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# Convergence to time-periodic solutions in time-periodic Hamilton-Jacobi equations on the circle

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**ABSTRACT.** — . The goal of this paper is to give a simple proof of the convergence to time-periodic states of the solutions of time-periodic Hamilton-Jacobi equations on the circle with convex Hamiltonian. Note that the period of limiting solutions may be greater than the period of the Hamiltonian.

**RESUMÉ.** — . On donne une preuve simple de la convergence vers des états périodiques en temps pour les solutions d'équations de Hamilton-Jacobi sur le cercle avec un Hamiltonien convexe et périodique en temps. Il est à noter que la périodes des solutions limites peut être plus grande que la période du Hamiltonien.

## 1. Introduction

We consider the Hamilton-Jacobi equation

$$(1.1) \quad u_t + H(t, x, u_x) = 0, \quad x \in \mathbf{T}$$

where  $\mathbf{T}$  is the unit circle. The Hamiltonian  $H(t, x, p) : \mathbb{R} \times \mathbf{T} \times \mathbb{R} \mapsto \mathbb{R}$  is  $C^2$ , 1-periodic in  $t$ , and satisfies the following classical hypotheses:

- Strict convexity:  $H_{pp}(t, x, p) > 0$  for all  $(t, x, p) \in \mathbb{R} \times \mathbf{T} \times \mathbb{R}$ .
- Super-linearity:  $H(t, x, p)/p \rightarrow \infty$  as  $|p| \rightarrow \infty$  for each  $(t, x) \in \mathbb{R} \times \mathbf{T}$ .
- Completeness: The Hamiltonian vector-field

$$X(t, x, p) = (H_p(t, x, p), -H_x(t, x, p))$$

is complete, *i.e.* for all  $(t_0, x_0, p_0)$ , there exists a  $C^2$  curve  $\gamma(t) = (x(t), p(t)) : \mathbb{R} \rightarrow \mathbf{T} \times \mathbb{R}$  such that  $(x(t_0), p(t_0)) = (x_0, p_0)$  and  $\dot{\gamma}(t) = X(t, \gamma(t))$  for all  $t \in \mathbb{R}$ .

The first two assumptions are classical in the viscosity solutions theory; see [18]. The last one was introduced in Mather [16]; note that it is satisfied if there exists a constant  $C$  such that  $|H_t| \leq C(1 + H)$ .

Under the above three assumptions, the Cauchy Problem for (1.1) is well posed in the viscosity sense: given a time  $s \in \mathbb{R}$  and a continuous function  $u_0 : \mathbf{T} \rightarrow \mathbb{R}$ ,

equation (1.1) has a unique viscosity solution  $u(t, x) : [s, +\infty[ \times \mathbf{T} \longrightarrow \mathbb{R}$ , such that  $u(s, \cdot) = u_0$ . It will be denoted by  $\mathcal{T}(s, t)u_0$ . See [18], for instance.

It is known - and this is not specific to the one-dimensional setting - that there exists a real number  $\lambda$  such that  $u(t, x) + \lambda t$  is bounded for all viscosity solution  $u : [s, +\infty[ \times \mathbf{T} \longrightarrow \mathbb{R}$  of (1.1). The real number  $\lambda$  has various different names: It is the critical value of Mañe, see [17], [7] or the value  $\alpha(0)$ , where  $\alpha : \mathbb{R} \longrightarrow \mathbb{R}$  is the Mather function, see [16], or the averaged Hamiltonian [19]. Note that the number  $\lambda$  may also be viewed as the eigenvalue of the Hopf-Lax-Oleinik operator in the sense of min-plus algebra, see [24]. We are interested in proving the following result.

**Theorem 1.1** *Let  $u(t, x) : [s, +\infty[ \times \mathbf{T} \longrightarrow \mathbb{R}$  be a viscosity solution of (1.1). There exist an integer  $T$  and a viscosity solution  $l(t, x) = \phi(t, x) - \lambda t : \mathbb{R} \times \mathbf{T} \longrightarrow \mathbb{R}$  such that  $\phi$  is  $T$ -periodic in  $t$  and*

$$\lim_{t \rightarrow \infty} \|u(t, \cdot) - l(t, \cdot)\|_{\infty} = 0.$$

In the following, we will always assume that  $\lambda = 0$ , which can be obtained by replacing the Hamiltonian  $H$  by  $H - \lambda$ .

It is known that there always exists a viscosity solution of (1.1) which is 1-periodic in time (such a solution is not unique in general). However, it is not hard to build examples of viscosity solutions of equations of the form (1.1) which do not converge to 1-periodic solutions, see [4] and [13]. More precisely, one can build solutions which are periodic in time, but of minimal period greater than one. Hence one cannot expect to have always  $T = 1$  in the theorem.

For time-independent Hamiltonians, convergence to steady states is known: a particular nontrivial multidimensional case (including some non-strictly convex situations) is studied in [20], and the general result in one dimension for strictly convex Hamiltonians is given in [23]. Recall - [4] - that non strict convexity may result in a failure of the convergence result. The general multidimensional result in the strictly convex case is due to Fathi [10], with dynamical systems arguments. A purely PDE proof, encompassing both assumptions of [10] and [20], is provided in [4]. A proof mixing PDE and dynamical systems arguments is given in [22], and proves the above-stated theorem in some cases (rational rotation number; see below).

The situation is not so clear when the Hamiltonian is time-periodic. In order to be more precise, it is useful to recall that one can associate to the equation (1.1) a rotation number  $\rho \in \mathbb{R}$ , see section 4., which is the rotation number of extremals. We then have the following

**Addendum** *The period  $T$  in the theorem is 1 if the rotation number  $\rho$  is irrational and is not greater than  $q$  if  $\rho$  is a rational  $p/q$ .*

The theorem and its addendum have been proved in [6] from the dynamical system point of view, that is from the study of extremals. Let us mention also that, in the case of an irrational rotation number, there is a single periodic viscosity solution, (up to an additive constant) as was proved in [8] and [24].

As said before, the result in the case of a rational rotation number had been previously obtained by a method relying on the dynamic programming principle in

[22]. It turns out that the ideas of this paper can be exploited further to provide a simpler proof of the general case. It is our aim to present this proof here.

We shall first recall the general properties of viscosity solutions, in Section 2.. In Section 3., we introduce some dynamics, define the Aubry-Mather sets, and recall specific observations concerning  $\omega$ -limit solutions, mostly taken from [22]. All the results in these sections are general, and remain true if one considers equation (1.1) on any compact manifold. We complement these general observations by specific one-dimensional arguments in Section 4. to conclude the proof.

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## 2. General properties and a large time behaviour candidate

Let us start with some well understood properties of viscosity solutions without proof, see [18], [9] or [13]. It is useful to introduce the Lagrangian

$$L(t, x, v) = \max_{p \in \mathbb{R}} (pv - H(t, x, p)),$$

which is also convex and superlinear. We have the following fundamental result of the calculus of variations (see a proof in the appendix of [16]):

Let us fix a time interval  $[t, t']$ , and two points  $X, X'$  in  $\mathbf{T}$ . If the Lagrangian is convex and superlinear, the action integral

$$\int_t^{t'} L(s, x(s), \dot{x}(s)) ds$$

reaches its minimum on the set of absolutely continuous curves  $x(s) : [t, t'] \rightarrow \mathbf{T}$  which satisfy  $x(t) = X$  and  $x(t') = X'$ . If in addition the Hamiltonian flow is complete, then the minimum is reached by  $C^2$  curves which satisfy the Euler-Lagrange equations. Denoting  $p(s) = L_v(q(s), \dot{q}(s), s)$ , these equations can be written

$$\dot{p}(s) = L_x(s, x(s), \dot{x}(s)) = -H_x(s, x(s), p(s))$$

hence the curve  $(x(s), p(s))$  is a trajectory of the Hamiltonian vector-field.

Let us define, for all  $t > s$ , the Hopf-Lax-Oleinik operator  $\mathcal{T}(s, t)$ , which, to any continuous function  $u_0 : \mathbf{T} \rightarrow \mathbb{R}$ , associates the continuous function

$$(2.1) \quad \mathcal{T}(s, t)u_0(x) = \inf_{\gamma} \left( u_0(\gamma(s)) + \int_s^t L(\sigma, \gamma, \dot{\gamma}) d\sigma \right),$$

where the infimum is taken on the set of absolutely continuous curves  $\gamma : [s, t] \rightarrow \mathbf{T}$  satisfying  $\gamma(t) = x$ .

The viscosity solution  $u(t, x) \in C([s, +\infty[ \times \mathbf{T}, \mathbb{R})$  of (1.1) with initial condition  $u(s, \cdot) = u_0$  is given by  $u(t, \cdot) = \mathcal{T}(0, t)u_0$ . This may be taken as the definition of

viscosity solutions in the present paper. This definition is equivalent to the standard one in our setting. This fact is classical, see a proof in [13] for instance.

We have the Markov property

$$\mathcal{T}(t, t') \circ \mathcal{T}(s, t) = \mathcal{T}(s, t')$$

for  $s \leq t \leq t'$ , hence the mappings  $\mathcal{T}(0, n) = \mathcal{T}(0, 1)^n$ ,  $n \in \mathbb{N}$  form a discrete semi-group. We will note  $\mathcal{T}$  for  $\mathcal{T}(0, 1)$  for simplicity. Here are some additional simple properties of  $\mathcal{T}$ . They are mainly proved in the literature in the autonomous case, but their extensions to time-dependent hamiltonians are straightforward.

- The mappings  $\mathcal{T}(s, t)$  are contractions,

$$\|\mathcal{T}(s, t)u - \mathcal{T}(s, t)v\|_\infty \leq \|u - v\|_\infty.$$

This is a well-known property; see [18].

- The mappings  $\mathcal{T}(s, t)$  are order-preserving and satisfy  $\mathcal{T}(s, t)(c + u) = c + \mathcal{T}(s, t)(u)$  for all real  $c$ . This is also well-known, see [18].

- The mappings  $\mathcal{T}(s, t)$  are compact. More precisely, there exists a positive non-increasing function  $K(\epsilon) : ]0, \infty[ \rightarrow ]0, \infty[$  such that  $\mathcal{T}(s, t)u$  is  $K(\epsilon)$ -Lipschitz for all continuous function  $u$  and all  $t \geq s + \epsilon$ . This fact is a regularizing effect related to the strict convexity assumption; see [1], [9], [11].

The following proposition will be useful in the sequel. Note that it implies the existence of a fixed point of the semi-group generated by  $\mathcal{T}$ , *i.e.* the existence of a 1-periodic viscosity solution of (1.1).

**Proposition 2.1** *Let  $u(t, x) \in C([s, \infty[ \times \mathbf{T}, \mathbb{R})$  be a viscosity solution of (1.1); recall that it is bounded on  $[s, \infty[ \times \mathbf{T}$ . Then the 1-periodic function*

$$\phi(t, x) = \liminf_{n \rightarrow +\infty} u(t + n, x)$$

*is a viscosity solution of (1.1).*

From the Barles-Perthame lemma [2],  $\phi$  is a viscosity sub-solution. Much more, due to the convexity of the Hamiltonian, it is a solution - Barron-Jensen [5]. We thank the referee for pointing out to us the last item. For the sake of self-containedness, we provide a simple proof using the Hopf-Lax-Oleinik expression of solutions.

PROOF. We have to prove that  $\mathcal{T}(s, t)\phi(s, \cdot) = \phi(t, \cdot)$  for all  $s \leq t$ . Let us first prove that  $\phi$  is a sub-solution, *i. e.* that  $\mathcal{T}(s, t)\phi(s, \cdot) \leq \phi(t, \cdot)$ . In order to do so, we fix  $(t, x)$  and consider an increasing sequence  $n_k$  of integers such that  $u(t + n_k, x) \rightarrow \phi(t, x)$ . There exists a sequence of curves  $\gamma_k : [s, t] \rightarrow \mathbf{T}$  such that

$$u(t + n_k, x) = u(s + n_k, \gamma_k(0)) + \int_s^t L(\sigma, \gamma_k(\sigma + t), \dot{\gamma}_k(\sigma + t)) d\sigma.$$

The sequence  $\gamma_k$  is compact for the  $C^1$  topology, and we will assume by possibly taking a subsequence in  $n_k$  that it is convergent, and note  $\gamma$  the limit. Taking the  $\liminf$  in the equality above gives

$$\phi(t, x) \geq \phi(s, \gamma(0)) + \int_s^t L(\sigma, \gamma(\sigma + t), \dot{\gamma}(\sigma + t)) d\sigma \geq \mathcal{T}(s, t)\phi(s, \cdot)(x).$$

We have used that the functions  $u(t, \cdot)$  are equicontinuous to conclude that  $\liminf u(s + n_k, \gamma_k(0)) = \liminf u(s + n_k, \gamma(0)) \geq \phi(s, \gamma(0))$ .

We now prove that  $\phi$  is a super-solution, *i. e.* that  $\mathcal{T}(s, t)\phi(s, \cdot) \geq \phi(t, \cdot)$ . Note that for all curve  $\gamma : [s, t] \rightarrow \mathbf{T}$ , we have

$$u(t + n, x) \leq u(s + n, \gamma(0)) + \int_s^t L(\sigma, \gamma(\sigma + t), \dot{\gamma}(\sigma + t)) d\sigma.$$

Taking the  $\liminf$ , we obtain

$$\phi(t, x) \leq \phi(s, \gamma(0)) + \int_s^t L(\sigma, \gamma(\sigma + t), \dot{\gamma}(\sigma + t)) d\sigma$$

for each curve  $\gamma$ , hence  $\phi(t, \cdot) \leq \mathcal{T}(s, t)\phi(s, \cdot)$ , which is the desired inequality.  $\square$

The basic objects to understand in order to study the asymptotic behaviour of solutions of (1.1) are the  $\omega$ -limit solutions. Recall that a solution  $u(t, x) : \mathbb{R} \times \mathbf{T} \rightarrow \mathbb{R}$  is called an  $\omega$ -limit solution if there exists a solution  $v : [s, \infty[ \times \mathbf{T} \rightarrow \mathbb{R}$  and an increasing sequence  $n_k$  of integers such that

$$u(t, x) = \lim_{k \rightarrow \infty} v(t + n_k, x).$$

In other words,  $\omega$ -limit solutions are solutions whose initial value  $u(0, \cdot)$  is an  $\omega$ -limit of the semi-group  $\mathcal{T}$ . If  $u(t, x) : \mathbb{R} \times \mathbf{T} \rightarrow \mathbb{R}$  is an  $\omega$ -limit solution of (1.1), then it is bounded, and the functions  $u(t, \cdot), t \in \mathbb{R}$  are equilipschitz.

### 3. Calibrated curves and Uniqueness set

In this section, we give some salient features of  $\omega$ -limit solutions that do not depend of the dimension of the ambient space. The key differentiability result is stated and used as a black box; the other results are proved, although their proofs are already in [9] or [22].

Let  $u : [s, \infty[ \times \mathbf{T} \rightarrow \mathbb{R}$  be a viscosity solution of (1.1). A curve  $\gamma : [s, \infty[ \supset [t, t'] \rightarrow \mathbf{T}$  is said calibrated by  $u$  if

$$u(t', \gamma(t')) = u(t, \gamma(t)) + \int_t^{t'} L(\sigma, \gamma, \dot{\gamma}) d\sigma.$$

Note that if  $\gamma(s)$  is a calibrated curve, then the curve  $(\gamma(s), p(s))$  is a trajectory of the Hamiltonian flow, where  $p(s) = \partial_v L(s, \gamma(s), \dot{\gamma}(s))$ . We have the following regularity result

**Theorem 3.1** (*Fathi, [9]*). *If  $\gamma : [t, t'] \rightarrow \mathbf{T}$  is calibrated by  $u$ , then  $u_x$  exists at each point  $(s, \gamma(s)), s \in [t, t'[$  and satisfies*

$$(3.2) \quad \begin{aligned} u_x(s, \gamma(s)) &= p(s) = L_v(s, \gamma(s), \dot{\gamma}(s)) \\ \iff \dot{\gamma}(s) &= H_p(s, \gamma(s), u_x(s, \gamma(s))). \end{aligned}$$

It was already proved in [14] that this relation holds at differentiability points of  $u$ . The regularity of  $u$  on calibrated curves was noticed in [21] for the equation  $|\nabla u| = f(x)$  on the sphere,  $f$  nonnegative with nonempty zero set. It was not, however, made as systematic as in [9].

We now choose once and for all a 1-periodic solution  $\phi(t, x)$  of (1.1). First, let us note that a classical compactness argument gives the existence of curves  $\gamma : \mathbb{R} \rightarrow \mathbf{T}$  which are calibrated by  $\phi$  on all compact interval.

As a consequence of Theorem 3.1, two such curves cannot intersect. Indeed, if  $\gamma_1$  and  $\gamma_2 : \mathbb{R} \rightarrow \mathbf{T}$  are calibrated by  $\phi$ , and if there exists a  $t$  such that  $\gamma_1(t) = \gamma_2(t)$ , then  $u_x$  exists at the point  $(t, \gamma_1(t))$ , and, setting  $p_i(s) = \partial L(s, \gamma_i(s), \dot{\gamma}_i(s))$ , we have

$$p_1(t) = u_x(t, \gamma_1(t)) = u_x(t, \gamma_2(t)) = p_2(t).$$

The curves  $(\gamma_i(s), p_i(s))$  are then two trajectories of the Hamiltonian flow which are equal at time  $t$ , so that they coincide for all times.

Let

$$\mathcal{D} \subset \mathbb{R} \times \mathbf{T}$$

be the union of the graphs of these orbits, and  $\mathcal{D}_0 \subset \mathbf{T}$  be the set of points  $\gamma(0)$ , where  $\gamma : \mathbb{R} \rightarrow \mathbf{T}$  is calibrated. This is a nonempty compact set. For each  $t$ , we define the mapping  $S^t : \mathcal{D}_0 \rightarrow \mathbf{T}$  which associates to each  $x \in \mathcal{D}_0$ , the value  $\gamma(t)$ , where  $\gamma : \mathbb{R} \rightarrow \mathbf{T}$  is the unique calibrated curve satisfying  $\gamma(0) = x$ . It is a homeomorphism onto its image. Let us mention for completeness that this homeomorphism is bi-Lipschitz, as can be obtained from refined versions of Theorem 3.1. The first results in that direction are due to Mather.

Clearly,  $S^1$  is a homeomorphism of  $\mathcal{D}_0$ . Let us note  $\mathcal{M}_0$  its  $\omega$ -limit. This is the closure in  $\mathbf{T}$  of the set of points  $x \in \mathcal{D}_0$  which are the limit of a sequence  $S^{n_k}(y)$  with  $y \in \mathcal{D}_0$  and  $n_k$  an increasing sequence of integers. The set  $\mathcal{M}_0$  is non-empty and compact. We call  $\mathcal{M}$  the union, in  $\mathbb{R} \times \mathbf{T}$ , of the graphs of curves  $S^t(x)$ ,  $x \in \mathcal{M}_0$ .

### 3.1. The behaviour of global solutions on global extremals

Recall that we have fixed a periodic viscosity solution  $\phi$ . The following remark, noticed in [22], is the key point to the convergence proof:

**Lemma 3.1** *Let  $u_1$  and  $u_2(t, x) : [s, \infty[ \times \mathbf{T} \rightarrow \mathbb{R}$  be viscosity solutions of (1.1), and let  $\gamma(t) : [s, s'] \rightarrow \mathbf{T}$  be a curve calibrated by  $u_2$ . Then the function  $t \mapsto u_1(t, \gamma(t)) - u_2(t, \gamma(t))$  is non-increasing.*

PROOF. For every  $t \leq t'$  we have:

$$u_1(t', \gamma(t')) - u_1(t, \gamma(t)) \leq \int_t^{t'} L(s, \gamma(s), \dot{\gamma}(s)) \, ds$$

by the Lax-Oleinik formula, and

$$u_2(t', \gamma(t')) - u_2(t, \gamma(t)) = \int_t^{t'} L(s, \gamma(s), \dot{\gamma}(s)) \, ds.$$

Hence we have  $(u_1 - u_2)(t', \gamma(t')) \leq (u_1 - u_2)(t, \gamma(t))$ .  $\square$

The important consequence below is also proved in [22] (recall that  $\phi$  is a prescribes periodic viscosity solution):

**Corollary 3.1** *Let  $u(t, x)$  be an  $\omega$ -limit viscosity solution of (1.1), and let  $x \in \mathcal{M}_0$ , then the function  $t \mapsto u(t, S^t(x)) - \phi(t, S^t(x))$  is constant.*

PROOF. Choose  $t_0 > 0$ . Let  $u_0 \in C(\mathbf{T})$  be such that  $u$  is an  $\omega$ -limit solution to  $u_0$ . Consider  $x_0 \in \mathcal{D}_0$  and  $(n_k)_k$  a sequence going to  $+\infty$  such that  $(S^{n_k}(x_0))_k$  converges to  $x$ ; we may always assume that  $(\mathcal{T}^{n_k}u_0)_k$  converges to  $\psi_1$ . Consider a sequence  $(p_k)_k$  going to  $+\infty$  such that  $\mathcal{T}(0, t + n_k + p_k)u_0$  converges uniformly to  $u(t)$ .

An application of Lemma 3.1 tells us that, for all  $\sigma \in \mathbb{R}$ , the function

$$s > 0 \mapsto \mathcal{T}(0, s + \sigma)u_0(S^s(x_0)) - \phi(s, S^s(x_0))$$

is non-increasing, hence has a finite limit  $l(\sigma)$  as  $s \rightarrow +\infty$ ; in particular for every  $\sigma > 0$  the function

$$s \mapsto \mathcal{T}(0, \sigma + s + t)u_0(S^{s+t}(x_0)) - \phi(s + t, S^{s+t}(x_0))$$

has the same limit  $l(\sigma)$  for all  $t \in \mathbb{R}$ ; we set  $l_k = l(p_k)$ . We may assume, up to the extraction of a subsequence, that the sequence  $(l_k)_k$  has a limit.

On the other hand the weak contraction property implies, for all  $t \in [-t_0, t_0]$ :

$$\|\mathcal{T}(0, t + p_k + n_k)u_0 - \mathcal{T}(0, t + p_k)\psi_1\|_\infty \leq \|\mathcal{T}(0, t + n_k)u_0 - \mathcal{T}(0, t)\psi_1\|_\infty;$$

hence  $(\mathcal{T}(0, t + n_k + p_k)u_0 - \mathcal{T}(0, t + p_k)\psi_1)_k$  converges to 0 uniformly on  $[-t_0, t_0] \times \mathbf{T}$ . Therefore we have, for all  $k$ :

$$\mathcal{T}(0, p_k + t)\psi_1(S^t(x)) - \phi(t, S^t(x)) = l_k;$$

letting  $k \rightarrow +\infty$  implies:  $u(t, S^t(x)) - \phi(t, S^t(x)) = l$ . We conclude by saying that  $t_0$  is arbitrary.  $\square$

It follows that the curve  $S^t(x)$  is calibrated by  $u$ , hence, by Theorem 3.1 we obtain:

**Corollary 3.2** *Let  $u(t, x)$  be an  $\omega$ -limit viscosity solution of (1.1). Then the derivatives  $u_x$  and  $\phi_x$  exist on  $\mathcal{M}$ , and they are equal,  $u_x(t, x) = \phi_x(t, x)$  for all  $(t, x) \in \mathcal{M}$ .*

Before we continue, let us give an important remark. All the objects constructed in this section, the sets  $\mathcal{D}$  and  $\mathcal{M}$  and the mappings  $S^t$ , depend on the periodic solution  $\phi$  that was chosen in the beginning. Let us note  $\mathcal{D}_\phi$  and  $\mathcal{M}_\phi$  and  $S_\phi^t$  in order to emphasise this dependence. If  $\psi$  is another 1-periodic viscosity solution, then we see from Corollary 3.1 that the orbits of  $\mathcal{M}_\psi$  are calibrated by  $\phi$ . It follows that the set

$$\mathcal{A} = \bigcap_{\phi} \mathcal{D}_\phi,$$

is not empty, where the intersection is taken on the set of 1-periodic viscosity solutions. This set is usually called the Aubry set. The mappings  $S_{\phi|_{\mathcal{A}_0}}^t$  do not depend on  $\phi$ , and for all  $\phi$ , we have  $\mathcal{M}_\phi \subset \mathcal{A}$ .



### 3.2. Uniqueness set

We mean a set such that two global solutions of (1.1) coinciding on this set coincide everywhere. We formulate the results in the general, non-autonomous, setting.

**Proposition 3.2** *Let  $u(t, x) : \mathbb{R} \times \mathbf{T} \longrightarrow \mathbb{R}$  be a global and bounded viscosity solution of (1.1). such that the functions  $u(t, \cdot), t \in \mathbb{R}$  are equicontinuous and let  $\phi(t, x) : \mathbb{R} \times \mathbf{T} \longrightarrow \mathbb{R}$  be a 1-periodic viscosity solution of (1.1). If  $u = \phi$  on  $\mathcal{M}$ , then  $u = \phi$ .*

Note that the condition of equicontinuity is satisfied by  $\omega$ -limit solutions.

PROOF. Let us fix a point  $(s, q) \in \mathbb{R} \times \mathbf{T}$ . There exists a curve  $x(t) : (-\infty, s] \longrightarrow \mathbf{T}$  which is calibrated by  $\phi$  and satisfies  $x(s) = q$ . In view of Lemma 3.1, the function  $t \longmapsto u(t, x(t)) - \phi(t, x(t))$  is non-increasing hence it has a limit as  $t \longrightarrow -\infty$ . Since  $\gamma$  is calibrated by  $\phi$ , there exists an increasing sequence  $n_k$  such that  $x(-n_k)$  has a limit  $x \in \mathcal{M}_0$ . We obtain:

$$\begin{aligned} \lim_{t \longrightarrow -\infty} u(t, x(t)) - \phi(t, x(t)) &= \lim_{k \longrightarrow \infty} u(-n_k, x(-n_k)) - \phi(-n_k, x(-n_k)) \\ &= \lim u(-n_k, x) - \phi(-n_k, x) = 0 \end{aligned}$$

where we have used the equicontinuity of the functions  $u(t, \cdot)$  and the fact that  $u = \phi$  on  $\mathcal{M}$ . It follows that  $u(t, x(t)) - \phi(t, x(t)) \leq 0$  for all  $t \leq s$  hence  $u(s, q) \leq \phi(s, q)$ . The reversed inequality is obtained in exactly the same way, with the help of Lemma 3.2 below, using a curve  $\gamma(t) : (-\infty, s] \longrightarrow \mathbf{T}$  calibrated by  $u$  and ending at  $q$ .

**Lemma 3.2** *Let  $u(t, x) : \mathbb{R} \times \mathbf{T} \longrightarrow \mathbb{R}$  be a global and bounded viscosity solution of (1.1) and  $\gamma(t) : ]-\infty, s] \longrightarrow \mathbf{T}$  be a curve calibrated by  $u$ . Then there exists an increasing sequence  $n_k$  of integers such that  $\gamma(-n_k) \longrightarrow x \in \mathcal{M}_0$ .*

PROOF. Let us consider a periodic viscosity solution  $\phi$ . In view of Lemma 3.1, the function  $t \longmapsto u(t, \gamma(t)) - \phi(t, \gamma(t))$  is non-increasing, and bounded. Hence this function has a limit as  $t \longrightarrow -\infty$ . Let us choose an increasing sequence  $t_k$  of integers such that the curves  $\gamma(t - t_k)$  are converging uniformly on compact sets to a limit  $\gamma_\infty : \mathbb{R} \longrightarrow \mathbf{T}$ . The following calculations show that this curve is calibrated by  $\phi$ :

$$\begin{aligned} \phi(t', \gamma_\infty(t')) - \phi(t, \gamma_\infty(t)) &= \lim \left( \phi(t', \gamma(t' - t_k)) - \phi(t, \gamma(t - t_k)) \right) \\ &= \lim \left( \phi(t' - t_k, \gamma(t' - t_k)) - \phi(t - t_k, \gamma(t - t_k)) \right) \\ &= \lim \left( u(t' - t_k, \gamma(t' - t_k)) - u(t - t_k, \gamma(t - t_k)) \right) \\ &= \lim \int_t^{t'} L(\sigma, \gamma(\sigma - t_k), \dot{\gamma}(\sigma - t_k)) d\sigma = \int_t^{t'} L(\sigma, \gamma_\infty(\sigma), \dot{\gamma}_\infty(\sigma)) d\sigma. \end{aligned}$$

As a consequence, the curve  $t \longmapsto (t, \gamma_\infty(t))$  is asymptotic to  $\mathcal{M}$ , so that there exists an increasing sequences  $m_k$  of integers such that  $\gamma_\infty(m_k) \longrightarrow x \in \mathcal{M}_0$ . Possibly taking a subsequence of  $t_k$ , we obtain that  $\gamma(m_k - t_k) \longrightarrow x \in \mathcal{M}$  and that

$n_k = t_k - m_k$  is increasing. □

Proposition 3.2 is of course very useful to obtain uniqueness result for 1-periodic viscosity solutions. As an example, we obtain that, if there exists a periodic solution  $\phi$  such that  $S_{\mathcal{M}_0}^1$  has a dense orbit, then all the other 1-periodic solutions are of the form  $c + \phi$ . It is a classical result from one-dimensional dynamics that this holds on the circle if the rotation number is irrational, see Section 4.. This yields the uniqueness result of E and Sobolevskii ([8] and [24]).

In the autonomous case, where the Hamiltonian  $H$  does not depend on the variable  $t$ , the above remarks imply that any  $\omega$ -limit viscosity solution  $u$  is independent of  $t$  on  $\mathcal{M} = \mathbb{R} \times \mathcal{M}_0$ . One can then conclude that the solution  $u$  is independent of  $t$ .

The time-periodic case however is more complicated, and we are not able to give a description of  $\omega$ -limit orbits without using some specific features of the low dimension. This will be done in the next section.

## 4. Rotation number and convergence

In this section, we shall take advantage of the low dimension. More precisely, we shall make use of Poincaré theory of homeomorphisms of the circle, see [15] for example. We have constructed in the previous section a closed subset  $\mathcal{A}$  of  $\mathbb{R} \times \mathbb{T}$ , which is the disjoint union of graphs of calibrated curves. These calibrated curves will be called the orbits of  $\mathcal{A}$  from now on. It will be useful to consider the standard projection  $\pi : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{T}$ , and the subset  $\overline{\mathcal{A}} = \pi^{-1}(\mathcal{A})$ . Since  $\mathcal{A}$  is the disjoint union of the graphs of its orbits  $\gamma : \mathbb{R} \rightarrow \mathbb{T}$ ,  $\overline{\mathcal{A}}$  is the disjoint union of graphs of continuous curves  $\overline{\gamma} : \mathbb{R} \rightarrow \mathbb{R}$ . These curves  $\overline{\gamma}$  are called orbits of  $\overline{\mathcal{A}}$ , they satisfy  $\pi \circ \overline{\gamma} = \gamma$ . By straightforward extension of Poincaré Theory of homeomorphisms of the circle, we obtain the following result:

The limit  $\rho = \lim_{t \rightarrow \infty} \overline{\gamma}(t)/t$  exists and does not depend on the orbits  $\overline{\gamma}$ . The number  $\rho$  is called the rotation number.

PROOF OF THE THEOREM. Let  $u(t, x)$  be an  $\omega$ -limit viscosity solution. Let  $\phi(t, x) = \liminf u(t+n, x)$  be as in Proposition 2.1. We will prove that  $u = \phi$  in the two cases  $\rho = 0$  and  $\rho$  irrational. The general case of a rational rotation number  $\rho = a/b$ ,  $a \neq 0$  can be reduced to the case  $\rho = 0$  by considering the Hamiltonian

$$\tilde{H}(t, x, p) = aH(at, bx - at, \frac{p}{b})$$

and noticing that the function  $u(t, x)$  is a solution of the equation (1.1) with Hamiltonian  $H$  if and only if the function  $\tilde{u}(t, x) = u(at, bx - at)$  is a solution of the equation (1.1) with Hamiltonian  $\tilde{H}$ , and that the rotation number associated to this second equation is  $\rho = 0$ .

In view of Proposition 3.2, it is enough to prove equality on  $\mathcal{M}_\phi$  (that will be denoted  $\mathcal{M}$  from now on). Let us note  $d = u - \phi$  and recall from the previous discussions the main properties of this function  $d$ .

- The derivative  $d_x$  exists on  $\mathcal{M}$ , and  $d_x(t, x) = 0$  for all  $(t, x) \in \mathcal{M}$ .
- The function  $d$  is constant on the graph of orbits of  $\mathcal{M}$ .
- The functions  $d(t, \cdot), t \in \mathbb{R}$  are equilipschitz.
- We have  $\liminf d(n, x) = 0$  for each  $x \in \mathbf{T}$ .

**Case 1.**  $\rho = 0$ . In this case, the orbits  $\gamma$  of  $\mathcal{A}$  are of two kinds:

(I). Orbits of period 1.

(II). Heteroclinic orbits connecting orbits of type (I). This means that if  $\gamma(t)$  is an orbit of type (II), there exist orbits  $\gamma^+$  and  $\gamma^-$  of type (I) such that  $\lim_{t \rightarrow \infty} (\gamma(t) - \gamma^+(t)) = 0$  and  $\lim_{t \rightarrow -\infty} (\gamma(t) - \gamma^-(t)) = 0$ .

It follows that the  $\omega$ -limit  $\mathcal{M}$  is the union of the graphs of the orbits of  $\mathcal{A}$  of type (I). Let  $\gamma$  be such an orbit. The function  $d$  is constant on the graph of  $\gamma$ . Since in addition we have  $0 = \liminf d(n, \gamma(0)) = \liminf d(n, \gamma(n))$ , this constant is zero. As a consequence,  $d = 0$  on  $\mathcal{M}$ .

**Case 2.**  $\rho$  is irrational. In this case, there is no periodic orbit in  $\mathcal{A}$ , hence in  $\mathcal{M}$ . We have to describe the complement of  $\mathcal{M}$ :

**Lemma 4.1** *Each connected component  $\bar{U}$  of the complement of  $\bar{\mathcal{M}}$  in  $\mathbb{R} \times \mathbb{R}$  is of the form*

$$\{(t, x) \text{ such that } \bar{\gamma}^-(t) \leq x \leq \bar{\gamma}^+(t)\},$$

where  $\bar{\gamma}^\pm(t)$  are two orbits of  $\bar{\mathcal{M}}$  which satisfy  $\lim_{t \rightarrow \pm\infty} |\bar{\gamma}^+(t) - \bar{\gamma}^-(t)| = 0$ .

**PROOF.** It is quite clear that each connected component  $\bar{U}$  of the complement of  $\bar{\mathcal{M}}$  in  $\mathbb{R} \times \mathbb{R}$  is of the form  $\bar{\gamma}^-(t) \leq x \leq \bar{\gamma}^+(t)$ . In order to prove that  $\lim_{t \rightarrow \pm\infty} |\bar{\gamma}^+(t) - \bar{\gamma}^-(t)| = 0$ , let us consider the interval  $\bar{I}_k = ]\bar{\gamma}^-(k), \bar{\gamma}^+(k)[ \subset \mathbb{R}$ . We claim that the intervals  $I_k = \pi(\bar{I}_k)$  are all disjoint in  $\mathbf{T}$ . Indeed, recalling that  $\mathcal{M}_0$  is the set of points  $x \in \mathbf{T}$  such that  $(0, x) \in \mathcal{M}$ , we see that each of the intervals  $I_k$  is a connected component in  $\mathbf{T}$  of the complement of  $\mathcal{M}_0$ . Now suppose that  $I_k$  and  $I_l$  have nonempty intersection. then the boundaries of  $I_k \cap I_l$  are points of  $\mathcal{M}_0$ , (since the boundaries of  $I_k$  and  $I_l$  are) so they are contained neither in  $I_k$  nor in  $I_l$ , which is possible only if  $I_k = I_l$ . Now assume that  $I_k = I_l$  with  $k < l$ . This implies that  $\bar{\gamma}^+(l) = \bar{\gamma}^+(k)$ , hence  $\bar{\gamma}^+(t)$  is periodic, which is a contradiction. So the intervals  $I_k$  are all disjoint, hence the sum  $\sum_k (\bar{\gamma}^+(k) - \bar{\gamma}^-(k))$  of their lengths is finite; hence the result.  $\square$

The proof that  $d|_{\mathcal{M}} = 0$  is similar to the proof of Proposition 6.5. in [6]. Let us set  $\bar{d} = d \circ \pi$ . Let  $\bar{U}$  be a connected component of the complement of  $\bar{\mathcal{M}}$ , and let  $\bar{\gamma}^\pm$  be its boundary curves. Clearly,  $\bar{d}(t, \bar{\gamma}^+(t))$  and  $\bar{d}(t, \bar{\gamma}^-(t))$  are constants  $c^+$  and  $c^-$ . In addition, since  $\lim_{t \rightarrow \pm\infty} (\bar{\gamma}^+(t) - \bar{\gamma}^-(t)) = 0$  and since the functions  $\bar{d}(t, \cdot), t \in \mathbb{R}$  are equilipschitz, we have  $c^+ = c^-$ . Since this holds for every connected component  $\bar{U}$  of the complement, there exists a continuous function  $D : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  which coincides with  $\bar{d}$  on  $\bar{\mathcal{M}}$  and is constant on the complement of  $\bar{\mathcal{M}}$ . Since  $d_x = 0$  on  $\mathcal{M}$ , the derivative  $D_x$  exists and vanishes at each point. It follows that the function  $D$  is constant, hence  $\bar{d}$  is constant on  $\bar{\mathcal{M}}$ , hence  $d$  is constant on  $\mathcal{M}$ . Recalling that

$\liminf d(n, x) = 0$  for  $x \in \mathcal{M}_0$ , we obtain  $d = 0$  on  $\mathcal{M}$ .  $\square$

**Remark.** The discussions above in case 2 also yields that, if  $\phi$  and  $\psi$  are two periodic viscosity solutions, then  $d = \psi - \phi$  is constant on  $\mathcal{M}$ , hence  $\psi - d$  and  $\phi$  are two periodic viscosity solutions which are equal on  $\mathcal{M}$ , hence equal by Proposition 3.2. This provides a proof of the uniqueness result of E and Sobolevskii. Note however that this result more simply follows from the fact that all the orbits of  $S_{\mathcal{M}_0}^1$  are dense in  $\mathcal{M}_0$ .

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