Multivariate optimization for multifractal-based texture segmentation

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MULTIVARIATE OPTIMIZATION FOR MULTIFRACTAL-BASED TEXTURE SEGMENTATION

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ABSTRACT

This work aims at segmenting a texture into different regions, each characterized by a priori unknown multifractal properties. The multifractal properties are quantified using the multiscale function $C_{1,j}$ that quantifies the evolution along analysis scales 2^j of the empirical mean of the log of the wavelet leaders. The segmentation procedure, applied on $(C_{1,j})_{1\leq j\leq J}$, involves a multivariate Mumford-Shah relaxation formulated as a convex optimization problem involving a structure tensor penalization and an efficient algorithmic solution based on primal-dual proximal algorithm. The performances are evaluated over simulated data.

Index Terms— Local regularity, multifractal spectrum, segmentation, convex optimization, wavelet Leaders

1. INTRODUCTION

Multifractal texture characterization. Multidimensional multifractal analysis is now considered as a classical tool for texture characterization (cf. e.g. [1]). It notably permits to capture in a refined manner the detailed fluctuations of regularity of a texture along space and thus ground texture characterization on the measurement of global and local smoothness. Local regularity is technically measured via the concept of Hölder exponent, and the multifractal spectrum provides practitioners with a global and geometrical characterization of the statistical fluctuations of Hölder exponents measured across the texture of interest. Multifractal tools have been used to characterize real-world textures from a large variety of applications of different natures ranging from biomedical (cf. e.g., [2]) to art investigations [3, 4]. It is also well documented that multifractal analysis should be grounded on wavelet leaders, consisting of local supremum of 2D wavelet transform coefficients taken across all finer scales [5, 1].

Texture segmentation. However, in its current formulation, multifractal analysis, as most texture characterization procedures, assumes a priori that the texture to analyze consists of a single piece with homogeneous properties, that is, texture properties of any subpart of the image available are identical. In most applications, however, texture analysis combines two different issues: Segmentation of the images into pieces or regions, of a priori unknown boundaries, where texture properties can be considered as homogeneous and characterization of texture properties in each different homogeneous parts.

Solution for image segmentation has been envisaged in many different ways [6, 7, 8, 9, 10, 11]. However, none of these algorithmic solutions appears to be robust to noise, unsupervised and exploiting the correlations through different components. To derive a new algorithmic solution satisfying all these constraints, we focus on a segmentation procedure derived from [12].

Goals and contributions. The present contribution elaborates on earlier works [13, 14], aiming to segment a texture into local regularity piecewise constant regions, by proposing one of the first and rare attempt to segment a texture into regions, with unknown boundaries, and within multifractal properties can be considered homogeneous. The proposed procedure thus aims at segmenting a texture into different regions, each characterized by a different a priori unknown multifractal spectrum. To that end, wavelet leader based characterization of texture is first recalled in Section 2. In the present work, multifractal properties are quantified using multiscale quantity $C_{1,j}$ that quantifies the evolution along analysis scales 2^j of the empirical mean of the log of the wavelet leaders at a given scale. This function is deeply related to the average or global regularity of the texture. Thus, it does not account for the entire multifractal properties of the texture, but provides us with a satisfactory partial description of multifractal properties that can be involved into texture segmentation. The multivariate segmentation procedure is detailed in Section 3. It consists in a multivariate Mumford-Shah relaxation formulated as a convex optimization problem involving a tensor structure penalization and an efficient algorithmic solution based on primal-dual proximal algorithm is proposed in Section 4. In Section 5 preliminary results are conducted on synthetic texture, designed to have piecewise constant multifractal properties.

2. MULTIFRACTAL ANALYSIS

Local regularity and multifractal spectrum. We denote $X = (X_{\underline{\ell}})_{1\leq \underline{\ell}\leq N}$ the image to analyze having N pixels.

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Its local regularity around position $\underline{\ell}$ can be quantified by the Hölder exponent $h_{\underline{\ell}}$: while large $h_{\underline{\ell}}$ points to a locally smooth portion of the field, low $h_{\underline{\ell}}$ indicates local high irregularity. Texture regularity fluctuations can be described by the so-called $\mathcal{D}(h)$ that describes the fluctuations of $h_{\underline{\ell}}$ along space in a global and geometrical manner (cf. e.g., [15, 5, 1] for details). For practical purposes, the multifractal spectrum can often be approximated as a parabola: $\mathcal{D}(h) = 2+(h-c_1)^2/(2c_2)$. The practical estimation of $\mathcal{D}(h)$ requires the use of wavelet leaders.

Wavelet leaders. We denote $d^{(m)}_{j,\underline{k}} = \langle X, \psi^{(m)}_{j,\underline{k}} \rangle$ the (L^1-normalized) 2D discrete wavelet coefficients of X at location $\underline{k} = 2^{-j}\underline{\ell}$, at scale 2^j with $j \in \{1, \ldots, J\}$, and where m stands for the horizontal/vertical/diagonal subband. For a detailed definition of the 2D-DWT, readers are referred to e.g., [16].

Wavelet leaders were recently introduced [15, 5] to permit an accurate characterization of the multifractal properties of a texture. The wavelet leader $L_{j,\underline{k}}$, located around position $\underline{\ell} = 2^j\underline{k}$, is defined as the local supremum of all wavelet coefficients taken within a spatial neighborhood across all finer scales $2^{j'} \leq 2^j$, that is,

$$L_{j,\underline{k}} = \sup_{\substack{m=1,2,3,\\ \lambda_{j',\underline{k}'} \subset \Lambda_{j,\underline{k}}}} |d^{(m)}_{j',\underline{k}'}|, \tag{1}$$

with $\lambda_{j,\underline{k}} = [\underline{k}2^j, (\underline{k}+1)2^j)$ and $\Lambda_{j,\underline{k}} = \bigcup_{p\in\{-1,0,1\}^2} \lambda_{j,\underline{k}+p}$ [15, 5].

Multifractal analysis. We define $C_{1,j} \in \mathbb{R}^N$ and $C_{2,j} \in \mathbb{R}^N$ as the sample estimates of the mean and variance of the variable $\ln L_j$, i.e., averages across space $\underline{\ell}$ at each given scale 2^j. It has been shown that functions $C_{1,j}$ and $C_{2,j}$ are related to the multifactal spectrum $\mathcal{D}(h)$ via the coefficients c_1 and c_2 involved in it approximate expansion [1]:

$$\mathbb{E}C_{1,j} = c_1^0 + c_1 \ln 2^j, \tag{2}$$

$$\mathbb{E}C_{2,j} = c_2^0 + c_2 \ln 2^j. \tag{3}$$

Multifractal segmentation. To segment textures, one could naturally consider estimating $C_{1,j}$ and $C_{2,j}$ locally in a neighborhood of each pixel $\underline{\ell}$, estimate the corresponding local parameters $c_{1,\underline{\ell}}$ and $c_{2,\underline{\ell}}$, and then perform a multivariate segmentation of $\{c_{1,\underline{\ell}}, c_{2,\underline{\ell}}\}$. This however relies on the strong assumption that real-world textures follow precisely the scaling behaviors prescribed in Eqs. (2) and (3) above. In the present contribution, it has been chosen to relax this requirement. Therefore, the proposed segmentation relies on the multiscale function $C_{1,j} = \left(C_{1,j,\underline{\ell}}\right)_{1\leq\underline{\ell}\leq N}$ as a function of scales 2^j, defined as a local sample mean estimate,

$$C_{1,j,\underline{\ell}} = \frac{1}{|S_{j,\underline{\ell}}|} \sum_{\underline{k}' \in S_{j,\underline{\ell}}} \ln L_{j,\underline{k}'}, \tag{4}$$

where $S_{j,\underline{\ell}}$ denotes a spatial (small) neighborhood of $\underline{\ell} = 2^j\underline{k}$ at scale 2^j and $|S_{j,\underline{\ell}}|$ the number of coefficients in that neighborhood.

3. MULTIVARIATE SEGMENTATION

Original formulation. The original Mumford-Shah problem consists in labeling an image Y into Q distinct areas having a mean value of v_q, with by convention $v_q \leq v_{q+1}$. The minimization problem is

$$\min_{\Omega_1,\ldots,\Omega_Q} \sum_{q=1}^{Q} \int_{\Omega_q} (Y - v_q)^2 dx + \frac{1}{2}\sum_{q=1}^{Q} \mathrm{Per}(\Omega_q)$$
$$\text{subj. to} \quad \begin{cases} \bigcup_{q=1}^{Q} \Omega_q = \Omega, \\ (\forall q \neq p),\ \Omega_q \cap \Omega_p = \emptyset, \end{cases} \tag{5}$$

where the penalization $\mathrm{Per}(\Omega_q)$ imposes the solution to have a minimal perimeter and the constraints over the Q areas ensure non-overlapping of the partition.

In several work [17, 18], a relevant convexification of this criterion has been proposed. The resulting minimization problem is specified for our study where the usual univariate quantity Y is replaced by the multivariate $(C_{1,j})_{1\leq j\leq J}$, so that Eq. (5) consists in labeling $C_{1,j}$ by estimating, for every $q \in \{1, \ldots, Q+1\}$, $\Theta_q = (\theta_{q,j})_{1\leq j\leq J} \in \mathbb{R}^{JN}$ such that

$$\underset{\Theta_1,\ldots,\Theta_{Q+1}}{\text{minimize}} \sum_{j=1}^{J}\sum_{q=1}^{Q} (\theta_{q,j} - \theta_{q+1,j})^\top (C_{1,j} - v_{q,j})^2$$
$$+ \lambda \sum_{j=1}^{J}\sum_{q=1}^{Q} \mathrm{TV}(\theta_{q,j})$$
$$\text{subj. to} \quad \begin{cases} \Theta_1 = 1, \\ \Theta_{Q+1} = 0, \\ 1 \geq \Theta_2 \geq \ldots \geq \Theta_Q \geq 0, \end{cases} \tag{6}$$

where $\lambda > 0$ and where TV denotes the usual total-variation penalization as defined in [19], i.e., for every $\theta \in \mathbb{R}^N$,

$$\mathrm{TV}(\theta) = \sum_{\underline{\ell}=1}^{N} \|(D\theta)_{\underline{\ell}}\|_2 \tag{7}$$

where $D \in \mathbb{R}^{2N\times N}$ denotes the discrete horizontal/vertical difference operator and thus $(D\theta)_{\underline{\ell}} \in \mathbb{R}^2$. The choice of $v_{q,j} \in \mathbb{R}$ will be discussed later. It clearly appears that this criterion, separable over j, does not impose coupling between the scales 2^j.

Proposed solution. We propose to introduce correlations by modifying the criterion as

$$\underset{\Theta_1,\ldots,\Theta_{Q+1}}{\text{minimize}} \sum_{q=1}^{Q}\sum_{j=1}^{J} (\theta_{q,j} - \theta_{q+1,j})^\top (C_{1,j} - v_{q,j})^2$$
$$+ \lambda \sum_{q=1}^{Q} \mathrm{STV}(\Theta_q)$$
$$\text{subj. to} \quad \begin{cases} \Theta_1 = 1, \\ \Theta_{Q+1} = 0, \\ 1 \geq \Theta_2 \geq \ldots \geq \Theta_Q \geq 0, \end{cases} \tag{8}$$

where, for every $\underline{\ell} \in \{1,\dots,N\}$, the structure tensor penalization is defined as

$$\mathrm{STV}(\Theta_q) = \sum_{\underline{\ell}=1}^{N} \|\zeta_{q,\underline{\ell}}\|_p \quad \text{where} \quad \zeta_{q,\underline{\ell}} = (\zeta_{q,\underline{\ell},1}, \zeta_{q,\underline{\ell},2}) \in \mathbb{R}^2$$

with $p \geq 1$ and where, for every $q \in \{1,\dots,Q+1\}$ and $j \in \{1,\dots,J\}$,

$$u_{q,j} = D\theta_{q,j} \in \mathbb{R}^{2N} \tag{9}$$

and, for every $\underline{\ell} \in \{1,\dots,N\}$,

$$u_{q,\cdot,\underline{\ell}} = U_{q,\underline{\ell}} X_{q,\underline{\ell}} (V_{q,\underline{\ell}})^\top \in \mathbb{R}^{J\times 2} \tag{10}$$

be the singular value decomposition of $u_{q,\cdot,\underline{\ell}} \in \mathbb{R}^{J\times 2}$ where

$$\begin{cases} (U_{q,\underline{\ell}})^\top U_{q,\underline{\ell}} = \mathrm{Id}_J \\ V_{q,\underline{\ell}} (V_{q,\underline{\ell}})^\top = \mathrm{Id}_2 \\ X_{q,\underline{\ell}} = \begin{pmatrix} \zeta_{q,\underline{\ell},1} & 0 & \dots & \dots & 0 \\ 0 & \zeta_{q,\underline{\ell},2} & 0 & \dots & 0 \end{pmatrix}^\top \end{cases} \tag{11}$$

This multivariate formulation could be interpreted as a discrete version of the relaxation proposed in [12].

4. PRIMAL-DUAL ALGORITHM

Reformulation To propose an efficient algorithm for minimizing such a criterion, we first rewrite (8) as

$$\underset{\Theta=(\Theta_2,\dots,\Theta_Q)}{\text{minimize}} \sum_{q=2}^{Q}\sum_{j=1}^{J} \theta_{q,j}^\top \Big((C_{1,j} - v_{q,j})^2 - (C_{1,j} - v_{q-1,j})^2 \Big) + \sum_{q=2}^{Q} \mathrm{STV}(\Theta_q) + \iota_{E_0}(\Theta) + \iota_{E_1}(\Theta) + \iota_{E_2}(\Theta) \tag{12}$$

where, for every $k \in \{0,1,2\}$, ι_{E_k} denoted the indicator function of the non-empty closed convex set $E_k \subset \mathbb{R}^{(Q-1)JN}$, that is $\iota_{E_k}(\Theta) = 0$ if $\Theta \in E_k$ and $+\infty$ otherwise. E_0 denotes a dynamic range constraint that imposes Θ to leave in $[0,1]^{(Q-1)JN}$ that is

$$E_0 = \{\Theta \in [0,1]^{(Q-1)JN}\}$$

and where

$$E_1 = \Big\{ \Theta \in \mathbb{R}^{(Q-1)JN} \,|\, \Theta_{2q} - \Theta_{2q+1} \geq 0, \quad (\forall q \in \{1,\dots,\lfloor (Q-1)/2 \rfloor\}) \Big\} \tag{13}$$

and

$$E_2 = \Big\{ \Theta \in \mathbb{R}^{(Q-1)JN} \,|\, \Theta_{2q+1} - \Theta_{2q+2} \geq 0, \quad (\forall k \in \{1,\dots,\lfloor (Q-2)/2 \rfloor\}) \Big\}. \tag{14}$$

The criterion (12) is a sum of five convex, lower-semicontinuous and proper functions, possibly non-smooth, and whose structure tensor penalization involves a linear operator. We thus propose iterations resulting from the proximal algorithm introduced in [20, 21]. The iterations are summarized in Algorithm 1. Under some technical assumptions insuring the existence of a solution, the iterates $\big(\Theta^{[n]}\big)_{n\in\mathbb{N}}$ converges to a minimizer of (12).

Algorithm 1 Multivariate segmentation algorithm.

Initialization
- Set $\tau > 0$ and $\sigma \in \Big]0, \big(\tau \max_{1\leq j\leq J} \{\|D^\top D\|\} + 3\big)^{-1}\Big[$.
- Set $\Theta^{[0]} = (\theta_{q,j}^{[0]})_{2\leq q\leq Q, 1\leq j\leq J} \in \mathbb{R}^{(Q-1)JN}$
- Set $y^{[0]} \in \mathbb{R}^{(Q-1)J(2N)}$ and $\widetilde{y}^{[0]}, \bar{y}^{[0]}, \bar{\bar{y}}^{[0]} \in \mathbb{R}^{(Q-1)JN}$

For $n = 0, 1, \dots$
- *Primal steps: update the variable $\theta^{[n+1]}$*
- For every $q \in \{2,\dots,Q\}$
  - For every $j \in \{1,\dots,J\}$
    - $z_{q,j}^{[n]} = \theta_{q,j}^{[n]} - \tau\big(D^\top y_{q,j}^{[n]} - \widetilde{y}_{q,j}^{[n]} - \bar{y}_{q,j}^{[n]} - \bar{\bar{y}}_{q,j}^{[n]}\big)$
- $\Theta^{[n+1]} = P_{E_0} z^{[n]}$
- $\widetilde{\Theta}^{[n+1]} = 2\Theta^{[n+1]} - \Theta^{[n]}$
- *Dual steps:*
- *update the variables $y^{[n+1]}$, $\widetilde{y}^{[n+1]}$, $\bar{y}^{[n+1]}$, $\bar{\bar{y}}^{[n+1]}$*
- For every $q \in \{2,\dots,Q\}$
  - For every $j \in \{1,\dots,J\}$
    - $u_{q,j}^{[n+1]} = y_{q,j}^{[n]} + \sigma D\widetilde{\theta}_{q,j}^{[n+1]}$
- $\widetilde{u}^{[n+1]} = \widetilde{y}^{[n]} + \sigma\widetilde{\Theta}^{[n+1]}$
- $\bar{u}^{[n+1]} = \bar{y}^{[n]} + \sigma\widetilde{\Theta}^{[n+1]}$
- $\bar{\bar{u}}^{[n+1]} = \bar{\bar{y}}^{[n]} + \sigma\widetilde{\Theta}^{[n+1]}$
- For every $q \in \{2,\dots,Q\}$
  - For every $\underline{\ell} \in \{1,\dots,N\}$
    - Compute $\zeta_{q,\underline{\ell},1}^{[n+1]}$ and $\zeta_{q,\underline{\ell},2}^{[n+1]}$ from $u_{q,\cdot,\underline{\ell}}^{[n+1]}$ (cf. (10))
    - $\eta_{q,\underline{\ell},\cdot}^{[n+1]} = \zeta_{q,\underline{\ell},\cdot}^{[n+1]} - \sigma \mathrm{prox}_{\sigma^{-1}\|\cdot\|_p}(\sigma^{-1}\zeta_{q,\underline{\ell},\cdot}^{[n+1]})$
    - Compute $y_{q,\underline{\ell}}^{[n+1]}$ from $\eta_{q,\underline{\ell},\cdot}^{[n+1]}$ (cf. (10))
  - For every $j \in \{1,\dots,J\}$
    - $\widetilde{y}_{q,j}^{[n+1]} = \widetilde{u}_{q,j}^{[n+1]} - \sigma \mathrm{prox}_{\sigma^{-1}\psi_{q,j}}(\sigma^{-1}\widetilde{u}_{q,j}^{[n+1]})$
- $\bar{y}^{[n+1]} = \bar{u}^{[n+1]} - \sigma P_{E_1}(\sigma^{-1}\bar{u}^{[n+1]})$
- $\bar{\bar{y}}^{[n+1]} = \bar{\bar{u}}^{[n+1]} - \sigma P_{E_2}(\sigma^{-1}\bar{\bar{u}}^{[n+1]})$

Proximity operator. In Algorithm 1, the notation prox denoted the proximity operator [22]. The proximity operator is defined for a convex, lower semi-continuous convex function φ from $\mathbb{R}^M$ to $]-\infty,+\infty]$, denoted prox_φ, is defined as, for every $u \in \mathbb{R}^M$, $\mathrm{prox}_\varphi(u) = \arg\min_{v\in\mathbb{R}^M} \frac{1}{2}\|u-v\|^2 + \varphi(v)$. When $\varphi = \iota_C$ with C being a non-empty closed convex subset of $\mathbb{R}^M$ then the proximity operator reduces to the projection, denoted P_C, onto the convex set.

The proximity operators involved in Algorithm 1 have a closed-form expression. Indeed, the closed form expression for $\mathrm{prox}_{\|\cdot\|_p}$ with $p = 2$ is given in [23], while the case $p = 1$ reduces to the soft-thresholding operator. Note that when $p = 2$, the singular value decomposition step could be

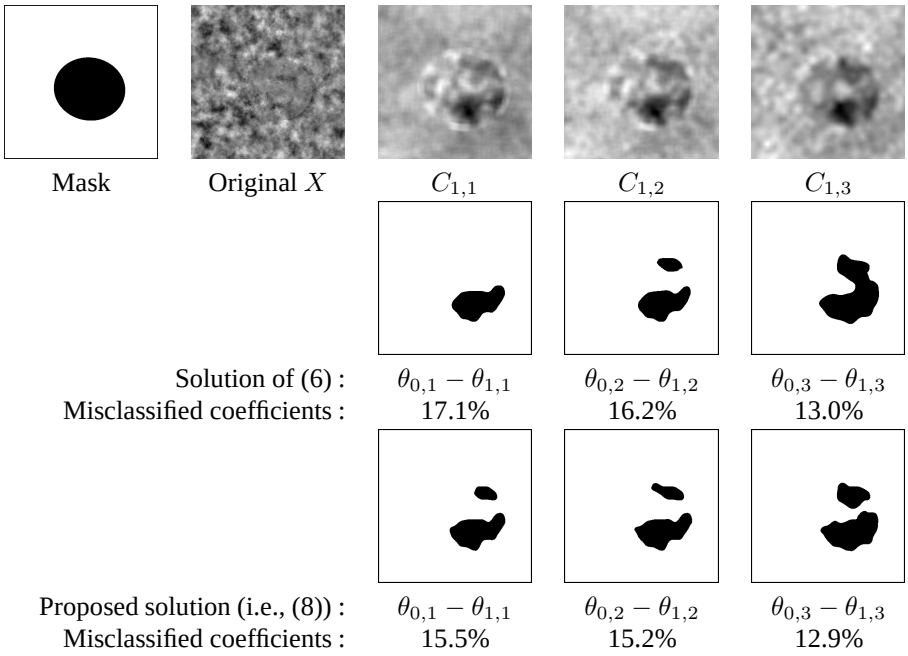

Fig. 1. Results of the proposed multivariate segmentation against a segmentation procedure done for each component separately. 1st line (left to right): mask allowing to generate the data, original data, estimates of the mean of $C_{1,j}$ for $j = 1$, $j = 2$, and $j = 3$. 2nd line (left to right): Results of the segmentation procedure described in (6) for $j = 1$, $j = 2$, and $j = 3$. 3rd line (left to right): Results of the proposed segmentation procedure described in (8) for $j = 1$, $j = 2$, and $j = 3$.

avoided [24]. On the other hand, we have denoted

$$(\forall \theta \in \mathbb{R}^N) \quad \psi_{q,j}(\theta) = \theta^\top \Big((C_{1,j} - v_{q,j})^2 - (C_{1,j} - v_{q-1,j})^2 \Big)$$

whose proximity operator reduces to

$$\mathrm{prox}_{\sigma^{-1}\psi_{q,j}} \theta = \theta - \sigma^{-1} \Big((C_{1,j} - v_{q,j})^2 - (C_{1,j} - v_{q-1,j})^2 \Big)$$

Finally, the projections onto E_0, E_1, and E_2 reduce to projection onto hyperslabs [25, Example 28.17]

Some other primal-dual solution should have been proposed such as the one derived in [26, 27]. For a summary on primal-dual strategy, the reader could refer to [28].

5. EXPERIMENTS

Performance of the proposed segmentation procedures are assessed on synthetic data, numerically produced by inclusion of a patch of 2D MRW [29] into a background of 2D-MRW with different multifractal parameters, $(c_1, c_2) = (0.8, -0.005)$ and $(0.5, -0.05)$ respectively. Patch and background have been normalized to ensure that the local variance does not depend on the image location. An example of such texture is shown in Figure 1.

Our simulations are performed using a standard 2D DWT with orthonomal tensor product Daubechies mother wavelets with 2 vanishing moments over $J = 3$ scales. We propose to compare the performance of the proposed multivariate solution against a segmentation proceeded for each $C_{1,j}$ separately. In our simulations $Q = 2$, $\lambda = 20$, and $p = 2$. For every scale $j \in \{1, \ldots, J\}$, $(v_{q,j})_{1 \leq q \leq Q}$ are chosen to be equally distributed between the minimum and maximum values of $C_{1,j}$. The proposed solution, whose result is depicted in Fig. 1-(bottom line), achieves a smaller rate of misclassified coefficients for each scale, which illustrate the interest of such a multivariate approach. The information of each scale can then be combined to achieve a segmentation of the original texture X. Segmentation have been performed over several realizations and similar conclusions can be done.

6. CONCLUSIONS AND PERSPECTIVES

Elaborating on our previous works aiming to segment textures into local regularity piecewise constant regions, the contribution of the present work is twofold : (i) it constitutes a first attempt to achieve texture segmentation into regions, each characterized with homogeneous multifractal properties and (ii) it proposes a multivariate segmentation procedure to take into account correlations between several components. Instead of making direct use of multifractal attributes parametrizing the multifractal spectrum (c_1, c_2,...), it has been chosen here to recourse to the multiscale quantities $C_{1,j}$ from which c_1 can theoretically be extracted. We have shown that the multivariate (multiple scales) segmentation of $C_{1,j}$ permits to detect the change of texture through the scales in order to identify regions with homogeneous multifractal properties.

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