Bulk Forces and Interface Forces in Assemblies of Magnetized Pieces of Matter

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When two magnets are stuck together, where do magnetic forces operate and which formulas should one apply to compute them? Such frequently asked questions do not find immediate answers in the literature on forces, mainly because the force field is obtained, by the Virtual Power Principle (VPP), as a (mathematical, vector-valued) *distribution,* not as a plain vector field, which would be more convenient for practical computation. We intend to show, in a few important cases of contact (between two linear materials with different permeabilities, between magnet and magnetizable metal, linear or not, between two hard magnets, etc.), how to represent this single distribution by *two* vector fields, one of them borne by the bulk of the matter, the other one localized at material interfaces where discontinuities of permeability, of magnetization, etc., do occur. Suitable extensions of the classical Maxwell tensor play an important role in the computation (by the so-called 'pillbox trick') of this interface vector field.

***Index Terms*—Magnetostatics, magnetic force, contact forces, virtual power principle, Maxwell's tensor.**

I. Introduction

CONSIDER a piece of matter with reluctivity ν (independent on the magnetic field) plunged into the field of a DC coil that lies some distance away.[1] Suppose ν insensitive also to the local strain (this is to avoid for the moment the difficulties of magnetostriction), but possibly non-uniform inside the domain D occupied by the matter. One will learn, from various sources (e.g., [1], [2], [3], etc.), that the force field inside D, or 'bulk force', is ${}^1\!/_2\,|B|^2\,\nabla\nu$. This vanishes for uniform ν. Yet the piece is attracted by the coil, so there must be a force, which cannot reside elsewhere than at the air–matter interface S, the boundary of D (Fig. 1). Indeed, it can be shown (we do it in detail below) that this surface force is

$$F_S = {}^1\!/_2\,(|H_\tau|^2[\mu] - |B_n|^2[\nu])\,n, \tag{1}$$

where n is the outward unit normal, H_τ and B_n the tangential part of H and normal part of B (both continuous across S), and $[\mu]$ and $[\nu]$ the jumps of μ and ν across the surface. (Note that $[\mu] > 0$, as a rule, and hence $[\nu] < 0$. Look at Fig. 1 for the sign conventions about the jump.)

There are several ways to prove (1). One of them consists in taking ${}^1\!/_2\,|B|^2\,\nabla\nu$ 'in the sense of distributions'. I shall explain in detail what this means, but let us first see how the Virtual Power Principle (VPP) yields the magnetic force as a distribution, by its very nature.

Let the symbol v (letter 'vee', not to be confused with ν for 'nu') denote the velocity of a virtual motion, in which a particle sitting at point x in the reference configuration (the one for which we want to compute forces) is displaced to the point $x + tv(x)$ at time t. We take v smooth and compactly supported (i.e., null outside some bounded region, called the support of v). Call $\Psi_v(t, B)$ the magnetic energy the system *would* contain at time t of this virtual motion, if the magnetic induction *was* B at this instant. Then—a well-known result; cf. [4] for a detailed proof—the virtual power at time 0 is minus the partial derivative of Ψ_v with respect to t, for $t = 0$ and $B = B(0)$, its value at time 0. Hence a linear function of v. It may happen that this function (which yields the virtual power at $t = 0$ for the virtual motion associated with v) has the form $\int F\cdot v$, where F is a vector field, which is then, by definition, the force field. (All integrals of this kind, where the integration domain is left unspecified, are over all space.) But most often the map $v \to -\partial_t\Psi_v(0, B(0)$ is just that: a map, linear in v, with the kind of continuity with respect to v that qualifies it as a distribution. (It's a *vector*-valued distribution, since the test functions v are themselves vector-valued.)

For instance, in the case just evoked of a piece with reluctivity ν, the linear map one finds *cannot* be written as $\int F\cdot v$, where F would be ${}^1\!/_2\,|B|^2\nabla\nu$ at all points where this vector field is well defined. This would exclude S, across which both ν and $|B|^2$ are discontinuous, and thus would make us miss the surface force. The convenient notation ${}^1\!/_2\,|B|^2\nabla\nu$ will be used nonetheless for the force distribution, but it will denote a different object than F. Which object, exactly, is what we need to make clear, and the proof of (1) will come as a by-product.

This exercise will be followed by a more difficult one, the case of hard magnets with $B = \mu_0(H + M)$ as B–H law. Several possibilities exist for how M depends on the deformation of matter. One of them (the simplest, according to which M rotates with matter, but does not depend on local strain) was addressed in [4], where the force field 'in the sense of distributions' was found to be

$$F = -\nabla M\cdot B - {}^1\!/_2\,\mathrm{rot}(H\times B), \tag{2}$$

where the meaning of $\nabla M\cdot B$ will soon be made precise. There is again, hidden in (2), a system of forces borne by S (the air-

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magnet interface, or the magnet-magnet interface where M can be discontinuous), a part of which is normal to S and the other part tangential. We shall generalize all that to a large category of non-linear B–H laws.

On notations: Given vector fields X and Y, we define $X\cdot\nabla Y$ as the vector field with components $X^j\,\partial_j Y^i$ in an orthonormal frame, using Einstein's convention. In contrast, $\nabla Y\cdot X$ is $\partial_i Y^j\,X^j$, as in (2). One will check that $\nabla Y\cdot X - X\cdot\nabla Y$ = $X\times \operatorname{rot} Y$. The 'dyadic product' $X\otimes Y$ is the 2-tensor T with components $T^{ij} = X^i Y^j$. Other examples of 2-tensors are the above ∇Y and the Kronecker δ. The classical Maxwell tensor is then ${}^M T = H\otimes B - {}^1\!/_2\,(H\cdot B)\,\delta$. We shall have use for the right-side divergence $\operatorname{div} T$, in components $\partial_j T^{ij}$, of a 2-tensor T, and for its product $T\cdot X$, in components $T^{ij}X^j$, with a vector X. Note that $\operatorname{div}\delta = 0$ and $\operatorname{div}(f\delta) = \nabla f$ if f is a scalar field.

II. Distributions 101

What follows is an elementary introduction to aspects of distribution theory relevant here. Let us for a moment distance ourselves from electromagnetism and deal with two scalar functions f and g. (Later, they will become ν and $|B|^2$.) The test functions, smooth and compactly supported, are called φ when scalar-valued, v when vector-valued.

If f is just integrable, without more regularity, the map $\varphi \to \int f\,\varphi$ does qualify as a distribution. But for a distribution such as the 'Dirac mass' $\varphi \to \varphi(a)$, where a is a given spatial point, there is *no* function δ_a such that $\varphi(a) = \int \delta_a\,\varphi$. Thus, distributions encompass functions and generalize them [7].

When f is not differentiable, its gradient 'in the sense of distributions' exists nonetheless. It's the distribution $v \to -\int f \operatorname{div} v$, to which one extends the notation ∇f. Then one understands $\int \nabla f\cdot v$ as $-\int f \operatorname{div} v$. A notational abuse, of course, but which makes sense: if f were differentiable all over space, one would have $-\int f\operatorname{div} v = \int \nabla f\cdot v$, indeed. Now suppose f smooth inside both D and the outer region D', but discontinuous across their common boundary S, with a jump $[f]$. Then, integrating by parts on D and D',

$$\int \nabla f\cdot v \,\widehat{=}\, -\int f\operatorname{div} v = -\int_S [f]\,n\cdot v + \int_{D\cup D'} (\nabla^s f)\cdot v, \quad (3)$$

where $\nabla^s f$ denotes the 'strong' gradient of f, well-defined in D and D', but not on S. The capped equal sign means that the integral expression to its left is *defined as* the one to its right.

So the vector-valued distribution denoted by ∇f in (3) can be represented by two ordinary vector fields, the almost everywhere defined $\nabla^s f$, living on 3D space, and $[f]\,n$, living on S only. We find them, as a rule, in the roles of body force and interface force, in all situations evoked here.

Now, let's try and interpret in the sense of distributions the product $g\,\nabla f$. If both f and g are smooth all over and compactly supported, one has $\int g\,\nabla f\cdot v = -\int f\operatorname{div}(gv)$. So we may take that as a definition of $g\,\nabla f$, provided $\int f\operatorname{div}(gv)$ makes sense, which requires $[g]$ = 0. Then,

$$\int (g\,\nabla f)\cdot v \,\widehat{=}\, -\int_S g\,[f]\,n\cdot v + \int_{D\cup D'} g\,(\nabla^s f)\cdot v. \quad (4)$$

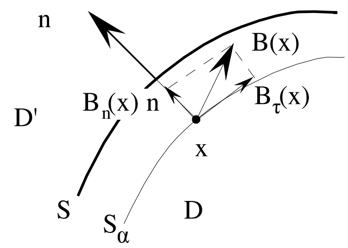

Fig. 1. The jump $[g]$ of a scalar quantity g across S is its value on the 'upstream' side of S minus its value on the 'downstream' side, as both defined by the direction of the normal field n. By convention, n goes from D to D' here. Also shown, one of the surfaces S_α of the foliation described in the text (S is S_0) and suggested, the orthogonal decomposition of the field B into normal and tangential parts.

The constraint $[g]$ = 0 is not a surprise, since the product of two distributions (here g and ∇f) does not exist unconditionally. But our goal, to find an interpretation of $|B|^2\nabla\nu$ as a distribution, is thwarted, since $[|B|^2] \neq 0$ as a rule. Neither can we handle $-|H|^2\nabla\mu$ that way, since $[|H|^2] \neq 0$ as well. Fortunately, a suitable combination of these two expressions will work.

III. Proving (1)

Suppose the interface S presented as the locus of points x for which $s(x) = 0$, for some smooth real function s. (Having that *locally* is enough.) Then, the surfaces S_α = $\{x : s(x) = \alpha\}$, for α in a neighborhood of zero, say $-\delta < \alpha < \delta$, make a foliation of a neighborhood of S, call it D_δ. Call d the function on D_δ defined by $d(x) = \alpha$ when x belongs to S_α. To each such point x, assign the unit vector $(\nabla d)(x)/|(\nabla d)(x)|$, hence a field n which extends the field of unit normals to S considered so far. Pick also two unit vectors, anchored at x, tangent to $S_{d(x)}$, mutually orthogonal, both smoothly depending on x. This way, we have a smooth system of orthonormal frames, 'adapted' to S in an obvious sense. Any smooth vector field X will have (when restricted to D_δ) an orthogonal decomposition of the form X = $X_n\,n + X_\tau$, normal part plus tangential part. When X is smooth in D and D' separately, but discontinuous across S, one may talk of the jumps $[X_n]\,n$ and $[X_\tau]$ across S of these two parts.

Now, let's apply this to the physical fields H and B, for which $[H_\tau]$ = 0 and $[B_n]$ = 0. Over D_δ, except on S where discontinuities occur, we have

$$|B|^2\nabla\nu = B_n^2\nabla\nu + |B_\tau|^2\nabla\nu$$

$$= B_n^2\nabla\nu + |\mu H_\tau|^2\nabla\nu = B_n^2\nabla\nu + |H_\tau|^2\mu^2\nabla\nu$$

$$= B_n^2\nabla\nu - |H_\tau|^2\nabla\mu, \quad (5)$$

since $\mu\nabla\nu = -\nu\nabla\mu$ as entailed by $\nu\mu = 1$. Thus, $|B|^2\nabla\nu$ appears as the difference of two terms, $B_n^2\nabla\nu$ and $|H_\tau|^2\nabla\mu$, both of the form $g\nabla f$ with $[g]$ = 0 on which we worked previously. Applying (4) to both terms yields (1).

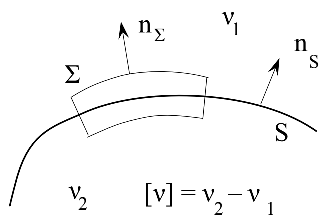

Fig. 2. Surface Σ here is astride a part of the material interface S. Integrating $T_M n$ over Σ gives the magnetic force on this part. The top and bottom surfaces of the pillbox need not coincide with surfaces of the foliation S_α of Fig. 1 (but should be close to S on the side of ν_i in case ν_i, $i = 1$, 2, is not uniform).

IV. Maxwell's tensor and the 'pillbox' method

There is another approach to the previous result, by using the identity, valid when $H = \nu B$ and $\mathrm{div} B = 0$,

$$\mathrm{div}({}^M T) = (\mathrm{rot} H) \times B + {}^1\!/_2\, |B|^2 \nabla \nu. \tag{6}$$

This is just algebra when ν, H and B are smooth, and we allow discontinuities by taking (6) in the sense of distributions. Consider now the vector field ${}^M T \cdot n_\Sigma$, i.e., $({}^M T)^{ij} n_\Sigma^j$ in coordinates, where n_Σ is the normal field living on the surface Σ of Fig. 2. Integrate it over the pillbox surface Σ. The result is a vector. According to Gauss's theorem and our previously acquired knowledge (cf., e.g., [4]) that $(\mathrm{rot} H) \times B + {}^1\!/_2\, |B|^2 \nabla \nu$ is the bulk force, this vector is the total force upon the matter inside Σ, and it converges towards the force on the part of the interface within Σ when the top and bottom surfaces of the pillbox are moved towards S. (Piecewise smoothness is enough for proving this convergence.)

The passage to the limit thus suggested leads to (1) by a computation one may find more intuitive than the previous one. The reason for that is visible in (6): As a distribution, ${}^M T$ is more regular (it's an ordinary tensor field, actually, almost everywhere defined) than its divergence $(\mathrm{rot} H) \times B$ + ${}^1\!/_2\, |B|^2 \nabla \nu$. This is an example of the general and well known fact that integration has a smoothing effect, contrary to differentiation. (Think of the vector potential A, more regular than the field B that derives from it: Here, ${}^M T$ is a 'tensor potential' from which the right-hand side of (6) derives. The analogy is apt in many respects.)

It thus appears that the knowledge of a 2-tensor from which the force, as a distribution, derives, is desirable. Hence our working programme from this point on: In more and more complex situations (non-linearity of the B–H law, anisotropy, magnetostrictive behavior), find the force F as a distribution by applying the VPP, then find a 2-tensor T such that $\mathrm{div}\, T$ = F (clearly, T need not be unique), then integrate $T \cdot n_\Sigma$ over the surface of a suitable pillbox to reveal the interface forces.

V. Magnetic energy during a virtual motion

Let us assume that we face a problem in magnetoelasticity in which the magnetic energy density of a piece of matter depends on its state of deformation β and of the ambient magnetic field b. (This excludes magnetic hysteresis and a lot of complex mechanical behaviors, such as those of ferrofluids and other so-called polar materials.) So we have a function $\psi(x, \beta, b)$ where β is a linear map (the one that sends a material vector ξ anchored at point x to its new value $\beta(\xi)$ in the deformed state), and b a vector. This function, we assume, is convex with respect to b, and the B–H law is simply expressed by $H(x) = \partial_b \psi(x, \beta(x), B(x))$. The magnetic energy density (later denoted by $\widehat{\psi}$) is then $\widehat{\psi}(x) = \psi(x, \beta(x), B(x))dx$, and the total magnetic energy is $\int \widehat{\psi}(x)dx$, an integral over all space, with dx as volume element.

Note that β can be considered as a 2-tensor, but not a symmetric one. By polar decomposition, one can write $\beta = rs$, where r is a rotation and s a symmetric map, the one known as *strain* in Mechanics. As a very general rule, constitutive laws are not affected by rotations, which in our case implies that $\psi(x, rs, rb) = \psi(x, s, b)$ whatever the rotation r. It is therefore enough to know $\psi(x, s, b)$, where s is a symmetric 2-tensor. We shall have use for the three partial derivatives of ψ, also as functions of x, s and b, denoted by $\partial_x \psi$, $\partial_s \psi$, $\partial_b \psi$. (Occasionally, we may write $\nabla \psi$ instead of $\partial_x \psi$, since ∇ connotes spatial derivation usually.) We note that $\partial_s \psi$ has the dimension of a *stress,* which justifies calling σ_M, with M for 'magnetostriction', its value in the reference state.

During a virtual motion $x \to x + tv(x)$, the magnetic energy changes, even though B is frozen at its reference value, because the material configuration changes. So we call $\psi_v(t, x, B(x))$ the value of the magnetic energy density at time t and point x in the ambient field $B(x)$ during such a virtual motion. (No mention of β: it is determined by v. Beware that ψ and ψ_v are very different objects.) Recall (cf. [4]) that the virtual power we aim at is the partial derivative in t, at $t = 0$, of (minus) the total magnetic energy $\Psi_v(t, B) = \int \psi_v(t, x, B(x))dx$, with B frozen in all space at its value at $t = 0$.

We start from

$$\psi_v(t, x + tv(x), b) = \psi(x, s_v(t, x), [r_v(t, x)]^{-1} b), \tag{7}$$

where s_v and r_v are the elements of the deformation (strain and rotation) all over. What (7) says is this: "The particle that was at point x at time 0 has reached the point $x + tv(x)$ at time t, and because of the deformation $r_v(t) s_v(t)$ that it has sustained, the magnetic energy density it bears in field b is what it would have been at x under the strain s_v in the field $[r_v(t, x)]^{-1} b$."

Next, we obtain the time-derivative of $-\psi_v$ at time 0 by differentiating both sides of (7) in t, using the chain rule. (Hence the virtual power, by summing up $-\partial_{t=0} \psi_v$ over all space.) One makes use of the approximations $s_v(t, x) \simeq 1 + t \nabla_{sym} v$ and $r_v(t, x) b \simeq 1 + t/2\, (\mathrm{rot} v) \times b$, where $\nabla_{sym} v$ is the symmetrized 2-tensor $(\partial_i v^j + \partial_j v^i)/2$. This amounts to neglecting terms in t^2 and higher order in the Taylor expansion of $-\psi_v$ about $t = 0$, which doesn't affect the result:

$$-\partial_t \psi_{v|_{t=0}} = v \cdot [\nabla \psi + \mathrm{div}(\sigma_M) - {}^1\!/_2\, \mathrm{rot}(H \times B)], \tag{8}$$

as obtained when one substitutes $B(x)$ for b. The appearance of H there is due to the fact that $H = \partial_b \psi$. The operators rot and $-$div come from standard integration by parts, as adjuncts of the rot in "rot v" and of the ∇_{sym} in "$\nabla_{sym} v$".

Remarkably, the result is generic: Whatever the B–H law, one has the same three terms in the expression of the magnetic force (in addition to the $(\text{rot}H)\times B$ force): An *inhomogeneity* term $\nabla\psi$ (generalizing the $|B|^2/2\,\nabla\nu$ we had when ψ was $\psi(x,s,b) = {}^1\!/_2\,\nu(x)|b|^2$ whatever s), a *magnetostrictive* term $\text{div}(\sigma_M)$, and an *anisotropy* term $-{}^1\!/_2\,\text{rot}(H\times B)$. All of these, of course, to be interpreted as distributions, so we aim now at a Maxwell-like tensor[2] whose divergence would be the term between square brackets in (8).

VI. An augmented Maxwell tensor

Let's recall a few identities: For H and B not necessarily the physical fields by this name,

$$(\text{rot}\,H)\times B = B\cdot\nabla H - \nabla H\cdot B. \tag{9}$$

One remarks that

$$\text{div}(H\otimes B) = H\,\text{div}B + B\cdot\nabla H, \tag{10}$$

$$\text{div}[H\otimes B - B\otimes H] = \text{rot}(H\times B). \tag{11}$$

Let now B and H be the physical fields, linked by the law $\psi(B(x)) + \varphi(H(x)) = B(x)\cdot H(x)$, where φ is the coenergy density dual to ψ, with $\text{div}\,B = 0$. (We'll say that B and H make a 'magnetic pair'.) To avoid confusion, we denote the $\nabla\psi$ of (8), that is to say the partial derivative of ψ with respect to position in presence of the ambient field B, by $\nabla\psi(\cdot,B)$. Same convention for $\nabla\varphi(\cdot,H)$. Recall that $\nabla\psi(\cdot,B) = -\nabla\varphi(\cdot,H)$ for such a magnetic pair. Let us denote by $\widehat{\varphi}(H)$ the coenergy density $\varphi(x,H(x))$, a function of x only (the H is just a label). By the chain rule,

$$\nabla\widehat{\varphi}(H) = \nabla\varphi(\,\cdot\,,H) + \nabla H\cdot B. \tag{12}$$

We may now assert:

Proposition 1. *If B and H form a magnetic pair, then*

$$\text{div}[{}^1\!/_2\,(H\otimes B + B\otimes H) - \widehat{\varphi}(H)\,\delta] = \tag{13}$$
$$(\text{rot}\,H)\times B - \nabla\varphi(\,\cdot\,,H) - {}^1\!/_2\,\text{rot}(H\times B).$$

Proof of (13): Thanks to (9), $(\text{rot}\,H)\times B - \nabla\varphi(\,\cdot\,,H) = B\cdot\nabla H - (\nabla\varphi(\,\cdot\,,H) + \nabla H\cdot B) = B\cdot\nabla H - \nabla\widehat{\varphi}(H)$, taking (12) into account. But this is $\text{div}(H\otimes B - \widehat{\varphi}(H)\,\delta)$, thanks to (10) and to the fact that $\text{div}(g\,\delta) = \nabla g$. So one has

$$\text{div}(H\otimes B - \widehat{\varphi}(H)\,\delta) = (\text{rot}\,H)\times B - \nabla\varphi(\,\cdot\,,H). \tag{14}$$

Subtracting from (14), on both sides, half of (11), one does find (13). ⋄

Hence a Maxwell tensor that generalizes Sanchez et al.'s proposal in [5] and [6] (which was $H\otimes B - \widehat{\varphi}(H)\,\delta$, if expressed in our notation) by accounting for the anisotropy forces (whose omission may make observable difference, as shown in [8]). By adding σ_M to it, we obtain the following augmented Maxwell tensor,

$${}^A T = {}^1\!/_2\,(H\otimes B + B\otimes H) - \widehat{\varphi}(H)\,\delta + \sigma_M, \tag{15}$$

which also accounts for magnetostriction.

[2] For lack of space, Section VI that follows had to be excised from the published version, in which the expression (15) for the augmented Maxwell tensor was given without proof.

VII. Interface terms

Let us conclude by giving the results of some pillbox computations. In the nonlinear but isotropic case, we find $F_S = ([\widehat{\varphi}(H)] - [H_n]B_n)n$ by using the adapted system of frames of Fig. 1 ([ ] denotes the jump). There is an equivalent 'B-oriented' formula, $F_S = (-[\widehat{\psi}(B)] + [B_\tau]\cdot H_\tau)n$, showing that there is no "preference for the coenergy" in the whole approach. (Details on this will appear in [9].)

In case of anisotropy, the term $-{}^1\!/_2\,\text{rot}(H\times B)$ in (2) stands for a distribution represented by two fields. One is $-{}^1\!/_2\,\text{rot}^s(H\times B)$, with the strong form of the curl, the bulk force. The other one, borne by S, is expressed by one half the jump $[n\times(H\times B)]$. By the double cross product formula, this jump is $[(n\cdot B)H - (n\cdot H)B]$, which equals $n\times[n\cdot H\,n\times B]$. This can be seen to reduce to $-{}^1\!/_2\,[B_\tau H_n]$, a tangential field living on S, by using the S-adapted system of frames.

Especially interesting is the case $B = \mu_0(H+M)$. In the adapted frame system, $M = M_n n + M_\tau$, both M_n and M_τ smooth in D and D', but discontinuous, with jumps $[M_n]$ and $[M_\tau]$ across S. Substituting $\nu_0 B - M$ for H and $\mu_0(H+M)$ for B, one finds, after a short calculation,

$$[n\times(H\times B)] = \mu_0[M_n]H_\tau - B_n[M_\tau] + \mu_0[M_n M_\tau], \tag{16}$$

to be multiplied by $1/2$ to get the tangential part of the interface force F_S. Last, the term $-\nabla M\cdot B$ of (2) is found, by the technique of Section II, to contribute to F_S the normal field

$$\{[M_n]B_n + \mu_0[M_\tau]\cdot H_\tau + {}^1\!/_2\,\mu_0[|M_\tau|^2]\}n. \tag{17}$$

It would remain to explore the large realm of magnetostriction by using (15). For example, assume $B = \mu_0(H + M(s))$. Then $\sigma_M = -\partial_s(M(s)\cdot B)$. One finds the same surface forces as above, with $M = M(1)$, *plus* a term in $[\sigma_M\cdot n]$. There is also a bulk force $\text{div}(\sigma_M)$, even if M is uniform (because B is not, as a rule). Knowing only $M(1)$, the magnetization in the reference configuration, would not give access to these terms, thus leaving the question "What is the force inside a permanent magnet?" ill-posed.

To sum up: Interface forces can be obtained by the VPP, but as distributions, and these are a delicate tool. More robust (and fully equivalent) is the use of Maxwell-like tensors allied with the pillbox trick. But such tensors, built from the VPP-computed forces, not the other way round, may differ from the classical Maxwell tensor, which should not be construed as a primary concept in the question of forces, but as just a helpful analytical tool.

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