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Are the validities of modal logic analytic?

Or analyticity again, through information , proof, modal logic and Hintikka

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Abstract

Dans la philosophie de Hintikka (e.g. Hintikka (1973), Hintikka (2007)) la notion d'analyticité occupe une place particulière ; plus précisément, le philosophe finlandais distingue deux notions d'analyticité : l'une qui est basée sur la notion d'information, l'autre sur la notion de preuve. Alors que ces deux notions ont été largement utilisées pour étudier la logique propositionnelle et la logique du premier ordre, aucun travail a été développé pour la logique modale. Cet article se propose de combler cette lacune et ainsi d'examiner l'analyticité des validités de la logique modale.

Abstract

In Hintikka's philosophy (e.g. Hintikka (1973), Hintikka (2007)) the notion of analyticity occupies a special place; in particular, the Finnish thinker distinguishes two notions of analyticity: one which is based on the concept of information, the other on the concept of proof. While these two notions have been broadly used to analyze propositional logic and first-order logic, no work has been done in modal propositional logic. This paper aims at filling this gap and thus studying the analyticity of the validities of modal logic.

1 Introduction

The title of this paper "Are the validities of modal logic analytic?" makes an explicit reference to Hintikka's famous article "Are logical truths analytic?" published in the *Philosophical Review* in 1965. In this article Hintikka distinguishes for the first time two ways of understanding the word *analytic*¹

1. A sentence is said to be analytic when it does not convey any factual information.
2. A sentence is said to be analytic when it can be shown to be valid by strictly analytic methods.

With respect to these notions of analyticity, Hintikka raises two questions. The first is: are these two notions equivalent? The second is: are the validities of first-order logic, for short *FOL*, analytic in either of the two senses of analyticity? As concerns the first question, the answer is negative: the two definitions are not equivalent and this is illustrated by Hintikka with a proof that 1 and 2 are not extensionally equivalent. The fact that notions 1 and 2 of analyticity are not extensionally equivalent is in its turn proved precisely by examining the second question. Indeed, by considering the validities of *FOL*, one can prove that these are analytic according to sense 1 of analyticity, but not according to sense 2 of analyticity.

In this paper we have the following aim: we want to consider the main normal systems of modal logic (which is to say, those systems that form the cube of modal logic, see Blackburn et al. (2001) or Poggiolesi (2010), e.g. systems *K*, *KT*, *KB*, ... *S4* and *S5*) and examine whether their validities are analytic in the two senses of analyticity introduced by Hintikka. This will shed light not only on the analyticity of modal logic – which is a question that has been, as far as we know, largely neglected – but also on the notions of information and proof which are central for the two senses of analyticity considered here.

In order to develop our aim, we will organize the paper as follows. In the next section we will analyze the main normal systems of modal logic with respect to notion 1 of analyticity, while, in Section 3, we will analyze them with respect to notion 2 of analyticity. In Section 4, we will end the paper with some general conclusions.

2 Modal logic and the conveyance of information

The task of this section is to check whether the validities of modal logic are analytic according to sense 1 of analyticity; in other words, the task of this section is to test whether the validities of modal logic convey any information at all. In order to accomplish this task, we firstly have to deal with two points.

¹The distinction between different senses of analyticity is investigated by Hintikka in many articles, chapters and books (e.g. Hintikka (1973), Hintikka (2007)).

The first point concerns the notion of information: we need to clearly specify the sense in which we will use this notion, if we want our argument to proceed straightforwardly. According to a long tradition of philosophers (e.g. Wittgenstein, Carnap, Bar-Hillel and Hintikka), one can define the information a sentence conveys in terms of the possibilities it excludes. More precisely, the more possibilities the sentence excludes, the more information it conveys; the more possibilities the sentence includes, the less information it conveys. As Wittgenstein (1974) put it: "I know nothing about the weather if I know that it is raining or not raining". In the rest of the paper, we will use the word *information* in this precise sense.

Once the notion of information is clarified, let us move to the second point, which concerns the strategy we want to adopt in order to fulfill the task of this section. The idea is to proceed by the following three stages. Firstly, we will explain how Wittgenstein (1974) established that the validities of propositional logic are analytic according to sense 1 of analyticity; secondly, we will explain how Hintikka established that the validities of FOL are analytic according to sense 1 of analyticity. Finally, by relying on the first two stages, we will test the analyticity of the validities of modal logic.

Propositional Logic. Let us start from the first stage of our strategy, i.e. the explanation of how Wittgenstein proved that propositional logic is analytic according to sense 1 of analyticity. For this, consider a simple propositional validity like $q \rightarrow (p \rightarrow q)$, and call this validity A ; then focus on the non-logical resources employed in A , namely the propositional atoms p and q . Now suppose that one wants to list all the alternatives concerning the world that can be expressed by means of these resources. The following four formulas do the trick: $p \wedge q, p \wedge \neg q, \neg p \wedge q, \neg p \wedge \neg q$. Assuming that p stands for "it is raining" and q for "it is windy", then the four formulas describe the four possible situations: "it is raining and it is windy", "it is raining and it is not windy", "it is not raining and it is windy", and "it is neither raining nor windy", respectively. These four formulas represent all and only those alternatives concerning the world that can be expressed by means of p and q . From now on formulas of this sort will be called *constituents*. More precisely a constituent can be defined as follows.

Definition 2.1. Let p_i ($i = 1, 2, \dots, n$) be atomic propositional sentences. Constituents are conjunctions which, for each i , contain either p_i or its negation $\neg p_i$.

Propositional constituents are thus defined with respect to certain resources - in this case the propositional atoms p and q - and with respect to a certain framework - in this case propositional logic. Each constituent represents the description of an alternative concerning the world according to the resources and the framework at our disposal.

Construct now the disjunction of the four constituents $p \wedge q, p \wedge \neg q, \neg p \wedge q, \neg p \wedge \neg q$ and call this disjunction A' . By means of the *distributive normal form theorem* (DTNF theorem for short) for propositional logic, one can prove that

the propositional validity A is logically equivalent to A' . So A is a logical consequence of A' ; since A' is a disjunction, A is a logical consequence of each disjunct of A' . But then, since each disjunct is nothing but an alternative concerning the world, this means that A is compatible with any of these alternatives concerning the world, which is to say A does not exclude any of them. According to the definition of information that we have given at the beginning of this section, the fact that A does not rule out any possibility amounts to the fact that A does not convey any genuine information. So, if A is shown to (be equivalent to A' and hence to) be a logical consequence of A' , then A is shown not to convey any factual information and so to be analytic according to sense 1 of analyticity.

By repeating this reasoning for all tautologies of propositional logic, Wittgenstein concluded that propositional logic is analytic according to sense 1 of analyticity. Let us sum up his argument with the following table:

Figure 1.

Analyticity of Propositional Logic	
1	consider a validity A
2	focus on its non-logical resources
3	list all the constituents constructed from those resources
4	form the disjunction A' of all such constituents
5	prove that A is a logical consequence of A' by means of the DTNF theorem
6	conclude that A is analytic

As the reader can easily see, in order to establish the analyticity of propositional logic, one can use a six-step strategy, where each step seems indispensable for the successive one. In order to show that the validities of FOL are analytic according to sense 1 of analyticity, Hintikka tried to use the same six-step strategy. We will describe his attempt here.

First-order logic. Consider a simple validity of FOL like $\exists x\exists y(Px, y) \rightarrow \neg\forall x\forall y\neg(Px, y)$ and call this validity B ; then focus on the non-logical resources employed in B , namely the dyadic predicate P . Now consider the attempt to list all the alternatives concerning the world which can be expressed by means of these resources in the framework of FOL. A little reflection suffices to realize that there is no hope of accomplishing such a task. First of all, each alternative concerning the world constructed by means of P in the framework of FOL would be nothing but an infinite conjunction (or an infinite set). Indeed not only would we have to specify which individuals stand in the relation P , but also which individuals stand in the relation P with other individuals that already stand in the relation P , and also which individuals stand in the relation P with other individuals that already stand in the relation P with other individuals that already stand in the relation P and so on . . .

Secondly, the list of all these (infinite) alternatives concerning the world would itself be infinite; indeed, if we had a finite list of all alternatives concerning the world which can be expressed in the framework of FOL, FOL would be

decidable. But FOL is not decidable, and so the list is infinite. Thus it seems that there is no way of drawing a finite list of finite alternatives concerning the world in the framework of FOL and thus that we are stuck at point 3 of the following table.

Analyticity of FOL / Part 1	
1	consider a validity B
2	focus on its non-logical resources
3	X

In order to overcome this major obstacle and reach his goal, namely the analogue of point 6 of Figure 1, Hintikka proceeds in the following way. Firstly, he introduces the notion of *degree* through the notion of *depth*.

Definition 2.2. The depth of a formula A , denoted $d(A)$, can be inductively defined in the following way:

$$d(A) = 0, \text{ if } A \text{ is an atomic formula or an identity}$$

$$d(\neg A) = d(A)$$

$$d(A_1 \wedge A_2) = d(A_1 \vee A_2) = \max((d(A_1), d(A_2)))$$

$$d(\exists x A) = d(\forall x A) = d(A) + 1$$

Definition 2.3. The degree of a formula A , $dg(A)$, corresponds to the sum of its depth plus the number of free individual symbols occurring in it (constants or free variables).

Once the notion of degree is introduced, Hintikka uses it to introduce the notion of *quantificational constituent* which represents a rough formal counterpart of the idea of alternative concerning the world that circumvents the aforementioned issue. The notion of quantificational constituent has been broadly used and discussed in the literature, e.g. Hodges (1993); Rantala (1987). Its definition however is quite long and laborious; therefore we have decided not to present it here. The interested reader can see Hintikka (1953); Hodges (1993). We will instead give an example of several quantificational constituents, thus the reader can grasp the intuitive idea without being burdened with too many technical details.

In order to give an example of quantificational constituents, we will use monadic predicates. This choice could be seen as controversial since monadic first-order logic is that part of first-order logic for which the notion of degree, which is central in the notion of quantificational constituent, is less required.² Nevertheless monadic first-order predicates represent the easiest way to give

²This point is strongly underlined by Hintikka (1965). In monadic first-order logic we do not need the notion of degree since in this logic we can draw finite lists of finite quantificational constituents.

an idea of what quantificational constituents are (even Hintikka (1965) uses monadic first-order predicates); in this sense we feel that our choice is justified.

Let us consider two monadic predicates P and Q . The technique for constructing quantificational constituents out of these predicates is fairly simple: firstly, one needs to list the possible kinds of individuals that can be specified by means of the predicates P and Q ; then, for each kind of individual, one needs to indicate whether individuals of that kind exists or not. Suppose then that P stands for "x is red" and Q for "x is round". Then we have four kinds of individuals that can be specified by means of "x is red" and "x is round", namely: " x is red and round", " x is not red but is round", " x is red but is not round" and " x is not red nor round". In order to indicate whether these kinds of individuals exists or not, we existentially quantify over them. So for example we have $\exists x(Px \wedge Qx) \wedge \exists x(\neg Px \wedge Qx) \wedge \exists x(Px \wedge \neg Qx) \wedge \exists x(\neg Px \wedge \neg Qx)$, but also $\neg \exists x(Px \wedge Qx) \wedge \exists x(\neg Px \wedge Qx) \wedge \exists x(Px \wedge \neg Qx) \wedge \exists x(\neg Px \wedge \neg Qx)$, and so on. These formulas are examples of quantificational constituents of degree 1. As the reader can easily see quantificational constituents represent the rough, since limitative, formal counterpart of the idea of alternative concerning the world.

Thanks to the notion of degree, we thus have found a way to formally and finitely describe the notion of alternative concerning the world at the first-order level. We can use also the notion of degree to limit the number of quantificational constituents that need to be taken into account with respect to a validity B . Consider indeed our first-order formula B which has a degree n ; it can be proved that B is equivalent to a disjunction of some (perhaps all) quantificational constituents of degree n . In other words, thanks to the notion of degree, given a formula B (of degree n), it can be finally proved that B has a distributive normal form composed by quantificational constituents of degree n . This is precisely the result that we wanted to obtain. Such a result can be found in Hintikka (1953, 1964).

Let us then sum up what we have just seen in the following table

Analyticity of FOL / Part 2	
1	consider a validity B
2	focus on its non-logical resources
3a	introduce the notion of degree and determine the degree n of B
4a	form the disjunction B' of all quantificational constituents of degree n
5a	prove that B is a logical consequence of B' by means of the DTNF theorem

Given this result, a question naturally arises: can Hintikka bring his strategy to an end by passing to step 6? In other words, can Hintikka conclude that the validities of FOL are analytic according to notion 1 of analyticity? The answer is affirmative as long as the notion of degree is *justified*. Indeed, since the whole strategy used to infer the analyticity of FOL heavily relies on the notion of degree, it must be shown that such a notion has not been introduced *ad hoc* for limiting the infinity that looms in the notion of alternative concerning the

world in the framework of FOL, but is on the contrary a natural and intuitive concept.³

Hintikka clearly sees this point and elaborates the following defense of the notion of degree. First of all, Hintikka emphasizes that FOL is a logic suited for talking about individuals. But how are individuals introduced in formulas of FOL? Basically in two ways: either by free individual symbols, or by quantifiers – “the existential quantifier should be read somewhat as follows: ‘there is at least one individual (call it x), such that’, and the universal quantifier should be read: ‘each individual (call it x) is such that’. These translations make it clear that, although the bound variable ‘ x ’ does not stand for any particular individual, each quantifier invites us to consider one individual in addition to the other ones which may have been introduced earlier” (Hintikka, 1973, p. 138-139). The formal notion of degree represents the way of counting individuals introduced in sentences of FOL in either of these two ways: more precisely, the notion of depth counts the number of (embedded) quantifiers and then the number of individual symbols is added to reach the notion of degree.

Let us emphasize that, by means of this notion of degree, one does not consider any individual whatsoever, but relations and individuals involved in these relations. This is the reason why the scopes of the quantifiers which are counted must overlap.

But why should one limit oneself in considering individuals and their relations? “Does not a general sentence speak of all individuals of the domain (universe of discourse)? Is not the number of individuals considered in such a sentence therefore infinite if the domain is infinite?” (Hintikka, 1965, p. 187). Hintikka provides the following answers to these crucial questions. Of course a general sentence speaks in some sense of all the individuals in the domain, but it never speaks of all the individuals in their relations to each other. Consider for example the sentence “All men admire Buddha”. In this case

We are, so to speak, considering each man at a time and saying something about *his* relation to the great Gautama. Hence the number of individuals considered in their relation to each other in this sentence is two, which is just its degree. (Hintikka, 1965, p. 187)

Another example that Hintikka makes is that of the sentence “John has at least one brother and John has at least one sister”. In this sentence one considers the relation between John and one of his brothers and between John and one of his sisters, but nothing is said about the relation between the three of them.

³For the sake of clarity let us underline that after having defended his notion of degree, Hintikka spends several pages (Hintikka, 1965, pp. 188-190) on defending the idea that quantificational constituents represent alternatives concerning the world in as a clear-cut sense as do the constituents of propositional logic; he also shows that validities of first-order logic are related to quantificational constituents in the same way as validities of propositional logic are related to constituents. These are important points if one wants to argue for the analyticity of FOL. Nevertheless, since they do not occupy any central role in our paper, we do not dwell on them.

Hence the number of individuals considered together at any given time in the sentence is only two, which is again exactly its degree. (Hintikka, 1965, p. 188)

Given these arguments, Hintikka concludes that in our first-order formulas one never speaks of all individuals, but just of individuals and their relations; the notion of degree precisely counts this number. Thus the notion of degree is completely intuitive and natural and its use in the definition of the notion of quantificational constituent, as well as in the theorem of distributive normal form, is reasonable and justified. As a consequence, the validities of FOL are analytic in sense 1 of analyticity.

Figure 2.

Analyticity of FOL	
1	consider a validity B
2	focus on its non-logical resources
3a	introduce the notion of degree and determine the degree n of B
4a	form the disjunction B' of all quantificational constituents of degree n
5a	prove that B is a logical consequence of B' by means of the DTNF theorem
5b	justify the notion of degree
6	conclude that B is analytic

We have thus completed the first two stages of our task, namely we have shown the analyticity of both propositional logic and first-order logic.

Intermezzo. Before passing to the third stage of our strategy, namely before checking whether the validities of modal logic are analytic according to sense 1 of analyticity, let us underline the following two important points. The first concerns the notion of *inconsistent quantificational constituent* that Hintikka broadly discusses in his paper and which is completely omitted here. The reason for this is that such a notion seems irrelevant for the discussion of the modal case. Thus a detailed exposition of it would have needlessly burdened the paper. (For a detailed description of inconsistent quantificational constituents, see, e.g., (Hintikka, 1973, 163-174))

The second point concerns the strategy adopted in Figures 1 and 2 for showing the analyticity of propositional logic and FOL, respectively. Someone could argue that in both cases one is making much ado about nothing. Indeed it seems that, by simply using the validity theorem for propositional and first-order logic, one could show that these logics are analytic according to sense 1 of analyticity. The argument would proceed as follows. (We just show it for the case of FOL, the case of propositional logic is analogous). Consider a validity S of FOL which, by definition, is a sentence true in any model of FOL. But a model of FOL can be seen as an alternative concerning the world described with the resources of FOL. Thus the fact that S is true in any model of FOL implies that S is compatible with any alternative concerning the world described with the resources of FOL. Hence S does not exclude any alternative concern-

ing the world; this amounts to say that S is analytic according to sense 1 of analyticity.

As Hintikka (1959) explicitly claims, the reason for not using this simple and shorter argument for showing the analyticity of FOL according to sense 1 of analyticity is fairly simple: if one used such a shorter and simple argument, one would move the discussion from a purely linguistic level to a purely semantic one (i.e. a level where domains of individuals need to be assumed); this move has quite controversial consequences. In 1953 Bar-Hillel and Carnap (1953) tried to show the analyticity of FOL according to sense 1. of analyticity by precisely working on a semantic level, i.e by assuming infinite domains of individuals; their solution turned out to be a source of endless controversies (for the details see (Hintikka, 1973, 153-163)). In order avoid these difficulties, Hintikka changes the level of discussion and thus draw a conclusion that looks stronger and more stable. In order to deal with the modal case, we will follow Hintikka's strategy and work within a purely linguistic level. This will allow us to avoid any unpleasant consequence.

Modal Logic. Consider a simple validity of modal logic like $\Box p \leftrightarrow \neg \Diamond \neg p$ and call this validity C ; then focus on the non-logical resources employed in C , namely the atomic sentence p . Now consider the attempt to list all the alternatives concerning the world which can be expressed by means of these resources in the framework of modal logic. On the one hand, each alternative concerning the world that can be constructed by means of p in the framework of modal logic cannot but be an infinite conjunction or an infinite set of formulas. Let us describe this in more formal terms by using *maximal consistent sets* of formulas. A maximal consistent set of formulas M is a set which does not have consistent proper extensions. In other words, a consistent set M is maximal whenever, if $M \subseteq M'$ and M' is consistent, then $M' = M$, i.e. $M' = M$. Given a maximal consistent set M , let us draw from it a *basic maximal consistent set* $\mathcal{P}(M)$. A basic maximal consistent set $\mathcal{P}(M)$ is a subset of the maximal consistent set M that contains all the formulas belonging to M which can be obtained from atoms by means of the symbols \neg , \wedge and \Diamond . Basic maximal consistent sets represent the infinite proper formal counterpart of the idea of alternative concerning the world in the framework of modal logic. An example of a basic maximal consistent set that can be constructed from p is the set composed by: $\{p, \Diamond p, \Diamond \neg p, \Diamond(p \wedge \Diamond p \wedge \Diamond \neg p), \Diamond(p \wedge \Diamond(p \wedge \Diamond p \wedge \Diamond \neg p)) \dots \}$.

Thus, on the one hand, we have that alternatives concerning the world in modal logic are, as in the case of first-order logic, infinite; moreover, they can be properly represented by basic maximal consistent sets. On the other hand, we have that, once more as in the case of first-order logic, we cannot make a finite list of them. As is well-known (e.g. see Blackburn et al. (2001)), once we consider maximal consistent sets, or, equivalently, basic maximal consistent sets, these are infinite.⁴ Therefore, even in the case of modal logic, if we try to

⁴Some readers could be puzzled by this claim for the following reason. Modal logic is known to be decidable (see Blackburn et al. (2001)); in the case of first-order logic, we seem to have established a link between finite lists of alternative concerning the world and decidability; but now

follow the six-step strategy of Figure 1, it seems that we are forced to stop at point 3.

Analyticity of ML / Part 1	
1	consider a validity B
2	focus on its non-logical resources
3	X

In order to overcome this major obstacle and reach our goal, namely the analogue of point 6 of Figure 1, one way to proceed might be that of imitating Hintikka's strategy. More precisely, we could firstly find the modal counterpart of the notion of degree; then, by using this notion, we could find the modal counterpart of the notion of quantificational constituent. Finally, by exploiting these two notions, it should be possible to prove the distributive normal form theorem and thus the analyticity, according to sense 1 of analytic, of modal logic.

If we decide to follow this path, things seem to be pretty easy. In a famous article of 1975 Fine (1975) develops the aforementioned steps for us. Let us then see Fine's work in detail. First of all, let us introduce the notion of *modal degree*, which is the modal counterpart of the notion of degree.

Definition 2.4. The modal degree of a formula A , $dg_m(A)$, is the length of the longest string of nested \diamond -occurrences in A :

$$\begin{aligned}
 dg_m(A) &= 0, \text{ if } A \text{ is atomic;} \\
 dg_m(\neg A) &= dg_m(A); \\
 dg_m(A_1 \wedge A_2) &= dg_m(A_1 \vee A_2) = \max(dg_m(A_1), dg_m(A_2)); \\
 dg_m(\diamond A) &= dg_m(\Box A) = dg_m(A) + 1
 \end{aligned}$$

Once the notion of modal degree is introduced, Fine uses it to introduce the notion of *modal constituent* which represents the modal counterpart of the notion of quantificational constituent and the finite formal counterpart of the notion of basic maximal consistent sets.

Definition 2.5. Given the atomic sentences p_i ($i = 1, \dots, n$), we define the modal constituents of degree n , by induction on n :

- $n = 0$. Each propositional constituent constructed from p_i is a modal constituent.

we affirm that in modal logic we cannot draw a finite list of basic maximal consistent sets, i.e. of alternatives concerning the world. The doubts can be dissipated as follows. The link between decidability and finite lists of alternatives concerning the world only holds in one direction, namely: if a logic L is undecidable, then the list of alternatives concerning the world that can be constructed in that logic L is infinite. Nothing is said on what follows from the fact that a logic L is decidable. Thus a logic L , like modal logic, can be decidable and still encounters the problem of infinite lists of alternatives concerning the world that can be constructed in L .

- $n > 0$. Suppose that $B_{n-1}^0, \dots, B_{n-1}^k$ are modal constituents. Then $C \wedge \pi \diamond B_{n-1}^0 \wedge \pi \diamond B_{n-1}^1 \cdots \wedge \pi \diamond B_{n-1}^k$, where C is a propositional constituent and π stands for either a blank or a negation, is a modal constituent of degree n .
- Nothing else is a modal constituent.

Let us dwell for a moment on this definition. Its intuitive idea is fairly simple: firstly, one needs to list the possible kinds of situations that can be specified by means of the atomic sentences p_i ; then, for each kind of situation, one needs to indicate whether it is possible or not. Let us make a simple example of what we have just said. For this, let us use the atomic sentences p . As before, let us assume that p stands for "it is raining"; then the two situations that can be specified by means of "it is raining" are: "it is raining" and "it is not raining". For each of these two situations, we need to specify whether they are possible or they are not. So, for example, we might have $p \wedge (\diamond p \wedge \diamond \neg p)$, but also $p \wedge (\neg \diamond p \wedge \diamond \neg p)$ but also $p \wedge \diamond(p \wedge (\diamond p \wedge \diamond \neg p))$ and also $\neg p \wedge \diamond(p \wedge (\neg \diamond p \wedge \diamond \neg p))$ and so on. These formulas are examples of modal constituents: the first two formulas are modal constituents of degree 1, while the latter two formulas are modal constituents of degree 2. As the reader can easily see modal constituents represent the finite counterpart of basic maximal consistent sets.

Thanks to the notion of modal degree and modal constituent, we thus have found a way to finitely describe the notion of alternative concerning the world at the modal level. Fine then further uses these notions to prove the distributive normal form theorem for the main systems of modal logic. Consider our modal formula C of degree n ; it can be proved that C is equivalent to a disjunction of some (perhaps all) modal constituents of degree n . In other words, thanks to the notion of modal degree, given a formula C (of degree n), it can be finally proved that C has a distributive normal form composed by modal constituents of degree n . This is precisely the result that we wanted to obtain. Such a result can be found in Fine (1975).

Let us then sum up what we have just seen in the following table

Analyticity of ML / Part 2	
1	consider a validity B
2	focus on its non-logical resources
3a	introduce the notion of modal degree and determine the degree n of B
4a	form the disjunction B' of all modal constituents of degree n
5a	prove that B is a logical consequence of B' by means of the DTNF theorem

Given this situation, one question naturally arises: can we bring our argument to a close by passing to point 6? In other words, can we draw the conclusion that the validities of modal logic are analytic according to notion 1. of analyticity? In order to answer these questions in the affirmative, we need to justify the notion of modal degree, which is to say we need to establish the analogue of point 5b of Figure 2. Unfortunately, in this case, it seems pretty

hard to provide such a justification. Let us spend a bit of time to explain why; for this, we will make a parallel with FOL.

Consider again the sentence "John has one brother and John has one sister" and call this sentence E . Once formalized in FOL, E has degree two. How can one conceptually justify this degree two with respect to the sentence E ? To answer this question, think of the ways one can read E . On the one hand, one can read it as a sentence referring to three individuals, namely John, his brother and his sister; on the other hand, one can read it as a sentence referring to individuals and their relations, namely the relation between John and his brother and the relation between John and his sister. In this second case, E speaks of two individuals at the time and this is exactly its degree. Since the second way of reading the sentence is the correct one, Hintikka can conclude that the notion of degree formalizes the correct way of reading E and thus it is a warranted and justified notion with respect to E .

Consider now the following sentence "it is possible that it rains and it is possible that it is windy" and call this sentence F . Once formalized in modal logic, this sentence has modal degree one. How can one conceptually justify this degree with respect to the sentence F ? If one interprets F by using Kripke semantics, then F is a sentence that talks about three possible worlds: the actual world, a second world accessible from the actual world where it is raining and a third world accessible from the actual world where it is windy. These last two worlds are at distance one from the actual world and this is exactly the modal degree of the sentence. Thus the modal degree captures the relations between the possible worlds by means of which the sentence F can be interpreted; this can then be seen as the conceptual justification of the modal degree with respect to the sentence F .

Though attractive, this answer seems much less solid than the one given in the case of FOL. In the case of FOL the defense of the notion of degree did not rely on any particular interpretation of FOL, but it was on the contrary built by using the objects which the sentences of FOL talk about, namely individuals and their relations. In the case of modal logic the justification of the modal degree is entirely based on possible worlds of Kripke semantics, which are not what sentences of modal logic talk about, but just one of their semantic interpretations. Thus, while the defense of the notion of degree is a general and hence robust one, the defense of the notion of modal degree is based on a particular point of view on modal logic and hence it lacks strength and conviction.

In order to get a solid defense of the notion of modal degree, one should abandon the privileged point of view represented by Kripke semantics and embrace a broader perspective, i.e. a perspective independent of the semantic or syntactic interpretation of modal logic. On the other hand, it is easy to see that, as soon as the privileged point of view represented by Kripke semantics is abandoned, it is no longer possible to give a defense of the notion of modal degree. What kind of conceptual link can one indeed establish between the sentence F and the modal degree one? The obvious answer is none. Thus, the notion of modal degree just looks like an emulation of the notion of degree, which automatically limits the number of possible consecutive diamonds in a

modal sentence, but is conceptually unjustified.

The notion of modal degree is not as warranted as its first-order correspondent. Therefore the analogue of point 5b of Figure 2 cannot be established. By consequence, we cannot draw the conclusion that the validities of modal logic are analytic according to sense 1 of analyticity. Given this situation, a question naturally arises: is there another way to show that the validities of modal logic are analytic according to notion 1 of analyticity, or shall we draw the conclusion that these validities are in fact synthetic? Intuition seems to suggest that the first answer is the good one. We will then spend the rest the section trying to develop it.

Let us start by recalling what exactly we have to prove in order to show that the validities of modal logic are analytic according to notion 1 of analyticity. Let us do this by means of an example. Consider again the modal validity C , namely the formula $\Box p \leftrightarrow \neg \Diamond \neg p$, and focus on its non-logical resource, namely the propositional sentence p . Then take into account all alternatives that can be formed from the resource p in the framework of modal logic. These are, as already emphasized, nothing but basic maximal consistent sets. In order to show that C is analytic according to sense 1. of analyticity, i.e. that C does not convey any information, we need to prove that C does not rule out any basic maximal consistent set. To show that C does not rule out any basic maximal consistent set amounts to showing that any basic maximal consistent set is compatible with C , i.e. that C is a logical consequence of each basic maximal consistent set. So this is what we need to demonstrate. We will do it in the following way.

First, recall that Fine (1975) has shown that any validity of degree n is (equivalent to and hence) a logical consequence of the disjunction of all modal constituents of degree n . For the case of C , this means that C is a logical consequence of the disjunction of all modal constituents of degree 1. It follows that C is a logical consequence of each modal constituent of degree 1. As explained above, the modal constituents of degree 1, relying as they do on the notion of modal degree, are not sufficiently warranted to build an argument for analyticity: it needs to be shown that C is a logical consequence of every alternative concerning the world that can be build from p using the resources of modal logic, not just of every modal constituent of degree 1. However, alternatives concerning the world that can be build from p using the resources of modal logic are basic maximal consistent sets. Recall that each such set is consistent, that it contains sentences that can be formed from certain non-logical resources (in this case p) using conjunctions, negations and diamonds, and that is maximal in the sense that no new sentence can be added to the set without yielding an inconsistent set. It follows that each basic maximal consistent set must contain exactly one modal constituent of degree 1 (which, as is clear from Definition 2.5, is formed using conjunctions, negations and diamonds). Hence, for each basic maximal consistent set, there is exactly one modal constituent of degree 1 that is a logical consequence of it. It thus follows by transitivity of logical consequence that, for each basic maximal consistent set, C is a logical consequence of it. Hence C is compatible with each basic maximal consistent

set, i.e. with each alternative concerning the world, and thus is analytic according to sense 1 of analyticity, as required. By repeating this reasoning for all other validities of the main systems of modal logic, we can finally conclude that these validities are analytic according to sense 1 of analyticity.

Note that, whilst this argument uses the notion of modal constituent of a given modal degree, and hence the notion of modal degree, it does not require it to have any particular philosophical import. It merely plays a technical role to show a relation of entailment between two notions that do have philosophical import and are directly involved in the discussion of analyticity in sense 1: namely the validities of modal logic and the alternatives concerning the world. Thus our argument seems to be simple but sound and it leads us to our desired conclusion.

Before closing the section, let us make a last brief but important remark. We have shown the analyticity, according to sense 1 of analytic, of propositional logic, first-order logic and modal logic. While the case of propositional logic has turned out to be rather straightforward, the cases of first-order logic and modal logic have required more work. This said, there are two parts of first-order logic and modal logic, respectively, for which the proof of their analyticity would more closely follow that of propositional logic. We are here thinking to the case of monadic first-order logic and the modal systems S4 and S5. The case of monadic first-order logic has already been commented on by Hintikka (1965); as for the modal systems S4 and S5, this is due to the fact that in S4 there are at most fourteen different modalities and in S5 there are at most 6 of them (see Blackburn et al. (2001)), hence any recourse to the infinity, which is necessary, as we have seen, for the other normal systems of modal logic, is in these two cases blocked.

The cases of monadic predicate logic, S4 and S5 are special cases that show that, as (Hintikka, 1965, p.191) says, "our notion of information has interesting applications outside of propositional logic".

3 Modal logics and analytic proof methods

Let us now turn to the second sense of analyticity, which is:

- 2 A sentence is said to be analytic when it can be shown to be true by strictly analytic methods.

In order to fully understand this definition, we must, first of all, clarify the expression "strictly analytic methods". Traditionally, there are several ways of distinguishing the analytic method from the synthetic one. Let us consider two. (i) The first way can be explained as follows, e.g. Heath (1981). While the synthetic method amounts to the deductive method, so that in a synthetic proof one starts from axiom(s) and goes down to theorems, in the analytic method the starting point is the sentence to prove and the proof is seen as a procedure that allows one to reach the axiom(s). (ii) A second way of understanding the

difference between analytic and synthetic method is tied with geometry. A geometrical proof is said to be analytic if no new construction, in the sense of new line, new circle or new point, is carried out in the proof; a geometric proof is said to be synthetic if such constructions are used.

Hintikka retains and works with this latter sense of the distinction between analytic and synthetic method. Indeed Hintikka aims at coming close to Kant's use of the term analytic, and, according to his interpretation of Kant (e.g. see Hintikka (1965, 1959, 1973)), Kant delineates the distinction between analytic and synthetic by using the concept of analytic proof method, and, more precisely, of analytic proof method defined according to (ii). As usual, we will follow Hintikka in his choice.

So we have a definition of the distinction analytic-synthetic method tied to geometry. The issue is how to adapt such a definition to logic, since we want to know whether the validities of the main normal systems of modal logic are analytic according to sense 2 of analyticity. The first thought is to adapt notion 2-(ii) of analyticity to logic by identifying it with the *subformula property*. To introduce this property, one needs to consider a logic from the point of view of a proof-theoretical system, i.e. as a calculus. A calculus is said to satisfy the subformula property if, and only if, every provable formula possesses a derivation such that every formula which occurs in it is a subformula of the conclusion. Standardly a calculus is proved to have the subformula property by means of the cut-elimination theorem (in case it is a sequent calculus) or by means of the normalization theorem (in case it is a natural deduction calculus).⁵

At first glance the subformula property could be taken as the adequate logical transposition of definition (ii). As definition (ii) says that no new line or point or circle should be constructed in a proof, in the same way the subformula property states that no new formula should be added in an analytic proof. The subformula property is thus certainly the first condition to respect in order to adapt definition (ii) to the logic realm; on the other hand, by itself, it cannot do the whole job. Why? Because FOL standardly enjoys the normalization theorem (and equivalently the cut-elimination theorem), and hence the subformula property, but, nevertheless, Hintikka claims that FOL is not analytic according to notion 2-(ii) of analyticity. Thus the subformula property is not the full answer to the problem.

Someone at this point could argue that we are making much ado about nothing. Indeed, for Hintikka to draw the conclusion that FOL is not analytic according to sense 2-(ii) of analyticity, he must have already formulated a logical transposition of definition 2-(ii) and we could easily borrow this transposition to examine the case of modal logic. Let us then have a brief look to Hintikka's result.

2* A proof of q from p is analytic if in none of the intermediate stages

⁵Strictly speaking, the cut-elimination theorem (or the normalization theorem) by itself does not ensure that the subformula property is satisfied, e.g. see Poggiolesi (2010). One also needs that in every rule of the calculus the formula(s) constituting the premise(s) are subformula(s) of the formula constituting the conclusion. On the other hand, in order not to burden the paper, we will generally take this second condition for granted.

of the proof the conjunction of the formulas occurring up to that stage has a degree higher than that of p or that of q .

2^* is Hintikka's logical counterpart of definition 2-(ii). As we have already hinted, FOL is not analytic according to 2^* . Consider for example the tableaux calculus for FOL. Such a calculus contains a rule called *existential instantiation* (EI), which has the following form:

$$\begin{array}{c} \exists xPx \\ \vdots \\ P[a/x] \end{array}$$

where a is a free individual symbol that should not have been used elsewhere and $P(a/x)$ is the result of replacing x by a in P (wherever it is bound to the initial quantifier $\exists xPx$). The rule of existential instantiation is a synthetic rule since it violates condition 2^* . To clearly see this, examine the following proof D :

$$\begin{array}{c} \neg(\exists x\exists yP(x, y) \rightarrow \exists y\exists xP(x, y)) \\ | \\ \exists x\exists yP(x, y), \neg\exists y\exists xP(x, y) \\ | \\ \exists yP(c, y) \\ | \\ \underline{P(c, d)} \\ | \\ \neg\exists xP(x, c), \neg\exists xP(x, d) \\ | \\ \neg P(c, c), \neg P(d, c), \underline{\neg P(c, d)}, \neg P(d, d) \\ X \end{array}$$

Consider for example the conjunction of the formulas of the first three steps of D (looking at D bottom up): the degree of this conjunction is higher than the degree of the conclusion and this is because of the introduction of two new individual constants c and d ; the new individual constants c and d are introduced by means of two applications of the rule of existential instantiation. Thus existential instantiation is the rule that increases the number of individuals, i.e. the degree, considered at one and the same time. Therefore those sentences of FOL that are provable by using the rule of existential instantiation are synthetic because they cannot be proved by strictly analytic methods.

Given this situation, one could naturally wonder whether the conclusion that has just been drawn is not an accidental peculiarity of tableaux calculi: it might indeed be the case that in other proof-theoretic frameworks there is no rule such as the rule of existential instantiation and thus that FOL is analytic according to sense 2 of analyticity. The answer to these doubts is actually negative, as Hintikka strongly emphasizes. Each calculus for FOL, like the natural

deduction calculus or the sequent calculus, contains a rule of the type of the rule *EI* that violates definitions 2*; this is so because one needs a rule of such a type for proving the completeness of first-order logic.⁶ Therefore first-order logic is unavoidably synthetic according to sense 2-(ii) of analyticity, definition 2* being the logical counterpart of definition 2-(ii).

Let us now turn to the validities of the main normal systems of modal logic. A natural move for us now would be to apply definition 2* to these validities to finally find out whether they are analytic or synthetic according to sense 2-(ii) of analyticity. For obvious reasons this cannot be done. Indeed notion 2* can certainly be seen as a logical counterpart of notion 2-(ii), but, on the other hand, it also seems a definition that is specific to FOL. It crucially relies on the concept of degree, which in its turn relies on the concepts of individuals and their relations and these concepts are specific to first-order logic, not necessarily to other logics. Thus a quick reflection is enough to realize that it makes little sense to check whether in the proofs of modal validities the degree of the formulas involved in the proofs does not change from the premisses to the conclusion. Another possibility is that of adapting definition 2* to the case of modal logic by substituting the notion of degree by the notion of modal degree. Even this idea does not seem very adequate. On the one hand, the notion of modal degree can hardly be seen as capturing the concepts expressed in definition 2(b); on the other hand, as we have already discussed in the previous section, the notion of modal degree seems to lack any justification and conceptual foundation.⁷

So finding a logic counterpart of definition 2-(ii) is not an easy enterprise and the question about the analyticity of modal logic is still open. Let us then take a more general stance on the problem and follow Hintikka in recounting a brief history of the meaning of the term analytic in sense 2-(ii) of analyticity. This term seems to have been linked to two other terms, the term *exposition* and the term *exhibition*. The part of the proof of an Euclidean theorem in which figures were introduced (drawn for the first time) was called *echtesis* or exposition. The same term was used in Aristotle syllogism to refer to an inference rule similar to the rule *EI* of the deduction calculus for first-order logic. In Kant the term analytic is associated with the term "exhibition" or construction

This observation on the nature of mathematics gives us a clue to the first and highest condition of its possibility, which is that some [. . .] visualizations must form its basis, in which all its concepts can be exhibited or constructed (Kant, 1913, p. 281)

And also

Philosophical knowledge is the knowledge gained by reason from

⁶For a precise and detailed description of the situation, see (Hintikka, 1965, p. 197)

⁷Poggiolesi (2015) was our first paper on the analyticity of modal logic. At the time we were not aware of the notion of modal degree. Thus we *erroneously* tried to adapt the definition of degree of quantification theory to modal logic. We drew erroneous conclusions. Only after its publication, we realized our mistake.

concepts, mathematical knowledge is the knowledge gained by reason from the construction of concepts. To *construct* a concept means to *exhibit* a priori the intuition which corresponds to the concept. (Kant, 1929, A 713)

According to Hintikka, by “intuition” Kant means in the first place the representative of an individual. Hence a construction is for him the introduction of a representative of an individual to illustrate a general concept. By consequence, for Kant (at least in Hintikka’s interpretation), what makes an argument synthetic is the *exhibition of a singular idea to represent a general concept*. If this is the case, knowing that the ultimate goal is to come close to Kant’s usage of the analytic-synthetic distinction, we can explain two main points. Firstly, we can explain why, beyond the notion of degree provided by Hintikka, the rule *EI* of FOL is synthetic: in such rule (and rules similar to *EI*) one precisely introduces a singular idea - namely a new constant - to represent a general concept - namely the existential. Secondly, we can explain why Hintikka is able to formulate a logical counterpart of definition 2-(ii) suited for FOL, while we cannot: FOL comes equipped with the means for inserting a new singular idea, namely the variables or the constants, while modal logic does not. Given these remarks, we can conclude that in order to check whether a (modal) logic is analytic in sense 2-(ii) of analyticity, one needs, first of all, to verify that the logic satisfies the subformula property and then, that no rule introduces any new element to exemplify a general concept.

Consider now the main normal systems of modal logic. Quite recently the proof theory for these systems has attracted the attention of many proof-theorists. Several different calculi for them have been proposed (see Poggiolesi (2010)); amongst these, the most well-known are the display calculi (Belnap (1982)), the tree-hypersequents calculi (Poggiolesi (2009)) and the multiple sequents calculi (Indrezejczak (1997)). It does not really matter which proof-theoretical framework one chooses to work with, in any of them: (i) the calculi for all systems of the cube of modal logic satisfy the subformula property; (ii) there does not exist a rule where a new singular idea is introduced for representing a general concept.⁸ Indeed a little reflection suffices to see that in modal logic is not possible to formulate such a rule since the language of modal logic does not possess the symbols to indicate a specific objects or a specific individual. Thus we can conclude the the validities of the main normal systems of modal logic are analytic according to sense 2. of analyticity.

Before ending the section, let us make a final remark on the conclusion that we have just drawn. Indeed someone might rightly raise the following point. If one considers the rule that eliminates the diamond in the framework of modal logic, one cannot not notice a striking analogy with the rule that eliminates

⁸For the sake of clearness, labelled calculi Russo (1996) are the one exception to what we have said, for in the rule that introduces the symbol \Box on the right side of the labelled sequent a new singular variable is introduced for representing the box itself. On the other hand, one might really wonder to what extent labelled calculi can be considered calculi for modal logic, given their massive use of variables and renaming as in quantification theory. For a deeper discussion on this issue see Poggiolesi (2010).

the existential quantifier in the framework of first-order logic: the two rules act in a parallel way. Thus it seems quite weird to accept the conclusion that the rule *EI* is synthetic according to sense 2. of analyticity while the rule that eliminates the diamond is analytic. This difference calls for an explanation.

The explanation that we offer is the following. On the one hand, we certainly acknowledge that there exists a strict analogy between the rule *EI* and the rule that eliminates the diamond in modal logic. This fact is widely witnessed in the literature (e.g. see Blackburn et al. (2001), Brünnler (2010)) and we certainly do not want to deny it. On the other hand, we would like to attract the reader's attention on a difference amongst the two rules that turns out to be quite important in this context. While in the rule *EI* we pass from an existential quantifier to a specific individual which needs to be new and exemplifies the quantifier, in the rule that eliminates the diamond no such passage need to be made. Indeed in many proof-theoretic frameworks like display calculi, tree-hypersequent calculi, multiple-sequents calculi, in the rule that eliminates the diamond, the diamond is not instantiated by a singular new element but substituted by some meta-linguistic structure. Now a little reflection suffices to realize that in all the classical rules for the classical connectives, connectives are generally substituted by some meta-linguistic structure; the conjunction and the disjunction connectives are for example substituted by a comma. Therefore, as the classical rules are unanimously considered as analytic according to sense 2 of analyticity, so must be the rule that eliminates the diamond. This conclusion cannot be drawn for the rule that eliminates the existential quantifier: in this case the existential quantifier is not substituted by a meta-linguistic structure but by a specific individual. For the reasons that we have exposed along this section, this is precisely what makes the rule synthetic. Thus the rule *EI* and the rule that eliminates the diamond are similar but not the same; in particular, in this context, the former is synthetic, while the latter analytic.

4 Conclusions

In this paper we consider two different notions of analyticity: the first notion is linked with the concept of information, while the second notion is linked with the concept of proof. Both these notions have been deeply studied and investigated by Hintikka. The first notion has actually given rise to a broad literature, which takes the name of *theory of information* (e.g. Allo (2011); Floridi and D'agostino (2009); Primiero (2007)). The second notion has been mostly ignored by the philosophical tradition. In this paper we test the validities of the main normal systems of modal logic with respect to these two notions of analyticity: it turns out that modal logic is analytic according to both senses of analyticity.

As a kind of conclusion of the paper, we would like to tackle two issues that seem quite important for the general understanding of the paper. The first issue concerns the strategy adopted for proving the analyticity of modal logic according the two senses 1 and 2 of analyticity; the second issue concerns the

interest of studying the analyticity of modal logic according to senses 1 and 2 of analytic. Let us start analyzing the first issue.

With respect to the work developed in the paper, someone could raise the following objection. In view of the fact that the analyticity of first-order logic with respect to senses 1 and 2 of analyticity has already been investigated and that there exists a formal embedding of modal logic into first-order logic van Benthem (1984), it seems weird for the author not to use these two facts for automatically drawing the appropriate conclusions about the analyticity of modal logic. Actually what seems even more in need of a clarification is the claim that modal logic is analytic according to sense 2 of analyticity, while first-order logic is actually synthetic according to sense 2 of analyticity.

To both these objections we would like to reply with an unique but general answer: a standard embedding from a logic L into a logic L' does not ensure by itself that the properties of analytic or synthetic enjoyed by logic L' are also (and sort of automatically) enjoyed by logic L . This can be shown by means of a very simple example. Let us indeed consider the case of intuitionistic logic. As is famously known, intuitionistic logic is embeddable into modal logic $S4$. This was proved by Gödel (1986) in a famous article of 1933. Consider now the case of modal logic $S4$; according to the standard translation cited above, modal logic $S4$ is embeddable into first-order logic. But first-order logic, as it has been widely explained previously, is synthetic according to sense 2 of analyticity. Thus, according to the objections raised above, even $S4$ should be synthetic according to sense 2 of analyticity; but then also intuitionistic logic should be synthetic according to sense 2 of analyticity. Now nobody who has even a vague idea of the notion of analyticity and of what intuitionistic logic is, would ever accept the conclusion that intuitionistic logic is synthetic according to sense 2 of analyticity; intuitionistic logic is indeed analytic according to sense 2 of analyticity. Thus a logic L can be embeddable into a logic L' without necessarily inheriting its properties of being analytic or being synthetic. Therefore the legitimated doubts raised by the precedent objections must disappear and our conclusions, as well as our arguments that support our conclusions, reassess.

Let us now move to the second issue linked to this paper, that of the importance of the study of the analyticity of modal logic, according to senses 1 and 2 of analyticity. Let us focus on the notions of information and analytic proof. Propositional logic constitutes a proof-theorist's and information-theorist's paradise. The notions of information and of analytic proof both admit of a sharp and clear-cut definitions in this framework. "The question now becomes as to whether, and if so how, the treatment of the concept of information [and of analytic proof] can be extended from propositional languages to others." (Hintikka, 2007, p. 193). This paper can be thought of as a partial solution to such a question for modal logic.

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