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Boris Haspot, Ewelina Zatorska

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Abstract

We consider the one-dimensional Cauchy problem for the Navier-Stokes equations with degenerate viscosity coefficient in highly compressible regime. It corresponds to the compressible Navier-Stokes system with large Mach number equal to $\frac{1}{\sqrt{\varepsilon}}$ for ε going to 0. When the initial velocity is related to the gradient of the initial density, a solution to the continuity equation– ρ_ε converges to the unique solution to the porous medium equation [13, 14]. For viscosity coefficient $\mu(\rho_\varepsilon) = \rho_\varepsilon^\alpha$ with $\alpha > 1$, we obtain a rate of convergence of ρ_ε in $L^\infty(0,T;H^{-1}(\mathbb{R}))$; for $1 < \alpha \leq \frac{3}{2}$ the solution ρ_ε converges in $L^\infty(0,T;L^2(\mathbb{R}))$. For compactly supported initial data, we prove that most of the mass corresponding to solution ρ_ε is located in the support of the solution to the porous medium equation. The mass outside this support is small in terms of ε.

1 Introduction

The compressible Navier-Stokes equations in the multidimensional case with the Mach number equal to $\varepsilon^{-1/2}$ read:

$$\begin{cases} \partial_t \varrho_\varepsilon + \operatorname{div}(\varrho_\varepsilon u_\varepsilon) = 0, \\ \partial_t(\varrho_\varepsilon u_\varepsilon) + \operatorname{div}(\varrho_\varepsilon u_\varepsilon \otimes u_\varepsilon) - \operatorname{div}(2\mu(\varrho_\varepsilon)\mathrm{D}(u_\varepsilon)) - \nabla(\lambda(\varrho_\varepsilon)\operatorname{div} u_\varepsilon) + \varepsilon\nabla P(\varrho_\varepsilon) = 0, \\ \varrho_\varepsilon(0,x) = \varrho_0(x), \quad \varrho_\varepsilon u_\varepsilon(0,x) = m_0(x), \end{cases} \tag{1}$$

where $\varrho_\varepsilon = \varrho_\varepsilon(t,x)$ and $u_\varepsilon = u_\varepsilon(t,x)$ denote the unknown density and the velocity vector field, respectively, $P(\varrho_\varepsilon) = \varrho_\varepsilon^\gamma$, $\gamma > 1$, denotes the pressure, $\mathrm{D}(u_\varepsilon) = \frac{\nabla u_\varepsilon + \nabla^t u_\varepsilon}{2}$ denotes the symmetric part of the gradient of u_ε, $\mu(\varrho_\varepsilon)$ and $\lambda(\varrho_\varepsilon)$ denote the two Lamé viscosity coefficients satisfying

$$\mu(\varrho_\varepsilon) > 0, \quad 2\mu(\varrho_\varepsilon) + N\lambda(\varrho_\varepsilon) \geq 0,$$

where N is the space dimension.

Our purpose in this paper is to study the asymptotic behaviour of the global weak solutions to the one-dimensional Cauchy problem for (1), when ε goes to 0, which corresponds to the

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highly compressible limit. It is the opposite to the low Mach number limit leading to the incompressible Navier-Stokes equations. The later has been intensively studied, especially in the case of constant viscosity coefficients. The first works in the framework of global weak solutions are due to B. Desjardins and E. Grenier [9] and P.-L. Lions and N. Masmoudi [24] (see also [10]). The case of global strong solution with small initial data in critical spaces for scaling of equations was considered by R. Danchin in [8].

Up to our knowledge the highly compressible regime has not been so well studied. The only results we are aware of concern Euler equations, see for example [7], [12]. The main difficulty to pass to the limit $\varepsilon \to 0$ in (1) with constant viscosity coefficients is due to lack of uniform estimates on $\{\varrho_\varepsilon\}_{\varepsilon>0}$ (only the L^1 norm is conserved). In this case, it is not clear whether the sequence of weak solutions to (1) converges to solution to the pressureless system. However, in the case of viscosities satisfying particular algebraic relation

$$\lambda(\varrho_\varepsilon) = 2\varrho\mu'(\varrho_\varepsilon) - 2\mu(\varrho_\varepsilon), \tag{2}$$

additional compactness information is available. The first author proved in [13, 14] that there exists a limit of the sequence $(\rho_\varepsilon, u_\varepsilon)$, called *a quasi-solution*, satisfying the pressureless system when $N \geq 2$. Moreover, the density of this system corresponds to the solution to the fast diffusion, the heat or the porous medium equation, depending on the choice of the viscosity coefficients. Relation (2) introduced by D. Bresch and B. Desjardins in [1,2] provides a new entropy structure which ensures estimate on the gradient of the density. In particular, it gives enough compactness information to pass to the limit when ε goes to 0.
Let us also mention that relation (2) is verified by the viscosity coefficients of the so-called *shallow water system*, for which the proof of existence of global weak solutions has been recently delivered by A. Vasseur and C. Yu [29], the extension to the case (2) can be found in [30], see also [11, 17]. Some ideas using *the cold pressure* which is singular at the vacuum can be found in [3,28,37] and in [4] using the notion of κ-entropy solutions. Stability of global weak solutions to (1) with (2) for ε fixed was proved by A. Mellet and A. Vasseur [25] using new energy estimate improving the integrability of the velocity.

In the following paper, we extend [13,14] by proving rate of convergence of ϱ_ε to a solution to the porous medium equation in a suitable functional framework. We restrict to the one-dimensional Euclidean space $\mathbb{R}$, for which system (1) may be rewritten with a single general viscosity coefficient $\mu(\varrho) > 0$ in the following form

$$\begin{cases} \partial_t \varrho_\varepsilon + \partial_x(\varrho_\varepsilon u_\varepsilon) = 0 \\ \partial_t\left(\varrho_\varepsilon u_\varepsilon\right) + \partial_x(\varrho_\varepsilon u_\varepsilon^2) - \partial_x(\mu(\varrho_\varepsilon)\partial_x u_\varepsilon) + \varepsilon\partial_x P(\varrho_\varepsilon) = 0. \end{cases} \tag{3}$$

Following the formulation used in [15] (see also [16] and [4]), we rewrite (3) in terms of the *effective velocity* $v_\varepsilon = u_\varepsilon + \partial_x\varphi(\varrho_\varepsilon)$ where $\varphi'(\varrho_\varepsilon) = \frac{\mu(\varrho_\varepsilon)}{\varrho_\varepsilon^2}$, it gives

$$\begin{cases} \partial_t \varrho_\varepsilon - \partial_x\left(\dfrac{\mu(\varrho_\varepsilon)}{\varrho_\varepsilon}\partial_x\varrho_\varepsilon\right) + \partial_x(\varrho_\varepsilon v_\varepsilon) = 0, \\ \partial_t\left(\varrho_\varepsilon v_\varepsilon\right) + \partial_x(\varrho_\varepsilon u_\varepsilon v_\varepsilon) + \varepsilon\partial_x P(\varrho_\varepsilon) = 0. \end{cases} \tag{4}$$

New regularizing parabolic effects on the density were observed in [16] in the framework of strong solution for initial density far away from the vacuum. The same concept was recently used in [4, 5] to construct regular solutions approximating global weak solutions to the full low Mach number limit system and the compressible system.

Our starting observation is that when $v_\varepsilon(0,\cdot)=0$, the sequence $\{\rho_\varepsilon v_\varepsilon\}_{\varepsilon>0}$ converges to 0 in a suitable functional framework. It heuristically implies that the limit $\widetilde{\rho}=\lim_{\varepsilon\to 0}\rho_\varepsilon$ solves the fast diffusion, the heat or the porous medium equation

$$\begin{cases}\partial_t\widetilde{\rho}-\partial_x\left(\dfrac{\mu(\widetilde{\rho})}{\widetilde{\rho}}\partial_x\widetilde{\rho}\right)=0,\\ \widetilde{\rho}(0,x)=\rho_0(x),\end{cases}\tag{5}$$

where

$$\mu(\widetilde{\rho})=\widetilde{\rho}^{\alpha}\ \text{ for }\ \alpha>0.\tag{6}$$

The rate of convergence of $(\rho_\varepsilon-\widetilde{\rho})$ can be obtained by employing a duality method from [32], used by J. L. Vázquez to prove uniqueness of the very weak solutions to the porous medium equation.

In the present paper we restrict to the case $\alpha>\frac{1}{2}$ because we are particularly interested in dealing with compactly supported initial data. When $0<\alpha<\frac{1}{2}$ the entropy discovered by D. Bresch and B. Desjardins (see [2]) allows to bound the density from below. Indeed, in [26] A. Mellet and A. Vasseur proved the existence of global strong solution to (3) for initial density far away from the vacuum. The main ingredient of the proof was to use entropy from [2] to estimate $\partial_x(\rho^{\alpha-\frac{1}{2}})$ in $L^\infty(0,T;L^2(\mathbb{R}))$ for all $T>0$. This implies boundedness of $\frac{1}{\rho}$ in $L^\infty(0,T;L^\infty(\mathbb{R}))$ which allows to show sufficient regularity of solution in order to prove the uniqueness. This result has been recently extended by the first author to the case of the shallow water system ($\alpha=1$) [15]. The main idea was to prove L^∞ bound of v_ε appearing in (4) by using the structure of the transport equation for v_ε. It allows to bound $\frac{1}{\rho}$ by application of the maximum principle to the continuity equation.

Finally, let us mention several important results concerning existence of global weak and strong solutions to system (3) with initial density close to vacuum. Presenting the exhaustive overview of the theory in this field is beyond of the scope of this article.

The existence of global weak solutions to system (3) was proven by Q. Jiu and Z. Xin in [20]. The main difficulty was to construct approximate regular solutions to (3) which verify all the entropies used in [25]. To construct such an approximation, Q. Jiu and Z. Xin employed the result on existence of global strong solution on bounded domain $[-M,M]$ with Dirichlet boundary conditions. The latter result was proven by H-L. Li., J. Li and Z. Xin [23] who used energy estimate to control the Lipschitz norm of the velocity.

There are also several interesting papers on system (3) with free boundary corresponding to the interface with the vacuum. The first result in this direction is due to D. Hoff and D. Serre in [18] who proved the existence of global strong solution with discontinuous density at the interface for constant viscosity coefficients. An interesting extension of this result to the Neumann boundary conditions is due to P. B. Mucha [27] and for the reactive system see [22]. S. Jiang, Z. Xin and P. Zhang in [19] obtained similar results in the case of degenerate viscosity coefficients. Note that the initial density considered in [19] is discontinuous at the interface and therefore is not compatible with the entropy introduced in [2]. We shall discuss this problem with more details later, in Section 5.

The free boundary problem with initial density continuously connecting to the vacuum was analyzed by T. Yang and H. Zhao in [35]. They proved the existence of weak solution in a finite time interval in Lagrangian coordinates by showing that for sufficiently small T the $L^\infty(0,T;L^\infty(\Omega))$ norm of $\rho^{\alpha+1}\partial_x u$ is bounded. Unfortunately, this is not sufficient to come back to the Eulerian description since it is not clear how to solve the ordinary differential equation describing the evolution of the free boundary. In other words, the change of coordinates may not be rigorously justified, see also [33, 34, 36].

The article is structured in the following way. In Section 2 we formulate the notion of the weak solution to problem (3) for ε fixed and we state our main results. Next, in Section 3 we recall the result from [20] on existence of global in time weak solutions and derive the estimates independent of ε. In the second part of this section we recall basic properties of the solution to the one-dimensional porous medium equation taken over from the monograph of J.-L. Vazquez [32]. In Section 4 we prove the main results of the paper and in the end, in Section 5 we explain difficulties arising in the study of the free boundary problems and discuss possible extensions of our results.

Notation: The letter C denotes generic constant, whose value may change from line to line.

2 Main result

Below we give the definition of a global weak solution to problem (3) which we denote by $(\rho_\varepsilon, u_\varepsilon)$ or by (ρ, u) when no confusion can arise. The existence of such solution was proven by Q. Jiu and Z. Xin in [20]. We then formulate our main theorems in which we compare ρ_ε with $\widetilde{\rho}$ the solution to the porous medium equation.

Let us start with introducing the hypothesis on the initial data, we assume that

$$\begin{aligned} &\rho_0 \geq 0,\ m_0 = 0 \ \text{ a.e. on } \ \{x \in \mathbb{R};\, \rho_0(x) = 0\},\\ &\rho_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}), \quad \partial_x \rho_0^{\alpha-\frac{1}{2}} \in L^2(\mathbb{R})\\ &\frac{m_0^2}{\rho_0} \in L^1(\mathbb{R}), \quad \frac{|m_0|^{2+\nu}}{\rho_0^{1+\nu}} \in L^1(\mathbb{R}), \end{aligned} \tag{7}$$

where $\alpha > \frac{1}{2}$ and $\nu > 0$ arbitrary small.

Definition 1 *Let ε be fixed. We say that a pair of functions (ρ, u) is a weak solution to system (3) provided that:*

- *The density $\rho \geq 0$ a.e., and the following regularity properties hold*

$$\begin{aligned} &\rho \in L^\infty(0,T; L^1(\mathbb{R}) \cap L^\gamma(\mathbb{R})) \cap C([0,+\infty), (W^{1,\infty}(\mathbb{R}))'),\\ &\partial_x \rho^{\alpha-\frac{1}{2}} \in L^\infty(0,T; L^2(\mathbb{R})), \sqrt{\rho} u \in L^\infty(0,T; L^2(\mathbb{R})), \end{aligned}$$

  where $(W^{1,\infty}(\mathbb{R}))'$ is the dual space of $W^{1,\infty}(\mathbb{R})$.

- *For any $t_2 \geq t_1 \geq 0$ and any $\psi \in C^1([t_1, t_2] \times \mathbb{R})$, the continuity equation is satisfied in the following sense:*

$$\int_{\mathbb{R}} \rho\psi(t_2)\,\mathrm{d}x - \int_{\mathbb{R}} \rho\psi(t_1)\,\mathrm{d}x = \int_{t_1}^{t_2} \int_{\mathbb{R}} (\rho\partial_t\psi + \rho u \partial_x \psi)\,\mathrm{d}x\,\mathrm{d}t. \tag{8}$$

  Moreover, for $\rho v = \rho u - \rho\partial_x\varphi(\rho)$, where $\varphi'(\rho) = \frac{\mu(\rho)}{\rho^2}$, the following equality is satisfied

$$\int_{\mathbb{R}} \rho\psi(t_2)\,\mathrm{d}x - \int_{\mathbb{R}} \rho\psi(t_1)\,\mathrm{d}x = \int_{t_1}^{t_2} \int_{\mathbb{R}} (\rho\partial_t\psi + \rho v \partial_x \psi - \rho\partial_x\varphi(\rho)\partial_x\psi)\,\mathrm{d}x\,\mathrm{d}t. \tag{9}$$

- *For any $\psi \in C_0^\infty([0,T)\times\mathbb{R})$ the momentum equation is satisfied in the following sense:*

$$\int_{\mathbb{R}} m_0\psi(0)\,\mathrm{d}x + \int_0^T\int_{\mathbb{R}} \left(\sqrt{\rho}(\sqrt{\rho}u)\partial_t\psi + ((\sqrt{\rho}u)^2 + \varepsilon\rho^\gamma)\partial_x\psi\right)\,\mathrm{d}x\,\mathrm{d}t + \langle\rho^\alpha\partial_x u, \partial_x\psi\rangle = 0, \tag{10}$$

where the diffusion term is defined as follows:

$$\langle\rho^\alpha\partial_x u, \partial_x\psi\rangle = -\int_0^T\int_{\mathbb{R}} \rho^{\alpha-\frac{1}{2}}\sqrt{\rho}u\partial_x\psi\,\mathrm{d}x\,\mathrm{d}t - \frac{2\alpha}{2\alpha-1}\int_0^T\int_{\mathbb{R}} \partial_x(\rho^{\alpha-\frac{1}{2}})\sqrt{\rho}u\psi\,\mathrm{d}x\,\mathrm{d}t. \tag{11}$$

Below we give our first main result on the convergence of $(\rho_\varepsilon, u_\varepsilon)$ to a solution to the associated porous medium equation.

Theorem 1 *Let $\gamma > 1$, $\alpha > 1$. Moreover, assume that the initial data (ρ_0, m_0) satisfy (7) and that*

$$\frac{m_0}{\sqrt{\rho_0}} = -\sqrt{\rho_0}\partial_x\varphi(\rho_0). \tag{12}$$

1. System (3) admits a global weak solution $(\rho_\varepsilon, u_\varepsilon)$ in the sense of Definition 1. In addition, ρ_ε converges strongly to $\widetilde{\rho}$ – the strong solution to the porous medium equation

$$\begin{cases} \partial_t\widetilde{\rho} - \dfrac{1}{\alpha}\partial_{xx}\widetilde{\rho}^\alpha = 0, \\ \widetilde{\rho}(0,x) = \rho_0(x), \end{cases} \tag{13}$$

in the following sense, there exists a constant $C > 0$ depending on ρ_0 such that

$$\|(\rho - \rho_\varepsilon)(t)\|_{H^{-1}(\mathbb{R})} \le C\varepsilon^{\frac{1}{2}}t^{\frac{1}{2}}. \tag{14}$$

2. For $1 < \alpha \le \frac{3}{2}$ there exists a constant $C > 0$ depending on ρ_0 such that

$$\|(\rho_\varepsilon - \widetilde{\rho})(t)\|_{L^2(\mathbb{R})} \le C\varepsilon^{\frac{1}{4}}(1 + t^{\frac{1}{4}}\mathbf{1}_{t\ge1}), \tag{15}$$

for all $t \ge 0$.
If in addition $\partial_x\rho_0^{\alpha-1} \in L^\infty(\mathbb{R})$ we have

$$\|(\rho_\varepsilon - \widetilde{\rho})(t)\|_{L^2(\mathbb{R})} \le C\varepsilon^{\frac{1}{4}}t^{\frac{1}{4}}. \tag{16}$$

for all $t \ge 0$.

The next main result is the following.

Theorem 2 *Let $\gamma > 1$, $1 < \alpha \le \frac{3}{2}$. Assume that the initial conditions satisfy (7) and that there exist two constants $-\infty < a < b < +\infty$ such that*

$$\mathrm{supp}[\rho_0] \subset [a,b].$$

Then, there exist constants $C > 0$ and $-\infty < a_1 < b_1 < \infty$ such that

$$\Omega_t := \mathrm{supp}[\widetilde{\rho}(t,\cdot)] \subset [a_1 - Ct^{\frac{1}{\alpha+1}}, b_1 + Ct^{\frac{1}{\alpha+1}}]. \tag{17}$$

Moreover, there exists a constant $C > 0$ depending only on ρ_0 such that

$$\|\rho_\varepsilon(t)\mathbf{1}_{\Omega_t^c}\|_{L^1(\mathbb{R})} \leq C\varepsilon^{\frac{1}{4}}(t^{\frac{1}{4}}\mathbf{1}_{t\geq 1} + 1)(1 + t^{\frac{1}{2(\alpha+1)}}), \tag{18}$$

where Ω_t^c denotes the complement of Ω_t.
If in addition $\partial_x \rho_0^{\alpha-1} \in L^\infty(\mathbb{R})$ then

$$\|\rho_\varepsilon(t)\mathbf{1}_{\Omega_t^c}\|_{L^1(\mathbb{R})} \leq C\varepsilon^{\frac{1}{4}} t^{\frac{1}{4}}(1 + t^{\frac{1}{2(\alpha+1)}}). \tag{19}$$

Remark 1 *The previous result is restricted to the case $1 < \alpha \leq \frac{3}{2}$ for technical reasons. This assumption allows to bound $\partial_x \widetilde{\rho}$ and $\partial_x \rho_\varepsilon$ in $L^\infty(0, T; L^2(\mathbb{R}))$.*

Remark 2 *Theorem 1 provides a rate of convergence of $(\rho_\varepsilon - \widetilde{\rho})$ and so it extends the results of the first author proven in [13,14] when $N \geq 2$. Theorem 2 implies that the mass distributed outside Ω_t–the support of solution to the porous medium equation, is small of order $\varepsilon^{\frac{1}{4}}$. The evolution of the support of Ω_t is known since the interfaces are described by the Darcy law (see Theorem 6). However, ρ_ε may not be a solution to the free boundary problem. In other words, there is no reason for $\mathrm{supp}\rho_\varepsilon(t, \cdot)$ to remain compact for all times $t > 0$. However, the mass that may be spread outside of Ω_t remains small. In this sense $\widetilde{\rho}(t, \cdot)$ is a good approximation of $\rho_\varepsilon(t, \cdot)$ (for any $t \leq \max(1, c(\varepsilon)^{-\frac{1}{1+\frac{2}{1+\alpha}}})$ for c small) and we can think about the "quasi finite propagation" of the mass.*

3 Overview of known results

This section is devoted to the overview and summary of results that will be used to prove our main theorems.

3.1 Weak solutions to system (3)

We first recall the result on global in time existence of weak solutions to system (3) with ε being fixed, whose proof can be found in the paper of Q. Jiu, Z. Xin ([20], Theorem 2.1.)

Theorem 3 (Existence of weak solutions) *Let $\gamma > 1$, $\alpha > 1/2$ and let $\varepsilon > 0$ be fixed. Assume that the initial conditions satisfy (7). Then, system (3) possesses a global in time weak solution (ρ, u) in the sense of Definition 1. Moreover, this solution satisfies the following inequalities uniformly with respect to ε*

$$\rho \in C((0, T) \times \mathbb{R}), \tag{20}$$

$$\sup_{t\in[0,T]} \int_{\mathbb{R}} \rho \, \mathrm{d}x + \max_{(t,x)\in[0,T]\times\mathbb{R}} \rho \leq c, \tag{21}$$

$$\sup_{t\in[0,T]} \int_{\mathbb{R}} \left(|\sqrt{\rho}u|^2 + \partial_x(\rho^{\alpha-\frac{1}{2}}) + \frac{\varepsilon}{\gamma - 1}\rho^\gamma \right) \mathrm{d}x + \int_0^T \int_{\mathbb{R}} \left(\varepsilon[\partial_x(\rho^{\frac{\gamma+\alpha-1}{2}})]^2 + \Lambda^2 \right) \mathrm{d}x \, \mathrm{d}t \leq c, \tag{22}$$

where c depends only on the initial data and $\Lambda \in L^2((0, T) \times \mathbb{R})$ is a function satisfying:

$$\int_0^T \int_{\mathbb{R}} \Lambda\varphi \, \mathrm{d}x \, \mathrm{d}t = -\int_0^T \int_{\mathbb{R}} \rho^{\alpha-\frac{1}{2}} \sqrt{\rho} u \partial_x \varphi \, \mathrm{d}x \, \mathrm{d}t - \frac{2\alpha}{2\alpha - 1} \int_0^T \int_{\mathbb{R}} \partial_x(\rho^{\alpha-\frac{1}{2}}) \sqrt{\rho} u\varphi \, \mathrm{d}x \, \mathrm{d}t. \tag{23}$$

Proof. The only element that we have to check is verification of the artificial formulation of the continuity equation (9). The rest was already proven by Q. Jiu, Z. Xin in [20] (see Theorem 2.1). Their proof combines the existence of global solutions on bounded domain $[-M, M]$, proven in [23] with the diagonal procedure allowing to let $M \to \infty$. Therefore, we omit the details of construction of this approximate solution and focus only on justfication that the formula (9) is valid.

Let $\{\rho_\delta, u_\delta\}_{\delta>0}$, $\rho_\delta \geq C(\delta)$ be a sequence smooth approximate solutions to (3) constructed in [20] on the bounden interval $\Omega = [-M, M]$ for M large, with the initial conditions

$$\rho_\delta(0) = \rho_{\delta,0}, \quad \rho_\delta(0)u_\delta(0) = m_{\delta,0}$$

and the boundary condition

$$u_\delta|_{\partial\Omega} = 0.$$

Assume that $\rho_{\delta,0}$, $m_{\delta,0}$ converge to ρ_0, m_0 in the following sense

$$\begin{aligned}
&\rho_{\delta,0} \to \rho_0, \quad \text{strongly in } L^1(\Omega),\\
&\partial_x(\rho_{\delta,0}^{\alpha-\frac{1}{2}}) \to \partial_x(\rho_0^{\alpha-\frac{1}{2}}) \quad \text{strongly in } L^2(\Omega)\\
&\frac{m_{\delta,0}^2}{\rho_{\delta,0}} \to \frac{m_0^2}{\rho_0} \quad \text{strongly in } L^1(\Omega),\\
&\frac{|m_{\delta,0}|^{2+\nu}}{\rho_{\delta,0}^{1+\nu}} \to \frac{|m_0|^{2+\nu}}{\rho_0^{1+\nu}} \in L^1(\mathbb{R}) \quad \text{strongly in } L^1(\Omega).
\end{aligned} \tag{24}$$

We show that (9) is weakly sequentially stable when $\delta \to 0$. To this purpose let us recall the a-priori estimates derived in [23] following the strategy developed by A. Mellet and A. Vasseur in [25] for the multidimensional case.

First recall that the classical energy balance gives for every $T > 0$ the following inequality

$$\begin{aligned}
\int_\Omega \left(\rho_\delta \frac{u_\delta^2}{2} + \frac{\varepsilon}{\gamma-1}\rho_\delta^\gamma\right)(T)\ \mathrm{d}x + \int_0^T\!\!\int_\Omega \rho_\delta^\alpha |\partial_x u_\delta|^2\ \mathrm{d}x\,\mathrm{d}t\\
\leq \int_\Omega \left(\rho_\delta \frac{u_\delta^2}{2} + \frac{\varepsilon}{\gamma-1}\rho_\delta^\gamma\right)(0)\ \mathrm{d}x.
\end{aligned} \tag{25}$$

In addition, the BD entropy gives rise to the estimates

$$\begin{aligned}
\int_\Omega \left(\rho_\delta \frac{(u_\delta + \partial_x\varphi(\rho_\delta))^2}{2} + \frac{\varepsilon}{\gamma-1}\rho_\delta^\gamma\right)(T)\ \mathrm{d}x + \frac{\alpha}{\alpha-1}\varepsilon\int_0^T\!\!\int_\Omega \partial_x\rho_\delta^\gamma \partial_x\rho_\delta^{\alpha-1}\ \mathrm{d}x\,\mathrm{d}t\\
\leq \int_\Omega \left(\rho_\delta \frac{(u_\delta + \partial_x\varphi(\rho_\delta))^2}{2} + \frac{\varepsilon}{\gamma-1}\rho_\delta^\gamma\right)(0)\ \mathrm{d}x.
\end{aligned} \tag{26}$$

Therefore, we obtain the following bounds

$$\begin{aligned}
&\|\sqrt{\rho_\delta}u_\delta\|_{L^\infty(0,T;L^2(\Omega))} + \|\partial_x(\rho_\delta^{\alpha-\frac{1}{2}})\|_{L^\infty(0,T;L^2(\Omega))} + \varepsilon\|\rho_\delta\|^\gamma_{L^\infty(0,T;L^\gamma(\Omega))}\\
&\quad + \varepsilon\|\partial_x(\rho_\delta^{\frac{\gamma+\alpha-1}{2}})\|^2_{L^2(0,T;L^2(\Omega))} + \|\rho_\delta^{\frac{\alpha}{2}}\partial_x u_\delta\|_{L^2(0,T;L^2(\Omega))} \leq C,
\end{aligned} \tag{27}$$

where the constant is independent of δ.

Following [23] we obtain

$$\|\rho_\delta\|_{L^\infty(0,T;L^\infty(\Omega))} \leq C. \tag{28}$$

This estimate can be then used to derive the improved estimate for the velocity, exactly as in [25]. By testing the momentum equation by $u_\delta|u_\delta|^\nu$ for $\nu > 0$ sufficiently small, we obtain

$$\sup_{t\in[0,T]} \|\rho_\delta|u_\delta|^{2+\nu}\|_{L^1(\Omega)} \leq C. \tag{29}$$

Having these estimates, we can follow the compactness arguments from [25] and from [23] to prove the convergence of the approximate solution (ρ_δ, u_δ) to some weak solution $(\rho_\varepsilon, u_\varepsilon)$ as $\delta \to 0$. Namely

$$\begin{aligned}
&\rho_\delta \to \rho_\varepsilon \quad \text{in } C([0,T]\times\overline{\Omega}),\\
&\partial_x\left(\rho_\delta^{\alpha-\frac{1}{2}}\right) \to \partial_x\left(\rho_\varepsilon^{\alpha-\frac{1}{2}}\right) \quad \text{weakly in } L^2(0,T;L^2(\Omega)),\\
&\sqrt{\rho_\delta}u_\delta \to \sqrt{\rho_\varepsilon}u_\varepsilon, \quad \rho_\delta^\alpha u_\delta \to \rho_\varepsilon^\alpha u_\varepsilon \quad \text{strongly in } L^{2+\frac{\nu}{2}}(0,T;L^{2+\frac{\nu}{2}}(\Omega)),\\
&\rho_\delta^\alpha \partial_x u_\delta \to \Lambda \quad \text{weakly in } L^2(0,T;L^2(\Omega)),
\end{aligned} \tag{30}$$

where Λ satisfies (23).

At this point, sequential stability of equations (8) and (10) is verified. To justify the limit passage in (9), one has to prove the convergence of the term $\rho_\delta v_\delta$. Observe that we have

$$\rho_\delta v_\delta = \rho_\delta u_\delta + \rho_\delta \partial_x \varphi(\rho_\delta) = \rho_\delta u_\delta + \frac{2}{2\alpha-1}\rho_\delta^{\frac{1}{2}}\partial_x\left(\rho_\delta^{\alpha-\frac{1}{2}}\right). \tag{31}$$

Due to (30) the first term converges to $\rho_\varepsilon u_\varepsilon$ strongly in $L^2(0,T;L^2(\Omega))$, while the second term converges to $\frac{2}{2\alpha-1}\rho_\varepsilon^{\frac{1}{2}}\partial_x\left(\rho_\varepsilon^{\alpha-\frac{1}{2}}\right)$ weakly in $L^2(0,T;L^2(\Omega))$. In consequence

$$\rho_\delta v_\delta \to \rho_\varepsilon v_\varepsilon \quad \text{weakly in } L^2(0,T;L^2(\Omega)), \tag{32}$$

and so one can pass to the limit in (9).

This is the final argument to prove the sequential stability of weak solutions to (4) on the bounded domain $\Omega = [-M, M]$ and with the Dirichlet boundary condition for u_δ. In order to let $M \to \infty$, we combine the diagonal procedure with the convergence of the initial data (24) as it was done in [20].

3.2 The porous medium equation

In order to understand the qualitative properties of the limit solution to (3), we recall several important features of the porous medium equations. The majority of them is taken from the excellent books of J. L. Vázquez [32], [31]. The porous medium equation can be written as follows:

$$\begin{cases} \partial_t\widetilde{\rho} - \partial_{xx}\widetilde{\rho}^\alpha = 0,\\ \widetilde{\rho}(0,x) = \widetilde{\rho}_0(x), \end{cases} \tag{33}$$

with $\alpha > 1$.

In the sequel we shall set $Q = (0,+\infty)\times\mathbb{R}$. Let us recall the notion of global strong solution for the porous medium equation (33) (see [32] Chapter 9 for more details).

Definition 2 *We say that a function $\widetilde{\rho} \in C([0,+\infty), L^1(\mathbb{R}))$ positive is a strong L^1 solution to problem (33) if:*

- $\widetilde{\rho}^{\alpha} \in L^1_{loc}(0,+\infty;L^1(\mathbb{R}))$ *and* $\widetilde{\rho}_t, \partial_{xx}\rho^{\alpha} \in L^1_{loc}((0,+\infty)\times\mathbb{R})$
- $\widetilde{\rho}_t = \mu\partial_{xx}\widetilde{\rho}^{\alpha}$ *in distribution sense.*
- $\widetilde{\rho}(t) \to \widetilde{\rho}_0$ *as* $t \to 0$ *in* $L^1(\mathbb{R})$.

The existence of global strong solution to (33) is guaranteed by the following theorem (see [32] page 197).

Theorem 4 *Let* $\alpha > 1$ *For every non-negative function* $\widetilde{\rho}_0 \in L^1(\mathbb{R}) \cap L^{\infty}(\mathbb{R})$ *there exists a unique global strong solution* $\widetilde{\rho} \geq 0$ *of (33). Moreover,* $\partial_t\widetilde{\rho} \in L^p_{loc}(Q)$ *for* $1 \leq p < \frac{(\alpha+1)}{\alpha}$ *and:*

$$\partial_t\widetilde{\rho} \geq -\frac{\widetilde{\rho}}{(\alpha-1)t} \quad in\ \mathcal{D}'(Q),$$
$$\|\partial_t\widetilde{\rho}(t)\|_{L^1(\mathbb{R})} \leq \frac{2\|\widetilde{\rho}_0\|_{L^1(\mathbb{R})}}{(\alpha-1)t}.$$

Let ρ_1 *and* ρ_2 *be two strong solutions of (33) in* $(0,T)\times\mathbb{R}$ *then for every* $0 \leq \tau < t$

$$\|\big(\rho_1-\rho_2\big)_+(t)\|_{L^1(\mathbb{R})} \leq \|\big(\rho_1-\rho_2\big)_+(\tau)\|_{L^1(\mathbb{R})}. \tag{34}$$

If ρ_1 *and* ρ_2 *are two strongs solution with initial data* $(\rho_1)_0$ *and* $(\rho_2)_0$*, such that* $(\rho_1)_0(x) \leq (\rho_2)_0(x)$ *in* $\mathbb{R}$*, then* $\rho_1(t,x) \leq \rho_2(t,x)$ *for all* $(t,x) \in (0,+\infty)\times\mathbb{R}$.

Likewise above, there exists a theory of global weak solutions with initial data being bounded measures. It covers the case of very important case of self similar solutions, the so called *Barrenblatt solutions* $U(t,x,M)$, of the form

$$U(t,x,M) = t^{\frac{1}{m+1}}F(xt^{\frac{-1}{m+1}}),\ F(\xi) = (C-\kappa\xi^2)_+^{\frac{1}{\alpha-1}}, \tag{35}$$

where $\kappa = \frac{\alpha-1}{2\alpha(\alpha+1)}$ and $C > 0$ with $C = cM^{\frac{1}{\gamma}}$ with $\gamma = \frac{\alpha+1}{2(\alpha-1)}$ and c depends only on $\alpha > 1$. The Barenblatt solutions verify (33) for $t > 0$ in the sense of distributions and with initial data $\widetilde{\rho}_0 = M\delta_0$, where δ_0 is the Dirac mass.
Analogue solutions exist also in the case of fast diffusion equations corresponding to (33) with $\alpha < 1$.

Remark 3 *The comparison principle from Theorem 4 ensures a finite speed of propagation of solutions to the porous medium equation with compactly supported initial data, see for example [6, 32]. In other words, the solution to the porous medium equation with compactly supported initial data remains compactly supported all along the time. Indeed, it suffices to compare such solution with the Barrenblatt solutions.*

Let us now recall the so called $L^1 - L^{\infty}$ smoothing effect (see [31] page 202).

Theorem 5 *Let* $\widetilde{\rho}_0 \in L^1(\mathbb{R})$. *For every* $t > 0$ *we have:*

$$\widetilde{\rho}(t,x) \leq C\|\widetilde{\rho}_0\|^{\sigma}_{L^1(0,T;\mathbb{R}^N)}t^{-\beta},$$

with $\sigma = \frac{2}{N(\alpha-1)+2}$, $\beta = \frac{1}{(\alpha-1)+2}$ *and* $C > 0$ *depends only on* α *and* N. *The exponents are sharp.*
When $\widetilde{\rho}_0$ *belongs also to* $L^{\infty}(\mathbb{R})$ *the maximum principle holds.*

Let us now denote by $\widetilde{v}$ the pressure in the sense of porous medium equation, i.e.

$$\widetilde{v} = \frac{\alpha}{\alpha - 1}\widetilde{\rho}^{\alpha-1}, \quad \widetilde{v}_0 = \frac{\alpha}{\alpha - 1}\widetilde{\rho}_0^{\alpha-1}.$$

We finish this section by recalling several results on regularity of $\widetilde{\rho}$ (see [32] pages 358, 359 and 361).

Proposition 1 *Let $\widetilde{\rho}_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with $\widetilde{\rho}_0 \geq 0$. Then, $\widetilde{\rho}$ and $\partial_x \widetilde{\rho}^\alpha$ are continuous in Q. In addition, for every $t > 0$, $\widetilde{v}(t, \cdot)$ is Lipschitz continuous in $\mathbb{R}$. More precisely we have*

$$|\partial_x \widetilde{v}(t, x)| \leq \sqrt{\frac{2\|\rho_0\|_{L^\infty}}{\alpha + 1}} t^{\frac{-1}{2}}, \quad \text{for } (t, x) \in Q. \tag{36}$$

If moreover $\partial_x \widetilde{v}_0 \in L^\infty(\mathbb{R})$, we have

$$|\partial_x \widetilde{v}(t, x)| \leq \|\partial_x \widetilde{v}_0\|_{L^\infty}. \tag{37}$$

3.2.1 Behavior of the interfaces for compactly supported solution

We recall an important theorem (see [32] page 376) which gives a precise behaviour of the interfaces when $\widetilde{\rho}_0$ has compact support.

Theorem 6 *Let $x = s(t)$ denote the right interface of the solution $\widetilde{\rho}$ to (33) emanating from the compactly supported initial data $\widetilde{\rho}_0 \in L^1(\mathbb{R})$. For all $t > 0$ there exist the one side limits*

$$D_x^- v_1(t, s(t)) = \lim_{x \to s(t)^-} \partial_x v_1(t, x), \;\; D^+ s(t) = \lim_{h \to 0^+} \frac{1}{h}[s(t + h) - s(t)]. \tag{38}$$

Moreover, the Darcy law holds in the form

$$D_t^+ s(t) = -D_x^- v_1(t, s(t)). \tag{39}$$

The same result is true for the left interface.

4 Proof of main results

Below we present the proofs of our main results stated in Theorem 1 and Theorem 2. We start by proving the rate of convergence from (15), the convergence of the approximate solutions $(\rho_\varepsilon, u_\varepsilon)$ to the solutions of the corresponding porous medium equation is a consequence of this estimate, the details can be found in [14].

Proof of Theorem 1

First let us note that thanks to Theorem 3, the continuity equation of system (3) after the change of variables $v_\varepsilon = u_\varepsilon + \partial_x \varphi(\rho_\varepsilon)$ with $\partial_x \varphi(\rho_\varepsilon) = \frac{\mu(\rho_\varepsilon)}{\rho_\varepsilon^2} \partial_x \rho_\varepsilon$ reads

$$\partial_t \rho_\varepsilon - \frac{1}{\alpha} \partial_{xx} \rho_\varepsilon^\alpha + \partial_x(\rho_\varepsilon v_\varepsilon) = 0, \tag{40}$$

and it is satisfied in the sense of distributions.
From (12) and (26) it also follows that

$$\sup_{t \in [0,T]} \|\sqrt{\rho_\varepsilon} v_\varepsilon(t)\|_{L^2(\mathbb{R})} \leq \frac{\varepsilon^{\frac{1}{2}}}{(\gamma - 1)^{\frac{1}{2}}} \|\rho_0\|_{L^\gamma(\mathbb{R})}^{\frac{\gamma}{2}} \leq C\varepsilon^{\frac{1}{2}}. \tag{41}$$

Next, we consider $\widetilde{\rho}$ – the solution to the corresponding porous medium equation with the same initial data ρ_0

$$\begin{cases} \partial_t \widetilde{\rho} - \dfrac{1}{\alpha}\partial_{xx}\widetilde{\rho}^\alpha = 0, \\ \widetilde{\rho}(0,\cdot) = \rho_0, \end{cases} \tag{42}$$

whose main properties were recalled in Section 3.2.
We now set

$$R_\varepsilon = \rho_\varepsilon - \widetilde{\rho}.$$

It follows from (40) and (42) that R_ε satisfies the equation

$$\begin{cases} \partial_t R_\varepsilon - \dfrac{1}{\alpha}\partial_{xx}(\rho_\varepsilon^\alpha - \widetilde{\rho}^\alpha) + \partial_x(\rho_\varepsilon v_\varepsilon) = 0, \\ R_\varepsilon(0,x) = 0, \quad x \in \mathbb{R}, \end{cases} \tag{43}$$

at least in the sense of distributions.

Our goal is to estimate a relevant norm of R_ε in terms of ε. To this purpose, we use duality technic in the spirit of J. L. Vázquez (see [32] Section 6.2.1). Testing (43) by $\psi \in C_c^\infty((0,T]\times\mathbb{R})$, we obtain

$$\int_0^T \int_{\mathbb{R}} \big(R_\varepsilon \partial_t \psi + \frac{1}{\alpha}(\rho_\varepsilon^\alpha - \widetilde{\rho}^\alpha)\partial_{xx}\psi\big)\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\mathbb{R}} (\rho_\varepsilon v_\varepsilon)\partial_x\psi\,\mathrm{d}x\,\mathrm{d}t - \int_{\mathbb{R}} (R_\varepsilon\psi)(T)\,\mathrm{d}x = 0. \tag{44}$$

Let us now define

$$a(t,x) = \begin{cases} \frac{\rho_\varepsilon^\alpha - \widetilde{\rho}^\alpha}{\rho_\varepsilon - \widetilde{\rho}} & \text{if} \quad \rho_\varepsilon \neq \widetilde{\rho} \\ 0 & \text{if} \quad \rho_\varepsilon = \widetilde{\rho}. \end{cases}$$

This definition implies in particular that $\rho_\varepsilon^\alpha - \widetilde{\rho}^\alpha = aR_\varepsilon$. We can hence rewrite (44) as

$$\int_0^T \int_{\mathbb{R}} \big(R_\varepsilon(\partial_t\psi + \frac{1}{\alpha}a\partial_{xx}\psi\big)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\mathbb{R}} (\rho_\varepsilon v_\varepsilon)\partial_x\psi\,\mathrm{d}x\,\mathrm{d}t - \int_{\mathbb{R}} (R_\varepsilon\psi)(T)\,\mathrm{d}x = 0. \tag{45}$$

The next step consists of solving the inverse problem in the interval $[-M,M]\subset\mathbb{R}$

$$\begin{cases} \partial_t\psi + \dfrac{1}{\alpha}a_n\partial_{xx}\psi = 0, \quad (t,x)\in[0,T]\ \times\ (-M,M), \\ \psi(t,-M) = \psi(t,M) = 0, \quad t\in[0,T], \\ \psi(T,x) = \theta(x), \quad x\in(-M,M), \end{cases} \tag{46}$$

where $\theta \in C_0^\infty((-M,M))$. Above, a_n is a smooth approximation of a such that $0<\eta\le a_n\le K<+\infty$ (it will be precisely defined later on). This assumption guarantees that system (46) is parabolic in the reverse time $t' = T-t$ and it admits unique smooth solution φ on $[0,T]$.

Since ψ is a solution to (46), we can use (45) to obtain

$$\int_{\mathbb{R}} R_\varepsilon(T)\theta\,\mathrm{d}x = \frac{1}{\alpha}\int_0^T\int_{\mathbb{R}} R_\varepsilon(a-a_n)\partial_{xx}\psi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\mathbb{R}}(\rho_\varepsilon v_\varepsilon)\partial_x\psi\,\mathrm{d}x\,\mathrm{d}t, \tag{47}$$

therefore

$$\left|\int_{\mathbb{R}} R_\varepsilon(T)\theta\,\mathrm{d}x\right| \le \left|\int_0^T\int_{\mathbb{R}}(\rho_\varepsilon v_\varepsilon)\partial_x\psi\,\mathrm{d}x\,\mathrm{d}t\right| + \int_0^T\int_{\mathbb{R}} |R_\varepsilon|\,|(a-a_n)|\,|\partial_{xx}\psi|\,\mathrm{d}x\,\mathrm{d}t. \tag{48}$$

We now need to obtain the a priori estimate for $\partial_{xx}\psi$. To this purpose, we multiply equation (46) by $\zeta\partial_{xx}\psi$, where $\zeta = \zeta(t)$ is a smooth and positive function such that $\frac{1}{2} \leq \zeta \leq 1$ and $\partial_t\zeta \geq c > 0$, we get

$$\int_0^T \int_{\mathbb{R}} \partial_t\psi\zeta\partial_{xx}\psi \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{\alpha}\int_0^T \int_{\mathbb{R}} \zeta a_n(\partial_{xx}\psi)^2 \,\mathrm{d}x\,\mathrm{d}t = 0. \tag{49}$$

Integrating the first term by parts and using the fact that $\psi(t,\cdot) \in C_0^\infty([-M,M])$ (see the general theory of [21], page 341), we obtain

$$\begin{aligned}\int_0^T \int_{\mathbb{R}} \partial_t\psi\zeta\partial_{xx}\psi \,\mathrm{d}x\,\mathrm{d}t &= -\int_0^T \int_{\mathbb{R}} \zeta\partial_x\psi\partial_x\partial_t\psi \,\mathrm{d}x\,\mathrm{d}t \\ &= -\frac{1}{2}\int_0^T \int_{\mathbb{R}} \partial_t(\partial_x\psi)^2\zeta \,\mathrm{d}x\,\mathrm{d}t, \\ &= \frac{1}{2}\int_0^T \int_{\mathbb{R}} (\partial_x\psi)^2\partial_t\zeta \,\mathrm{d}x\,\mathrm{d}t - \frac{1}{2}\int_{\mathbb{R}} ((\partial_x\psi)^2\zeta)(T)\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}} ((\partial_x\psi)^2\zeta)(0)\,\mathrm{d}x.\end{aligned}$$

Hence, it follows that

$$\int_0^T \int_{\mathbb{R}} \partial_t\psi\zeta\partial_{xx}\psi \,\mathrm{d}x\,\mathrm{d}t \geq \frac{1}{2}\int_0^T \int_{\mathbb{R}} (\partial_x\psi)^2\partial_t\zeta \,\mathrm{d}x\,\mathrm{d}t - \frac{1}{2}\int_{\mathbb{R}} ((\partial_x\psi)^2\zeta)(T)\,\mathrm{d}x. \tag{50}$$

Plugging it into (49) we obtain

$$\frac{1}{2}\int_0^T \int_{\mathbb{R}} (\partial_x\psi)^2\partial_t\zeta \,\mathrm{d}x\,\mathrm{d}t - \frac{1}{2}\int_{\mathbb{R}} ((\partial_x\theta)^2\zeta)(T)\,\mathrm{d}x + \frac{1}{\alpha}\int_0^T \int_{\mathbb{R}} \zeta a_n(\partial_{xx}\psi)^2 \mathrm{d}x\,\mathrm{d}t \leq 0. \tag{51}$$

In particular, the assumptions on ζ imply that we have

$$\int_0^T \int_{\mathbb{R}} (\partial_x\psi)^2 \mathrm{d}x\,\mathrm{d}t + \frac{1}{\alpha}\int_0^T \int_{\mathbb{R}} a_n(\partial_{xx}\psi)^2 \mathrm{d}x\,\mathrm{d}t \leq c\|\partial_x\theta\|^2_{L^2(\mathbb{R})}. \tag{52}$$

Coming back to (48) and recalling that $\operatorname{supp}\psi$ and $\operatorname{supp}\theta$ are included included in $[0,T]\times[-M,M]$ and $[-M,M]$, respectively, we obtain

$$\left|\int_{\mathbb{R}} R_\varepsilon(T)\theta \,\mathrm{d}x\right| \leq c\|\partial_x\theta\|_{L^2(\mathbb{R})}\left(\left(\int_0^T \int_{-M}^M \frac{[a-a_n]^2}{a_n}|R_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}} + \|\rho_\varepsilon v_\varepsilon\|_{L^2(0,T;L^2(\mathbb{R}))}\right). \tag{53}$$

At this stage following ideas from [32] we construct the approximation a_n that is sufficiently regular and verifies $\eta \leq a_n \leq K$ for $\eta > 0$ small and for $K > \eta$ large enough. This requires two steps of approximation. The first consists of taking $a_{K,\eta} = \min(K, \max(\eta, a))$ with $0 < \eta < K$, in other words

$$\begin{cases} a_{K,\eta}(t,x) = K \ \text{ on } \{a(t,x) > K\}, \\ a_{K,\eta} = a \ \text{ on } \{\eta \leq a \leq K\}, \\ a_{K,\eta} = \eta \ \text{ on } \{\eta < a\}. \end{cases}$$

Since $a_{K,\eta}$ is $L^\infty((0,T)\times\mathbb{R})$, we know that $a_{K,\eta}$ is also in $L^p((0,T)\times(-M,M))$ for $1 \leq p \leq +\infty$. Therefore, in the next step, we can consider a smooth approximation $a_n =$

$a_{K,\eta} * \psi_n$ (with ψ_n a standard regularizing kernel) such that for $n \to \infty$ $a_n \to a_{K,\eta}$ strongly in $L^p((0,T)\times(-M,M))$ for all $1 \le p < +\infty$.

It remains to estimate the r.h.s. of (53). Since $a_{K,\eta} \ge \eta$ we deduce that also $a_n \ge \eta$ for any n, therefore

$$\begin{aligned}&\int_0^T \int_{-M}^M \frac{|a-a_n|^2}{a_n} |R_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t \le \frac{1}{\eta}\int_0^T \int_{-M}^M |a-a_n|^2 |R_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\le \frac{2}{\eta}\left(\int_0^T \int_{-M}^M |a_{K,\eta}-a_n|^2 |R_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{-M}^M ((a-K)_+ + \eta)^2 |R_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t\right).\end{aligned} \tag{54}$$

We call the two integrals on the r.h.s. I_1 and I_2, respectively. The integrand of I_2 is pointwise bounded by

$$a^2 |R_\varepsilon|^2 1_{a>K} + \eta^2 \mathbf{1}_{a<\eta} |R_\varepsilon|^2 = (\widetilde{\rho}^\alpha - \rho_\varepsilon^\alpha)^2 \mathbf{1}_{a>K} + \eta^2 \mathbf{1}_{a<\eta} |R_\varepsilon|^2,$$

thus, since $(\widetilde{\rho}^\alpha - \rho_\varepsilon^\alpha)^2$ is in $L^1(0,T;L^1(\mathbb{R}))$ (which follows from (21) and (7)), using the dominated convergence theorem we justify that $\int_0^T \int_{-M}^M (\widetilde{\rho}^\alpha - \rho_\varepsilon^\alpha)^2 1_{a>K} \,\mathrm{d}x\,\mathrm{d}t \to 0$ for $K \to +\infty$. It implies that for K large enough, we have

$$\int_0^T \int_{-M}^M \left((a-K)_+ + \eta\right)^2 |R_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t \le 2\eta^2, \tag{55}$$

where we used the fact that R_ε is uniformly bounded in $L^2((0,T)\times\mathbb{R})$ for $0 < T < +\infty$. To estimate I_1, it suffices to observe that R_ε^2 is actually uniformly bounded in $L^{p'}((0,T)\times\mathbb{R})$ for any $1 \le p' \le +\infty$ due to (21) and due to L^∞ bound on the solution to the porous medium equation. We now use the fact that $a_n \to a_{K,\eta}$ when $n \to +\infty$ strongly in $L^p((0,T)\times(-M,M))$ to obtain that $\int_0^T \int_{-M}^M |a-a_n|^2 |R_\varepsilon|^2 \mathrm{d}x\,\mathrm{d}t \to 0$. Using this, (55) and (54) we deduce that for $n = n(\eta)$ large enough we have

$$\int_0^T \int_{-M}^M \frac{|a-a_n|^2}{a_n} |R_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t \le 8\eta. \tag{56}$$

Coming back to (53) and using the Hölder inequality we obtain

$$\left|\int_{\mathbb{R}} R_\varepsilon(T)\theta \,\mathrm{d}x\right| \le C\|\partial_x\theta\|_{L^2(\mathbb{R})}\big(\eta + T^{\frac12}\|\sqrt{\rho_\varepsilon}\|_{L^\infty(0,T;L^\infty(\mathbb{R}))}\|\sqrt{\rho_\varepsilon}v_\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{R}))}\big). \tag{57}$$

From (28) and (41) we thus deduce that

$$\begin{aligned}\left|\int_{\mathbb{R}} R_\varepsilon(T)\theta \,\mathrm{d}x\right| &\le C\|\partial_x\theta\|_{L^2(\mathbb{R})} \frac{\|\rho_0\|_{L^\gamma}^{\frac{\gamma}{2}}}{\gamma-1} \varepsilon^{\frac12} T^{\frac12} \|\sqrt{\rho_\varepsilon}\|_{L^\infty(0,T;L^\infty(\mathbb{R}))} \\ &\le C\|\partial_x\theta\|_{L^2(\mathbb{R})} \varepsilon^{\frac12} T^{\frac12}.\end{aligned} \tag{58}$$

To conclude, let us observe that the above inequality holds for any θ from $C_0^\infty((-M,M))$ with arbitrary $M > 0$ and that the constant C is independent of M. Thus, letting M to $+\infty$ we obtain

$$\|R_\varepsilon(T)\|_{H^{-1}(\mathbb{R})} \le C\varepsilon^{\frac12} T^{\frac12}, \tag{59}$$

for any $T > 0$.

Concerning the second part of Theorem 1, for $1 < \alpha \leq \frac{3}{2}$ we have

$$\|\partial_x \widetilde{\rho}(T)\|_{L^2(\mathbb{R})} \leq C\|\widetilde{\rho}^{2-\alpha}(T)\|_{L^2(\mathbb{R})}\|\partial_x \widetilde{\rho}^{\alpha-1}(T)\|_{L^\infty(\mathbb{R})} \tag{60}$$

for all $T > 0$. Next, using Proposition 1, we obtain

$$\|\partial_x \widetilde{\rho}^{\alpha-1}(T)\|_{L^\infty(\mathbb{R})} \leq \sqrt{\frac{2\|\rho_0^{\alpha-1}\|_{L^\infty}}{\alpha+1}} T^{-\frac{1}{2}}. \tag{61}$$

By interpolation and using the mass conservation, we deduce the L^∞ bound on $\widetilde{\rho}$ provided that $2(2-\alpha) \geq 1$ which is equivalent to restriction $\alpha \leq \frac{3}{2}$. In the end, we obtain

$$\|\partial_x \widetilde{\rho}(T)\|_{L^2(\mathbb{R})} \leq CT^{-\frac{1}{2}}. \tag{62}$$

Similarly, since $\partial_x \rho_\varepsilon^{\alpha-\frac{1}{2}}$ is uniformly bounded in $L^\infty(0,T;L^2(\mathbb{R}))$ and ρ_ε is uniformly bounded in $L^\infty(0,T;L^\infty(\Omega))$ we deduce that $\partial_x \rho_\varepsilon$ is uniformly bounded in $L^\infty(0,T;L^2(\mathbb{R}))$. By interpolation since ρ_ε, ρ are also uniformly bounded in $L^\infty(0,T;L^2(\mathbb{R}))$ and by (59), we conclude that

$$\|R_\varepsilon(T)\|_{L^2(\mathbb{R})} \leq C\varepsilon^{\frac{1}{4}}(\mathbf{1}_{T\leq 1} + T^{\frac{1}{4}}\mathbf{1}_{T\geq 1}). \tag{63}$$

The rate of convergence in $L^\infty(0,T;L^p(\mathbb{R}))$ for $1 < p < \infty$ can be proven by interpolation.

Assuming in addition that $\partial_x \rho_0^{\alpha-1}$ belongs to $L^\infty(\mathbb{R})$ and proceeding as previously, we may improve the rate of convergence, namely

$$\|R_\varepsilon(T)\|_{L^2(\mathbb{R})} \leq C\varepsilon^{\frac{1}{4}} T^{\frac{1}{4}}. \tag{64}$$

This finishes the proof of Theorem 1. □

Proof of Theorem 2

Having proven the above rate of convergence, our next aim will be to provide some more information about the behaviour of ρ_ε on the support of the solution to the corresponding porous medium equation.

Under assumptions of Theorem 2 and from the properties of the solutions to the one-dimensional porous medium equation presented in Section 3.2, we deduce that the $\operatorname{supp}\widetilde{\rho}(t,\cdot)$ remains compact for all time $t > 0$ and it satisfies (17). Indeed, since $\operatorname{supp}\rho_0$ is included in $[a,b]$, we can consider the Barrenblatt solution $U(t_1, M)$ at time $t_1 > 0$ such that

$$\|\rho_0\|_{L^\infty(\mathbb{R})} \leq \|U(t_1, M)\|_{L^\infty(\mathbb{R})}.$$

Last inequality is verified if M is large enough depending on $\|\rho_0\|_{L^\infty}$ and $\frac{1}{t_1}$ (see formula (35)). Using the maximum principle we deduce that:

$$\|\widetilde{\rho}(t)\|_{L^\infty(\mathbb{R})} \leq \|U(t_1 + t, M)\|_{L^\infty(\mathbb{R})}.$$

In particular $\mathrm{supp}\widetilde{\rho}(t,\cdot)$ is included in $[a_1 - C(t+t_1)^{\frac{1}{\alpha+1}}, b_1 + C(t+t_1)^{\frac{1}{\alpha+1}}]$ for some constants $-\infty < a_1 < b_1 < \infty$ and $C > 0$.
Now, we may write the following sequence of equalities and inequalities

$$\begin{aligned}
\|\rho_\varepsilon(t)\mathbf{1}_{\Omega_t^c}\|_{L^1(\mathbb{R})} &\leq \|\rho_0\|_{L^1(\mathbb{R})} - \|\rho_\varepsilon(t)\mathbf{1}_{\Omega_t}\|_{L^1(\mathbb{R})} \quad \text{(from the mass conservation)} \\
&\leq \|\widetilde{\rho}(t)\mathbf{1}_{\Omega_t}\|_{L^1(\mathbb{R})} - \|\rho_\varepsilon(t)\mathbf{1}_{\Omega_t}\|_{L^1(\mathbb{R})} \\
&\leq \|(\widetilde{\rho}(t) - \rho_\varepsilon(t))\mathbf{1}_{\Omega_t}\|_{L^1(\mathbb{R})} \\
&\leq |\Omega_t|^{\frac{1}{2}}\|\widetilde{\rho}(t) - \rho_\varepsilon(t)\|_{L^2(\mathbb{R})} \qquad \text{(from the Hölder inequality)} \\
&\leq C\varepsilon^{\frac{1}{4}}(t^{\frac{1}{4}}1_{t\geq 1} + 1)(1 + t^{\frac{1}{2(\alpha+1)}}) \qquad \text{(from (17) and (64))}.
\end{aligned} \tag{65}$$

In a similar way, when $\partial_x \rho_0^{\alpha-1} \in L^\infty(\mathbb{R})$ we show

$$\begin{aligned}
\|\rho_\varepsilon(t)\mathbf{1}_{\Omega_t^c}\|_{L^1(\mathbb{R})} &\leq |\Omega_t|^{\frac{1}{2}}\|\widetilde{\rho}(t) - \rho_\varepsilon(t)\|_{L^2(\mathbb{R})} \qquad \text{(from the Hölder inequality)} \\
&\leq C\varepsilon^{\frac{1}{4}}t^{\frac{1}{4}}(1 + t^{\frac{1}{2(\alpha+1)}}) \qquad \text{(from (17) and (64))}.
\end{aligned} \tag{66}$$

This finishes the proof of Theorem 2. □

5 Where are the interfaces?

A natural question arising in the analysis of one-dimensional problems with compactly supported initial data and free boundary conditions is propagation of the interface. From the classical theory for porous medium equation (see Section 3.2) we know that the interface moves with a finite speed. Moreover, according to (18) and (19) we can estimate the mass corresponding to ρ_ε inside the support of $\widetilde{\rho}$. This could suggest that the interfaces of free-boundary Navier-Stokes equations behave similarly to the interfaces of the porous medium equation. However, the issue of existence of global weak solutions to the free-boundary Navier-Stokes equations remains open.

Below, we present an overview of partial results in this topic and explain the main problems.

1. A case of compactly supported initial density with jump discontinuity at the interface was studied in [19]. In this article S. Jiang, Z. Xin, and P. Zhang proved existence of global strong solution to (3) written in Lagrangian coordinates. Extending their solution outside the free-boundary domain by imposing the Rankine Hugoniot condition on the velocity, allows to control $\partial_x u_\varepsilon$ in $L^1(0,T;L^\infty(\mathbb{R}))$. It enables in particular to obtain global strong solution on the whole $\mathbb{R}$. It is also sufficient to define the free boundary which verifies an ordinary differential equation. In this case one can pass from the Eulerian to the Lagrangian coordinates and conversely.

   It would be therefore interesting to compare the density ρ_ε from [19] with solution to the porous medium equation $\widetilde{\rho}$ emanating from the same initial data ρ_0. However, since ρ_0 is discontinuous, it is not compatible with assumptions $\sqrt{\rho_0}\partial_x\varphi(\rho_0) \in L^2(\mathbb{R})$ and $\rho_0 \in L^1(\mathbb{R})$. As a consequence, we loose the a priori estimate on $\sqrt{\rho_\varepsilon}v_\varepsilon$ (26) which was necessary to show damping of $\partial_x(\rho_\varepsilon v_\varepsilon)$ in (4). In fact, the behaviour of ρ_ε is completely different than behaviour of $\widetilde{\varrho}$. The discontinuity at the interface of ρ_ε exists all along the time, while $\widetilde{\rho}$ becomes continuous in arbitrary finite time (see Remark 3).

2. The free-boundary problem with compact initial density, continuously connected to the vacuum was studied in [35]. There, T. Yang, Z-a. Yao, and C. Zhu proved the local in time existence of weak solutions to (3) in Lagrangian coordinates. Although this setting seems to be more adequate for our considerations, the existence of interface corresponding to this solution is not clear. Indeed, in order to determine the evolution of the interface one needs to come back to the Eulerian coordinates. Unfortunately, result obtained in [35] does not guarantee boundedness of the gradient of the velocity in $L^1(0,T;L^\infty(\mathbb{R}))$. This lack of control on $\partial_x u_\varepsilon$ appears at the boundary, where the density vanishes.

   For the porous medium equation, the behavior of the interface in the 1D case is well understood since it verifies the Darcy law (see Theorem 6). It could be also appropriate to replace the usual free boundary condition for compressible Navier-Stokes equations by the Darcy law at the interfaces. Indeed in this case we could give a sense to $u_\varepsilon(t, s_1(t))$ (where s_1 denotes the right interface) by considering the limit $\lim_{x\to s(t)^-} u_\varepsilon(t,x)$ on the left hand side of the free boundary s_1. However with such condition the Lagrangian change of coordinates is a priori not possible.

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