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# Extended Hermite Subdivision Schemes

Jean-Louis Merrien\*, Tomas Sauer†

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## Abstract

Subdivision schemes are efficient tools for building curves and surfaces. For vector subdivisions, it is not so straightforward to prove more than the Hölder regularity of the limit function. On the other hand, Hermite subdivision schemes produce function vectors that consist of derivatives of a certain function, so that the notion of convergence automatically includes regularity of the limit. In this paper, we establish an equivalence between spectral condition and operator factorizations, then we study how such schemes with smooth limit functions can be extended into ones with higher regularity. We conclude by a new approach to cardinal splines.

## 1 Introduction and Notations

In recent decades, subdivision schemes [1, 10] have resulted in efficient methods to generate smooth curves and surfaces, for example, for the purpose of computer aided design.

In *vector subdivision schemes* [20], one starts with a set of vector valued data at level 0, and iteratively applies a stationary subdivision rule to generate a new sets of vectors on the finer and finer grids  $2^{-n}\mathbb{Z}$ . In this process, the data  $\mathbf{f}_n$  is interpreted as an approximation of a limit function  $\mathbf{f} : \mathbb{R} \rightarrow \mathbb{R}^p$  on the grid, i.e.,

$$\mathbf{f}_n(\alpha) \simeq \mathbf{f}(2^{-n}\alpha), \quad \alpha \in \mathbb{Z}.$$

Under suitable conditions on the subdivision rule, this sequence of vectors then converges to a limit curve or surface, either uniformly or in some  $L_p$

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norm. In this context, the results are vector valued functions with some Hölder regularity related to the convergence rate of the process, but essentially they are “simply” continuous or  $L_p$  functions. The smoothness of the limit functions in the sense of differentiability implies additional conditions for the subdivision scheme which cannot be easily verified, cf. [2, 3, 20].

Hermite subdivision schemes where the components of the vector valued function are the successive exact or approximated derivatives of the first component, approach subdivision from a different point of view [8, 9, 12, 13, 14, 15]. Besides the fact that now the subdivision operator varies with the level of iteration, the regularity of the limit function is part of the definition of convergence and, therefore, the verification of smoothness properties is already included in the convergence proof.

Convergence of subdivision schemes is usually described by identifying certain special eigenvectors of the subdivision operator which imply rise to a factorization of the subdivision operator by means of an operator that annihilates this invariance space. In the scalar case, this annihilator is simply a difference operator, in the vector case, it is a more intricate difference based on the *rank* of the subdivision scheme and in Hermite subdivision it is the *Taylor operator*, cf. [7, 17]. All of these operators are in a one-to-one relationship with the sum rules or spectral conditions for the subdivision operator.

In this paper, we study these relationships for Hermite schemes. We show that spectral condition is equivalent to a factorization, once a certain normalization property is satisfied; in a previous paper we had only proved that spectral condition implies factorization. After that, we explore how the regularity of the limit can be increased by an *extended scheme* that transfers a given Hermite scheme to one of higher order. After a first example, the extension process and the factorization is studied in detail.

As an example, we propose a new approach to cardinal splines [21]. Starting with transformations to a scalar subdivision scheme [19], we describe extended Hermite schemes of different orders, we give new eigenvalues and eigenpolynomial properties and we prove that the maximum regularity of the spline can be recovered by this new process.

The paper is organized as follows: in Section 2, we recall the definition of a Hermite subdivision scheme and we define the key to transform this scheme into a vector subdivision scheme, namely spectral condition and show that it characterizes factorization in the presence of normalization. In Section 3, we introduce extended schemes. Then two extensions are studied, one from de Rham scheme [5, 6] in Section 4, one from cardinal splines in Section 5.

As for notation, vectors in  $\mathbb{R}^r$  will be labeled by lowercase boldface letters:  $\mathbf{y} = [y_j]_{j=0,\dots,r-1}$  or  $\mathbf{y} = [y^{(j)}]_{j=0,\dots,r-1}$  and matrices in  $\mathbb{R}^{r \times r}$  will be written as uppercase boldface letters, such as  $\mathbf{A} = [a_{jk}]_{j,k=0,\dots,r-1}$ . The space of polynomials in one variable of degree at most  $n$  will be written as  $\mathcal{P}_n$ . Vector sequences will be considered as functions from  $\mathbb{Z}$  to  $\mathbb{R}^r$  and the vector space of all such functions will be denoted by  $\ell(\mathbb{Z}, \mathbb{R}^r)$  or  $\ell^r(\mathbb{Z})$ . For  $\mathbf{y}(\cdot) \in \ell(\mathbb{Z}, \mathbb{R}^r)$ , the *forward difference* is defined as  $\Delta \mathbf{y}(\alpha) := \mathbf{y}(\alpha+1) - \mathbf{y}(\alpha)$ ,  $\alpha \in \mathbb{Z}$ , and iterated to  $\Delta^{i+1} \mathbf{y} := \Delta \Delta^i \mathbf{y} = \Delta^i \mathbf{y}(\cdot + 1) - \Delta^i \mathbf{y}(\cdot)$ , for  $i \geq 0$ .

By  $\mathcal{A} \in \ell^{r \times r}(\mathbb{Z})$ , we denote a sequence of matrices, that is, for  $\alpha \in \mathbb{Z}$  the sequence element  $\mathbf{A}(\alpha) = [a_{jk}(\alpha)]_{j,k=0,\dots,r-1}$  is an  $r \times r$  matrix. Any such sequence will be called a *mask* provided that it is finitely supported, that is, there exists  $N \in \mathbb{N}$  such that

$$\text{supp} \mathcal{A} := \{\alpha \in \mathbb{Z} : \mathbf{A}(\alpha) \neq 0\} \subseteq [-N, N].$$

To any mask  $\mathcal{A}$ , we associate the *stationary vector subdivision operator*  $S_{\mathcal{A}} : \ell^r(\mathbb{Z}) \rightarrow \ell^r(\mathbb{Z})$ , defined as

$$S_{\mathcal{A}} \mathbf{c}(\alpha) := \sum_{\beta \in \mathbb{Z}} \mathbf{A}(\alpha - 2\beta) \mathbf{c}(\beta), \quad \alpha \in \mathbb{Z}, \quad \mathbf{c} \in \ell^r(\mathbb{Z}). \quad (1)$$

and also its *symbol* which is the Laurent polynomial

$$\mathcal{A}^*(z) := \sum_{\alpha \in \mathbb{Z}} \mathbf{A}(\alpha) z^\alpha. \quad (2)$$

We also recall the *shift relation*

$$(S_{\mathcal{A}}) \mathbf{c}(\cdot + 2) = S_{\mathcal{A}}(\mathbf{c}(\cdot + 1)). \quad (3)$$

The *stationary subdivision scheme* associated with the subdivision operator  $S_{\mathcal{A}}$  is given by the repeated application of the operator, starting with any initial sequence in  $\ell_{\infty}^r(\mathbb{Z})$  *i.e.*

$$\mathbf{c}_0 \in \ell_{\infty}^r(\mathbb{Z}), \quad \mathbf{c}_{n+1} := S_{\mathcal{A}} \mathbf{c}_n, \quad n > 0, \quad (4)$$

If we let the masks vary with the iteration level  $n$  (*non stationary subdivision scheme*), we get a level dependent scheme of the form

$$\mathbf{c}_0 \in \ell_{\infty}^r(\mathbb{Z}), \quad \mathbf{c}_{n+1}(\alpha) := \sum_{\beta \in \mathbb{Z}} \mathbf{A}_n(\alpha - 2\beta) \mathbf{c}_n(\beta), \quad \mathbf{c} \in \ell^r(\mathbb{Z}). \quad (5)$$

A simple form of such a level dependence occurs in Hermite subdivision schemes where

$$\mathbf{A}_n(\alpha) = \mathbf{D}^{-n-1} \mathbf{A}(\alpha) \mathbf{D}^n, \quad \alpha \in \mathbb{Z},$$

as we will point out in the next section.

## 2 Hermite subdivision schemes

A special type of a level dependent vector subdivision scheme is given by *Hermite subdivision schemes*  $H_{\mathcal{A}}$  of degree  $d$ . Starting with  $\mathbf{f}_0 \in \ell^{d+1}(\mathbb{Z})$ , for  $n \in \mathbb{N}$ , we define  $\mathbf{f}_{n+1} \in \ell^{d+1}(\mathbb{Z})$  by

$$\mathbf{D}^{n+1} \mathbf{f}_{n+1}(\alpha) = S_{\mathcal{A}} \mathbf{D}^n \mathbf{f}_n(\alpha) = \sum_{\beta \in \mathbb{Z}} \mathbf{A}(\alpha - 2\beta) \mathbf{D}^n \mathbf{f}_n(\beta), \quad \alpha \in \mathbb{Z}, \quad (6)$$

where

$$\mathbf{D} := \begin{bmatrix} 1 & & & \\ & \frac{1}{2} & & \\ & & \ddots & \\ & & & \frac{1}{2^d} \end{bmatrix}$$

is the diagonal matrix with diagonal entries  $2^{-j}$ , respectively,  $j = 0, \dots, d$ , that represents the influence of scaling on the successive derivatives.

**Remark 2.1** *It is important to observe that (6) can also be written as*

$$\mathbf{f}_{n+1}(\alpha) = S_{\mathcal{A}_n} \mathbf{f}_n(\alpha) = \sum_{\beta \in \mathbb{Z}} \mathbf{D}^{-(n+1)} \mathbf{A}(\alpha - 2\beta) \mathbf{D}^n \mathbf{f}_n(\beta), \quad (7)$$

where the Hermite subdivision scheme is actually non-stationary since the matrices involved in the linear combination are level dependent. Indeed at level  $n$ , the mask  $\mathcal{A}_n := \{\mathbf{D}^{-(n+1)} \mathbf{A}(\alpha) \mathbf{D}^n\}_{\alpha \in \mathbb{Z}}$  is used.

**Remark 2.2** *We notice that if  $\varphi \in C^d(\mathbb{R})$  and  $\varphi_n(x) := \varphi(x/2^n)$ , then*

$$\begin{bmatrix} \varphi_n(x) \\ \varphi_n'(x) \\ \vdots \\ \varphi_n^{(d)}(x) \end{bmatrix} = \mathbf{D}^n \begin{bmatrix} \varphi(x/2^n) \\ \varphi'(x/2^n) \\ \vdots \\ \varphi^{(d)}(x/2^n) \end{bmatrix}.$$

For the iterated application of the Hermite subdivision scheme, the first component  $f_n^{(0)}(\beta)$  can be interpreted as the approximation of  $\varphi(\beta/2^n)$ , while the next one,  $f_n^{(1)}(\beta)$ , describes the approximation of the derivative  $\varphi'(\beta/2^n)$ , and so on, up to the last one,  $f_n^{(d)}(\beta)$  which is an approximation of  $\varphi^{(d)}(\beta/2^n)$ .

Finally,  $H_{\mathcal{A}}$  will be called *interpolatory* if it starts with any  $\mathbf{f}_0 \in \ell^{d+1}(\mathbb{Z})$  at each step,  $\mathbf{f}_{n+1}(2\cdot) = \mathbf{f}_n(\cdot)$ . This is equivalent to  $\mathbf{A}(0) = \mathbf{D}$  and  $\mathbf{A}(2\beta) = \mathbf{0}$  for any  $\beta \in \mathbb{Z}$ ,  $\beta \neq 0$ .

## 2.1 Spectral condition

To any function  $\varphi \in C^d(\mathbb{R})$ , we associate the vector sequence  $\mathbf{v}_\varphi \in \ell^{d+1}(\mathbb{Z})$  of the samples of the function and its derivatives up to order  $d$  on  $\mathbb{Z}$ , defined by

$$\mathbf{v}_\varphi(\alpha) := \begin{bmatrix} \varphi(\alpha) \\ \varphi'(\alpha) \\ \vdots \\ \varphi^{(d)}(\alpha) \end{bmatrix}, \quad \alpha \in \mathbb{Z}. \quad (8)$$

$H_{\mathcal{A}}$  reproduces a function  $\varphi \in C^d(\mathbb{R})$  if for the initial value  $\mathbf{f}_0 = \mathbf{v}_\varphi$  we obtain  $\mathbf{f}_n(\cdot) = \mathbf{v}_\varphi(\cdot/2^n)$  for any  $n \in \mathbb{N}$ . At this point, it should be recalled that a  $C^d$ -convergent interpolatory scheme of degree  $d$  as defined in Definition 2.13 reproduces any polynomial of  $\mathcal{P}_d$  (see Theorem 2. of [9]).

A fundamental property of Hermite subdivision schemes is the *spectral condition* introduced in [7], which requires the existence of particular *polynomial* eigenvalues of the stationary subdivision operator  $S_{\mathcal{A}}$  that affect the behavior of the Hermite scheme in a crucial way.

**Definition 2.3** *A mask  $\mathcal{A}$  or its associated subdivision operator  $S_{\mathcal{A}}$  satisfies the spectral condition of order  $\ell$  if for  $j = 0, \dots, \ell$ , there exist polynomials  $p_j$  of exact degree  $j$  such that*

$$S_{\mathcal{A}}\mathbf{v}_{p_j} = \frac{1}{2^j}\mathbf{v}_{p_j}. \quad (9)$$

If (9) holds true, we will always assume that  $p_j$  is normalized such that  $p_j(x) = \frac{1}{j!}x^j + q_j(x)$  with  $q_j \in \mathcal{P}_{j-1}$  for  $j > 0$  and  $q_0 = 0$ .

**Remark 2.4** *Though spectral condition is a property of the stationary subdivision scheme,  $S_{\mathcal{A}}$ , it is linked with the derivatives of polynomials. Moreover, it was proved in [7] that spectral condition is also equivalent to the sum rule introduced by Bin Han and his collaborators in [13, 14].*

The following result is easily obtained by using linearity to ensure the reproduction of any basis of polynomials and especially the  $p_j(x) := x^j/j!$  for  $j = 0, \dots, \ell$ .

**Proposition 2.5** *If  $H_{\mathcal{A}}$  reproduces a basis of  $\mathcal{P}_\ell$  then  $S_{\mathcal{A}}$  satisfies spectral condition of order  $\ell$  for the polynomials  $p_j(x) := \frac{x^j}{j!}$ ,  $j = 0, \dots, \ell$ .*

**Definition 2.6** The Taylor operator  $T_d$  and the complete Taylor operator  $\tilde{T}_d$  of order  $d$ , acting on  $\ell^{(d+1)}(\mathbb{Z})$  are defined as

$$T_d := \begin{bmatrix} \Delta & -1 & \cdots & -\frac{1}{(d-1)!} & -\frac{1}{d!} \\ & \Delta & \ddots & \vdots & \vdots \\ & & \ddots & -1 & \vdots \\ & & & \Delta & -1 \\ & & & & 1 \end{bmatrix}, \quad \tilde{T}_d := \begin{bmatrix} \Delta & -1 & \cdots & -\frac{1}{(d-1)!} & -\frac{1}{d!} \\ & \Delta & \ddots & \vdots & \vdots \\ & & \ddots & -1 & \vdots \\ & & & \Delta & -1 \\ & & & & \Delta \end{bmatrix}.$$

**Proposition 2.7** For any  $p \in \mathcal{P}_d$ , we have  $T_d \mathbf{v}_p = [0, \dots, 0, p^{(d)}]^T \in \mathbb{R}^{d+1}$  and  $\tilde{T}_d \mathbf{v}_p = \mathbf{0} \in \mathbb{R}^{d+1}$ .

**Proof:** If  $p \in \mathcal{P}_d$ , then its  $d + 1$ -th derivative is 0. Thus for  $j = 0, \dots, d - 1$ , the Taylor expansion of  $p^{(j)}(\alpha + 1)$  at point  $\alpha$  is  $p^{(j)}(\alpha + 1) = \sum_{i=0}^{d-j} \frac{1}{i!} p^{(j+i)}(\alpha)$ , which gives that the  $j$ -th row of  $T_d \mathbf{v}_p(\alpha)$  or  $\tilde{T}_d \mathbf{v}_p$  is 0. The last row of  $T_d \mathbf{v}_p(\alpha)$  or  $\tilde{T}_d \mathbf{v}_p(\alpha)$  is  $p^{(d)}(\alpha)$  or  $p^{(d)}(\alpha + 1) - p^{(d)}(\alpha) = 0$ , respectively.  $\square$

The following factorization result has mostly been proved, among others, in [17]. We add a normalization property that will be used for the converse.

**Theorem 2.8** If the mask  $\mathcal{A} \in \ell^{(d+1) \times (d+1)}(\mathbb{Z})$  satisfies the spectral condition of order at least  $d$ , then there exist two finitely supported masks  $\mathcal{B}$  and  $\tilde{\mathcal{B}}$  in  $\ell^{(d+1) \times (d+1)}(\mathbb{Z})$  such that

$$T_d S_{\mathcal{A}} = 2^{-d} S_{\mathcal{B}} T_d, \quad \text{and} \quad \tilde{T}_d S_{\mathcal{A}} = 2^{-d} S_{\tilde{\mathcal{B}}} \tilde{T}_d. \quad (10)$$

Moreover,

$$S_{\mathcal{B}} \mathbf{e}_d = \mathbf{e}_d := [0, \dots, 0, 1]^T \in \mathbb{R}^{d+1}. \quad (11)$$

**Proof:** The new property is the normalization one i.e.  $S_{\mathcal{B}} \mathbf{e}_d = \mathbf{e}_d$ . Let  $p_d \in \mathcal{P}_d$  be the normalized polynomial such that  $\frac{1}{2^d} \mathbf{v}_{p_d} = S_{\mathcal{A}} \mathbf{v}_{p_d}$ . In Proposition 2.7, we have seen that  $T_d \mathbf{v}_{p_d} = [0, \dots, 0, 1]^T = \mathbf{e}_d$ . The normalization follows from  $\mathbf{e}_d = T_d \mathbf{v}_{p_d} = 2^d T_d S_{\mathcal{A}} \mathbf{v}_{p_d} = S_{\mathcal{B}} T_d \mathbf{v}_{p_d} = S_{\mathcal{B}} \mathbf{e}_d$ .  $\square$

Next, we will give a converse to Theorem 2.8, stated and proved in Theorem 2.11, that shows that the incomplete Taylor factorization does in fact characterize spectral condition of order  $d$  as long as the additional normalization condition (11) from Theorem 2.8 is satisfied. A few preliminary results are required.

**Proposition 2.9** *A sequence  $\mathbf{v} \in \ell^{d+1}(\mathbb{Z})$  satisfies  $\tilde{T}_d \mathbf{v} = \mathbf{0}$  if and only if  $\mathbf{v} = \mathbf{v}_p$  for some  $p \in \mathcal{P}_d$ .*

**Proof:** Since the direction “ $\Leftarrow$ ” is given in Proposition 2.7, we only need to prove “ $\Rightarrow$ ” for which we use an idea from [7, Lemma 4]. Let  $\mathbf{v} \in \ell^{d+1}(\mathbb{Z})$  such that  $\tilde{T}_d \mathbf{v} = \mathbf{0}$ . For a given  $\alpha \in \mathbb{Z}$ , we define

$$p(x) = \sum_{k=0}^d \frac{v_k(\alpha)}{k!} (x - \alpha)^k$$

and the corresponding sampling vector  $\mathbf{v}_p$ . Since for  $k = 0, \dots, d$ ,  $p^{(k)}(\alpha) = v_k(\alpha)$ , we notice that  $\mathbf{v}_p(\alpha) - \mathbf{v}(\alpha) = \mathbf{0}$ .

The hypothesis gives  $\tilde{T}_d \mathbf{v} = \mathbf{0}$  and by Proposition 2.7 also  $\tilde{T}_d \mathbf{v}_p = \mathbf{0}$ . Thus, for  $\mathbf{u} = \mathbf{v} - \mathbf{v}_p$  we obtain that  $\tilde{T}_d \mathbf{u} = \mathbf{0}$  or

$$u_k(\cdot + 1) = \sum_{j=k}^d \frac{u_j(\cdot)}{(j-k)!}, \quad k = 0, \dots, d.$$

Then inductive reasoning, starting from  $\mathbf{u}(\alpha) = \mathbf{0}$ , yields that  $\mathbf{u}(\beta) = \mathbf{0}$  or  $\mathbf{u}(\beta) = \mathbf{v}_p(\beta)$  for any integer  $\beta \geq \alpha$ .

If we choose  $\alpha'$  instead of  $\alpha$  and corresponding polynomial  $q$  instead of  $p$ , we obtain for any  $\beta \geq \max(\alpha, \alpha')$  that  $\mathbf{v}(\beta) = \mathbf{v}_p(\beta) = \mathbf{v}_q(\beta)$ , hence  $p(\beta) = q(\beta)$  for any sufficiently large  $\beta$  and therefore, for all  $\beta \in \mathbb{Z}$ . Consequently,  $p = q$  and especially  $\mathbf{v}(\alpha') = \mathbf{v}_p(\alpha') = \mathbf{v}_q(\alpha')$ . Now we fix  $\alpha = 0$ . Then  $\mathbf{v}(\beta) = \mathbf{v}_p(\beta)$  for any  $\beta \geq 0$ . If  $\beta < 0$ , we fix  $\alpha' = \beta$  and we have seen that  $\mathbf{v}(\alpha') = \mathbf{v}_p(\alpha')$ . Hence, we can conclude that  $\mathbf{v} = \mathbf{v}_p$ .  $\square$

**Lemma 2.10** *Let  $p_\ell := (\cdot)^\ell / \ell!$ ,  $0 \leq \ell \leq d$ . Then*

$$\Delta_\beta \mathbf{v}_{p_\ell} := \mathbf{v}_{p_\ell}(\cdot + \beta) - \mathbf{v}_{p_\ell} = \sum_{k=0}^{\ell-1} \frac{\beta^{\ell-k}}{(\ell-1-k)!} \mathbf{v}_{p_k}. \quad (12)$$

**Proof:** For  $j \leq \ell$  we have

$$\begin{aligned} \frac{(x + \beta)^{\ell-j}}{(\ell-j)!} - \frac{x^{\ell-j}}{(\ell-j)!} &= \frac{1}{(\ell-j)!} \sum_{k=0}^{\ell-j-1} \binom{\ell-j}{k} \beta^{\ell-j-k} x^k \\ &= \sum_{k=0}^{\ell-j-1} \frac{\beta^{\ell-j-k}}{(\ell-j-k)!} \frac{x^k}{k!} =: q_j^*(x), \end{aligned}$$



hence, with  $q_j^* = 0$ ,  $j = \ell + 1, \dots, d$ ,

$$\mathbf{v}_p(\cdot + \beta) - \mathbf{v}_p = \begin{bmatrix} q_0^* \\ \vdots \\ q_d^* \end{bmatrix}.$$

On the other hand, for  $r \leq \ell - j - 1$ ,

$$\frac{d^r}{dx^r} q_j^*(x) = \sum_{k=r}^{\ell-j-1} \frac{\beta^{\ell-j-k} x^{k-r}}{(k-r)!(\ell-j-k)!} = \sum_{k=0}^{\ell-j-1-r} \frac{\beta^{\ell-j-k-r} x^k}{(\ell-j-k-r)! k!} = q_{j-r}^*(x),$$

and setting  $q_{\beta,\ell} := q_0^*$  yields

$$q_{\beta,\ell} = \sum_{k=0}^{\ell-1} \frac{\beta^{\ell-k}}{(\ell-1-k)!} \frac{(\cdot)^k}{k!} = \sum_{k=0}^{\ell-1} \frac{\beta^{\ell-k}}{(\ell-1-k)!} p_k$$

and proves the claim.  $\square$

**Theorem 2.11** *Let  $\mathcal{A} \in \ell^{(d+1) \times (d+1)}(\mathbb{Z})$ . If*

1.  $S_{\mathcal{A}} \mathbf{v}_p = \mathbf{v}_q$ ,  $\deg q \leq \deg p$ , for any  $p \in \mathcal{P}_{d-1}$ ,
2. There exists a finitely supported  $\mathcal{B} \in \ell^{(d+1) \times (d+1)}(\mathbb{Z})$  such that (10) holds true, i.e.,  $T_d S_{\mathcal{A}} = 2^{-d} S_{\mathcal{B}} T_d$ ,
3.  $S_{\mathcal{B}} \mathbf{e}_d = \mathbf{e}_d$ ,

then  $\mathcal{A}$  satisfies spectral condition of order  $d$ .

**Proof:** With the terminology of Lemma 2.10, set  $\mathbf{v} = \mathbf{v}_{p_d}$ . Then

$$T_d \mathbf{v} = \mathbf{e}_d = S_{\mathcal{B}} \mathbf{e}_d = S_{\mathcal{B}} T_d \mathbf{v} = 2^d T_d S_{\mathcal{A}} \mathbf{v},$$

i.e.,  $T_d (S_{\mathcal{A}} \mathbf{v} - 2^{-d} \mathbf{v}) = 0$ , which yields that  $S_{\mathcal{A}} \mathbf{v} = 2^{-d} \mathbf{v} + \mathbf{v}'$  with

$$0 = T_d \mathbf{v}' = \begin{bmatrix} \tilde{T}_{d-1} & * \\ 0 & 1 \end{bmatrix} \mathbf{v}'.$$

Hence,  $v'_d = 0$  and  $\tilde{T}_{d-1} \mathbf{v}'_{1:d-1} = 0$ , so that Proposition 2.9 implies that there exists  $\tilde{q}_{d-1} \in \mathcal{P}_{d-1}$  such that

$$S_{\mathcal{A}} \mathbf{v}_{p_d} = 2^{-d} \mathbf{v}_{p_{d-1}} + \mathbf{v}_{\tilde{q}_{d-1}}. \quad (13)$$

Since  $S_{\mathcal{A}}$  is a stationary subdivision operator with scaling factor 2, we have  $S_{\mathcal{A}}\mathbf{c}(\cdot + 2) = S_{\mathcal{A}}(\mathbf{c}(\cdot + 1))$ , hence, by Lemma 2.10,

$$\Delta_2 S_{\mathcal{A}}\mathbf{v} = S_{\mathcal{A}}(\mathbf{v}(\cdot + 1) - \mathbf{v}) = S_{\mathcal{A}}\Delta\mathbf{v}_{p_d} = S_{\mathcal{A}}\mathbf{v}_{p_{d-1}} + \sum_{j=0}^{d-2} \frac{1}{(d-1-j)!} S_{\mathcal{A}}\mathbf{v}_{p_j}$$

On the other hand, Lemma 2.10 also yields that there exists  $q \in \mathcal{P}_{d-2}$  such that

$$\begin{aligned} \Delta_2 \left( 2^{-d}\mathbf{v}_{p_{d-1}} + \mathbf{v}_{\tilde{q}_{d-1}} \right) &= 2^{-d} \sum_{j=0}^{d-1} \frac{2^{d-k}}{(d-1-j)!} \mathbf{v}_{p_j} + \mathbf{v}_q \\ &= 2^{-d+1}\mathbf{v}_{p_{d-1}} + 2^{-d} \sum_{j=0}^{d-2} \frac{2^{d-k}}{(d-1-j)!} \mathbf{v}_{p_j} + \mathbf{v}_q. \end{aligned}$$

Substituting these two identities into (13), we thus obtain that

$$S_{\mathcal{A}}\mathbf{v}_{p_{d-1}} = 2^{-d+1}\mathbf{v}_{p_{d-1}} + 2^{-d} \sum_{j=0}^{d-2} \frac{2^{d-k} - 1}{(d-1-j)!} \mathbf{v}_{p_j} + \mathbf{v}_q,$$

hence,

$$S_{\mathcal{A}}\mathbf{v}_{p_{d-1}} = 2^{-d+1}\mathbf{v}_{p_{d-1}} + \mathbf{v}_{\tilde{q}_{d-1}}, \quad \tilde{q}_{d-1} \in \mathcal{P}_{d-2}, \quad (14)$$

which is (13) with  $d$  replaced by  $d-1$ . Since the argument was only based on the stationarity of  $S_{\mathcal{A}}$ , we can repeat this process and find that

$$S_{\mathcal{A}}\mathbf{v}_{p_j} = 2^{-j}\mathbf{v}_{p_j} + \mathbf{v}_{\tilde{q}_j}, \quad q_j \in \mathcal{P}_{j-1}, \quad j = 0, \dots, d. \quad (15)$$

Since  $\mathcal{P}_j$  is spanned by  $p_0, \dots, p_j$ , (15) means that there exists a lower triangular matrix  $L$  with diagonal  $1, \dots, 2^{-d+1}, 2^{-d}$  such that

$$S_{\mathcal{A}}[\mathbf{v}_{p_0}, \dots, \mathbf{v}_{p_d}] = [\mathbf{v}_{p_0}, \dots, \mathbf{v}_{p_d}]L = [\mathbf{v}_{p_0}, \dots, \mathbf{v}_{p_d}] \begin{bmatrix} 1 & & & & \\ * & 2^{-1} & & & \\ \vdots & \ddots & \ddots & & \\ * & \dots & * & 2^{-d} & \end{bmatrix}. \quad (16)$$

Using the lower triangular matrix  $S$  which factorizes  $L$  into

$$L = SDS^{-1}, \quad D = \begin{bmatrix} 1 & & & & \\ & 2^{-1} & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & 2^{-d} \end{bmatrix},$$

we now define

$$[\mathbf{v}_0, \dots, \mathbf{v}_d] = [\mathbf{v}_{\widehat{p}_0}, \dots, \mathbf{v}_{\widehat{p}_d}] = [\mathbf{v}_{p_0}, \dots, \mathbf{v}_{p_d}] S$$

and obtain by a straightforward computation that

$$\begin{aligned} S_{\mathcal{A}}[\mathbf{v}_0, \dots, \mathbf{v}_d] &= S_{\mathcal{A}}[\mathbf{v}_{p_0}, \dots, \mathbf{v}_{p_d}] S \\ &= [\mathbf{v}_{p_0}, \dots, \mathbf{v}_{p_d}] L S \\ &= [\mathbf{v}_{p_0}, \dots, \mathbf{v}_{p_d}] S D \\ &= [\mathbf{v}_0, \dots, \mathbf{v}_d] D, \end{aligned}$$

hence  $S_{\mathcal{A}}$  satisfies spectral condition of order  $d$ .  $\square$

**Corollary 2.12** *For a mask  $\mathcal{A} \in \ell^{(d+1) \times (d+1)}(\mathbb{Z})$  the following statements are equivalent:*

1.  $\mathcal{A}$  satisfies a spectral condition of order  $d$ .
2. There exist masks  $\mathcal{B}_k$ ,  $k = 0, \dots, d$  such that

$$\begin{bmatrix} T_k & \\ & I \end{bmatrix} S_{\mathcal{A}} = 2^{-k} S_{\mathcal{B}_k} \begin{bmatrix} T_k & \\ & I \end{bmatrix}, \quad S_{\mathcal{B}_k} \mathbf{e}_k = \mathbf{e}_k.$$

3. There exists  $\mathcal{B}$  such that

$$T_d S_{\mathcal{A}} = 2^{-d} S_{\mathcal{B}} T_d, \quad S_{\mathcal{B}} \mathbf{e}_d = \mathbf{e}_d.$$

**Proof:** “1)  $\Rightarrow$  2)” has been the iterative step in [17] and the normalization is deduced as in Theorem 2.8, “2)  $\Rightarrow$  3)” is trivial and “3)  $\Rightarrow$  1)” is the statement of Theorem 2.11.  $\square$

## 2.2 Convergence

**Definition 2.13** *Let  $\mathcal{A} \in \ell^{(d+1) \times (d+1)}(\mathbb{Z})$  be a mask and  $H_{\mathcal{A}}$  the associated Hermite subdivision scheme on  $\ell^{d+1}(\mathbb{Z})$  as defined in (6). The scheme is called convergent if for any data  $\mathbf{f}_0 \in \ell^{d+1}(\mathbb{Z})$  and the corresponding sequence of refinements  $\mathbf{f}_n$ , there exists a **vector function**  $\Phi = [\phi_i]_{i=0, \dots, d} \in C(\mathbb{R}, \mathbb{R}^{d+1})$  such that for any compact  $K \subset \mathbb{R}$  there exists a sequence  $\varepsilon_n$  with limit 0 which satisfies*

$$\max_{i=0, \dots, d} \max_{\alpha \in \mathbb{Z} \cap 2^n K} \left| f_n^{(i)}(\alpha) - \phi_i(2^{-n} \alpha) \right| \leq \varepsilon_n. \quad (17)$$

The scheme  $H_{\mathcal{A}}$  is said to be  $C^\ell$ -convergent if moreover  $\phi_0 \in C^\ell(\mathbb{R}, \mathbb{R})$  and

$$\frac{d^i \phi_0}{dx^i} = \phi_i, \quad i = 0, \dots, \min\{\ell, d\}.$$

We recall the result on convergence proved in [17].

**Theorem 2.14** *Let  $\mathcal{A} \in \ell^{d+1}(\mathbb{Z})$  be a mask which satisfies spectral condition of order at least  $d$ . Suppose that for any data  $\mathbf{f}_0 \in \ell^{d+1}(\mathbb{Z})$  and associated refinement sequence  $\mathbf{f}_n$  of the Hermite scheme  $H_{\mathcal{A}}$ ,*

1. *the sequence  $\mathbf{f}_n(0)$  converges to a limit  $\mathbf{y} \in \mathbb{R}^{d+1}$ ,*
2. *(at least) one of the following two properties holds true:*
  - (a) *the associated Taylor subdivision scheme  $S_{\mathcal{B}}$  is  $C^0$ -convergent and for any initial data  $\mathbf{g}_0 = T_d \mathbf{f}_0$ , the limit function  $\Psi = \Psi_{\mathbf{g}} \in C(\mathbb{R}, \mathbb{R}^{d+1})$  satisfies*

$$\Psi = \begin{bmatrix} \mathbf{0} \\ \psi_d \end{bmatrix}, \quad \psi_d \in C(\mathbb{R}, \mathbb{R}). \quad (18)$$

- (b) *the associated complete Taylor subdivision scheme  $S_{\tilde{\mathcal{B}}}$  is contractive, that is, it is  $C^0$ -convergent and for any initial data  $\mathbf{g}_0 = \tilde{T}_d \mathbf{f}_0$ , the limit function  $\Psi = \Psi_{\mathbf{g}} \in C(\mathbb{R}, \mathbb{R}^{d+1})$  satisfies  $\Psi = \mathbf{0}$ .*

Then  $H_{\mathcal{A}}$  is  $C^d$ -convergent.

We conclude the section with an important corollary whose proof follows the lines of the proof of [17, Corollary 5].

**Corollary 2.15** *Let  $\mathcal{A} \in \ell^{d+1}(\mathbb{Z})$  be a given mask which satisfies the spectral condition of order at least  $d$ . Suppose that the associated Taylor subdivision scheme  $S_{\mathcal{B}}$  is  $C^k$ -convergent with the conditions of Theorem 2.14, then  $H_{\mathcal{A}}$  is  $C^{d+k}$ -convergent.*

### 3 Extended schemes

#### 3.1 A first extension

From a given a Hermite scheme with mask  $\mathcal{A}$ , we build a new scheme called *extended scheme* and labelled  $\mathcal{A}_+$ , whose definition is based on the approximation of a derivative by a slope. Precisely for a function  $\varphi \in C^{d+1}(\mathbb{R})$  and

$\alpha \in \mathbb{Z}$ , we use the fact that

$$\begin{aligned}\varphi^{(d+1)}\left(\frac{2\alpha}{2^{n+1}}\right) &= \varphi^{(d+1)}\left(\frac{\alpha}{2^n}\right) \simeq 2^{n-1}\left(\varphi^{(d)}\left(\frac{\alpha+1}{2^n}\right) - \varphi^{(d)}\left(\frac{\alpha-1}{2^n}\right)\right) \\ \varphi^{(d+1)}\left(\frac{2\alpha+1}{2^{n+1}}\right) &\simeq 2^n\left(\varphi^{(d)}\left(\frac{\alpha+1}{2^n}\right) - \varphi^{(d)}\left(\frac{\alpha}{2^n}\right)\right).\end{aligned}$$

Starting from any  $\mathbf{f}_0$ , the previous approximations give the construction for the additional components  $f_{n+1}^{(d+1)}$  as

$$\begin{aligned}f_{n+1}^{(d+1)}(2\alpha) &= 2^{n-1}\left(f_n^{(d)}(\alpha+1) - f_n^{(d)}(\alpha-1)\right) \\ f_{n+1}^{(d+1)}(2\alpha+1) &= 2^n\left(f_n^{(d)}(\alpha+1) - f_n^{(d)}(\alpha)\right).\end{aligned}$$

The new mask is then defined in the usual way, with just  $d$  replaced by  $d+1$  so that  $\mathbf{A}_+(\alpha) \in \mathbb{R}^{(d+2) \times (d+2)}$  has the entries

$$\begin{aligned}\mathbf{A}_+(-2) &= \left[ \begin{array}{c|c} \mathbf{A}(-2) & \mathbf{0} \\ \mathbf{0} & 2^{-d-2} \end{array} \middle| \begin{array}{c} \mathbf{0} \\ 0 \end{array} \right], & \mathbf{A}_+(-1) &= \left[ \begin{array}{c|c} \mathbf{A}(-1) & \mathbf{0} \\ \mathbf{0} & 2^{-d-1} \end{array} \middle| \begin{array}{c} \mathbf{0} \\ 0 \end{array} \right] \\ \mathbf{A}_+(1) &= \left[ \begin{array}{c|c} \mathbf{A}(1) & \mathbf{0} \\ \mathbf{0} & 2^{-d-1} \end{array} \middle| \begin{array}{c} \mathbf{0} \\ 0 \end{array} \right], & \mathbf{A}_+(-2) &= \left[ \begin{array}{c|c} \mathbf{A}(-2) & \mathbf{0} \\ \mathbf{0} & 2^{-d-2} \end{array} \middle| \begin{array}{c} \mathbf{0} \\ 0 \end{array} \right] \\ \mathbf{A}_+(\alpha) &= \left[ \begin{array}{c|c} \mathbf{A}(\alpha) & \mathbf{0} \\ \mathbf{0} & \end{array} \middle| \begin{array}{c} \mathbf{0} \\ 0 \end{array} \right] & \text{for } \alpha \notin \{-2, -1, 1, 2\}.\end{aligned} \quad (19)$$

**Proposition 3.1** *Suppose that the scheme with mask  $\mathbf{A}$  of order  $d$  satisfies the spectral condition of order  $d+1$  with the polynomials  $p_0, p_1, \dots, p_{d+1}$ . Then the extension with mask  $\mathbf{A}_+(\alpha)$  defined by (19) also satisfies spectral condition of order  $d+1$  with the polynomials  $p_0, p_1, \dots, p_{d+1}$ , thus  $T_{d+1}S_{\mathbf{A}_+} = 2^{-d-1}S_{\mathbf{B}_+}T_{d+1}$  and  $\tilde{T}_{d+1}S_{\mathbf{A}_+} = 2^{-d-1}S_{\tilde{\mathbf{B}}_+}\tilde{T}_{d+1}$ .*

**Proof:** Since  $\mathbf{A}$  satisfies spectral condition of order  $d+1$ , for any  $i \in \{0, \dots, d+1\}$  and any real numbers  $x, y$  and  $u$ , since  $p_i \in \mathcal{P}_{d+1}$ , we obtain

$$p_i^{(d)}(x+u) - p_i^{(d)}(x) = up_i^{(d+1)}(y) = \begin{cases} 0 & \text{for } i = 0, \dots, d, \\ u & \text{for } i = d+1, \end{cases}$$

since  $p_i \in \mathcal{P}_i$  and  $p_{d+1}(x) = x^{d+1}/d! + \dots$

If we define  $\mathbf{v}_i(x) = [p_i(x), \dots, p_i^{(d)}(x), p_i^{(d+1)}(x)]^T$  for  $i = 0, \dots, d+1$  then

$$\begin{aligned}
& S_{\mathcal{A}_+} \mathbf{v}_i(2\alpha) \\
&= \sum_{\beta \in \mathbb{Z}} \mathbf{A}_+(2\alpha - 2\beta) \mathbf{v}_i(\beta) = \sum_{\beta \in \mathbb{Z}} \begin{bmatrix} \mathbf{A}(2\alpha - 2\beta) \begin{bmatrix} p_i(\beta) \\ \vdots \\ p_i^{(d)}(\beta) \end{bmatrix} \\ a_{d+1,d}(2\alpha - 2\beta) p_i^{(d)}(\beta) \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2^i} \begin{bmatrix} p_i(2\alpha) \\ \vdots \\ p_i^{(d)}(2\alpha) \end{bmatrix} \\ 1/2^{d+2} (p_i^{(d)}(\alpha + 1) - p_i^{(d)}(\alpha - 1)) \end{bmatrix} = \begin{bmatrix} \frac{1}{2^i} \begin{bmatrix} p_i(2\alpha) \\ \vdots \\ p_i^{(d)}(2\alpha) \end{bmatrix} \\ 1/2^{d+1} p_i^{(d+1)}(2\alpha) \end{bmatrix} = \frac{1}{2^i} \mathbf{v}_i(2\alpha).
\end{aligned}$$

Similarly,  $S_{\mathcal{A}_+} \mathbf{v}_i(2\alpha + 1) = \frac{1}{2^i} \mathbf{v}_i(2\alpha + 1)$  and therefore, spectral condition of order  $d+1$  is satisfied by the extended scheme  $\mathcal{A}_+$ . Therefore, we can compute the masks  $\mathcal{B}_+$  and  $\widetilde{\mathcal{B}}_+$  such that  $T_{d+1} S_{\mathcal{A}_+} = 2^{-d-1} S_{\mathcal{B}_+} T_{d+1}$  and  $\widetilde{T}_{d+1} S_{\mathcal{A}_+} = 2^{-d-1} S_{\widetilde{\mathcal{B}}_+} \widetilde{T}_{d+1}$ .  $\square$

Examples of such extensions are given in the next section.

### 3.2 Factorization of extensions

We now suppose that  $\mathcal{A} \in \ell^{d \times d}(\mathbb{Z})$  is a mask which satisfies a spectral condition of order  $d' > d$  and extend this mask in such a way that the extended scheme  $\mathcal{A}' \in \ell^{d' \times d'}(\mathbb{Z})$  satisfies a Taylor extension of order  $d'$ . We begin with a simple observation concerning the Taylor operators, namely,

$$T_{d'}(z) = \begin{bmatrix} \widetilde{T}_d(z) & -W \\ 0 & T_{\bar{d}}(z) \end{bmatrix} \text{ and } \widetilde{T}_{d'}(z) = \begin{bmatrix} \widetilde{T}_d(z) & -W \\ 0 & \widetilde{T}_{\bar{d}}(z) \end{bmatrix}, \quad \bar{d} := d' - d - 1, \quad (20)$$

where

$$W = \begin{bmatrix} \frac{1}{(d+1)!} & \cdots & \frac{1}{(d+d)!} \\ \vdots & \ddots & \vdots \\ 1 & \cdots & \frac{1}{(d+1)!} \end{bmatrix} \in \mathbb{R}^{d \times \bar{d}}.$$

From (20) one can verify by straightforward computation that

$$\begin{aligned} T_{d'}^{-1}(z) &= \begin{bmatrix} \tilde{T}_d^{-1}(z) & \tilde{T}_d^{-1}(z) W T_{\bar{d}}^{-1}(z) \\ 0 & T_{\bar{d}}^{-1}(z) \end{bmatrix}, \\ \tilde{T}_{d'}^{-1}(z) &= \begin{bmatrix} \tilde{T}_d^{-1}(z) & \tilde{T}_d^{-1}(z) W \tilde{T}_{\bar{d}}^{-1}(z) \\ 0 & \tilde{T}_{\bar{d}}^{-1}(z) \end{bmatrix}, \end{aligned} \quad (21)$$

respectively. Since  $\det T_d(z) = (z^{-1} - 1)^{d-1}$  and  $\det \tilde{T}_d(z) = (z^{-1} - 1)^d$ , the matrices in (21) are well-defined for  $z \neq 1$ . Now we extend  $\mathbf{A}(z)$  to the matrix mask

$$\mathbf{A}'(z) := \begin{bmatrix} \mathbf{A}(z) & 0 \\ \mathbf{U}(z) & \mathbf{V}(z) \end{bmatrix}$$

and compute

$$\begin{aligned} T_{d'}(z) \mathbf{A}'(z) &= \begin{bmatrix} \tilde{T}_d(z) \mathbf{A}(z) - W \mathbf{U}(z) & -W \mathbf{V}(z) \\ T_{\bar{d}}(z) \mathbf{U}(z) & T_{\bar{d}}(z) \mathbf{V}(z) \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{B}(z) \tilde{T}_d(z^2) - W \mathbf{U}(z) & -W \mathbf{V}(z) \\ T_{\bar{d}}(z) \mathbf{U}(z) & T_{\bar{d}}(z) \mathbf{V}(z) \end{bmatrix}, \end{aligned}$$

hence,

$$\begin{aligned} &T_{d'}(z) \mathbf{A}'(z) T_{d'}^{-1}(z^2) \\ &= \begin{bmatrix} \mathbf{B}(z) - W \mathbf{U}(z) \tilde{T}_d^{-1}(z^2) & (\mathbf{B}(z) - W \mathbf{U}(z) T_d^{-1}(z^2)) W T_{\bar{d}}^{-1}(z^2) - W \mathbf{V}(z) T_{\bar{d}}^{-1}(z^2) \\ T_{\bar{d}}(z) \mathbf{U}(z) T_d^{-1}(z^2) & T_{\bar{d}}(z) \mathbf{U}(z) T_d^{-1}(z^2) W T_{\bar{d}}^{-1}(z^2) + T_{\bar{d}}(z) \mathbf{V}(z) T_{\bar{d}}^{-1}(z^2) \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{B}(z) - W \mathbf{U}(z) T_d^{-1}(z^2) & (\mathbf{B}(z) W - W \mathbf{U}(z) T_d^{-1}(z^2) W - W \mathbf{V}(z)) T_{\bar{d}}^{-1}(z^2) \\ T_{\bar{d}}(z) \mathbf{U}(z) T_d^{-1}(z^2) & T_{\bar{d}}(z) (\mathbf{U}(z) T_d^{-1}(z^2) W + \mathbf{V}(z)) T_{\bar{d}}^{-1}(z^2) \end{bmatrix} \end{aligned}$$

This suggests choosing  $\mathbf{U}(z) := \tilde{\mathbf{U}}(z) T_d(z^2)$  and introducing  $\tilde{\mathbf{V}}(z) := \tilde{\mathbf{U}}(z) W + \mathbf{V}(z)$ , so that we get the simplified expression

$$\begin{aligned} &T_{d'}(z) \mathbf{A}'(z) T_{d'}^{-1}(z^2) \\ &= \begin{bmatrix} \mathbf{B}(z) - W \tilde{\mathbf{U}}(z) & (\mathbf{B}(z) W - W \tilde{\mathbf{U}}(z) W - W \mathbf{V}(z)) T_{\bar{d}}^{-1}(z^2) \\ T_{\bar{d}}(z) \tilde{\mathbf{U}}(z) & T_{\bar{d}}(z) (\tilde{\mathbf{U}}(z) W + \mathbf{V}(z)) T_{\bar{d}}^{-1}(z^2) \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{B}(z) - W \tilde{\mathbf{U}}(z) & (\mathbf{B}(z) W - W \tilde{\mathbf{V}}(z)) T_{\bar{d}}^{-1}(z^2) \\ T_{\bar{d}}(z) \tilde{\mathbf{U}}(z) & T_{\bar{d}}(z) \tilde{\mathbf{V}}(z) T_{\bar{d}}^{-1}(z^2) \end{bmatrix}. \end{aligned}$$

Thus, Corollary 2.12 yields the following conclusion.

**Proposition 3.2**  $\mathcal{A}'$  satisfies spectral condition of order  $d'$ , i.e.,  $T_{d'}S_{\mathcal{A}'} = S_{\mathcal{B}'}T_{d'}$  and  $S_{\mathcal{B}'}\mathbf{e}_{d'} = \mathbf{e}_{d'}$ , if and only if

1.  $\tilde{\mathbf{V}}$  satisfies a spectral condition of order  $\bar{d} - 1$ ,
2.  $(\mathbf{B}(z)W - W\tilde{\mathbf{V}}(z))T_{\bar{d}}^{-1}(z)\mathbf{e}_{\bar{d}} = 0$ .

Since  $T_{\bar{d}}(z)\mathbf{e}_{\bar{d}} = \mathbf{e}_{\bar{d}}$ , hence also  $T_{\bar{d}}^{-1}(z)\mathbf{e}_{\bar{d}} = \mathbf{e}_{\bar{d}}$ , the second condition in Proposition 3.2 is equivalent to  $\mathbf{B}(z)W\mathbf{e}_{\bar{d}} = W\mathbf{V}(z)\mathbf{e}_{\bar{d}}$  or

$$\mathbf{B}(z) \begin{bmatrix} \frac{1}{d!} \\ \vdots \\ \frac{1}{(d'-d)!} \end{bmatrix} = W\mathbf{v}_{\bar{d}}(z), \quad (22)$$

where  $\mathbf{v}_{\bar{d}}$  denotes the last column of  $\mathbf{V}$ .

## 4 The Example of de Rham schemes

### 4.1 Construction and first properties

The construction was already proposed in [6] and some of the results and proofs are in [5]. With de Rham scheme, we lost the interpolating property but the challenge is the proof of higher regularity of the limit function.

For a Hermite subdivision scheme  $H_{\mathcal{A}}$  of degree  $d$ , with mask  $\{\mathbf{A}(\alpha)\}_{\alpha \in \mathbb{Z}}$ , let us define a new (dual) Hermite subdivision scheme by taking de Rham transform of  $\mathcal{A}$  and itself. From the initial state of the scheme  $\bar{\mathbf{f}}_0 = \mathbf{f}_0 : \mathbb{Z} \rightarrow \mathbb{R}^{d+1}$ , we define the sequence  $\bar{\mathbf{f}}_n$  for  $n > 0$  by

$$\begin{aligned} D^{n+1}\mathbf{g}(\beta) &= \sum_{\gamma \in \mathbb{Z}} \mathbf{A}(\beta - 2\gamma)D^n\bar{\mathbf{f}}_n(\gamma), \quad \beta \in \mathbb{Z} \\ D^{n+2}\mathbf{h}(\alpha) &= \sum_{\beta \in \mathbb{Z}} \mathbf{A}(\alpha - 2\beta)D^{n+1}\mathbf{g}(\beta), \quad \alpha \in \mathbb{Z} \\ \bar{\mathbf{f}}_{n+1}(\alpha) &= \mathbf{h}(2\alpha + 1), \quad \alpha \in \mathbb{Z}. \end{aligned}$$

Then it is easy to prove that  $D^{n+1}\bar{\mathbf{f}}_{n+1}(\alpha) = \sum_{\gamma \in \mathbb{Z}} \bar{\mathbf{A}}(\alpha - 2\gamma)D^n\bar{\mathbf{f}}_n(\gamma)$  where

$$\bar{\mathbf{A}}(\alpha) = D^{-1} \sum_{\beta \in \mathbb{Z}} \mathbf{A}(2\alpha + 1 - 2\beta)\mathbf{A}(\beta), \quad \alpha \in \mathbb{Z}. \quad (23)$$

**Definition 4.1** Let  $S_{\mathcal{A}}$  be a subdivision scheme. De Rham transform  $S_{\bar{\mathcal{A}}}$  of  $S_{\mathcal{A}}$  is the subdivision scheme whose mask  $\bar{\mathcal{A}}$  is defined by (23).



**Remark 4.2** Note that if the support of  $S_{\mathcal{A}}$  is  $[\sigma, \sigma']$ , then the support of its de Rham transform  $S_{\bar{\mathcal{A}}}$  is contained in  $[(3\sigma - 1)/2, (3\sigma' - 1)/2]$ .

The following Theorem and Corollary are proved in [5].

**Theorem 4.3** Let  $S_{\mathcal{A}}$  be a Hermite subdivision scheme which satisfies spectral condition of order  $\ell$ , then de Rham transform  $S_{\bar{\mathcal{A}}}$  satisfies the corresponding spectral condition  $\sum_{\beta \in \mathbb{Z}} \bar{\mathbf{A}}(\alpha - 2\beta) \bar{\mathbf{v}}_j(\beta) = \bar{\mathbf{v}}_j(\alpha)/2^j$  where  $\bar{\mathbf{v}}_j(\alpha) = [\bar{p}_j(\alpha), \bar{p}'_j(\alpha), \dots, \bar{p}_j^{(d)}(\alpha)]^T$  for an appropriate sequence of polynomials  $\bar{p}_j$  of degree  $j$ ,  $j = 0, \dots, \ell$ .

**Corollary 4.4** Let there be a Hermite scheme of degree  $d$  with mask  $\mathcal{A}$  which reproduces a basis of  $\mathcal{P}_\ell$ , then its de Rham transform with mask  $\bar{\mathcal{A}}$  satisfies the corresponding spectral condition with the sequence of polynomials  $\bar{p}_k(x) = (x - 1/2)^k/k!$ ,  $k = 0, \dots, \ell$ .

**Remark 4.5** From [9], we know that a convergent Hermite interpolatory scheme of degree  $d$  reproduces any polynomial of degree at most  $d$ .

## 4.2 An extension of a Hermite subdivision scheme of degree 1

We start with the interpolatory Hermite subdivision scheme of order  $d = 1$  proposed by Merrien [15] depending on two parameters  $\lambda, \mu$ . The non zero matrices of its mask are  $\mathbf{A}(-1)$ ,  $\mathbf{A}(0)$ ,  $\mathbf{A}(1)$ ,

$$\frac{1}{4} \begin{bmatrix} 2 & 4\lambda \\ 2(1-\mu) & \mu \end{bmatrix}, \quad \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad \frac{1}{4} \begin{bmatrix} 2 & -4\lambda \\ -2(1-\mu) & \mu \end{bmatrix}. \quad (24)$$

For any values of the parameters  $\lambda$  and  $\mu$ , the scheme reproduces polynomials of degree 1. Moreover it reproduces  $\mathcal{P}_2$  if and only if  $\lambda = -1/8$  and  $\mathcal{P}_3$  if also  $\mu = -0.5$ .

Applying de Rham transform, we end up with the dual approximating Hermite subdivision scheme  $\bar{\mathcal{A}}$ . The non zero entries of the mask,  $\bar{\mathbf{A}}(-2)$ ,  $\bar{\mathbf{A}}(-1)$ ,  $\bar{\mathbf{A}}(0)$ ,  $\bar{\mathbf{A}}(1)$ , are computed as (see also [17])

$$\begin{aligned} & \frac{1}{8} \begin{bmatrix} 2 + 4\lambda(1-\mu) & 4\lambda + 2\lambda\mu \\ 4 - 2\mu - 2\mu^2 & \mu^2 + 8\lambda(1-\mu) \end{bmatrix}, \quad \frac{1}{8} \begin{bmatrix} 6 - 4\lambda(1-\mu) & 8\lambda - 2\lambda\mu \\ 4 - 2\mu - 2\mu^2 & \mu^2 - 8\lambda(1-\mu) + 2\mu \end{bmatrix}, \\ & \frac{1}{8} \begin{bmatrix} 6 - 4\lambda(1-\mu) & -8\lambda + 2\lambda\mu \\ -4 + 2\mu + 2\mu^2 & \mu^2 - 8\lambda(1-\mu) + 2\mu \end{bmatrix}, \quad \frac{1}{8} \begin{bmatrix} 2 + 4\lambda(1-\mu) & -4\lambda - 2\lambda\mu \\ -4 + 2\mu + 2\mu^2 & \mu^2 + 8\lambda(1-\mu) \end{bmatrix}, \end{aligned} \quad (25)$$

respectively. Since  $H_{\bar{\mathcal{A}}}$  satisfies spectral condition of order  $\ell = 1$ , we can construct masks  $\mathcal{B}$  and  $\tilde{\mathcal{B}}$  supported on  $[-1, 1]$  such that

$$\begin{bmatrix} z^{-1} - 1 & -1 \\ 0 & 1 \end{bmatrix} \bar{\mathcal{A}}^*(z) = \frac{1}{2} \mathcal{B}^*(z) \begin{bmatrix} z^{-2} - 1 & -1 \\ 0 & 1 \end{bmatrix} \quad (26)$$

and

$$\begin{bmatrix} z^{-1} - 1 & -1 \\ 0 & z^{-1} - 1 \end{bmatrix} \bar{\mathcal{A}}^*(z) = \frac{1}{2} \tilde{\mathcal{B}}(z) \begin{bmatrix} z^{-2} - 1 & -1 \\ 0 & z^{-2} - 1 \end{bmatrix}. \quad (27)$$

From the above, we see that the mask  $\tilde{\mathcal{B}}$  is the difference mask of  $\mathcal{B}$  since the common eigenvector of  $\mathcal{B}_e$  and  $\tilde{\mathcal{B}}_e$  is  $(0, 1)^T$ . Hence,

$$\tilde{\mathcal{B}}^*(z) = \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} - 1 \end{bmatrix} \mathcal{B}^*(z)$$

The non-zero entries of mask  $\tilde{\mathcal{B}}$  are

$$\begin{aligned} \tilde{\mathcal{B}}(-1) &= \frac{1}{4} \begin{bmatrix} 2 + 4\lambda(1 - \mu) & 2\lambda(1 + \mu) \\ 4 - 2\mu - 2\lambda^2 & \mu^2 + 8\lambda(1 - \mu) \end{bmatrix} \\ \tilde{\mathcal{B}}(0) &= \frac{1}{4} \begin{bmatrix} 2\mu + 2\mu^2 - 8\lambda(1 - \mu) & -4\lambda(1 - \mu) - \mu^2 \\ 0 & 2\mu - 16\lambda(1 - \mu) \end{bmatrix}, \\ \tilde{\mathcal{B}}(1) &= \frac{1}{4} \begin{bmatrix} -2 + 4\lambda(1 - \mu) + 2\mu + 2\mu^2 & -6\lambda\mu - \mu^2 - 2\mu + 2 \\ -4 + 2\mu + 2\mu^2 & 4 - 2\mu - \mu^2 + 8\lambda(1 - \mu) \end{bmatrix}. \end{aligned} \quad (28)$$

Our aim is to show that the subdivision scheme associated with  $\bar{\mathcal{A}}$  can be  $C^2$  and therefore, we are going to show that  $\mathcal{B}$  can be  $C^1$ .

For our case, we fix  $\lambda = -1/8$  so that  $\mathcal{A}$  reproduces any polynomial of  $\mathcal{P}_2$ . Using Corollary 4.4, de Rham scheme  $\mathcal{A}$  satisfies spectral condition of order  $\ell = 2$  with the polynomials  $\bar{p}_0(x) = 1$ ,  $\bar{p}_1(x) = x - 1/2$  and  $\bar{p}_2(x) = 1/2(x - 1/2)^2$  and so does  $\bar{\mathcal{A}}_+$  (Definition in (19)). The computation of  $\widetilde{\mathcal{B}}_+$  gives

$$\begin{aligned} \widetilde{\mathcal{B}}_+(-1) &= \frac{1}{8} \begin{bmatrix} 6 + 2\mu & -2 - \mu & 0 \\ 16 - 8\mu - 8\mu^2 & -4 + 4\mu + 4\mu^2 & 0 \\ 0 & 4 & 0 \end{bmatrix}, \\ \widetilde{\mathcal{B}}_+(0) &= \frac{1}{4} \begin{bmatrix} 2 + 2\mu + 4\mu^2 & -\mu - 2\mu^2 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 0 \end{bmatrix} \\ \widetilde{\mathcal{B}}_+(1) &= \frac{1}{8} \begin{bmatrix} -10 + 10\mu + 8\mu^2 & 4 - 5\mu - 4\mu^2 & 1 \\ -16 + 8\mu + 8\mu^2 & 4 - 4\mu - 4\mu^2 & 4 \\ 0 & -4 & 4 \end{bmatrix}, \\ \widetilde{\mathcal{B}}_+(2) &= \frac{1}{4} \begin{bmatrix} 0 & -1 & 1 \\ 0 & -2 & 2 \\ 0 & -2 & 2 \end{bmatrix} \end{aligned} \quad (29)$$

Now following [1], to prove the contractivity of the scheme with mask  $\widetilde{\mathcal{B}}_+$ , we build the two matrices

$$U_0 = \begin{bmatrix} \widetilde{\mathcal{B}}_+(2) & \widetilde{\mathcal{B}}_+(0) & \mathbf{0} \\ \mathbf{0} & \widetilde{\mathcal{B}}_+(1) & \widetilde{\mathcal{B}}_+(-1) \\ \mathbf{0} & \widetilde{\mathcal{B}}_+(2) & \widetilde{\mathcal{B}}_+(0) \end{bmatrix} \text{ and } U_1 = \begin{bmatrix} \widetilde{\mathcal{B}}_+(1) & \widetilde{\mathcal{B}}_+(-1) & \mathbf{0} \\ \widetilde{\mathcal{B}}_+(2) & \widetilde{\mathcal{B}}_+(0) & \mathbf{0} \\ \mathbf{0} & \widetilde{\mathcal{B}}_+(1) & \widetilde{\mathcal{B}}_+(-1) \end{bmatrix}$$

And for a given positive interger  $p$ , we compute all the matrices  $M_{\mathbf{u}} = \prod_{k=1}^p U_0^{u_k} U_1^{1-u_k}$ ,  $u_k \in \{0,1\}$  and  $\mathbf{u} = [u_k]_{k=1,\dots,p}$ , then the value  $\rho_p = \max_{\mathbf{u}} \|M_{\mathbf{u}}\|_1^{1/p}$ . If  $\rho_p < 1$  then the scheme  $S_{\widetilde{\mathcal{B}}_+}$  is contractive and the Hermite scheme  $H_{\widetilde{\mathcal{A}}_+}$  is  $C^2$ . See Figures 1 and 2 where the  $C^2$ -convergence is obtained for  $\mu \in [-0.8, 0.37]$ .

### 4.3 An extension of a Hermite subdivision scheme of degree 2

In this second case, we start with the interpolatory Hermite subdivision scheme of order  $d = 2$  proposed by Merrien in [15]. This is a 2 point subdivision scheme depending on several parameters. The non zero matrices of its mask are  $\mathbf{A}(-1)$ ,  $\mathbf{A}(0)$ ,  $\mathbf{A}(1)$ ,

$$D \times \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix}, \quad D, \quad D \times \begin{bmatrix} \alpha_1 & -\alpha_2 & \alpha_3 \\ -\beta_1 & \beta_2 & -\beta_3 \\ \gamma_1 & -\gamma_2 & \gamma_3 \end{bmatrix} \text{ where } D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}. \quad (30)$$

To guarantee reproduction of degree 3 polynomials, among the free parameters, the following constraints are assumed (see [12])

$$\alpha_1 = \frac{1}{2}, \quad \gamma_1 = 0, \quad \beta_2 = \frac{1-\beta_1}{2}, \quad \gamma_3 = \frac{1-\gamma_2}{2}, \quad \alpha_3 = \frac{-1-8\alpha_2}{16}, \quad \beta_3 = \frac{2\beta_1-3}{24}. \quad (31)$$

Since the mask under consideration depends only on  $\alpha_2$ ,  $\beta_1$ ,  $\gamma_2$ , for the sake of brevity, we rename the free parameters as  $\alpha$ ,  $\beta$ ,  $\gamma$ , respectively and denote by  $\mathcal{R}$  the range of variation for them generating a  $C^2$  scheme given in [12]. Therefore, the mask we consider reads as

$$\begin{bmatrix} \frac{1}{2} & \alpha & \frac{-1-8\alpha}{16} \\ \frac{\beta}{2} & \frac{1-\beta}{4} & \frac{2\beta-3}{48} \\ 0 & \frac{\gamma}{4} & \frac{1-\gamma}{8} \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}, \quad \begin{bmatrix} \frac{1}{2} & -\alpha & \frac{-1-8\alpha}{16} \\ -\frac{\beta}{2} & \frac{1-\beta}{4} & -\frac{2\beta-3}{48} \\ 0 & -\frac{\gamma}{4} & \frac{1-\gamma}{8} \end{bmatrix}, \quad (\alpha, \beta, \gamma) \in \mathcal{R} \quad (32)$$

We apply de Rham transform for the dual approximating Hermite subdivision scheme  $\mathcal{A}$ . The non zero entries of the mask are given by

$$\begin{aligned}
\bar{\mathcal{A}}(-2) &= \begin{bmatrix} \frac{1+2\alpha\beta}{4} & \frac{\alpha(3-\beta)}{4} - \frac{(1+8\alpha)\gamma}{64} & \frac{\alpha(2\beta-3)}{48} - \frac{(1+8\alpha)(5-\gamma)}{128} \\ \frac{(3-\beta)\beta}{4} & \alpha\beta + \frac{(1-\beta)^2}{8} + \frac{\gamma(2\beta-3)}{96} & -\frac{\beta(1+8\alpha)}{16} + \frac{(3-2\beta-\gamma)(2\beta-3)}{192} \\ \frac{\beta\gamma}{2} & \frac{(3-2\beta-\gamma)\gamma}{8} & \frac{(2\beta-3)\gamma}{48} + \frac{(1-\gamma)^2}{16} \end{bmatrix} \\
\bar{\mathcal{A}}(-1) &= \begin{bmatrix} \frac{3-2\alpha\beta}{4} & \frac{\alpha(3+\beta)}{4} - \frac{(1+8\alpha)\gamma}{64} & -\frac{\alpha(2\beta-3)}{48} - \frac{(1+8\alpha)(7-\gamma)}{128} \\ \frac{(3-\beta)\beta}{4} & -\alpha\beta + \frac{(1-\beta)(3-\beta)}{8} - \frac{(2\beta-3)\gamma}{96} & \frac{\beta(1+8\alpha)}{16} + \frac{(3-2\beta+\gamma)(2\beta-3)}{192} \\ -\frac{\beta\gamma}{2} & \frac{(3+2\beta-\gamma)\gamma}{8} & -\frac{(2\beta-3)\gamma}{48} + \frac{(1-\gamma)(3-\gamma)}{16} \end{bmatrix} \\
\bar{\mathcal{A}}(0) &= \begin{bmatrix} \frac{3-2\alpha\beta}{4} & -\frac{\alpha(3+\beta)}{4} + \frac{\gamma(1+8\alpha)}{64} & -\frac{\alpha(2\beta-3)}{48} - \frac{(1+8\alpha)(7-\gamma)}{128} \\ \frac{(\beta-3)\beta}{4} & -\alpha\beta + \frac{(1-\beta)(3-\beta)}{8} - \frac{(2\beta-3)\gamma}{96} & -\frac{\beta(1+8\alpha)}{16} + \frac{(2\beta-3-\gamma)(2\beta-3)}{192} \\ -\frac{\beta\gamma}{2} & \frac{(\gamma-3-2\beta)\gamma}{8} & -\frac{(2\beta-3)\gamma}{48} + \frac{(1-\gamma)(3-\gamma)}{16} \end{bmatrix} \\
\bar{\mathcal{A}}(1) &= \begin{bmatrix} \frac{1+2\alpha\beta}{4} & \frac{\alpha(\beta-3)}{4} + \frac{(1+8\alpha)\gamma}{64} & +\frac{\alpha(2\beta-3)}{48} - \frac{(1+8\alpha)(5-\gamma)}{128} \\ \frac{(\beta-3)\beta}{4} & \alpha\beta + \frac{(1-\beta)^2}{8} + \frac{\gamma(2\beta-3)}{96} & \frac{(1+8\alpha)\beta}{16} - \frac{(2\beta-3)(3-2\beta-\gamma)}{192} \\ \frac{\beta\gamma}{2} & \frac{\gamma(2\beta+\gamma-3)}{8} & \frac{(2\beta-3)\gamma}{48} + \frac{(1-\gamma)^2}{16} \end{bmatrix}
\end{aligned}$$

Now, the parameters for interpolating scheme  $\mathcal{A}$  have been chosen in such a way that it reproduces polynomials in  $\mathcal{P}_3$ . Again, using Corollary 4.4, de Rham scheme  $\bar{\mathcal{A}}$  satisfies spectral condition of order 3 with the polynomials  $\bar{p}_0(x) = 1$ ,  $\bar{p}_1(x) = x - 1/2$ ,  $\bar{p}_2(x) = 1/2(x - 1/2)^2$  and  $\bar{p}_3(x) = 1/6(x - 1/2)^3$  and so does the extended scheme  $\bar{\mathcal{A}}_+$ .

$$\text{We recall that } \tilde{\mathcal{T}}_3^*(z) = \begin{bmatrix} z^{-1} - 1 & -1 & \frac{-1}{2} & \frac{-1}{6} \\ 0 & z^{-1} - 1 & -1 & \frac{-1}{2} \\ 0 & 0 & z^{-1} - 1 & -1 \\ 0 & 0 & 0 & z^{-1} - 1 \end{bmatrix}, \text{ and the}$$

computation of  $\widetilde{\mathcal{B}}_+$  satisfying  $\widetilde{T}_3 S_{\mathcal{A}} = 2^{-3} S_{\widetilde{\mathcal{B}}_+} \widetilde{T}_3$  gives

$$\begin{aligned}
\widetilde{\mathcal{B}}(-1) &= 8 \begin{bmatrix} \frac{1+2\alpha\beta}{4} & \frac{144\alpha-3\gamma-48\alpha\beta-24\alpha\gamma}{192} & -\frac{-16\alpha\beta-24\alpha\gamma+15-3\gamma+144\alpha}{384} & 0 \\ \frac{\beta(-\beta+3)}{4} & \frac{96\alpha\beta+12-24\beta+12\beta^2+2\beta\gamma-3\gamma}{96} & -\frac{96\alpha\beta+9+4\beta^2+2\beta\gamma-3\gamma}{192} & 0 \\ \frac{\beta\gamma}{2} & \frac{\gamma(3-2\beta-\gamma)}{8} & -\frac{-2\beta\gamma-3-3\gamma^2+9\gamma}{48} & 0 \\ 0 & 0 & \frac{1}{16} & 0 \end{bmatrix}, \\
\widetilde{\mathcal{B}}(0) &= 8 \begin{bmatrix} \frac{-4\alpha\beta-\beta\gamma+\beta^2-3\beta+2}{4} & \frac{-24\beta^2+48\beta-24-96\alpha\beta+12\gamma^2-30\gamma+20\beta\gamma}{192} & \dots \\ -\frac{\beta\gamma}{2} & +\frac{-24\beta+24-192\alpha\beta-30\gamma+20\beta\gamma+12\gamma^2}{96} & \dots \\ -\beta\gamma & \frac{4\gamma\beta}{8} & \dots \\ 0 & 0 & \dots \\ \dots & \frac{8\beta^2+30\gamma-4-4\beta\gamma+160\alpha\beta-12\gamma^2}{384} & 0 \\ \dots & \frac{-12\gamma^2+24\beta+192\alpha\beta-4\beta\gamma+30\gamma-18}{192} & 0 \\ \dots & \frac{3-4\beta\gamma}{48} & 0 \\ \dots & \frac{1}{16} & 0 \end{bmatrix}, \\
\widetilde{\mathcal{B}}(1) &= 8 \begin{bmatrix} \frac{2\alpha\beta+1+\beta^2-3\beta+\beta\gamma}{4} & \frac{144\alpha\beta+96\beta-24\beta^2-20\beta\gamma+24\alpha\gamma-39\gamma-24+12\gamma^2-144\alpha}{192} & \dots \\ \frac{\beta(\beta-3+2\gamma)}{4} & \frac{-39\gamma+12+12\gamma^2+96\alpha\beta+48\beta-12\beta^2-22\beta\gamma}{96} & \dots \\ \frac{\beta\gamma}{2} & \frac{\gamma(-3-2\beta+\gamma)}{8} & \dots \\ 0 & 0 & \dots \\ \dots & -\frac{24\alpha\gamma+48\beta+176\alpha\beta-8\beta^2-4\beta\gamma-7-144\alpha-39\gamma+12\gamma^2}{384} & \frac{1}{384} \\ \dots & -\frac{-39\gamma+24\beta-6\beta\gamma+12\gamma^2+15+96\alpha\beta-4\beta^2}{192} & \frac{1}{64} \\ \dots & -\frac{-9\gamma+3\gamma^2-2\beta\gamma+3}{48} & \frac{1}{16} \\ \dots & -\frac{1}{16} & \frac{1}{16} \end{bmatrix}, \\
\widetilde{\mathcal{B}}(2) &= 8 \begin{bmatrix} 0 & 0 & -\frac{1}{96} & \frac{1}{96} \\ 0 & 0 & -\frac{1}{32} & \frac{1}{32} \\ 0 & 0 & -\frac{1}{16} & \frac{1}{16} \end{bmatrix}.
\end{aligned}$$

We use the method of the previous subsection to prove that de Rham scheme is  $C^3$  see Figures 4 and 4.

## 5 Cardinal spline functions and Hermite schemes

In this section, we rewrite the well known cardinal splines (cf. [21]) in terms of a scalar subdivision scheme (or vector scheme of dimension 1 or Hermite scheme of degree 0). Then we extend this first subdivision scheme into Hermite schemes of different degrees.

## 5.1 Construction

Our presentation is based on a construction detailed by Michelli in [19]. Let

$$\varphi_0(x) = \chi_{[0,1]} = \begin{cases} 1 & \text{if } x \in [0, 1], \\ 0 & \text{if } x \notin [0, 1]. \end{cases}$$

For  $r = 1, 2, \dots$ , we build  $\varphi_r$  by means of autoconvolution as  $\varphi_r = \varphi_0 * \varphi_{r-1}$  or  $\varphi_r(x) = \int_{x-1}^x \varphi_{r-1}(t) dt$ .

By recursion, it is easily seen that  $\sigma_r := \text{supp}(\varphi_r) = [0, r + 1]$ , that  $\varphi_r$  is a  $C^{r-1}$  piecewise polynomial of degree  $r$  and that the functions  $\varphi_r(\cdot - \alpha)$ ,  $\alpha \in \mathbb{Z}$ , form a nonnegative partition of unity, i.e.  $\sum_{\alpha \in \mathbb{Z}} \varphi_r(x - \alpha) = 1$  and  $\varphi_r(x - \alpha) \geq 0$ . Moreover,

$$\varphi_r(x) = \frac{1}{2^r} \sum_{\alpha \in \mathbb{Z}} \binom{r+1}{\alpha} \varphi_r(2x - \alpha), \quad \binom{i}{j} = \begin{cases} \frac{i!}{j!(i-j)!} & \text{if } 0 \leq j \leq i, \\ 0 & \text{otherwise.} \end{cases}$$

Considering  $v(x) = \sum_{\alpha \in \mathbb{Z}} f_0^{(0)}(\alpha) \varphi_r(x - \alpha)$ , which is a finite sum for any  $x \in \mathbb{R}$  since  $\varphi_r(x - \alpha) = 0$  if  $x - \alpha \notin [0, r + 1]$ , we deduce for  $n \in \mathbb{N}_0$  that  $v(x) = \sum_{\alpha \in \mathbb{Z}} f_n^{(0)}(\alpha) \varphi_r(2^n x - \alpha)$  where

$$f_{n+1}^{(0)}(\alpha) = \frac{1}{2^r} \sum_{\beta \in \mathbb{Z}} \binom{r+1}{\alpha - 2\beta} f_n^{(0)}(\beta) =: \sum_{\beta \in \mathbb{Z}} a_r(\alpha - 2\beta) f_n^{(0)}(\beta), \quad \alpha \in \mathbb{Z}, \quad (33)$$

that is,

$$a_r(\alpha) = \frac{1}{2^r} \binom{r+1}{\alpha}, \quad \alpha \in \mathbb{Z}. \quad (34)$$

Moreover, the well-known derivative formula for cardinal B-spline yields

$$\frac{d^i v}{dx^i}(x) = \sum_{\alpha \in \mathbb{Z}} 2^{ni} \Delta^i f_n^{(0)}(\alpha - i) \varphi_{r-i}(2^n x - \alpha), \quad i = 0, \dots, r-1. \quad (35)$$

We have a particular case when  $i = r-1$ . Since the function  $\varphi_1$  is piecewise linear with  $\varphi_1(\alpha) = \delta_{1\alpha}$ , we obtain  $\frac{d^{r-1} v}{dx^{r-1}}(\beta/2^n) = 2^{ni} \Delta^i f_n^{(0)}(\beta - r + 1)$ .

## 5.2 Spectral properties of $S_{a_r}$

We look for the eigenpolynomials of the scheme, namely the polynomials  $p$  such that  $S_{a_r} p = \lambda p$ . Before the proposition, we have a lemma which was already given in [1] in the multidimensional case.

**Lemma 5.1** *Let  $S_a$  be a stationary scalar subdivision scheme. If  $p$  and  $q := S_a p$  are both polynomials then*

$$q(x) = \frac{1}{2} \sum_{\beta \in \mathbb{Z}} a(\beta) p\left(\frac{x - \beta}{2}\right), \quad x \in \mathbb{R}. \quad (36)$$

**Proof:** For any  $\alpha \in \mathbb{Z}$  we have  $q(2\alpha) = \sum_{\beta} a(2\alpha - 2\beta) p(\beta) = \sum_{\gamma} a(2\gamma) p(\alpha - \gamma)$ . Since the two polynomials in  $t$ ,  $q(2t)$  and  $\sum_{\gamma} a(2\gamma) p(t - \gamma)$  coincide for all integers, they are equal. With  $x = 2t$ , we obtain

$$q(x) = \sum_{\gamma} a(2\gamma) p\left(\frac{x - 2\gamma}{2}\right), \quad x \in \mathbb{R}. \quad (37)$$

Similarly,  $q(2\alpha + 1) = \sum_{\beta} a(2\alpha - 2\beta + 1) p(\beta) = \sum_{\gamma} a(2\gamma + 1) p(\alpha - \gamma)$ , from which we deduce that  $q(2t + 1) = \sum_{\gamma} a(2\gamma + 1) p(t - \gamma)$ ,  $t \in \mathbb{R}$ , and with  $x = 2t + 1$ , we obtain

$$q(x) = \sum_{\gamma} a(2\gamma + 1) p\left(\frac{x - 2\gamma - 1}{2}\right), \quad x \in \mathbb{R}. \quad (38)$$

Computing the average of (37) and (38),

$$\begin{aligned} q(x) &= \frac{1}{2} \sum_{\gamma \in \mathbb{Z}} a(2\gamma) p\left(\frac{x - 2\gamma}{2}\right) + a(2\gamma + 1) p\left(\frac{x - 2\gamma - 1}{2}\right) \\ &= \frac{1}{2} \sum_{\beta \in \mathbb{Z}} a(\beta) p\left(\frac{x - \beta}{2}\right), \quad x \in \mathbb{R}, \end{aligned}$$

yields (36).  $\square$

We also recall the following simple fact that can be proved by standard subdivision techniques.

**Lemma 5.2** *For any polynomial  $p \in \mathcal{P}_r$  also  $S_{a_r} p \in \mathcal{P}_r$ .*

**Proof:** Since  $\Delta^{r+1} S_{a_r} = 2^{-r} \Delta^{r+1}$ , we have for any  $p \in \mathcal{P}_r$  that  $\Delta^{r+1} S_{a_r} p = 2^{-r} \Delta^{r+1} p = 0$  which is equivalent to  $S_{a_r} p \in \mathcal{P}_r$ .  $\square$

**Proposition 5.3** *For a given integer  $r > 0$ , let  $\ell_r(x) = \frac{1}{r!} \prod_{j=1}^r (x + j)$ , then*

$$S_{a_r} \ell_r^{(i)} = \frac{1}{2^{r-i}} \ell_r^{(i)}, \quad i = 0, \dots, r. \quad (39)$$

**Proof:** *Case  $i = 0$ :* Let  $q_r(\alpha) := S_{a_r} \ell_r(\alpha) = \sum_{\beta \in \mathbb{Z}} a_r(\alpha - 2\beta) \ell_r(\beta)$ .

By Lemma 5.2,  $\ell_r$  and  $q_r$  are two polynomials of degree at most  $r$ . Thus it is sufficient to verify the identity  $\ell_r = 2^r q_r$  at  $r + 1$  points, namely  $\alpha = 0, -1, \dots, -r$ .

To begin, we notice that  $a_r(\alpha - 2\beta) = 0$  as soon as  $\alpha - 2\beta < 0$  or  $\alpha - 2\beta > r + 1$ . For  $\alpha \in \{-1, -2, \dots, -r\}$ , this gives  $a_r(\alpha - 2\beta) = 0$  as soon as  $\beta \geq 0$  or  $\beta < -r$ . On the other hand, for  $\beta \in [-r, 0) \cap \mathbb{Z}$ , we have  $\ell_r(\beta) = 0$  so that all the terms in the sum  $q_r$  are vanishing and therefore,  $\ell_r(\alpha) = 2^r q_r(\alpha) = 0$ ,  $-r \leq \alpha < 0$ . Finally, for  $\alpha = 0$ , an analogous support argument shows that almost all terms of the sum are vanishing except one, yielding  $q_r(0) = a_r(0) \ell_r(0) = 2^{-r} \ell_r(0)$ .

To sum up,  $2^{-r} \ell_r(\alpha) = q_r(\alpha)$  for  $\alpha = 0, -1, \dots, -r$  which concludes the case.

*Case  $i > 0$ :* Since  $\frac{1}{2^r} \ell_r = S_{a_r} \ell_r$  by Lemma 5.1, we deduce that

$$\frac{1}{2^r} \ell_r(x) = \frac{1}{2} \sum_{\beta \in \mathbb{Z}} a_r(\beta) \ell_r\left(\frac{x - \beta}{2}\right), \quad x \in \mathbb{R}.$$

Differentiating  $i$  times, this gives

$$\frac{1}{2^r} \ell_r^{(i)}(x) = \frac{1}{2} \frac{1}{2^i} \sum_{\beta \in \mathbb{Z}} a_r(\beta) \ell_r^{(i)}\left(\frac{x - \beta}{2}\right), \quad x \in \mathbb{R}. \quad (40)$$

Since  $S_{a_r} \ell_r^{(i)}$  is a polynomial, Lemma 5.1 and (40) yield

$$S_{a_r} \ell_r^{(i)}(\alpha) = \frac{1}{2} \sum_{\beta \in \mathbb{Z}} a_r(\beta) \ell_r^{(i)}\left(\frac{\alpha - \beta}{2}\right) = \frac{1}{2^{r-i}} \ell_r^{(i)}(\alpha), \quad \alpha \in \mathbb{Z}$$

which completes the proof of (39).  $\square$

### 5.3 Extension to Hermite subdivision schemes

We define Hermite subdivision schemes of degree  $d \leq r$  with mask  $\{A(\alpha)\}$  and support  $[0, r + d + 1]$  which are extensions of  $S_{a_r}$  by applying differences to the mask  $a_r$ , yielding

$$\mathbf{A}(\alpha) = \begin{bmatrix} a_r(\alpha) & 0 & \dots & 0 \\ \Delta a_r(\alpha - 1) & 0 & \dots & 0 \\ \Delta^2 a_r(\alpha - 2) & 0 & \dots & 0 \\ \vdots & & & \\ \Delta^d a_r(\alpha - d) & 0 & \dots & 0 \end{bmatrix} \quad \text{or} \quad \mathbf{A}^*(z) = \frac{(1+z)^{r+1}}{2^r} \begin{bmatrix} 1 & 0 & \dots & 0 \\ (1-z) & 0 & \dots & 0 \\ (1-z)^2 & 0 & \dots & 0 \\ \vdots & & & \\ (1-z)^d & 0 & \dots & 0 \end{bmatrix}.$$



We begin with  $\mathbf{f}_0 \in \ell^{d+1}(\mathbb{Z})$  and  $\mathbf{f}_n \in \ell^{d+1}(\mathbb{Z})$  defined by (6) and notice that  $2^{-(n+1)}f_{n+1}^{(1)}(\alpha) = \sum_{\beta \in \mathbb{Z}} \Delta a_r(\alpha - 1 - 2\beta)f_n^{(0)}(\beta) = \Delta f_{n+1}^{(0)}(\alpha - 1)$  so that for  $n \geq 1$ ,

$$f_n^{(1)}(\alpha) = 2^n \Delta f_n^{(0)}(\alpha - 1). \quad (41)$$

Similarly for  $i = 2, \dots, d$ :

$$f_n^{(i)}(\alpha) = 2^{in} \Delta^i f_n^{(0)}(\alpha - i). \quad (42)$$

Now with (33) and (35), for  $n > 0$ ,

$$\frac{d^i v}{dx^i}(x) = \sum_{\alpha \in \mathbb{Z}} f_n^{(i)}(\alpha) \varphi_{r-i}(2^n x - \alpha), \quad i = 0, \dots, d.$$

Generally, this Hermite scheme does not satisfy spectral conditions, see [18] for an example with  $r = 3$  and  $d = 2$  but it is possible to get a modified scheme.

**Proposition 5.4** *For given integers  $r > 0$  and  $d \leq r$ , there exists an upper triangular matrix  $\mathbf{R}_{r,d} \in \mathbb{R}^{(d+1) \times (d+1)}$  with 1 on the diagonal and 0 on the first row except the first term such that spectral condition is satisfied for scheme  $S_{\bar{\mathbf{A}}}$  with mask  $\bar{\mathbf{A}}(\alpha) := \mathbf{R}_{r,d}^{-1} \mathbf{A}(\alpha) \mathbf{R}_{r,d}$  with the polynomials  $p_{r-i} := \ell_r^{(i)}$  and corresponding eigenvalue  $1/2^{r-i}$ ,  $i = r$  down to  $r - d$ .*

**Proof:** Let  $\mathbf{R}_{r,d} := \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & r_{12} & \dots & r_{1d} \\ \vdots & \ddots & \ddots & \ddots & \dots \\ 0 & \dots & 0 & 1 & r_{d-1,d} \\ 0 & \dots & \dots & 0 & 1 \end{bmatrix} \in \mathbb{R}^{(d+1) \times (d+1)}.$

Since  $S_{a_r p_j} = 2^{-j} p_j$  and  $v_{p_j}^{(0)} = p_j$ , we deduce that

$$\sum_{\beta \in \mathbb{Z}} \mathbf{A}(\alpha - 2\beta) \mathbf{R}_{r,d} \mathbf{v}_{p_j}(\beta) = \begin{bmatrix} \sum_{\beta} a_r(\alpha - 2\beta) v_{p_j}^{(0)}(\beta) \\ \sum_{\beta} \Delta a_r(\alpha - 1 - 2\beta) v_{p_j}^{(0)}(\beta) \\ \vdots \\ \sum_{\beta} \Delta^d a_r(\alpha - d - 2\beta) v_{p_j}^{(0)}(\beta) \end{bmatrix} = \frac{1}{2^j} \begin{bmatrix} p_j(\alpha) \\ \Delta p_j(\alpha - 1) \\ \vdots \\ \Delta^p p_j(\alpha - d) \end{bmatrix}$$

Then

$$\begin{aligned}
S_{\mathbf{A}} v_{p_j} = \frac{1}{2^j} v_{p_j} &\Leftrightarrow \sum_{\beta \in \mathbb{Z}} \mathbf{A}(\alpha - 2\beta) \mathbf{R}_{r,d} v_{p_j}(\beta) = \frac{1}{2^j} \mathbf{R}_{r,d} v_{p_j} \\
&\Leftrightarrow \begin{bmatrix} p_j(\alpha) \\ \Delta p_j(\alpha - 1) \\ \vdots \\ \Delta^p p_j(\alpha - d) \end{bmatrix} = \mathbf{R}_{r,d} \begin{bmatrix} p_j(\alpha) \\ p_j'(\alpha) \\ \vdots \\ p_j^{(d)}(\alpha) \end{bmatrix}.
\end{aligned}$$

To obtain the components of  $\mathbf{R}_{r,d}$ , we remark that for any polynomial  $p \in \mathcal{P}_d$ , by the computation of Taylor expansions at point  $\alpha$ ,  $\Delta^i p(\alpha - i) = p^{(i)}(\alpha) + \sum_{k=i+1}^d r_{ik} p^{(k)}(\alpha)$ .  $\square$

**Lemma 5.5** *For any  $p \in \mathcal{P}_d$  and  $i \leq d$ , we have*

$$\Delta^i p(\alpha - i) = p^{(i)}(\alpha) + \sum_{k=i+1}^d p^{(k)}(\alpha) \sum_{j=1}^i \frac{(-1)^i}{i!} \binom{i}{j} (-1)^j. \quad (43)$$

**Proof:** We fix  $\alpha$  and set

$$\tilde{p} = p - \sum_{j=0}^{i-1} \frac{p^{(j)}(\alpha)}{j!} (\cdot - \alpha)^j$$

so that  $p - \tilde{p} \in \mathcal{P}_{i-1}$  and  $\tilde{p}^{(k)}(\alpha) = 0$ ,  $k = 0, \dots, i-1$  as well as  $\tilde{p}^{(k)}(\alpha) = p^{(k)}(\alpha)$ ,  $k = i, \dots, d$ . Then, by a Taylor expansion,

$$\begin{aligned}
\Delta^i p(\alpha - i) &= \Delta^i \tilde{p}(\alpha - i) = \sum_{j=0}^i \binom{i}{j} (-1)^j \sum_{k=0}^d \frac{\tilde{p}^{(k)}(\alpha)}{k!} (-j)^k \\
&= \sum_{j=0}^i \binom{i}{j} (-1)^j \sum_{k=i}^d \frac{\tilde{p}^{(k)}(\alpha)}{k!} (-j)^k \\
&= \sum_{k=i}^d \frac{(-1)^k}{k!} p^{(k)}(\alpha) \sum_{j=0}^i \binom{i}{j} (-1)^j j^k =: \sum_{k=i}^d r_{ik} p^{(k)}(\alpha),
\end{aligned}$$

where the coefficients  $r_{jk}$  are independent of  $p$ . If we specifically set  $p = \ell_r = \frac{1}{r!} x^r + \dots$  and note that

$$\Delta \ell_r = \frac{1}{r!} \prod_{j=2}^r (\cdot + j) ((x + r + 1) - (x + 1)) = \ell_{r-1}(\cdot + 1),$$

it follows that

$$\ell_{r-i} = \sum_{k=1}^d r_{ik} \ell_r^{(k)}$$

and a comparison of coefficients with respect to  $x^{r-i}$  yields  $r_{ii} = 1$  and thus (43).  $\square$

A nice byproduct of Lemma 5.5 is the combinatorial identity

$$\sum_{j=1}^k \binom{k}{j} (-1)^j j^k = (-1)^k k!.$$

In the following subsections, we give examples of modified schemes that satisfy spectral conditions and are also extensions of the initial subdivision scheme.

#### 5.4 Cardinal Spline with $r = 4$ , $d = 2$

From the Hermite scheme with mask  $\mathbf{A}$  whose coefficients in the first column are given according to the following table:

$\alpha$	0	1	2	3	4	5	6	7
$16a_{00}(\alpha)$	1	5	10	10	5	1	0	0
$16a_{10}(\alpha)$	1	4	5	0	-5	-4	-1	0
$16a_{20}(\alpha)$	1	3	1	-5	-5	1	3	1

and  $\mathbf{R}_{4,2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$ ,

we define the Hermite scheme with mask  $\bar{\mathbf{A}}(\alpha) = \mathbf{R}_{4,2}^{-1} \mathbf{A}(\alpha) \mathbf{R}_{4,2}$ .

With  $p_0(x) = 1$ ,  $p_1(x) = x + \frac{5}{2}$ ,  $p_2(x) = \frac{1}{2}x^2 + \frac{5}{2}x + \frac{35}{12}$  and we obtain that

$$S_{\mathbf{A}} \mathbf{R}_{4,2} \mathbf{b}_j = 2^{-j} \mathbf{R}_{4,2} \mathbf{b}_j \text{ or } S_{\bar{\mathbf{A}}} \mathbf{b}_j = 2^{-j} \mathbf{b}_j \text{ where } \mathbf{b}_j = \begin{bmatrix} p_j \\ p'_j \\ p''_j \end{bmatrix}, \text{ for } j = 0, 1, 2$$

#### 5.5 Cardinal spline with $r = 4$ , $d = 3$

Now, we build another extension, beginning with mask  $\mathbf{A}_+(\alpha) \in \mathbb{R}^{4 \times 4}$  with

$\alpha$	0	1	2	3	4	5	6	7	8
$16a_{00}(\alpha)$	1	5	10	10	5	1	0	0	0
$16a_{10}(\alpha)$	1	4	5	0	-5	-4	-1	0	0
$16a_{20}(\alpha)$	1	3	1	-5	-5	1	3	1	0
$16a_{30}(\alpha)$	1	2	-2	-6	0	6	2	-2	-1

and

$$\mathbf{R}_{4,3} = \mathbf{R}_+ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{6} \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then  $\bar{\mathbf{A}}_+(\alpha) = \mathbf{R}_+^{-1} \mathbf{A}_+(\alpha) \mathbf{R}_+$ .

With  $p_0(x) = 1$ ,  $p_1(x) = x + \frac{5}{2}$ ,  $p_2(x) = \frac{1}{2}x^2 + \frac{5}{2}x + \frac{35}{12}$ ,  $p_3(x) = \frac{1}{6}x^3 + \frac{5}{4}x^2 + \frac{35}{12}x + \frac{25}{12}$ , we obtain that  $S_{\mathbf{A}_+} \mathbf{R}_+ \mathbf{b}_j = 2^{-j} \mathbf{R}_+ \mathbf{b}_j$  or  $S_{\bar{\mathbf{A}}_+} \mathbf{b}_j = 2^{-j} \mathbf{b}_j$

where  $\mathbf{b}_j = \begin{bmatrix} p_j \\ p_j' \\ p_j'' \\ p_j^{(3)} \end{bmatrix}$ , for  $j = 0, 1, 2, 3$

For  $\bar{\mathbf{A}}_+$ , some straightforward computations yield the *symbol representation*,

$$\bar{\mathbf{A}}_+^*(z) = \frac{(1+z)^5}{96} \begin{bmatrix} 6 & 0 & 0 & 0 \\ (z-1)(11-7z+2z^2) & 0 & 0 & 0 \\ 6(z-1)^2(z-2) & 0 & 0 & 0 \\ 6(z-1)^3 & 0 & 0 & 0 \end{bmatrix}$$

and we obtain that  $\tilde{\mathbf{B}}_+$  defined by  $\tilde{\mathbf{T}}_3 \bar{\mathbf{A}}_+ = 2^{-3} \tilde{\mathbf{B}}_+ \tilde{\mathbf{T}}_3$  has coefficients  $\tilde{\mathbf{B}}_+(0) = \mathbf{0}$  and

$$\tilde{\mathbf{B}}_+(7) = \frac{1}{72} \begin{bmatrix} -36 & 36 & -18 & 6 \\ -66 & 66 & -33 & 11 \\ -72 & 72 & -36 & 12 \\ -36 & 36 & -18 & 6 \end{bmatrix} = \tilde{\mathbf{B}}_+(8), \quad \tilde{\mathbf{B}}_+(5) = \frac{1}{72} \begin{bmatrix} 108 & -72 & 0 & 24 \\ 198 & -132 & 0 & 44 \\ 216 & -144 & 0 & 48 \\ 108 & -72 & 0 & 24 \end{bmatrix} = \tilde{\mathbf{B}}_+(6),$$

$$\tilde{\mathbf{B}}_+(3) = \frac{1}{72} \begin{bmatrix} -108 & 36 & 18 & 6 \\ -198 & 66 & 33 & 11 \\ -216 & 72 & 36 & 12 \\ -108 & 36 & 18 & 6 \end{bmatrix} = \tilde{\mathbf{B}}_+(4), \quad \tilde{\mathbf{B}}_+(1) = \frac{1}{72} \begin{bmatrix} 36 & 0 & 0 & 0 \\ 66 & 0 & 0 & 0 \\ 72 & 0 & 0 & 0 \\ 36 & 0 & 0 & 0 \end{bmatrix} = \tilde{\mathbf{B}}_+(2).$$

We use the same tools as in the previous section, building the two matrices in  $\mathbb{R}^{32 \times 32}$ :

$$\begin{aligned}
U_0 &:= \begin{bmatrix} \tilde{\mathbf{B}}_+(1) & \tilde{\mathbf{B}}_+(-1) & \mathbf{0} & \dots & \mathbf{0} \\ \tilde{\mathbf{B}}_+(2) & \tilde{\mathbf{B}}_+(0) & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & \vdots & & & \vdots \\ \tilde{\mathbf{B}}_+(7) & \tilde{\mathbf{B}}_+(5) & \tilde{\mathbf{B}}_+(3) & \dots & \mathbf{0} \\ \tilde{\mathbf{B}}_+(8) & \tilde{\mathbf{B}}_+(6) & \tilde{\mathbf{B}}_+(4) & \dots & \mathbf{0} \end{bmatrix}, \\
U_1 &:= \begin{bmatrix} \tilde{\mathbf{B}}_+(2) & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \tilde{\mathbf{B}}_+(3) & \tilde{\mathbf{B}}_+(1) & \mathbf{0} & & \\ \vdots & \vdots & & & \vdots \\ \tilde{\mathbf{B}}_+(8) & \tilde{\mathbf{B}}_+(6) & \tilde{\mathbf{B}}_+(4) & \dots & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{B}}_+(7) & \tilde{\mathbf{B}}_+(5) & \dots & \mathbf{0} \end{bmatrix}
\end{aligned}$$

Again for a positive integer  $p$ , if  $M_{\mathbf{u}} := \prod_{k=1}^p \mathbf{U}_0^{u_k} \mathbf{U}_1^{1-u_k}$ , for all  $u_k \in \{0, 1\}$  and  $\mathbf{u} =: [u_k]_{k=1, \dots, p}$ , if  $\rho_p := \max_{\mathbf{u} \in \{0, 1\}^p} \|M_{\mathbf{u}}\|_1^{1/p}$ , then a numerical computation gives  $\rho_8 = 0.9173$ . Therefore, the operator  $S_{\tilde{\mathbf{B}}_+}$  is contractive, proving that the Hermite scheme  $S_{\tilde{\mathbf{A}}_+}$  is  $C^3$ . Now, since  $\mathbf{R}_{4,3}$  is triangular with 1 on its diagonal, it is easy to deduce that for  $r = 4$  and  $d = 3$ , the scheme  $S_{\mathbf{A}}$  and  $S_{a_4}$  are  $C^3$  which is a well known property of cardinal splines.

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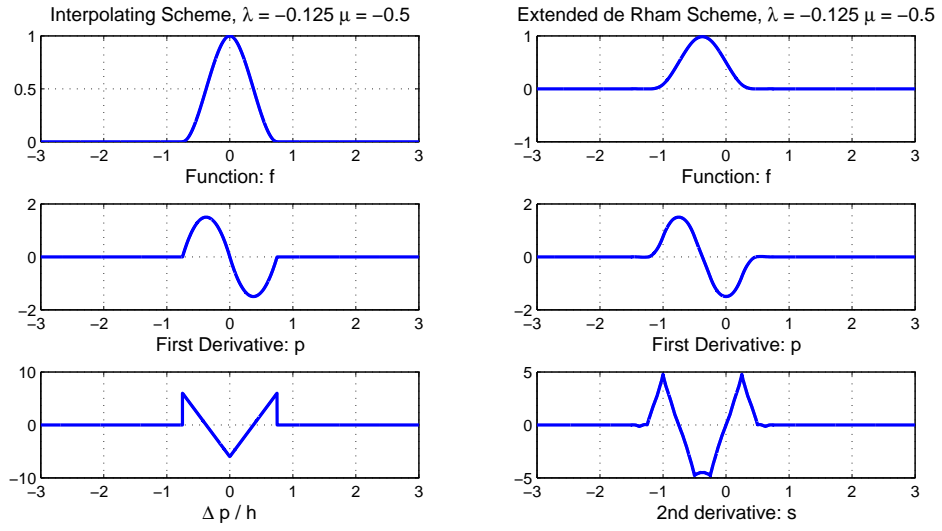


Figure 1: Interpolating and Extended de Rham Schemes

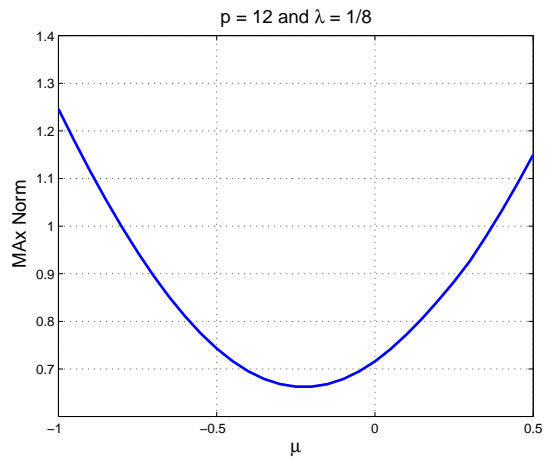


Figure 2:  $C^2$ -convergence when  $MaxNorm < 1$ .



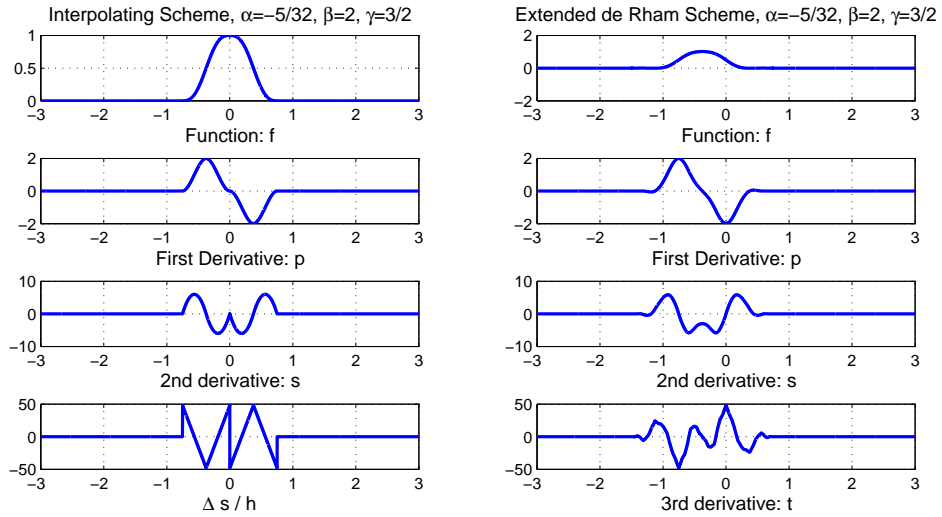


Figure 3: Interpolating and Extended de Rham Schemes

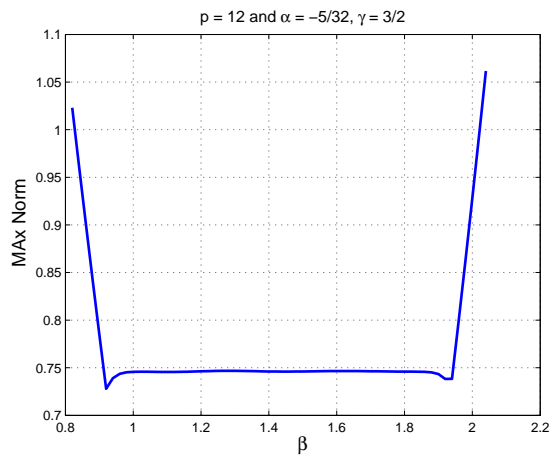


Figure 4: The  $C^3$ -convergence is obtained for  $\beta \in [0.82, 2.05]$