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► **To cite this version:**

Anna Aksamit, Monique Jeanblanc, Marek Rutkowski. PREDICTABLE REPRESENTATION PROPERTY FOR PROGRESSIVE ENLARGEMENTS OF A POISSON FILTRATION. 2015. hal-01249662

HAL Id: hal-01249662

<https://hal.science/hal-01249662v1>

Preprint submitted on 2 Jan 2016

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PREDICTABLE REPRESENTATION PROPERTY FOR PROGRESSIVE ENLARGEMENTS OF A POISSON FILTRATION

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10 December 2015

Abstract

We study problems related to the predictable representation property for a progressive enlargement \mathbb{G} of a reference filtration \mathbb{F} through observation of a finite random time τ . We focus on cases where the avoidance property and/or the continuity property for \mathbb{F} -martingales do not hold and the reference filtration is generated by a Poisson process. Our goal is to find out whether the predictable representation property (PRP), which is known to hold in the Poisson filtration, remains valid for a progressively enlarged filtration \mathbb{G} with respect to a judicious choice of \mathbb{G} -martingales.

Keywords: predictable representation property, Poisson process, random time, progressive enlargement

Mathematics Subjects Classification (2010): 60H99

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1 Introduction

We study various problems associated with a progressive enlargement $\mathbb{G} = (\mathcal{G}_t)_{t \in \mathbb{R}_+}$ of a reference filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in \mathbb{R}_+}$ through observation of the occurrence of a finite random time τ . We focus on the cases where the so-called *avoidance property* and/or the continuity property for \mathbb{F} -martingales do not hold. Under the assumptions that \mathbb{F} is the Brownian filtration, the probability distribution of τ is continuous, and the filtration \mathbb{F} is *immersed* in its progressive enlargement \mathbb{G} , it was shown by Kusuoka [20] that any \mathbb{G} -martingale can be decomposed as a stochastic integral with respect to the Brownian motion and a stochastic integral with respect to the compensated martingale of the indicator process $H_t := \mathbb{1}_{\{\tau \leq t\}}$. We are interested in various extensions of Kusuoka's result, in particular, to the case where the filtration \mathbb{F} is generated by a Poisson process N . The main goal is to examine whether the predictable representation property (PRP), which is well known to hold for the Poisson process and its natural filtration, is also valid in the progressively enlarged filtration \mathbb{G} with respect to judiciously chosen family of \mathbb{G} -martingales. In contrast to the classical versions of the predictable representation property in a Brownian filtration, we attempt here to derive explicit expressions for the integrands, rather than to establish the existence and uniqueness of integrands. As a main input, we postulate the knowledge of the Azéma supermartingale of τ with respect to the filtration generated by a Poisson process.

The paper is structured as follows. In Section 2, we introduce the set-up and notation. We also review briefly the classic results regarding the \mathbb{G} -semimartingale decomposition of \mathbb{F} -martingales stopped at τ . It is worth noting that no general result on a \mathbb{G} -semimartingale decomposition after τ of an \mathbb{F} -martingale is available in the existing literature. Moreover, most papers in this area are devoted to either the case of *honest times* (see, for instance, Jeulin and Yor [18, 19]) or the case where the *density hypothesis* holds (see Jeanblanc and Le Cam [12]).

In Section 3.1, we examine the case of a general filtration \mathbb{F} . We postulate that the immersion property holds between \mathbb{F} and \mathbb{G} and thus the Azéma supermartingale of a random time τ is a decreasing process, which is also assumed to be \mathbb{F} -predictable. We derive several alternative integral representations for a \mathbb{G} -martingale stopped at τ (see Propositions 3.2 and 3.3).

In Section 3.2, we maintain the assumption that the immersion property holds, but we no longer postulate that the Azéma supermartingale of τ is \mathbb{F} -predictable. Under the assumption that the reference filtration is generated by the Poisson process, we obtain an explicit integral representation for particular \mathbb{G} -martingales stopped at τ in terms of the compensated martingale of the Poisson process and the compensated martingale of the indicator process of τ (see Proposition 3.4).

In Section 4, the immersion property is relaxed, but we still assume that the filtration is generated by the Poisson process. Since we work with the Poisson filtration, all \mathbb{F} -martingales are necessarily processes of finite variation and thus any \mathbb{F} -martingale is manifestly a \mathbb{G} -semi-martingale; in other words, the so-called *hypothesis* (H') is satisfied. We examine the validity of the predictable representation property for the Poisson filtration without making any additional assumptions, except for the fact that we still consider \mathbb{G} -martingales stopped at time τ . We also attempt to identify cases where the *multiplicity* ([6, 7]) of the enlarged filtration with respect to the class of all \mathbb{G} -martingales stopped at τ equals two.

The main result of this work, Theorem 4.1, offers a general representation formula for any \mathbb{G} -martingale stopped at τ in terms of the compensated martingale of the Poisson process, the compensated martingale of the indicator process of a random time and an additional \mathbb{G} -martingale, which can be seen as a “correction term”, which presence is due to the possibility of an overlap of jumps of the Poisson process and a single jump the indicator process H associated with a random time τ . We also show that this additional term has a natural interpretation as an optional stochastic integral (as opposed to the usual predictable stochastic integrals). It is worth noting that the Azéma supermartingale of a random time does not uniquely determine all the properties of τ with respect to a filtration \mathbb{F} (see Jeanblanc and Song [13, 14] and Li and Rutkowski [21, 22]). In our context, due to only a partial information about the random time τ and possible joint jumps of the Poisson process N and the indicator process H , more explicit computations seem to be out of reach.

2 Preliminaries

We first introduce the notation for an abstract set-up in which a reference filtration \mathbb{F} is progressively enlarged through observations of a random time. Let $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$ be a probability space where \mathbb{F} is an arbitrary filtration satisfying the usual conditions and such that $\mathcal{F}_\infty \subseteq \mathcal{G}$. Let τ be a *random time*, that is, a strictly positive, finite random variable on $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$. The *indicator process* $H := \mathbb{1}_{[\tau, \infty[}$ is a raw increasing process. We denote by \mathbb{G} the *progressive enlargement* of the filtration \mathbb{F} with the random time τ , that is, the smallest right-continuous and \mathbb{P} -completed filtration \mathbb{G} such the inclusion $\mathbb{F} \subset \mathbb{G}$ holds and the process H is \mathbb{G} -adapted, so that τ is a \mathbb{G} -stopping time. In other words, the progressive enlargement is the smallest filtration satisfying the usual conditions that renders τ a stopping time.

For a filtration $\mathbb{K} = \mathbb{F}$ or $\mathbb{K} = \mathbb{G}$, we denote by $B^{p, \mathbb{K}}$ (resp., $B^{o, \mathbb{K}}$) the dual \mathbb{K} -predictable (resp., the dual \mathbb{K} -optional) projection of a process B of finite variation, whereas ${}^{p, \mathbb{K}}U$ (resp., ${}^{o, \mathbb{K}}U$) stands for the \mathbb{K} -predictable (resp., \mathbb{K} -optional) projection of a process U . The \mathbb{K} -predictable covariation process of two semi-martingales X and Y is denoted by $\langle X, Y \rangle^{\mathbb{K}}$. Recall that $\langle X, Y \rangle^{\mathbb{K}}$ is defined as the dual \mathbb{K} -predictable projection of the covariation process $[X, Y]$, that is, $\langle X, Y \rangle^{\mathbb{K}} := [X, Y]^{p, \mathbb{K}}$. The càdlàg, bounded \mathbb{F} -supermartingale Z given by

$$Z_t := \mathbb{P}(\tau > t | \mathcal{F}_t) = {}^{o, \mathbb{F}}(\mathbb{1}_{[0, \tau]})_t$$

is called the *Azéma supermartingale* (see Azéma [4]) of the random time τ . Note that since τ is assumed to be finite, we have that $Z_\infty := \lim_{t \rightarrow \infty} Z_t = 0$. The process Z admits a unique *Doob-Meyer decomposition* $Z = \mu - A^p$ where μ is an \mathbb{F} -martingale and A^p is an \mathbb{F} -predictable, increasing process. Specifically, A^p is the dual \mathbb{F} -predictable projection of the process H , that is, $A^p = H^{p, \mathbb{F}}$, whereas the positive \mathbb{F} -martingale μ is given by the equality $\mu_t = \mathbb{E}(A_\infty^p | \mathcal{F}_t)$. It is well known (see, e.g., Jeulin [17, p. 64]) that for any bounded, \mathbb{G} -predictable process U , the process (by convention, we denote $\int_s^t = \int_{]s, t]}$)

$$U_\tau H_t - \int_0^{t \wedge \tau} \frac{U_s}{Z_{s-}} dA_s^p \quad (2.1)$$

is a uniformly integrable \mathbb{G} -martingale. In particular, the process $M^{(\tau)}$ given by

$$M_t^{(\tau)} := H_t - \int_0^{t \wedge \tau} \frac{dA_s^p}{Z_{s-}} \quad (2.2)$$

is a uniformly integrable \mathbb{G} -martingale. Also, it is worth noting that

$$U_\tau H_t - \int_0^{t \wedge \tau} \frac{U_s}{Z_{s-}} dA_s^p = \int_0^t U_s dM_s^{(\tau)}.$$

We shall also use the *optional decomposition* $Z = m - A^o$ where $A^o := H^{o, \mathbb{F}}$ is the dual \mathbb{F} -optional projection of H and the positive \mathbb{F} -martingale m is given by $m_t := \mathbb{E}(A_\infty^o | \mathcal{F}_t)$. Since τ is strictly positive, we have that $\mu_0 = m_0 = 1$. In the special case where τ avoids all \mathbb{F} -stopping times, that is, $\mathbb{P}(\tau = \sigma) = 0$ for any \mathbb{F} -stopping time σ , the equality $A^p = A^o$ holds and thus also $\mu = m$.

Although most existing results regarding the \mathbb{G} -semimartingale decomposition of an \mathbb{F} -martingale are formulated in terms of the Azéma supermartingale Z , it is also possible to use for this purpose the supermartingale \tilde{Z} , which is given by

$$\tilde{Z}_t := \mathbb{P}(\tau \geq t | \mathcal{F}_t) = {}^{o, \mathbb{F}}(\mathbb{1}_{[0, \tau]})_t.$$

Note that the Azéma supermartingale Z is a càdlàg process, but the supermartingale \tilde{Z} fails to be càdlàg, in general. It is also worth mentioning (see Jeulin and Yor [18, p. 79]) that $\tilde{Z} = m - H_-^{o, \mathbb{F}} = m - A_-^o$ and thus $\tilde{Z} = Z_- + \Delta m$.

It is known that the processes Z and $Z_- = \tilde{Z}_-$ do not vanish before τ . Specifically, the random sets $\{\tilde{Z} = 0\}$ and $\{Z_- = 0\} = \{\tilde{Z}_- = 0\}$ are disjoint from $]0, \tau]$, so that the set $\{Z = 0\}$ is disjoint

from $]0, \tau[$. Moreover, the sets $\{Z = 0\}$, $\{\tilde{Z} = 0\}$ and $\{Z_- = 0\} = \{\tilde{Z}_- = 0\}$ are known to have the same début, which is the \mathbb{F} -predictable stopping time R given by

$$R := \inf \{t > 0 : Z_t = 0\} = \inf \{t > 0 : Z_{t-} = 0\} = \inf \{t > 0 : \tilde{Z}_{t-} = 0\}$$

where the middle equality holds since Z is a non-negative, càdlàg supermartingale and the last one is obvious.

For the reader's convenience, we first recall some results on progressive enlargement of a generic filtration \mathbb{F} and an arbitrary random time τ . As was already mentioned, no general result furnishing a \mathbb{G} -semimartingale decomposition after τ of an \mathbb{F} -martingale is available in the existing literature, although such results were established for particular classes of random times, such as: the *honest times* (see Jeulin and Yor [19]), the random times satisfying the *density hypothesis* (see Jeanblanc and Le Cam [12]), as well as for some classes of random times obtained through various extensions of the so-called *Cox construction* of a random time (see, e.g., [9, 13, 14, 21, 22, 25]). In this work, we will focus on decomposition results for \mathbb{F} -martingales stopped at τ and thus we first quote some classical results regarding this case. For part (i) in Proposition 2.1, the reader is referred to Jeulin [17, Proposition 4.16]; the second part is borrowed from Jeulin and Yor [18, Théorème 1, pp. 87–88]).

Proposition 2.1. (i) For any \mathbb{F} -local martingale X , the process

$$\hat{X}_t := X_{t \wedge \tau} - \int_0^{t \wedge \tau} \frac{1}{Z_{s-}} d\langle X, m \rangle_s^{\mathbb{F}} \quad (2.3)$$

is a \mathbb{G} -local martingale (stopped at τ).

(ii) For any \mathbb{F} -local martingale X , the process

$$\bar{X}_t := X_{t \wedge \tau} - \int_0^{t \wedge \tau} \mathbb{1}_{\{Z_{s-} < 1\}} \frac{1}{Z_{s-}} (d\langle X, \mu \rangle_s^{\mathbb{F}} + d\bar{J}_s) \quad (2.4)$$

where $\bar{J} := (H\Delta X_\tau)^{p, \mathbb{F}}$, is a \mathbb{G} -local martingale (stopped at τ).

By comparing parts (i) and (ii) in Proposition 2.1, we obtain the following well known result, which will be used in the proof of the main result of this note (see Theorem 4.1).

Corollary 2.1. The following equality holds for any \mathbb{F} -local martingale X

$$\int_0^{t \wedge \tau} \frac{1}{Z_{s-}} d\bar{J}_s = \int_0^{t \wedge \tau} \frac{1}{Z_{s-}} d\langle X, m - \mu \rangle_s^{\mathbb{F}}. \quad (2.5)$$

Proof. For the sake of completeness, we provide the proof of the corollary. We note that the processes

$$\int_0^{\cdot \wedge \tau} \frac{1}{Z_{s-}} d\langle X, m \rangle_s^{\mathbb{F}}$$

and

$$\int_0^{\cdot \wedge \tau} \mathbb{1}_{\{Z_{s-} < 1\}} \frac{1}{Z_{s-}} (d\langle X, \mu \rangle_s^{\mathbb{F}} + d\bar{J}_s)$$

are \mathbb{G} -predictable. Hence equations (2.3) and (2.4) yield two Doob-Meyer decompositions of the special \mathbb{G} -semi-martingale $X_{\cdot \wedge \tau}$. The uniqueness of the Doob-Meyer decomposition leads to the equality $\hat{X} = \bar{X}$, and thus also to

$$\begin{aligned} 0 &= \int_0^{t \wedge \tau} \mathbb{1}_{\{Z_{s-} < 1\}} \frac{1}{Z_{s-}} (d\langle X, \mu \rangle_s^{\mathbb{F}} + d\bar{J}_s) - \int_0^{t \wedge \tau} \frac{1}{Z_{s-}} d\langle X, m \rangle_s^{\mathbb{F}} \\ &= \int_0^{t \wedge \tau} \frac{1}{Z_{s-}} (d\langle X, \mu - m \rangle_s^{\mathbb{F}} + d\bar{J}_s) - \int_0^{t \wedge \tau} \mathbb{1}_{\{Z_{s-} = 1\}} (d\langle X, \mu \rangle_s^{\mathbb{F}} + d\bar{J}_s) \\ &= \int_0^{t \wedge \tau} \frac{1}{Z_{s-}} (d\langle X, \mu - m \rangle_s^{\mathbb{F}} + d\bar{J}_s) \end{aligned}$$

where the last equality follows from Lemme 4(b) in Jeulin and Yor [18]. We conclude that (2.5) is valid. \square

3 The PRP under the Immersion Hypothesis

Let us introduce the following notation for the class of \mathbb{G} -martingales studied in this work

$$\mathcal{M}(\mathbb{G}, \tau) := \{Y_t^h := \mathbb{E}(h_\tau | \mathcal{G}_t) : h \in \mathcal{H}^o(\mathbb{F}, \tau)\}$$

where $\mathcal{H}^o(\mathbb{F}, \tau) := \{h : h \text{ is an } \mathbb{F}\text{-optional process and } \mathbb{E}|h_\tau| < \infty\}$. Note that $\mathcal{M}(\mathbb{G}, \tau)$ is in fact the set of all \mathbb{G} -martingales stopped at τ . We denote by $\mathcal{H}^p(\mathbb{F}, \tau)$ the set of all processes h from $\mathcal{H}^o(\mathbb{F}, \tau)$ that are \mathbb{F} -predictable, rather than merely \mathbb{F} -optional.

Our main question reads as follows: under which assumptions any process Y^h from the class $\mathcal{M}(\mathbb{G}, \tau)$ admits an integral representation with respect to some “fundamental” \mathbb{G} -martingales?

Of course, a part of the problem is a judicious specification of the fundamental \mathbb{G} -martingales, which will serve as integrators in the representation results. Our goal is to characterize the dynamics of the processes $Y^h \in \mathcal{M}(\mathbb{G}, \tau)$ and to obtain sufficient conditions for the PRP to hold in the filtration \mathbb{G} with respect to the \mathbb{G} -martingale $M^{(\tau)}$ given by equation (2.2) and some auxiliary \mathbb{F} - (or \mathbb{G} -) martingales. Of course, the choice of these martingales depend on the set-up at hand. Somewhat surprisingly, it is not easy to obtain the dynamics of Y^h by direct computations in a general set-up and thus we will first focus on the case when the immersion property holds. Recall that the *immersion property* between \mathbb{F} and \mathbb{G} , which is also known as the hypothesis (H), means that any \mathbb{F} -local martingale is a \mathbb{G} -local martingale. Specifically, in Section 3.1 we will work under Assumption 3.1 and we will assume that \mathbb{F} is an arbitrary filtration. Next, in Section 3.2, we will examine some examples of representation theorems when the immersion holds for the filtration generated by a Poisson process and the Azéma supermartingale is either an \mathbb{F} -predictable process, or merely an \mathbb{F} -optional process.

In Section 3.1, we will work under the following assumption.

Assumption 3.1. We postulate that the process h belongs to $\mathcal{H}^o(\mathbb{F}, \tau)$, the random time τ is such that the filtration \mathbb{F} is immersed in its progressive enlargement \mathbb{G} , and the Azéma supermartingale Z is decreasing and \mathbb{F} -predictable, so that $Z = 1 - A^p$.

If the filtration \mathbb{F} is immersed in \mathbb{G} , then Z is a decreasing process. It is also known (see Lemma 6.3 in Nikeghbali [24]) that if τ avoids \mathbb{F} -stopping times (respectively, if all \mathbb{F} -martingales are continuous), then the process $A^o = A^p$ is continuous (respectively, the process A^o is \mathbb{F} -predictable and thus $A^o = A^p$).

Proposition 3.1. *If either (i) the process h belongs to $\mathcal{H}^p(\mathbb{F}, \tau)$ or (ii) Assumption 3.1 holds, then the \mathbb{G} -martingale Y^h satisfies*

$$Y_t^h = \mathbb{1}_{\{\tau \leq t\}} h_\tau + \mathbb{1}_{\{t < \tau\}} \frac{1}{Z_t} \mathbb{E} \left(\int_t^\infty h_s dA_s^p \middle| \mathcal{F}_t \right) = H_t h_\tau + (1 - H_t) X_t^h (Z_t)^{-1} \quad (3.1)$$

where we denote

$$X_t^h := \mathbb{E} \left(\int_t^\infty h_s dA_s^p \middle| \mathcal{F}_t \right) = \mu_t^h - \int_0^t h_s dA_s^p \quad (3.2)$$

and μ^h stands for the following uniformly integrable \mathbb{F} -martingale

$$\mu_t^h := \mathbb{E} \left(\int_0^\infty h_s dA_s^p \middle| \mathcal{F}_t \right). \quad (3.3)$$

Proof. Under assumption (i), equality (3.2) was established in Elliott et al. [8]. They first show that (see Lemma 3.1 in [8]) for an \mathbb{F} -optional process h we have

$$Y_t^h = \mathbb{1}_{\{\tau \leq t\}} h_\tau + \mathbb{1}_{\{t < \tau\}} \frac{1}{Z_t} \mathbb{E}(h_\tau \mathbb{1}_{\{t < \tau\}} | \mathcal{F}_t). \quad (3.4)$$

Next, in Section 3.4, they show that the properties of the dual \mathbb{F} -predictable projection imply that

$$\mathbb{E}(h_\tau \mathbb{1}_{\{t < \tau\}} | \mathcal{F}_t) = \mathbb{E}\left(\int_t^\infty h_s dA_s^p \mid \mathcal{F}_t\right). \quad (3.5)$$

for any process $h \in \mathcal{H}^p(\mathbb{F}, \tau)$. By combining (3.4) with (3.5), we obtain (3.1).

In the second part of the proof, we suppose that Assumption 3.1 is satisfied. We claim that $H^{o, \mathbb{F}} = {}^{o, \mathbb{F}}H = A^p$. Indeed, under hypothesis (H), τ is a pseudo-stopping time and thus, by Theorem 1 in [25], the equality $H^{o, \mathbb{F}} = {}^{o, \mathbb{F}}H$ holds. Moreover, by our assumption

$$Z = {}^{o, \mathbb{F}}(1 - H) = 1 - {}^{o, \mathbb{F}}H = 1 - A^p,$$

so that the equality ${}^{o, \mathbb{F}}H = A^p$ is valid as well. Using the properties of the dual \mathbb{F} -optional projection and the equality $H^{o, \mathbb{F}} = A^p$, we obtain, for any process $h \in \mathcal{H}^o(\mathbb{F}, \tau)$,

$$\mathbb{E}(h_\tau \mathbb{1}_{\{t < \tau\}} | \mathcal{F}_t) = \mathbb{E}\left(\int_t^\infty h_s dH_s^{o, \mathbb{F}} \mid \mathcal{F}_t\right) = \mathbb{E}\left(\int_t^\infty h_s dA_s^p \mid \mathcal{F}_t\right). \quad (3.6)$$

By combining the last equality with (3.4), we conclude once again that (3.1) holds. \square

Remark 3.1. Let us observe that if $Z = 1 - A^p$ (without assuming that the immersion property holds), then the equality $H^{p, \mathbb{F}} = {}^{p, \mathbb{F}}H$ is valid. Indeed, it is well known that the equality ${}^{p, \mathbb{F}}({}^{o, \mathbb{F}}X) = {}^{p, \mathbb{F}}X$ holds for any bounded measurable process (see property (1.26) on p. 14 in Jacod [11]). From the equality $Z = 1 - A^p$, we obtain ${}^{o, \mathbb{F}}H = A^p$, and thus

$${}^{p, \mathbb{F}}({}^{o, \mathbb{F}}H) = {}^{p, \mathbb{F}}(A^p) = A^p = H^{p, \mathbb{F}}$$

where the second equality is obvious, since A^p is an \mathbb{F} -predictable process. We conclude that under Assumption 3.1, all four projections of H are identical (as classes of equivalences of measurable stochastic processes), that is,

$${}^{o, \mathbb{F}}H = H^{o, \mathbb{F}} = H^{p, \mathbb{F}} = {}^{p, \mathbb{F}}H.$$

It is well known that the immersion property implies that $\mathbb{P}(\tau > t | \mathcal{F}_t) = \mathbb{P}(\tau > t | \mathcal{F}_\infty)$ for all $t \in \mathbb{R}_+$. Therefore, under immersion, the Azéma supermartingale Z is an \mathbb{F} -adapted, decreasing process and, obviously, the \mathbb{F} -martingale μ^h is also a \mathbb{G} -martingale. Let us mention that if Z is an \mathbb{F} -predictable, decreasing process, then it is not necessarily true that the immersion property holds. In fact, it was shown by Nikeghbali and Yor [25] (see Theorem 1 therein) that, if all \mathbb{F} -martingales are continuous, then the property that Z is an \mathbb{F} -predictable, decreasing process is equivalent to the property that τ is an \mathbb{F} -pseudo-stopping time (the latter property is weaker than the immersion property between \mathbb{F} and \mathbb{G}). Finally, it was recently shown by Aksamit and Li [3] that \tilde{Z} is càglàd and decreasing if and only if τ is an \mathbb{F} -pseudo-stopping time.

3.1 The PRP for the Enlargement of a General Filtration

In this subsection, we work with a general right-continuous and \mathbb{P} -completed filtration \mathbb{F} and we search for an integral representation of Y^h in terms of the \mathbb{G} -martingales $M^{(\tau)}$ and μ^h , which are given by equations (2.2) and (3.3), respectively. We start by noting that for any càdlàg \mathbb{F} -adapted process U , the jump $\Delta U = U - U_-$ process is a thin process, i.e., there exists a sequence of \mathbb{F} -stopping times S_n such that $\{\Delta U \neq 0\} \subset \bigcup_n \llbracket S_n \rrbracket$ (see Definition 7.39 in [10]). As a consequence, if Z is a finite variation process then

$$\int_0^t \Delta \left(\frac{X_s^h}{Z_s} \right) \frac{dZ_s}{Z_{s-}} = \sum_{0 < s \leq t} \Delta \left(\frac{X_s^h}{Z_s} \right) \frac{\Delta Z_s}{Z_{s-}}, \quad \forall t \in \mathbb{R}_+, \quad (3.7)$$

for an arbitrary process h from the class $\mathcal{H}^o(\mathbb{F}, \tau)$ and the associated \mathbb{F} -martingale μ^h given by equation (3.3) and the process X^h given by (3.2).

We are in a position to prove the following result yielding an integral representation for any process Y^h from $\mathcal{M}(\mathbb{G}, \tau)$.

Proposition 3.2. *If Assumption 3.1 holds, then $Y^h \in \mathcal{M}(\mathbb{G}, \tau)$ admits the following representation*

$$dY_t^h = \left(h_t - \frac{X_t^h}{Z_t} \right) dM_t^{(\tau)} + \frac{1 - H_{t-}}{Z_{t-}} d\mu_t^h. \quad (3.8)$$

Proof. Since $Z = 1 - A^p$, it is clear that $dZ_t = -dA_t^p$. If we denote $Y = Y^h$ and $X = X^h$, then the Itô formula yields

$$\begin{aligned} dY_t &= h_t dH_t + (1 - H_{t-}) d\left(\frac{X_t}{Z_t}\right) - \frac{X_{t-}}{Z_{t-}} dH_t - \Delta\left(\frac{X_t}{Z_t}\right) \Delta H_t \\ &= \left(h_t - \frac{X_{t-}}{Z_{t-}} \right) dH_t + (1 - H_{t-}) d\left(\frac{X_t}{Z_t}\right) - \Delta\left(\frac{X_t}{Z_t}\right) \Delta H_t \end{aligned}$$

and thus, since $\mu = 0$,

$$d\left(\frac{X_t}{Z_t}\right) = \frac{1}{Z_{t-}} dX_t + X_{t-} \left(-\frac{1}{Z_{t-}^2} dZ_t + \frac{1}{Z_{t-}^2} \Delta Z_t + \Delta\left(\frac{1}{Z_t}\right) \right) + \Delta X_t \Delta\left(\frac{1}{Z_t}\right).$$

From equation (3.2), we have $dX_t = d\mu_t^h - h_t dA_t^p$ and thus, using $dZ_t = -dA_t^p$,

$$\frac{1}{Z_{t-}} dX_t - \frac{X_{t-}}{Z_{t-}^2} dZ_t = \frac{1}{Z_{t-}} d\mu_t^h - \left(h_t - \frac{X_{t-}}{Z_{t-}} \right) \frac{dA_t^p}{Z_{t-}},$$

which leads to

$$dY_t = \left(h_t - \frac{X_{t-}}{Z_{t-}} \right) dM_t^{(\tau)} + \frac{1 - H_{t-}}{Z_{t-}} d\mu_t^h + dL_t \quad (3.9)$$

where

$$\begin{aligned} dL_t &:= (1 - H_{t-}) \left(\frac{X_{t-}}{Z_{t-}^2} \Delta Z_t + X_{t-} \Delta\left(\frac{1}{Z_t}\right) + \Delta X_t \Delta\left(\frac{1}{Z_t}\right) \right) - \Delta\left(\frac{X_t}{Z_t}\right) \Delta H_t \\ &= -(1 - H_{t-}) \Delta\left(\frac{X_t}{Z_t}\right) \frac{\Delta Z_t}{Z_{t-}} - \Delta\left(\frac{X_t}{Z_t}\right) \Delta H_t \\ &= -(1 - H_{t-}) \Delta\left(\frac{X_t}{Z_t}\right) \left(\frac{dZ_t}{Z_{t-}} \right) - \Delta\left(\frac{X_t}{Z_t}\right) dH_t = -\Delta\left(\frac{X_t}{Z_t}\right) dM_t^{(\tau)} \end{aligned}$$

where the penultimate equality follows from (3.7) and the last one holds since $dZ_t = -dA_t^p$. The asserted equality (3.8) now easily follows from (3.9). \square

It is not obvious that the first term in the right-hand side of (3.8) is a \mathbb{G} -martingale, since the integrand is not necessarily \mathbb{G} -predictable. However, this is indeed true, since Y^h is a \mathbb{G} -martingale, the integrand in the second term in right-hand side of (3.8) is \mathbb{G} -predictable and, due to the immersion property, the \mathbb{F} -martingale μ^h is also a \mathbb{G} -martingale.

Remark 3.2. Assume that Z is decreasing, but not necessarily \mathbb{F} -predictable, so that $Z = \mu - A^p$, where the \mathbb{F} -martingale μ is of finite variation (for an explicit example, see Lemma 3.2 below). A slight modification of the proof of Proposition 3.2 yields

$$dY_t^h = \left(h_t - \frac{X_t^h}{Z_t} \right) dM_t^{(\tau)} + \frac{1 - H_{t-}}{Z_{t-}} d\mu_t^h - \frac{(1 - H_{t-})X_t^h}{Z_t Z_{t-}} d\mu_t. \quad (3.10)$$

It is unclear, however, whether the first and the last terms in the right-hand side of (3.10) are \mathbb{G} -martingales, even though it is obvious that their sum is a \mathbb{G} -martingale.

In the proof of the next result, we will need the following elementary lemma in which we implicitly assume that the integrals are well defined.

Lemma 3.1. *Let Assumption 3.1 be valid. If V is a process of finite variation*

$$\int_0^t \Delta\mu_s^h dV_s = 0, \quad \forall t \in \mathbb{R}_+, \quad (3.11)$$

then

$$\int_0^t \left(h_s - \frac{X_s^h}{Z_s} \right) dV_s = \int_0^t \frac{Z_{s-}}{Z_s} \left(h_s - \frac{X_{s-}^h}{Z_{s-}} \right) dV_s, \quad \forall t \in \mathbb{R}_+.$$

Proof. We write $Y = Y^h$ and $X = X^h$. Noting that

$$\Delta X_t = \Delta\mu_t^h - h_t \Delta A_t^p = \Delta\mu_t^h + h_t \Delta Z_t$$

and using (3.11), we obtain

$$\begin{aligned} \left(h_t - \frac{X_t}{Z_t} \right) dV_t &= \left(h_t - \frac{X_{t-}}{Z_{t-}} - \Delta \left(\frac{X_t}{Z_t} \right) \right) dV_t \\ &= \left(h_t - \frac{X_{t-}}{Z_{t-}} - \frac{(X_{t-} + \Delta\mu_t^h + h_t \Delta Z_t) Z_{t-} - X_{t-} Z_t}{Z_t Z_{t-}} \right) dV_t \\ &= \frac{h_t Z_t Z_{t-} - (X_{t-} + h_t \Delta Z_t) Z_{t-}}{Z_t Z_{t-}} dV_t \\ &= \frac{Z_{t-}}{Z_t} \left(h_t - \frac{X_{t-}}{Z_{t-}} \right) dV_t, \end{aligned}$$

which is the desired equality. \square

By combining Proposition 3.2 with Lemma 3.1, we obtain the following result yielding an alternative representation for martingales from the class $\mathcal{M}(\mathbb{G}, \tau)$.

Corollary 3.1. *Let Assumption 3.1 be valid. If*

$$\int_0^t \Delta\mu_s^h dM_s^{(\tau)} = 0, \quad \forall t \in \mathbb{R}_+, \quad (3.12)$$

then $Y^h \in \mathcal{M}(\mathbb{G}, \tau)$ admits the following representation

$$dY_t^h = \frac{Z_{t-}}{Z_{t-} - \Delta A_t^p} \left(h_t - \frac{X_{t-}^h}{Z_{t-}} \right) dM_t^{(\tau)} + \frac{1 - H_{t-}}{Z_{t-}} d\mu_t^h. \quad (3.13)$$

Proof. From Proposition 3.2, we know that (3.8) is valid. If, in addition, condition (3.12) is satisfied then, using Lemma 3.1, we obtain

$$\left(h_t - \frac{X_t^h}{Z_t} \right) dM_t^{(\tau)} = \frac{Z_{t-}}{Z_t} \left(h_t - \frac{X_{t-}^h}{Z_{t-}} \right) dM_t^{(\tau)} = \frac{Z_{t-}}{Z_{t-} - \Delta A_t^p} \left(h_t - \frac{X_{t-}^h}{Z_{t-}} \right) dM_t^{(\tau)}$$

since $M^{(\tau)}$ is a process of finite variation and $Z_t = Z_{t-} + \Delta Z_t = Z_{t-} - \Delta A_t^p$. Hence representations (3.8) and (3.13) of Y^h are equivalent under the present assumptions. \square

Remark 3.3. In view of (2.2), to establish (3.12), it suffices to show that

$$\int_0^t \Delta\mu_s^h dA_s^p = 0 = \int_0^t \Delta\mu_s^h dH_s, \quad \forall t \in \mathbb{R}_+. \quad (3.14)$$

In fact, the first equality is true if the filtration \mathbb{F} is quasi-left-continuous and the second one is satisfied under the avoidance property. It is also obvious that both equalities are satisfied when all \mathbb{F} -martingales are continuous.

Let us now assume, in addition, that there exists a d -dimensional \mathbb{F} -martingale M , which has the predictable representation property with respect to the filtration \mathbb{F} . The following result, which is an immediate consequence of Proposition 3.2, shows that the $(d + 1)$ -dimensional \mathbb{G} -martingale $(M^{(\tau)}, M)$ generates any \mathbb{G} -martingale stopped at time τ of the form $\mathbb{E}(h_\tau | \mathcal{G}_t)$ for some process $h \in \mathcal{H}^o(\mathbb{F}, \tau)$.

Proposition 3.3. *Let Assumption 3.1 be valid. If an \mathbb{F} -martingale M has the PRP with respect to \mathbb{F} , then $Y^h \in \mathcal{M}(\mathbb{G}, \tau)$ admits the following representation*

$$dY_t^h = \left(h_t - \frac{X_t^h}{Z_t} \right) dM_t^{(\tau)} + \frac{1 - H_{t-}}{Z_{t-}} \phi_t^h dM_t \quad (3.15)$$

for some \mathbb{F} -predictable process ϕ^h . If, in addition, condition (3.12) holds then also

$$dY_t^h = \frac{Z_{t-}}{Z_{t-} - \Delta A_t^p} \left(h_t - \frac{X_{t-}^h}{Z_{t-}} \right) dM_t^{(\tau)} + \frac{1 - H_{t-}}{Z_{t-}} \phi_t^h dM_t. \quad (3.16)$$

Remark 3.4. Let us note that this framework was studied by Kusuoka [20] under the additional assumption that the filtration \mathbb{F} is generated by a Brownian motion. Note that if \mathbb{F} is a Brownian filtration, then all \mathbb{F} -martingales (in particular, the martingale μ^h defined by (3.3)) are continuous, so that condition (3.12) is trivially satisfied and thus Proposition 3.3 is valid when $M = W$ is a Brownian motion.

Remark 3.5. It is also worth stressing that Proposition 3.3 is not covered by the recent results of Jeanblanc and Song [15], where the authors prove that if the hypothesis (H') holds between \mathbb{F} and \mathbb{G} and the PRP for the filtration \mathbb{F} is valid with respect to an \mathbb{F} -martingale M , then the following conditions are equivalent:¹

- (i) $M_\tau \in \mathcal{F}_{\tau-}$ and the immersion property between \mathbb{F} and \mathbb{G} holds,
- (ii) the PRP with respect to M and $M^{(\tau)}$ holds and the equality $\mathcal{G}_\tau = \mathcal{G}_{\tau-}$ is satisfied.

For an example of the set-up when the immersion property between \mathbb{F} and \mathbb{G} holds and the property that M_τ is $\mathcal{F}_{\tau-}$ -measurable is not valid, see the case of the filtration \mathbb{F} generated by a Poisson process, which is examined in Section 3.2.

3.2 The PRP for the Enlargement of the Poisson Filtration

Let N be a standard Poisson process with the sequence of jump times denoted as $(T_n)_{n=1}^\infty$ and the constant intensity λ . We take \mathbb{F} to be the filtration generated by the Poisson process N and we assume, as usual, that $\mathcal{F}_\infty \subseteq \mathcal{G}$. We now denote by $M_t := N_t - \lambda t$ the compensated Poisson process, which is an \mathbb{F} -martingale. From the well known predictable representation property of the compensated Poisson process (see, for instance, Proposition 8.3.5.1 in [16]), the \mathbb{F} -martingales μ and μ^h admit the integral representations

$$\mu_t = 1 + \int_0^t \phi_s dM_s, \quad \mu_t^h = \mu_0^h + \int_0^t \phi_s^h dM_s \quad (3.17)$$

for some \mathbb{F} -predictable processes ϕ and ϕ^h .

Remark 3.6. Observe that condition (3.12) is satisfied when the filtration \mathbb{F} is generated by a Poisson process N and the random time τ is independent of \mathbb{F} , so that the immersion property holds and the Azéma supermartingale Z is a decreasing, deterministic function (hence an \mathbb{F} -predictable process). Indeed, from (3.17), we deduce that

$$\{\Delta \mu^h \neq 0\} \subset \{\Delta M > 0\} = \{\Delta N > 0\}. \quad (3.18)$$

Moreover,

$$\int_0^t \Delta N_s dZ_s = \sum_{0 < s \leq t} \Delta N_s \Delta Z_s = 0 = \int_0^t \Delta N_s dH_s, \quad \forall t \in \mathbb{R}_+, \quad (3.19)$$

¹Recall that $\mathcal{F}_{\tau-}$ is the σ -field generated by \mathbb{F} -predictable processes stopped at τ .

where the second equality holds since the jumps of Z occur at deterministic times. The last equality follows from the fact that $\int_0^t \Delta N_s dH_s$ is non-negative and that, due to the independence of N and τ , we have

$$\mathbb{E} \left(\int_0^\infty \Delta N_s dH_s \right) = \mathbb{E} (\Delta N_\tau) = \int_0^\infty \mathbb{P}(\Delta N_t = 1) d\mathbb{P}(\tau \leq t) = 0.$$

In the remaining part of this section, we work under the following postulate.

Assumption 3.2. The probability space $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$ supports a random variable Θ with the unit exponential distribution and such that Θ is independent of the filtration \mathbb{F} generated by the Poisson process N . The random time τ is given through the *Cox construction*, that is, by the formula

$$\tau = \inf \{ t \in \mathbb{R}_+ : \Lambda_t \geq \Theta \} \quad (3.20)$$

where Λ is an \mathbb{F} -adapted, increasing process such that $\Lambda_0 = 0$ and $\Lambda_\infty := \lim_{t \rightarrow \infty} \Lambda_t = \infty$.

Under Assumption 3.2, the immersion property holds for the filtrations \mathbb{F} and \mathbb{G} . Furthermore, the Azéma supermartingale Z equals $Z_t = \mathbb{P}(\tau > t | \mathcal{F}_t) = e^{-\Lambda_t}$ and thus it is a decreasing and an \mathbb{F} -adapted (but not necessarily \mathbb{F} -predictable), process. It is also worth noting that it may happen, for instance, that $\{\Delta H > 0\} \subset \{\Delta N > 0\}$ (see equation (3.21)), so that the validity of the second equality in (3.14) is not ensured, in general.

3.2.1 Predictable Azéma's Supermartingale

Let us first consider the situation where the process Λ in Assumption 3.2 is \mathbb{F} -predictable. Then we have the following corollary to Proposition 3.3, which covers, in particular, the case where τ is independent of N (see Remark 3.6).

Corollary 3.2. *Let Assumption 3.2 be valid with an \mathbb{F} -predictable process Λ . Then for any process $Y^h \in \mathcal{M}(\mathbb{G}, \tau)$ representations (3.15) and (3.16) hold with $M_t = N_t - \lambda t$.*

Proof. Under the present assumptions, the immersion property holds and the Azéma supermartingale Z is decreasing and \mathbb{F} -predictable, so that $Z = 1 - A^p$. It thus suffices to show that condition (3.12) holds for the \mathbb{F} -martingale μ^h . Note that as Poisson filtration is quasi-left-continuous, the martingale μ^h can only jump at totally inaccessible times. Since Z is predictable, the processes μ^h and Z cannot jump together and thus the first equality in (3.14) holds.

It remains to show that the second equality in (3.14) holds for all $t \in \mathbb{R}_+$. We observe that

$$\mathbb{E} \left(\int_0^\infty \Delta N_s dH_s \right) = \int_0^\infty \mathbb{P}(\Delta N_\tau = 1 | \Theta = \theta) d\mathbb{P}(\Theta \leq \theta) = 0$$

where the last equality holds since, for any fixed θ , the random time τ is \mathbb{F} -predictable and the jump times of the Poisson process are \mathbb{F} -totally inaccessible. Since $\int_0^\infty \Delta N_s dH_s$ is non-negative and (3.18) is valid, we conclude that the equality $\int_0^t \Delta \mu_s^h dH_s = 0$ is satisfied for all $t \in \mathbb{R}_+$. The statement now follows from Proposition 3.3. \square

3.2.2 Non-Predictable Azéma Supermartingale

We continue the study of the Poisson filtration and the Cox construction of a random time, but we no longer assume that the Azéma supermartingale of τ is \mathbb{F} -predictable. Hence condition (3.12) is not satisfied, in general, and thus Corollary 3.2 no longer applies. Despite the fact that we will still postulate the immersion property between \mathbb{F} and \mathbb{G} , it seems to us that a general representation result is rather hard to establish. Therefore, we postpone an attempt to derive a general result to the foregoing section, where we will work without postulating the immersion property.

Our immediate goal is merely to show that explicit representations are still available when the Azéma supermartingale Z is not \mathbb{F} -predictable, at least for some particular martingales from the

class $\mathcal{M}(\mathbb{G}, \tau)$. In contrast to the preceding subsection, these representations are derived by means of direct computations, rather than through an application of the predictable representation property of the compensated Poisson process. Hence we will be able to compute explicitly the integrand ϕ^h arising in suitable variants of representation (3.15). For the sake of concreteness, we also specialize Assumption 3.2 by postulating that $\Lambda = N$.

Assumption 3.3. The probability space $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$ supports a random variable Θ with the unit exponential distribution and such that Θ is independent of the filtration \mathbb{F} generated by the Poisson process N . The random time τ is given by the formula

$$\tau = \inf \{ t \in \mathbb{R}_+ : N_t \geq \Theta \}. \quad (3.21)$$

Under Assumption 3.3, the Azéma supermartingale equals $Z_t = e^{-N_t}$ for all $t \in \mathbb{R}_+$ and thus it is decreasing, but not \mathbb{F} -predictable. As in the preceding subsection, the filtration \mathbb{F} is immersed in \mathbb{G} and thus the compensated Poisson \mathbb{F} -martingale M is also a \mathbb{G} -martingale. It is crucial to observe that the inclusion $[[\tau]] \subset \cup_n [[T_n]]$ holds, meaning that a jump of the process H may only occur when the Poisson process N has a jump, that is, $\{\Delta H > 0\} \subset \{\Delta N > 0\}$. We first compute explicitly the Doob-Meyer decomposition of Z and we show that the compensator of H is continuous.

Lemma 3.2. *Let $Z = e^{-N}$ where N is the Poisson process. Then the following assertions hold:*
 (i) Z admits the Doob-Meyer decomposition $Z = \mu - A^p$ where

$$\mu_t = 1 - \int_0^t \gamma e^{-N_{s-}} dM_s, \quad A_t^p = \int_0^t \gamma \lambda e^{-N_s} ds \quad (3.22)$$

and $\gamma = 1 - \frac{1}{e} > 0$;

(ii) the process $M_t^{(\tau)} := H_t - \gamma \lambda (t \wedge \tau)$ is a \mathbb{G} -martingale and thus the random time τ given by (3.21) is a totally inaccessible \mathbb{G} -stopping time.

Proof. The proof of part (i) is elementary, since it relies on the standard Stieltjes integration and thus is omitted. For the second part, the \mathbb{G} -martingale property of the process $M^{(\tau)}$ is a consequence of (2.2), and the fact that τ is a totally inaccessible follows from the continuity of the compensator A^p of H . \square

Recall that the process X^h is defined by (3.2) and note that, due to the form of A^p ,

$$\{\Delta M^{(\tau)} > 0\} = \{\Delta H > 0\} \subset \{\Delta N > 0\} = \{\Delta M > 0\}. \quad (3.23)$$

It is also worth stressing that here the equality $M_\tau = M_{\tau-}$ is not satisfied (see Remark 3.5). The following result shows that when \mathbb{F} is the Poisson filtration, the immersion property holds, but the Azéma supermartingale Z is not \mathbb{F} -predictable, then an extension of Proposition 3.3 is still feasible in some circumstances.

Proposition 3.4. *Let Assumption 3.3 be valid. Consider the \mathbb{G} -martingale $Y^h \in \mathcal{M}(\mathbb{G}, \tau)$ where the process $h \in \mathcal{H}^p(\mathbb{F}, \tau)$ is given by $h_t = h(N_{t-})$ for some Borel function $h : \mathbb{R} \rightarrow \mathbb{R}$. Then Y^h admits the following representation*

$$dY_t^h = (h_t - \tilde{h}(N_{t-} + 1)) dM_t^{(\tau)} + (1 - H_{t-})(\tilde{h}(N_{t-} + 1) - \tilde{h}(N_{t-})) dM_t \quad (3.24)$$

or, equivalently,

$$dY_t^h = (h_t - \tilde{h}(N_{t-})) dM_t^{(\tau)} + (1 - H_{t-})(\tilde{h}(N_{t-} + 1) - \tilde{h}(N_{t-})) d\bar{M}_t^{(\tau)} \quad (3.25)$$

where the processes $M_t^{(\tau)} := H_t - \gamma \lambda (t \wedge \tau)$ and $\bar{M}_t^{(\tau)} := M_t - M_t^{(\tau)}$ are orthogonal \mathbb{G} -martingales and the function \tilde{h} is given by (3.28)

Proof. We write $X = X^h$ and $Y = Y^h$. In view of Lemma 3.2 and equation (3.1), we have

$$Y_t = \int_0^t h_s dH_s + (1 - H_t) \frac{1}{Z_t} \mathbb{E} \left(\int_t^\infty \gamma \lambda h_s e^{-N_s} ds \middle| \mathcal{F}_t \right). \quad (3.26)$$

Using the independence of increments of the Poisson process N , we obtain

$$X_t = \mathbb{E} \left(\int_t^\infty \gamma \lambda h(N_s) e^{-N_s} ds \middle| \mathcal{F}_t \right) = \tilde{h}(N_t) e^{-N_t} = \tilde{h}(N_t) Z_t \quad (3.27)$$

where

$$\tilde{h}(x) := \mathbb{E} \left(\int_0^\infty \gamma \lambda h(N_s + x) e^{-N_s} ds \right). \quad (3.28)$$

Equations (3.26)–(3.27) yield

$$Y_t = \int_0^t h_s dH_s + (1 - H_t) \tilde{h}(N_t)$$

and thus, since $\{\Delta H > 0\} \subset \{\Delta N > 0\}$,

$$\begin{aligned} dY_t &= (h_t - \tilde{h}(N_{t-})) dH_t + (1 - H_{t-}) d\tilde{h}(N_t) - \Delta H_t \Delta \tilde{h}(N_t) \\ &= (h_t - \tilde{h}(N_{t-})) dH_t + (1 - H_{t-}) (\tilde{h}(N_{t-} + 1) - \tilde{h}(N_{t-})) dN_t - (\tilde{h}(N_{t-} + 1) - \tilde{h}(N_{t-})) dH_t \\ &= (h_t - \tilde{h}(N_{t-} + 1)) dH_t + (1 - H_{t-}) (\tilde{h}(N_{t-} + 1) - \tilde{h}(N_{t-})) dN_t. \end{aligned}$$

Consequently,

$$\begin{aligned} dY_t &= (h_t - \tilde{h}(N_{t-} + 1)) dM_t^{(\tau)} + (1 - H_{t-}) (\tilde{h}(N_{t-} + 1) - \tilde{h}(N_{t-})) dM_t \\ &\quad + (1 - H_t) \lambda \gamma (h(N_{t-}) - \tilde{h}(N_{t-} + 1)) dt + (1 - H_t) \lambda (\tilde{h}(N_{t-} + 1) - \tilde{h}(N_{t-})) dt. \end{aligned}$$

To complete the proof, it suffices to show that the following equality is satisfied for all $x \geq 0$

$$\gamma h(x) + (1 - \gamma) \tilde{h}(x + 1) - \tilde{h}(x) = 0. \quad (3.29)$$

To establish (3.29), we first observe that

$$\tilde{h}(x + 1) = \mathbb{E} \left(\int_0^\infty \gamma \lambda h(N_s + x + 1) e^{-N_s} ds \right) = e \mathbb{E} \left(\int_0^\infty \gamma \lambda h(N_s + x + 1) e^{-(N_s + 1)} ds \right).$$

If we denote by T_1 the moment of the first jump of N , then we obtain

$$\begin{aligned} \tilde{h}(x) &= \mathbb{E} \left(\int_0^\infty h(N_s + x) e^{-N_s} ds \right) = \mathbb{E} \left(\int_0^{T_1} \gamma \lambda h(x) ds \right) + \mathbb{E} \left(\int_{T_1}^\infty \gamma \lambda h(N_s + x) e^{-N_s} ds \right) \\ &= \gamma h(x) + \mathbb{E} \left(\int_0^\infty \gamma \lambda h(N_s + x + 1) e^{-(N_s + 1)} ds \right) \\ &= \gamma h(x) + e^{-1} \tilde{h}(x + 1) = \gamma h(x) + (1 - \gamma) \tilde{h}(x + 1). \end{aligned}$$

We conclude that (3.29) holds and thus the proof of (3.24) is completed. Representation (3.25) is an easy consequence of equation (3.24) and thus we omit the details. Let us finally observe that the orthogonality of $M^{(\tau)}$ and $\bar{M}^{(\tau)}$ follows from the fact that $M^{(\tau)}$ and N are pure jump martingales with jumps of size 1. \square

Remark 3.7. Proposition 3.4 will be revisited in Section 4.2 (see Example 4.1), where we will re-derive representation (3.25) using the general representation formula.

4 The PRP Beyond the Immersion Hypothesis

In this section, we work with the filtration \mathbb{F} generated by a Poisson process N , but we no longer postulate that the immersion property between \mathbb{F} and \mathbb{G} holds. Recall that the equality $Z = \mu - A^p$ is the Doob-Meyer decomposition of the Azéma supermartingale Z associated with a random time τ . It is worth noting that the Azéma supermartingale of any random time τ is necessarily a process of finite variation when the filtration \mathbb{F} is generated by a Poisson process.

We argue that the main difficulty in establishing the PRP for a progressive enlargement is due to the fact that the jumps of \mathbb{F} -martingales may overlap with the jump of the process H . It appears that even when the filtration \mathbb{F} is generated by a Poisson process N , the validity of the PRP for the progressive enlargement of \mathbb{F} with a random time τ is a challenging problem for the part after τ if no additional assumptions are made.

Since the inclusion $\{\Delta H > 0\} \subset \{\Delta N > 0\}$ may fail to hold, in general, it is hard to control a possible overlap of jumps of processes N and H when the only information about the \mathbb{F} -conditional distribution of the random time τ is its Azéma supermartingale Z . Nevertheless, in the main result of this section (Theorem 4.1) we offer a general representation formula for a \mathbb{G} -martingale stopped at τ . It is fair to acknowledge that we need to introduce for this purpose an additional martingale to compensate for a potential mismatch of jumps of H and N . Subsequently, we illustrate our general result by considering some special cases. We conclude this note by emphasizing the role of the optional stochastic integral in our general representation result and by obtaining in Corollary 4.1 an equivalent representation of Y^h in terms of the optional and predictable stochastic integrals.

Remark 4.1. From part (i) in Proposition 2.1, we deduce that the compensated martingale $M_t := N_t - \lambda t$ stopped at τ admits the following semimartingale decomposition with respect to \mathbb{G}

$$\widehat{M}_t = M_{t \wedge \tau} - \int_0^{t \wedge \tau} \frac{d\langle M, m \rangle_s^{\mathbb{F}}}{Z_{s-}}. \quad (4.1)$$

Note that the compensated Poisson process $M_t = N_t - \lambda t$ is an \mathbb{F} -adapted (hence also \mathbb{G} -adapted) process of finite variation, so that it is manifestly a \mathbb{G} -semimartingale for any choice of a random time τ . Consequently, due to the PRP of the compensated Poisson process with respect to its natural filtration, no additional assumptions regarding the random time τ are needed to ensure that the hypothesis (H') is satisfied, that is, any \mathbb{F} -martingale is always a \mathbb{G} -semimartingale.

4.1 Main Result

We are in a position to prove the main result of this note. It should be stressed that the \mathbb{G} -martingales $M^{(\tau)}$ and \widehat{M} in the statement of the following Theorem 4.1 are universal, in the sense that they do not depend on the choice of the process $h \in \mathcal{H}^p(\mathbb{F}, \tau)$ (see equations (2.2) and (4.1)). By contrast, the \mathbb{G} -martingale \widetilde{M}^h , which is defined by (4.3), is clearly dependent on h . In the next subsection, we present some special cases of representation established in Theorem 4.1 in which the process \widetilde{M}^h does not appear, despite the fact that they are valid for any process $h \in \mathcal{H}^o(\mathbb{F}, \tau)$.

In the statement of Theorem 4.1, we will use the following lemma, which extends a result quoted in Jeulin [17] (see Remark 4.5 therein or equation (2.1) with $U = 1$). Note that equation (2.2) can be obtained as a special case of (4.2) by setting $\xi = \kappa = 1$.

Lemma 4.1. *Let the process B be given by the formula $B = \xi H$ where ξ is an integrable and \mathcal{G}_τ -measurable random variable. Then the process M , which is given by the equality*

$$\widetilde{M}_t = B_t - \int_0^t \frac{1 - H_{s-}}{Z_{s-}} dB_s^{p, \mathbb{F}}, \quad (4.2)$$

is a purely discontinuous \mathbb{G} -martingale stopped at τ . Moreover, the dual \mathbb{F} -predictable projection of B satisfies $B_t^{p, \mathbb{F}} = \int_0^t \kappa_s dA_s^p$ where κ is an \mathbb{F} -predictable process such that the equality $\kappa_\tau = \mathbb{E}(\xi | \mathcal{F}_{\tau-})$ holds.

Proof. Let the process B be given by $B = \xi H$, where the integrable random variable ξ is \mathcal{G}_τ -measurable, and let $B^{p,\mathbb{F}}$ be its dual \mathbb{F} -predictable projection. On the one hand, we have, for any $u \geq t$,

$$\begin{aligned}\mathbb{E}(B_u - B_t | \mathcal{G}_t) &= \mathbb{E}(\xi \mathbf{1}_{\{u \geq \tau > t\}} | \mathcal{G}_t) = \mathbf{1}_{\{\tau > t\}} (Z_t)^{-1} \mathbb{E}(\xi \mathbf{1}_{\{u \geq \tau > t\}} | \mathcal{F}_t) \\ &= \mathbf{1}_{\{\tau > t\}} (Z_t)^{-1} \mathbb{E}(B_u^{p,\mathbb{F}} - B_t^{p,\mathbb{F}} | \mathcal{F}_t).\end{aligned}$$

On the other hand, we obtain

$$\begin{aligned}\mathbb{E}\left(\int_t^u \frac{1 - H_{s-}}{Z_{s-}} dB_s^{p,\mathbb{F}} \middle| \mathcal{G}_t\right) &= \mathbb{E}\left(\mathbf{1}_{\{\tau > t\}} \int_t^{u \wedge \tau} \frac{1}{Z_{s-}} dB_s^{p,\mathbb{F}} \middle| \mathcal{G}_t\right) \\ &= \mathbf{1}_{\{\tau > t\}} \frac{1}{Z_t} \mathbb{E}\left(\mathbf{1}_{\{\tau > t\}} \int_t^{u \wedge \tau} \frac{1}{Z_{s-}} dB_s^{p,\mathbb{F}} \middle| \mathcal{F}_t\right).\end{aligned}$$

We define the \mathbb{F} -predictable process Λ by setting $\Lambda_t = \int_0^t \frac{1}{Z_{s-}} dB_s^{p,\mathbb{F}}$. Then we obtain

$$\begin{aligned}\mathbb{E}\left(\mathbf{1}_{\{\tau > t\}} \int_t^{u \wedge \tau} \frac{1}{Z_{s-}} dB_s^{p,\mathbb{F}} \middle| \mathcal{F}_t\right) &= \mathbb{E}\left(\mathbf{1}_{\{\tau > u\}} \int_t^u \frac{1}{Z_{s-}} dB_s^{p,\mathbb{F}} + \mathbf{1}_{\{u \geq \tau > t\}} \int_t^\tau \frac{1}{Z_{s-}} dB_s^{p,\mathbb{F}} \middle| \mathcal{F}_t\right) \\ &= \mathbb{E}\left(Z_u \int_t^u \frac{1}{Z_{s-}} dB_s^{p,\mathbb{F}} + \int_t^u \int_t^v \frac{1}{Z_{s-}} dB_s^{p,\mathbb{F}} dA_v^p \middle| \mathcal{F}_t\right) \\ &= \mathbb{E}\left(Z_u(\Lambda_u - \Lambda_t) + \int_t^u (\Lambda_s - \Lambda_t) dA_s^p \middle| \mathcal{F}_t\right) \\ &= \mathbb{E}\left(Z_u(\Lambda_u - \Lambda_t) - \int_t^u (\Lambda_s - \Lambda_t) dZ_s \middle| \mathcal{F}_t\right) \\ &= \mathbb{E}(B_u^{p,\mathbb{F}} - B_t^{p,\mathbb{F}} | \mathcal{F}_t)\end{aligned}$$

where the last equality is a consequence of the following elementary computation

$$\begin{aligned}Z_u(\Lambda_u - \Lambda_t) - \int_t^u (\Lambda_s - \Lambda_t) dZ_s &= Z_u(\Lambda_u - \Lambda_t) + \Lambda_t(Z_u - Z_t) - \int_t^u \Lambda_s dZ_s \\ &= Z_u \Lambda_u - \Lambda_t Z_t - \Lambda_u Z_u + \Lambda_t Z_t + \int_t^u Z_{s-} d\Lambda_s = B_u^{p,\mathbb{F}} - B_t^{p,\mathbb{F}}.\end{aligned}$$

That completes the proof of the first statement in Lemma 4.1. For the second statement, we note that, from the definition of the σ -algebra $\mathcal{F}_{\tau-}$, the equality $\mathbb{E}(\xi | \mathcal{F}_{\tau-}) = \kappa_\tau$ holds for some \mathbb{F} -predictable process κ . Hence for any bounded, \mathbb{F} -predictable process X , we obtain (note that X_τ is $\mathcal{F}_{\tau-}$ -measurable)

$$\begin{aligned}\mathbb{E}\left(\int_0^\infty X_s dB_s^{p,\mathbb{F}}\right) &= \mathbb{E}\left(\int_0^\infty X_s dB_s\right) = \mathbb{E}\left(\int_0^\infty \xi X_s dH_s\right) = \mathbb{E}(\xi X_\tau) = \mathbb{E}(X_\tau \mathbb{E}(\xi | \mathcal{F}_{\tau-})) \\ &= \mathbb{E}(X_\tau \kappa_\tau) = \mathbb{E}\left(\int_0^\infty X_s \kappa_s dH_s\right) = \mathbb{E}\left(\int_0^\infty X_s \kappa_s dA_s^p\right),\end{aligned}$$

since $A^p = H^{p,\mathbb{F}}$. We conclude that $B_t^{p,\mathbb{F}} = \int_0^t \kappa_s dA_s^p$ for all $t \in \mathbb{R}_+$. \square

The following theorem establishes the integral representation with predictable integrands for an arbitrary \mathbb{G} -martingale Y^h associated with a process h from the class $\mathcal{H}^p(\mathbb{F}, \tau)$.

Theorem 4.1. *If the process h belongs to the class $\mathcal{H}^p(\mathbb{F}, \tau)$, then the \mathbb{G} -martingale Y^h stopped at τ admits the following predictable representation*

$$dY_t^h = \frac{Z_{t-}}{Z_{t-} - \Delta A_t^p} \left(h_t - \frac{X_{t-}^h}{Z_{t-}}\right) dM_t^{(\tau)} + \frac{1 - H_{t-}}{Z_{t-} + \phi_t} \left(\phi_t^h - \phi_t \frac{X_{t-}^h}{Z_{t-}}\right) d\widehat{M}_t + \frac{1}{Z_{t-} + \phi_t} d\widetilde{M}_t^h$$

where the \mathbb{F} -predictable processes ϕ and ϕ^h are given by (3.17) and the \mathbb{G} -martingale \widetilde{M}^h equals

$$\widetilde{M}_t^h := B_t^h - \int_0^t \frac{1 - H_{s-}}{Z_{s-}} dB_s^{h,p,\mathbb{F}} \quad (4.3)$$

where $B^h := \xi^h H$ and the \mathcal{G}_τ -measurable random variable ξ^h is given by

$$\xi^h := \Delta\mu_\tau \frac{X_{\tau-}^h}{Z_{\tau-}} - \Delta\mu_\tau^h. \quad (4.4)$$

Proof. As usual, we write $X = X^h$ and $Y = Y^h$. By proceeding as in the proof of Proposition 3.2 and recalling that $dZ_t = d\mu_t - dA_t^p$, we obtain

$$dY_t = \left(h_t - \frac{X_{t-}}{Z_{t-}} \right) dM_t^{(\tau)} + \frac{1 - H_{t-}}{Z_{t-}} \left(d\mu_t^h - \frac{X_{t-}}{Z_{t-}} d\mu_t \right) - (1 - H_{t-}) \Delta \left(\frac{X_t}{Z_t} \right) \frac{\Delta Z_t}{Z_{t-}} - \Delta \left(\frac{X_t}{Z_t} \right) \Delta H_t.$$

Recall that $\Delta Z_t = \Delta\mu_t - \Delta A_t^p$ and $\Delta X_t = \Delta\mu_t^h - h_t \Delta A_t^p$. Moreover, the \mathbb{F} -martingales μ and μ^h satisfy (3.17), so that $\{\Delta\mu \neq 0\} \subset \{\Delta N > 0\}$ and $\{\Delta\mu^h \neq 0\} \subset \{\Delta N > 0\}$. Since the process A^p is \mathbb{F} -predictable and the jump times of N are \mathbb{F} -totally inaccessible, we obtain

$$\Delta\mu_t \Delta A_t^p = \Delta\mu_t^h \Delta A_t^p = 0.$$

Therefore,

$$\Delta \left(\frac{X_t}{Z_t} \right) = \frac{Z_{t-} \Delta X_t - X_{t-} \Delta Z_t}{Z_t Z_{t-}} = \frac{Z_{t-} \Delta\mu_t^h - X_{t-} \Delta\mu_t}{(Z_{t-} + \phi_t) Z_{t-}} - \left(h_t - \frac{X_{t-}}{Z_{t-}} \right) \frac{\Delta A_t^p}{(Z_{t-} - \Delta A_t^p)}$$

and thus also

$$\Delta \left(\frac{X_t}{Z_t} \right) \frac{\Delta Z_t}{Z_{t-}} = \frac{Z_{t-} \Delta\mu_t^h - X_{t-} \Delta\mu_t}{(Z_{t-} + \phi_t) Z_{t-}} \frac{\Delta\mu_t}{Z_{t-}} + \left(h_t - \frac{X_{t-}}{Z_{t-}} \right) \frac{\Delta A_t^p}{(Z_{t-} - \Delta A_t^p)} \frac{dA_t^p}{Z_{t-}}.$$

Consequently, using the definition of $M^{(\tau)}$, we obtain

$$\begin{aligned} dY_t &= \left(h_t - \frac{X_{t-}}{Z_{t-}} \right) \left(1 + \frac{\Delta A_t^p}{Z_{t-} - \Delta A_t^p} \right) dM_t^{(\tau)} + \frac{\xi^h}{Z_{t-} + \phi_t} dH_t \\ &\quad + \frac{1 - H_{t-}}{Z_{t-}} \left(d\mu_t^h - \frac{X_{t-}}{Z_{t-}} d\mu_t - \frac{\Delta\mu_t}{Z_{t-} + \phi_t} \Delta\mu_t^h + \frac{\Delta\mu_t}{Z_{t-} + \phi_t} \frac{X_{t-}}{Z_{t-}} \Delta\mu_t \right). \end{aligned}$$

Let us set

$$U_t := \int_0^t \frac{X_{s-}}{Z_{s-}} d\mu_s - \mu_t^h$$

so that $\Delta_\tau U = \xi^h$. By applying Corollary 2.1 to the \mathbb{F} -local martingale U , we obtain

$$\begin{aligned} \frac{1 - H_{t-}}{Z_{t-}} d(\xi^h H)_t^{p,\mathbb{F}} &= \frac{1 - H_{t-}}{Z_{t-}} d\langle U, m - \mu \rangle_t^{\mathbb{F}} \\ &= \frac{1 - H_{t-}}{Z_{t-}} \left(\frac{X_{t-}}{Z_{t-}} d\langle \mu, m \rangle_t^{\mathbb{F}} - \frac{X_{t-}}{Z_{t-}} d\langle \mu, \mu \rangle_t^{\mathbb{F}} - d\langle \mu^h, m \rangle_t^{\mathbb{F}} + d\langle \mu^h, \mu \rangle_t^{\mathbb{F}} \right) \end{aligned}$$

and thus

$$dY_t = \frac{Z_{t-}}{Z_{t-} - \Delta A_t^p} \left(h_t - \frac{X_{t-}}{Z_{t-}} \right) dM_t^{(\tau)} + \frac{1}{Z_{t-} + \phi_t} d\widetilde{M}_t^h + \frac{1 - H_{t-}}{Z_{t-} + \phi_t} d\widehat{K}_t \quad (4.5)$$

where

$$\begin{aligned} d\widehat{K}_t &:= \frac{1}{Z_{t-}} \left((Z_{t-} + \phi_t) d\mu_t^h - d\langle \mu^h, m \rangle_t^{\mathbb{F}} - \Delta\mu_t \Delta\mu_t^h + d\langle \mu^h, \mu \rangle_t^{\mathbb{F}} \right) \\ &\quad + \frac{1 - H_{t-}}{Z_{t-} (Z_{t-} + \phi_t)} \frac{X_{t-}}{Z_{t-}} \left(- (Z_{t-} + \phi_t) d\mu_t + d\langle \mu, m \rangle_t^{\mathbb{F}} + (\Delta\mu_t)^2 - d\langle \mu, \mu \rangle_t^{\mathbb{F}} \right) \\ &= \frac{1}{Z_{t-}} \left((Z_{t-} + \phi_t) \phi_t^h dM_t - \phi_t^h d\langle M, m \rangle_t^{\mathbb{F}} - d[\mu, \mu^h]_t + d\langle \mu^h, \mu \rangle_t^{\mathbb{F}} \right) \\ &\quad - \frac{X_{t-}}{Z_{t-}^2} \left((Z_{t-} + \phi_t) \phi_t dM_t - \phi_t d\langle M, m \rangle_t^{\mathbb{F}} - d[\mu, \mu]_t + \langle \mu, \mu \rangle_t^{\mathbb{F}} \right) \end{aligned}$$

since, in view of (3.17), we have $d(\Delta\mu_t\Delta\mu_t^h) = d[\mu, \mu^h]_t$ and $d(\Delta\mu_t)^2 = d[\mu, \mu]_t$. To obtain the asserted formula from equation (4.5), it remains to show that

$$d\widehat{K}_t = \left(\phi_t^h - \phi_t \frac{X_{t-}^h}{Z_{t-}} \right) d\widehat{M}_t$$

where \widehat{M} is given by (4.2). Using again (3.17), we get

$$d[\mu, \mu^h]_t - d\langle \mu^h, \mu \rangle_t^{\mathbb{F}} = \phi_t^h \phi_t dM_t, \quad d[\mu, \mu]_t - d\langle \mu, \mu \rangle_t^{\mathbb{F}} = \phi_t^2 dM_t,$$

so that finally

$$\begin{aligned} d\widehat{K}_t &= \frac{1}{Z_{t-}} \left((Z_{t-} + \phi_t) \phi_t^h dM_t - \phi_t^h d\langle M, m \rangle_t^{\mathbb{F}} - \phi_t^h \phi_t dM_t \right) \\ &\quad - \frac{X_{t-}^h}{Z_{t-}^2} \left((Z_{t-} + \phi_t) \phi_t d\mu_t - \phi_t d\langle M, m \rangle_t^{\mathbb{F}} - \phi_t^2 dM_t \right) \\ &= \left(\phi_t^h - \phi_t \frac{X_{t-}^h}{Z_{t-}} \right) d\widehat{M}_t, \end{aligned}$$

which is the desired equality. Hence the proof of the proposition is completed. \square

Remark 4.2. Let us set $H_t^* = \int_0^t \Delta N_s dH_s$ and $\sigma = \inf \{t \geq 0 : H_t^* = 1\}$ so that $H_t^* = \mathbf{1}_{\{\sigma \leq t\}}$ and the jump times of the process $H - H^*$ are disjoint from the sequence T_n . The \mathbb{G} -adapted, increasing process H^* stopped at τ admits a \mathbb{G} -predictable compensator and thus there exists an \mathbb{F} -predictable, increasing process Λ^* such that the process $M_t^* := H_t^* - \Lambda_{t \wedge \tau}^*$ is a \mathbb{G} -martingale stopped at τ . Since $\{\Delta\mu^h \neq 0\} \subset \{\Delta N > 0\}$ and $\{\Delta\mu \neq 0\} \subset \{\Delta N > 0\}$, it is clear that

$$\left(\Delta\mu_\tau^h - \Delta\mu_\tau \frac{X_{\tau-}^h}{Z_{\tau-}} \right) dH_t = \left(\Delta\mu_\tau^h - \Delta\mu_\tau \frac{X_{\tau-}^h}{Z_{\tau-}} \right) dH_t^* = \left(\phi_t^h - \phi_t \frac{X_{t-}^h}{Z_{t-}} \right) dH_t^*$$

and thus

$$d\widetilde{M}_t^h = dB_t - \frac{1 - H_{t-}}{Z_{t-}} dB_t^{h,p,\mathbb{F}} = \left(\phi_t^h - \phi_t \frac{X_{t-}^h}{Z_{t-}} \right) dM_t^*.$$

Unfortunately, an explicit formula for the \mathbb{G} -compensator Λ^* is not available, in general. However, this argument formally gives the PRP for the triplet $(M^{(\tau)}, \widehat{M}, M^*)$ of \mathbb{G} -martingales, meaning that any process $Y^h \in \mathcal{M}(\mathbb{G}, \tau)$ can be represented as follows

$$dY_t^h = \frac{Z_{t-}}{Z_{t-} - \Delta A_t^p} \left(h_t - \frac{X_{t-}^h}{Z_{t-}} \right) dM_t^{(\tau)} + \frac{1 - H_{t-}}{Z_{t-} + \phi_t} \left(\phi_t^h - \phi_t \frac{X_{t-}^h}{Z_{t-}} \right) (d\widehat{M}_t + dM_t^*).$$

It is worth stressing that none of the processes $M^{(\tau)}, \widehat{M}, M^*$ depends on h .

4.2 Variants of the Predictable Representation Formula

We present here some special cases of the integral representation for the process Y^h , which can be deduced from Theorem 4.1.

(i) Suppose first that τ is independent of the natural filtration \mathbb{F} of a Poisson process N . Then $\Delta N \Delta H = 0$ and thus also $\Delta\mu_\tau^h = \Delta\mu_\tau = 0$. Then the random variable ξ^h given by (4.4) satisfies $\xi^h = 0$, so that $B^h = 0$ for any process $h \in \mathcal{H}^0(\mathbb{F}, \tau)$. Since $\mu = 1$, we have $\phi = 0$ and thus for any process $Y^h \in \mathcal{M}(\mathbb{G}, \tau)$, we obtain

$$dY_t^h = \frac{Z_{t-}}{Z_{t-} - \Delta A_t^p} \left(h_t - \frac{X_{t-}^h}{Z_{t-}} \right) dM_t^{(\tau)} + \frac{1 - H_{t-}}{Z_{t-}} \phi_t^h d\widehat{M}_t. \quad (4.6)$$

Due to the postulated independence of τ and \mathbb{F} , the Azéma supermartingale Z is a decreasing deterministic function, so that it is \mathbb{F} -predictable. Recall that this set-up was also covered by

Corollary 3.2. As expected, representation (4.6) coincides with equation (3.16), since under the present assumptions the equality $\widehat{M} = M$ holds.

(ii) Let us now assume that a random time τ avoids all \mathbb{F} -stopping times, that is, $\mathbb{P}(\tau = \sigma) = 0$ for any \mathbb{F} -stopping time σ . Then $\Delta N \Delta H = 0$ and thus $B^h = 0$ for any process $h \in \mathcal{H}^o(\mathbb{F}, \tau)$. Moreover, the process A^p in the Doob-Meyer decomposition of Z is known to be continuous. Therefore, any process $Y^h \in \mathcal{M}(\mathbb{G}, \tau)$ satisfies

$$dY_t^h = \left(h_t - \frac{X_{t-}^h}{Z_{t-}} \right) dM_t^{(\tau)} + \frac{1 - H_{t-}}{Z_{t-} + \phi_t} \left(\phi_t^h - \phi_t \frac{X_{t-}^h}{Z_{t-}} \right) d\widehat{M}_t. \quad (4.7)$$

Hence the multiplicity of the filtration \mathbb{G} with respect to martingales stopped at τ equals two, where by the *multiplicity* of the filtration \mathbb{G} , we mean here the minimal number of mutually orthogonal martingales needed to represent all martingales stopped at τ as stochastic integrals. For the concept of multiplicity of a filtration, see Davis and Varaiya [7] and the survey paper by Davis [6] and the references therein.

(iii) Under the assumption that the graph of the random time τ is included in $\cup_n \llbracket T_n \rrbracket$, that is, when $\{\Delta H > 0\} \subset \{\Delta N > 0\}$, we obtain

$$dB_t^h = \left(\Delta \mu_\tau^h - \Delta \mu_\tau \frac{X_{\tau-}^h}{Z_{\tau-}} \right) dH_t = \left(\phi_t^h - \phi_t \frac{X_{t-}^h}{Z_{t-}} \right) dH_t$$

and thus

$$d\widehat{M}_t^h = dB_t^h - \frac{1 - H_{t-}}{Z_{t-}} dB_t^{h,p,\mathbb{F}} = \left(\phi_t^h - \phi_t \frac{X_{t-}^h}{Z_{t-}} \right) dM_t^{(\tau)}.$$

Consequently, any process $Y^h \in \mathcal{M}(\mathbb{G}, \tau)$ admits the following representation

$$dY_t^h = \frac{Z_{t-}}{Z_{t-} - \Delta A_t^p} \left(h_t - \frac{X_{t-}^h}{Z_{t-}} \right) dM_t^{(\tau)} + \frac{1 - H_{t-}}{Z_{t-} + \phi_t} \left(\phi_t^h - \phi_t \frac{X_{t-}^h}{Z_{t-}} \right) d\widehat{M}_t^{(\tau)} \quad (4.8)$$

where the \mathbb{G} -martingales $M^{(\tau)}$ and $\widehat{M}^{(\tau)} := \widehat{M} - M^{(\tau)}$ are orthogonal. We thus see that the multiplicity of the filtration \mathbb{G} with respect to martingales stopped at τ is equal to two.

Example 4.1. To illustrate part (iii), we will re-examine the set-up introduced in Section 3.2.2. Recall that it was assumed in Section 3.2.2 that $h_t = h(N_{t-})$, $Z_t = e^{-N_t}$ and the immersion property holds, so that the equality $\widehat{M} = M$ holds. From (3.22) and the equality (see (3.27))

$$\mu_t^h = X_t^h + \int_0^t h_s dA_s^p = \tilde{h}(N_t)Z_t - \int_0^t \gamma \lambda h(N_s)Z_s ds$$

we deduce that the processes ϕ and ϕ^h appearing in (3.17) are given by

$$\phi_t = -\gamma e^{-N_{t-}} = -\gamma Z_{t-}, \quad \phi_t^h = ((1 - \gamma)\tilde{h}(N_{t-} + 1) - \tilde{h}(N_{t-}))Z_{t-}$$

where \tilde{h} is given by (3.28). By substituting these processes into (4.8), we obtain (recall that, from Lemma 3.2, we have $\Delta A^p = 0$)

$$\begin{aligned} dY_t^h &= \left(h_t - \frac{X_{t-}^h}{Z_{t-}} \right) dM_t^{(\tau)} \\ &\quad + \frac{(1 - H_{t-})}{(1 - \gamma)Z_{t-}} \left(((1 - \gamma)\tilde{h}(N_{t-} + 1) - \tilde{h}(N_{t-}))Z_{t-} + \gamma \tilde{h}(N_{t-})Z_{t-} \right) d(M_t - M_t^{(\tau)}) \\ &= (h_t - \tilde{h}(N_{t-})) dM_t^{(\tau)} + (1 - H_{t-})(\tilde{h}(N_{t-} + 1) - \tilde{h}(N_{t-})) d\bar{M}_t^{(\tau)} \end{aligned}$$

where $\bar{M}^{(\tau)} := M - M^{(\tau)} = \widehat{M} - M^{(\tau)} =: \widehat{M}^{(\tau)}$. This result coincides with representation (3.25), which was previously established in Proposition 3.4 by means of more direct computations.

4.3 Optional Representation Formula for the Poisson Filtration

In this final subsection, we make use of the *optional* stochastic integral, in order to derive an integral representation for any process Y^h in terms of the \mathbb{G} -martingales $M^{(\tau)}$ and \widehat{M} (see Corollary 4.1). In essence, the idea of an optional stochastic integral is to extend the notion of the Itô stochastic integral from predictable to optional integrands by ensuring that the integral of an optional integrand with respect to a local martingale is uniquely defined and follows a local martingale. It is worth noting that the optional stochastic integral was used in the recent paper of Aksamit et al. [2] in the context of arbitrage properties of a financial model endowed with a progressively enlarged filtration.

Several alternative approaches to the *optional* stochastic integral were proposed in the literature (see, e.g., [23, 26, 27]). We follow here the exposition presented in Chapter III of Jacod [11] where the optional stochastic integral is introduced as a special case of a stochastic integral with respect to a random measure. Let \mathbb{G} be an arbitrary filtration satisfying the usual conditions and let X be an arbitrary \mathbb{G} -local martingale null at time 0. The integer-valued random measure μ^X on $\Omega \times \mathbb{R}_+ \times \mathbb{R}$ associated with the jumps of X is given by the following expression (see Example 3.22 in [11])

$$\mu^X(dt, dx) := \sum_{s>0} \mathbf{1}_{\{\Delta X_s \neq 0\}} \delta_{(s, \Delta X_s)}(dt, dx). \quad (4.9)$$

We denote by ν^X the dual \mathbb{G} -predictable projection of the random measure μ^X ; the existence of ν^X is established in Theorem 3.15 in [11]).

Let $W : \Omega \times \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ and $V : \Omega \times \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ be some mappings. For any mapping $W \in G_{loc}^1(\mu^X)$ (resp., $V \in H_{loc}^1(\mu^X)$), we denote by $W \star (\mu^X - \nu^X)$ (resp., $V \star \mu^X$) the stochastic integral of the first (resp., second) kind with respect to a random measure $\mu^X - \nu^X$ (resp., μ^X), which is given by Definition 3.63 (resp., Definition 3.73) in [11]. For the definitions of the spaces $G_{loc}^1(\mu^X)$ and $H_{loc}^1(\mu^X)$ of integrands, the reader is referred to pages 98 and 101 in [11], respectively. We will need the property that, by definition, the integrals $W \star (\mu^X - \nu^X)$ and $V \star \mu^X$ belong to the space \mathcal{M}_{loc}^d of purely discontinuous \mathbb{G} -local martingales. To be a bit more specific, for any process $V \in H_{loc}^1(\mu^X)$, the integral $V \star \mu^X$ is defined as a unique process from \mathcal{M}_{loc}^d such that

$$\Delta(V \star \mu^X)_t = V(t, \Delta X_t) \mathbf{1}_D(t), \quad \forall t \in \mathbb{R}_+, \quad (4.10)$$

where $D := \{\Delta X \neq 0\}$.

For the definition and properties of the optional stochastic integral with respect to a local martingale X , which is denoted hereafter as $K \odot X$ for any process K belonging to the space ${}^oL_{loc}^1(X)$ of optional integrands, the reader is referred to Jacod [11] (see pages 106–108 therein). Let us only mention here that, for any process $K \in {}^oL_{loc}^1(X)$, the optional stochastic integral $K \odot X$ is the unique local martingale such that

$$(K \odot X)^c = ({}^{p,\mathbb{F}}K) \cdot X^c, \quad \Delta(K \odot X) = K \Delta M - {}^{p,\mathbb{F}}(K \Delta M), \quad (4.11)$$

where, as usual, X^c stands for the continuous martingale part of X . Note that when K is an \mathbb{G} -predictable process, then the conditions above reduce to the following conditions

$$(K \odot X)^c = K \cdot X^c, \quad \Delta(K \odot X) = K \Delta M,$$

which are known to uniquely characterize the classic concept of *predictable* stochastic integral (see Definition 2.46 in [11]). The next result, which is merely a restatement of Theorem 3.84 from Jacod [11], furnishes a link between the optional stochastic integral with respect to a \mathbb{G} -local martingale X and stochastic integrals with respect to the associated random measures $\mu^X - \nu^X$ and μ^X .

Theorem 4.2. (i) *Let X be a \mathbb{G} -local martingale null at time 0 and let μ^X be the associated random measure given by (4.9). Then the set of optional stochastic integrals*

$$\{K \odot X : K \in {}^oL_{loc}^1(X)\}$$

coincides with the following set

$$\{U \cdot X^c + W \star (\mu^X - \nu^X) + V \star \mu^X : U \in L_{loc}^1(X^c), W \in G_{loc}^1(\mu^X), V \in H_{loc}^1(\mu^X)\}.$$

(ii) Let $U \in L_{loc}^1(X^c)$, $W \in G_{loc}^1(\mu^X)$, $V \in H_{loc}^1(\mu^X)$ and let the process K be given by the following expression

$$K_t = U_t \mathbb{1}_{D^c \cap J^c}(t) + \frac{1}{\Delta X_t} (W(t, \Delta X_t) + V(t, \Delta X_t)) \mathbb{1}_D(t)$$

where $J := \{\nu^X(t, \mathbb{R}) > 0\}$. Then K belongs to the space ${}^oL_{loc}^1(X)$ of optional integrands and thus the optional stochastic integral $K \odot X$ is well defined. Moreover, the following equality holds

$$K \odot X = U \cdot X^c + W \star (\mu^X - \nu^X) + V \star \mu^X.$$

In our application of Theorem 4.2, we consider the progressive enlargement of the Poisson filtration and we focus on the \mathbb{G} -martingale \widetilde{M}^h given by equation (4.3) or, equivalently, by the following expressions

$$\widetilde{M}_t^h = \xi^h H_t - \int_0^t \frac{1 - H_{s-}}{Z_{s-}} \kappa_s^h dA_s^p = (\xi^h - \kappa_\tau^h) H_t - \int_0^t \kappa_s^h dM_s^{(\tau)} \quad (4.12)$$

where the random variable ξ^h is given by (4.4) and κ^h is an \mathbb{F} -predictable process such that the equality $\kappa_\tau^h = \mathbb{E}(\xi^h | \mathcal{F}_{\tau-})$ holds. It is clear from (4.3) and (4.12) that the process $M^\xi := (\xi^h - \kappa_\tau^h) H$ is a purely discontinuous \mathbb{G} -martingale. Our goal is to derive the optional integral representation of M^ξ with respect to $M^{(\tau)}$.

Recall that we denote by $Z = \mu - A^p$ the Doob-Meyer decomposition of the Azéma supermartingale Z of τ . Let (S_n) be a sequence of \mathbb{F} -predictable stopping times exhausting the jumps of the \mathbb{F} -predictable, increasing process A^p . Since

$$\Delta M_t^{(\tau)} = \Delta H_t - (1 - H_{t-})(Z_{t-})^{-1} \Delta A_t^p,$$

it is clear that the random measure $\mu^\tau := \mu^{M^{(\tau)}}$ associated with the jumps of the \mathbb{G} -martingale $M^{(\tau)}$ equals

$$\mu^\tau(dt, dx) = \delta_{(\tau, 1)}(dt, dx) + \sum_n \mathbb{1}_{\{S_n \leq \tau\}} \delta_{(S_n, -(Z_{S_n-})^{-1} \Delta A_{S_n}^p)}(dt, dx).$$

Hence the dual \mathbb{G} -predictable projection ν^τ of μ^τ is given by the following expression

$$\nu^\tau(dt, dx) = \delta_1(dx) \frac{1 - H_{t-}}{Z_{t-}} dA_t^p + \sum_n \mathbb{1}_{\{S_n \leq \tau\}} \delta_{(S_n, -(Z_{S_n-})^{-1} \Delta A_{S_n}^p)}(dt, dx).$$

Let us first consider the special case where the process A^p is assumed to be continuous.

Lemma 4.2. *Assume that the process A^p is continuous, so that the \mathbb{G} -martingale $M^{(\tau)}$ is quasi-left-continuous. Then $M^\xi = L^h \odot M^{(\tau)}$ where $L^h = M^\xi$.*

Proof. To alleviate notation, we write $\xi = \xi^h$ and $\kappa = \kappa^h$ in the proof. The assertion follows immediately from Theorem 4.2 applied to $X = M^{(\tau)}$, as $(\xi - \kappa_\tau)H = V \star \mu^\tau$ with $V(s, x) = (\xi - \kappa_\tau)H_s$ since $\mu^\tau(dt, dx) = \delta_{(\tau, 1)}(dt, dx)$ when A^p is continuous. The property that $V \in H_{loc}^1(\mu^X)$ is a consequence of the equality $\kappa_\tau = \mathbb{E}(\xi | \mathcal{F}_{\tau-})$.

Alternatively, one can establish the lemma directly. To this end, we observe that the martingale $M^\xi \odot M^{(\tau)}$ has a null continuous martingale part and the jump process given by

$$(\xi - \kappa_\tau) \Delta H_t - {}^{p, \mathbb{G}}((\xi - \kappa_\tau) \Delta H_t) = (\xi - \kappa_\tau) \Delta H_t$$

since τ is a totally inaccessible \mathbb{G} -stopping time. The process M^ξ is a purely discontinuous \mathbb{G} -martingale with the same jump process and thus we conclude that $M^\xi = M^\xi \odot M^{(\tau)}$. \square

In the next result, the continuity assumption for A^p is relaxed.

Lemma 4.3. *We have $M^\xi = L^h \odot M^{(\tau)}$ where the process L^h equals, for all $t \in \mathbb{R}_+$,*

$$L_t^h = \frac{\Delta M_t^{(\tau)} Z_{\tau-} + \Delta A_\tau^p}{\Delta M_t^{(\tau)} (Z_{\tau-} + \Delta A_\tau^p)} (\xi^h - \kappa_\tau^h) H_t \mathbb{1}_{\{\Delta M_t^{(\tau)} \neq 0\}}. \quad (4.13)$$

Proof. As before, we denote $\xi = \xi^h$ and $\kappa = \kappa^h$. Let us set

$$V(s, x) := \frac{x Z_{\tau-} + \Delta A_\tau^p}{Z_{\tau-} + \Delta A_\tau^p} (\xi - \kappa_\tau) H_s.$$

Then $(\xi - \kappa_\tau)H = V \star \mu^\tau$, since

$$V \star \mu_t^\tau = V(\tau, 1)H_t + \sum_n V\left(S_n, -\frac{\Delta A_{S_n}^p}{Z_{S_n-}}\right) \mathbb{1}_{\{S_n \leq \tau\}} \mathbb{1}_{\{S_n \leq t\}} = (\xi - \kappa_\tau)H_t.$$

The asserted equality (4.13) can now be deduced from Theorem 4.2. Once again, one can check that $V \in H_{loc}^1(\mu^X)$ since $\kappa_\tau = \mathbb{E}(\xi | \mathcal{F}_{\tau-})$. \square

The final result of this note shows that the optional and predictable stochastic integrals can be coupled in order to obtain an integral representation of any \mathbb{G} -martingale stopped at τ in terms of two universal martingales: the \mathbb{G} -martingale $M^{(\tau)}$ associated with τ and the \mathbb{G} -martingale part \widehat{M} of the compensated \mathbb{F} -martingale M of the Poisson process N . To establish Corollary 4.1, it suffices to combine Theorem 4.1 with Lemma 4.3.

Corollary 4.1. *For any process $Y^h \in \mathcal{M}(\mathbb{G}, \tau)$, the following integral representation is valid*

$$Y^h = K^h \odot M^{(\tau)} + \frac{1 - H_-}{Z_- + \phi} \left(\phi^h - \phi \frac{X_-}{Z_-} \right) \cdot \widehat{M}$$

where the process K^h equals

$$K_t^h = \frac{Z_{t-}}{Z_{t-} - \Delta A_t^p} \left(h_t - \frac{X_{t-}}{Z_{t-}} \right) + L_t^h$$

where the process L^h is given by (4.13).

4.4 Optional Decomposition Property

Representations (2.3) and (2.4) can be termed *predictable* decompositions. It appears that a particular *optional* decomposition is also available (see (4.14)). Recall that $\widetilde{Z}_t := \mathbb{P}(\tau \geq t | \mathcal{F}_t)$. Let the \mathbb{F} -stopping time \widetilde{R} be given by²

$$\widetilde{R} := R_{\{\widetilde{Z}_{\widetilde{R}} = 0 < Z_{\widetilde{R}-}\}} = \inf \{t > 0 : \widetilde{Z}_t = 0, Z_{t-} > 0\}.$$

For any process X , we denote

$$\widetilde{J} := (\widetilde{H} \Delta X_{\widetilde{R}})^{p, \mathbb{F}}$$

where $\widetilde{H} := \mathbb{1}_{[\widetilde{R}, \infty[}$. The following optional decomposition result was established by Aksamit [1] (see Theorem 7.1 therein).

Proposition 4.1. *For any \mathbb{F} -local martingale X , the process \widetilde{X} given by*

$$\widetilde{X}_t := X_{t \wedge \tau} - \int_0^{t \wedge \tau} \frac{1}{\widetilde{Z}_s} d[X, m]_s + \widetilde{J}_{t \wedge \tau} \quad (4.14)$$

is a \mathbb{G} -local martingale (stopped at time τ).

²For any event C and any random time σ , we set $\sigma_C := \sigma \mathbb{1}_C + \infty \mathbb{1}_{\{\Omega \setminus C\}}$.

Remark 4.3. After completing this paper, we learnt about the innovative work by Choulli et al. [5] where the authors introduced the following \mathbb{G} -martingale

$$\widetilde{M}_t^{(\tau)} := H_t - \int_0^{t \wedge \tau} \frac{dA_s^o}{\widetilde{Z}_s}$$

and established several decompositions of a process $Y^h \in \mathcal{M}(\mathbb{G}, \tau)$ based, in particular, on representations of some \mathbb{G} -martingales as Lebesgue-Stieltjes integrals with respect to $\widetilde{M}^{(\tau)}$ and with optional integrands.

Acknowledgements. The research of Anna Aksamit was supported by the Chaire Marchés en Mutation (Fédération Bancaire Française) and by the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013) / ERC grant agreement no. 335421. The research of Monique Jeanblanc was supported by the Chaire Marchés en Mutation (Fédération Bancaire Française). The research of Marek Rutkowski was supported by the Chaire Marchés en Mutation (Fédération Bancaire Française) and the Australian Research Council's Discovery Projects funding scheme (DP120100895).

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