tau(x)(0, \psi_0(\mu(x)), \psi_1(\mu(x)), \psi_2(\mu(x)), \dots)^t$$

and try to find some functions  $x \mapsto \tau(x)$  and  $x \mapsto \mu(x)$  so that  $iH_0U = \lambda U$  for a given  $\lambda \in i\mathbb{R}$ . Such a representation comes from a direct interpretation of a Dirac mass

$$u(x, v) = \tau(x)\delta(v - \mu(x))$$

in terms of a moment representation: indeed the moments of  $u$  are

$$\int_{\mathbb{R}} u(x, v)\psi_n(v)dv = \tau(x)\psi_n(\mu(x)).$$

Therefore the equation  $iH_0U = \lambda U$  can be interpreted as a solution of  $v\partial_x u - E_0(x)u = 0$ . Geometrically it means that  $u$  is a Dirac mass at a varying vertical position

$$v = \mu(x), \quad (3.17)$$

with a weight  $\tau(x)$  that accounts for the local curvature of the characteristic line. This is the KvN methodology. The algebra is detailed in the next proposition which focuses on the construction in a small interval  $(x_1, x_2) \in I$ .

PROPOSITION 9. Assume that

$$-(\tau(x)\mu(x))' = \lambda\tau(x), \quad x_1 < x < x_2 \quad (3.18)$$

with the characteristic equation

$$\frac{1}{2}\mu(x)^2 - \varphi(x) = e \in \mathbb{R}, \quad x_1 < x < x_2. \quad (3.19)$$

Then  $U_\lambda(x) = \tau(x)U_{\mu(x)}$  is solution of

$$(iH_0U_\lambda)(x) = \lambda U_\lambda(x) \quad x_1 < x < x_2. \quad (3.20)$$

*Proof.* By construction  $AU(x) = \mu(x)U(x)$ . So

$$\begin{aligned} iH_0U_\lambda(x) &= -\partial_x(\tau(x)\mu(x)U_{\mu(x)}) + E_0(x)\tau(x)BU_{\mu(x)} \\ &= -(\tau(x)\mu(x))'U_{\mu(x)} - \tau(x)\mu(x)\mu'(x)(\partial_\lambda U_\lambda)_{\lambda=\mu(x)} + E_0(x)\tau(x)BU_{\mu(x)}. \end{aligned}$$

One checks by using (3.4) that  $\partial_\lambda U_\lambda = -BU_\lambda$ . So

$$\begin{aligned} iH_0U_\lambda(x) &= -(\tau(x)\mu(x))'U_{\mu(x)} + \tau(x)[\mu(x)\mu'(x) + E_0(x)]BU_{\mu(x)} \\ &= \lambda\tau(x)U_{\mu(x)} + \tau(x)\left(\frac{1}{2}\mu(x)^2 - \varphi(x)\right)U_{\mu(x)} = \lambda U_\lambda(x), \quad x_1 < x < x_2. \end{aligned}$$

The proof is ended.  $\square$

The next steps is to extend the construction in the entire interval  $I$  with a correct treatment of the periodic boundary condition, and to verify the integrability condition  $U_\lambda \in V_{\text{loc}}$ . At inspection of the figure 3.1 one distinguishes three zones in the construction/verification.

**3.3.1. Zone +.** Consider zone + where the characteristics lines are mono-valued functions of the  $x$  variable. In view of (3.17-3.19) it corresponds to  $e > -\varphi_-$  and positive velocities. Then  $\mu$  in (3.19) is directly calculated: we write it with an index  $e$  to make clear the dependence with respect to the height of the characteristics

$$\mu_e(x) = \sqrt{2(e + \varphi(x))} \geq \sqrt{2(e + \varphi_-)} > 0. \quad (3.21)$$

It is a positive and periodic function in zone  $I$ . Let us define the new variable

$$y(x) = \int_0^x \frac{ds}{\mu_e(s)}.$$

Since  $\mu_e$  is the dimension of a velocity in view of (3.17), then  $y$  has the dimension of a time. It is therefore the time variable spent by a particle which travels along the characteristics. The total time spent by the particle along the characteristic is

$$t_e = \int_0^1 \mu_e(s)^{-1} ds > 0. \quad (3.22)$$

Equipped with these notations, one can now find the solutions of equation (3.18-3.19).

PROPOSITION 10. *In zone +, the periodic solutions of (3.18-3.19) are given by (3.21) and*

$$\tau_{e,k}(x) = \frac{1}{t_e \mu_e(x)} \exp\left(2i\pi k \frac{\int_0^x \mu_e(s)^{-1} ds}{t_e}\right). \quad (3.23)$$

REMARK 11. *The scaling  $\frac{1}{t_e \mu_e(x)}$  is convenient since it is equal to 1 if  $\varphi$  vanishes. In all cases,  $\int_0^1 \frac{dx}{t_e \mu_e(x)} = 1$ . If  $\varphi \equiv 0$ , then  $\frac{\int_0^x \mu_e(s)^{-1} ds}{t_e} = x$  for all  $e$ : in this case the function  $\exp\left(2i\pi k \frac{\int_0^x \mu_e(s)^{-1} ds}{t_e}\right) = \exp(2i\pi k x)$  is the usual Fourier mode.*

*Proof.* Equation (3.19) has already been discussed. Equation (3.18) is  $-\mu(\tau\mu_e)'(x) = \lambda(\tau\mu_e)$  that is with the time variable

$$-\frac{d}{dy}(\tau\mu_e) = \lambda(\tau\mu_e).$$

The general solution is an exponential. Since we look for periodic functions, one obtains

$$(\tau\mu_e)(y) = \frac{1}{L} \exp(2i\pi ky/y(1)), \quad \lambda = -2i\pi k/y(1)$$

where  $k \in \mathbb{Z}$  and  $L$  is an arbitrary factor. Taking  $L = t_e^+$  and going back to the original variable, the proof of the claim is ended.  $\square$

One obtains a first family of generalized eigenvector as

$$U_{e,k}^+(x) = \frac{1}{t_e \mu_e(x)} \exp\left(2i\pi k \frac{\int_0^x \mu_e(s)^{-1} ds}{t_e}\right) U_{\mu_e(x)}. \quad (3.24)$$

PROPOSITION 12. *The infinite vectorial function  $U_{e,k}^+$  is in  $V_{\text{loc}}$ , and is a generalized eigenvector*

$$(iH_0)U_{e,k}^+ = \lambda_{e,k}U_{e,k}^+, \quad \lambda_{e,k} = -\frac{2i\pi k}{t_e} \in i\mathbb{R}.$$

PROPOSITION 13. *In the interval  $(-\varphi_-, \infty)$  the function  $e \mapsto t_e$  is monotone decreasing from  $t_{(-\varphi_-)^+}$  to 0.*

*Proof.* One has by definition  $t_e = \int_0^1 \frac{dx}{\sqrt{2(\varepsilon + \varphi(x))}}$ . So  $\frac{d}{d\varepsilon} t_e = -\int_0^1 \frac{dx}{(2(\varepsilon + \varphi(x)))^{\frac{3}{2}}} < 0$ . The limits are evident. The proof is ended.  $\square$

PROPOSITION 14. *Assume  $\varphi''(0) > 0$  and  $\varphi \in C^3(I)_{\text{per}}$ . Then  $t_{(-\varphi_-)^+} = +\infty$  and there exists  $C_\varphi \in \mathbb{R}$  such that*

$$t_{-\varphi_- + \varepsilon} + \frac{1}{\sqrt{\varphi''(0)}} \log \varepsilon + C_\varphi = O(\varepsilon). \quad (3.25)$$

*Proof.* One has  $t_{-\varphi_- + \varepsilon} = \int_0^1 \frac{dz}{\sqrt{2(\varepsilon + \varphi(z) - \varphi_-)}}$ . Consider first  $A = \int_0^{\frac{1}{2}} \frac{dz}{\sqrt{2(\varepsilon + \varphi(z) - \varphi_-)}}$ . One has

$$A = \underbrace{\int_0^{\frac{1}{2}} \frac{dz}{\sqrt{2(\varepsilon + \frac{1}{2}\varphi''(0)z^2)}}}_{=A_1} + \underbrace{\int_0^1 \left( \frac{1}{\sqrt{2(\varepsilon + \varphi(z) - \varphi_-)}} - \frac{1}{\sqrt{2(\varepsilon + \frac{1}{2}\varphi''(0)z^2)}} \right) dz}_{=A_2}.$$

The first term is

$$\begin{aligned} A_1 &= \int_0^{\frac{1}{2}} \frac{dz}{\sqrt{2(\varepsilon + \frac{1}{2}\varphi''(0)z^2)}} = \frac{1}{\sqrt{\varphi''(0)}} \int_0^{1/(2\sqrt{2\varepsilon/\varphi''(0)})} \frac{dy}{\sqrt{1+y^2}} \\ &= \frac{1}{\sqrt{\varphi''(0)}} \int_0^1 \frac{dy}{\sqrt{1+y^2}} + \frac{1}{\sqrt{\varphi''(0)}} \int_1^{1/(2\sqrt{2\varepsilon/\varphi''(0)})} \left( \frac{1}{\sqrt{1+y^2}} - \frac{1}{y} \right) dy + \frac{1}{\sqrt{\varphi''(0)}} \log \left( 1/(2\sqrt{2\varepsilon/\varphi''(0)}) \right). \end{aligned}$$

At inspection of the various terms,  $A_1$  is such that  $A_1 = \frac{1}{\sqrt{\varphi''(0)}} \left(-\frac{1}{2} \log \varepsilon\right) + q_1(\varepsilon)$  where  $q_1$  is a function which admits a finite limit when  $\varepsilon \Rightarrow 0^+$ . More precisely the second integral is convergent so the reminder can be bounded

$$\begin{aligned} &\frac{1}{\sqrt{\varphi''(0)}} \int_1^{1/(2\sqrt{2\varepsilon/\varphi''(0)})} \left( \frac{1}{\sqrt{1+y^2}} - \frac{1}{y} \right) dy \\ &= \frac{1}{\sqrt{\varphi''(0)}} \int_1^\infty \left( \frac{1}{\sqrt{1+y^2}} - \frac{1}{y} \right) dy - \frac{1}{\sqrt{\varphi''(0)}} \int_{1/(2\sqrt{2\varepsilon/\varphi''(0)})}^\infty \left( \frac{1}{\sqrt{1+y^2}} - \frac{1}{y} \right) dy. \end{aligned}$$

Since  $\left| \frac{1}{\sqrt{1+y^2}} - \frac{1}{y} \right| = \frac{1}{y\sqrt{1+y^2}(y+\sqrt{1+y^2})} \leq \frac{1}{2y^3}$ , the last integral is bounded as

$$\left| \int_{1/(2\sqrt{2\varepsilon/\varphi''(0)})}^\infty \left( \frac{1}{\sqrt{1+y^2}} - \frac{1}{y} \right) dy \right| \leq (2\sqrt{2\varepsilon/\varphi''(0)})^2 = O(\varepsilon).$$

The second term admits a limit since the function under the integral is

$$F_\varepsilon(z) = \frac{1}{\sqrt{2(\varepsilon + \varphi(z) - \varphi_-)}} - \frac{1}{\sqrt{2(\varepsilon + \frac{1}{2}\varphi''(0)z^2)}} \\ = \frac{2\left(\frac{1}{2}\varphi''(0)z^2 - \varphi(z) + \varphi_-\right)}{\sqrt{2(\varepsilon + \varphi(z) - \varphi_-)}\sqrt{2(\varepsilon + \frac{1}{2}\varphi''(0)z^2)}\left(\sqrt{2(\varepsilon + \varphi(z) - \varphi_-)} - \sqrt{2(\varepsilon + \frac{1}{2}\varphi''(0)z^2)}\right)}.$$

The numerator is  $\frac{1}{2}\varphi''(0)z^2 - \varphi(z) + \varphi_- = \frac{1}{2}\varphi''(0)z^2 - \varphi(z) + \varphi(0) = O(z^3)$ . Since the denominator is also  $O(z^3)$  for  $\varepsilon = 0$ , the limit  $F_0$  is finite. Therefore one can write  $A_2 = q_2(\varepsilon)$  where  $q_2$  is a function which admits a finite limit when  $\varepsilon \Rightarrow 0^+$ . More precisely  $q_2(\varepsilon) = q_2(0) + O(\varepsilon)$ .

But symmetry the other term  $A_1 = \int_{\frac{1}{2}}^1 \frac{dz}{\sqrt{2(\varepsilon + \frac{1}{2}\varphi''(0)z^2)}}$  has a similar behavior. The proof is ended after summation of all contributions to  $A$ .  $\square$

**3.3.2. Zone -.** The construction is the same, it is sufficient to take the negative root of the characteristic equation (3.19), still with the restriction  $e > -\varphi_-$ . For simplicity of notations, we keep the same positive definition (3.21) for  $\mu_e$ . The family writes

$$U_{e,k}^-(x) = \frac{1}{t_e \mu_e(x)} \exp\left(-2i\pi k \frac{\int_0^x \mu_e(s)^{-1} ds}{t_e}\right) U_{-\mu_e(x)}. \quad (3.26)$$

One obtains readily.

PROPOSITION 15. *The infinite vectorial function  $U_{e,k}^-$  is in  $V_{\text{loc}}$ , and is a generalized eigenvector*

$$(iH_0)U_{e,k}^- = \lambda_{e,k}U_{e,k}^-, \quad \lambda_{e,k} = -\frac{2i\pi k}{t_e}.$$

*Proof.* Just consider (3.18) is now written as

$$-\left(\frac{1}{t_e \mu_e(x)} \exp\left(-2i\pi k \frac{\int_0^x \mu_e(s)^{-1} ds}{t_e}\right) (-\mu_e(x))\right)' = -\frac{2i\pi k}{t_e} \left(\frac{1}{t_e \mu_e(x)} \exp\left(-2i\pi k \frac{\int_0^x \mu_e(s)^{-1} ds}{t_e}\right)\right).$$

$\square$

**3.3.3. Zone c.** In view of the fact that the characteristic lines are closed (see figure 3.1), a modification is needed to define correctly the generalized eigenvectors in the central zone.

Firstly one must restrict the range

$$e \in (-\varphi_-, -\varphi_+), \quad \varphi_- = \min_{x \in I} \varphi(x), \quad \varphi_+ = \max_{x \in I} \varphi(x). \quad (3.27)$$

We look for eigenvectors under the form

$$U(x) = \tau^+(x)U_{\mu(x)} - \tau^-(x)U_{-\mu(x)} \quad (3.28)$$

where

$$\mu_e(x) = \sqrt{2 \max(e + \varphi(x), 0)} \geq 0$$

and  $\tau^\pm$  are unknown functions to be determined. Notice that  $\mu_e(x) = 0$  for  $e + \varphi(x) \leq 0$ . Considering (3.27), the function  $\mu_e$  vanishes at two endpoints  $0 < a_e < b_e < 1$  defined by  $\mu_e(a) = \mu_e(b) = 0$ . So

$$\mu_e(x) > 0 \text{ for } a_e < x < b_e$$

and

$$\mu_e(x) = 0 \text{ for } 0 \leq x \leq a_e \text{ and } b_e \leq x \leq 1.$$

We have the idea to impose the condition that  $\tau^+(x) = \tau^-(x) = 0$  for  $0 \leq x < a_e$  and  $b_e < x \leq 1$ . It will insure that  $U$  described by (3.28) satisfies

$$U(x) = 0 \text{ for } 0 \leq x < a_e \text{ or } b_e < x \leq 1.$$

In other word, the function vanishes identically (in  $x$ ) outside the support (in  $x$ ) of the characteristic line with level (or label)  $e$ . This idea is the main difference with the construction in zones  $+$  and  $-$ .

Let us take

$$\tau^+(x) = \frac{1}{\mu_e(x)} \exp\left(-\lambda \int_{a_e}^x \mu_e(s)^{-1} ds\right), \quad a_e < x < b_e.$$

So one has locally

$$(iH_0)\tau^+(x)U_{\mu_e(x)} = \lambda\tau^+(x)U_{\mu_e(x)}, \quad a_e < x < b_e.$$

Similarly take

$$\tau^-(x) = \frac{1}{\mu_e(x)} \exp\left(\lambda \int_{a_e}^x \mu_e(s)^{-1} ds\right), \quad a_e < x < b_e,$$

so that

$$(iH_0)\tau^-(x)U_{-\mu_e(x)} = \lambda\tau^-(x)U_{-\mu_e(x)}, \quad a_e < x < b_e.$$

Therefore  $U$  in (3.28) satisfies

$$(iH_0)U(x) = \lambda U(x), \quad x \neq a_e, b_e. \quad (3.29)$$

There is a divergence at  $x = a_e$  and  $x = b_e$ , since  $\mu_e(a_e) = \mu_e(b_e) = 0$ . More precisely for  $a_e < x < b_e$

$$\mu_e(x) = \sqrt{2(\varphi(x) - \varphi(a_e))} \approx \sqrt{2\varphi'(a_e)}\sqrt{x - a_e}, \text{ for } x \approx a_e^+ \quad (3.30)$$

and

$$\mu_e(x) = \sqrt{2(\varphi(x) - \varphi(b_e))} \approx \sqrt{-2\varphi'(b_e)}\sqrt{b_e - x}, \text{ for } x \approx b_e^-. \quad (3.31)$$

In particular  $U$  might not be even in  $V_{\text{loc}}$ . For the simplicity of the proof, we make a stronger assumption with respect to the hypothesis 8.

ASSUMPTION 16. *The electric potential is a 1-periodic function, is smooth  $\varphi \in C^2(I)$ , and is strictly monotone*

$$\begin{cases} \varphi'(x) > 0 \text{ for } 0 < x < x_0, \\ \varphi'(x) < 0 \text{ for } x_0 < x < 1. \end{cases} \quad (3.32)$$

PROPOSITION 17. *Make assumption 16. Let  $e$  be in the range (3.27). Consider*

$$\lambda_{k,e} = \frac{-2i\pi k}{t_e} \text{ for } k \in \mathbb{Z}, \quad t_e = 2 \int_{a_e}^{b_e} \mu(s)^{-1} ds$$

and

$$\begin{aligned} U_{e,k}(x) &= \frac{1}{t_e \mu_e(x)} \exp\left(2i\pi k \frac{\int_{a_e}^x \mu_e(s)^{-1} ds}{t_e}\right) U_{\mu_e(x)} \\ &+ \frac{1}{t_e \mu_e(x)} \exp\left(-2i\pi k \frac{\int_{a_e}^x \mu_e(s)^{-1} ds}{t_e}\right) U_{-\mu_e(x)}, \quad x \in (a_e, b_e), \end{aligned} \quad (3.33)$$

with  $U_{e,k}(x) = 0$  for  $x \in [0, a_e) \cup (b_e, 1]$ .

Then one has the regularity  $U_{e,k}|_n \in L^p(I)$  for all  $n \in \mathbb{N}$  and  $1 \leq p < \frac{1}{2}$  and  $AU_{e,k} \in C^0(I)^\mathbb{N}$ . Moreover  $U_{e,k}$  is a generalized eigenvector, more precisely one has that

$$(iH_0)U_{e,k}(x) = \lambda_{k,\varepsilon}U_{e,k}(x) \quad (3.34)$$

in  $L^p(I)^\mathbb{N}$  for  $1 \leq p < \frac{1}{2}$ .

*Proof.* The proof proceeds in three steps.

- For  $x \in (a, b)$ , one has  $U_{e,k} = (0, \alpha_0, \alpha_1, \dots)_{n \in \mathbb{N}}$  with

$$\alpha_n(x) = \frac{1}{t_e \mu_e(x)} \exp\left(\frac{2i\pi k \int_{a_e}^x \mu_e(s)^{-1} ds}{t_e}\right) \psi_n(\mu_e(x)) + \frac{1}{t_e \mu_e(x)} \exp\left(-\frac{2i\pi k \int_{a_e}^x \mu_e(s)^{-1} ds}{t_e}\right) \psi_n(-\mu_e(x))$$

and  $\alpha_n(x) = 0$  for  $x$  in the complement, that is for  $x \in [0, a_e) \cup (b_e, 1]$ . Since  $\mu_e$  has at most a square root singularity at  $a_e$  and  $b_e$ , it is clear that  $\alpha_n \in L^p(I)$  for all  $n \in \mathbb{N}$  and  $1 \leq p < \frac{1}{2}$ .

- Next one has that

$$AU_{e,k}(x) = \frac{1}{t_e} \exp\left(\frac{2i\pi k \int_{a_e}^x \mu_e(s)^{-1} ds}{t_e}\right) U_{\mu_e(x)} - \frac{1}{t_e} \exp\left(-\frac{2i\pi k \int_{a_e}^x \mu_e(s)^{-1} ds}{t_e}\right) U_{-\mu_e(x)}, \quad x \in (a_e, b_e). \quad (3.35)$$

So

$$\lim_{x \rightarrow a_e^+} AU_{e,k}(x) = \frac{1}{t_e} (U_{\mu_e(a_e)} - U_{-\mu_e(a_e)}) = \frac{1}{t_e} (U_0 - U_0) = 0.$$

Since  $U_{e,k}(x) = 0$  for  $x < a_e$ , one has the continuity of  $AU_{e,k}$  at  $a_e$ . Similarly

$$\begin{aligned} \lim_{x \rightarrow b_e^-} AU_{e,k}(x) &= \frac{1}{t_e} \left( \exp\left(\frac{2i\pi k \int_{a_e}^{b_e} \mu_e(s)^{-1} ds}{t_e}\right) U_{\mu_e(b_e)} - \exp\left(-\frac{2i\pi k \int_{a_e}^{b_e} \mu_e(s)^{-1} ds}{t_e}\right) U_{-\mu_e(b_e)} \right) \\ &= \frac{1}{t_e} ((-1)^k U_0 - (-1)^k U_0) = 0. \end{aligned}$$

Since  $U_{e,k}(x) = 0$  for  $b_e < x$ , one has the continuity of  $AU_{e,k}$  at  $b_e$ . Therefore one has indeed  $AU_{e,k} \in C^0(I)^\mathbb{N}$ .

- Since  $AU_{e,k}$  is continuous, the derivative in the sense of distribution of  $AU_{e,k}$  is almost everywhere equal to the point wise derivative. At inspection of (3.35), the  $x$ -derivative has the regularity of the inverse of the square root function. Therefore  $A\partial_x U_{e,k} \in L^p(I)^\mathbb{N}$  for  $1 \leq p < \frac{1}{2}$ . One also has directly that  $E_0(x)BU_{e,k}$  has, component by component, the regularity of the derivative of the square root function. That is  $E_0(x)BU_{e,k} \in L^p(I)^\mathbb{N}$  in the same range  $1 \leq p < \frac{1}{2}$ . One also has of course  $\lambda_{e,k}U_{e,k} \in L^p(I)^\mathbb{N}$ . Considering the eigenvalue equation is satisfied almost everywhere (3.29), it show the last result. The proof is ended.  $\square$

For the simplicity of the analysis, we add one condition on the electric potential  $\varphi$ .

ASSUMPTION 18. *The electric potential is strictly concave at the maximum, that is  $\varphi''(x_0) < 0$ .*

PROPOSITION 19. *In the interval  $(-\varphi_+, -\varphi_-)$  the function  $e \mapsto t_e$  goes from  $t_{(-\varphi_+)}^+ = 2\pi/\sqrt{2}$  to  $t_{(-\varphi_-)}^+$ .*

*Proof.* We only have to determine the value of  $t_{(-\varphi_+)}^+$ . Make assumption 18. For  $e = -\varphi_+ + \varepsilon$ , we linearize the function  $\varphi$  at second order near its maximum and write without giving the non essential details

$$\begin{aligned} t_e(-\varphi_+ + \varepsilon) &= \frac{2}{\sqrt{2}} \int_{\varepsilon + \frac{1}{2}\varphi''(x_0)s^2 > 0} \frac{ds}{\sqrt{\varepsilon + \frac{1}{2}\varphi''(x_0)s^2}} + o(\varepsilon) \\ &= \frac{2}{\sqrt{2}} \int_{1-s^2 > 0} \frac{ds}{\sqrt{1-s^2}} + o(\varepsilon) = \frac{4}{\sqrt{2}} \sin^{-1}(1) + \dots = \frac{2\pi}{\sqrt{2}} + o(\varepsilon). \end{aligned}$$

The proof is ended.  $\square$

The next property will be essential to show the structural well-posedness of an original integral equation derived in the sequel.

PROPOSITION 20. *Assume the function  $x \mapsto \sqrt{\varphi_+ - \varphi(x)}$  is strictly concave separately in  $[0, x_0)$  and in  $(x_0, 1]$ . Then  $t_e$  is strictly increasing in the interval  $(-\varphi_+, -\varphi_-)$ .*

*Proof.* One has the decomposition

$$\sqrt{2}t_e = \int_{a_e}^{x_0} \frac{dx}{\sqrt{e + \varphi(x)}} + \int_{x_0}^{b_e} \frac{dx}{\sqrt{e + \varphi(x)}}, \quad e + \varphi(a_e) = e + \varphi(b_e) = 0.$$

We firstly study the second term  $\int_{x_0}^{b_e} \frac{dx}{\sqrt{e + \varphi(x)}}$ . To simplify the notations we set  $\bar{e} = e + \varphi_+ \in (0, \varphi_+ - \varphi_-)$  and  $\psi(z) = \varphi(x) - \varphi_+ = \varphi(x) - \varphi(x_0) \leq 0$ . With these notations the second term is

$$\int_{x_0}^{b_e} \frac{dx}{\sqrt{e + \varphi(x)}} = H(\bar{e}) := \int_{x_0}^{b_e} \frac{dx}{\sqrt{\bar{e} + \psi(x)}}.$$

Set the change of variable  $\psi(x) = -\bar{e}u^2$  for  $x_0 \leq x \leq 1$  and  $0 \leq u \leq 1$ . One has  $\psi'(x)dx = -2\bar{e}udu$ . So

$$H(\bar{e}) = \int_0^1 \frac{2u}{\sqrt{1-u^2}} g_u(\bar{e}) du, \quad g_u(\bar{e}) = \frac{\sqrt{\bar{e}}}{-\psi'(\psi^{-1}(-\bar{e}u^2))}.$$

The derivative is

$$\frac{d}{d\bar{e}} g_u(\bar{e}) = \frac{1}{-2\sqrt{\bar{e}}\psi'(z)} + \sqrt{\bar{e}} \frac{\psi''(z)\psi'(z)^{-1}(-u^2)}{\psi'(z)^2}, \quad \psi(z) = -\bar{e}u^2,$$

that is

$$\frac{d}{d\bar{e}} g_u(\bar{e}) = \frac{1}{-2\sqrt{\bar{e}}\psi'(z)} \left( 1 - 2 \frac{\psi''(z)\psi(z)}{\psi'(z)^2} \right)$$

At the same time one has  $\sqrt{-\psi'} = -\frac{\psi'}{2\sqrt{-\psi}}$  so

$$2\sqrt{-\psi}'' = -\frac{\psi''}{\sqrt{-\psi}} + \frac{1}{2} \frac{(\psi')^2}{(-\psi)^{\frac{3}{2}}} = -\frac{1}{2} \frac{(\psi')^2}{(-\psi)^{\frac{3}{2}}} \left( 1 - 2 \frac{\psi''\psi}{(\psi')^2} \right).$$

Therefore the strict concavity of  $\sqrt{-\psi} = \sqrt{\varphi(x_0) - \varphi(x)}$  yields an increasing  $e \mapsto g_u(e)$  which turns into a strictly increasing  $H(e)$ . The same result for the second integral  $\int_{a_e}^{x_0} \frac{dx}{\sqrt{e + \varphi(x)}}$ . The proof is ended.  $\square$

### 3.4. Completeness.

We claim the following.

**THEOREM 21.** *Let  $U \in V$ . Then the spectral decomposition holds*

$$\begin{aligned} U &= U(\cdot) \cdot e_0 e_0 + \sum_{k \in \mathbb{Z}} \int_{e=-\varphi_-}^{\infty} (U, U_{e,k}^+) U_{e,k}^+ t_e de \\ &+ \sum_{k \in \mathbb{Z}} \int_{e=-\varphi_-}^{\infty} (U, U_{e,k}^-) U_{e,k}^- t_e de + \sum_{k \in \mathbb{Z}} \int_{-\varphi_+}^{-\varphi_-} (U, U_{e,k}) U_{e,k} t_e de, \end{aligned} \quad (3.36)$$

with the Plancherel relation

$$\begin{aligned} \|U\|_V^2 &= (U, e_0)^2 + \sum_{k \in \mathbb{Z}} \int_{-\varphi_-}^{\infty} |(U, U_{e,k}^+)|^2 t_e de \\ &+ \sum_{k \in \mathbb{Z}} \int_{-\varphi_-}^{\infty} |(U, U_{e,k}^-)|^2 t_e de + \sum_{k \in \mathbb{Z}} \int_{-\varphi_+}^{-\varphi_-} |(U, U_{e,k})|^2 t_e de. \end{aligned} \quad (3.37)$$

*Proof.* The proof is performed for  $U \in W_0$  by a direct computation. By density of  $W_0$  in  $V$ , it will prove the result in  $V$ . Notice that  $U \in W_0$  has only a finite number of non zero components which are all in  $H^1(I)$ : since  $U_{e,k} \in L^p(I)^{\mathbb{N}}$  for  $1 \leq p < \frac{1}{2}$ , it is enough to get the integrability and summability needed to give sense to  $(U, U_{e,k})$ . Note also that the formulas (3.36) and (3.37) are equivalent, we will prove only the first one. A last remark is that (3.37) can be checked for different lines independently one to the other: indeed  $U, U_{e,k}^+, \dots$

are infinite vectors. The method of the proof is by successive changes of variables and the final use of the completeness of the Hermite functions.

We begin with the analysis of

$$\begin{aligned} K_1 &= \sum_{k \in \mathbb{Z}} \int_{-\varphi_-}^{\infty} (U, U_{e,k}^+) U_{e,k}^+(x) t_e de \\ &= \int_{-\varphi_-}^{\infty} \left( \sum_{k \in \mathbb{Z}} \int_0^1 U(y) \cdot U_{\mu_e(y)} \exp \left( 2i\pi k \frac{\int_0^x \frac{ds}{\mu_e(s)} - \int_0^y \frac{ds}{\mu_e(s)}}{t_e} \right) \frac{dy}{t_e \mu_e(y)} \right) U_{\mu_e(x)} \frac{de}{\mu_e(x)}. \end{aligned}$$

The notation  $\cdot$  is understood as the weak product of the vector  $U(y)$  which has only a finite number of non zero components since  $U \in W_0$  and the infinite vector  $U_{\mu_e(y)}$ . We make a change of variable  $y \rightarrow \hat{y}_e$  such that

$$\hat{y}_e = \frac{1}{t_e} \int_0^y \frac{ds}{\mu_e(s)}, \quad d\hat{y}_e = \frac{dy}{t_e \mu_e(y)}.$$

We also set  $\hat{x}_e = \frac{1}{t_e} \int_0^x \frac{ds}{\mu_e(s)}$ . One has that

$$\begin{aligned} &\sum_{k \in \mathbb{Z}} \int_0^1 U(y) \cdot U_{\mu_e(y)} \exp \left( -2i\pi k \frac{\int_0^x \frac{ds}{\mu_e(s)} - \int_0^y \frac{ds}{\mu_e(s)}}{t_e} \right) \frac{dy}{t_e \mu_e(y)} \\ &= \sum_{k \in \mathbb{Z}} \int_0^1 U(y) \cdot U_{\mu_e(y)} \exp(-2i\pi k(\hat{x}_e - \hat{y}_e)) d\hat{y}_e = U(x) \cdot U_{\mu_e(x)} \end{aligned} \quad (3.38)$$

using standard properties of the Fourier transform, such as the Dirac comb formula

$$\sum_{k \in \mathbb{Z}} \exp(2i\pi k(x - y)) = \delta(x - y). \quad (3.39)$$

A lengthier verification of the identity is possible with smooth test functions. One obtains

$$K_1 = \int_{-\varphi_-}^{\infty} U(x) \cdot U_{\mu_e(x)} \frac{1}{\mu_e(x)} U_{\mu_e(x)} de.$$

The definition of  $\mu_e$ , namely  $\frac{1}{2}\mu_e(x)^2 - \varphi(x) = e$ , yields  $d\mu_e(x) = \frac{de}{\mu_e(x)}$  for every  $x$ . Since it is independent of  $x$ , we will write  $d\lambda = d\mu_e(x)$ . Therefore one can recast as

$$K_1 = \int_{\sqrt{\varphi(x)-\varphi_-}}^{\infty} U(x) \cdot U_{\lambda} U_{\lambda} d\lambda$$

where  $U_{\lambda}$  has been defined in (3.13). With exactly the same method one obtains for the third term in (3.36)

$$K_2 = \sum_{k \in \mathbb{Z}} \int_{-\varphi_-}^{\infty} (U, U_{e,k}^-) U_{e,k}^-(x) \frac{de}{\mu_e(x)} = \int_{\sqrt{\varphi(x)-\varphi_-}}^{\infty} U(x) \cdot U_{-\lambda} U_{-\lambda} d\lambda = \int_{-\infty}^{-\sqrt{\varphi(x)-\varphi_-}} U(x) \cdot U_{\lambda} U_{\lambda} d\lambda.$$

With a little more technicalities detailed in the next proposition, one obtains also

$$K_3 = \int_{-\sqrt{\varphi(x)-\varphi_-}}^{\sqrt{\varphi(x)-\varphi_-}} U(x) \cdot U_{\lambda} U_{\lambda} d\lambda. \quad (3.40)$$

Therefore using the orthonormality properties of the Hermite functions, one gets

$$K_1 + K_2 + K_3 = \int_{-\infty}^{\infty} U(x) \cdot U_{\lambda} U_{\lambda} d\lambda = U(x) - U(x) \cdot e_0 e_0$$

from which the claim is proved.  $\square$

PROPOSITION 22. *The formula (3.35) holds.*

*Proof.* To be more precise, we note  $0 < a_e < b_e < 1$  the boundaries of the interval described in (3.30-3.31). Since  $U_{\mu_e}$  is now defined in (3.33) as the difference of two terms, full expansion yields four contributions. So

$$\begin{aligned}
K_3 &= \int_{-\varphi_+}^{-\varphi_-} \left( \underbrace{\sum_{k \in \mathbb{Z}} \int_{a_e}^{b_e} U(y) \cdot U_{\mu_e(y)} \exp \left( 2i\pi k \frac{\int_{a_e}^x \frac{ds}{\mu_e(s)} - \int_{a_e}^y \frac{ds}{\mu_e(s)}}{t_e} \right) \frac{dy}{t_e \mu_e(y)}}_{=D_1} \right) U_{\mu_e(x)} \frac{de}{\mu_e(x)} \\
&+ \int_{-\varphi_+}^{-\varphi_-} \left( \underbrace{\sum_{k \in \mathbb{Z}} \int_{a_e}^{b_e} U(y) \cdot U_{-\mu_e(y)} \exp \left( -2i\pi k \frac{\int_{a_e}^x \frac{ds}{\mu_e(s)} - \int_{a_e}^y \frac{ds}{\mu_e(s)}}{t_e} \right) \frac{dy}{t_e \mu_e(y)}}_{=D_2} \right) U_{-\mu_e(x)} \frac{de}{\mu_e(x)} \\
&+ \int_{-\varphi_+}^{-\varphi_-} \left( \underbrace{\sum_{k \in \mathbb{Z}} \int_{a_e}^{b_e} U(y) \cdot U_{-\mu_e(y)} \exp \left( 2i\pi k \frac{\int_{a_e}^x \frac{ds}{\mu_e(s)} + \int_{a_e}^y \frac{ds}{\mu_e(s)}}{t_e} \right) \frac{dy}{t_e \mu_e(y)}}_{=D_3} \right) U_{\mu_e(x)} \frac{de}{\mu_e(x)} \\
&+ \int_{-\varphi_+}^{-\varphi_-} \left( \underbrace{\sum_{k \in \mathbb{Z}} \int_{a_e}^{b_e} U(y) \cdot U_{\mu_e(y)} \exp \left( -2i\pi k \frac{\int_{a_e}^x \frac{ds}{\mu_e(s)} + \int_{a_e}^y \frac{ds}{\mu_e(s)}}{t_e} \right) \frac{dy}{t_e \mu_e(y)}}_{=D_4} \right) U_{-\mu_e(x)} \frac{de}{\mu_e(x)}.
\end{aligned}$$

We consider the change of variable  $\hat{y}_e = \frac{2}{t_e} \int_{a_e}^y \frac{ds}{\mu_e(s)}$ , with  $d\hat{y}_e = \frac{2dy}{t_e \mu_e(y)}$ . Notice that  $\hat{y}_e \in [0, 1]$ . The same notation is used for  $\hat{x}_e \in [0, 1]$ .

- It yields for the first term

$$D_1 = \frac{1}{2} \sum_{k \in \mathbb{Z}} \int_0^1 U(y) \cdot U_{\mu_e(y)} \exp(i\pi k(\hat{x}_e - \hat{y}_e)) d\hat{y}_e.$$

Splitting between even and odd  $k$ , one has

$$\begin{aligned}
D &= \frac{1}{2} \sum_{k \in \mathbb{Z}} \int_0^1 U(y) \cdot U_{\mu_e(y)} \exp(2i\pi k(\hat{x}_e - \hat{y}_e)) d\hat{y}_e \\
&+ \frac{1}{2} \exp(-i\pi \hat{x}_e) \sum_{k \in \mathbb{Z}} \int_0^1 U(y) \cdot U_{\mu_e(y)} \exp(i\pi \hat{y}_e) \exp(2i\pi k(\hat{x}_e - \hat{y}_e)) d\hat{y}_e.
\end{aligned}$$

Use the Dirac comb (3.39) on both terms. It yields

$$D_1 = \frac{1}{2} U(x) \cdot U_{\mu_e(x)} + \frac{1}{2} \exp(-i\pi \hat{x}_e) (U(x) \cdot U_{\mu_e(x)} \exp(i\pi \hat{x}_e)) = U(x) \cdot U_{\mu_e(x)}.$$

- One has for similar reasons  $D_2 = U(x) \cdot U_{-\mu_e(x)}$ .
- The same change of variable in the third term yields

$$D_3 = \frac{1}{2} \sum_{k \in \mathbb{Z}} \int_0^1 U(y) \cdot U_{\mu_e(y)} \exp(i\pi k(\hat{x}_e + \hat{y}_e)) d\hat{y}_e$$

$$\begin{aligned}
&= \frac{1}{2} \sum_{k \in \mathbb{Z}} \int_0^1 U(y) \cdot U_{\mu_e(y)} \exp(2i\pi k(-(1 - \hat{x}_e) + \hat{y}_e)) d\hat{y}_e \\
&+ \frac{1}{2} \exp(i\pi \hat{x}_e) \sum_{k \in \mathbb{Z}} \int_0^1 U(y) \cdot U_{\mu_e(y)} \exp(i\pi \hat{y}_e) \exp(2i\pi k(-(1 - \hat{x}_e) + \hat{y}_e)) d\hat{y}_E.
\end{aligned}$$

Notice that the change of variable is such that there exists  $\bar{x} \in (a_e, b_e)$  such that

$$\hat{x}_e = \frac{2}{t_e} \int_{b_e + a_e - \bar{x}}^{b_e} \frac{ds}{\mu_e(s)}$$

so that

$$1 - \hat{x}_e = 1 - \frac{2}{t_e} \int_{b_e + a_e - \bar{x}}^{b_e} \frac{ds}{\mu_e(s)} = \frac{2}{t_e} \int_{a_e}^{b_e + a_e - \bar{x}} \frac{ds}{\mu_e(s)} = \frac{2}{t_e} \int_{a_e}^{b_e} \frac{ds}{\mu_e(s)} - \frac{2}{t_e} \int_{b_e + a_e - \bar{x}}^{b_e} \frac{ds}{\mu_e(s)} = \frac{2}{t_e} \int_{a_e}^{b_e + a_e - \bar{x}} \frac{ds}{\mu_e(s)}.$$

In other words  $1 - \hat{x}_e$  corresponds to  $b_e + a_e - \bar{x}$ . Therefore one can write with the Dirac comb technique

$$\begin{aligned}
D_3 &= \frac{1}{2} U(b_e + a_e - \bar{x}) \cdot U_{\mu_e(b_e + a_e - \bar{x})} \\
&+ \frac{1}{2} \exp(i\pi \hat{x}_e) U(b_e + a_e - \bar{x}) \cdot U_{\mu_e(b_e + a_e - \bar{x})} \exp(+i\pi(1 - \hat{x}_e)) = 0
\end{aligned}$$

after simplifications.

- Similarly  $D_4 = 0$ .
- So one can write

$$\begin{aligned}
K_3 &= \int_{-\varphi_+}^{-\varphi_-} U(x) \cdot U_{\mu_e(x)} U_{\mu_e(x)} \frac{de}{\mu_e(x)} + \int_{-\varphi_+}^{-\varphi_-} U(x) \cdot U_{-\mu_e(x)} U_{-\mu_e(x)} \frac{de}{\mu_e(x)} \\
&= \int_0^{\sqrt{\varphi(x) - \varphi_-}} U(x) \cdot U_\lambda U_\lambda d\lambda + \int_{-\sqrt{\varphi(x) - \varphi_-}}^0 U(x) \cdot U_\lambda U_\lambda d\lambda = \int_{-\sqrt{\varphi(x) - \varphi_-}}^{\sqrt{\varphi(x) - \varphi_-}} U(x) \cdot U_\lambda U_\lambda d\lambda.
\end{aligned}$$

The results proceeds by summation of  $D_1$ ,  $D_2$ ,  $D_3$  and  $D_4$ . It ends the proof.  $\square$

The previous result can easily be transformed into a spectral representation formula for  $iH_0$ . We just give the result. One has the result

$$\begin{aligned}
iH_0 U &= \sum_{k \in \mathbb{Z}} \int_{e=-\varphi_-}^{\infty} \left( U, U_{e,k}^+ \right) \lambda_{e,k} U_{e,k}^+ t_e de \\
&+ \sum_{k \in \mathbb{Z}} \int_{e=-\varphi_-}^{\infty} \left( U, U_{e,k}^- \right) \lambda_{e,k} U_{e,k}^- t_e de + \sum_{k \in \mathbb{Z}} \int_{-\varphi_+}^{-\varphi_-} \left( U, U_{e,k} \right) \lambda_{e,k} U_{e,k} t_e de.
\end{aligned} \tag{3.41}$$

One has an associated Plancherel relation

$$\begin{aligned}
\|H_0 U\|_V^2 &= \sum_{k \in \mathbb{Z}} \int_{-\varphi_-}^{\infty} \left| \left( U, U_{e,k}^+ \right) \right|^2 |\lambda_{e,k}|^2 t_e de \\
&+ \sum_{k \in \mathbb{Z}} \int_{-\varphi_-}^{\infty} \left| \left( U, U_{e,k}^- \right) \right|^2 |\lambda_{e,k}|^2 t_e de + \sum_{k \in \mathbb{Z}} \int_{-\varphi_+}^{-\varphi_-} \left| \left( U, U_{e,k} \right) \right|^2 |\lambda_{e,k}|^2 t_e de.
\end{aligned} \tag{3.42}$$

This formula can actually be used to give a meaning to the largest space such that summability and integrability hold in all previous formulas.

**4. Generalized eigenvectors of  $iH$ .** We now construct the eigenvectors of  $iH$

$$iHV_{e,k}^+ = \lambda_{e,k}V_{e,k}^+$$

by a perturbation method. We start from the representation formula (3.36) rewritten as

$$U = U(\cdot) \cdot e_0 e_0 + \sum_z \sum_{k \in \mathbb{Z}} \int (U, U_{e,k}^z) U_{e,k}^z t_e^z de^z \quad (4.1)$$

where  $z$  is the index of the zone, that is  $z = +$ ,  $z = -$  or  $z = c$  in the central zone.

The main idea is to assume a representation under the form

$$V_{e,k}^+ = U_{e,k}^+ + a(x)e_0 + \sum_z \sum_{p \in \mathbb{Z}} \int b_{s,p}^z U_{s,p}^z t_s^z ds^z. \quad (4.2)$$

where  $a$  and  $b_{s,p}^z$  are unknown. Note that  $a$  and  $b_{s,p}^z$  depend also on  $e, k$  and the index of the zone  $z = +$ . One has

$$\begin{aligned} iHV_{e,k}^+ &= iH_0V_{e,k}^+ + iKV_{e,k}^+ = \\ &= \lambda_{e,k}U_{e,k}^+ + \sum_z \sum_{p \in \mathbb{Z}} \int b_{s,p}^z \lambda_{s,p}^z U_{s,p}^z t_s^z ds^z + iKa(x)e_0 + iKU_{e,k}^+ + iK \sum_z \sum_{p \in \mathbb{Z}} \int b_{s,p}^z U_{s,p}^z t_s^z ds^z. \end{aligned}$$

More precisely since  $K = \exp(\varphi/2)D$

$$\begin{aligned} iHV_{e,k}^+ &= \lambda_{e,k}U_{e,k}^+ + \sum_z \sum_{p \in \mathbb{Z}} \int b_{s,p}^z \lambda_{s,p}^z U_{s,p}^z t_s^z ds^z - \alpha \exp(\varphi(x)/2) a(x)e_2 \\ &+ \alpha \left( U_{e,k}^+ \cdot e_2 + \sum_z \sum_{p \in \mathbb{Z}} \int b_{s,p}^z U_{s,p}^z \cdot e_2 t_s^z ds^z \right) e_0. \end{aligned}$$

On the other hand one has

$$\lambda_{e,k}V_{e,k}^+ = \lambda_{e,k}U_{e,k}^+ + \lambda_{e,k}a(x)e_0 + \sum_z \sum_{p \in \mathbb{Z}} \int b_{s,p}^z \lambda_{e,k} U_{s,p}^z t_s^z ds^z.$$

The eigenequation yields

$$\begin{aligned} \sum_z \sum_{p \in \mathbb{Z}} \int b_{s,p}^z (\lambda_{s,p}^z - \lambda_{e,k}) U_{s,p}^z t_s^z ds^z &= \alpha \exp(\varphi(x)/2) a(x)e_2 \\ + \alpha \exp(\varphi(x)/2) \left( \frac{1}{\alpha} \exp(-\varphi(x)/2) \lambda_{e,k} a(x) - U_{e,k}^+ \cdot e_2 - \sum_z \sum_{p \in \mathbb{Z}} \int b_{s,p}^z U_{s,p}^z \cdot e_2 t_s^z ds^z \right) &e_0. \end{aligned}$$

The right hand side is a linear combination of  $e_0$  and  $e_2$ , so we identify this relation has an equality between functions in  $V$  for which (3.36) applies. The equality of the coefficients writes

$$\begin{cases} b_{s,p}^x (\lambda_{s,p}^x - \lambda_{e,k}) = (\alpha \exp(\varphi(x)/2) a(x)e_2, U_{s,p}^z), \\ \frac{1}{\alpha} \exp(-\varphi(x)/2) \lambda_{e,k} a(x) - \sum_{p \in \mathbb{Z}} \int_{e=-\varphi-}^{\infty} b_{s,p}^+ U_{s,p}^+ \cdot e_2 t_s ds = U_{e,k}^+ \cdot e_2. \end{cases} \quad (4.3)$$

From now on, the main question is the solvability of this system. To continue the discussion, we make a difference between  $\lambda_{e,k} \neq 0$  and  $\lambda_{e,k=0}$ . In the homogeneous case, it will be clear that vanishing eigenvalues play no role for linear Landau damping since they are naturally eliminated from the discussion.

**4.1. An integral equation.** One eliminates  $b_{s,p}^z$  in (4.3) with

$$b_{s,p}^x = \frac{(\alpha \exp(\varphi(x)/2) a(x) e_2, U_{s,p}^z)}{\lambda_{s,p}^x - \lambda_{e,k}}$$

and plug in the second equation. It yields the integral equation

$$\begin{aligned} \frac{1}{\alpha} \exp(-\varphi(x)/2) \lambda_{e,k} a(x) - \sum_z \sum_{p \neq 0} P.V. \int \frac{(\alpha \exp(\varphi(x)/2) a(x) e_2, U_{s,p}^z)}{\lambda_{s,p}^z - \lambda_{e,k}} U_{s,p}^z \cdot e_2 t_s^z ds^z \\ - \frac{1}{\lambda_{e,k}} \sum_z \int (\alpha \exp(\varphi(x)/2) a(x) e_2, U_{s,0}^z) U_{s,0}^z \cdot e_2 t_s^z ds^z = U_{e,k}^+ \cdot e_2. \end{aligned} \quad (4.4)$$

The singular integrals are interpreted principal values [29], similar to the Hilbert transform. The other integral for  $p = 0$  are convergent. The case where  $\lambda_{e,k} \rightarrow 0$  is not as singular as it appears at first sight, see discussion below and remark 24.

**4.2. Another family of eigenvector in the null space.** The null space is defined by  $HV = 0$ . All generalized eigenvectors with  $\lambda_{e,k}^+ = 0$  are in the null space. However in view of the fact that all functions  $a(x)e_0 \in V$  are in the null space of  $H_0$ , it is natural to think that similar eigenvectors exist for  $H$ . It is the case as shown below.

Start from the representation  $V(x) = f(x)e_0 + \alpha g(x)e_1$ . A direct computation shows that

$$\begin{aligned} 0 = iHV &= \left( -A\partial_x + E_0(x)B + \sqrt{n_0(x)}D \right) (f(x)e_0 + g(x)e_1) \\ &= \sqrt{n_0(x)}Df(x)e_0 + (-A\partial_x + E_0(x)B)g(x)e_1. \end{aligned}$$

In view of the definition of the matrices, one gets  $0 = -\alpha\sqrt{n_0(x)}f(x)e_2 - \alpha g'(x)e_2 - \frac{1}{2}E_0(x)\alpha g(x)e_2 = \alpha \left( -\sqrt{n_0(x)}f(x) - g'(x) - \frac{1}{2}E_0(x)g(x) \right) e_2$  rewritten as  $\exp(\varphi(x)/2) f(x) = -g'(x) + \frac{1}{2}\varphi'(x)g(x)$ . Since  $f, g$  and  $\varphi$  are 1-periodic functions, the solution is

$$f = -(\exp(-\varphi/2)g)'$$

Such vectors will be denoted as

$$V_g = -(\exp(-\varphi/2)g)'(\cdot)e_0 + \alpha g(\cdot)e_1. \quad (4.5)$$

This family is directly linked to the Gauss law.

**PROPOSITION 23.** *Assume  $U$  is orthogonal to  $V_g$  for all  $g \in H^1(I)_{\text{per}}$ . Then  $U$  satisfies the Gauss law.*

*Proof.* By definition (3.6) of  $U$ , one has that

$$\begin{aligned} 0 = (U, V_g) &= \int_I \left( -(\exp(-\varphi(x)/2)g(x))' F(x) + \alpha g(x)u_0(x) \right) dx \\ &= \int_I \left( \exp(-\varphi(x)/2)g(x)F'(x) + \alpha g(x) \int_{\mathbb{R}} \frac{1}{\alpha} \exp\left(-\frac{v^2}{4}\right) u(x, v) dv \right) dx. \end{aligned}$$

Since it is true for all  $g$ , it shows that  $F'(x) = -\exp(\varphi(x)/2) \int_{\mathbb{R}} \frac{1}{\alpha} \exp\left(-\frac{v^2}{4}\right) u(x, v) dv$ , which is the Gauss law (2.6). The proof is ended.  $\square$

**4.3. Comment on the structural well posedness of the integral equations.** With the proposed approach, the construction of the generalized eigenvectors relies on the solution of an integral equation. To have a better understanding we write them more explicitly in the case of study with the three zones.

One has the explicit formulas

$$\left\{ \begin{array}{l} \text{zone +: } U_{s,p}^+(x) \cdot e_2 = \frac{1}{t_s \alpha} \exp\left(-2i\pi k \frac{\int_0^x \mu_s(t)^{-1} dt}{t_s}\right) \exp\left(\frac{-\mu_e(x)^2}{4}\right), \\ \text{zone -: } U_{s,p}^-(x) \cdot e_2 = \frac{1}{t_s \alpha} \exp\left(2i\pi k \frac{\int_0^x \mu_s(t)^{-1} dt}{t_s}\right) \exp\left(\frac{-\mu_e(x)^2}{4}\right), \\ \text{zone c: } U_{s,p}(x) \cdot e_2 = \mathbf{I}_{\{a_e < x < b_e\}} \frac{1}{t_s \alpha} \exp\left(-2i\pi k \frac{\int_{a_e}^x \mu_s(t)^{-1} dt}{t_s}\right) \exp\left(\frac{-\mu_e(x)^2}{4}\right) \\ \quad + \mathbf{I}_{\{a_e < x < b_e\}} \frac{1}{t_s \alpha} \exp\left(2i\pi k \frac{\int_{a_e}^x \mu_s(t)^{-1} dt}{t_s}\right) \exp\left(\frac{-\mu_e(x)^2}{4}\right) \\ \quad = \mathbf{I}_{\{a_e < x < b_e\}} \frac{2}{t_s \alpha} \cos\left(2\pi k \frac{\int_{a_e}^x \mu_s(t)^{-1} dt}{t_s}\right) \exp\left(\frac{-\mu_e(x)^2}{4}\right). \end{array} \right. \quad (4.6)$$

Using that  $\frac{1}{2}\mu_e(x) - \varphi(x) = e$  (it is the characteristic equation (3.19)), one gets

$$\begin{aligned} \lambda_{e,k} a(x) + \sum_{p \in \mathbb{Z}} P.V. \int_{e=-\varphi_-}^{\infty} \frac{\int_I a(y) \exp\left(-2i\pi p \frac{\int_x^y \mu_s(t)^{-1} dt}{t_s}\right) dy \exp(-s) ds}{\lambda_{s,p} - \lambda_{e,k}} \frac{1}{t_s} \\ + \sum_{p \in \mathbb{Z}} P.V. \int_{e=-\varphi_-}^{\infty} \frac{\int_I a(y) \exp\left(2i\pi p \frac{\int_x^y \mu_s(t)^{-1} dt}{t_s}\right) dy \exp(-s) ds}{\lambda_{s,p} - \lambda_{e,k}} \frac{1}{t_s} \\ + \sum_{p \in \mathbb{Z}} P.V. \int_{e=-\varphi_+}^{-\varphi_-} \frac{\int_{a_e}^{b_e} 2a(y) \cos\left(2\pi p \frac{\int_0^y \mu_s(t)^{-1} dt}{t_s}\right) dy}{\lambda_{s,p} - \lambda_{e,k}} \cos\left(2\pi p \frac{\int_0^x \mu_s(t)^{-1} dt}{t_s}\right) \frac{\exp(-s) ds}{t_s} \\ = i \frac{1}{t_e} \exp\left(-2i\pi k \frac{\int_0^x \mu_s(t)^{-1} dt}{t_s}\right) \exp(-e/2). \end{aligned} \quad (4.7)$$

REMARK 24. Note that  $\lambda_{e,k} = -\frac{2i\pi k}{t_e}$ . In the case  $k \neq 0$ , it seems that there is a degeneracy for  $t_e \rightarrow +\infty$ . But this degeneracy can be eliminated since the same  $1/(t_e)$  show up in the right hand side.

REMARK 25. To construct  $V_{e,k}^-$  and  $V_{e,k}^c$ , it is sufficient to change the right hand side.

Nevertheless two conditions seem necessary for this equation to make sense: one needs that the integrals written as principal values [29] make sense individually (we call this structural well-posedness); one needs the sums to be convergent. To save place we made no distinction between  $p = 0$  and  $p \neq 0$ , so that all integrals are written as principal values.

PROPOSITION 26 (Structural well-posedness). *Make the assumption 18. Then the equation (4.7) is structurally well-posed.*

*Proof.* One has to check the integrability at the boundaries of the integrals, and the monotony of  $\lambda_{s,p}$ .

Propositions 13 and 19 show that the weights  $\frac{\exp(-s) ds}{t_s}$  are integrable near the limit points of the integral.

Except for  $p = 0$  which poses no real problem, the variation of  $\lambda_{s,p} - \lambda_{e,k}$  is monotone in the two first integrals so the principal values make sense immediatly. Concerning the third integral one can invoke proposition 20, so  $t_e$  is strictly monotone in the interval  $(-\varphi_+, -\varphi_-)$ . Moreover the weight  $1/(t_e)$  vanishes at  $-\varphi_-$  and the integral vanishes at  $-\varphi_+$  since in this case  $a_e = b_e = x_0$ . Therefore all integrals/principal values make sense individually. The proof is ended.  $\square$

For  $i\mu = \lambda$  and  $\mu \in \mathbb{R}$ , let us define the operator

$$\begin{aligned} T_\mu(a) = \sum_{p \in \mathbb{Z}} P.V. \int_{e=-\varphi_-}^{\infty} \frac{\int_I a(y) \exp\left(-2i\pi p \frac{\int_x^y \mu_s(t)^{-1} dt}{t_s}\right) dy \exp(-s) ds}{\lambda_{s,p} - i\mu} \frac{1}{t_s} \\ + \sum_{p \in \mathbb{Z}} P.V. \int_{e=-\varphi_-}^{\infty} \frac{\int_I a(y) \exp\left(2i\pi p \frac{\int_x^y \mu_s(t)^{-1} dt}{t_s}\right) dy \exp(-s) ds}{\lambda_{s,p} - i\mu} \frac{1}{t_s} \end{aligned}$$

$$+ \sum_{p \in \mathbb{Z}} P.V. \int_{e=-\varphi_+}^{-\varphi_-} \frac{\int_I 2a(y) \cos\left(2\pi p \frac{\int_0^y \mu_s(t)^{-1} dt}{t_s}\right) dy}{\lambda_{s,p} - i\mu} \cos\left(2\pi p \frac{\int_0^x \mu_s(t)^{-1} dt}{t_s}\right) \frac{\exp(-s) ds}{t_s}$$

and the right hand side

$$b_\mu^k(x) = i \frac{1}{t_e} \exp\left(-2i\pi k \frac{\int_0^x \mu_s(t)^{-1} dt}{t_s}\right) \exp(-e/2), \quad i\mu = \frac{-2i\pi k}{t_e}.$$

The equation writes

$$i\mu a_\mu + T_\mu(a_\mu) = b_\mu^k. \quad (4.8)$$

REMARK 27. *The equation (4.8) poses important difficulties. It is nevertheless possible in certain cases to show the summability in  $\mathcal{L}(L^2(\mathbb{R}))$  of the series that define  $T_\mu$ . In this work we do not pursue in this direction and focus more on the exact solution in the homogeneous case where everything becomes much simpler. Moreover some of the methods which can be used to show the summability in the general situation are by comparison with the homogeneous case which is detailed below.*

**5. The homogeneous case  $\varphi \equiv 0$ .** If  $\varphi(x) = 0$  for all  $x$ , the problem is homogeneous in space and all calculations become explicit because of the orthogonality of usual Fourier modes. The third integral vanishes since  $\varphi_- = \varphi_+$ . Moreover  $t_s = \frac{1}{\sqrt{2s}}$  and

$$\frac{1}{t_s} \int_0^x \frac{ds}{\mu(s)} = x \quad \forall s.$$

PROPOSITION 28. *The solution of the integral equation is  $a(x) = \gamma \exp(2i\pi kx)$  with*

$$\left(4\pi^2 k^2 + P.V. \int_{-\infty}^{\infty} \frac{\mu}{\mu - \lambda} \exp\left(-\frac{\mu^2}{2}\right) d\mu\right) \gamma = 2i\pi k \exp\left(-\frac{\lambda^2}{4}\right). \quad (5.1)$$

REMARK 29. *The principal value is bounded [6]*

$$\left| P.V. \int_{-\infty}^{\infty} \frac{\mu}{\mu - \lambda} \exp\left(-\frac{\mu^2}{2}\right) d\mu \right| \leq (2\pi)^{\frac{1}{2}}, \quad \forall \lambda \in \mathbb{R}.$$

So  $\gamma$  and  $a$  are well defined for  $k \neq 0$ .

*Proof.* Plug in the integral equation (4.4) and obtain after simplification due to the orthogonality of usual Fourier modes

$$\begin{aligned} & -2i\pi k \sqrt{2e} \gamma - P.V. \int_0^\infty \frac{\gamma \sqrt{2s}}{-2i\pi k \sqrt{2s} - (-2i\pi k) \sqrt{2e}} \exp(-s) ds \\ & + P.V. \int_0^\infty \frac{i\gamma \sqrt{2s}}{2i\pi k \sqrt{2s} - (-2i\pi k) \sqrt{2e}} \exp(-s) ds = \sqrt{2e} \exp(-e/2) \end{aligned}$$

that is

$$\begin{aligned} & -4\pi^2 k^2 \sqrt{2e} \gamma - \left( P.V. \int_0^\infty \frac{\sqrt{2s}}{\sqrt{2s} - \sqrt{2e}} \exp(-s) ds \right) \gamma \\ & - \left( P.V. \int_0^\infty \frac{\sqrt{2s}}{-\sqrt{2s} - \sqrt{2e}} \frac{\exp(-s) ds}{t_s} \right) \gamma = -2i\pi k \sqrt{2e} \exp(-e/2). \end{aligned}$$

Make the change of variable  $\sqrt{2e} = \lambda$ ,  $\sqrt{2s} = \mu > 0$  in the first integral and  $-\sqrt{2s} = \mu < 0$  in the second integral

$$4\pi^2 k^2 \lambda \gamma + \left( P.V. \int_0^\infty \frac{\mu}{\mu - \lambda} \exp\left(-\frac{\mu^2}{2}\right) \mu d\mu \right) \gamma + \left( P.V. \int_0^\infty \frac{\mu}{-\mu - \lambda} \exp\left(-\frac{\mu^2}{2}\right) \mu d\mu \right) \gamma = 2i\pi k \lambda \exp\left(-\frac{\lambda^2}{4}\right).$$

One obtains

$$\left(4\pi^2 k^2 \lambda + P.V. \int_{-\infty}^{\infty} \frac{\mu^2}{\mu - \lambda} \exp\left(-\frac{\mu^2}{2}\right) d\mu\right) \gamma = 2i\pi k \lambda \exp\left(-\frac{\lambda^2}{4}\right).$$

Note that

$$P.V. \int_{-\infty}^{\infty} \frac{\mu^2}{\mu - \lambda} \exp\left(-\frac{\mu^2}{2}\right) d\mu = \lambda P.V. \int_{-\infty}^{\infty} \frac{\mu}{\mu - \lambda} \exp\left(-\frac{\mu^2}{2}\right) d\mu$$

One gets after simplification by  $\lambda$  the simpler expression

$$\left(4\pi^2 k^2 + P.V. \int_{-\infty}^{\infty} \frac{\mu}{\mu - \lambda} \exp\left(-\frac{\mu^2}{2}\right) d\mu\right) \gamma = 2i\pi k \exp\left(-\frac{\lambda^2}{4}\right).$$

The proof is ended.  $\square$

So, up to the fact that the problem is posed in  $I = [0, 1]$  instead of  $[0, 2\pi]$ , the generalized eigenvectors are the same as the ones in in [6] (moreover with  $k$  changed in  $-k$  due to a different normalization).

PROPOSITION 30. *The generalized eigenvectors satisfy*

$$\begin{aligned} & \left(4\pi^2 k^2 + P.V. \int_{-\infty}^{\infty} \frac{\mu}{\mu - \lambda} \exp\left(-\frac{\mu^2}{2}\right) d\mu\right) V_{e,k}^+ \\ &= \left(4\pi^2 k^2 + P.V. \int_{-\infty}^{\infty} \frac{\mu}{\mu - \lambda} \exp\left(-\frac{\mu^2}{2}\right) d\mu\right) U_\lambda \exp(2i\pi k x) \\ &+ 2i\pi k \exp\left(-\frac{\lambda^2}{4}\right) \exp(2i\pi k x) - \exp\left(-\frac{\lambda^2}{4}\right) \left(P.V. \int_{\mathbb{R}} \frac{\mu}{\mu - \lambda} \exp\left(-\frac{\mu^2}{4}\right) U_\mu d\mu\right) \exp(2i\pi k x) \end{aligned} \quad (5.2)$$

and

$$\begin{aligned} & \left(4\pi^2 k^2 + P.V. \int_{-\infty}^{\infty} \frac{\mu}{\mu + \lambda} \exp\left(-\frac{\mu^2}{2}\right) d\mu\right) V_{e,k}^- \\ &= \left(4\pi^2 k^2 + P.V. \int_{-\infty}^{\infty} \frac{\mu}{\mu + \lambda} \exp\left(-\frac{\mu^2}{2}\right) d\mu\right) U_\lambda \exp(2i\pi k x) \\ &+ 2i\pi k \exp\left(-\frac{\lambda^2}{4}\right) \exp(2i\pi k x) - \exp\left(-\frac{\lambda^2}{4}\right) \left(P.V. \int_{\mathbb{R}} \frac{\mu}{\mu - \lambda} \exp\left(-\frac{\mu^2}{4}\right) U_\mu d\mu\right) \exp(2i\pi k x). \end{aligned} \quad (5.3)$$

*Proof.* Considering the eigenvector (4.2), one has that

$$\begin{aligned} V_{e,k}^+ &= U_{\sqrt{2e}} \exp(2i\pi k x) + \gamma e_0 \exp(2i\pi k x) \\ &+ \left(P.V. \int_0^\infty \frac{\gamma \mu \exp\left(-\frac{\mu^2}{4}\right)}{-2i\pi k (2s)^{\frac{1}{2}} + 2i\pi k (2e)^{\frac{1}{2}}} U_{\sqrt{2s}} \frac{1}{\sqrt{2s}} ds\right) \exp(2i\pi k x) \\ &+ \left(\int_0^\infty \frac{\gamma \mu \exp\left(-\frac{\mu^2}{4}\right)}{2i\pi k (2s)^{\frac{1}{2}} + 2i\pi k (2e)^{\frac{1}{2}}} \exp(-s) U_{-\sqrt{2s}} \frac{1}{\sqrt{2s}} ds\right) \exp(2i\pi k x). \end{aligned}$$

The same change of variable  $\lambda = \sqrt{2e}$  (and  $\mu = \sqrt{2s}$ ) yields

$$V_{e,k}^+ = U_\lambda \exp(2i\pi k x) + \gamma e_0 \exp(2i\pi k x)$$

$$\begin{aligned}
& -\frac{\gamma}{2i\pi k} \left( P.V. \int_0^\infty \frac{\mu}{\mu-\lambda} \exp\left(-\frac{\mu^2}{4}\right) U_\mu d\mu \right) \exp(2i\pi kx) - \frac{\gamma}{2i\pi k} \left( \int_0^\infty \frac{\mu}{-\mu-\lambda} \exp\left(-\frac{\mu^2}{4}\right) U_{-\mu} d\mu \right) \exp(2i\pi kx) \\
& = U_\lambda \exp(2i\pi kx) + \gamma e_0 \exp(2i\pi kx) - \frac{\gamma}{2i\pi k} \left( P.V. \int_{\mathbb{R}} \frac{\mu}{\mu-\lambda} \exp\left(-\frac{\mu^2}{4}\right) U_\mu d\mu \right) \exp(2i\pi kx).
\end{aligned}$$

Another more explicit form is possible with (5.1). After multiplication by  $4\pi^2 k^2 + P.V. \int_{-\infty}^\infty \frac{\mu}{\mu-\lambda} \exp\left(-\frac{\mu^2}{2}\right) d\mu$ , it writes

$$\begin{aligned}
& \left( 4\pi^2 k^2 + P.V. \int_{-\infty}^\infty \frac{\mu}{\mu-\lambda} \exp\left(-\frac{\mu^2}{2}\right) d\mu \right) V_{e,k}^+ \tag{5.4} \\
& = \left( 4\pi^2 k^2 + P.V. \int_{-\infty}^\infty \frac{\mu}{\mu-\lambda} \exp\left(-\frac{\mu^2}{2}\right) d\mu \right) U_\lambda \exp(2i\pi kx) \\
& + 2i\pi k \exp\left(-\frac{\lambda^2}{4}\right) \exp(2i\pi kx) - \exp\left(-\frac{\lambda^2}{4}\right) \left( P.V. \int_{\mathbb{R}} \frac{\mu}{\mu-\lambda} \exp\left(-\frac{\mu^2}{4}\right) U_\mu d\mu \right) \exp(2i\pi kx)
\end{aligned}$$

which is the claim. A similar verification yields the other formula. The proof is ended.  $\square$

The spectral completeness of the eigenvectors  $V_{e,k}^\pm$  can be proved, the difficulty being to find the measure. Hopefully we can use a formula proved in [24]. To simplify and unify the notations, we note the right hand side of (5.2-5.3)

$$\begin{aligned}
V_{\lambda,k} & = \left( 4\pi^2 k^2 + P.V. \int_{-\infty}^\infty \frac{\mu}{\mu-\lambda} \exp\left(-\frac{\mu^2}{2}\right) d\mu \right) U_\lambda \\
& + 2i\pi k \exp\left(-\frac{\lambda^2}{4}\right) e_0 - \exp\left(-\frac{\lambda^2}{4}\right) \left( P.V. \int_{\mathbb{R}} \frac{\mu}{\mu-\lambda} \exp\left(-\frac{\mu^2}{4}\right) U_\mu d\mu \right). \tag{5.5}
\end{aligned}$$

This definition is for the Fourier mode  $k \neq 0$  and  $\lambda \in \mathbb{R}$ .

Set  $\varepsilon_I^k(\lambda) = \frac{1}{4\pi^2 k^2} \lambda \exp\left(-\frac{\lambda^2}{2}\right)$ ,

$$\varepsilon_R^k(\lambda) = 1 + \frac{1}{\pi} P.V. \int_{\mathbb{R}} \frac{\varepsilon_I(\mu)}{\mu-\lambda} d\mu = \frac{1}{4\pi^2 k^2} \left( 4\pi^2 k^2 + P.V. \int_{-\infty}^\infty \frac{\mu}{\mu-\lambda} \exp\left(-\frac{\mu^2}{2}\right) d\mu \right),$$

$$|\varepsilon^k(\lambda)|^2 = |\varepsilon_R^k(\lambda)|^2 + |\varepsilon_I^k(\lambda)|^2$$

and  $v_{0,k} = (-2i\pi k, \alpha, 0, 0, \dots)^t$  which is exactly (4.5) for  $g = \exp(2i\pi kx)$ . Notice that for all  $\lambda$

$$\begin{aligned}
V_{\lambda,k} \cdot v_{0,k} & = -4\pi^2 k^2 \exp\left(-\frac{\lambda^2}{4}\right) + \alpha \left( 4\pi^2 k^2 + P.V. \int_{-\infty}^\infty \frac{\mu}{\mu-\lambda} \exp\left(-\frac{\mu^2}{2}\right) d\mu \right) \frac{1}{\alpha} \exp\left(-\frac{\lambda^2}{4}\right) \\
& - \alpha \exp\left(-\frac{\lambda^2}{4}\right) \left( P.V. \int_{\mathbb{R}} \frac{\mu}{\mu-\lambda} \frac{1}{\alpha} \exp\left(-\frac{\mu^2}{4}\right) \exp\left(-\frac{\mu^2}{4}\right) d\mu \right) = 0
\end{aligned}$$

which is interpreted as the standard orthogonality condition between eigenvectors (with different eigenvalues) of an hermitian operator.

**THEOREM 31.** *For all  $k \neq 0$ , and for all  $W, Z \in l^2$  one has*

$$W \cdot W = \frac{|W \cdot v_{0,k}|^2}{4\pi^2 k^2 + \alpha^2} + \int_{\mathbb{R}} \frac{|W \cdot V_{\lambda,k}|^2}{(2\pi k)^4 |\varepsilon^k(\lambda)|^2} d\lambda.$$

**REMARK 32.** *This formula is identified as a representation of the identity for the spectral representation of unbounded hermitian operators, which is a key object in scattering theory.*

The purely technical proof, based on the inverse formula in [24], is postponed to the appendix.

**6. Moller wave operators.** Once the eigenvectors are constructed, it is possible to define in the general case formally a Moller wave operator as

$$LU = \sum_z \sum_p \int q_{s,p}^z (U, V_{s,p}^z) U_{s,p}^z t_s^z d_s^z \quad (6.1)$$

where  $q_{s,p}^z$  is a weight which guarantees the summability of the formula. Under this form the Moller operator is just a change of basis formula.

REMARK 33. *The wave operators are traditionally defined as unitary operators [15, 19]. In the homogeneous case, this can be achieved with a convenient choice of the weight  $q_{s,k}^z$  dictated by the completeness Theorem 31. Nevertheless we do not pursue in this direction since it is not needed for the analysis below.*

The main formal property is the following.

THEOREM 34. *Assume  $U$  is solution to  $\partial_t U = iHU$ . Then  $W = LU$  is solution to  $\partial_t W = iH_0W$ .*

*Proof.* Indeed

$$\begin{aligned} \partial_t W &= \sum_z \sum_p \int q_{s,p}^z (iHU, V_{s,p}^z) U_{s,p}^z t_s^z d_s^z = i \sum_z \sum_p \int q_{s,p}^z (U, HV_{s,p}^z) U_{s,p}^z t_s^z d_s^z \\ &= i \sum_z \sum_p \int q_{s,p}^z (U, V_{s,p}^z) \lambda_{s,p}^z U_{s,p}^z t_s^z d_s^z = \sum_z \sum_p \int q_{s,p}^z (U, V_{s,p}^z) iH_0 U_{s,p}^z t_s^z d_s^z = iH_0 W. \end{aligned}$$

The proof is ended.  $\square$

**7. Morrison transform and linear Landau damping.** In the homogeneous case, the Moller wave operator defined with the general change of basis function boils down to a simple integro-differential operator for a kinetic function. Consider  $u(x, v) = U(x) \cdot U_v$  and  $\mathcal{L}u(x, v) = (LU)(x) \cdot U_v$ .

PROPOSITION 35. *Consider the homogeneous case. Take  $q_{s,k}^z = 4\pi^2 k^2 + P.V. \int_{-\infty}^{\infty} \frac{\mu}{\mu - \lambda} \exp\left(-\frac{\mu^2}{2}\right) d\mu$ . Assume the Gauss law  $\partial_x F(x) = -\int u(x, v) M(v) dv$  is satisfied.*

*Then one has the formula*

$$\begin{aligned} \mathcal{L}u &= \left( -\partial_{xx} + \left( 4\pi^2 p^2 + P.V. \int_{-\infty}^{\infty} \frac{\mu}{\mu - v} \exp\left(-\frac{\mu^2}{2}\right) d\mu \right) \right) u(x, v) \\ &\quad - v \exp\left(-\frac{v^2}{4}\right) \left( P.V. \int_{\mathbb{R}} \frac{1}{\mu - v} \exp\left(-\frac{\mu^2}{4}\right) u(x, \mu) d\mu \right) \end{aligned}$$

*which is the Morrison transform [6, 24].*

*Proof.* The Fourier decomposition holds in the homogeneous case. Using (5.5) one gets

$$LU = \sum_p \int_{\mathbb{R}} (U, V_{\lambda,p}) U_{\lambda} \exp(2ipx) d\lambda.$$

One has

$$\begin{aligned} (U, V_{\lambda,p}) &= \left( U, \left( 4\pi^2 p^2 + P.V. \int_{-\infty}^{\infty} \frac{\mu}{\mu - \lambda} \exp\left(-\frac{\mu^2}{2}\right) d\mu \right) U_{\lambda} \exp(2ipx) \right. \\ &\quad \left. + 2i\pi k \exp\left(-\frac{\lambda^2}{4}\right) \exp(2ipx) e_0 - \exp\left(-\frac{\lambda^2}{4}\right) \left( P.V. \int_{\mathbb{R}} \frac{\mu}{\mu - \lambda} \exp\left(-\frac{\mu^2}{4}\right) U_{\mu} \exp(2ipx) d\mu \right) \right). \end{aligned}$$

Using the Gauss law one gets

$$\begin{aligned} (U, V_{\lambda,p}) &= \left( U, \left( 4\pi^2 p^2 + P.V. \int_{-\infty}^{\infty} \frac{\mu}{\mu - \lambda} \exp\left(-\frac{\mu^2}{2}\right) d\mu \right) U_{\lambda} \exp(2ipx) \right. \\ &\quad \left. - \lambda \exp\left(-\frac{\lambda^2}{4}\right) \left( P.V. \int_{\mathbb{R}} \frac{1}{\mu - \lambda} \exp\left(-\frac{\mu^2}{4}\right) U_{\mu} \exp(2ipx) d\mu \right) \right) \end{aligned}$$

$$\begin{aligned}
&= \left( 4\pi^2 p^2 + P.V. \int_{-\infty}^{\infty} \frac{\mu}{\mu - \lambda} \exp\left(-\frac{\mu^2}{2}\right) d\mu \right) (U, U_\lambda \exp(2ipx)) \\
&\quad - \lambda \exp\left(-\frac{\lambda^2}{4}\right) \left( P.V. \int_{\mathbb{R}} \frac{1}{\mu - \lambda} \exp\left(-\frac{\mu^2}{4}\right) (U, U_\mu \exp(2ipx)) d\mu \right).
\end{aligned}$$

Therefore

$$\begin{aligned}
LU &= \int_{\mathbb{R}} \sum_p \left[ \left( 4\pi^2 p^2 + P.V. \int_{-\infty}^{\infty} \frac{\mu}{\mu - \lambda} \exp\left(-\frac{\mu^2}{2}\right) d\mu \right) (U, U_\lambda \exp(2i\pi p x)) \right. \\
&\quad \left. - \lambda \exp\left(-\frac{\lambda^2}{4}\right) \left( P.V. \int_{\mathbb{R}} \frac{1}{\mu - \lambda} \exp\left(-\frac{\mu^2}{4}\right) \sum_p \int_{\mathbb{R}} (U, U_\mu \exp(2i\pi p x)) d\mu \right) \right] U_\lambda \exp(2ipx) d\lambda,
\end{aligned}$$

that is

$$\begin{aligned}
\mathcal{L}u(x, \lambda) &= \sum_p \left( 4\pi^2 p^2 + P.V. \int_{-\infty}^{\infty} \frac{\mu}{\mu - \lambda} \exp\left(-\frac{\mu^2}{2}\right) d\mu \right) (U, U_\lambda \exp(2i\pi p x)) \exp(2ipx) \\
&\quad - \sum_p \lambda \exp\left(-\frac{\lambda^2}{4}\right) \left( P.V. \int_{\mathbb{R}} \frac{1}{\mu - \lambda} \exp\left(-\frac{\mu^2}{4}\right) \sum_p \int_{\mathbb{R}} (U, U_\mu \exp(2i\pi p x)) d\mu \right) \exp(2ipx) \\
&= \left( 4\pi^2 p^2 + P.V. \int_{-\infty}^{\infty} \frac{\mu}{\mu - \lambda} \exp\left(-\frac{\mu^2}{2}\right) d\mu \right) U \cdot U_\lambda - \lambda \exp\left(-\frac{\lambda^2}{4}\right) \left( P.V. \int_{\mathbb{R}} \frac{1}{\mu - \lambda} \exp\left(-\frac{\mu^2}{4}\right) U \cdot U_\mu d\mu \right) \\
&= \left( 4\pi^2 p^2 + P.V. \int_{-\infty}^{\infty} \frac{\mu}{\mu - \lambda} \exp\left(-\frac{\mu^2}{2}\right) d\mu \right) u(x, \lambda) - \lambda \exp\left(-\frac{\lambda^2}{4}\right) \left( P.V. \int_{\mathbb{R}} \frac{1}{\mu - \lambda} \exp\left(-\frac{\mu^2}{4}\right) u(x, \mu) d\mu \right).
\end{aligned}$$

The proof is ended.  $\square$

The kinetic counterpart of the formal proposition 34 is as follows.

**PROPOSITION 36.** *Take  $u$  that satisfies*

$$\partial_t u + v \partial_x u + v M E = 0$$

together with the Gauss law  $\partial_x F(t, x) + \int_{\mathbb{R}} u(t, x, v) M(v) dv = 0$ . Then  $h = Lu$  satisfies the transport equation in free space

$$\partial_t h + v \partial_x h = 0. \tag{7.1}$$

*Proof.* Let us start from  $\partial_t u + v \partial_x u + v M(v) E = 0$ . Since  $\partial_t L = L \partial_t$  one has  $\partial_t h + L(v \partial_x u) + L(v M(v) E) = 0$ . The second term is

$$\begin{aligned}
L(v \partial_x u) &= v \partial_x ((-\partial_{xx} + q(v))u) - v M(v) P.V. \int_{\mathbb{R}} \frac{1}{w - v} w \partial_x u(x, w) M(w) dw \\
&= v \partial_x Lu - v M(v) \int_{\mathbb{R}} \partial_x u(x, w) M(w) dw = v \partial_x Lu - v M(v) \partial_x \int_{\mathbb{R}} u(x, w) M(w) dw
\end{aligned}$$

where we have used that  $\frac{w}{w-v} = \frac{v}{w-v} + 1$ . The last term is

$$L(v M(v) E) = v M(v) (-\partial_{xx} + q(v)) E - E v M(v) P.V. \int_{\mathbb{R}} \frac{w}{w-v} M(w)^2 dw$$

$$= vM(v)(-\partial_{xx} + q(v))E - EvM(v)q(v) = -vM(v)\partial_{xx}E.$$

The Gauss law  $\partial_x E + \int_{\mathbb{R}} u(w)M(v)dw = 0$  implies cancellations in the sum of these two terms which simplifies into  $L(v\partial_x u) + L(vM(v)E) = v\partial_x Lu = v\partial_x h$ . The proof is ended.  $\square$

This formula finally yields a proof of the linear Landau damping phenomenon for an initial data with no contribution on the Fourier mode  $k = 0$ . In this case it is known that one can write

$$E(t, x) = \int_I \int_{\mathbb{R}} K_x(y, v)u(t, y, v)dydv$$

where  $K_x$  is a smooth kernel [6] in the range of  $L^*$ . One gets formally

$$\begin{aligned} E(t, x) &= (u, K_x) = (Lu, L^{-*}K_x) = (\exp(iH_0t) Lu_0, L^{-*}K_x) \\ &= \int_I \int_{\mathbb{R}} u_0(t, y - vt, v)(L^{-*}K_x)(y, v)dydv. \end{aligned}$$

It is easy to prove under this form that the electric field tends to zero for  $t \rightarrow +\infty$ , with a rate that is function of the regularity of  $Lu_0$ . Details are in [25, 24, 6].

REMARK 37. *In the general case, the use of the Moller operator is ultimately a consequence of the well-posedness of the integral equation (4.3) or (4.7). A necessary condition seems to be that the travel time is a strictly monotone function of the characteristic label. Further research will devoted to the study of this equation.*

REMARK 38. *The restriction of a Boltzmannian stationary state can probably be relaxed using other orthogonal polynomials [6]. The multivalued case, that is  $f_0(x, v) = F(v^2/2 - \varphi(x))$  with a multivalued  $F$ , needs probably different ideas.*

REMARK 39. *The use of the methods pursued in this work in conjunction with non linear techniques such as [20, 21, 22, 23, 26, 28, 10, 11] is an open problem. We also quote possible connections with other techniques such as [8, 3].*

### Appendix A. Proof of Theorem 31.

The proof is performed in several steps.

- The vector  $W \in l^2$  is decomposed as  $W = \frac{W \cdot v_{0,k}}{4\pi^2 k^2 + \alpha^2} v_{0,k} + \widehat{W}$ . The orthogonality  $W \cdot v_{0,k} = 0$  yields that

$$W \cdot W = \frac{|W \cdot v_{0,k}|^2}{4\pi^2 k^2 + \alpha^2} + \widehat{W} \cdot \widehat{W}. \quad (\text{A.1})$$

So it remains to prove that  $\widehat{W} \cdot \widehat{W} = \int_{\mathbb{R}} \frac{|W \cdot V_{\lambda,k}|^2}{(2\pi k)^4 |\varepsilon^k(\lambda)|^2} d\lambda$ .

- This is actually a consequence of Theorem (G4) in [24]. Let us use the notations of the reference and restrict the proof to its algebraic aspects.

Define the Hilbert transform  $H[f](\lambda) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{1}{\mu - \lambda} f(\mu) d\mu$  and the operators  $G$  and  $\widehat{G}$  by

$$G(g) = \varepsilon_R^k g + \varepsilon_I^K H[g] \quad \text{and} \quad \widehat{G}(g) = \frac{1}{|\varepsilon^k|^2} (\varepsilon_R^k g - \varepsilon_I^K H[g]).$$

The formula of Theorem (G4) writes  $\widehat{G}(G(g)) = g$  from which we deduce  $G(\widehat{G}(g)) = g$ . It yields the integral form

$$\int_{\mathbb{R}} \widehat{G}(g)(\lambda) \overline{G^*(h)(\lambda)} d\lambda = \int_{\mathbb{R}} g(\lambda) \overline{h(\lambda)} d\lambda \quad (\text{A.2})$$

with natural integrability conditions for which we refer to [24].

- Define  $\widehat{u}(\lambda) = \widehat{W} \cdot U_\lambda$  and  $g(\lambda) = \exp\left(-\frac{\lambda^2}{4}\right) \widehat{u}(\lambda)$ . One has that

$$\begin{aligned} \widehat{G}(g)(\lambda) &= \frac{1}{|\varepsilon^k|^2} (\varepsilon_R^k g - \varepsilon_I^K H[g]) \\ &= \frac{\exp\left(-\frac{\lambda^2}{4}\right)}{4\pi^2 k^2 |\varepsilon^k|^2} \left( \left( 4\pi^2 k^2 + P.V. \int_{-\infty}^{\infty} \frac{\mu}{\mu - \lambda} \exp\left(-\frac{\mu^2}{2}\right) d\mu \right) \widehat{u}(\lambda) \right) \end{aligned}$$

$$\begin{aligned}
& -\lambda \exp\left(-\frac{\lambda^2}{4}\right) P.V. \int_{\mathbb{R}} \frac{1}{\mu - \lambda} \exp\left(-\frac{\mu^2}{4}\right) \widehat{u}(\mu) d\mu \\
&= \frac{\exp\left(-\frac{\lambda^2}{4}\right)}{|\varepsilon^k|^2} \left( \left( 4\pi^2 k^2 + P.V. \int_{-\infty}^{\infty} \frac{\mu}{\mu - \lambda} \exp\left(-\frac{\mu^2}{2}\right) d\mu \right) \widehat{u}(\lambda) - \right. \\
&\quad \left. \exp\left(-\frac{\lambda^2}{4}\right) P.V. \int_{\mathbb{R}} \frac{\mu}{\mu - \lambda} \exp\left(-\frac{\mu^2}{4}\right) \widehat{u}(\mu) d\mu + \exp\left(-\frac{\lambda^2}{4}\right) \int_{\mathbb{R}} \exp\left(-\frac{\mu^2}{4}\right) \widehat{u}(\mu) d\mu \right) \\
&= \frac{\exp\left(-\frac{\lambda^2}{4}\right)}{4\pi^2 k^2 |\varepsilon^k|^2} \left( \widehat{W} \cdot \left( V_{\lambda,k} - 2i\pi k \exp\left(-\frac{\lambda^2}{4}\right) e_0 \right) + \exp\left(-\frac{\lambda^2}{4}\right) \int_{\mathbb{R}} \exp\left(-\frac{\mu^2}{4}\right) \widehat{u}(\mu) d\mu \right).
\end{aligned}$$

Since one has that

$$\exp\left(-\frac{\lambda^2}{4}\right) \int \exp\left(-\frac{\mu^2}{4}\right) u(\mu) d\mu = \exp\left(-\frac{\lambda^2}{4}\right) \int \exp\left(-\frac{\mu^2}{4}\right) W \cdot U_{\lambda} d\mu = \exp\left(-\frac{\lambda^2}{4}\right) \alpha W \cdot e_1$$

it yields

$$\begin{aligned}
& \widehat{W} \cdot \left( -2i\pi k \exp\left(-\frac{\lambda^2}{4}\right) e_0 \right) + \exp\left(-\frac{\lambda^2}{4}\right) \int \exp\left(-\frac{\mu^2}{4}\right) u(\mu) d\mu \\
&= \exp\left(-\frac{\lambda^2}{4}\right) W \cdot (-2i\pi k e_0 + \alpha e_1) = \exp\left(-\frac{\lambda^2}{4}\right) W \cdot v_{0,k} = 0.
\end{aligned}$$

Therefore

$$\widehat{G}(g)(\lambda) = \frac{\exp\left(-\frac{\lambda^2}{4}\right)}{4\pi^2 k^2 |\varepsilon^k|^2} \widehat{W} \cdot V_{\lambda,k}. \quad (\text{A.3})$$

- Define  $h(\lambda) = \exp\left(+\frac{\lambda^2}{4}\right) \widehat{u}(\lambda)$ . Then

$$\begin{aligned}
G^*(h) &= \frac{\exp\left(\frac{\lambda^2}{4}\right)}{4\pi k^2} \left( \left( 4\pi^2 k^2 + P.V. \int_{-\infty}^{\infty} \frac{\mu}{\mu - \lambda} \exp\left(-\frac{\mu^2}{2}\right) d\mu \right) \widehat{u}(\lambda) \right. \\
&\quad \left. - \exp\left(-\frac{\lambda^2}{4}\right) P.V. \int \frac{\mu}{\mu - \lambda} \exp\left(-\frac{\mu^2}{4}\right) \widehat{u}(\mu) d\mu \right)
\end{aligned}$$

that is

$$G(g)(\lambda) = \frac{\exp\left(+\frac{\lambda^2}{4}\right)}{4\pi^2 k^2} \widehat{W} \cdot V_{\lambda,k}. \quad (\text{A.4})$$

- Plug (A.3-A.4) in (A.2)

$$\begin{aligned}
& \int_{\mathbb{R}} \frac{1}{(2\pi k)^4 |\varepsilon^k|^2} \left| \widehat{W} \cdot V_{\lambda,k} \right|^2 d\lambda = \int g(\lambda) \overline{h(\lambda)} d\lambda \\
&= \int \exp\left(-\frac{\lambda^2}{4}\right) \widehat{u}(\lambda) \overline{\exp\left(+\frac{\lambda^2}{4}\right) \widehat{u}(\lambda)} d\lambda = \int \left| \widehat{W} \cdot V_{\lambda} \right|^2 d\lambda = \widehat{W} \cdot \widehat{W}
\end{aligned}$$

using the orthogonality of the Hermite functions (3.14). Plug in (A.1). The proof is ended.

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