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BV-regularity of the Boltzmann equation in Non-Convex Domains

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Abstract We consider the Boltzmann equation in a general non-convex domain with the diffuse boundary condition. We establish optimal BV estimates for such solutions. Our method consists of a new $W^{1,1}$ -trace estimate for the diffuse boundary condition and a delicate construction of an ε -tubular neighborhood of the singular set.

Keywords Boltzmann equation · Boundary condition

1 Introduction

Boundary effects play an important role in the dynamics of solutions of the following so-called Boltzmann equation

$$\partial_t F + v \cdot \nabla_x F = Q(F, F), \quad (1)$$

where $F(t, x, v) \geq 0$ denotes the particle distribution in the phase space $\Omega \times \mathbb{R}^3$. Here t stands for the time variable, x for the space variables, and v for the

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velocity variables. Throughout this paper, the collision operator takes the form

$$\begin{aligned} Q(F_1, F_2) &:= Q_{\text{gain}}(F_1, F_2) - Q_{\text{loss}}(F_1, F_2) \\ &= \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |v - u|^\kappa q_0(\theta) [F_1(u')F_2(v') - F_1(u)F_2(v)] d\omega du, \end{aligned} \quad (2)$$

where $u' = u + [(v - u) \cdot \omega]\omega$, $v' = v - [(v - u) \cdot \omega]\omega$ and $0 \leq \kappa \leq 1$ (hard potential) and $0 \leq q_0(\theta) \leq C|\cos \theta|$ (angular cutoff) with $\cos \theta = \frac{v-u}{|v-u|} \cdot \omega$ with $\omega \in \mathbb{S}^2$. We denote the global Maxwellian

$$\mu(v) = \exp\left(-\frac{|v|^2}{2}\right).$$

Throughout this paper we assume that Ω is a bounded open subset of \mathbb{R}^3 . The boundary $\partial\Omega$ is locally a graph of a given C^2 function: for each point $x_0 \in \partial\Omega$ there exist $r > 0$ and a C^2 function $\eta : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that, up to a rotation and relabeling, we have

$$\begin{aligned} \partial\Omega \cap B(x_0; r) &= \{x \in B(x_0; r) : x_3 = \eta(x_1, x_2)\}, \\ \Omega \cap B(x_0; r) &= \{x \in B(x_0; r) : x_3 > \eta(x_1, x_2)\}. \end{aligned} \quad (3)$$

The boundary of the phase space $\Omega \times \mathbb{R}^3$ is

$$\gamma := \{(x, v) \in \partial\Omega \times \mathbb{R}^3\}. \quad (4)$$

We denote $n = n(x)$ the outward normal unit vector at $x \in \partial\Omega$. We decompose γ as

$$\begin{aligned} \gamma_- &= \{(x, v) \in \partial\Omega \times \mathbb{R}^3 : n(x) \cdot v < 0\}, & (\text{the incoming set}), \\ \gamma_+ &= \{(x, v) \in \partial\Omega \times \mathbb{R}^3 : n(x) \cdot v > 0\}, & (\text{the outgoing set}), \\ \gamma_0 &= \{(x, v) \in \partial\Omega \times \mathbb{R}^3 : n(x) \cdot v = 0\}, & (\text{the grazing set}). \end{aligned}$$

It is important to point out that the boundary condition is imposed only for the incoming set γ_- for general kinetic PDEs. We consider *the diffuse boundary condition* in this paper: for $(x, v) \in \gamma_-$

$$F(t, x, v) = c_\mu \mu(v) \int_{n(x) \cdot u > 0} F(t, x, u) \{n(x) \cdot u\} du, \quad (5)$$

where the constant c_μ is chosen to satisfies $c_\mu \int_{n(x) \cdot u > 0} \mu(u) \{n(x) \cdot u\} du = 1$.

Despite extensive developments in the study of the Boltzmann equation, many basic questions regarding solutions in a physical bounded domain, such as their regularity, have remained largely open. This is partly due to the characteristic nature of boundary conditions in kinetic theory: Consider the simple transport equation $v \cdot \nabla_x f(x, v) = 0$ with the given boundary condition $f|_{\gamma_-} = g$. Then we solve $f(x, v) = g(x_{\mathbf{b}}(x, v), v) = g(x - t_{\mathbf{b}}(x, v)v, v)$ where $t_{\mathbf{b}}(x, v)$ is the *backward exit time* defined as

$$\begin{aligned} t_{\mathbf{b}}(x, v) &:= \sup(\{0\} \cup \{\tau > 0 : x - sv \in \Omega \text{ for all } 0 < s < \tau\}), \\ x_{\mathbf{b}}(x, v) &:= x - t_{\mathbf{b}}(x, v)v. \end{aligned} \quad (6)$$

Similarly the *forward exit time* $t_{\mathbf{f}}$ is defined as

$$\begin{aligned} t_{\mathbf{f}}(x, v) &:= \sup(\{0\} \cup \{\tau > 0 : x + sv \in \Omega \text{ for all } 0 < s < \tau\}), \\ x_{\mathbf{f}}(x, v) &:= x + t_{\mathbf{f}}(x, v)v. \end{aligned} \quad (7)$$

Since $x_{\mathbf{b}}(x, v)$ has singular behavior (even not continuous) if $n(x_{\mathbf{b}}(x, v)) \cdot v = 0$, we expect f to be singular on *the singular set*:

$$\mathfrak{S}_B := \{(x, v) \in \bar{\Omega} \times \mathbb{R}^3 : n(x_{\mathbf{b}}(x, v)) \cdot v = 0\}, \quad (8)$$

which is the collection of all the characteristics emanating from the grazing set γ_0 .

In [5], it is shown that in convex domains, the solutions of the Boltzmann equation are continuous away from the grazing set γ_0 . On the other hand, in [7], it is shown that the singularity (discontinuity) does occur for Boltzmann solutions in a non-convex domain, and such singularity propagates along the singular set \mathfrak{S}_B . Very recently in [6] the authors were able to establish weighted C^1 estimates in convex domains for all basic boundary conditions. The main purpose of this paper is to establish the first BV regularity estimate for the Boltzmann solution in non-convex domains.

We denote $\|\cdot\|_{\infty}$ the $L^{\infty}(\bar{\Omega} \times \mathbb{R}^3)$ norm, while $\|\cdot\|_p$ is the $L^p(\Omega \times \mathbb{R}^3)$ norm. We denote $|\cdot|_p$ the $L^p(\partial\Omega \times \mathbb{R}^3, dS_x dv)$ norm and $|\cdot|_{\gamma, p}$ the $L^p(\partial\Omega \times \mathbb{R}^3) = L^p(\partial\Omega \times \mathbb{R}^3, d\gamma)$ norm where $d\gamma = |n(x) \cdot v| dS_x dv$ with the surface measure dS_x on $\partial\Omega$. We write $|\cdot|_{\gamma_{\pm}, p} = |\cdot|_{\gamma, p}$. For a function f on $\Omega \times \mathbb{R}^3$, we denote f_{γ} its trace on γ whenever it exists.

A function $f \in L^1(\Omega \times \mathbb{R}^3)$ has *bounded variation* in $\Omega \times \mathbb{R}^3$ if

$$\sup \left\{ \iint_{\Omega \times \mathbb{R}^3} f \operatorname{div} \varphi dx dv : \varphi \in C_c^1(\Omega \times \mathbb{R}^3; \mathbb{R}^3 \times \mathbb{R}^3), |\varphi| \leq 1 \right\} < \infty.$$

We define

$$\|f\|_{BV} := \|f\|_{L^1(\Omega \times \mathbb{R}^3)} + \|f\|_{\widehat{BV}},$$

where

$$\|f\|_{\widehat{BV}} := \sup \left\{ \iint_{\Omega \times \mathbb{R}^3} f \operatorname{div} \varphi dx dv : \varphi \in C_c^1(\Omega \times \mathbb{R}^3; \mathbb{R}^3 \times \mathbb{R}^3), |\varphi| \leq 1 \right\} < \infty.$$

Now we are ready to state the main theorem.

Theorem 1 *Let Ω be a bounded open subset of \mathbb{R}^3 with C^2 boundary $\partial\Omega$ as in (3). Assume that $0 \leq \kappa \leq 1$ in (2), $F_0 = \sqrt{\mu} f_0 \geq 0$, $f_0 \in BV(\Omega \times \mathbb{R}^3)$, and $\|e^{\theta|v|^2} f_0\|_{\infty} < +\infty$ for $0 < \theta < \frac{1}{4}$. Then there exists $T = T(\|e^{\theta|v|^2} f_0\|_{\infty}) > 0$ such that $F = \sqrt{\mu} f$ solves the Boltzmann equation (1) with the diffuse boundary condition (5) and $f \in L^{\infty}([0, T]; BV(\Omega \times \mathbb{R}^3))$ and $\nabla_{x,v} f d\gamma$ is a Radon measure on $\partial\Omega \times \mathbb{R}^3$.*

Moreover, for all $0 \leq t \leq T$,

$$\|f(t)\|_{BV} \lesssim_{T, \Omega} \|f_0\|_{BV} + P(\|e^{\theta|v|^2} f_0\|_{\infty}), \quad (9)$$

for some polynomial P and $\nabla_{x,v} f_{\gamma}(t)$ is a Radon measure σ_t on $\partial\Omega \times \mathbb{R}^3$ such that $\int_0^T |\sigma_t(\partial\Omega \times \mathbb{R}^3)| dt \lesssim_{T, \Omega} \|f_0\|_{BV} + P(\|e^{\theta|v|^2} f_0\|_{\infty})$.

We remark that the result holds even without any size restriction for the initial datum within a small time. On the other hand, if $\|e^{\theta|v|^2}g_0\|_\infty \ll 1$ for $F_0 = \mu + \sqrt{\mu}g_0 \geq 0$ ($g_0 = f_0 - \sqrt{\mu}$), then Theorem 1 holds for $g(t)$ for all $t \geq 0$:

$$\|g(t)\|_{BV} \lesssim_{t,\Omega} \|g_0\|_{BV} + P(\|e^{\theta|v|^2}g_0\|_\infty).$$

Due to our weight function $e^{\theta|v|^2}$ for $0 < \theta < \frac{1}{4}$, these estimates imply that $f \sim e^{-\theta|v|^2} \sim \mu(v)^{2\theta}$.

Moreover the BV regularity (even in the bulk) is the best regularity we can expect. The reason is that in general the singular set \mathfrak{S}_B is a co-dimension 1 subset in the phase space $\Omega \times \mathbb{R}^3$.

Remark 1 Assume that the domain Ω is non-convex, i.e., there exist at least one point $x_0 \in \partial\Omega$ and $u \in \mathbb{R}^3$ and $(u_1, u_2) \neq 0$ such that (3) and

$$\sum_{i,j=1,2} u_i u_j \partial_i \partial_j \eta(x_0) < 0, \quad (\text{strictly non-convex point}). \quad (10)$$

Then the singular set \mathfrak{S}_B is a co-dimension 1 subset of $\Omega \times \mathbb{R}^3$. Moreover if we restrict the singular set to the characteristics emanating from the strictly non-convex points

$$\{(x, v) \in \mathfrak{S}_B : (x_b(x, v), v) \text{ is a strictly non-convex point}\},$$

then this set is a co-dimension 1 submanifold of $\Omega \times \mathbb{R}^3$.

We prove Remark 1 in the appendix. Since discontinuous solutions were constructed for non-convex domains in [7], this remark shows that the Boltzmann solutions are singular on the co-dimensional 1 subset \mathfrak{S}_B . Then it is standard to conclude that the best possible regularity space is indeed the BV space ([3]), which implies that Theorem 1 is optimal.

The equation for $f = F/\sqrt{\mu}$ where F solves (1) is

$$\partial_t f + v \cdot \nabla_x f + \nu(\sqrt{\mu}f)f = \Gamma_{\text{gain}}(f, f), \quad \text{in } \Omega \times \mathbb{R}^3, \quad (11)$$

where

$$\begin{aligned} \Gamma_{\text{gain}}(f_1, f_2) &:= \frac{1}{\sqrt{\mu}} Q_{\text{gain}}(\sqrt{\mu}f_1, \sqrt{\mu}f_2), \\ \nu(\sqrt{\mu}f_1)f_2 &:= \frac{1}{\sqrt{\mu}} Q_{\text{loss}}(\sqrt{\mu}f_1, \sqrt{\mu}f_2). \end{aligned} \quad (12)$$

The boundary condition for $f = F/\sqrt{\mu}$ where F satisfies (5) is

$$\begin{aligned} f(t, x, v) &= c_\mu \sqrt{\mu(v)} \int_{n(x) \cdot u > 0} f(t, x, u) \sqrt{\mu(u)} \{n(x) \cdot u\} du, \\ &\quad \text{on } (x, v) \in \gamma_-. \end{aligned} \quad (13)$$

The local-in-time existence of the solution f with $\sup_{0 \leq t \leq T} \|e^{\theta|v|^2} f(t)\|_\infty \lesssim \|e^{\theta'|v|^2} f_0\|_\infty$ for $0 < \theta < \theta' < \frac{1}{4}$ is standard (e.g. Lemma 6 in [6]).

We now illustrate the main ideas of the proof of Theorem 1. For simplicity we assume that f solves the following simpler problem

$$\partial_t f + v \cdot \nabla_x f + \nu f = H, \quad f|_{t=0} = f_0, \quad (14)$$

with the boundary condition (13), and where $\nu = \nu(t, x, v) \geq 0$, H , and ν are smooth enough. We note that in general solutions f of (14) are discontinuous on \mathfrak{S}_B and (distributional) derivatives do not exist [7]. In order to take (distributional) derivatives we employ the following *approximation scheme* using some smooth cut-off function $\chi_\varepsilon(x, v)$ vanishing on some open neighborhood of \mathfrak{S}_B :

$$\begin{aligned} \partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon + \nu f^\varepsilon &= \chi_\varepsilon H \text{ in } (x, v) \in \Omega \times \mathbb{R}^3, \\ f^\varepsilon|_{t=0} &= \chi_\varepsilon f_0 \text{ in } (x, v) \in \Omega \times \mathbb{R}^3, \\ f^\varepsilon(t, x, v) &= \chi_\varepsilon c_\mu \sqrt{\mu(v)} \int_{n(x) \cdot u > 0} f^\varepsilon(t, x, u) \sqrt{\mu(u)} \{n(x) \cdot u\} du, \\ &\text{on } (x, v) \in \gamma_-. \end{aligned} \quad (15)$$

Due to the cut-off χ_ε , the solution of (15) f^ε vanishes on some open subset of $\bar{\Omega} \times \mathbb{R}^3$ containing the singular set \mathfrak{S}_B defined in (8). Therefore f^ε is smooth. Once we can show that f^ε is uniformly bounded in L^∞ and ∂f^ε is uniformly bounded in $L^1(\Omega \times \mathbb{R}^3)$ then we conclude that f^ε converges to f strongly in L^1 up to a subsequence. Combining this L^1 convergence and the uniform L^∞ bound we conclude that $f \in BV$ solves (14) with (13). We apply (distributional) derivatives $\partial \in \{\nabla_x, \nabla_v\}$ to the equation and obtain

$$|\partial_t \partial f^\varepsilon + v \cdot \nabla_x \partial f^\varepsilon + \nu \partial f^\varepsilon| \leq |\partial f^\varepsilon| + |\partial \nu f^\varepsilon| + |\partial \chi_\varepsilon H| + |\chi_\varepsilon \partial H|.$$

On the other hand at the boundary we use an orthonormal transformation $\mathcal{T}(x)$ flattening the boundary in order to remove a x -dependence of the integration range: $\{n(x) \cdot u > 0\} \mapsto \{(\mathcal{T}^{-1}u)_3 > 0\}$ (see (17) – (21) in [6]). Instead the geometric x -dependence enters into the velocity component and hence after differentiating in x tangentially we have an extra v -derivative. For the normal derivative in x we use the equation. Overall the derivatives of the boundary terms are bounded as in [6]:

$$|\partial f^\varepsilon| \sim |\partial \chi_\varepsilon| + \frac{1}{|n \cdot v|} \int_{n \cdot u > 0} |\partial f^\varepsilon| \{n \cdot u\} du + \frac{1}{|n \cdot v|} \{|H| + |\nu f|\}, \quad \text{on } \gamma_-.$$

We then apply the energy-type estimate (Green's identity, Lemma 8) and the above boundary control to have

$$\begin{aligned}
& \|\partial f^\varepsilon(t)\|_1 + \int_0^t |\partial f^\varepsilon|_{\gamma_+,1} + \int_0^t \|\nu \partial f^\varepsilon(s)\| ds \\
& \lesssim \|\partial \chi_\varepsilon f(0)\|_1 + \int_0^t |\partial f^\varepsilon|_{\gamma_-,1} + \int_0^t \|\partial \chi_\varepsilon H\|_1 + \text{"good terms"} \\
& \lesssim_t \underbrace{\|\partial \chi_\varepsilon\|_1 + |\partial \chi_\varepsilon|_{\gamma_-,1}}_{(A)} + \underbrace{C \int_0^t |\partial f^\varepsilon|_{\gamma_+,1}}_{(B)} + \text{"good terms"}.
\end{aligned}$$

The first main difficulty is to *construct a smooth cut-off function* χ_ε which vanishes on an open neighborhood of \mathfrak{S}_B and makes (A) be finite at the same time. We carefully construct, in Lemma 1, an open neighborhood \mathcal{O}_ε of \mathfrak{S}_B . More precisely \mathcal{O}_ε is a collection of ε -tubular neighborhoods of forward trajectories emanating from the grazing set γ_0 . Also we show that \mathcal{O}_ε contains all points whose distance from \mathfrak{S}_B is less than ε . Such ε -thickness is important for constructing cut-off functions. In fact we construct smooth cut-off functions χ_ε by convoluting the characteristic function $\mathbf{1}_{\bar{\Omega} \times \mathbb{R}^3 \setminus \mathcal{O}_\varepsilon}$ with some standard mollifier. Moreover the ε -thickness guarantees that the cut-off function vanishes around \mathfrak{S}_B . Fortunately χ_ε satisfies the desired bound (A) $< \infty$, that is, χ_ε is uniformly bounded in $W^{1,1}$. (Lemma 2 and Proposition 1, whose proofs are delicate) Since χ_ε is a standard ε -mollification of $\mathbf{1}_{\bar{\Omega} \times \mathbb{R}^3 \setminus \mathcal{O}_\varepsilon}$ we have $\partial \chi_\varepsilon \sim \partial[1 - \chi_\varepsilon] \sim \frac{1}{\varepsilon} (\frac{1}{\varepsilon^6} \mathbf{1}_{|x|+|v|<\varepsilon}) * \mathbf{1}_{\mathcal{O}_\varepsilon}$. For example a desired estimate for $|\partial \chi_\varepsilon|_{\gamma_-,1}$ is

$$\int_{(x,v) \in \gamma_-, |v| \lesssim 1} \mathbf{1}_{\mathcal{O}_\varepsilon}(x,v) |n(x) \cdot v| dv dS_x \sim \varepsilon.$$

Let us denote \mathcal{O}_ε a collection of ε -tubular neighborhoods of forward trajectories emanating from γ_0 . Unfortunately there could be infinitely many grazing trajectories passing by x , which might lead to

$$\begin{aligned}
& \int_{(x,v) \in \gamma_-, |v| \lesssim 1} \mathbf{1}_{\mathcal{O}_\varepsilon}(x,v) |n(x) \cdot v| dv dS_x \\
& \sim \{\text{number of grazing at } x\} \times \int_{|v| \lesssim 1} \mathbf{1}_{\varepsilon\text{-tubular neighborhood}}(v) |n(x) \cdot v| dv \\
& \sim \infty.
\end{aligned}$$

Instead we establish the geometric Lemma 3 to show that $|n(x) \cdot v| \lesssim \sqrt{\varepsilon}$ if $(x,v) \in \mathcal{O}_\varepsilon$. For the proof, we decompose \mathcal{O}_ε carefully in position and velocity with varying grazing trajectories. We remark that $|\partial \chi_\varepsilon|_{\gamma_+,1} < \infty$ may not be true in general.

The second main difficulty is to *control the outgoing term* (B). We denote the (outgoing) almost grazing set

$$\gamma_+^\delta := \{(x,v) \in \gamma_+ : v \cdot n(x) < \delta \text{ or } |v| > 1/\delta\}, \quad (16)$$

and the (outgoing) non-grazing set

$$\gamma_+ \setminus \gamma_+^\delta = \{(x, v) \in \gamma_+ : v \cdot n(x) \geq \delta \text{ and } |v| \leq 1/\delta\}. \quad (17)$$

In fact the $\gamma_+ \setminus \gamma_+^\delta$ contribution can be controlled by the bulk integration and the initial data by the trace theorem. However the γ_+^δ contribution cannot be bounded by the bulk integration nor $\int_0^t |\partial f^\varepsilon|_{\gamma_{+,1}}$ in the LHS of the energy-type estimate since the constant $C > 0$ of (B) can be large in general. The new idea is to use the Duhamel formula along the trajectory once again (*Double iteration scheme*) to extract an extra small constant to close the estimate. We evaluate ∂f^ε along the characteristics and use the bound of ∂f^ε on γ_- to have

$$\begin{aligned} & \int_0^t |\partial f^\varepsilon|_{\gamma_{+,1}^\delta} \\ &= \int_0^t \int_{(x,v) \in \gamma_+^\delta} |\partial f^\varepsilon(s, x, v)| \{n(x) \cdot v\} dS_x dv ds \\ &\sim \int_0^t ds \int_{(x,v) \in \gamma_+^\delta} |\partial f^\varepsilon(s - t_{\mathbf{b}}(x, v), x_{\mathbf{b}}(x, v), v)| \{n(x) \cdot v\} dS_x dv \\ &\sim \int_0^t \int_{(x,v) \in \gamma_+^\delta} \{n(x) \cdot v\} |\partial \chi_\varepsilon(x_{\mathbf{b}}(x, v), v)| dS_x dv ds \end{aligned} \quad (18)$$

$$\begin{aligned} &+ \int_0^t \int_{(x,v) \in \gamma_+^\delta} \frac{n(x) \cdot v}{n(x_{\mathbf{b}}(x, v)) \cdot v} \\ &\quad \times \int_{n(x_{\mathbf{b}}) \cdot u > 0} |\partial f^\varepsilon(x_{\mathbf{b}}(x, v), u)| \{n(x_{\mathbf{b}}(x, v)) \cdot u\} du dS_x dv ds. \end{aligned} \quad (19)$$

In Lemma 4, we establish a crucial change of variables $(x, v) \mapsto (x_{\mathbf{b}}(x, v), v)$ with $|n(x) \cdot v| dS_x dv \lesssim |n(x_{\mathbf{b}}) \cdot v| dS_{x_{\mathbf{b}}} dv$. Clearly (18) is bounded by $|\partial \chi_\varepsilon|_{\gamma_{-,1}}$. For (19) we use Lemma 4 to convert x -integration into $x_{\mathbf{b}}$ -integration and remove the singular factor $\frac{n(x) \cdot v}{n(x_{\mathbf{b}}(x, v)) \cdot v}$. Furthermore, since $x \in \partial\Omega$ we have $x = x_{\mathbf{b}}(x_{\mathbf{b}}, -v)$ and $(x, v) \in \gamma_+^\delta$ which implies $(x_{\mathbf{b}}(x_{\mathbf{b}}, -v), v) \in \gamma_+^\delta$. Then we can bound the last term by

$$\sup_{x_{\mathbf{b}} \in \partial\Omega} \int \mathbf{1}_{(x_{\mathbf{b}}(x_{\mathbf{b}}, -v), v) \in \gamma_+^\delta} dv \times \int_0^t |\partial f^\varepsilon|_{\gamma_{+,1}}.$$

Using the covering lemma of [5] (Lemma 9), we are able to extract an extra small constant from $\sup_{x_{\mathbf{b}} \in \partial\Omega} \int \mathbf{1}_{(x_{\mathbf{b}}(x_{\mathbf{b}}, -v), v) \in \gamma_+^\delta} dv$.

Finally in order to study the nonlinear problem with diffuse boundary condition we employ some *approximation scheme*. On each sequence the problem is a linear problem with given boundary data but the solutions are vanishing on the singular set \mathfrak{S}_B . Thanks to the crucial properties of the smooth cut-off function χ_ε we are able to achieve uniform estimates via energy-type estimates with the new estimate for the outgoing term. The quadratic nonlinear terms are controllable due to the known pointwise estimates of the solutions ([5, 6]).

The plan of the paper is the following: In Section 2 we construct the desired ε -neighborhood of the singular set and its smooth cut-off functions. Then we prove the quantitative estimates of the cut-off functions and their derivatives in the bulk and on the boundary. In Section 3 we establish the new trace theorem using double iteration. In Appendix A we recall some basic geometric results. In the Appendix B we show that the singular set is co-dimension 1 in general.

2 The ε -Neighborhood of the Singular set

In this section, we construct, in Lemma 1, an open covering of the singular set \mathfrak{S}_B of (8) and construct a smooth function that cuts off the open covering. Moreover, we prove the crucial properties of such cut-off functions in Lemma 2, and Proposition 1.

2.1 Construction of Neighborhoods

Lemma 1 *For $0 < \varepsilon \leq \varepsilon_1 \ll 1$ and $\theta > 0$, we construct an open set $\mathcal{O}_{\varepsilon, \varepsilon_1} \subset \mathbb{R}^3 \times \mathbb{R}^3$, such that,*

$$\mathfrak{S}_B \subset \mathcal{O}_{\varepsilon, \varepsilon_1}. \quad (20)$$

There exists $C_ = C_*(\Omega) \gg 1$ such that for any $0 < \varepsilon \leq \varepsilon_1 \ll 1$*

$$\overline{\mathcal{O}_{\varepsilon, \varepsilon_1}} \subset \mathcal{O}_{\varepsilon, C_* \varepsilon_1}. \quad (21)$$

Moreover there exist $C_1 = C_1(\theta, \Omega, C_) > 0$, $C_2 = C_2(\Omega, C_*) > 0$, such that*

$$\iint_{\Omega \times \mathbb{R}^3} \mathbf{1}_{\mathcal{O}_{\varepsilon, C_* \varepsilon}}(x, v) e^{-\theta|v|^2} dv dx < C_1 \varepsilon, \quad (22)$$

and

$$\text{dist}(\bar{\Omega} \times \mathbb{R}^3 \setminus \mathcal{O}_{\varepsilon, C_* \varepsilon}, \mathfrak{S}_B) > C_2 \varepsilon. \quad (23)$$

Proof Construction of $\mathcal{O}_{\varepsilon, \varepsilon_1}$: Let us fix $\delta > 0$ (δ will be chosen later in (26)). Since the boundary $\partial\Omega$ is locally a graph of smooth functions, there exists a finite number $M_{\Omega, \delta}$ of small open balls $\mathcal{U}_1, \mathcal{U}_2, \dots, \mathcal{U}_{M_{\Omega, \delta}} \subset \mathbb{R}^3$ with $\text{diam}(\mathcal{U}_m) < 2\delta$ for all m , such that

$$\partial\Omega \subset \bigcup_{m=1}^{M_{\Omega, \delta}} [\mathcal{U}_m \cap \partial\Omega] \quad \text{with} \quad M_{\Omega, \delta} = O\left(\frac{1}{\delta^2}\right), \quad (24)$$

and for every m , inside \mathcal{U}_m the boundary $\mathcal{U}_m \cap \partial\Omega$ is exactly described by a smooth function η_m defined on a (small) open set $\mathcal{A}_m \subset \mathbb{R}^2$.

For all m , without loss of generality (up to rotations and translations *depending on m* , and up to reducing the size of the ball \mathcal{U}_m) we will always assume that

$$\begin{aligned}\mathcal{U}_m \cap \partial\Omega &= \{(x_1, x_2, \eta_m(x_1, x_2)) \in \mathcal{A}_m \times \mathbb{R}\}, \\ \mathcal{U}_m \cap \Omega &= \{(x_1, x_2, x_3) \in \mathcal{A}_m \times \mathbb{R} : x_3 > \eta_m(x_1, x_2)\},\end{aligned}\quad (25)$$

and

$$\begin{aligned}(0, 0) &\in \mathcal{A}_m \subset_{\text{open}} [-\delta, \delta] \times [-\delta, \delta], \\ \partial_1 \eta_m(0, 0) &= 0 = \partial_2 \eta_m(0, 0).\end{aligned}$$

Therefore

$$\begin{aligned}&n(0, 0, \eta_m(0, 0)) \\ &= \frac{1}{\sqrt{1 + |\partial_1 \eta_m(0, 0)|^2 + |\partial_2 \eta_m(0, 0)|^2}} (\partial_1 \eta_m(0, 0), \partial_2 \eta_m(0, 0), -1) \\ &= (0, 0, -1).\end{aligned}$$

Recall that $\partial\Omega$ is locally C^2 . Then we can choose $\delta > 0$ small enough to satisfy for all $m \in \{1, \dots, M_{\Omega, \delta}\}$

$$\begin{aligned}&|\partial_1 \eta_m(x_1, x_2) - \partial_1 \eta_m(0, 0)| + |\partial_2 \eta_m(x_1, x_2) - \partial_2 \eta_m(0, 0)| \\ &= |\partial_1 \eta_m(x_1, x_2)| + |\partial_2 \eta_m(x_1, x_2)| \leq \frac{1}{8} \quad \text{for } (x_1, x_2) \in \mathcal{A}_m,\end{aligned}\quad (26)$$

and

$$|\partial_1^2 \eta_m(x_1, x_2)| + |\partial_2^2 \eta_m(x_1, x_2)| + |\partial_1 \partial_2 \eta_m(x_1, x_2)| \leq C_\eta \quad \text{for } (x_1, x_2) \in \mathcal{A}_m. \quad (27)$$

Now we define the lattice point on \mathcal{A}_m as

$$c_{m, i, j, \varepsilon} := (\varepsilon i, \varepsilon j) \quad \text{for } -N_\varepsilon \leq i, j \leq N_\varepsilon = O\left(\frac{\delta}{\varepsilon}\right). \quad (28)$$

Then we define the (i, j) -rectangular $\mathcal{R}_{m, i, j, \varepsilon, \varepsilon_1}$ which is centered at $c_{m, i, j, \varepsilon}$ and whose side is $2\varepsilon_1$:

$$\mathcal{R}_{m, i, j, \varepsilon, \varepsilon_1} := \left\{ (x_1, x_2) : \varepsilon i - \varepsilon_1 < x_1 < \varepsilon i + \varepsilon_1 \text{ and } \varepsilon j - \varepsilon_1 < x_2 < \varepsilon j + \varepsilon_1 \right\} \cap \mathcal{A}_m. \quad (29)$$

Note that if $\varepsilon_1 \geq \varepsilon$ then $\{\mathcal{R}_{m, i, j, \varepsilon, \varepsilon_1}\}$ is open covering of \mathcal{A}_m , i.e.

$$\mathcal{A}_m \subset \bigcup_{-N_\varepsilon \leq i, j \leq N_\varepsilon} \mathcal{R}_{m, i, j, \varepsilon, \varepsilon_1} \quad \text{with } N_\varepsilon = O\left(\frac{\delta}{\varepsilon}\right). \quad (30)$$

For each rectangle we define the representative outward normal

$$n_{m, i, j, \varepsilon} := \frac{1}{\sqrt{1 + |\partial_1 \eta_m(c_{m, i, j, \varepsilon})|^2 + |\partial_2 \eta_m(c_{m, i, j, \varepsilon})|^2}} \begin{pmatrix} \partial_1 \eta_m(c_{m, i, j, \varepsilon}) \\ \partial_2 \eta_m(c_{m, i, j, \varepsilon}) \\ -1 \end{pmatrix}.$$

Let $\{\hat{x}_{1,m,i,j,\varepsilon}, \hat{x}_{2,m,i,j,\varepsilon}\} \subset \mathbb{S}^2$ be an orthonormal basis of the tangent space of $\partial\Omega$ at $(c_{m,i,j,\varepsilon}, \eta_m(c_{m,i,j,\varepsilon}))$. Remark that the three vectors $\hat{x}_{1,m,i,j,\varepsilon}$, $\hat{x}_{2,m,i,j,\varepsilon}$, and $n_{m,i,j,\varepsilon}$ are fixed for each m, i, j, ε and that $\{\hat{x}_{1,m,i,j,\varepsilon}, \hat{x}_{2,m,i,j,\varepsilon}, n_{m,i,j,\varepsilon}\}$ is an orthonormal basis of \mathbb{R}^3 .

We split the tangent velocity space at $(c_{m,i,j,\varepsilon}, \eta_m(c_{m,i,j,\varepsilon})) \in \partial\Omega$ as

$$\{v \in \mathbb{R}^3 : v \cdot n_{m,i,j,\varepsilon} = 0\} \subseteq \bigcup_{\ell=0}^{L_\varepsilon} \Theta_{m,i,j,\varepsilon,\varepsilon_1,\ell}, \quad \text{with } L_\varepsilon = O\left(\frac{1}{\varepsilon}\right),$$

where

$$\begin{aligned} \Theta_{m,i,j,\varepsilon,\varepsilon_1,\ell} &:= \left\{ r_v \cos \theta_v \cos \phi_v \hat{x}_{1,m,i,j,\varepsilon} + r_v \sin \theta_v \cos \phi_v \hat{x}_{2,m,i,j,\varepsilon} + r_v \sin \phi_v n_{m,i,j,\varepsilon} \in \mathbb{R}^3 : \right. \\ &\quad |r_v \sin \phi_v| < 8C_\eta \varepsilon_1 \text{ for } r_v \in [0, 1], \quad |\sin \phi_v| < 8C_\eta \varepsilon_1 \text{ for } r_v \in [1, \infty), \\ &\quad \left. |\theta_v - \varepsilon \ell| < \varepsilon_1 \text{ for } r_v \in [0, \infty) \right\}, \end{aligned} \quad (31)$$

with the constant $C_\eta > 0$ from (27).

Remark that for $\varepsilon_1 \geq \varepsilon$,

$$\bigcup_{\ell=0}^{L_\varepsilon} \Theta_{m,i,j,\varepsilon,\varepsilon_1,\ell} = \left\{ v \in \mathbb{R}^3 : \begin{array}{l} |v \cdot n_{m,i,j,\varepsilon}| < 8C_\eta \varepsilon_1 \text{ for } |v| \leq 1, \\ \left| \frac{v}{|v|} \cdot n_{m,i,j,\varepsilon} \right| < 8C_\eta \varepsilon_1 \text{ for } |v| \geq 1 \end{array} \right\}. \quad (32)$$

Now we are ready to construct the desired open cover corresponding to $\mathcal{R}_{m,i,j,\varepsilon,\varepsilon_1} \times \Theta_{m,i,j,\varepsilon,\varepsilon_1,\ell}$ as

$$\mathcal{O}_{m,i,j,\varepsilon,\varepsilon_1,\ell} := \left[\bigcup_{x \in \mathcal{X}_{m,i,j,\varepsilon,\varepsilon_1,\ell}} B_{\mathbb{R}^3}(x; \varepsilon_1) \right] \times \Theta_{m,i,j,\varepsilon,\varepsilon_1,\ell}, \quad (33)$$

where

$$\begin{aligned} \mathcal{X}_{m,i,j,\varepsilon,\varepsilon_1,\ell} &:= \left\{ (x_1, x_2, \eta_m(x_1, x_2)) + \tau [\cos \theta \hat{x}_{1,m,i,j,\varepsilon} + \sin \theta \hat{x}_{2,m,i,j,\varepsilon}] + s n_{m,i,j,\varepsilon} \in \mathbb{R}^3 : \right. \\ &\quad (x_1, x_2) \in \mathcal{R}_{m,i,j,\varepsilon,\varepsilon_1}, \quad \theta \in (\varepsilon \ell - \varepsilon_1, \varepsilon \ell + \varepsilon_1), \quad s \in (-\varepsilon_1, \varepsilon_1) \\ &\quad \left. \tau \in [0, t_f((x_1, x_2, \eta_m(x_1, x_2)), \cos \theta \hat{x}_{1,m,i,j,\varepsilon} + \sin \theta \hat{x}_{2,m,i,j,\varepsilon})) \right\}. \end{aligned} \quad (34)$$

We note that $\mathcal{O}_{m,i,j,\varepsilon,\varepsilon_1,\ell}$ is an infinite union of open sets and hence is an open set.

Finally we define

$$\mathcal{O}_{\varepsilon,\varepsilon_1} := \bigcup_{m,i,j,\ell} \mathcal{O}_{m,i,j,\varepsilon,\varepsilon_1,\ell} \cup [\mathbb{R}^3 \times B_{\mathbb{R}^3}(0; \varepsilon_1)], \quad (35)$$

where $1 \leq m \leq M_{\Omega, \delta} = O(\frac{1}{\delta^2})$, $-N_\varepsilon \leq i, j \leq N_\varepsilon = O(\frac{\delta}{\varepsilon})$, $0 \leq \ell \leq L_\varepsilon = O(\frac{1}{\varepsilon})$. Since $\mathcal{O}_{\varepsilon, \varepsilon_1}$ is a union of open sets, it is an open set.

Proof of (20): Suppose there exists $(x, v) \in \mathfrak{S}_B$. By the definition of \mathfrak{S}_B in (8) there exists $y = x_{\mathbf{b}}(x, v) \in \partial\Omega$, such that $x = y + t_{\mathbf{b}}(x, v)v$ and $v \cdot n(y) = 0$ from (6) and (7). Then $y \in \mathcal{U}_m$ for some m . Without loss of generality (up to rotations and translations) we may assume that $y = (y_1, y_2, \eta_m(y_1, y_2))$ and $(y_1, y_2) \in \mathcal{R}_{m, i, j, \varepsilon, \varepsilon_1}$ for some i, j .

Firstly we consider the case of $|v| \geq 1$. Then we check that

$$\begin{aligned} & \left| n_{m, i, j, \varepsilon} \cdot \frac{v}{|v|} \right| \\ & \leq \left| n(y_1, y_2, \eta_m(y_1, y_2)) \cdot \frac{v}{|v|} \right| + \left| [n_{m, i, j, \varepsilon} - n(y_1, y_2, \eta_m(y_1, y_2))] \cdot \frac{v}{|v|} \right| \\ & \leq 0 + |n(c_{m, i, j, \varepsilon}, \eta_m(c_{m, i, j, \varepsilon})) - n(y_1, y_2, \eta_m(y_1, y_2))| \\ & \leq |\nabla \eta_m(c_{m, i, j, \varepsilon}) - \nabla \eta_m(y_1, y_2)| \\ & \quad + \frac{|\sqrt{1 + |\nabla \eta_m(y_1, y_2)|^2} - \sqrt{1 + |\nabla \eta_m(c_{m, i, j, \varepsilon})|^2}|}{\sqrt{1 + |\nabla \eta_m(c_{m, i, j, \varepsilon})|^2}} \\ & \leq 2|\nabla \eta_m(c_{m, i, j, \varepsilon}) - \nabla \eta_m(y_1, y_2)|, \end{aligned}$$

where we used the Taylor expansion at the last line. Using (27), we have

$$\begin{aligned} |\nabla \eta_m(c_{m, i, j, \varepsilon}) - \nabla \eta_m(y_1, y_2)| & \leq 4\varepsilon_1 \times \|\eta_m\|_{C^2(\mathcal{R}_{m, i, j, \varepsilon, \varepsilon_1})} \\ & \leq 4\varepsilon_1 \times \|\eta_m\|_{C^2(\mathcal{A}_m)} \\ & \leq 4C_\eta \varepsilon_1. \end{aligned}$$

Therefore we conclude

$$\left| n_{m, i, j, \varepsilon} \cdot \frac{v}{|v|} \right| \leq 8C_\eta \varepsilon_1.$$

By (32), $v \in \bigcup_{\ell=0}^{L_\varepsilon} \Theta_{m, i, j, \varepsilon, \varepsilon_1, \ell}$ and hence $(x, v) \in \mathcal{O}_{\varepsilon, \varepsilon_1}$.

Secondly we consider the case of $|v| \leq 1$. Then from (27) and following the similar estimate of $|v| \geq 1$ case

$$\begin{aligned} |v \cdot n_{m, i, j, \varepsilon}| & \leq |v \cdot n(y)| + |v \cdot (n(y) - n_{m, i, j, \varepsilon})| \\ & \leq 4\varepsilon_1 \|\eta\|_{C^2(\mathcal{R}_{m, i, j, \varepsilon, \varepsilon_1})} \leq 4\varepsilon_1 \|\eta\|_{C^2(\mathcal{A}_m)} \\ & \leq 8C_\eta \varepsilon_1. \end{aligned}$$

By the statement of (32), $v \in \bigcup_{\ell=0}^{L_\varepsilon} \Theta_{m, i, j, \varepsilon, \varepsilon_1, \ell}$ and hence $(x, v) \in \mathcal{O}_{m, i, j, \varepsilon, \varepsilon_1, \ell} \subset \mathcal{O}_{\varepsilon, \varepsilon_1}$.

Proof of (21): It suffices to show that there exists a constant $C_* \gg 1$ such that if $(x, v) \in \mathcal{O}_{\varepsilon, \varepsilon_1}$ then $(x, v) \in \mathcal{O}_{\varepsilon, C_* \varepsilon_1}$.

Since in the definition (35) the union on m, i, j, ℓ is finite, we have

$$\begin{aligned}\overline{\mathcal{O}_{\varepsilon, \varepsilon_1}} &= \bigcup_{m, i, j, \ell} \overline{\mathcal{O}_{m, i, j, \varepsilon, \varepsilon_1, \ell}} \cup [\mathbb{R}^3 \times \{v \in \mathbb{R}^3 : |v| \leq \varepsilon_1\}] \\ &= [\mathbb{R}^3 \times \{v \in \mathbb{R}^3 : |v| \leq \varepsilon_1\}] \\ &\quad \cup \bigcup_{m, i, j, \ell} \left[\underbrace{\left(\bigcup_{x \in \mathcal{X}_{m, i, j, \varepsilon, \varepsilon_1, \ell}} B_{\mathbb{R}^3}(x; \varepsilon_1) \right)}_{\text{underbraced set}} \times \overline{\Theta_{m, i, j, \varepsilon, \varepsilon_1, \ell}} \right].\end{aligned}$$

First we define an open set including the underbraced set (a closed set). For $0 < \varsigma$, we define

$$\begin{aligned}&\bigcup_{x \in \mathcal{X}_{m, i, j, \varepsilon, \varepsilon_1, \ell}} \bigcup_{y \in B_{\mathbb{R}^3}(x; \varepsilon_1)} B_{\mathbb{R}^3}(y; \varsigma) \\ &= \left\{ z \in \mathbb{R}^3 : \text{there exists } x \in \mathcal{X}_{m, i, j, \varepsilon, \varepsilon_1, \ell} \text{ and } y \in B_{\mathbb{R}^3}(x; \varepsilon_1) \right. \\ &\quad \left. \text{such that } z \in B_{\mathbb{R}^3}(y; \varsigma) \right\}.\end{aligned}\tag{36}$$

Since it is an infinite union of open balls, (36) is open and the underbraced set is contained in (36) for any $\varsigma > 0$.

Now we claim that, there exists $C_* = C_*(\Omega) \gg 1$ such that for $0 < \varepsilon \leq \varepsilon_1 \ll 1$, there exists $0 < \varsigma = \varsigma(\varepsilon_1, C_*) \ll 1$ such that

$$\bigcup_{x \in \mathcal{X}_{m, i, j, \varepsilon, \varepsilon_1, \ell}} \bigcup_{y \in B_{\mathbb{R}^3}(x; \varepsilon_1)} B_{\mathbb{R}^3}(y; \varsigma) \subset \bigcup_{x \in \mathcal{X}_{m, i, j, \varepsilon, C_* \varepsilon_1, \ell}} B_{\mathbb{R}^3}(x; C_* \varepsilon_1).\tag{37}$$

Choose $z \in \bigcup_{x \in \mathcal{X}_{m, i, j, \varepsilon, \varepsilon_1, \ell}} \bigcup_{y \in B_{\mathbb{R}^3}(x; \varepsilon_1)} B_{\mathbb{R}^3}(y; \varsigma)$. From (36) there exist $x \in \mathcal{X}_{m, i, j, \varepsilon, \varepsilon_1, \ell}$ and $y \in B_{\mathbb{R}^3}(x; \varepsilon_1)$ such that $z \in B_{\mathbb{R}^3}(y; \varsigma)$. If we choose $\varsigma < \varepsilon_1$ then $|x - z| \leq |x - y| + |y - z| \leq 2\varepsilon_1 < C_* \varepsilon_1$ and therefore $z \in B_{\mathbb{R}^3}(x; C_* \varepsilon_1)$. Clearly $x \in \mathcal{X}_{m, i, j, \varepsilon, C_* \varepsilon_1, \ell}$. This proves our claim (37).

On the other hand, from (31), $C_* \gg 1$ and the fact that the vectors $\hat{x}_{1, m, i, j, \varepsilon}$, $\hat{x}_{2, m, i, j, \varepsilon}$, and $n_{m, i, j, \varepsilon}$ are fixed for given m, i, j ,

$$\begin{aligned}&\overline{\Theta_{m, i, j, \varepsilon, \varepsilon_1, \ell}} \\ &= \left\{ v = r_v \cos \theta_v \cos \phi_v \hat{x}_{1, m, i, j, \varepsilon} + r_v \sin \theta_v \cos \phi_v \hat{x}_{2, m, i, j, \varepsilon} \right. \\ &\quad \left. + r_v \sin \phi_v n_{m, i, j, \varepsilon} \in \mathbb{R}^3 : |r_v \sin \phi_v| \leq 8C_\eta \varepsilon_1 \text{ for } r_v \in [0, 1], \right. \\ &\quad \left. |\sin \phi_v| \leq 8C_\eta \varepsilon_1 \text{ for } r_v \in [1, \infty), |\theta_v - \varepsilon \ell| \leq \varepsilon_1 \text{ for } r_v \in [0, \infty) \right\} \\ &\subset \left\{ v = r_v \cos \theta_v \cos \phi_v \hat{x}_{1, m, i, j, \varepsilon} + r_v \sin \theta_v \cos \phi_v \hat{x}_{2, m, i, j, \varepsilon} \right. \\ &\quad \left. + r_v \sin \phi_v n_{m, i, j, \varepsilon} \in \mathbb{R}^3 : |r_v \sin \phi_v| < 8C_\eta C_* \varepsilon_1 \text{ for } r_v \in [0, 1], \right. \\ &\quad \left. |\sin \phi_v| < 8C_\eta C_* \varepsilon_1 \text{ for } r_v \in [1, \infty), |\theta_v - \varepsilon \ell| < C_* \varepsilon_1 \text{ for } r_v \in [0, \infty) \right\} \\ &= \Theta_{m, i, j, \varepsilon, C_* \varepsilon_1, \ell}.\end{aligned}\tag{38}$$

Finally we conclude, from (37) and (38),

$$\begin{aligned}\overline{\mathcal{O}_{\varepsilon, \varepsilon_1}} &\subset \bigcup_{m, i, j, \ell} \left[\bigcup_{x \in \mathcal{X}_{m, i, j, \varepsilon, C_* \varepsilon_1, \ell}} B_{\mathbb{R}^3}(x; C_* \varepsilon_1) \times \Theta_{m, i, j, \varepsilon, C_* \varepsilon_1, \ell} \right] \\ &\quad \cup [\mathbb{R}^3 \times B_{\mathbb{R}^3}(0; C_* \varepsilon_1)] \\ &= \mathcal{O}_{\varepsilon, C_* \varepsilon_1}.\end{aligned}$$

Proof of (22): From (35), we deduce

$$\begin{aligned}&\iint_{\Omega \times \mathbb{R}^3} \mathbf{1}_{\mathcal{O}_{\varepsilon, C_* \varepsilon}}(x, v) e^{-\theta|v|^2} dv dx \\ &\leq \sum_{m, i, j, \ell} \iint_{\Omega \times \mathbb{R}^3} \mathbf{1}_{\mathcal{O}_{m, i, j, \varepsilon, C_* \varepsilon, \ell}}(x, v) e^{-\theta|v|^2} dv dx + m_3(\Omega) O(|\varepsilon|^3) \\ &\leq M_{\Omega, \delta} (2N_\varepsilon)^2 L_\varepsilon \times \sup_{m, i, j, \ell} \iint_{\Omega \times \mathbb{R}^3} \mathbf{1}_{\mathcal{O}_{m, i, j, \varepsilon, C_* \varepsilon, \ell}}(x, v) e^{-\theta|v|^2} dv dx \\ &\quad + m_3(\Omega) O(|\varepsilon|^3) \\ &\lesssim_\Omega O\left(\frac{1}{\varepsilon^3}\right) \times \sup_{m, i, j, \ell} \iint_{\Omega \times \mathbb{R}^3} \mathbf{1}_{\mathcal{O}_{m, i, j, \varepsilon, C_* \varepsilon, \ell}}(x, v) e^{-\theta|v|^2} dv dx + O(|\varepsilon|^3).\end{aligned}$$

Therefore, to prove (22), it suffices to show

$$\sup_{m, i, j, \ell} \iint_{\Omega \times \mathbb{R}^3} \mathbf{1}_{\mathcal{O}_{m, i, j, \varepsilon, C_* \varepsilon, \ell}}(x, v) e^{-\theta|v|^2} dv dx \lesssim_{\delta, \Omega} \varepsilon^4. \quad (39)$$

From (31),

$$\begin{aligned}&\int_{\mathbb{R}^3} \mathbf{1}_{\mathcal{O}_{m, i, j, \varepsilon, C_* \varepsilon, \ell}}(v) e^{-\theta|v|^2} dv \\ &= \int_{|v| \leq 1} + \int_{|v| \geq 1} \\ &\leq \int_{|r_v \sin \phi_v| \leq 8C_\eta C_* \varepsilon} d|r_v \sin \phi_v| \\ &\quad \times \int_0^\infty |r_v \cos \phi_v| e^{-\theta|r_v \cos \phi_v|^2} d|r_v \cos \phi_v| \int_{|\theta_v - \varepsilon \ell| < C_* \varepsilon} d\theta_v \\ &\quad + \int_1^\infty |r_v|^2 e^{-\theta|r_v|^2} dr_v \int_{|\sin \phi_v| < 8C_\eta C_* \varepsilon} d\phi_v \int_{|\theta_v - \varepsilon \ell| < C_* \varepsilon} d\theta_v \\ &\lesssim_\Omega \varepsilon^2.\end{aligned}$$

Now we claim that, for $\varepsilon_1 \geq \varepsilon$,

$$m_3\left(\bigcup_{x \in \mathcal{X}_{m, i, j, \varepsilon, \varepsilon_1, \ell}} B_{\mathbb{R}^3}(x; \varepsilon_1)\right) \lesssim_\Omega \varepsilon_1^2. \quad (40)$$

Without loss of generality we assume that $i = j = 0$ and $l = 0$. Therefore $c_{m,i,j,\varepsilon} = 0$ in (28) and

$$\begin{aligned} & \mathcal{X}_{m,i,j,\varepsilon,\varepsilon_1,\ell} \\ & \subset \left\{ (x_1, x_2, \eta_m(x_1, x_2)) + \tau[\cos \theta \mathbf{e}_1 + \sin \theta \mathbf{e}_2] + s \mathbf{e}_3 \in \mathbb{R}^3 : \right. \\ & \quad (x_1, x_2) \in (-\varepsilon_1, \varepsilon_1)^2, \theta \in (-\varepsilon_1, \varepsilon_1), \\ & \quad \left. \tau \in [0, t_{\mathbf{f}}((x_1, x_2, \eta(x_1, x_2)), \cos \theta \mathbf{e}_1 + \sin \theta \mathbf{e}_2)], s \in (-\varepsilon_1, \varepsilon_1) \right\}. \end{aligned}$$

Since Ω is bounded, we have that $\text{diam}(\Omega) = \sup_{x,y \in \Omega} |x - y| < +\infty$ and hence

$$t_{\mathbf{f}}((x_1, x_2, \eta(x_1, x_2)), \cos \theta \mathbf{e}_1 + \sin \theta \mathbf{e}_2) \leq \text{diam}(\Omega).$$

We have

$$\bigcup_{x \in \mathcal{X}_{m,i,j,\varepsilon,\varepsilon_1,\ell}} B_{\mathbb{R}^3}(x; \varepsilon_1) \subset \bigcup_{\tau=0}^{2\text{diam}(\Omega)} B_{\mathbb{R}^3}(\tau \mathbf{e}_1; [10 + \|\eta\|_{C^1(\mathcal{A}_m)} + \tau \|\eta\|_{C^2(\mathcal{A}_m)}] \varepsilon_1).$$

More precisely $\bigcup_{x \in \mathcal{X}_{m,i,j,\varepsilon,\varepsilon_1,\ell}} B_{\mathbb{R}^3}(x; \varepsilon_1)$ is included in the truncated cone with height $\text{diam}(\Omega)$, top radius $[10 + \|\eta\|_{C^1(\mathcal{A}_m)}] \varepsilon_1$, and the bottom radius $[10 + \|\eta\|_{C^1(\mathcal{A}_m)} + \text{diam}(\Omega) \|\eta\|_{C^2(\mathcal{A}_m)}] \varepsilon_1$.

Therefore, using (26) and (27), we conclude (40)

$$\begin{aligned} & m_3 \left(\bigcup_{x \in \mathcal{X}_{m,i,j,\varepsilon,\varepsilon_1,\ell}} B_{\mathbb{R}^3}(x; \varepsilon_1) \right) \\ & \leq m_3 \left(\bigcup_{\tau=0}^{2\text{diam}(\Omega)} B_{\mathbb{R}^3}(\tau \mathbf{e}_1; [10 + \|\eta\|_{C^1(\mathcal{A}_m)} + \tau \|\eta\|_{C^2(\mathcal{A}_m)}] \varepsilon_1) \right) \\ & \leq 3 \text{diam}(\Omega) \left[10 + \|\eta\|_{C^1(\mathcal{A}_m)} + \text{diam}(\Omega) \|\eta\|_{C^2(\mathcal{A}_m)} \right]^2 \times (\varepsilon_1)^2 \\ & \leq 3 \text{diam}(\Omega) \left[10 + \frac{1}{8} + C_\eta \text{diam}(\Omega) \right]^2 (\varepsilon_1)^2 \\ & \lesssim_{\Omega} \varepsilon_1^2. \end{aligned}$$

Finally selecting $\varepsilon_1 = C_* \varepsilon$ in (40) we conclude (39) as

$$\begin{aligned} & m_3 \left(\bigcup_{x \in \mathcal{X}_{m,i,j,\varepsilon,C_*\varepsilon,\ell}} B_{\mathbb{R}^3}(x; C_* \varepsilon) \right) \times \int_{\mathbb{R}^3} \mathbf{1}_{\Theta_{m,i,j,\varepsilon,C_*\varepsilon,\ell}}(v) e^{-\theta|v|^2} dv \\ & \lesssim m_3 \left(\bigcup_{x \in \mathcal{X}_{m,i,j,\varepsilon,C_*\varepsilon,\ell}} B_{\mathbb{R}^3}(x; \varepsilon_1) \right) \times (\varepsilon)^2 \\ & \lesssim \varepsilon^4. \end{aligned}$$

Proof of (23): Due to (20), it suffices to show that there exists $C_2 = C_2(C_*) > 0$ such that

$$\text{dist}(\bar{\Omega} \times \mathbb{R}^3 \setminus \mathcal{O}_{\varepsilon,C_*\varepsilon}, \overline{\mathcal{O}_{\varepsilon,\varepsilon}}) > C_2 \varepsilon. \quad (41)$$

By the definition of $\mathcal{O}_{\varepsilon,\varepsilon}$ in (35),

$$\begin{aligned}
& \text{dist}(\bar{\Omega} \times \mathbb{R}^3 \setminus \mathcal{O}_{\varepsilon,C_*\varepsilon}, \overline{\mathcal{O}_{\varepsilon,\varepsilon}}) \\
&= \inf \left\{ |(x,v) - (y,u)| : (x,v) \in (\mathcal{O}_{\varepsilon,C_*\varepsilon})^c, (y,u) \in \overline{\mathcal{O}_{\varepsilon,\varepsilon}} \right\} \\
&= \inf_{m,i,j,\ell} \inf \left\{ |(x,v) - (y,u)| : (x,v) \in (\mathcal{O}_{\varepsilon,C_*\varepsilon})^c, \right. \\
&\quad \left. (y,u) \in \overline{\mathcal{O}_{m,i,j,\varepsilon,\ell}} \cup [\mathbb{R}^3 \times \overline{B_{\mathbb{R}^3}(0;\varepsilon)}] \right\} \\
&\geq \inf_{m,i,j,\ell} \inf \left\{ |(x,v) - (y,u)| : (x,v) \in (\mathcal{O}_{m,i,j,\varepsilon,C_*\varepsilon,\ell})^c \cap [\mathbb{R}^3 \times B_{\mathbb{R}^3}(0;C_*\varepsilon)^c], \right. \\
&\quad \left. (y,u) \in \overline{\mathcal{O}_{m,i,j,\varepsilon,\ell}} \cup [\mathbb{R}^3 \times B_{\mathbb{R}^3}(0;\varepsilon)] \right\} \\
&= \inf_{m,i,j,\ell} \min \left\{ \inf \left\{ |(x,v) - (y,u)| : (y,u) \in \mathbb{R}^3 \times B_{\mathbb{R}^3}(0;\varepsilon), \right. \right. \\
&\quad \left. \left. (x,v) \in (\mathcal{O}_{m,i,j,\varepsilon,C_*\varepsilon,\ell})^c \cap [\mathbb{R}^3 \times B_{\mathbb{R}^3}(0;C_*\varepsilon)^c] \right\}, \right. \\
&\quad \left. \inf \left\{ |(x,v) - (y,u)| : (x,v) \in (\mathcal{O}_{m,i,j,\varepsilon,C_*\varepsilon,\ell})^c \cap [\mathbb{R}^3 \times B_{\mathbb{R}^3}(0;C_*\varepsilon)^c], \right. \right. \\
&\quad \left. \left. (y,u) \in \overline{\mathcal{O}_{m,i,j,\varepsilon,\ell}} \cap [\mathbb{R}^3 \times B_{\mathbb{R}^3}(0;\varepsilon)^c] \right\} \right\}. \quad (43)
\end{aligned}$$

Clearly,

$$\begin{aligned}
(42) &\geq \inf \left\{ |(x,v) - (y,u)| : (x,v) \in \mathbb{R}^3 \times B_{\mathbb{R}^3}(0;C_*\varepsilon)^c, \right. \\
&\quad \left. (y,u) \in \mathbb{R}^3 \times B_{\mathbb{R}^3}(0;\varepsilon) \right\} \\
&\geq \inf \left\{ |v - u| : v \in B_{\mathbb{R}^3}(0;C_*\varepsilon)^c, u \in B_{\mathbb{R}^3}(0;\varepsilon) \right\} \\
&= (C_* - 1)\varepsilon.
\end{aligned}$$

Now we claim that (43) is bounded below by the minimum of (44) and (45):

$$\begin{aligned}
& (43) \\
&\geq \min \left(\inf \left\{ |(x,v) - (y,u)| : \right. \right. \\
&\quad (x,v) \in \bigcup_{x \in \mathcal{X}_{m,i,j,\varepsilon,C_*\varepsilon,\ell}} B_{\mathbb{R}^3}(x;C_*\varepsilon) \times [(\mathcal{O}_{m,i,j,\varepsilon,C_*\varepsilon,\ell})^c \setminus B_{\mathbb{R}^3}(0;C_*\varepsilon)], \\
&\quad (y,u) \in \left[\bigcup_{x \in \mathcal{X}_{m,i,j,\varepsilon,\frac{C_*}{2}\varepsilon,\ell}} B_{\mathbb{R}^3}(x;\frac{C_*}{2}\varepsilon) \right] \times [\overline{\mathcal{O}_{m,i,j,\varepsilon,\ell}} \setminus B_{\mathbb{R}^3}(0;\varepsilon)] \left. \right\}, \quad (44) \\
&\quad \inf \left\{ |(x,v) - (y,u)| : \right. \\
&\quad (x,v) \in \left[\bigcap_{x \in \mathcal{X}_{m,i,j,\varepsilon,C_*\varepsilon,\ell}} (B_{\mathbb{R}^3}(x;C_*\varepsilon))^c \right] \times [\mathbb{R}^3 \setminus B_{\mathbb{R}^3}(0;C_*\varepsilon)], \\
&\quad (y,u) \in \left[\bigcup_{x \in \mathcal{X}_{m,i,j,\varepsilon,\frac{C_*}{2}\varepsilon,\ell}} B_{\mathbb{R}^3}(x;\frac{C_*}{2}\varepsilon) \right] \times [\overline{\mathcal{O}_{m,i,j,\varepsilon,\ell}} \setminus B_{\mathbb{R}^3}(0;\varepsilon)] \left. \right\} \Big). \quad (45)
\end{aligned}$$

Firstly, we divide $\{(x, v) \in (\mathcal{O}_{m,i,j,\varepsilon,C_*\varepsilon,\ell})^c\}$ in (43) into two parts: from the definition of $\mathcal{O}_{m,i,j,\varepsilon,C_*\varepsilon,\ell}$ in (33), we deduce that

$$\begin{aligned} (\mathcal{O}_{m,i,j,\varepsilon,C_*\varepsilon,\ell})^c &= \left[\bigcup_{x \in \mathcal{X}_{m,i,j,\varepsilon,C_*\varepsilon,\ell}} B_{\mathbb{R}^3}(x; C_*\varepsilon) \right] \times (\Theta_{m,i,j,\varepsilon,C_*\varepsilon,\ell})^c \\ &\cup \left[\bigcap_{x \in \mathcal{X}_{m,i,j,\varepsilon,C_*\varepsilon,\ell}} (B_{\mathbb{R}^3}(x; C_*\varepsilon))^c \right] \times \mathbb{R}^3. \end{aligned}$$

Therefore, (43) is bounded below by the minimum of the following two numbers:

$$\begin{aligned} &\inf \left\{ |(x, v) - (y, u)| : (x, v) \in \left[\bigcup_{x \in \mathcal{X}_{m,i,j,\varepsilon,C_*\varepsilon,\ell}} B_{\mathbb{R}^3}(x; C_*\varepsilon) \right] \right. \\ &\quad \times [(\Theta_{m,i,j,\varepsilon,C_*\varepsilon,\ell})^c \setminus B_{\mathbb{R}^3}(0; C_*\varepsilon)^c], \\ &\quad \left. (y, u) \in \overline{\mathcal{O}_{m,i,j,\varepsilon,\varepsilon,\ell}} \cap [\mathbb{R}^3 \times B_{\mathbb{R}^3}(0; \varepsilon)^c] \right\}, \\ &\inf \left\{ |(x, v) - (y, u)| : \right. \\ &\quad (x, v) \in \left[\bigcap_{x \in \mathcal{X}_{m,i,j,\varepsilon,C_*\varepsilon,\ell}} (B_{\mathbb{R}^3}(x; C_*\varepsilon))^c \right] \times [\mathbb{R}^3 \setminus B_{\mathbb{R}^3}(0; C_*\varepsilon)], \\ &\quad \left. (y, u) \in \overline{\mathcal{O}_{m,i,j,\varepsilon,\varepsilon,\ell}} \cap [\mathbb{R}^3 \times B_{\mathbb{R}^3}(0; \varepsilon)^c] \right\}. \end{aligned} \quad (46)$$

Secondly, we consider $\{(y, u) \in \overline{\mathcal{O}_{m,i,j,\varepsilon,\varepsilon,\ell}}\}$. From (37) with $\varepsilon_1 = \varepsilon$, for some $\varsigma = \varsigma(\varepsilon, C_*) > 0$

$$\begin{aligned} \overline{\bigcup_{x \in \mathcal{X}_{m,i,j,\varepsilon,\varepsilon,\ell}} B_{\mathbb{R}^3}(x; \varepsilon)} &\subset \bigcup_{x \in \mathcal{X}_{m,i,j,\varepsilon,\varepsilon,\ell}} \bigcup_{y \in B_{\mathbb{R}^3}(x; \varepsilon)} B_{\mathbb{R}^3}(y; \varsigma) \\ &\subset \bigcup_{x \in \mathcal{X}_{m,i,j,\varepsilon,\varepsilon,\ell}} B_{\mathbb{R}^3}(x; \frac{C_*}{2}\varepsilon), \end{aligned}$$

and from the definition of $\mathcal{O}_{m,i,j,\varepsilon,\varepsilon,\ell}$ in (33), we conclude

$$\begin{aligned} \overline{\mathcal{O}_{m,i,j,\varepsilon,\varepsilon,\ell}} &= \overline{\bigcup_{x \in \mathcal{X}_{m,i,j,\varepsilon,\varepsilon,\ell}} B_{\mathbb{R}^3}(x; \varepsilon)} \times \overline{\Theta_{m,i,j,\varepsilon,\varepsilon,\ell}} \\ &\subset \left[\bigcup_{x \in \mathcal{X}_{m,i,j,\varepsilon,\varepsilon,\ell}} B_{\mathbb{R}^3}(x; \frac{C_*}{2}\varepsilon) \right] \times \overline{\Theta_{m,i,j,\varepsilon,\varepsilon,\ell}}. \end{aligned}$$

Therefore, the first number of (46) is bounded below by (44) and the second of (46) by (45). This proves the claim.

Now we claim that

$$(44) \gtrsim \varepsilon, \quad \text{and} \quad (45) \gtrsim \varepsilon.$$

Firstly, we prove (44) $\gtrsim \varepsilon$. Let $v \in (\Theta_{m,i,j,\varepsilon,C_*\varepsilon,\ell})^c \setminus B_{\mathbb{R}^3}(0; C_*\varepsilon)$. By (31)

$$v = r_v \cos \theta_v \cos \phi_v \hat{x}_{1,m,i,j,\varepsilon} + r_v \sin \theta_v \cos \phi_v \hat{x}_{2,m,i,j,\varepsilon} + r_v \sin \theta_v \sin \phi_v n_{m,i,j,\varepsilon},$$

where

$$\begin{aligned} & |r_v \sin \phi_v| \geq 8C_\eta C_* \varepsilon \quad \text{and} \quad |r_v| \leq 1, \\ \text{or } & |\sin \phi_v| \geq 8C_\eta C_* \varepsilon \quad \text{and} \quad |r_v| \geq 1, \\ \text{or } & |\theta_v - \varepsilon \ell| \geq C_* \varepsilon. \end{aligned} \quad (47)$$

Let $u \in \overline{\Theta_{m,i,j,\varepsilon,\ell}} \setminus B_{\mathbb{R}^3}(0; \varepsilon)$. Again from (31) we have

$$u = r_u \cos \theta_u \cos \phi_u \hat{x}_{1,m,i,j,\varepsilon} + r_u \sin \theta_u \cos \phi_u \hat{x}_{2,m,i,j,\varepsilon} + r_u \sin \phi_u n_{m,i,j,\varepsilon},$$

where

$$\begin{aligned} & |\theta_u - \varepsilon \ell| \leq \varepsilon, \\ \text{and } & |r_u \sin \phi_u| \leq 8C_\eta \varepsilon \quad \text{for } |r_u| \leq 1, \\ \text{and } & |\sin \phi_u| \leq 8C_\eta \varepsilon \quad \text{for } |r_u| \geq 1. \end{aligned} \quad (48)$$

If $|\theta_v - \varepsilon \ell| \geq C_* \varepsilon$ then clearly $|v - u| \gtrsim \varepsilon$ since $|\theta_u - \varepsilon \ell| \leq \varepsilon$.

Now we consider the case of $|\theta_v - \varepsilon \ell| \leq C_* \varepsilon$.

If $|r_v| \leq 1$ (therefore $|r_v \sin \phi_v| \geq 8C_\eta C_* \varepsilon$ from (47)) and $|r_u| \leq 1$ (therefore $|r_u \sin \phi_u| \leq 8C_\eta \varepsilon$ from (48)) then

$$\begin{aligned} |v - u| & \geq |(v - u) \cdot n_{m,i,j,\varepsilon}| \geq |v \cdot n_{m,i,j,\varepsilon}| - |u \cdot n_{m,i,j,\varepsilon}| \\ & \geq |r_v \sin \phi_v| - |r_u \sin \phi_u| \geq 8C_\eta C_* \varepsilon - 8C_\eta \varepsilon \\ & \gtrsim \varepsilon. \end{aligned}$$

On the other hand if $|r_v| \geq 1$ and $|r_u| \leq 1$ (therefore $|\sin \phi_v| \geq 8C_\eta C_* \varepsilon$ from (47) and $|r_u \sin \phi_u| \leq 8C_\eta \varepsilon$ from (48)), then

$$\begin{aligned} |v - u| & \geq |(v - u) \cdot n_{m,i,j,\varepsilon}| \geq |r_v \sin \phi_v - r_u \sin \phi_u| \geq |r_v \sin \phi_v| - |r_u \sin \phi_u| \\ & \geq |\sin \phi_v| - 8C_\eta \varepsilon \\ & \geq 8C_\eta C_* \varepsilon - 8C_\eta \varepsilon \\ & \gtrsim \varepsilon. \end{aligned}$$

If $|r_v| \leq 1$ and $|r_u| \geq 1$, then $|r_v \sin \phi_v| \geq 8C_\eta C_* \varepsilon$ from (47) and $|\sin \phi_u| \leq 8C_\eta \varepsilon$ from (48).

Fix $0 < c_* \ll 1 \ll C_*$. If $C_* - c_* \geq |r_u|$, then

$$\begin{aligned} |v - u| & \geq |(v - u) \cdot n_{m,i,j,\varepsilon}| \geq |v \cdot n_{m,i,j,\varepsilon}| - |u \cdot n_{m,i,j,\varepsilon}| \\ & = |r_v \sin \phi_v| - |r_u \sin \phi_u| \\ & \geq 8C_\eta C_* \varepsilon - |r_u| \times 8C_\eta \varepsilon \geq 8C_\eta \varepsilon (C_* - |r_u|) \\ & \geq 8C_\eta \varepsilon \times c_*. \end{aligned}$$

On the other hand, if $C_* - c_* \leq |r_u|$, then

$$\begin{aligned}
|v - u| &\geq |[u - (u \cdot n_{m,i,j,\varepsilon})n_{m,i,j,\varepsilon}] - [v - (v \cdot n_{m,i,j,\varepsilon})n_{m,i,j,\varepsilon}]| \\
&\geq |r_u| |\cos \phi_u| - |r_v| |\cos \phi_v| \\
&\geq |r_u| \sqrt{1 - 64(C_\eta)^2 \varepsilon^2} - |\cos \phi_v| \\
&\geq (C_* - c_*) \sqrt{1 - 64(C_\eta)^2 \varepsilon^2} - 1 \\
&\gtrsim 1.
\end{aligned}$$

If $|r_v| \geq 1$ and $|r_u| \geq 1$ then $|\sin \phi_v| \geq 8C_\eta C_* \varepsilon$ and $|\sin \phi_u| \leq 8C_\eta \varepsilon$ from (47) and (48). Then

$$|v - u| \geq |(v - u) \cdot n_{m,i,j,\ell}| \gtrsim |r_v| |\sin \phi_v - \sin \phi_u| \gtrsim C_\eta (C_* - 1) \varepsilon.$$

Combining all cases, we deduce (44) $\gtrsim \varepsilon$.

Secondly, we prove (45) $\gtrsim \varepsilon$. The proof is due to

$$\begin{aligned}
(45) &\geq \inf \left\{ |x - y| : x \in \bigcap_{z \in \mathcal{X}_{m,i,j,\varepsilon,C_*\varepsilon,\ell}} (B_{\mathbb{R}^3}(z; C_*\varepsilon))^c, \right. \\
&\quad \left. y \in \bigcup_{z \in \mathcal{X}_{m,i,j,\varepsilon,\frac{C_*}{2}\varepsilon,\ell}} B_{\mathbb{R}^3}(z; \frac{C_*}{2}\varepsilon) \right\} \\
&\geq \inf \left\{ |x - y| : x \in \bigcap_{z \in \mathcal{X}_{m,i,j,\varepsilon,\frac{C_*}{2}\varepsilon,\ell}} (B_{\mathbb{R}^3}(z; C_*\varepsilon))^c, \right. \\
&\quad \left. y \in \bigcup_{z \in \mathcal{X}_{m,i,j,\varepsilon,\frac{C_*}{2}\varepsilon,\ell}} B_{\mathbb{R}^3}(z; \frac{C_*}{2}\varepsilon) \right\} \\
&\geq \inf_{z \in \mathcal{X}_{m,i,j,\varepsilon,\frac{C_*}{2}\varepsilon,\ell}} \inf \left\{ |x - y| : x \in (B_{\mathbb{R}^3}(z; C_*\varepsilon))^c, y \in B_{\mathbb{R}^3}(z; \frac{C_*}{2}\varepsilon) \right\} \\
&\geq \frac{C_*}{2} \varepsilon.
\end{aligned}$$

2.2 Construction of Cut-off functions

Recall the standard mollifier $\varphi : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow [0, \infty)$,

$$\begin{aligned}
\varphi(x, v) &:= C \exp \left(\frac{1}{|x|^2 + |v|^2 - 1} \right), \text{ for } \sqrt{|x|^2 + |v|^2} < 1, \\
\varphi(x, v) &:= 0, \text{ for } \sqrt{|x|^2 + |v|^2} \geq 1,
\end{aligned}$$

where the constant $C > 0$ is selected so that $\int_{\mathbb{R}^3 \times \mathbb{R}^3} \varphi(x, v) dv dx = 1$.

For each $\varepsilon > 0$, set

$$\varphi_\varepsilon(x, v) := (\varepsilon/\tilde{C})^{-6} \varphi\left(\frac{\sqrt{|x|^2 + |v|^2}}{\varepsilon/\tilde{C}}\right), \quad (49)$$

where $\tilde{C} \gg C_* \gg 1$. Clearly φ_ε is smooth and bounded and satisfies

$$\iint_{\mathbb{R}^3 \times \mathbb{R}^3} \varphi_\varepsilon(x, v) dv dx = 1, \quad \text{spt}(\varphi_\varepsilon) \subset B_{\mathbb{R}^3 \times \mathbb{R}^3}(0; \varepsilon/\tilde{C}).$$

Definition 1 We define a smooth cut-off function $\chi_\varepsilon : \bar{\Omega} \times \mathbb{R}^3 \rightarrow [0, 2]$ as

$$\begin{aligned} \chi_\varepsilon(x, v) &:= \mathbf{1}_{\bar{\Omega} \times \mathbb{R}^3 \setminus \mathcal{O}_{\varepsilon, C_* \varepsilon}} * \varphi_\varepsilon(x, v) \\ &= \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \mathbf{1}_{\bar{\Omega} \times \mathbb{R}^3 \setminus \mathcal{O}_{\varepsilon, C_* \varepsilon}}(y, u) \varphi_\varepsilon(x - y, v - u) dy du. \end{aligned} \quad (50)$$

The following properties of the cut-off function are crucial for our analysis.

Lemma 2 For $\theta > 0$, there exist $\tilde{C} \gg C_* \gg 1$ in (49) and (50) and $\varepsilon_0 = \varepsilon_0(\Omega) > 0$ such that if $0 < \varepsilon < \varepsilon_0$ then

$$\mathfrak{S}_B \subset \{(x, v) \in \bar{\Omega} \times \mathbb{R}^3 : \chi_\varepsilon(x, v) = 0\}, \quad (51)$$

and, for either $\partial = \nabla_x$ or $\partial = \nabla_v$,

$$\iint_{\Omega \times \mathbb{R}^3} [1 - \chi_\varepsilon(x, v)] e^{-\theta|v|^2} dv dx \lesssim_\Omega \varepsilon, \quad (52)$$

$$\iint_{\Omega \times \mathbb{R}^3} |\partial \chi_\varepsilon(x, v)| e^{-\theta|v|^2} dv dx \lesssim_\Omega 1. \quad (53)$$

Proof Firstly we prove (51). Let $(x, v) \in \mathfrak{S}_B$. Due to (49) if $|(x, v) - (y, u)| \geq \varepsilon/\tilde{C}$ then $\varphi_\varepsilon(x - y, v - u) = 0$. Therefore

$$(50) = \iint_{B_{\mathbb{R}^6}((x, v); \varepsilon/\tilde{C})} \mathbf{1}_{\bar{\Omega} \times \mathbb{R}^3 \setminus \mathcal{O}_{\varepsilon, C_* \varepsilon}}(y, u) \varphi_\varepsilon(x - y, v - u) dy du.$$

On the other hand, due to (23) with $\varepsilon_1 = \varepsilon$ and $\tilde{C} \gg C_*$, we have $(y, u) \in \mathcal{O}_{\varepsilon, C_* \varepsilon}$ and

$$\mathbf{1}_{\bar{\Omega} \times \mathbb{R}^3 \setminus \mathcal{O}_{\varepsilon, C_* \varepsilon}}(y, u) \equiv 0, \quad \text{on } (y, u) \in B_{\mathbb{R}^6}((x, v); \varepsilon/\tilde{C}).$$

Therefore we conclude $\chi_\varepsilon(x, v) = 0$ and (51).

Secondly we deduce (52). We use (22) with $\varepsilon_1 = \varepsilon$ to have

$$\begin{aligned} & \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} [1 - \mathbf{1}_{\bar{\Omega} \times \mathbb{R}^3 \setminus \mathcal{O}_{\varepsilon, C_* \varepsilon}}(y, u)] \varphi_\varepsilon(x - y, v - u) e^{-\theta|v|^2} du dy dv dx \\ & \leq \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \mathbf{1}_{\mathcal{O}_{\varepsilon, C_* \varepsilon}}(y, u) e^{-\frac{\theta}{2}|u|^2} du dy \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \varphi_\varepsilon(x - y, v - u) e^{\theta|v - u|^2} dv dx \\ & \leq C_1 \frac{\varepsilon}{2} \iint_{B_{\mathbb{R}^6}(0; \varepsilon/\tilde{C})} \varphi_\varepsilon(x, v) e^{\theta \varepsilon^2 / \tilde{C}^2} dv dx \\ & \lesssim \varepsilon, \end{aligned}$$

where we used

$$-\theta|v|^2 = \theta|v - u|^2 - \theta|v - u|^2 - \theta|v|^2 \leq \theta|v - u|^2 - \frac{\theta}{2}|u|^2. \quad (54)$$

Thirdly we prove (53). Note that from a standard scaling argument and (49)

$$|\partial\varphi_\varepsilon(x, v)| \lesssim \frac{\tilde{C}^6}{\varepsilon^7} \mathbf{1}_{B_{\mathbb{R}^6}(0; \varepsilon/\tilde{C})}(x, v).$$

We also note that $\partial\chi_\varepsilon = -\partial[1 - \chi_\varepsilon]$. Therefore, by Lemma 1,

$$\begin{aligned} & \iint_{\Omega \times \mathbb{R}^3} |\partial\chi_\varepsilon(x, v)| e^{-\theta|v|^2} dv dx \\ &= \iint \left| \iint [1 - \mathbf{1}_{\Omega \times \mathbb{R}^3 \setminus \mathcal{O}_{\varepsilon, C_*\varepsilon}}(y, u)] \partial\varphi_\varepsilon(x - y, v - u) e^{-\theta|v|^2} du dy \right| dv dx \\ &\leq \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \mathbf{1}_{\mathcal{O}_{\varepsilon, C_*\varepsilon}}(y, u) e^{-\frac{\theta}{2}|u|^2} du dy \iint_{\mathbb{R}^3 \times \mathbb{R}^3} O(\varepsilon^{-7} \tilde{C}^6) \mathbf{1}_{B_{\mathbb{R}^6}(0; \varepsilon/\tilde{C})}(x, v) dv dx \\ &\leq O(\varepsilon) \times O(\varepsilon^{-1}) \\ &\lesssim 1. \end{aligned}$$

Proposition 1 *With the same constants $\tilde{C} \gg C_* \gg 1$ as in Lemma 2 and $0 < \varepsilon < \varepsilon_0$,*

$$\mathfrak{S}_B \cap [\partial\Omega \times \mathbb{R}^3] \subset \{(x, v) \in \partial\Omega \times \mathbb{R}^3 : \chi_\varepsilon(x, v) = 0\}. \quad (55)$$

Moreover if $|(y, u)| \leq \varepsilon/\tilde{C}$ for $\tilde{C} \gg C_* \gg 1$ then

$$\int_{\partial\Omega} \int_{n(x) \cdot v < 0} \mathbf{1}_{\mathcal{O}_{\varepsilon, C_*\varepsilon}}(x - y, v - u) e^{-\theta|v - u|^2} |n(x - y) \cdot (v - u)| dv dS_x \lesssim \varepsilon, \quad (56)$$

and

$$\int_{\gamma_-} [1 - \chi_\varepsilon(x, v)] e^{-\theta|v|^2} d\gamma \lesssim_\Omega \varepsilon, \quad (57)$$

$$\int_{\gamma_-} |\partial\chi_\varepsilon(x, v)| e^{-\theta|v|^2} d\gamma \lesssim_\Omega 1. \quad (58)$$

The following fact is crucial to prove Proposition 1 and especially (56):

Lemma 3 *We fix $m_0 = 1, 2, \dots, M_{\Omega, \delta}$ in (24). From (25), we may assume (up to rotations and translations) there exists a C^2 -function $\eta_{m_0} : [-\delta, \delta] \times [-\delta, \delta] \rightarrow \mathbb{R}$, whose graph is the boundary $\mathcal{U}_{m_0} \cap \partial\Omega$.*

Let $(x_1, x_2) \in \mathcal{A}_{m_0} \cap [-\delta/2, \delta/2] \times [-\delta/2, \delta/2]$ and $(x_1, x_2) \in \mathcal{R}_{m_0, i_0, j_0, \varepsilon, C_\varepsilon}$ for $|i_0|, |j_0| \leq N_\varepsilon$. (see (28), (29), and (30))*

Suppose i) $|y| \leq \varepsilon/\tilde{C}$ and

$$((x_1, x_2, \eta_{m_0}(x_1, x_2)) - y, v) \in \mathcal{O}_{\varepsilon, C_*\varepsilon}, \quad (59)$$

and ii) for large but fixed $s_ \gg 1$,*

$$-1 \leq n_{m_0}(0, 0) \cdot \frac{v}{|v|} \leq -s_* C_2 \sqrt{\varepsilon}, \quad \text{with } C_2 := \sqrt{\frac{8C_*}{3}} [1 + \|\eta_{m_0}\|_{C^2(\mathcal{A}_{m_0})}]^{1/2}, \quad (60)$$

where $n_{m_0}(0, 0) = \frac{1}{\sqrt{1+|\nabla\eta_{m_0}(0,0)|^2}}(\nabla\eta_{m_0}(0,0), -1)$.

Then either $|v| < \varepsilon^{1/3}$ or there exists $(i, j) \in [-N_1 + i_0, N_1 + i_0] \times [-N_1 + j_0, N_1 + j_0]$, with

$$N_1 := \lfloor \frac{8C_3}{\sqrt{\varepsilon}} \rfloor, \quad C_3 := \frac{4C_* + 8C_*[1 + \|\eta_{m_0}\|_{C^1(\mathcal{A}_{m_0})}]^{1/2} + 2/\tilde{C}}{s_*C_2}, \quad (61)$$

such that

$$((x_1, x_2, \eta_{m_0}(x_1, x_2)) - y, v) \in \bigcup_{0 \leq \ell \leq L_\varepsilon} \mathcal{O}_{m_0, i, j, \varepsilon, C_*\varepsilon, \ell} \cap \bar{\Omega} \times \{v \in \mathbb{R}^3 : |v| \geq \varepsilon^{1/3}\},$$

and

$$\left| n_{m_0}(0, 0) \cdot \frac{v}{|v|} \right| \leq C_4 \sqrt{\varepsilon} \quad \text{with } C_4 = C_3[1 + \|\eta_{m_0}\|_{C^2(\mathcal{A}_{m_0})}]. \quad (62)$$

Remark that the constant N_1 in (61) does not depend on x, y, v .

Proof (Proof of Lemma 3) Without loss of generality (up to rotations and translations), we may assume

$$(i_0, j_0) = (0, 0) \quad \text{and} \quad \eta_{m_0}(0, 0) = 0 \quad \text{and} \quad \nabla\eta_{m_0}(0, 0) = 0. \quad (63)$$

Consider the case of $|v| \geq \varepsilon^{1/3}$. Since $((x_1, x_2, \eta_{m_0}(x_1, x_2)) - y, v) \in \mathcal{O}_{\varepsilon, C_*\varepsilon}$ we use the definition of $\mathcal{O}_{\varepsilon, C_*\varepsilon}$ in (35) to have

$$\text{either } \underbrace{|v| < C_*\varepsilon}_{(64)-(i)} \quad \text{or} \quad (x - y, v) \in \underbrace{\bigcup_{m, i, j, \ell} \mathcal{O}_{m, i, j, \varepsilon, C_*\varepsilon, \ell}}_{(64)-(ii)}. \quad (64)$$

For small $0 < \varepsilon \ll 1$, we can exclude the case of (64) – (i) since $|v| > \varepsilon^{1/3} \gg C_*\varepsilon$.

Consider the case of (64) – (ii). In this case, we claim that

$$((x_1, x_2, \eta_{m_0}(x_1, x_2)) - y, v) \in \bigcup_{i, j, \ell} \mathcal{O}_{m_0, i, j, \varepsilon, C_*\varepsilon, \ell}. \quad (65)$$

From (64) – (ii) and the definition of $\mathcal{O}_{m_0, i, j, \varepsilon, C_*\varepsilon, \ell}$ in (33), there exist m, i, j, ℓ such that

$$((x_1, x_2, \eta_{m_0}(x_1, x_2)) - y, v) \in \left[\bigcup_{p \in \mathcal{X}_{m, i, j, \varepsilon, C_*\varepsilon, \ell}} B_{\mathbb{R}^3}(p; C_*\varepsilon) \right] \times \Theta_{m, i, j, \varepsilon, C_*\varepsilon, \ell}.$$

In particular, there exists $p \in \mathcal{X}_{m, i, j, \varepsilon, C_*\varepsilon, \ell}$ satisfying

$$|p - ((x_1, x_2, \eta_{m_0}(x_1, x_2)) - y)| < C_*\varepsilon.$$

By the definition of $\mathcal{X}_{m, i, j, \varepsilon, C_*\varepsilon, \ell}$ in (34),

$$p = (\bar{p}_1, \bar{p}_2, \eta_m(\bar{p}_1, \bar{p}_2)) + \bar{\tau} [\cos \bar{\theta} \hat{x}_{1, m, i, j, \varepsilon} + \sin \bar{\theta} \hat{x}_{2, m, i, j, \varepsilon}] + \bar{s} n_{m, i, j, \varepsilon},$$

for some

$$\begin{aligned} (\bar{p}_1, \bar{p}_2) &\in \mathcal{R}_{m,i,j,\varepsilon,C_*\varepsilon}, \\ \bar{\theta} &\in (\varepsilon\ell - C_*\varepsilon, \varepsilon\ell + C_*\varepsilon), \\ \bar{\tau} &\in [0, t_{\mathbf{f}}((\bar{p}_1, \bar{p}_2, \eta_m(\bar{p}_1, \bar{p}_2)), \cos \bar{\theta} \hat{x}_{1,m,i,j,\varepsilon} + \sin \bar{\theta} \hat{x}_{2,m,i,j,\varepsilon})], \\ \bar{s} &\in [-\varepsilon, \varepsilon]. \end{aligned}$$

By the definition of $t_{\mathbf{f}}$ in (7),

$$z := p - \bar{s}n_{m,i,j,\varepsilon} = (\bar{p}_1, \bar{p}_2, \eta_m(\bar{p}_1, \bar{p}_2)) + \bar{\tau} [\cos \bar{\theta} \hat{x}_{1,m,i,j,\varepsilon} + \sin \bar{\theta} \hat{x}_{2,m,i,j,\varepsilon}] \in \Omega.$$

And

$$\begin{aligned} &|z - ((x_1, x_2, \eta_{m_0}(x_1, x_2)) - y)| \\ &\leq |z - p| + |p - ((x_1, x_2, \eta_{m_0}(x_1, x_2)) - y)| \\ &\leq 2C_*\varepsilon. \end{aligned} \tag{66}$$

From (63), (66), and $|y| \leq \varepsilon/\tilde{C}$, we deduce

$$\begin{aligned} &|z - (0, 0, \eta_{m_0}(0, 0))| \\ &\leq |z - ((x_1, x_2, \eta_{m_0}(x_1, x_2)) - y)| + |(x_1, x_2, \eta_{m_0}(x_1, x_2)) - (0, 0, \eta_{m_0}(0, 0))| + |y| \\ &\leq 2C_*\varepsilon + 4C_*\varepsilon(1 + \|\eta_{m_0}\|_{C^1(\mathcal{A}_{m_0})}) + \varepsilon/\tilde{C}. \end{aligned}$$

Denote $(\bar{z}_1, \bar{z}_2) = (\bar{p}_1, \bar{p}_2)$. By the definition of $t_{\mathbf{b}}$ and $t_{\mathbf{f}}$ in (6) and (7)

$$x_{\mathbf{b}}(z, \cos \bar{\theta} \hat{x}_{1,m,i,j,\varepsilon} + \sin \bar{\theta} \hat{x}_{2,m,i,j,\varepsilon} + 0n_{m,i,j,\varepsilon}) = (\bar{z}_1, \bar{z}_2, \eta_{m_0}(\bar{z}_1, \bar{z}_2)). \tag{67}$$

On the other hand, by the definition of $\Theta_{m,i,j,\varepsilon,C_*\varepsilon,\ell}$ in (31),

$$\begin{aligned} \frac{v}{|v|} &= \cos \theta_v \cos \phi_v \hat{x}_{1,m,i,j,\varepsilon} + \sin \theta_v \cos \phi_v \hat{x}_{2,m,i,j,\varepsilon} + \sin \phi_v n_{m,i,j,\varepsilon}, \\ &\text{with } |\theta_v - \varepsilon\ell| < C_*\varepsilon, \end{aligned} \tag{68}$$

and

$$\begin{aligned} &|v \cdot n_{m,i,j,\varepsilon}| < 8C_\eta C_*\varepsilon, \quad \text{for } \varepsilon^{1/3} \leq |v| \leq 1, \\ &\left| \frac{v}{|v|} \cdot n_{m,i,j,\varepsilon} \right| < 8C_\eta C_*\varepsilon, \quad \text{for } 1 \leq |v|. \end{aligned}$$

Therefore, for $0 < \varepsilon \ll 1$,

$$\left| \frac{v}{|v|} \cdot n_{m,i,j,\varepsilon} \right| = |\sin \phi_v| < \max \{8C_\eta C_*\varepsilon^{2/3}, 8C_\eta C_*\varepsilon\} \leq 16C_\eta C_*\varepsilon^{2/3}. \tag{69}$$

Now we estimate as

$$\begin{aligned} &n_{m_0}(0, 0) \cdot (\cos \bar{\theta} \hat{x}_{1,m,i,j,\varepsilon} + \sin \bar{\theta} \hat{x}_{2,m,i,j,\varepsilon} + 0n_{m,i,j,\varepsilon}) \\ &\leq n_{m_0}(0, 0) \cdot \frac{v}{|v|} + n_{m_0}(0, 0) \cdot \underbrace{\left(\frac{v}{|v|} - (\cos \bar{\theta} \hat{x}_{1,m,i,j,\varepsilon} + \sin \bar{\theta} \hat{x}_{2,m,i,j,\varepsilon} + 0n_{m,i,j,\varepsilon}) \right)}_{(\mathbf{a})}. \end{aligned}$$

We use (68), (69), and $\bar{\theta} \in (\varepsilon\ell - C_*\varepsilon, \varepsilon\ell + C_*\varepsilon)$ to conclude that, for $0 < \varepsilon \ll 1$,

$$\begin{aligned} (\mathbf{a}) &\leq 2\{|\cos\theta_v - \cos\bar{\theta}| + |\cos\theta_v|\cos\phi_v - 1| + |\sin\theta_v - \sin\bar{\theta}| \\ &\quad + |\sin\theta_v|\cos\phi_v - 1| + |\sin\phi_v|\} \\ &\leq 2\{4C_*\varepsilon + 16C_\eta C_*\varepsilon^{2/3} + 2(16)^2 C_\eta^2 C_*^2 \varepsilon^{4/3}\} \\ &\leq 200C_\eta C_*\varepsilon^{2/3}. \end{aligned}$$

Finally from (60), for $0 < \varepsilon \ll 1$,

$$\begin{aligned} -1 &\leq n_{m_0}(0,0) \cdot (\cos\bar{\theta}\hat{x}_{1,m,i,j,\varepsilon} + \sin\bar{\theta}\hat{x}_{2,m,i,j,\varepsilon} + 0n_{m,i,j,\varepsilon}) \\ &\leq -s_* \times C_2\sqrt{\varepsilon} + 400C_\eta C_*\varepsilon^{2/3} \\ &\leq -\frac{s_*C_2}{2}\sqrt{\varepsilon}. \end{aligned} \tag{70}$$

Now we are ready to prove the first claim (65). Denote

$$\hat{u} := \cos\bar{\theta}\hat{x}_{1,m,i,j,\varepsilon} + \sin\bar{\theta}\hat{x}_{2,m,i,j,\varepsilon}.$$

Recall that $|z| \leq (2C_* + 4C_*[1 + \|\eta_{m_0}\|_{C^1(\mathcal{A}_{m_0})}] + 1/\tilde{C})\varepsilon$ and $z \in \Omega$. Therefore for $0 < \varepsilon \ll 1$ the function η_{m_0} is defined around (z_1, z_2) and $z_3 > \eta_{m_0}(z_1, z_2)$.

We define, for $|\tau| \ll 1$,

$$\Phi(\tau) = z_3 - \hat{u}_3\tau - \eta_{m_0}(z_1 - \hat{u}_1\tau, z_2 - \hat{u}_2\tau). \tag{71}$$

Clearly $\Phi(0) > 0$. Expanding $\Phi(\tau)$ in τ , from $-\hat{u}_3 = n_{m_0}(0,0) \cdot (\cos\bar{\theta}\hat{x}_{1,m,i,j,\varepsilon} + \sin\bar{\theta}\hat{x}_{2,m,i,j,\varepsilon})$, and (70), we have

$$\begin{aligned} \Phi(\tau) &\leq -\hat{u}_3\tau + |z_3| + |\eta_{m_0}(z_1 - \hat{u}_1\tau, z_2 - \hat{u}_2\tau)| \\ &\leq -s_* \times \frac{C_2}{2}\sqrt{\varepsilon}\tau \\ &\quad + (2C_* + 4C_*[1 + \|\eta_{m_0}\|_{C^1(\mathcal{A}_{m_0})}] + 1/\tilde{C})\varepsilon \\ &\quad + \|\eta_{m_0}\|_{C^2(\mathcal{A}_{m_0})}(2C_* + 4C_*[1 + \|\eta_{m_0}\|_{C^1(\mathcal{A}_{m_0})}] + 1/\tilde{C})^2\varepsilon^2 \\ &\quad + \|\eta_{m_0}\|_{C^2(\mathcal{A}_{m_0})}|\tau|^2, \end{aligned}$$

where we have used

$$\begin{aligned} &\eta_{m_0}(z_1 - \hat{u}_1\tau, z_2 - \hat{u}_2\tau) \\ &= \eta_{m_0}(z_1, z_2) + \int_0^\tau \frac{d}{ds}\eta_{m_0}(z_1 - \hat{u}_1s, z_2 - \hat{u}_2s)ds \\ &= \eta_{m_0}(z_1, z_2) - (\hat{u}_1, \hat{u}_2) \cdot \nabla\eta_{m_0}(z_1, z_2)\tau \\ &\quad + \int_0^\tau \int_0^s \frac{d}{ds_1^2}\eta_{m_0}(z_1 - \hat{u}_1s_1, z_2 - \hat{u}_2s_1)ds_1ds \\ &\leq \|\eta_{m_0}\|_{C^2(\mathcal{A}_{m_0})}\frac{|z|^2}{2} - (\hat{u}_1, \hat{u}_2) \cdot \nabla\eta_{m_0}(0,0)|\tau| + \|\eta_{m_0}\|_{C^2(\mathcal{A}_{m_0})}|z||\tau| \\ &\quad + \|\eta_{m_0}\|_{C^2(\mathcal{A}_{m_0})}\frac{|\tau|^2}{2} \\ &\leq \|\eta_{m_0}\|_{C^2(\mathcal{A}_{m_0})}\left(|z|^2 + |\tau|^2\right). \end{aligned}$$

Now we plug $\tau = \frac{1}{s_*} \times C_3 \sqrt{\varepsilon}$ with the constant C_3 in (61) to have, for $s_* \gg 1$ and $0 < \varepsilon \ll 1$,

$$\begin{aligned} \Phi(\tau) &\leq -\left[\frac{C_2 C_3}{2} - (2C_* + 4C_*[1 + \|\eta_{m_0}\|_{C^1(\mathcal{A}_{m_0})}] + 1/\tilde{C})\right. \\ &\quad \left. - \frac{\|\eta_{m_0}\|_{C^2(\mathcal{A}_{m_0})} C_3^2}{(s_*)^2}\right] \varepsilon + O(\varepsilon^2) < 0. \end{aligned}$$

By the mean value theorem, there exists at least one $\tau \in (0, C_3 \sqrt{\varepsilon}]$ satisfying $\Phi(\tau) = 0$. We choose the smallest one of them and denote it as $\tau_0 \in (0, C_3 \sqrt{\varepsilon}]$. By this definition and (67), for $0 < \varepsilon \ll 1$,

$$\begin{aligned} x_{\mathbf{b}}(z, \hat{u}) &= x_{\mathbf{b}}(z, \cos \bar{\theta} \hat{x}_{1,m,i,j,\varepsilon} + \sin \bar{\theta} \hat{x}_{2,m,i,j,\varepsilon}) \\ &= z - \tau_0 \hat{u} \\ &= (z_1 - \tau_0 \hat{u}_1, z_2 - \tau_0 \hat{u}_2, z_3 - \tau_0 \hat{u}_3). \end{aligned}$$

Therefore, $x_{\mathbf{b}}(z, \hat{u}) \in \partial\Omega \cap \mathcal{U}_{m_0}$ and this proves (65).

For $0 < \varepsilon \ll 1$

$$\begin{aligned} |(z_1 - \tau_0 \hat{u}_1, z_2 - \tau_0 \hat{u}_2)| &\leq (2C_* + 4C_*(1 + \|\eta_{m_0}\|_{C^1(\mathcal{A}_m)} + 1/\tilde{C}))\varepsilon + C_3 \sqrt{\varepsilon} \\ &\leq 2C_3 \sqrt{\varepsilon}. \end{aligned}$$

Moreover,

$$(z_1 - \tau_0 \hat{u}_1, z_2 - \tau_0 \hat{u}_2) \in \mathcal{R}_{m_0, i, j, \varepsilon, C_* \varepsilon},$$

for

$$|i - i_0|, |j - j_0| \leq (2C_3 \sqrt{\varepsilon})/\varepsilon \leq 2C_3 \frac{1}{\sqrt{\varepsilon}} \leq N_1.$$

We only need to prove (62). From (69) and (61)

$$\begin{aligned} |n_{m_0}(0, 0) \cdot \frac{v}{|v|}| &\leq |n_{m_0, i, j, \varepsilon, C_* \varepsilon} \cdot \frac{v}{|v|}| + |(n_{m_0}(0, 0) - n_{m_0, i, j, \varepsilon, C_* \varepsilon}) \cdot \frac{v}{|v|}| \\ &\leq 16C_\eta C_* \varepsilon^{2/3} + \|n_{m_0}\|_{C^1(\mathcal{A}_{m_0})} |N_1 \varepsilon + C_* \varepsilon| \\ &\leq 16C_\eta C_* \varepsilon^{2/3} + \|n_{m_0}\|_{C^1(\mathcal{A}_{m_0})} \{2C_3 \sqrt{\varepsilon} + C_* \varepsilon\} \\ &\leq 10C_3 (1 + \|\eta_{m_0}\|_{C^2(\mathcal{A}_{m_0})}) \sqrt{\varepsilon} \\ &\leq C_4 \sqrt{\varepsilon}, \end{aligned}$$

and (62) follows.

Proof (Proof of Proposition 1)

The first statement (55) is clear from (51). Once we assume (56) then it is easy to prove (57), (58):

$$\begin{aligned}
& \iint_{x \in \partial \Omega, n(x) \cdot v < 0} [1 - \chi_\varepsilon(x, v)] e^{-\theta|v|^2} |n(x) \cdot v| dS_x dv \\
&= \iint_{x \in \partial \Omega, n(x) \cdot v < 0} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} [1 - \mathbf{1}_{\bar{\Omega} \times \mathbb{R}^3 \setminus \mathcal{O}_{\varepsilon, C_* \varepsilon}}(x - y, v - u)] \varphi_\varepsilon(y, u) e^{-\theta|v|^2} \\
&\quad dud y |n(x) \cdot v| dS_x dv \\
&\leq \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \varphi_\varepsilon(y, u) e^{\theta|u|^2} dud y \\
&\quad \times \iint_{x \in \partial \Omega, n(x) \cdot v < 0} \mathbf{1}_{\mathcal{O}_{\varepsilon, C_* \varepsilon}}(x - y, v - u) e^{-\frac{\theta}{2}|v - u|^2} |n(x) \cdot v| dS_x dv \\
&= \iint_{B_{\mathbb{R}^6}(0; \varepsilon/\bar{C})} \varphi_\varepsilon(y, u) e^{\frac{\theta}{2}|u|^2} dud y \\
&\quad \times \iint_{x \in \partial \Omega, n(x) \cdot v < 0} \mathbf{1}_{\mathcal{O}_{\varepsilon, C_* \varepsilon}}(x - y, v - u) e^{-\frac{\theta}{4}|v - u|^2} e^{-\frac{\theta}{2}|v|^2} |n(x) \cdot v| dS_x dv,
\end{aligned}$$
$$\begin{aligned} -\theta|v|^2 &\leq -\frac{\theta}{2}|v|^2 - \left(\frac{\theta}{2}|v|^2 - \frac{\theta}{4}|v-u|^2\right) - \frac{\theta}{4}|v-u|^2 \\ &\leq -\frac{\theta}{2}|v|^2 - \left(\frac{\theta}{2}|v|^2 - \frac{\theta}{2}|v|^2 - \frac{\theta}{2}|u|^2\right) - \frac{\theta}{4}|v-u|^2 \\ &\leq -\frac{\theta}{2}|v|^2 + \frac{\theta}{2}|u|^2 - \frac{\theta}{4}|v-u|^2. \end{aligned}$$
$$\begin{aligned} n(x) \cdot v &= n(x) \cdot v - n(x-y) \cdot (v-u) + n(x-y) \cdot (v-u) \\ &= n(x-y) \cdot (v-u) + [n(x) - n(x-y)] \cdot v + n(x-y) \cdot u \\ &= n(x-y) \cdot (v-u) + O\left(\frac{\varepsilon}{C}\right)(1+|v|). \end{aligned}$$
$$\begin{aligned}
(57) &\leq \iint_{B_{\mathbb{R}^6}(0;\varepsilon/\tilde{C})} \varphi_\varepsilon(y, u) e^{\frac{\theta}{2}|u|^2} \mathrm{d}u \mathrm{d}y \times \iint_{x \in \partial\Omega, n(x) \cdot v < 0} \mathbf{1}_{\mathcal{O}_{\varepsilon, C_*\varepsilon}}(x - y, v - u) \\
&\quad e^{-\frac{\theta}{4}|v-u|^2} e^{-\frac{\theta}{2}|v|^2} |n(x - y) \cdot (v - u)| \mathrm{d}S_x \mathrm{d}v \\
&\quad + O\left(\frac{\varepsilon}{\tilde{C}}\right) e^{\frac{\theta\varepsilon^2}{2\tilde{C}^2}} \times \mathrm{m}_3(\partial\Omega) \times \int_{\mathbb{R}^3} (1 + |v|) e^{-\frac{\theta}{2}|v|^2} \mathrm{d}v \\
&\lesssim_\Omega \varepsilon \times e^{\frac{\theta\varepsilon^2}{2(\tilde{C})^2}} \\
&\lesssim_\Omega \varepsilon.
\end{aligned}$$

Secondly we prove (58). Following the same proof of (57), we deduce

$$\begin{aligned}
& \left| \iint_{x \in \partial\Omega, n(x) \cdot v < 0} \partial \chi_\varepsilon(x, v) e^{-\theta|v|^2} |n(x) \cdot v| dS_x dv \right| \\
&= \left| \iint_{x \in \partial\Omega, n(x) \cdot v < 0} \partial [\chi_\varepsilon(x, v) - 1] e^{-\theta|v|^2} |n(x) \cdot v| dS_x dv \right| \\
&= \left| \iint_{x \in \partial\Omega, n(x) \cdot v < 0} \partial \left[\iint_{\mathbb{R}^3 \times \mathbb{R}^3} \mathbf{1}_{\mathcal{O}_{\varepsilon, C_* \varepsilon}}(y, u) \varphi_\varepsilon(x - y, v - u) du dy \right] \right. \\
&\quad \left. \times e^{-\theta|v|^2} |n(x) \cdot v| dS_x dv \right| \\
&\leq \left| \iint_{x \in \partial\Omega, n(x) \cdot v < 0} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \mathbf{1}_{\mathcal{O}_{\varepsilon, C_* \varepsilon}}(x - y, v - u) |\partial \varphi_\varepsilon(y, u)| du dy \right. \\
&\quad \left. \times |n(x) \cdot v| e^{-\theta|v|^2} dS_x dv \right| \\
&= \iint_{B_{\mathbb{R}^6}(0; \varepsilon/\tilde{C})} |\partial \varphi_\varepsilon(y, u)| e^{\frac{\theta}{2}|u|^2} du dy \\
&\quad \times \iint_{x \in \partial\Omega, n(x) \cdot v < 0} \mathbf{1}_{\mathcal{O}_{\varepsilon, C_* \varepsilon}}(x - y, v - u) e^{-\frac{\theta}{4}|v-u|^2} e^{-\frac{\theta}{2}|v|^2} |n(x) \cdot v| dS_x dv \\
&\lesssim \sup_{(y, u) \in B_{\mathbb{R}^6}(0; \varepsilon/\tilde{C})} \iint_{x \in \partial\Omega, n(x) \cdot v < 0} \mathbf{1}_{\mathcal{O}_{\varepsilon, C_* \varepsilon}}(x - y, v - u) \\
&\quad e^{-\frac{\theta}{4}|v-u|^2} e^{-\frac{\theta}{2}|v|^2} (1 + |v|) dS_x dv \\
&+ O\left(\frac{1}{\varepsilon}\right) \sup_{(y, u) \in B_{\mathbb{R}^6}(0; \varepsilon/\tilde{C})} \iint_{x \in \partial\Omega, n(x) \cdot v < 0} \mathbf{1}_{\mathcal{O}_{\varepsilon, C_* \varepsilon}}(x - y, v - u) \\
&\quad e^{-\frac{\theta}{4}|v-u|^2} e^{-\frac{\theta}{2}|v|^2} |n(x - y) \cdot (v - u)| dS_x dv \\
&\lesssim 1.
\end{aligned}$$

Proof of (56). Let $|(y, u)| \leq \varepsilon/\tilde{C}$. We use (24) to decompose

$$\begin{aligned}
(56) &\leq \sum_{m=1}^{M_{\Omega, \delta}} \int_{\mathcal{U}_m \cap \partial\Omega} \int_{n_m(x) \cdot v < 0} \mathbf{1}_{\mathcal{O}_{\varepsilon, C_* \varepsilon}}(x - y, v - u) e^{-\theta|v-u|^2} e^{-\frac{\theta}{2}|v|^2} \\
&\quad |n_m(x - y) \cdot (v - u)| dv dS_x \\
&\leq M_{\Omega, \delta} \times \sup_m \int_{\mathcal{U}_m \cap \partial\Omega} \int_{n_m(x) \cdot v < 0} \mathbf{1}_{\mathcal{O}_{\varepsilon, C_* \varepsilon}}(x - y, v - u) \\
&\quad e^{-\theta|v-u|^2} e^{-\frac{\theta}{2}|v|^2} |n_m(x - y) \cdot (v - u)| dv dS_x \\
&\lesssim_{\Omega} \frac{1}{\delta^2} \sup_m \int_{\mathcal{U}_m \cap \partial\Omega} \int_{n_m(x) \cdot v < 0} \mathbf{1}_{\mathcal{O}_{\varepsilon, C_* \varepsilon}}(x - y, v - u) \\
&\quad e^{-\theta|v-u|^2} e^{-\frac{\theta}{2}|v|^2} |n_m(x - y) \cdot (v - u)| dv dS_x.
\end{aligned}$$

For fixed $m = 1, 2, \dots, M_{\Omega, \delta}$, we use (25) and (30) again to decompose

$$\begin{aligned}
& \int_{\mathcal{U}_m \cap \partial\Omega} \int_{n_m(x) \cdot v < 0} \mathbf{1}_{\mathcal{O}_{\varepsilon, C_*\varepsilon}}(x - y, v - u) e^{-\theta|v-u|^2} e^{-\frac{\theta}{2}|v|^2} \\
& \quad |n_m(x - y) \cdot (v - u)| dv dS_x \\
&= \int_{\mathcal{A}_m} \int_{n_m(x_1, x_2) \cdot v < 0} \mathbf{1}_{\mathcal{O}_{\varepsilon, C_*\varepsilon}}(x_1 - y_1, x_2 - y_2, \eta_m(x_1, x_2) - y_3, v - u) \\
& \quad \times e^{-\theta|v-u|^2} e^{-\frac{\theta}{2}|v|^2} |n_m(x - y) \cdot (v - u)| dv \sqrt{1 + |\nabla \eta_m(x_1, x_2)|^2} dx_1 dx_2 \\
&\leq \sum_{-N_\varepsilon \leq i, j \leq N_\varepsilon} \int_{\mathcal{R}_{m, i, j, \varepsilon, C_*\varepsilon}} \int_{n_m(x_1, x_2) \cdot v < 0} \\
& \quad \mathbf{1}_{\mathcal{O}_{\varepsilon, C_*\varepsilon}}(x_1 - y_1, x_2 - y_2, \eta_m(x_1, x_2) - y_3, v - u) \\
& \quad \times e^{-\theta|v-u|^2} e^{-\frac{\theta}{2}|v|^2} |n_m(x - y) \cdot (v - u)| dv \sqrt{1 + |\nabla \eta_m(x_1, x_2)|^2} dx_1 dx_2 \\
&\lesssim \frac{\delta^2}{\varepsilon^2} \sup_{-N_\varepsilon \leq i, j \leq N_\varepsilon} \int_{\mathcal{R}_{m, i, j, \varepsilon, C_*\varepsilon}} \int_{n_m(x_1, x_2) \cdot v < 0} \\
& \quad \mathbf{1}_{\mathcal{O}_{\varepsilon, C_*\varepsilon}}(x_1 - y_1, x_2 - y_2, \eta_m(x_1, x_2) - y_3, v - u) \\
& \quad \times e^{-\theta|v-u|^2} e^{-\frac{\theta}{2}|v|^2} |n_m(x - y) \cdot (v - u)| dv \sqrt{1 + |\nabla \eta_m(x_1, x_2)|^2} dx_1 dx_2,
\end{aligned}$$

where $n_m(x_1, x_2) = \frac{1}{\sqrt{1 + |\partial_1 \eta_m(x_1, x_2)|^2 + |\partial_2 \eta_m(x_1, x_2)|^2}} \begin{pmatrix} \partial_1 \eta_m(x_1, x_2) \\ \partial_2 \eta_m(x_1, x_2) \\ -1 \end{pmatrix}$.

We fix i, j . Without loss of generality (up to rotations and translations), we may assume

$$c_{m, i, j, \varepsilon} = (0, 0), \quad \partial_1 \eta_m(0, 0) = 0 = \partial_2 \eta_m(0, 0), \quad n_{m, i, j, \varepsilon} = (0, 0, -1).$$

We claim

$$\begin{aligned}
& \int_{[-C_*\varepsilon, C_*\varepsilon]^2} \int_{n_m(x_1, x_2) \cdot (v+u) < 0} \\
& \quad \mathbf{1}_{\mathcal{O}_{\varepsilon, C_*\varepsilon}}(x_1 - y_1, x_2 - y_2, \eta_m(x_1, x_2) - y_3, v) \\
& \quad \times e^{-\theta|v|^2} e^{-\frac{\theta}{2}|v+u|^2} |n_m(x - y) \cdot v| dv \sqrt{1 + |\nabla \eta_m(x_1, x_2)|^2} dx_1 dx_2 \\
& \lesssim \varepsilon^3.
\end{aligned} \tag{72}$$

Once we prove (72), due to the above estimates for the decomposition, we conclude (56) directly.

For $(x_1, x_2) \in [-C_*\varepsilon, C_*\varepsilon]^2$, $|(y, u)| < \varepsilon/\tilde{C}$, and $n_m(x_1, x_2) \cdot (v + u) < 0$, we deduce

$$\begin{aligned}
& n_{m, i, j, \varepsilon} \cdot v \\
&= n_m(0, 0) \cdot v \\
&= n_m(x_1, x_2) \cdot (v + u) + [n_m(0, 0) \cdot v - n_m(x_1, x_2) \cdot (v + u)] \\
&< 0 + |n_m(x_1, x_2)| |u| + |n_m(0, 0) - n_m(x_1, x_2)| |v| \\
&\leq \varepsilon/\tilde{C} + 2C_*\varepsilon \|\eta_m\|_{C^2([-C_*\varepsilon, C_*\varepsilon]^2)} |v| \\
&\leq C_5(1 + |v|)\varepsilon,
\end{aligned} \tag{73}$$

where $C_5 = \max \{1/\tilde{C}, 2C_* \|\eta_m\|_{C^2([-C_*\varepsilon, C_*\varepsilon]^2)}\}$. Therefore

$$(72) \leq \int_{[-C_*\varepsilon, C_*\varepsilon]^2} \int_{n_m(i,j,\varepsilon) \cdot v < C_5(1+|v|)\varepsilon} \cdots.$$

According to Lemma 3, we decompose

$$\begin{aligned} & \int_{[-C_*\varepsilon, C_*\varepsilon]^2} \int_{n_m(0,0) \cdot v \leq C_5(1+|v|)\varepsilon} \\ & \quad \mathbf{1}_{\mathcal{O}_{\varepsilon, C_*\varepsilon}}(x_1 - y_1, x_2 - y_2, \eta_m(x_1, x_2) - y_3, v) \\ & \quad \times e^{-\theta|v|^2} e^{-\frac{\theta}{2}|v+u|^2} |n_m(x-y) \cdot v| dv \sqrt{1 + |\nabla \eta_m(x_1, x_2)|^2} dx_1 dx_2 \\ = & \underbrace{\int_{[-C_*\varepsilon, C_*\varepsilon]^2} \int_{\left\{-s_*C_2\sqrt{\varepsilon} \leq n_m(0,0) \cdot \frac{v}{|v|} \leq C_5 \frac{1+|v|}{|v|}\varepsilon\right\}} \cdots}_{(\text{I})} \\ & + \underbrace{\int_{[-C_*\varepsilon, C_*\varepsilon]^2} \int_{\left\{-1 \leq n_m(0,0) \cdot \frac{v}{|v|} \leq -s_*C_2\sqrt{\varepsilon}\right\}} \cdots}_{(\text{II})}. \end{aligned} \quad (74)$$

First we consider (I). If $-s_*C_2\sqrt{\varepsilon} \leq n_m(0,0) \cdot \frac{v}{|v|} \leq 0$ then $0 \leq v_3 = -n_m(0,0) \cdot v \leq s_*C_2|v|\sqrt{\varepsilon}$ and for $0 < \varepsilon \ll 1$

$$0 \leq v_3 \leq 2s_*C_2\sqrt{|v_1|^2 + |v_2|^2}\sqrt{\varepsilon}.$$

Moreover

$$\begin{aligned} |n_m(x-y) \cdot v| & \leq |n_m(0,0) \cdot v| + \|n_m\|_{C^1([-C_*\varepsilon, C_*\varepsilon]^2)}(C_* + 1/\tilde{C})|v|\varepsilon \\ & \leq s_*C_2|v|\sqrt{\varepsilon} + 4\|\eta_m\|_{C^2([-C_*\varepsilon, C_*\varepsilon]^2)}(C_* + 1/\tilde{C})|v|\varepsilon. \end{aligned}$$

If $n_m(0,0) \cdot \frac{v}{|v|} \leq C_5 \frac{1+|v|}{|v|}\varepsilon$ then for $0 < \varepsilon \ll 1$

$$|v_3| = |n_m(0,0) \cdot v| \leq 2C_5(1 + \sqrt{|v_1|^2 + |v_2|^2})\varepsilon.$$

Therefore,

$$\begin{aligned}
(\mathbf{I}) &= \int_{[-C_*\varepsilon, C_*\varepsilon]^2} \int_{0 \leq v_3 \leq 2s_*C_2\sqrt{|v_1|^2+|v_2|^2}\sqrt{\varepsilon}} \\
&\quad e^{-\theta|v|^2} \{s_*C_2|v|\sqrt{\varepsilon} + 4\|\eta_m\|_{C^2([-C_*\varepsilon, C_*\varepsilon]^2)}(C_* + 1/\tilde{C})|v|\varepsilon\} \\
&\quad + \int_{[-C_*\varepsilon, C_*\varepsilon]^2} \int_{|v_3| \leq 2C_5(1+\sqrt{|v_1|^2+|v_2|^2})\varepsilon} e^{-\theta|v|^2} \\
&\lesssim m_2([-C_*\varepsilon, C_*\varepsilon]^2) \times \left\{ \right. \\
&\quad \sqrt{\varepsilon} \iint_{\mathbb{R}^2} dv_1 dv_2 e^{-\frac{\theta}{2}|v_1|^2} e^{-\frac{\theta}{2}|v_2|^2} \int_0^{2s_*C_2\sqrt{|v_1|^2+|v_2|^2}\sqrt{\varepsilon}} dv_3 \\
&\quad \left. + \iint_{\mathbb{R}^2} dv_1 dv_2 e^{-\theta|v_1|^2} e^{-\theta|v_2|^2} \int_0^{2C_5(1+\sqrt{|v_1|^2+|v_2|^2})\varepsilon} dv_3 \right\} \\
&\lesssim \varepsilon^3.
\end{aligned} \tag{75}$$

We decompose (\mathbf{II}) , according to Lemma 3:

$$\begin{aligned}
(\mathbf{II}) &= \int_{[-C_*\varepsilon, C_*\varepsilon]^2} \int_{|v| < \varepsilon^{1/3}} \\
&\quad + \int_{[-C_*\varepsilon, C_*\varepsilon]^2} \int_{\left\{ -1 \leq n_m(0,0) \cdot \frac{v}{|v|} \leq -s_*C_2\sqrt{\varepsilon} \text{ and } |v| \geq \varepsilon^{1/3} \right\}}.
\end{aligned}$$

The first term is clearly bounded by $O(1)\varepsilon^3$. For the second term we use (62) to have

$$\begin{aligned}
&\left\{ -1 \leq n_m(0,0) \cdot \frac{v}{|v|} \leq -s_*C_2\sqrt{\varepsilon} \text{ and } |v| \geq \varepsilon^{1/3} \right\} \\
&\subset \left\{ |n_m(0,0) \cdot \frac{v}{|v|}| \leq C_4\sqrt{\varepsilon} \text{ and } |v| \geq \varepsilon^{1/3} \right\}.
\end{aligned}$$

Therefore we follow the same proof as for (75) to obtain

$$\begin{aligned}
(\mathbf{II}) &\lesssim \varepsilon^3 + \int_{[-C_*\varepsilon, C_*\varepsilon]^2} \int_{|v_3| \leq 2C_4\sqrt{|v_1|^2+|v_2|^2}\sqrt{\varepsilon}} \\
&\quad \times e^{-\theta|v|^2} \{C_4|v|\sqrt{\varepsilon} + 4\|\eta_m\|_{C^2([-C_*\varepsilon, C_*\varepsilon]^2)}(C_* + 1/\tilde{C})|v|\varepsilon\} \\
&\lesssim \varepsilon^3.
\end{aligned} \tag{76}$$

We conclude (74) from (75) and (76).

3 New Trace Theorem via the Double Iteration

In this section we prove the following geometric result. For the later purpose (this will be used in the approximation scheme for the nonlinear problem with diffuse BC) we state the result for the sequence of solutions.

Proposition 2 Let $h_0 \in L^1(\Omega \times \mathbb{R}^3)$. Let $(h^m)_{m \geq 0} \subset L^\infty([0, T]; L^1(\Omega \times \mathbb{R}^3)) \cap L^1([0, T]; L^1(\gamma_+, d\gamma))$ solve

$$\{\partial_t + v \cdot \nabla_x + \nu\} h^{m+1} = H^m, \quad h^{m+1}|_{t=0} = h_0, \quad (77)$$

where $\nu = \nu(t, x, v) \geq 0$, and such that the following inequality holds for all $(x, v) \in \gamma_-$

$$\begin{aligned} & |h^{m+1}(t, x, v)| \\ & \leq C_1 \sqrt{\mu(v)} \left(1 + \frac{\langle v \rangle}{|n(x) \cdot v|}\right) \int_{n(x) \cdot u > 0} |h^m(t, x, u)| \mu(u)^{\frac{1}{4}} \{n(x) \cdot u\} du \\ & + \left(1 + \frac{e^{-C_2|v|^2}}{|n(x) \cdot v|}\right) R^m, \end{aligned} \quad (78)$$

where $H^m \in L^1([0, T]; L^1(\Omega \times \mathbb{R}^3))$ and $R^m \in L^1([0, T]; L^1(\partial\Omega \times \mathbb{R}^3, \langle v \rangle dS_x dv))$.

Then for all $m \geq 1$, $h_{\gamma_-}^{m+1} \in L^1([0, T]; L^1(\gamma_-, d\gamma))$ and satisfies, for $t \in [0, T]$ and $0 < \delta \ll 1$,

$$\begin{aligned} & \int_0^t |h^{m+1}(s)|_{\gamma_-, 1} \\ & \leq O(\delta) \int_0^t |h^{m-1}(s)|_{\gamma_+, 1} + C_\delta \|h_0\|_1 \\ & + C_\delta \max_{i=m, m-1} \left\{ \int_0^t \|h^i(s)\|_1 + \int_0^t |\langle v \rangle R^i(s)|_1 + \int_0^t \|H^i(s)\|_1 \right\}. \end{aligned} \quad (79)$$

Our proof requires the following lemma:

Lemma 4 Let $\Omega \subset \mathbb{R}^3$ be an open bounded set with a smooth boundary $\partial\Omega$. For $k \in \mathbb{N}$, consider the map

$$\begin{aligned} \Phi_k : \quad & \{(x, v) \in \gamma_+ : n(x_{\mathbf{b}}(x, v)) \cdot v < -1/k\} \\ & \rightarrow \{(x_{\mathbf{b}}, v) \in \gamma_- : n(x_{\mathbf{b}}) \cdot v < -1/k\}, \\ & (x, v) \mapsto \Phi_k(x, v) := (\tilde{x}, v) := (x_{\mathbf{b}}(x, v), v). \end{aligned}$$

Then Φ_k is one-to-one and we have a change of variables formula for all $k \in \mathbb{N}$:

$$\mathbf{1}_{\{n(\tilde{x}) \cdot v < -1/k\}} |n(\tilde{x}) \cdot v| dv dS_{\tilde{x}} = \mathbf{1}_{\{n(x_{\mathbf{b}}(x, v)) \cdot v < -1/k\}} |n(x) \cdot v| dv dS_x.$$

Proof (Proof of Lemma 4)

Let $(x, v), (x', v') \in \gamma_+$ such that $n(x_{\mathbf{b}}(x, v)) \cdot v, n(x_{\mathbf{b}}(x', v')) \cdot v' < -1/k$. If $\Phi_k(x, v) = \Phi_k(x', v')$ then $v = v'$ and $x_{\mathbf{b}}(x, v) = x_{\mathbf{b}}(x', v)$. Since $x = x_{\mathbf{f}}(x_{\mathbf{b}}(x, v), v) = x_{\mathbf{f}}(x_{\mathbf{b}}(x', v), v) = x'$ the mapping Φ_k is one-to-one.

Now we prove the change of variables formula. It suffices to consider a small neighborhood of $\partial\Omega$ around x . Without loss of generality we may assume $x_3 = \eta(x_1, x_2)$ for some $\eta : \mathbb{R}^2 \rightarrow \mathbb{R}$. First we consider the case $(0, 0, 1) \cdot n(x_{\mathbf{b}}(x, v)) \neq 0$ so that

$$\tilde{x} = x_{\mathbf{b}}(x, v) = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = (\tilde{x}_1, \tilde{x}_2, \varphi(\tilde{x}_1, \tilde{x}_2)) \in \partial\Omega,$$

for some function $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$.

The change of variable is given by

$$\begin{aligned} dS_{\tilde{x}} dv &= \sqrt{1 + |\nabla \varphi|^2} d\tilde{x}_1 d\tilde{x}_2 dv \\ &= \frac{\sqrt{1 + |\nabla \varphi|^2}}{\sqrt{1 + |\nabla \eta|^2}} J \sqrt{1 + |\nabla \eta|^2} dx_1 dx_2 dv \\ &= \frac{\sqrt{1 + |\nabla \varphi|^2}}{\sqrt{1 + |\nabla \eta|^2}} J dS_x dv. \end{aligned} \quad (80)$$

where J is the Jacobian,

$$J = \left| \frac{\partial(\tilde{x}_1, \tilde{x}_2, v_1, v_2, v_3)}{\partial(x_1, x_2, v_1, v_2, v_3)} \right| = \left| \begin{array}{cc} \partial_{x_1} \tilde{x}_1 & \partial_{x_2} \tilde{x}_1 \\ \partial_{x_1} \tilde{x}_2 & \partial_{x_2} \tilde{x}_2 \end{array} \right|.$$

By the definition of $x_b(x, v)$, we have the following identity: $v|x - \tilde{x}| = |v|(x - \tilde{x})$, i.e.

$$\begin{aligned} &\{(x_1 - \tilde{x}_1)^2 + (x_2 - \tilde{x}_2)^2 + [\eta(x_1, x_2) - \varphi(\tilde{x}_1, \tilde{x}_2)]^2\}^{\frac{1}{2}} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \\ &= |v| \begin{pmatrix} x_1 - \tilde{x}_1 \\ x_2 - \tilde{x}_2 \\ \eta(x_1, x_2) - \varphi(\tilde{x}_1, \tilde{x}_2) \end{pmatrix}. \end{aligned} \quad (81)$$

Denote $D = \{(x_1 - \tilde{x}_1)^2 + (x_2 - \tilde{x}_2)^2 + [\eta(x_1, x_2) - \varphi(\tilde{x}_1, \tilde{x}_2)]^2\}$. Directly from (81)

$$\begin{aligned} &\begin{bmatrix} [(x_1 - \tilde{x}_1) + (\eta - \varphi)\partial_{\tilde{x}_1}\varphi] D^{-\frac{1}{2}} v_1 - |v| & [(x_2 - \tilde{x}_2) + (\eta - \varphi)\partial_{\tilde{x}_2}\varphi] D^{-\frac{1}{2}} v_1 \\ [(x_1 - \tilde{x}_1) + (\eta - \varphi)\partial_{\tilde{x}_1}\varphi] D^{-\frac{1}{2}} v_2 & [(x_2 - \tilde{x}_2) + (\eta - \varphi)\partial_{\tilde{x}_2}\varphi] D^{-\frac{1}{2}} v_2 - |v| \end{bmatrix} \\ &\times \begin{bmatrix} \partial_{x_1} \tilde{x}_1 & \partial_{x_2} \tilde{x}_1 \\ \partial_{x_1} \tilde{x}_2 & \partial_{x_2} \tilde{x}_2 \end{bmatrix} \\ &= \begin{bmatrix} v_1 D^{-\frac{1}{2}} ((x_1 - \tilde{x}_1) + (\eta - \varphi)\partial_{x_1}\eta) - |v| & v_1 D^{-\frac{1}{2}} ((x_2 - \tilde{x}_2) + (\eta - \varphi)\partial_{x_2}\eta) \\ v_2 D^{-\frac{1}{2}} ((x_1 - \tilde{x}_1) + (\eta - \varphi)\partial_{x_1}\eta) & v_2 D^{-\frac{1}{2}} ((x_2 - \tilde{x}_2) + \partial_{x_2}\eta(\eta - \varphi)) - |v| \end{bmatrix}. \end{aligned}$$

Direct computations yield

$$\begin{aligned} J &= \frac{|v| - D^{-\frac{1}{2}} [v_1(x_1 - \tilde{x}_1) + v_1(\eta - \varphi)\partial_{x_1}\eta + v_2(x_2 - \tilde{x}_2) + v_2(\eta - \varphi)\partial_{x_2}\eta]}{|v| - D^{-\frac{1}{2}} [v_2(x_2 - \tilde{x}_2) + v_2(\eta - \varphi)\partial_{\tilde{x}_2}\varphi + v_1(x_1 - \tilde{x}_1) + v_1(\eta - \varphi)\partial_{\tilde{x}_1}\varphi]} \\ &= \frac{|v|^2 - [(v_1)^2 + (v_2)^2 + (v_3)(v_1\partial_{x_1}\eta + v_2\partial_{x_2}\eta)]}{|v|^2 - [(v_1)^2 + (v_2)^2 + (v_3)(v_1\partial_{\tilde{x}_1}\varphi + v_2\partial_{\tilde{x}_2}\varphi)]} = \frac{(\partial_{x_1}\eta, \partial_{x_2}\eta, -1) \cdot v}{(\partial_{\tilde{x}_1}\varphi, \partial_{\tilde{x}_2}\varphi, -1) \cdot v} \\ &= \frac{\sqrt{1 + |\nabla \eta|^2}}{\sqrt{1 + |\nabla \varphi|^2}} \times \frac{n(x) \cdot v}{n(\tilde{x}) \cdot v}. \end{aligned}$$

Then we use (80) to conclude the proof.

Secondly we consider the case of $n_1(x_{\mathbf{b}}(x, v)) \neq 0$ or $n_2(x_{\mathbf{b}}(x, v)) \neq 0$. Without loss of generality we may assume $n_2(x_{\mathbf{b}}(x, v)) \neq 0$ so that

$$\tilde{x} = x_{\mathbf{b}}(x, v) = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = (\tilde{x}_1, \varphi(\tilde{x}_1, \tilde{x}_3), \tilde{x}_3),$$

for some function $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$. Notice that (80) still holds with \tilde{x}_2 replaced by \tilde{x}_3 . From the fact that $v|x - \tilde{x}| = |v|(x - \tilde{x})$ we have

$$\begin{aligned} & \{(x_1 - \tilde{x}_1)^2 + (x_2 - \varphi(\tilde{x}_1, \tilde{x}_3))^2 + [\eta(x_1, x_2) - \tilde{x}_3]^2\}^{\frac{1}{2}} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \\ &= |v| \begin{pmatrix} x_1 - \tilde{x}_1 \\ x_2 - \varphi(\tilde{x}_1, \tilde{x}_3) \\ \eta(x_1, x_2) - \tilde{x}_3 \end{pmatrix}. \end{aligned} \quad (82)$$

We define $\tilde{D} = \{(x_1 - \tilde{x}_1)^2 + (x_2 - \varphi(\tilde{x}_1, \tilde{x}_3))^2 + [\eta(x_1, x_2) - \tilde{x}_3]^2\}$. By direct computation

$$\begin{aligned} & \begin{bmatrix} [(x_1 - \tilde{x}_1) + (x_2 - \varphi)\partial_{\tilde{x}_1}\varphi]v_1\tilde{D}^{-\frac{1}{2}} - |v| & [(x_2 - \varphi)\partial_{\tilde{x}_3}\varphi + (\eta - \tilde{x}_3)]v_1\tilde{D}^{-\frac{1}{2}} \\ [(x_1 - \tilde{x}_1) + (x_2 - \varphi)\partial_{\tilde{x}_1}\varphi]v_3\tilde{D}^{-\frac{1}{2}} & [(x_2 - \varphi)\partial_{\tilde{x}_3}\varphi + (\eta - \tilde{x}_3)]v_3\tilde{D}^{-\frac{1}{2}} - |v| \end{bmatrix} \\ & \times \begin{bmatrix} \partial_{x_1}\tilde{x}_1 & \partial_{x_2}\tilde{x}_1 \\ \partial_{x_1}\tilde{x}_3 & \partial_{x_2}\tilde{x}_3 \end{bmatrix} \\ &= \begin{bmatrix} [(x_1 - \tilde{x}_1) + (\eta - \tilde{x}_3)\partial_{x_1}\eta]v_1\tilde{D}^{-1/2} - |v| & [(x_2 - \varphi) + (\eta - \tilde{x}_3)\partial_{x_2}\eta]v_1\tilde{D}^{-1/2} \\ [(x_1 - \tilde{x}_1) + (\eta - \tilde{x}_3)\partial_{x_1}\eta]v_3\tilde{D}^{-1/2} - |v|\partial_{x_1}\eta & [(x_2 - \varphi) + (\eta - \tilde{x}_3)\partial_{x_2}\eta]v_3\tilde{D}^{-1/2} - |v|\partial_{x_2}\eta \end{bmatrix}, \end{aligned}$$

and

$$\begin{aligned} & \det \begin{bmatrix} \partial_{x_1}\tilde{x}_1 & \partial_{x_2}\tilde{x}_1 \\ \partial_{x_1}\tilde{x}_3 & \partial_{x_2}\tilde{x}_3 \end{bmatrix} \\ &= \frac{|v|^2\partial_{x_2}\eta - [(x_1 - \tilde{x}_1) + (\eta - \tilde{x}_3)\partial_{x_1}\eta]v_1|v|\tilde{D}^{-\frac{1}{2}}\partial_{x_2}\eta + [(x_2 - \varphi) + (\eta - \tilde{x}_3)\partial_{x_2}\eta]\tilde{D}^{-\frac{1}{2}}|v|(v_1\partial_{x_1}\eta - v_3)}{|v|^2 - [(x_1 - \tilde{x}_1) + (x_2 - \varphi)\partial_{\tilde{x}_1}\varphi]|v|v_1\tilde{D}^{-\frac{1}{2}} - [(x_2 - \varphi)\partial_{\tilde{x}_3}\varphi + (\eta - \tilde{x}_3)]|v|v_3\tilde{D}^{-\frac{1}{2}}} \\ &= \frac{v_1\partial_{x_1}\eta + v_2\partial_{x_2}\eta - v_3}{-v_1\partial_{\tilde{x}_1}\varphi + v_2 - v_3\partial_{\tilde{x}_3}\varphi} = \frac{(\partial_{x_1}\eta, \partial_{x_2}\eta, -1) \cdot v}{-(\partial_{\tilde{x}_1}\varphi, -1, \partial_{\tilde{x}_3}\varphi) \cdot v} \\ &= -\frac{\sqrt{1 + |\nabla\eta|^2}}{\sqrt{1 + |\nabla\varphi|^2}} \times \frac{n(x) \cdot v}{n(\tilde{x}) \cdot v}. \end{aligned}$$

Then we use (80) (with \tilde{x}_2 replaced by \tilde{x}_3) to conclude the proof.

Proof (Proof of Proposition 2)

It suffices to prove the estimate (79).

Using (78), we obtain

$$\begin{aligned} \int_0^t |h^{m+1}(s)|_{\gamma_{-,1}} &:= \int_0^t \iint_{n(x) \cdot v < 0} |h^{m+1}(s, x, v)| |n(x) \cdot v| dS_x dv ds \\ &\lesssim (A) + (B), \end{aligned}$$

where

$$\begin{aligned} (A) &:= \int_0^t \iint_{n(x) \cdot v > 0} |h^m(s, x, v)| \mu(v)^{\frac{1}{4}} |n(x) \cdot v| dS_x dv ds, \\ (B) &:= \int_0^t \iint_{n(x) \cdot v < 0} |R^m(s, x, v)| \{1 + |n(x) \cdot v|\} dS_x dv ds. \end{aligned}$$

Clearly the last term (B) is bounded by the RHS of (79).

Focus on (A). Recall the almost grazing set γ_+^δ and the non-grazing set $\gamma_+ \setminus \gamma_+^\delta$ in (16) and (17). We split the outgoing part as

$$\gamma_+ = \gamma_+^\delta \cup \gamma_+ \setminus \gamma_+^\delta.$$

Due to Lemma 7 in Appendix A, the non-grazing part $\gamma_+ \setminus \gamma_+^\delta$ of the integral is bounded as

$$\begin{aligned} \int_0^t \iint_{\gamma_+ \setminus \gamma_+^\delta} &\lesssim_{t,\delta,\Omega} \|h_0\|_1 + \int_0^t \{ \|h^m(s)\|_1 + \|[\partial_t + v \cdot \nabla_x + \nu]h^m(s)\|_1 \} ds \\ &\lesssim_{t,\delta,\Omega} \|h_0\|_1 + \int_0^t \|h^m\|_1 + \int_0^t \|H^{m-1}\|_1. \end{aligned} \quad (83)$$

For the almost grazing set γ_+^δ , we claim that the following truncated term with a number $k \in \mathbb{N}$ is uniformly bounded in k as follows:

Claim:

$$\begin{aligned} &\int_0^t \iint_{\substack{x \in \partial\Omega, \\ n(x) \cdot v > 0}} \mathbf{1}_{\{(x,v) \in \gamma_+^\delta\}} \mathbf{1}_{\{1/k < |n(x_{\mathbf{b}}(x,v)) \cdot v|\}} \\ &\quad |h^m(s, x, v)| \mu(v)^{\frac{1}{4}} \{n(x) \cdot v\} dv dS_x ds \\ &\leq O(\delta) \int_0^t |h^{m-1}(s)|_{\gamma_+,1} \\ &\quad + C_\delta \left\{ \|h_0\|_1 + \int_0^t \|h^{m-1}(s)\|_1 + \int_0^t \|H^{m-1}\|_1 + t \|R^{m-1}\|_1 \right\}. \end{aligned} \quad (84)$$

Proof of Claim (84): In order to show (84) we use the Duhamel formula of the equation (77) together with (78): for $(x, v) \in \gamma_+^\delta$ and $\frac{1}{k} < |n(x_{\mathbf{b}}(x, v)) \cdot v|$

$$\begin{aligned} &|h^m(s, x, v)| \mathbf{1}_{\{(x,v) \in \gamma_+^\delta\}} \mathbf{1}_{\{1/k < |n(x_{\mathbf{b}}(x,v)) \cdot v|\}} \\ &\leq \mathbf{1}_{\{s < t_{\mathbf{b}}(x,v)\}} |h_0(x - sv, v)| + \int_{\max\{0, s - t_{\mathbf{b}}(x,v)\}}^s |H^{m-1}(\tau, x - (s - \tau)v, v)| d\tau \\ &\quad + \mathbf{1}_{\{s > t_{\mathbf{b}}(x,v)\}} \mathbf{1}_{\{1/k < |n(x_{\mathbf{b}}(x,v)) \cdot v|\}} C_1 \sqrt{\mu(v)} \left(1 + \frac{\langle v \rangle}{|n(x_{\mathbf{b}}(x, v)) \cdot v|} \right) \\ &\quad \times \int_{n(x_{\mathbf{b}}(x,v)) \cdot v_1 > 0} |h^{m-1}(s - t_{\mathbf{b}}(x, v), x_{\mathbf{b}}(x, v), v_1)| \mu(v_1)^{\frac{1}{4}} \{n(x_{\mathbf{b}}(x, v)) \cdot v_1\} dv_1 \\ &\quad + \mathbf{1}_{\{s > t_{\mathbf{b}}(x,v)\}} \mathbf{1}_{\{1/k < |n(x_{\mathbf{b}}(x,v)) \cdot v|\}} \left(1 + \frac{e^{-C_2|v|^2}}{|n(x_{\mathbf{b}}(x, v)) \cdot v|} \right) \\ &\quad \times |R^{m-1}(s - t_{\mathbf{b}}(x, v), x_{\mathbf{b}}(x, v), v)|. \end{aligned}$$

We plug this estimate into the left hand side of (84) to have

$$\begin{aligned} & \int_0^t \iint_{\substack{x \in \partial\Omega, \\ n(x) \cdot v > 0}} \mathbf{1}_{\{(x,v) \in \gamma_+^\delta\}} \mathbf{1}_{\{1/k < |n(x_{\mathbf{b}}(x,v)) \cdot v|\}} |h^m(s, x, v)| \mu(v)^{\frac{1}{4}} \{n(x) \cdot v\} dv dS_x ds \\ & \leq \int_0^t \iint_{\gamma_+^\delta} \mathbf{1}_{\{1/k < |n(x_{\mathbf{b}}(x,v)) \cdot v|\}} |h_0(x - sv, v)| \mu(v)^{\frac{1}{4}} |n(x) \cdot v| dS_x dv ds \end{aligned} \quad (85)$$

$$\begin{aligned} & + \int_0^t \iint_{\gamma_+^\delta} \mathbf{1}_{\{1/k < |n(x_{\mathbf{b}}(x,v)) \cdot v|\}} \mu(v)^{\frac{1}{4}} |n(x) \cdot v| \\ & \quad \times \int_{\max\{0, s - t_{\mathbf{b}}(x,v)\}}^s |H^{m-1}(\tau, x - (s - \tau)v, v)| d\tau dS_x dv ds \end{aligned} \quad (86)$$

$$\begin{aligned} & + \int_0^t \iint_{\gamma_+^\delta} \mathbf{1}_{\{1/k < |n(x_{\mathbf{b}}(x,v)) \cdot v|\}} \mu(v)^{\frac{1}{2}} \frac{|n(x) \cdot v|}{|n(x_{\mathbf{b}}(x,v)) \cdot v|} \int_{n(x_{\mathbf{b}}(x,v)) \cdot v_1 > 0} \mathbf{1}_{\{s > t_{\mathbf{b}}(x,v)\}} \\ & \quad \times |h^{m-1}(s - t_{\mathbf{b}}(x,v), x_{\mathbf{b}}(x,v), v_1)| \mu(v_1)^{\frac{1}{4}} \{n(x_{\mathbf{b}}(x,v)) \cdot v_1\} dv_1 dS_x dv ds \end{aligned} \quad (87)$$

$$\begin{aligned} & + \int_0^t \iint_{\gamma_+^\delta} \mathbf{1}_{\{s > t_{\mathbf{b}}(x,v)\}} \mathbf{1}_{\{1/k < |n(x_{\mathbf{b}}(x,v)) \cdot v|\}} \mu(v)^{\frac{1}{4}} \frac{|n(x) \cdot v|}{|n(x_{\mathbf{b}}(x,v)) \cdot v|} \\ & \quad \times |R^{m-1}(s - t_{\mathbf{b}}(x,v), x_{\mathbf{b}}(x,v), v)| dS_x dv ds. \end{aligned} \quad (88)$$

Estimate of (85): Note that $x \in \partial\Omega$ in (85). Without loss of generality we may assume that there exists $\eta : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that $x_3 = \eta(x_1, x_2)$. We apply the following change of variables: for fixed $v \in \mathbb{R}^3$,

$$\begin{aligned} (x_1, x_2; s) & \in \mathbb{R}^2 \times \{0 \leq s \leq t_{\mathbf{b}}(x, v)\} \\ & \mapsto y = (x_1 - sv_1, x_2 - sv_2, \eta(x_1, x_2) - sv_3) \in \bar{\Omega}. \end{aligned}$$

Clearly such mapping is one-to-one.

We compute the Jacobian:

$$\begin{aligned} \det \left(\frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, s)} \right) & = \det \begin{pmatrix} 1 & 0 & -v_1 \\ 0 & 1 & -v_2 \\ \partial_{x_1}\eta(x_1, x_2) & \partial_{x_2}\eta(x_1, x_2) & -v_3 \end{pmatrix} \\ & = v \cdot \begin{pmatrix} \partial_{x_1}\eta \\ \partial_{x_2}\eta \\ -1 \end{pmatrix} = v \cdot n \sqrt{1 + |\partial_{x_1}\eta|^2 + |\partial_{x_2}\eta|^2}. \end{aligned}$$

Therefore

$$\{v \cdot n(x)\} dS_x ds = \{v \cdot n(x)\} \sqrt{1 + |\partial_{x_1}\eta|^2 + |\partial_{x_2}\eta|^2} dx_1 dx_2 ds = dy = dy_1 dy_2 dy_3,$$

and

$$\begin{aligned} (85) & \leq \int_{\mathbb{R}^3} dv \int_0^t ds \int_{\partial\Omega} dS_x \\ & \quad \mathbf{1}_{\{(x,v) \in \gamma_+\}} \mathbf{1}_{\{1/k < |n(x_{\mathbf{b}}(x,v)) \cdot v|\}} |h_0(x - sv, v)| \mu(v)^{\frac{1}{4}} |n(x) \cdot v| \\ & \leq \int_{\mathbb{R}^3} dv \int_{\Omega} dy |h_0(y, v)| \mu(v)^{\frac{1}{4}} \\ & \leq \|h_0\|_1. \end{aligned} \quad (89)$$

Estimate of (86): Considering the region of $\{(\tau, s) \in [0, t] \times [0, t] : \max\{0, s - t_{\mathbf{b}}(x, v)\} \leq \tau \leq s\}$,

$$(86) \leq \int_{\mathbb{R}^3} dv \int_0^t d\tau \int_{\tau}^{\min\{t, \tau + t_{\mathbf{b}}(x, v)\}} ds \int_{\partial\Omega} dS_x |H^{m-1}(\tau, x - (s - \tau)v, v)| \mu(v)^{\frac{1}{4}} |n(x) \cdot v|. \quad (90)$$

Note that $x \in \partial\Omega$. Without loss of generality we may assume that $x_3 = \eta(x_1, x_2)$ for $\eta : \mathbb{R}^2 \rightarrow \mathbb{R}$. We apply the change of variables: for fixed $v \in \mathbb{R}^3$ and $\tau \in [0, t]$,

$$(x_1, x_2; s) \in \mathbb{R}^2 \times [\tau, \min\{t, \tau + t_{\mathbf{b}}(x, v)\}] \\ \mapsto y \equiv (x_1 - (s - \tau)v_1, x_2 - (s - \tau)v_2, \eta(x_1, x_2) - (s - \tau)v_3).$$

The Jacobian is $\{v \cdot n(x)\} \sqrt{1 + |\partial_{x_1} \eta|^2 + |\partial_{x_2} \eta|^2}$ and $\{v \cdot n(x)\} ds dS_x \leq dy$. Applying the change of variables to (90) we have

$$(86) \leq \int_0^t \int_{\mathbb{R}^3} \int_{\Omega} |H^{m-1}(\tau, y, v)| \mu(v)^{\frac{1}{4}} dy dv d\tau. \quad (91)$$

Estimate of (87): This part is the most delicate among (85)–(88). Rewrite (87) as

$$\int_0^t ds \int_{\partial\Omega} dS_x \int_{\mathbb{R}^3} dv \int_{\mathbb{R}^3} dv_1 \mathbf{1}_{\{(x, v) \in \gamma_+^s\}} \mathbf{1}_{\{n(x_{\mathbf{b}}(x, v)) \cdot v_1 > 0\}} \\ \mathbf{1}_{\{s > t_{\mathbf{b}}(x, v)\}} \mathbf{1}_{\{|n(x_{\mathbf{b}}(x, v)) \cdot v| > 1/k\}} \mu(v_1)^{\frac{1}{4}} \mu(v)^{\frac{1}{2}} \frac{|n(x) \cdot v|}{|n(x_{\mathbf{b}}(x, v)) \cdot v|} \\ |n(x_{\mathbf{b}}(x, v)) \cdot v_1| |h^{m-1}(s - t_{\mathbf{b}}(x, v), x_{\mathbf{b}}(x, v), v_1)|. \quad (92)$$

First we apply the following change of variables

$$s \in [0, t] \mapsto \tilde{s} = s - t_{\mathbf{b}}(x, v) \in [0, t - t_{\mathbf{b}}(x, v)], \quad (93)$$

where we have used the fact that s is integrated over $[t_{\mathbf{b}}(x, v), t]$. Clearly the Jacobian is 1 so that $d\tilde{s} = ds$ and hence

$$(92) \leq \int_0^t d\tilde{s} \int_{\partial\Omega} dS_x \int_{\mathbb{R}^3} dv \int_{\mathbb{R}^3} dv_1 \underbrace{\mathbf{1}_{\{(x, v) \in \gamma_+^s\}}}_{\mathbf{1}_{\{n(x_{\mathbf{b}}(x, v)) \cdot v_1 > 0\}}} \\ \mathbf{1}_{\{|n(x_{\mathbf{b}}(x, v)) \cdot v| > 1/k\}} \mu(v_1)^{\frac{1}{4}} \mu(v)^{\frac{1}{2}} \frac{|n(x) \cdot v|}{|n(x_{\mathbf{b}}(x, v)) \cdot v|} |n(x_{\mathbf{b}}(x, v)) \cdot v_1| \\ |h^{m-1}(\tilde{s}, x_{\mathbf{b}}(x, v), v_1)|. \quad (94)$$

Let us denote

$$\tilde{x} := x_{\mathbf{b}}(x, v). \quad (95)$$

Note that since $(x, v) \in \gamma_+$ and $|n(x_{\mathbf{b}}(x, v)) \cdot v| > 1/k$, from Lemma 4, the mapping $(x, v) \mapsto (\tilde{x}, v)$ is one-to-one and

$$\begin{aligned} t_{\mathbf{b}}(x, v) &= t_{\mathbf{b}}(x_{\mathbf{b}}(x, v), -v), \\ x &= x_{\mathbf{b}}(x, v) + t_{\mathbf{b}}(x, v)v = x_{\mathbf{b}}(x, v) + t_{\mathbf{b}}(x_{\mathbf{b}}(x, v), -v)v \\ &= x_{\mathbf{b}}(x, v) - t_{\mathbf{b}}(x_{\mathbf{b}}(x, v), -v)(-v) \\ &= \tilde{x} - t_{\mathbf{b}}(\tilde{x}, -v)(-v), \end{aligned}$$

and hence we can rewrite the underbraced term in (94) as

$$\mathbf{1}_{\{(x, v) \in \gamma_+^\delta\}} = \mathbf{1}_{\{0 < n(\tilde{x} - t_{\mathbf{b}}(\tilde{x}, -v)(-v)) \cdot v < \delta \text{ or } |v| > 1/\delta\}}. \quad (96)$$

Now we apply the change of variables of Lemma 4: for $(x, v) \in \gamma_+$ and $|n(x_{\mathbf{b}}(x, v)) \cdot v| = |n(\tilde{x}) \cdot v| > 1/k$, we apply the change of variables

$$(x, v) \mapsto (\tilde{x}, v) := (x_{\mathbf{b}}(x, v), v). \quad (97)$$

From Lemma 4, the Jacobian is

$$\begin{aligned} \det \left(\frac{\partial(\tilde{x}, v)}{\partial(x, v)} \right) &= \det \left(\frac{\partial \tilde{x}}{\partial x} \right) = \left| \frac{n(x) \cdot v}{n(\tilde{x}) \cdot v} \right| \frac{\sqrt{1 + |\nabla \eta|^2}}{\sqrt{1 + |\nabla \varphi|^2}}, \\ \text{and } dS_{\tilde{x}} &:= \left| \frac{n(x) \cdot v}{n(\tilde{x}) \cdot v} \right| dS_x. \end{aligned}$$

Then from (94) and (96),

$$\begin{aligned} (92) &\leq \\ &\int_0^t d\tilde{s} \int_{\mathbb{R}^3} dv_1 \int_{\mathbb{R}^3} dv \int_{\partial\Omega} dS_{\tilde{x}} \{ \mathbf{1}_{0 < n(\tilde{x} - t_{\mathbf{b}}(\tilde{x}, -v)(-v)) \cdot v < \delta} + \mathbf{1}_{|v| > 1/\delta} \} \\ &\quad \times \mathbf{1}_{\{n(\tilde{x}) \cdot v_1 > 0\}} \mathbf{1}_{\{|n(\tilde{x}) \cdot v| > 1/k\}} \mu(v)^{\frac{1}{2}} \mu(v_1)^{\frac{1}{4}} |n(\tilde{x}) \cdot v_1| |h^{m-1}(\tilde{s}, \tilde{x}, v_1)| \\ &\leq \int_0^t \iint_{\gamma_+} |h^{m-1}(\tilde{s}, \tilde{x}, v_1)| \mu(v_1)^{\frac{1}{4}} |n(\tilde{x}) \cdot v_1| dS_{\tilde{x}} dv_1 d\tilde{s} \\ &\quad \times \underbrace{\sup_{\tilde{x} \in \partial\Omega} \int_{\mathbb{R}^3} \mathbf{1}_{\{-\delta < n(\tilde{x} - t_{\mathbf{b}}(\tilde{x}, -v)(-v)) \cdot (-v) < 0\}} \mu(v)^{\frac{1}{2}} dv}_{\text{}} \\ &\quad + O(\delta) \int_0^t \iint_{\gamma_+} |h^{m-1}(\tilde{s}, \tilde{x}, v_1)| \mu(v_1)^{\frac{1}{4}} |n(\tilde{x}) \cdot v_1| dS_{\tilde{x}} dv_1 d\tilde{s}, \end{aligned} \quad (98)$$

where we extracted $O(\delta)$ from $\int_{\mathbb{R}^3} \mathbf{1}_{|v| > 1/\delta} \mu(v)^{\frac{1}{2}} dv \lesssim e^{-\frac{1}{10\delta}} \int_{\mathbb{R}^3} \mu(v)^{\frac{1}{4}} dv$.

We claim the following:

Claim : For any small $0 < \delta' \ll 1$, we can choose sufficiently small $0 < \delta \ll 1$ such that

$$\sup_{\tilde{x} \in \partial\Omega} \int_{\mathbb{R}^3} \mathbf{1}_{\{-\delta < n(\tilde{x} - t_{\mathbf{b}}(\tilde{x}, -v)(-v)) \cdot (-v) < 0\}} \mu(v)^{\frac{1}{2}} dv \leq \delta'. \quad (99)$$

This is a consequence of Lemma 9. For given $\delta' > 0$, we choose a sufficiently large $N \gg \frac{1}{\delta'}$ and we take $\delta_{\delta', N} > 0$ as in Lemma 9. Then we choose a sufficiently small $\delta = \delta(\delta', N) > 0$ such that $\delta \ll \delta_{\delta', N}$ in Lemma 9. Due to Lemma 9 and (121),

$$\begin{aligned} & \max_i \sup_{\tilde{x} \in B(x_i; r_i)} m_3\{v \in \mathbb{R}^3 : |v| \leq N, |n(x_{\mathbf{b}}(\tilde{x}, -v)) \cdot (-v)| \leq \delta\} \\ & \leq \max_i m_3(\mathcal{O}_{x_i}) \leq \delta'. \end{aligned}$$

Finally we conclude the claim (99) by

$$\begin{aligned} & \int_{\mathbb{R}^3} \mathbf{1}_{\{-\delta < n(x_{\mathbf{b}}(\tilde{x}, -v)) \cdot (-v) < 0\}} \mu(v)^{\frac{1}{2}} dv \\ & = \int_{|v| \geq N} + \int_{|v| \leq N} \\ & \leq e^{-N^2/4} + \max_i m_3(\mathcal{O}_i) \\ & \leq e^{-\frac{1}{4(\delta')^2}} + \delta'. \end{aligned}$$

Therefore, from (94), (98), (99), we have, for $0 < \delta, \delta' \ll 1$,

$$\begin{aligned} (87) & \lesssim [O(\delta) + O(\delta')] \\ & \times \int_0^t \int_{\gamma_+} |h^{m-1}(\tilde{s}, \tilde{x}, v_1)| \mu(v_1)^{\frac{1}{4}} |n(\tilde{x}) \cdot v_1| dS_{\tilde{x}} dv_1 d\tilde{s}. \end{aligned} \quad (100)$$

Estimate of (88): We apply the change of variables (93) and then apply (97) and use Lemma 4 to bound

$$(88) \lesssim \int_0^t \int_{\partial\Omega} \int_{\mathbb{R}^3} \mu(v)^{\frac{1}{4}} |R^{m-1}(\tilde{s}, \tilde{x}, v)| dS_{\tilde{x}} dv d\tilde{s}. \quad (101)$$

Finally from (89), (91), (100) and (101), we prove our claim (84).

The last step is to pass a limit $k \rightarrow \infty$. Clearly the sequence is non-decreasing in k :

$$0 \leq \mathbf{1}_{\{\frac{1}{k} < |n(x_{\mathbf{b}}(x, v)) \cdot v|\}} |h^m(s, x, v)| \leq \mathbf{1}_{\{\frac{1}{k+1} < |n(x_{\mathbf{b}}(x, v)) \cdot v|\}} |h^m(s, x, v)|.$$

We claim the following:

Claim : As $k \rightarrow \infty$,

$$\mathbf{1}_{\{\frac{1}{k} < |n(x_{\mathbf{b}}(x, v)) \cdot v|\}} \mu(v)^{\frac{1}{4}} |h^m(s, x, v)| \rightarrow \mu(v)^{\frac{1}{4}} |h^m(s, x, v)|,$$

a.e. $(x, v) \in \gamma_+$ with $d\gamma$. It suffices to show $\mathbf{1}_{\{\frac{1}{k} < |n(x_{\mathbf{b}}(x, v)) \cdot v|\}} \mu(v)^{\frac{1}{4}} \rightarrow \mu(v)^{\frac{1}{4}}$ a.e. on γ_+ . For $\varepsilon > 0$ and $N \gg \varepsilon^{-1} \gg 1$, choose $k \gg 1$ such that $\frac{1}{k} < \delta_{\varepsilon, N}$ in Lemma 9. Then

$$\begin{aligned} & \left[1 - \mathbf{1}_{\{|n(x_{\mathbf{b}}(x, v)) \cdot v| > \frac{1}{k}\}}(x, v) \right] \mu(v)^{\frac{1}{4}} \\ & \leq \max_{1 \leq i \leq l_{\varepsilon, N, \Omega}} \mathbf{1}_{B(x_i; r_i)}(x) \times \mathbf{1}_{\{|n(x_{\mathbf{b}}(x, v)) \cdot v| \leq \frac{1}{k}\}} \mu(v)^{\frac{1}{4}} \\ & \leq \max_{1 \leq i \leq l_{\varepsilon, N, \Omega}} \mathbf{1}_{B(x_i; r_i)}(x) \times \mathbf{1}_{\{|n(x_{\mathbf{b}}(x, v)) \cdot v| \leq \delta_{\varepsilon, N}\}} \mu(v)^{\frac{1}{4}} \\ & \leq \max_{1 \leq i \leq l_{\varepsilon, N, \Omega}} \mathbf{1}_{B(x_i; r_i)}(x) \times \left\{ \mathbf{1}_{\{|v| \leq N, v \in \mathcal{O}_i\}}(v) \mu(v)^{\frac{1}{4}} + \mathbf{1}_{\{|v| \geq N\}}(v) e^{-\frac{N^2}{16}} \mu(v)^{\frac{1}{8}} \right\}, \end{aligned}$$

and hence

$$\begin{aligned} & \int_{\gamma_+} |1 - \mathbf{1}_{\{|n(x_{\mathbf{b}}(x, v)) \cdot v| > \frac{1}{k}\}}(x, v)| \mu(v)^{\frac{1}{2}} d\gamma \\ & \leq \max_{1 \leq i \leq l_{\varepsilon, N, \Omega}} \int_{\partial\Omega} \int_{n \cdot v > 0} \mathbf{1}_{\{|v| \leq N, v \in \mathcal{O}_i\}} dv dS_x + O\left(\frac{1}{N}\right) \\ & \lesssim \varepsilon + O\left(\frac{1}{N}\right) \lesssim \varepsilon, \end{aligned}$$

which concludes the claim.

Now we use the monotone convergence theorem to conclude

$$\begin{aligned} & \int_0^t \int_{\gamma_+^\delta} \mathbf{1}_{\{1/k < |n(x_{\mathbf{b}}(x, v)) \cdot v|\}} |h^m(s, x, v)| \mu(v)^{\frac{1}{4}} d\gamma ds \\ & \rightarrow \int_0^t \int_{\gamma_+^\delta} |h^m(s, x, v)| \mu(v)^{\frac{1}{4}} d\gamma ds, \end{aligned}$$

as $k \rightarrow \infty$ and therefore $\int_0^t \int_{\gamma_+^\delta} |h^m(s, x, v)| \mu(v)^{\frac{1}{4}} d\gamma ds$ has the same upper bound of (84). Together with (83) we conclude (79).

4 Linear and Nonlinear Estimates

The main purpose of this section is to prove the main theorem (Theorem 1). To estimate solutions of the nonlinear equation (1) with the diffuse BC (5) we use following approximation scheme.

For $f_0 \in BV(\Omega \times \mathbb{R}^3)$ and $\|e^{\theta|v|^2} f_0\|_\infty < \infty$ we choose $f_0^\varepsilon \in BV(\Omega \times \mathbb{R}^3) \cap C^\infty(\Omega \times \mathbb{R}^3)$ satisfying $\|e^{\theta|v|^2} [f_0^\varepsilon - f_0]\|_\infty \rightarrow 0$ and $\|\nabla_{x,v} f_0^\varepsilon\|_1 \rightarrow \|f_0\|_{\widetilde{BV}}$.

Consider the sequence $f^{\varepsilon,m}$ defined by $f^{\varepsilon,0} = \chi_\varepsilon f_0^\varepsilon$ and for all $m \geq 0$,

$$\begin{aligned} \partial_t f^{\varepsilon,m+1} + v \cdot \nabla_x f^{\varepsilon,m+1} + \nu(\sqrt{\mu} f^{\varepsilon,m}) f^{\varepsilon,m+1} \\ = \chi_\varepsilon \Gamma_{\text{gain}}(f^{\varepsilon,m}, f^{\varepsilon,m}), \quad \text{in } \Omega \times \mathbb{R}^3, \\ f^{\varepsilon,m+1}(0, x, v) = \chi_\varepsilon f_0^\varepsilon(x, v), \quad \text{in } \Omega \times \mathbb{R}^3, \\ f^{\varepsilon,m+1}(t, x, v) = \chi_\varepsilon(x, v) c_\mu \sqrt{\mu(v)} \int_{n(x) \cdot u > 0} f^{\varepsilon,m}(t, x, u) \sqrt{\mu}\{n \cdot u\} du, \quad \text{on } \gamma_-, \end{aligned} \quad (102)$$

where χ_ε is defined in (50).

In order to study such sequences, we first consider a linear equation with the in-flow boundary condition

$$f(t, x, v)|_{\gamma_-} = g(t, x, v). \quad (103)$$

Let $\{\tau_1(x), \tau_2(x)\}$ be a basis of the tangent space at $x \in \partial\Omega$ (therefore $\{\tau_1(x), \tau_2(x), n(x)\}$ is an orthonormal basis of \mathbb{R}^3). Denote ∂_{τ_i} to be the (tangential) τ_i -directional derivative and ∂_n to be the normal derivative.

Lemma 5 *Assume \mathcal{U} is an open subset of $\mathbb{R}^3 \times \mathbb{R}^3$ such that $\mathfrak{S}_B \subset \mathcal{U}$. Assume $f_0(x, v) \equiv 0$, $g(t, x, v) \equiv 0$, $H(t, x, v) \equiv 0$, for $(t, x, v) \in [0, T] \times \{\mathcal{U} \cap (\bar{\Omega} \times \mathbb{R}^3)\}$.* (104)

Assume further that for $0 < \theta < \frac{1}{4}$,

$$e^{\theta|v|^2} f_0 \in L^\infty(\Omega \times \mathbb{R}^3), \quad e^{\theta|v|^2} g \in L^\infty([0, T] \times \gamma_-), \quad e^{\theta|v|^2} H \in L^\infty([0, T] \times \Omega \times \mathbb{R}^3),$$

and

$$\begin{aligned} \nabla_x f_0, \nabla_v f_0 &\in L^1(\Omega \times \mathbb{R}^3), \\ \partial_{\tau_i} g, \frac{1}{n(x) \cdot v} \left\{ -\partial_t g - \sum_i (v \cdot \tau_i) \partial_{\tau_i} g - \nu g + H \right\}, \\ \nabla_v g, e^{-\theta|v|^2} \nabla_x \nu, e^{-\theta|v|^2} \nabla_v \nu &\in L^1([0, T] \times \gamma_-), \\ \nabla_x H, \nabla_v H, e^{-\theta|v|^2} \nabla_x \nu, e^{-\theta|v|^2} \nabla_v \nu &\in L^1([0, T] \times \Omega \times \mathbb{R}^3). \end{aligned}$$

Then there exists a unique solution f to the transport equation (14) with in-flow boundary condition (103) such that $e^{\theta|v|^2} f \in C^0([0, T] \times \bar{\Omega} \times \mathbb{R}^3)$ and $\nabla_x f, \nabla_v f \in C^0([0, T]; L^1(\Omega \times \mathbb{R}^3))$ and the traces satisfy

$$\begin{aligned} \nabla_x f &= \nabla_x g, \quad \nabla_v f = \nabla_v g, \quad \text{on } \gamma_-, \\ \nabla_x f(0, x, v) &= \nabla_x f_0, \quad \nabla_v f(0, x, v) = \nabla_v f_0, \quad \text{in } \Omega \times \mathbb{R}^3, \end{aligned}$$

where $\nabla_x g$ is defined by

$$\nabla_x g = \sum_{i=1,2} \tau_i \partial_{\tau_i} g + \frac{n}{n \cdot v} \left\{ -\partial_t g - \sum_i (v \cdot \tau_i) \partial_{\tau_i} g - \nu g + H \right\}.$$

Moreover

$$\begin{aligned}
& \|\nabla_x f(t)\|_1 + \int_0^t |\nabla_x f|_{\gamma_+,1} + \int_0^t \|\nu \nabla_x f\|_1 \\
&= \|\nabla_x f_0\|_1 + \int_0^t |\nabla_x g|_{\gamma_-,1} + \int_0^t \iint_{\Omega \times \mathbb{R}^3} \operatorname{sgn}(\nabla_x f) \{\nabla_x H - \nabla_x \nu f\}, \quad (105) \\
& \|\nabla_v f(t)\|_1 + \int_0^t |\nabla_v f|_{\gamma_+,1} + \int_0^t \|\nu \nabla_v f\|_1 \\
&= \|\nabla_v f_0\|_1 + \int_0^t |\nabla_v g|_{\gamma_-,1} + \int_0^t \iint_{\Omega \times \mathbb{R}^3} \operatorname{sgn}(\nabla_v f) \{\nabla_v H - \nabla_x f - \nabla_v \nu f\}. \quad (106)
\end{aligned}$$

Proof We use the Duhamel formula of f :

$$\begin{aligned}
f(t, x, v) &= \mathbf{1}_{\{t < t_{\mathbf{b}}(x, v)\}} e^{-\int_0^t \nu(t-\tau, x-\tau v, v) d\tau} f_0(x - tv, v) \\
&\quad + \mathbf{1}_{\{t > t_{\mathbf{b}}(x, v)\}} e^{-\int_0^{t_{\mathbf{b}}(x, v)} \nu(t-\tau, x-\tau v, v) d\tau} g(t - t_{\mathbf{b}}(x, v), x_{\mathbf{b}}(x, v), v) \\
&\quad + \int_0^{\min\{t, t_{\mathbf{b}}(x, v)\}} e^{-\int_0^s \nu(t-\tau, x-\tau v, v) d\tau} H(t - s, x - sv, v) ds. \quad (107)
\end{aligned}$$

Following Proposition 1 of [6], we have, on $\{t \neq t_{\mathbf{b}}\}$

$$\begin{aligned}
& \nabla_x f(t, x, v) \mathbf{1}_{\{t \neq t_{\mathbf{b}}\}} \\
&= \mathbf{1}_{\{t < t_{\mathbf{b}}\}} e^{-\int_0^t \nu(t-\tau, x-\tau v, v) d\tau} \\
&\quad \times \left\{ \nabla_x f_0(x - tv, v) - \left(\int_0^t \nabla_x \nu(t-\tau, x-\tau v, v) d\tau \right) f_0(x - tv, v) \right\} \\
&\quad + \mathbf{1}_{\{t > t_{\mathbf{b}}\}} e^{-\int_0^{t_{\mathbf{b}}} \nu(t-\tau, x-\tau v, v) d\tau} \\
&\quad \times \left\{ \sum_{i=1}^2 \tau_i \partial_{\tau_i} g - \frac{n(x_{\mathbf{b}})}{v \cdot n(x_{\mathbf{b}})} \left\{ \partial_t g + \sum_{i=1}^2 (v \cdot \tau_i) \partial_{\tau_i} g + \nu g - H \right\} \right\} (t - t_{\mathbf{b}}, x_{\mathbf{b}}, v) \quad (108) \\
&\quad - \mathbf{1}_{\{t > t_{\mathbf{b}}\}} e^{-\int_0^{t_{\mathbf{b}}} \nu(t-\tau, x-\tau v, v) d\tau} \left(\int_0^{t_{\mathbf{b}}} \nabla_x \nu(t-\tau, x-\tau v, v) d\tau \right) g(t - t_{\mathbf{b}}, x_{\mathbf{b}}, v) \\
&\quad + \int_0^{\min(t, t_{\mathbf{b}})} e^{-\int_0^s \nu(t-\tau, x-\tau v, v) d\tau} \nabla_x H(t - s, x - vs, v) ds \\
&\quad - \int_0^{\min(t, t_{\mathbf{b}})} e^{-\int_0^s \nu(t-\tau, x-\tau v, v) d\tau} \\
&\quad \times \left(\int_0^s \nabla_x \nu(s-\tau, x-\tau v, v) d\tau \right) H(t - s, x - vs, v) ds,
\end{aligned}$$

$$\begin{aligned}
& \nabla_v f(t, x, v) \mathbf{1}_{\{t \neq t_{\mathbf{b}}\}} \\
&= \mathbf{1}_{\{t < t_{\mathbf{b}}\}} e^{-\int_0^t \nu(t-\tau, x-\tau v, v) d\tau} [-t \nabla_x f_0 + \nabla_v f_0](x - tv, v) \\
&\quad - \mathbf{1}_{\{t < t_{\mathbf{b}}\}} e^{-\int_0^t \nu(t-\tau, x-\tau v, v) d\tau} \int_0^t \{-\tau \nabla_x \nu + \nabla_v \nu\}(t - \tau, x - \tau v, v) d\tau f_0(x - tv, v) \\
&\quad - \mathbf{1}_{\{t > t_{\mathbf{b}}\}} t_{\mathbf{b}} e^{-\int_0^{t_{\mathbf{b}}} \nu(t-\tau, x-\tau v, v) d\tau} \\
&\quad \times \left\{ \sum_{i=1}^2 \tau_i \partial_{\tau_i} g - \frac{n(x_{\mathbf{b}})}{v \cdot n(x_{\mathbf{b}})} \left\{ \partial_t g + \sum_{i=1}^2 (v \cdot \tau_i) \partial_{\tau_i} g + \nu g - H \right\} \right\} (t - t_{\mathbf{b}}, x_{\mathbf{b}}, v) \\
&\quad + \mathbf{1}_{\{t > t_{\mathbf{b}}\}} e^{-\int_0^{t_{\mathbf{b}}} \nu(t-\tau, x-\tau v, v) d\tau} \left\{ \nabla_v g(t - t_{\mathbf{b}}, x_{\mathbf{b}}, v) \right\} \\
&\quad - \mathbf{1}_{\{t > t_{\mathbf{b}}\}} e^{-\int_0^{t_{\mathbf{b}}} \nu(t-\tau, x-\tau v, v) d\tau} \left\{ \int_0^{t_{\mathbf{b}}} \{-\tau \nabla_x \nu + \nabla_v \nu\}(t - \tau, x - \tau v, v) d\tau \right\} g(t - t_{\mathbf{b}}, x_{\mathbf{b}}, v) \\
&\quad + \int_0^{\min(t, t_{\mathbf{b}})} e^{-\int_0^s \nu(t-\tau, x-\tau v, v) d\tau} \{\nabla_v H - s \nabla_x H\}(t - s, x - vs, v) ds \\
&\quad - \int_0^{\min(t, t_{\mathbf{b}})} e^{-\int_0^s \nu(t-\tau, x-\tau v, v) d\tau} \\
&\quad \quad \left\{ \int_0^s \{-\tau \nabla_x \nu + \nabla_v \nu\}(t - \tau, x - \tau v, v) d\tau \right\} H(t - s, x - vs, v) ds
\end{aligned} \tag{109}$$

Therefore, we have

$$\begin{aligned}
\|\nabla_x f(t) \mathbf{1}_{\{t \neq t_{\mathbf{b}}\}}\|_1 &\lesssim \|\nabla_x f_0\|_1 + t \{ \|e^{\theta|v|^2} f_0\|_{\infty} + \|e^{\theta|v|^2} g\|_{\infty} \} \\
&\quad + \int_0^t \left| \sum_{i=1}^2 \tau_i \partial_{\tau_i} g - \frac{n}{v \cdot n} \left\{ \partial_t g + \sum_{i=1}^2 (v \cdot \tau_i) \partial_{\tau_i} g + \nu g - H \right\} \right|_{\gamma_-, 1} \\
&\quad + \int_0^t \|\nabla_x H(s)\|_1 + \int_0^t s \|e^{\theta|v|^2} H(s)\|_{\infty}
\end{aligned} \tag{110}$$

$$\begin{aligned}
\|\nabla_v f(t) \mathbf{1}_{\{t \neq t_{\mathbf{b}}\}}\|_1 &\lesssim t \|\nabla_x f_0\|_1 + \|\nabla_v f_0\|_1 + t \|e^{\theta|v|^2} f_0\|_{\infty} \\
&\quad + t \int_0^t \left| \sum_{i=1}^2 \tau_i \partial_{\tau_i} g - \frac{n}{v \cdot n} \left\{ \partial_t g + \sum_{i=1}^2 (v \cdot \tau_i) \partial_{\tau_i} g + \nu g - H \right\} \right|_{\gamma_-, 1} \\
&\quad + \int_0^t |\nabla_v g|_{\gamma_-, 1} + t^2 \sup_{0 \leq s \leq t} |e^{\theta|v|^2} g(s)|_{\gamma_-, \infty} \\
&\quad + \int_0^t \|\nabla_x H\|_1 + \int_0^t \|\nabla_v H\|_1 + C \int_0^t \|e^{\theta|v|^2} H\|_{\infty}.
\end{aligned}$$

From our assumptions, f_0 , g , and H have compact supports and the RHS are bounded. Therefore

$$\partial f \mathbf{1}_{\{t \neq t_{\mathbf{b}}\}} = [\partial_t f \mathbf{1}_{\{t \neq t_{\mathbf{b}}\}}, \nabla_x f \mathbf{1}_{\{t \neq t_{\mathbf{b}}\}}, \nabla_v f \mathbf{1}_{\{t \neq t_{\mathbf{b}}\}}] \in L^{\infty}([0, T]; L^1(\Omega \times \mathbb{R}^3)).$$

Since $\partial f \equiv 0$ around $\{t = t_{\mathbf{b}}\}$ clearly $\partial f \mathbf{1}_{\{t \neq t_{\mathbf{b}}\}}$ is the distributional derivative of f . Therefore $\nabla_x f$ and $\nabla_v f$ lie in $L^{\infty}([0, T]; L^1(\Omega \times \mathbb{R}^3))$; this allows us to apply Lemma 7 to compute the traces on the incoming boundary in $L^1([0, T]; L^1(\gamma_-, d\gamma))$ (by taking limits of the flow along the characteristics:

see the proof of Proposition 1 in [6] for details). Then, by Green's identity (Lemma 8) we know that $\nabla_x f$ and $\nabla_v f$ lie in $C^0([0, T]; L^1(\Omega \times \mathbb{R}^3))$ and we get (105) and (106).

Before going to the proof of main theorem we recall the standard estimate from [2]: Suppose $a_i \geq 0, D \geq 0$ and $A_i = \max\{a_i, \dots, a_{i-(k-1)}\}$ for fixed $k \in \mathbb{N}$.

$$\text{If } a_{m+1} \leq \frac{1}{8}A_m + D \text{ then } A_m \leq \frac{1}{8}A_0 + \left(\frac{8}{7}\right)^2 D \text{ for } m/k \gg 1. \quad (111)$$

Now we are ready to prove the main theorem.

Proof (Proof of Theorem 1) We consider the approximation scheme (102).

Now we consider the derivatives of the solution $f^{\varepsilon, m}$ of (102). Recall that $BV(\Omega \times \mathbb{R}^3)$ has *i*) a *compactness* property:

$$\begin{aligned} \text{Suppose } g^k \in BV \text{ and } \sup_k \|g^k\|_{BV} < \infty \\ \text{then } \exists g \in BV \text{ with } g^k \rightarrow g \text{ in } L^1 \text{ up to subsequence,} \end{aligned} \quad (112)$$

and *ii*) a *lower semicontinuity* property:

$$\text{Suppose } g^k \in BV \text{ and } g^k \rightarrow g \text{ in } L^1_{loc} \text{ then } \|f\|_{\widehat{BV}} \leq \liminf_{k \rightarrow \infty} \|g^k\|_{\widehat{BV}}. \quad (113)$$

Due to the smooth approximation f_0^ε of the initial datum f_0 and the cut-off χ_ε , $f^{\varepsilon, m}$ is smooth by Lemma 5. We take derivatives $\partial \in \{\nabla_x, \nabla_v\}$ to have

$$\begin{aligned} & [\partial_t + v \cdot \nabla_x + \nu(\sqrt{\mu} f^{\varepsilon, m})] \partial f^{\varepsilon, m+1} \\ &= -\partial v \cdot \nabla_x f^{\varepsilon, m+1} - \nu(\partial[\sqrt{\mu} f^{\varepsilon, m}]) f^{\varepsilon, m+1} + \partial \chi_\varepsilon \Gamma_{\text{gain}}(f^{\varepsilon, m}, f^{\varepsilon, m}) \\ & \quad + \chi_\varepsilon \partial[\Gamma_{\text{gain}}(f^{\varepsilon, m}, f^{\varepsilon, m})] + (\text{error}) \\ & \partial f^{\varepsilon, m+1}(0, x, v) = \partial \chi_\varepsilon f_0^\varepsilon(x, v) + \chi_\varepsilon \partial f_0^\varepsilon(x, v), \end{aligned}$$

where $(\text{error}) \leq e^{-\theta|v|^2} \partial \nu \|e^{\theta|v|^2} f^{\varepsilon, m}\|_\infty \|e^{\theta|v|^2} f^{\varepsilon, m+1}\|_\infty$. For all $(x, v) \in \gamma_-$,

$$\begin{aligned} & |\partial f^{\varepsilon, m+1}(t, x, v)| \\ & \lesssim \sqrt{\mu(v)} \left(1 + \frac{\langle v \rangle}{|n(x) \cdot v|}\right) \int_{n(x) \cdot u > 0} |\partial f^{\varepsilon, m}(t, x, u)| \mu(u)^{\frac{1}{4}} \{n(x) \cdot u\} du \\ & \quad + \frac{\langle v \rangle^\kappa e^{-C_\theta|v|^2}}{|n(x) \cdot v|} \|e^{\theta|v|^2} f_0\|_\infty + |\partial \chi_\varepsilon(x, v) \sqrt{\mu(v)}| P(\|e^{\theta|v|^2} f_0\|_\infty), \end{aligned}$$

for some polynomial P . Due to quadratic nonlinear term Γ we require $P(s) = s(1 + s)$.

Then by Proposition 5 and $\sqrt{\mu}f^{\varepsilon,m} \geq 0$,

$$\begin{aligned}
& \|\partial f^{\varepsilon,m+1}(t)\|_1 + \int_0^t |\partial f^{\varepsilon,m+1}(s)|_{\gamma_+,1} \\
& \lesssim \|e^{-\theta'|v|^2} \partial \chi_\varepsilon\|_1 \|e^{\theta'|v|^2} f_0\|_\infty + \|\partial f_0^\varepsilon\|_1 + \int_0^t |\partial f^{\varepsilon,m+1}(s)|_{\gamma_-} \\
& \quad + \int_0^t \|\partial f^{\varepsilon,m+1}(s)\|_1 ds + P(\|e^{\theta|v|^2} f^{\varepsilon,m}\|_\infty) \\
& \quad \times \left\{ t + t \iint_{\Omega \times \mathbb{R}^3} e^{-C_\theta|v|^2} |\partial \chi_\varepsilon| + \int_0^t \|\partial f^{\varepsilon,m}(s)\|_1 ds \right\},
\end{aligned} \tag{114}$$

where we have used Lemma 10 in Appendix A.

Applying Lemma 2 and Proposition 1 to (114), we obtain

$$\begin{aligned}
& \|\partial f^{\varepsilon,m+1}(t)\|_1 + \int_0^t |\partial f^{\varepsilon,m+1}(s)|_{\gamma_+,1} \\
& \lesssim \|e^{\theta'|v|^2} f_0\|_\infty + \|f_0\|_{BV} + \int_0^t |\partial f^{\varepsilon,m+1}(s)|_{\gamma_-,1} \\
& \quad + t[1 + P(\|e^{\theta'|v|^2} f_0\|_\infty)] \sup_{0 \leq s \leq t} \|\partial f^{\varepsilon,m+1}(s)\|_1 ds + tP(\|e^{\theta'|v|^2} f_0\|_\infty).
\end{aligned} \tag{115}$$

On the other hand, we apply Proposition 2 and Lemma 7 to bound

$$\begin{aligned}
& \int_0^t |\partial f^{\varepsilon,m+1}|_{\gamma_-,1} \\
& \lesssim O(\delta) \int_0^t |\partial f^{\varepsilon,m-1}|_{\gamma_+,1} + C_\delta \{ \|f_0\|_{BV} + tP(\|e^{\theta'|v|^2} f_0\|_\infty) \} \\
& \quad + C_\delta t[1 + P(\|e^{\theta'|v|^2} f_0\|_\infty)] \max_{i=m,m-1} \sup_{0 \leq s \leq t} \|\partial f^{\varepsilon,i}(s)\|_1.
\end{aligned} \tag{116}$$

Finally from (115) and (116), choosing $\delta \ll 1$ and $T := T(f_0)$ small enough, we have for all $0 \leq t \leq T$

$$\begin{aligned}
& \sup_{0 \leq s \leq t} \|\partial f^{\varepsilon,m+1}(s)\|_1 + \int_0^t |\partial f^{\varepsilon,m+1}(s)|_{\gamma,1} \\
& \leq C \{ \|f_0\|_{BV} + P(\|e^{\theta'|v|^2} f_0\|_\infty) \} \\
& \quad + \frac{1}{8} \max_{i=m,m-1} \left\{ \sup_{0 \leq s \leq t} \|\partial f^{\varepsilon,i}(s)\|_1 + \int_0^t |\partial f^{\varepsilon,i}|_{\gamma,1} \right\}.
\end{aligned}$$

Now using (111) we conclude

$$\sup_{0 \leq s \leq t} \|\partial f^{\varepsilon,m}(s)\|_1 + \int_0^t |\partial f^{\varepsilon,m}(s)|_{\gamma,1} \lesssim \|f_0\|_{BV} + P(\|e^{\theta|v|^2} f_0\|_\infty) \tag{117}$$

for all $m \in \mathbb{N}$.

Now we pass the to limit in m and then in ε to conclude the main theorem. From the compactness (112) and a lower semicontinuity (113) we conclude

$$\sup_{0 \leq s \leq t} \|f(s)\|_{BV} \lesssim \|f_0\|_{BV} + P(\|e^{\theta|v|^2} f_0\|_{\infty}).$$

On the other hand for a fixed $0 < \varepsilon \ll 1$, it is clear that $\{f^{\varepsilon,m}\}_{m=1}^{\infty}$ is Cauchy for the norm $\sup_{0 \leq t \leq T} \|e^{\theta'|v|^2} \cdot\|_{\infty}$ for $0 < \theta' < \theta < \frac{1}{4}$ and some $0 < T \ll 1$. The key element of the proof is to utilize the exponential weight in v to suppress the $|v|$ growth in the gain term estimate at least for some short time. For details, see Lemma 6 in [6]. Therefore $f^{\varepsilon,m} \rightarrow f^{\varepsilon}$ up to a subsequence for the norm $\sup_{0 \leq t \leq T} \|e^{\theta'|v|^2} \cdot\|_{\infty}$ and f^{ε} satisfies (102) with $f^{\varepsilon,m+1}$ and $f^{\varepsilon,m}$ replaced by f^{ε} by the trace theorem. Moreover since $|\chi_{\varepsilon}| \leq 1$ for $0 < \varepsilon \ll 1$, $\sup_{0 \leq t \leq T} \|e^{\theta'|v|^2} f^{\varepsilon}(t)\|_{\infty}$ is uniformly bounded in ε for $0 < \varepsilon \ll 1$ and $0 < T \ll 1$. Now we combine such uniform bounds in a weighted L^{∞} and L^1 convergence up to a subsequence to conclude that the limiting function f solves the Boltzmann equation (11) and the diffuse boundary condition (13).

For the boundary term we use the *weak compactness of measures*: If σ^k is a signed Radon measure on $\partial\Omega \times \mathbb{R}^3$ satisfying $\sup_k \sigma^k(\partial\Omega \times \mathbb{R}^3) < \infty$ then there exists a Radon measure σ such that $\sigma^k \rightharpoonup \sigma$ in \mathcal{M} .

More precisely we define, for almost-every s , and for any Lebesgue-measurable set $A \subset \partial\Omega \times \mathbb{R}^3$,

$$\begin{aligned} \sigma_s^{\varepsilon,m}(A) &= \left(\sigma_{s,x^1}^{\varepsilon,m}(A), \sigma_{s,x^2}^{\varepsilon,m}(A), \sigma_{s,x^3}^{\varepsilon,m}(A), \sigma_{s,v^1}^{\varepsilon,m}(A), \sigma_{s,v^2}^{\varepsilon,m}(A), \sigma_{s,v^3}^{\varepsilon,m}(A) \right)^T \\ &:= \int_A \nabla_{x,v} f^{\varepsilon,m}(s) d\gamma \in \mathbb{R}^6. \end{aligned}$$

Then there exists a Radon measure σ_s such that $\sigma_s^{\varepsilon,m} \rightharpoonup \sigma_s$ in \mathcal{M} , i.e.

$$\int_{\partial\Omega \times \mathbb{R}^3} g \partial f^{\varepsilon,m}(s) d\gamma \rightarrow \int_{\partial\Omega \times \mathbb{R}^3} g d\sigma_s \quad \text{for all } g \in C_c^0(\partial\Omega \times \mathbb{R}^3). \quad (118)$$

It is standard (Hahn's decomposition theorem) to decompose $\sigma_s = \sigma_{s,+} - \sigma_{s,-}$ with $\sigma_{s,\pm} \geq 0$. Denote $|\sigma_s|_{\mathcal{M}(\gamma)} = \sigma_{s,+}(\partial\Omega \times \mathbb{R}^3) + \sigma_{s,-}(\partial\Omega \times \mathbb{R}^3)$. Then by the lower semicontinuity property of measures we have $|\sigma_s|_{\mathcal{M}(\gamma)} \leq \liminf |\sigma_s^{\varepsilon,m}|_{\mathcal{M}(\gamma)} = \liminf |\partial f_s^{\varepsilon,m}|_{L^1(\gamma)}$, so that by (117) $\int_0^t |\sigma_s|_{\mathcal{M}(\gamma)} ds \lesssim \|f_0\|_{BV} + P(\|e^{\theta|v|^2} f_0\|_{\infty})$. Due to (118), the (distributional) derivatives $\nabla_{x,v} f(s)|_{\gamma}$ equal the Radon measure σ_s on $\partial\Omega \times \mathbb{R}^3$ in the sense of distributions.

A Some Basic Results

We collect some basic known results such as the derivatives of $t_{\mathbf{b}}$ and $x_{\mathbf{b}}$, the standard trace theorem, integration by parts formula, and the size of singular set.

Lemma 6 ([5, 2]) *If*

$$v \cdot n(x_{\mathbf{b}}(x, v)) < 0, \quad (119)$$

then $(t_{\mathbf{b}}(x, v), x_{\mathbf{b}}(x, v))$ are smooth functions of (x, v) such that

$$\begin{aligned} \nabla_x t_{\mathbf{b}} &= \frac{n(x_{\mathbf{b}})}{v \cdot n(x_{\mathbf{b}})}, & \nabla_v t_{\mathbf{b}} &= -\frac{t_{\mathbf{b}} n(x_{\mathbf{b}})}{v \cdot n(x_{\mathbf{b}})}, \\ \nabla_x x_{\mathbf{b}} &= I - \frac{n(x_{\mathbf{b}})}{v \cdot n(x_{\mathbf{b}})} \otimes v, & \nabla_v x_{\mathbf{b}} &= -t_{\mathbf{b}} I + \frac{t_{\mathbf{b}} n(x_{\mathbf{b}})}{v \cdot n(x_{\mathbf{b}})} \otimes v. \end{aligned}$$

Recall the almost grazing set γ_+^δ defined in (16). We first estimate the outgoing trace on $\gamma_+ \setminus \gamma_+^\delta$.

Lemma 7 (Outgoing trace theorem, [6]) *Assume that $\varphi \geq 0$. For any small parameter $\delta > 0$, there exists a constant $C_{\delta, T, \Omega} > 0$ such that for any h in $L^1([0, T] \times \Omega \times \mathbb{R}^3)$ with $\partial_t h + v \cdot \nabla_x h + \varphi h$ lying in $L^1([0, T] \times \Omega \times \mathbb{R}^3)$, we have for all $0 \leq t \leq T$,*

$$\int_0^t \int_{\gamma_+ \setminus \gamma_+^\delta} |h| d\gamma ds \leq C_{\delta, T, \Omega} \left[\|h_0\|_1 + \int_0^t \{ \|h(s)\|_1 + \|[\partial_t + v \cdot \nabla_x + \varphi]h(s)\|_1 \} ds \right].$$

Furthermore, for any (s, x, v) in $[0, T] \times \Omega \times \mathbb{R}^3$ the function $h(s + s', x + s'v, v)$ is absolutely continuous in s' in the interval $[-\min\{t_{\mathbf{b}}(x, v), s\}, \min\{t_{\mathbf{b}}(x, -v), T - s\}]$.

We remark that for the outgoing part, our estimate is global in time without cut-off, in contrast to the general trace theorem.

Lemma 8 (Green's Identity, [5, 2]) *For $p \in [1, \infty)$ assume that $f, \partial_t f + v \cdot \nabla_x f + \varphi f \in L^p([0, T] \times \Omega \times \mathbb{R}^3)$ with $\varphi \geq 0$ and $f_{\gamma_-} \in L^p([0, T] \times \partial\Omega \times \mathbb{R}^3; dtd\gamma)$. Then $f \in C^0([0, T]; L^p(\Omega \times \mathbb{R}^3))$ and $f_{\gamma_+} \in L^p([0, T] \times \partial\Omega \times \mathbb{R}^3; dtd\gamma)$ and for almost every $t \in [0, T]$:*

$$\|f(t)\|_p^p + \int_0^t \|f\|_{\gamma_+, p}^p = \|f(0)\|_p^p + \int_0^t \|f\|_{\gamma_-, p}^p + \int_0^t \iint_{\Omega \times \mathbb{R}^3} \{\partial_t f + v \cdot \nabla_x f + \varphi f\} p |f|^{p-2} f.$$

Lemma 9 (Lemma 17 and Lemma 18 of [5]) *Let $\Omega \subset \mathbb{R}^3$ be an open bounded set with a smooth boundary $\partial\Omega$. Then, for all $x \in \Omega$, we have*

$$m_3\{v \in \mathbb{R}^3 : n(x_{\mathbf{b}}(x, v)) \cdot v = 0\} = 0. \quad (120)$$

Moreover, for any $\varepsilon > 0$ and $N \gg 1$, there exist $\delta_{\varepsilon, N} > 0$ and $l = l_{\varepsilon, N, \Omega}$ balls $B(x_1; r_1), B(x_2; r_2), \dots, B(x_l; r_l)$ with $x_i \in \bar{\Omega}$ and covering $\bar{\Omega}$ (i.e. $\bar{\Omega} \subset \bigcup B(x_i; r_i)$), as well as l open sets $\mathcal{O}_{x_1}, \mathcal{O}_{x_2}, \dots, \mathcal{O}_{x_l} \subset B_N := \{v \in \mathbb{R}^3 : |v| \leq N\}$, with $m_3(\mathcal{O}_{x_i}) < \varepsilon$ for all $1 \leq i \leq l_{\varepsilon, N, \Omega}$, such that for any $x \in \bar{\Omega}$, there exists $i = 1, 2, \dots, l_{\varepsilon, N, \Omega}$ such that $x \in B(x_i; r_i)$ and

$$|v \cdot n(x_{\mathbf{b}}(x, v))| > \delta_{\varepsilon, N}, \quad \text{for all } v \notin \mathcal{O}_{x_i}.$$

In particular,

$$\mathcal{O}_{x_i} \supset \bigcup_{x \in B(x_i; r_i)} \{v \in B_N : |v \cdot n(x_{\mathbf{b}}(x, v))| \leq \delta_{\varepsilon, N}\}. \quad (121)$$

Proof The details of the proof are recorded in [5]. The proof of (120) is due to Sard's theorem: For fixed $x \in \bar{\Omega}$ we consider the following mapping

$$\phi_x : \partial\Omega \rightarrow \mathbb{S}^2, \quad \phi_x : y \in \partial\Omega \mapsto -\frac{y - x}{|y - x|}.$$

If $n(x_{\mathbf{b}}(x, v)) \cdot v = 0$ then $\frac{v}{|v|}$ is a critical value of ϕ_x at $y = x_{\mathbf{b}}(x, v)$. Then by Sard's theorem the Lebesgue measure of such set on \mathbb{S}^2 is zero.

Now we fix $0 < \varepsilon \ll 1$ and $x \in \bar{\Omega}$. Due to (120) there exists an open set $\mathcal{O}_x \in \mathbb{R}^3$ such that $m_3(\mathcal{O}_x) < \varepsilon$ and $|v \cdot n(x_{\mathbf{b}}(x, v))| \neq 0$ for $v \notin \mathcal{O}_x$. By Lemma 6, $v \mapsto v \cdot n(x_{\mathbf{b}}(x, v))$ is smooth on the compact set $\{\mathbb{R}^3 \setminus \mathcal{O}_x\} \cap B_N$. Then by the compactness we have a positive lower bound $2\delta_{\varepsilon, N, x} > 0$ of $|v \cdot n(x_{\mathbf{b}}(x, v))|$. Then by Lemma 6 again, there exists a ball $B(x; r_x)$ such that for all y in this ball and all $v \in \{\mathbb{R}^3 \setminus \mathcal{O}_x\} \cap B_N$ we have $|v \cdot n(x_{\mathbf{b}}(y, v))| \geq \delta_{\varepsilon, N, x}$. Then we use the compactness of $\bar{\Omega}$ to extract the finite covering which satisfies (121).

Lemma 10 (Lemma 5 of [6]) For any smooth function $g = g(x, v)$ and $\partial \in \{\nabla_x, \nabla_v\}$ and $0 < \theta < \frac{1}{4}$,

$$\begin{aligned} & \|\partial \Gamma_{\text{gain}}(g, g)\|_1 \\ & \lesssim \|e^{\theta|v|^2} g\|_\infty \left\{ |\partial x| \|\nabla_x g\|_1 + |\partial v| \|\nabla_v g\|_1 \right\} + \langle v \rangle^\kappa e^{-\theta|v|^2} \|\partial v\| \|e^{\theta|v|^2} g\|_\infty^2, \\ & \iint_{\Omega \times \mathbb{R}^3} |\nu(\partial[\sqrt{\mu}g])g| dv dx \\ & \lesssim \|e^{\theta|v|^2} g\|_\infty \int_{\Omega} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{-\frac{\theta}{4}|v-u|^2} |\partial g(u)| du dv dx \lesssim \|e^{\theta|v|^2} g\|_\infty \|\partial g\|_1. \end{aligned}$$

B The singular set \mathfrak{S}_B is a Co-Dimension 1 subset

We prove **Remark 1**. It suffices to show that $\mathfrak{S}_B \cap \bar{\Omega} \times \mathbb{R}^3$ is a co-dimension 1 submanifold of $\bar{\Omega} \times \mathbb{R}^3$. More precisely we will show that if $(x_0, v_0) \in \Omega \times \mathbb{R}^3$ satisfies $n(x_B(x_0, v_0)) \cdot v_0 = 0$ and the boundary is strictly non-convex (10) at $(x_B(x_0, v_0), v_0)$ then there exists $0 < \varepsilon \ll 1$ such that the following set is a 5 dimensional submanifold:

$$\{(x, v) \in \mathfrak{S}_B \cap B((x_0, v_0); \varepsilon) : x_B(x, v) \sim x_B(x_0, v_0)\} \subset \bar{\Omega} \times \mathbb{R}^3. \quad (122)$$

Without loss of generality we may assume $x_B(x_0, v_0) = (0, 0, 0) = \mathbf{0}$ and $v_0 = \mathbf{e}_1$ and $n(0, 0, 0) = -\mathbf{e}_3$ so that $\partial\Omega$ is locally a graph of a function $\eta : \mathbb{R}^2 \rightarrow \mathbb{R}$ and $\nabla\eta(0, 0) = \mathbf{0}$. Therefore the strictly non-convex condition (10) at $(x_B(x_0, v_0), v_0) = (\mathbf{0}, \mathbf{e}_1)$ implies

$$\partial_1 \partial_1 \eta(0, 0) \neq 0. \quad (123)$$

Clearly, (122) is contained in

$$\{(x + sv, v) \in \bar{\Omega} \times \mathbb{R}^3 : x \in \partial\Omega, n(x) \cdot v = 0, (x, v) \sim (x_0, v_0), s \in [0, \infty)\}. \quad (124)$$

Consider $(x, v) \sim (x_0, v_0)$. We choose a basis for the tangent space:

$$\begin{aligned} \tau_1 &= \frac{1}{\sqrt{1 + |\nabla\eta|^2}} \begin{pmatrix} 1 \\ 0 \\ \partial_1 \eta \end{pmatrix}, \\ \tau_2 &= \frac{1}{\sqrt{1 + |\nabla\eta|^2} \sqrt{1 + (\partial_1 \eta)^2}} \begin{pmatrix} -\partial_1 \eta \partial_2 \eta \\ 1 + (\partial_1 \eta)^2 \\ \partial_2 \eta \end{pmatrix}. \end{aligned}$$

For $(x_1, x_2, \theta, r_v, s) \in \mathbb{R}^2 \times [0, 2\pi) \times [0, \infty) \times [0, \infty)$ we write $(x + sv, v)$ in (124) as

$$\begin{aligned} X(x_1, x_2, \theta, r_v, s) &:= \begin{pmatrix} x_1 \\ x_2 \\ \eta(x_1, x_2) \end{pmatrix} + sr_v \cos \theta \tau_1(x_1, x_2) + sr_v \sin \theta \tau_2(x_1, x_2), \\ V(x_1, x_2, \theta, r_v, s) &:= r_v \cos \theta \tau_1(x_1, x_2) + r_v \sin \theta \tau_2(x_1, x_2). \end{aligned}$$

In order to prove Remark 1 it suffices to show that the followings are linearly independent

$$\begin{pmatrix} \partial_{x_1} X \\ \partial_{x_1} V \end{pmatrix}, \begin{pmatrix} \partial_{x_2} X \\ \partial_{x_2} V \end{pmatrix}, \begin{pmatrix} \partial_\theta X \\ \partial_\theta V \end{pmatrix}, \begin{pmatrix} \partial_s X \\ \partial_s V \end{pmatrix}, \begin{pmatrix} \partial_{r_v} X \\ \partial_{r_v} V \end{pmatrix} \in \mathbb{R}^6.$$

That is it suffices to show that the normal is non-vanishing:

$$\mathcal{N} := \det \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 & \mathbf{e}_5 & \mathbf{e}_6 \\ \partial_{x_1} X_1 & \partial_{x_1} X_2 & \partial_{x_1} X_3 & \partial_{x_1} V_1 & \partial_{x_1} V_2 & \partial_{x_1} V_3 \\ \partial_{x_2} X_1 & \partial_{x_2} X_2 & \partial_{x_2} X_3 & \partial_{x_2} V_1 & \partial_{x_2} V_2 & \partial_{x_2} V_3 \\ \partial_\theta X_1 & \partial_\theta X_2 & \partial_\theta X_3 & \partial_\theta V_1 & \partial_\theta V_2 & \partial_\theta V_3 \\ \partial_s X_1 & \partial_s X_2 & \partial_s X_3 & 0 & 0 & 0 \\ \partial_{r_v} X_1 & \partial_{r_v} X_2 & \partial_{r_v} X_3 & \partial_{r_v} V_1 & \partial_{r_v} V_2 & \partial_{r_v} V_3 \end{pmatrix}.$$

To compute the normal we need to know

$$\begin{aligned}
\partial_1 \tau_1(x_1, x_2) &= \frac{\partial_1^2 \eta}{[1 + (\nabla \eta)^2]^{3/2}} \begin{pmatrix} -\partial_1 \eta \\ 0 \\ 1 \end{pmatrix} + \frac{\partial_2 \eta}{[1 + (\nabla \eta)^2]^{3/2}} \begin{pmatrix} 0 \\ 0 \\ \partial_2 \eta \partial_1^2 \eta - \partial_1 \eta \partial_1 \partial_2 \eta \end{pmatrix}, \\
\partial_2 \tau_1(x_1, x_2) &= \frac{1}{[1 + (\nabla \eta)^2]^{1/2}} \begin{pmatrix} 0 \\ 0 \\ \partial_1 \partial_2 \eta \end{pmatrix} - \frac{1}{[1 + (\nabla \eta)^2]^{3/2}} \begin{pmatrix} \nabla \eta \cdot \nabla \partial_2 \eta \\ 0 \\ \partial_1 \eta \nabla \eta \cdot \nabla \partial_2 \eta \end{pmatrix}, \\
\partial_1(\tau_2)_1 &= \frac{(\partial_1 \eta)^2 \partial_2 \eta \partial_1^2 \eta}{[1 + (\partial_1 \eta)^2]^{3/2} [1 + |\nabla \eta|^2]^{1/2}} + \frac{(\partial_1 \eta)^2 \partial_2 \eta \partial_1^2 \eta + \partial_1 \eta (\partial_2 \eta)^2 \partial_1 \partial_2 \eta}{[1 + (\partial_1 \eta)^2]^{1/2} [1 + |\nabla \eta|^2]^{3/2}} \\
&\quad - \frac{\partial_1^2 \eta \partial_2 \eta + \partial_1 \eta \partial_1 \partial_2 \eta}{[1 + (\partial_1 \eta)^2]^{1/2} [1 + |\nabla \eta|^2]^{1/2}}, \\
\partial_2(\tau_2)_1 &= \frac{\partial_1 \eta \partial_2 \eta \partial_1 \partial_2 \eta}{[1 + (\partial_1 \eta)^2]^{3/2} [1 + |\nabla \eta|^2]^{1/2}} + \frac{(\partial_1 \eta)^2 \partial_2 \eta \partial_1 \partial_2 \eta + \partial_1 \eta (\partial_2 \eta)^2 \partial_2^2 \eta}{[1 + (\partial_1 \eta)^2]^{1/2} [1 + |\nabla \eta|^2]^{3/2}} \\
&\quad - \frac{\partial_1 \partial_2 \eta \partial_2 \eta + \partial_1 \eta \partial_2^2 \eta}{[1 + (\partial_1 \eta)^2]^{1/2} [1 + |\nabla \eta|^2]^{1/2}}, \\
\partial_1(\tau_2)_2 &= \frac{\partial_1 \eta \partial_1^2 \eta}{[1 + (\partial_1 \eta)^2]^{1/2} [1 + |\nabla \eta|^2]^{1/2}} - \frac{[1 + (\partial_1 \eta)^2]^{1/2} [\partial_1 \eta \partial_1^2 \eta + \partial_2 \eta \partial_1 \partial_2 \eta]}{[1 + |\nabla \eta|^2]^{3/2}}, \\
\partial_2(\tau_2)_2 &= \frac{\partial_1 \eta \partial_1 \partial_2 \eta}{[1 + (\partial_1 \eta)^2]^{1/2} [1 + |\nabla \eta|^2]^{1/2}} - \frac{[1 + (\partial_1 \eta)^2]^{1/2} [\partial_1 \eta \partial_1 \partial_2 \eta + \partial_2 \eta \partial_2^2 \eta]}{[1 + |\nabla \eta|^2]^{3/2}}, \\
\partial_1(\tau_2)_3 &= -\frac{\partial_1 \eta \partial_2 \eta \partial_1^2 \eta}{[1 + (\partial_1 \eta)^2]^{3/2} [1 + |\nabla \eta|^2]^{1/2}} - \frac{\partial_1 \eta \partial_2 \eta \partial_1^2 \eta + (\partial_2 \eta)^2 \partial_1 \partial_2 \eta}{[1 + (\partial_1 \eta)^2]^{1/2} [1 + |\nabla \eta|^2]^{3/2}} \\
&\quad + \frac{\partial_1 \partial_2 \eta}{[1 + (\partial_1 \eta)^2]^{1/2} [1 + |\nabla \eta|^2]^{1/2}}, \\
\partial_2(\tau_2)_3 &= -\frac{\partial_1 \eta \partial_2 \eta \partial_2^2 \eta}{[1 + (\partial_1 \eta)^2]^{3/2} [1 + |\nabla \eta|^2]^{1/2}} - \frac{\partial_1 \eta \partial_2 \eta \partial_1 \partial_2 \eta + (\partial_2 \eta)^2 \partial_2^2 \eta}{[1 + (\partial_1 \eta)^2]^{1/2} [1 + |\nabla \eta|^2]^{3/2}} \\
&\quad + \frac{\partial_2^2 \eta}{[1 + (\partial_1 \eta)^2]^{1/2} [1 + |\nabla \eta|^2]^{1/2}}.
\end{aligned}$$

We evaluate the normal at $(x_1, x_2, \theta, s, r_v) = (0, 0, 0, s, r_v)$. Since $\partial_1 \eta(0, 0) = 0 = \partial_2 \eta(0, 0)$,

$$\begin{aligned}
n(0, 0) &= \mathbf{e}_3, \quad \tau_1(0, 0) = \mathbf{e}_1, \quad \tau_2(0, 0) = \mathbf{e}_2, \\
\partial_1 \tau_1(0, 0) &= \partial_1 \partial_1 \eta(0, 0) \mathbf{e}_3, \quad \partial_2 \tau_1(0, 0) = \partial_1 \partial_2 \eta(0, 0) \mathbf{e}_3, \\
\partial_1 \tau_2(0, 0) &= \partial_1 \partial_2 \eta(0, 0) \mathbf{e}_3, \quad \partial_2 \tau_2(0, 0) = \partial_2 \partial_2 \eta(0, 0) \mathbf{e}_3.
\end{aligned}$$

Due to (123) we have

$$\mathcal{N}(0, 0, 0, s, r_v) = \det \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 & \mathbf{e}_5 & \mathbf{e}_6 \\ 1 & 0 & -s \partial_1 \partial_1 \eta & 0 & 0 & -r_v \partial_1 \partial_1 \eta \\ 0 & 1 & -s \partial_1 \partial_2 \eta & 0 & 0 & -r_v \partial_1 \partial_2 \eta \\ 0 & s & 0 & 0 & r_v & 0 \\ r_v & 0 & 0 & 0 & 0 & 0 \\ s & 0 & 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ r_v^2 \partial_1 \partial_1 \eta(0, 0) \\ 0 \\ 0 \\ sr_v \partial_1 \partial_1 \eta(0, 0) \end{pmatrix} \neq 0.$$

Therefore $\mathcal{N}(x_1, x_2, \theta, s, r_v) \neq 0$ for $(x_1, x_2, \theta) \sim (0, 0, 0)$. This proves the claim.

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