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BORROWING IN EXCESS OF NATURAL ABILITY TO REPAY¹

V. Filipe Martins-da-Rocha^a and Yiannis Vailakis^b

This paper aims at improving our understanding of self-enforcing debt in competitive dynamic economies without commitment when default induces a permanent loss of access to international credit markets. We show, by means of two examples, that sovereigns can sustain self-enforcing debt levels in excess of their natural ability to repay represented by the present value of future endowments. This is in sharp contrast with the standard results in the full commitment literature and shows that the future resources for repayment and the market value of time (i.e., the interest rates) are not the only relevant aspects of a sovereign’s borrowing capacity. Indeed, we reveal a new channel through which self-enforcing debt is sustained at equilibrium: creditworthiness in international credit markets may reflect the intermediation services of the debtors to alleviate the financial frictions of potential creditors.

KEYWORDS: Limited Commitment, Self-enforcing Debt, Natural Debt Limit.

JEL CLASSIFICATIONS: D50, D51, D53, F43, G13, H63.

1. INTRODUCTION

In competitive dynamic economies with infinite horizon and sequential trading of contingent claims, debt limits must be imposed to prevent Ponzi games. However, these debt limits should be sufficiently loose to permit the maximum expansion of risk-sharing without introducing unjustified financial frictions. When there is full commitment, the only requirement imposed on debt limits is that they should not bind at equilibrium. This implies that every agent’s wealth—defined as the present value of future endowment—is finite and equilibrium debt, contingent to any event, is bounded from above by the agent’s contingent wealth. This upper bound on debt, known as the “natural debt limit” (see for instance Ljungqvist and Sargent (2004), Acemoglu (2009) and Miao (2014)), corresponds to what an agent can repay by never consuming again and using all income for repayment. It reflects two relevant aspects of the borrowing capacity: the future resources for repayment and the market value of time (i.e., the interest rates).

When there is lack of commitment, borrowing constraints should also be consistent with repayment incentives. Formally, the constraints should be self-enforcing in the sense that the debt limits should be tight enough to prevent default at equilibrium. Intuition suggests that without commitment, debt should be lower than in the full commitment

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environment. In particular, one expects that a potential borrower should not be able to issue debt in excess of its natural ability to repay (defined by the natural debt limit). The objective of this paper is to show that this intuition is not correct.

Following Bulow and Rogoff (1989) and Hellwig and Lorenzoni (2009), we consider a general equilibrium model with lack of commitment in which default induces the loss of access to international borrowing.¹ Our contribution consists on showing that equilibrium self-enforcing debt levels can exceed a borrower's ability to repay out of his future resources.

The starting point of our analysis is a simple Markovian economy with two states of nature and three agents or countries (two of them are referred as *rich* and the third one as *poor*) sharing risks by trading one-period contingent bonds. In each period, one rich country receives a high endowment and the other one receives a low endowment. The endowments of the rich countries switch with some positive probability from one period to the next. At the initial event, the poor country has the high endowment level of the rich countries. If the state does not switch, its endowment remains the same. However, when the state switches, the poor country loses a fraction of its endowment. In particular, along paths with infinitely many switches, the endowment of the poor country vanishes.²

We show that there are primitives (preferences, transition probabilities and endowments) for which the economy admits a stationary Markovian competitive equilibrium where the poor country sustains positive levels of debt despite the fact that its natural ability to repay is finite. Moreover, along the path of successive switches, the poor country eventually borrows in excess of its natural debt limits.

An immediate question then arises: why potential creditors will ever accept to lend in excess of the debtor's natural ability to repay? We show that this is possible because the debtor provides an additional service that reduces the financial frictions, due to the lack of commitment, imposed on the other countries. Indeed, at equilibrium, the rich countries are not creditworthy (i.e., their debt limits are equal to zero) but they have strong incentives to trade with each other. The poor country acts as a pass-through intermediary, borrowing from one country and repaying the other one. In this way, debt is rolled over indefinitely and eventually exceeds the country's natural debt limit, with the difference reflecting the market value of the financial intermediation service the country provides.

Our analysis stands in contrast with the classic Impossibility Theorem of Bulow and

¹There is a growing literature in which a number of authors have studied models with limited enforceability of risk-sharing contracts where debt repudiation induces exclusion from borrowing but not from saving. It includes, among others, Cole and Kehoe (1995), Cole and Kehoe (1998), Kletzer and Wright (2000), Dutta and Kapur (2002), Kehoe and Perri (2002), Gul and Pesendorfer (2004), Krueger and Uhlig (2006), Amador (2012). We refer to Wright (2011) and Aguiar and Amador (2014) for a thorough discussion of this literature.

²This justifies why we refer to this country as *poor*.

Rogoff (1989). They showed that the threat of enjoining from ever borrowing in the future in no way induces incentives for repayment. The crucial difference is that Bulow and Rogoff (1989) impose the “ad hoc” assumption that debt limits must be tighter than natural debt limits. This immediately rules out the possibility of an agent to roll over its debt and act as financial intermediary by alleviating the financial frictions of other agents.

Our work complements Hellwig and Lorenzoni (2009)’s analysis of repayment incentives. They showed that debt limits are self-enforcing if, and only if, borrowers can exactly roll over these debt limits period by period. However, in the two-agents example they analyze, there is no issue on whether or not the equilibrium debt limits exceed the natural debt limits. This is because both agents have infinite wealth at equilibrium. We could infer from this that a sovereign can sustain positive levels of debt only if its wealth is infinite, or equivalently, only if interest rates are lower than its endowments growth rates.³ We instead show that a country’s “good reputation” for repayment is endogenously determined at equilibrium and is not necessarily dependent on whether interest rates are lower than its endowments’ growth rates. Indeed, in our example, the poor country sustains positive levels of debt (eventually larger than its natural ability to borrow) despite the fact that its wealth is finite at any contingency. However, it is worth noticing that the level of equilibrium interest rates does play a preeminent role. We clarify that interest rates matter to the extent they induce lenders to provide credit at infinite.⁴

An additional observation is that, for the same bond prices, there is a continuum of competitive Markovian equilibria with self-enforcing debt where the gains from trade between the rich countries are only partially intermediated by the poor country. This shows that there is indeterminacy of creditworthiness or good reputation which has real effects.

In the stochastic example we analyze, the equilibrium risk-less interest rate is equal to zero. To illustrate that our results do not depend on this property, we provide a second example where a country sustains debt levels in excess of its natural ability to repay but the risk-less interest rate is constant and strictly positive. This example makes the comparison with the Impossibility Theorem of Bulow and Rogoff (1989) more transparent since it illustrates that rolling over debt is compatible with positive interest rates.

The paper is organized as follows: Section 2 describes a stochastic dynamic competitive economy with lack of commitment where default amounts to exclusion from credit markets forever. Section 3 presents the two examples of economies exhibiting a competitive equilibrium in which one of the countries is able to issue debt in excess of his natural debt limits. It also contains some key observations related to the role of limited

³This is because if a non-negative and non-zero process satisfies exact roll-over, then it cannot be tighter than any process with finite present value.

⁴In our example, the rich countries (which are the lenders) have infinite wealth.

commitment and the level of interest rates for debt sustainability and the possibility of indeterminacy which has real effects. A technical result related to the necessity of a “market transversality condition” is presented in the appendix.

2. FUNDAMENTALS AND MARKETS

We present an infinite horizon general equilibrium model with lack of commitment and self-enforcing debt limits along the lines of Bulow and Rogoff (1989) and Hellwig and Lorenzoni (2009). Time and uncertainty are both discrete and there is a single non-storable consumption good. The economy consists of a finite set I of infinitely lived agents (countries) sharing risks in an environment where debtors cannot commit to their promises.

2.1. Uncertainty

We use an event tree Σ to describe time, uncertainty and the revelation of information over an infinite horizon. There is a unique initial date-0 event $s^0 \in \Sigma$ and for each date $t \in \{0, 1, 2, \dots\}$ there is a finite set $S^t \subset \Sigma$ of date- t events s^t . Each s^t has a unique predecessor $\sigma(s^t)$ in S^{t-1} and a finite number of successors s^{t+1} in S^{t+1} for which $\sigma(s^{t+1}) = s^t$. We use the notation $s^{t+1} \succ s^t$ to specify that s^{t+1} is a successor of s^t . Event $s^{t+\tau}$ is said to follow event s^t , also denoted $s^{t+\tau} \succ s^t$, if $\sigma^{(\tau)}(s^{t+\tau}) = s^t$. The set $S^{t+\tau}(s^t) := \{s^{t+\tau} \in S^{t+\tau} : s^{t+\tau} \succ s^t\}$ denotes the collection of all date- $(t+\tau)$ events following s^t . Abusing notation, we let $S^t(s^t) := \{s^t\}$. The subtree of all events starting from s^t is then

$$\Sigma(s^t) := \bigcup_{\tau \geq 0} S^{t+\tau}(s^t).$$

We use the notation $s^\tau \succeq s^t$ when $s^\tau \succ s^t$ or $s^\tau = s^t$. In particular, we have $\Sigma(s^t) = \{s^\tau \in \Sigma : s^\tau \succeq s^t\}$.

2.2. Endowments and Preferences

Agents’ endowments are subject to random shocks. We denote by $y^i = (y^i(s^t))_{s^t \in \Sigma}$ the agent i ’s process of positive endowments $y^i(s^t) > 0$ of the consumption good contingent to event s^t . Preferences over (non-negative) consumption processes $c = (c(s^t))_{s^t \in \Sigma}$ are represented by the lifetime discounted utility functional

$$U(c) := \sum_{s^t \in \Sigma} \beta^t \pi(s^t) u(c(s^t)),$$

where $\beta \in (0, 1)$ is the discount factor, $\pi(s^t)$ is the unconditional probability of s^t and $u : \mathbb{R}_+ \rightarrow [-\infty, \infty)$ is a Bernoulli function assumed to be strictly increasing, concave,

continuous on \mathbb{R}_+ , differentiable on $(0, \infty)$, bounded from above and satisfying Inada's condition at the origin.

Given an event s^t , we denote by $U(c|s^t)$ the lifetime continuation utility conditional on s^t , defined by

$$U(c|s^t) := \sum_{s^{t+\tau} \in \Sigma(s^t)} \beta^\tau \pi(s^{t+\tau}|s^t) u(c(s^{t+\tau}))$$

where $\pi(s^{t+\tau}|s^t) := \pi(s^{t+\tau})/\pi(s^t)$ is the conditional probability of $s^{t+\tau}$ given s^t .

A collection $(c^i)_{i \in I}$ of consumption processes is said to be resource feasible if $\sum_{i \in I} c^i = \sum_{i \in I} y^i$.

2.3. Markets

At every event s^t , agents can issue and trade a complete set of one-period contingent bonds, which promise to pay one unit of the consumption good contingent on the realization of any successor event $s^{t+1} \succ s^t$. Let $q(s^{t+1}) > 0$ denote the price, in units of consumption, at event s^t of the s^{t+1} -contingent bond. Agent i 's holding of this bond is $a^i(s^{t+1})$. The amount of state-contingent debt agent i can issue is observable and subject to state-contingent (non-negative) upper bounds (or debt limits) $D^i = (D^i(s^t))_{s^t \succ s^0}$. Given an initial financial claim $a^i(s^0)$, we denote by $B^i(D^i, a^i(s^0)|s^0)$ the budget set of all pairs (c^i, a^i) of consumption and bond holdings satisfying the following constraints: for every event $s^t \succeq s^0$,

$$(2.1) \quad c^i(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1}) a^i(s^{t+1}) \leq y^i(s^t) + a^i(s^t)$$

and

$$(2.2) \quad a^i(s^{t+1}) \geq -D^i(s^{t+1}).$$

Given some initial claim $b \in \mathbb{R}$ at an event s^t , we denote by $J^i(D^i, b|s^t)$ the largest continuation utility defined by

$$J^i(D^i, b|s^t) := \sup\{U(c^i|s^t) : (c^i, a^i) \in B^i(D^i, b|s^t)\},$$

where $B^i(D^i, b|s^t)$ is the set of all plans (c^i, a^i) satisfying $a^i(s^t) = b$, together with Equations (2.1) and (2.2) for every successor event $s^\tau \succeq s^t$.

Recall that Euler equations and the transversality condition are sufficient conditions for the optimality of agents' choices. Formally, consider a budget feasible plan (c^i, a^i) satisfying the flow constraints (2.1) with equality. If c^i is strictly positive, satisfies the Euler equations at every event $s^t \succ s^0$

$$(2.3) \quad (a^i(s^t) + D^i(s^t)) \left[q(s^t) - \beta \pi(s^t|s^{t-1}) \frac{u'(c^i(s^t))}{u'(c^i(s^{t-1}))} \right] = 0,$$

and the transversality condition

$$(2.4) \quad \liminf_{t \rightarrow \infty} \sum_{s^t \in S^t} \beta^t \pi(s^t) u'(c^i(s^t)) [a^i(s^t) + D^i(s^t)] = 0,$$

then (c^i, a^i) is optimal in the budget set $B^i(D^i, a^i(s^0)|s^0)$.

2.4. Default Punishment

We consider an environment where there is no commitment. Agents might not honor their debt obligations and decide to default. Such a decision depends on the consequences of default. Following Bulow and Rogoff (1989) (see also Hellwig and Lorenzoni (2009)), we assume that a defaulting agent starts with neither assets nor liabilities, is excluded from future credit but retains the ability to save (i.e., purchase bonds). Therefore, agent i 's default option at event s^t is

$$V^i(s^t) := J^i(0, 0|s^t).$$

Lenders have no incentives to provide credit contingent to some event if they anticipate that the borrower will default.⁵ The maximum amount of debt $D^i(s^t)$ at any event $s^t \succ s^0$ should reflect this property. If agent i 's initial financial claim at event s^t corresponds to the maximum debt $-D^i(s^t)$, then the agent prefers to repay its debt if, and only if, $J^i(D^i, -D^i(s^t)|s^t) \geq V^i(s^t)$. When a process of bounds satisfies the above inequality at every event $s^t \succ s^0$, it is called self-enforcing.⁶ Competition among lenders naturally leads to consider the largest self-enforcing bound $D^i(s^t)$ defined by the equation

$$(2.5) \quad J^i(D^i, -D^i(s^t)|s^t) = V^i(s^t).$$

Since the seminal contribution of Alvarez and Jermann (2000), the literature refers to such debt limits as “not-too-tight”.

2.5. Competitive Equilibrium

Fix an allocation $(a^i(s^0))_{i \in I}$ of initial financial claims that satisfies market clearing, i.e., $\sum_{i \in I} a^i(s^0) = 0$.⁷ A competitive equilibrium $(q, (c^i, a^i, D^i)_{i \in I})$ consists of state-contingent bond prices q , a resource feasible consumption allocation $(c^i)_{i \in I}$, a market clearing allocation of bond holdings $(a^i)_{i \in I}$ and an allocation of debt limits $(D^i)_{i \in I}$ such that, for each i , the plan (c^i, a^i) is optimal among budget feasible plans in $B^i(D^i, a^i(s^0)|s^0)$. A competitive equilibrium with self-enforcing debt is a competitive equilibrium for which debt limits are not-too-tight.

⁵Since the default punishment is independent of the default level, an agent either fully repays his debt or defaults totally. There is no partial default.

⁶Indeed, since the function $J^i(D^i, \cdot |s^t)$ is increasing, for any bond holding $a^i(s^t)$ satisfying the restriction $a^i(s^t) \geq -D^i(s^t)$, agent i prefers honoring his obligation than defaulting on $a^i(s^t)$.

⁷Bonds are in zero net supply.

2.6. Natural Ability to Borrow

Consider for a moment the benchmark environment with full commitment. In order to prevent Ponzi schemes, we need to impose debt limits on bond holdings, however these limits need not be self-enforcing. To ensure that the debt constraints do not introduce an additional imperfection into the model, the debt limits should be sufficiently large to permit all justified transfers of income. In other words, debt limits should never bind at equilibrium. When this is the case, the wealth of each agent—defined as the present value of future endowments—is finite at equilibrium. To state this result formally, we need to introduce some notation.

Given state-contingent bond prices $q = (q(s^t))_{s^t \succ s^0}$, we denote by $p(s^t)$ the associated date-0 price of consumption at s^t defined recursively by $p(s^0) = 1$ and $p(s^{t+1}) = q(s^{t+1})p(s^t)$ for every $s^{t+1} \succ s^t$. We use $\text{PV}(x|s^t)$ to denote the present value (at event s^t) of a process x restricted to the subtree $\Sigma(s^t)$ and defined by

$$\text{PV}(x|s^t) := \frac{1}{p(s^t)} \sum_{s^{t+\tau} \in \Sigma(s^t)} p(s^{t+\tau})x(s^{t+\tau}).$$

Agent i 's wealth $W^i(s^t)$ at event s^t is then defined as the present value of future income, i.e.,

$$W^i(s^t) := \text{PV}(y^i|s^t).$$

Observe that $W^i(s^t)$ could be infinite.⁸ However, if $(q, (c^i, a^i, D^i)_{i \in I})$ is a competitive equilibrium in which debt limits never bind, i.e.,

$$\forall i \in I, \quad \forall s^t \succ s^0, \quad a^i(s^t) > -D^i(s^t),$$

then $W^i(s^t)$ is finite and $a^i(s^t) > -W^i(s^t)$ for every $i \in I$ and any event s^t . It turns out that the wealth process $(W^i(s^t))_{s^t \succ s^0}$ is the natural candidate for the (non-binding) debt limits in an environment with full commitment. Such debt limits are known as the “natural debt limits” or the “natural ability to repay” (see for instance Ljungqvist and Sargent (2004), Acemoglu (2009) and Miao (2014)).

REMARK 2.1 Assume that agent i could trade at event s^t a complete set of bonds with all possible maturities. No arbitrage would imply that the price at event s^t of the bond with maturity at event $s^{t+\tau}$ is $p(s^{t+\tau})/p(s^t)$. In a such environment, country i could sell at event s^t the whole process of future endowments $(y(s^{t+\tau}))_{s^{t+\tau} \succeq s^t}$. The proceeds would then be

$$\frac{1}{p(s^t)} \sum_{s^{t+\tau} \succeq s^t} p(s^{t+\tau})y^i(s^{t+\tau}) = y^i(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1})W^i(s^{t+1}).$$

⁸However, if $W^i(s^0)$ is finite, then $W^i(s^t)$ is also finite at every $s^t \succ s^0$.

The term $q(s^{t+1})W^i(s^{t+1})$ is the “natural borrowing limit” interpreted as the maximum amount country i can borrow at event s^t by selling his future income conditional to the successor event s^{t+1} . This is different from the wealth level $W^i(s^{t+1})$ corresponding to the maximum amount of debt country i can issue contingent to event s^{t+1} .

2.7. Rolling Over Debt at Infinity

Bulow and Rogoff (1989) proved that debt limits cannot be simultaneously self-enforcing and tighter than natural debt limits. Formally, we have the following result.⁹

THEOREM 2.1 (Bulow and Rogoff) *Assume that agent i 's wealth is finite. If a debt limit process D^i is self-enforcing and tighter than natural debt limits, i.e.,*

$$\forall s^t \succ s^0, \quad J^i(D^i, -D^i(s^t)|s^t) \geq V^i(s^t) \quad \text{and} \quad D^i(s^t) \leq W^i(s^t)$$

then $D^i(s^t) = 0$ at every event $s^t \succ s^0$.

In other words, if an agent's wealth is finite and his debt capacity is bounded from above by his natural ability to repay, then the threat of credit exclusion is not sufficient to induce repayment incentives. Hellwig and Lorenzoni (2009) went further and characterized repayment incentives without assuming a priori that agents' wealth is finite. They proved the following connection between debt sustainability and rational bubbles on debt limits.

THEOREM 2.2 (Hellwig and Lorenzoni) *A debt limit process D^i is not-too-tight if, and only if, it allows for exact roll-over, in the sense that*

$$(ER) \quad \forall s^t \succ s^0, \quad D^i(s^t) = \sum_{s^{t+1} \succ s^t} q(s^{t+1})D^i(s^{t+1}).$$

This result implies that an agent can credibly promise to repay a positive amount of debt if, and only if, this debt can be rolled over at infinite. An important question is whether such roll-over property is compatible with the market clearing conditions at equilibrium. Indeed, if an agent is rolling over his debt at infinite, there must be

⁹The model in Bulow and Rogoff (1989) is slightly different than the one presented here. They analyzed repayment incentives of a small open economy borrowing from competitive, risk neutral foreign investors. The sovereign country trades at the initial event a complete set of state-contingent contracts that specify the net transfers to foreign investors in all future periods and events. Contracts are restricted to be compatible with repayment incentives and to allow investors to break even in present value terms (the environment is in the spirit of Kehoe and Levine (1993), but with a different default option). We show in Martins-da-Rocha and Vailakis (2014) that the proof of Theorem 2.1 consists of a straightforward generalization of the arguments in Bulow and Rogoff (1989).

other agents lending at infinite. In the benchmark environment with full commitment and non-binding debt limits, lending at infinite is not consistent with lenders' necessary transversality conditions. In contrast, Hellwig and Lorenzoni (2009) showed, by means of an example, that without commitment, the not-too-tight debt limits typically bind infinitely often and agents may have infinite wealth at equilibrium. This is in contrast with the Impossibility Theorem of Bulow and Rogoff (1989). Indeed, when the natural debt limits are infinite, the condition $D^i(s^t) \leq W^i(s^t)$ imposes no restriction on debt levels, and positive debt limits may be sustained at equilibrium.

3. BORROWING IN EXCESS OF NATURAL DEBT LIMITS

We present below two economies with three agents (countries) having a competitive equilibrium where one country has the same repayment incentives as in Bulow and Rogoff (1989)—interest rates are sufficiently high to imply finite wealth levels at every contingency—but succeeds to sustain positive levels of debt. The Impossibility Theorem of Bulow and Rogoff (1989) does not apply since we exhibit equilibria where one country borrows more than its natural ability to repay, that is, the country faces not-too-tight debt limits that exceed its wealth levels. This illustrates that the country's ability to borrow is not necessarily bounded from above by the present value of its future income (the natural ability to repay). This is also in sharp contrast with the full commitment environment in which equilibrium debt levels are necessarily tighter than the natural debt limits. We show that the excess of debt to wealth levels reflects the market value of a financial intermediation service the country provides to potential creditors.

3.1. Stochastic Economy with Zero Riskless Interest Rate

Our first example is an economy with stochastic endowments for which we exhibit a stationary Markovian equilibrium with zero risk-less rates.

EXAMPLE 3.1 We modify the example in Hellwig and Lorenzoni (2009) with two *rich* countries (agents R_1 and R_2) by adding a third *poor* country (agent P). The primitives $(\beta, u(\cdot))$ together with some probability $\pi \in (0, 1)$ are chosen such that there exists a pair (\underline{c}, \bar{c}) satisfying

$$(3.1) \quad 0 < \underline{c} < \bar{c}, \quad \underline{c} + \bar{c} = 1 \quad \text{and} \quad 1 - \beta(1 - \pi) = \beta\pi \frac{u'(\underline{c})}{u'(\bar{c})}.$$

We let (q^c, q^{nc}) be defined by

$$(3.2) \quad q^c := \beta\pi \frac{u'(\underline{c})}{u'(\bar{c})} \quad \text{and} \quad q^{nc} := \beta(1 - \pi).$$

Observe that $q^c + q^{nc} = 1$. We fix some arbitrary number $\delta > 0$ such that $q^c\delta < \underline{c}$ and let (\underline{y}, \bar{y}) be the pair defined by

$$\underline{y} := \underline{c} - q^c\delta \quad \text{and} \quad \bar{y} := \bar{c} + q^c\delta.$$

Observe that

$$0 < \underline{y} < \underline{c} < \bar{c} < \bar{y} \quad \text{and} \quad \underline{y} + \bar{y} = 1.$$

In each period, one of the rich countries receives the high endowment \bar{y} and the other receives the low endowment \underline{y} . The rich countries switch endowment with probability π from one period to the next. Formally, uncertainty is captured by the Markov process s_t with state space $\{z_1, z_2\}$ and symmetric transition probabilities

$$\pi := \text{Prob}(s_{t+1} = z_1 | s_t = z_2) = \text{Prob}(s_{t+1} = z_2 | s_t = z_1).$$

The event s^t corresponds to the sequence (s_0, s_1, \dots, s_t) and the endowments $y^{\text{Rk}}(s^t)$ only depend on the current realization of s_t , with

$$y^{\text{Rk}}(s^t) := \begin{cases} \bar{y}, & \text{if } s_t = z_k \\ \underline{y}, & \text{otherwise.} \end{cases}$$

Country's P endowment is defined by $y^{\text{P}}(s^0) := \bar{y}$ and for each event $s^t \succ s^0$,

$$y^{\text{P}}(s^t) := \begin{cases} y^{\text{P}}(s^{t-1}), & \text{if } s_t = s_{t-1} \\ \gamma y^{\text{P}}(s^{t-1}), & \text{otherwise,} \end{cases}$$

where $\gamma \in (0, 1)$ is chosen such that

$$(3.3) \quad \frac{u'(\gamma y^{\text{P}}(s^t))}{u'(y^{\text{P}}(s^t))} \leq \frac{u'(\underline{c})}{u'(\bar{c})}, \quad \text{for all } s^t.$$

REMARK 3.1 If we let u be such that $u(c) := \ln(c)$ in the interval $(0, \bar{y}]$ and extend this function on $[\bar{y}, \infty)$ such that the assumptions on u are satisfied, then the inequality (3.3) is true for any γ in the interval $[\underline{c}/\bar{c}, 1)$.

To focus on a stationary equilibrium, we assume that the economy begins in state $s^0 = s_0 = z_1$ (the rich country R_1 has the highest endowment) and the initial asset positions are

$$a^{\text{P}}(s^0) := -\delta, \quad a^{\text{R1}}(s^0) := 0 \quad \text{and} \quad a^{\text{R2}}(s^0) := \delta.$$

PROPOSITION 3.1 *The economy of Example 3.1 admits a competitive equilibrium with self-enforcing debt in which country P faces positive not-too-tight debt limits $D^P(s^t) = \delta$ although its natural debt limits $W^P(s^t)$ are finite at equilibrium. Moreover, there exists a path such that for t large enough, the debt limit is strictly larger than the country's natural debt limit. Formally, for the path $(\sigma_t)_{t \geq 0} := (z_1, z_2, z_1, z_2, \dots)$ we have*

$$\lim_{t \rightarrow \infty} D^P(\sigma^t) = \delta > 0 = \lim_{t \rightarrow \infty} W^P(\sigma^t).$$

PROOF: We first describe the equilibrium prices, debt limits and allocations.

Let the price process $(q(s^t))_{s^t \succ s^0}$ be as follows:

$$q(s^t) := \begin{cases} q^c, & \text{if } s_t \neq s_{t-1} \\ q^{nc}, & \text{otherwise.} \end{cases}$$

Since by assumption $q^c + q^{nc} = 1$, the risk-less interest rate is zero.

Consider the following debt limits:

$$D^{R1}(s^t) = D^{R2}(s^t) := 0 \quad \text{and} \quad D^P(s^t) := \delta.$$

It follows that debt limits are not-too-tight since they allow for exact roll-over (i.e., condition (ER) holds true).

Let (c^P, a^P) be defined as follows: $c^P(s^t) := y^P(s^t)$ and $a^P(s^t) := -\delta$ for every event s^t . At the initial period, the poor country repays the inherited debt δ by issuing the non-contingent debt δ . At the subsequent periods, instead of repaying, it rolls over this debt at infinite.

We also let (c^{Rk}, a^{Rk}) be defined as follows:

$$c^{Rk}(s^t) := \begin{cases} \bar{c}, & \text{if } s_t = z_k \\ \underline{c}, & \text{otherwise} \end{cases} \quad \text{and} \quad a^{Rk}(s^{t+1}) := \begin{cases} \delta, & \text{if } s_{t+1} \neq z_k \\ 0, & \text{otherwise.} \end{cases}$$

Each rich country saves to transfer resources against the low income shock. They do not issue debt since they are credit-constrained.

We next show that equilibrium allocations are indeed optimal.

Observe that (c^P, a^P) is optimal since it is budget feasible (with equality) and satisfies the Euler equations (2.3) (this follows from the definition of asset prices, i.e., conditions (3.2) and (3.3)) together with the transversality condition (2.4) (debt limits always bind). Most importantly, we have

$$\sum_{s^{t+1} \succ s^t} q(s^{t+1}) y^P(s^{t+1}) \leq (q^c \gamma + q^{nc}) y^P(s^t).$$

Since $\gamma \in (0, 1)$ and $q^c + q^{nc} = 1$, this implies that country P's wealth (and therefore its natural debt limit) is finite at any period. Indeed, if we let $\chi := (q^c \gamma + q^{nc})$, we can

show that $W^P(s^t) \leq y^P(s^t) \sum_{t \geq 0} \chi^t = y^P(s^t)/(1 - \chi)$. Moreover, for the path $(\sigma_t)_{t \geq 0} := (z_1, z_2, z_1, z_2, \dots)$, we have

$$W^P(\sigma^t) \leq y(\sigma^t) \sum_{\tau=0}^{\infty} \chi^\tau = \frac{\gamma^t \bar{y}}{1 - \chi} \xrightarrow{t \rightarrow \infty} 0.$$

The plan (c^{R_k}, a^{R_k}) is also optimal since it is budget feasible (with equality), it satisfies the Euler equations (2.3) (this follows from the definition of asset prices, i.e., condition (3.2)) and the transversality condition (2.4).¹⁰

Finally, all markets clear by construction.

Q.E.D.

In the equilibrium described above, country P sustains positive levels of debt even if its wealth is finite. This is surprising since in this case, the country P's repayment incentives seem to be the same as in Bulow and Rogoff (1989). The difference between our example and Theorem 2.1 is that Bulow and Rogoff (1989) restrict the sovereign's debt limits to be tighter than the natural debt limits. Our example illustrates that, even if a country's natural debt limits are finite, the level of "reputation debt" sustained at equilibrium can be greater than the natural debt limits. An obvious question then arises: why the standard result of the full commitment literature does not apply here? Or, equivalently, why investors accept to lend more than country P's natural ability to repay? The answer is that because they are credit constrained and they need the poor country to act as a financial intermediary. Indeed, as shown by Hellwig and Lorenzoni (2009), countries R₁ and R₂ would like to share risks by trading with each other. However, in the equilibrium we described, these two countries are not creditworthy (their debt limits are equal to zero) but country P turns out to have a "good reputation" as a credible borrower. The creditworthiness of the poor country (with an income process that vanishes along a path of successive negative shocks) stems from its intermediation role, helping countries R₁ and R₂ to smooth consumption. The poor country P acts as a pass-through intermediary and extracts the surplus δ .

Hellwig and Lorenzoni (2009) also construct an economy where two agents credibly issue positive levels of debt at equilibrium. Interest rates are sufficiently low such that each agent's wealth is infinite. The interpretation proposed by the authors is that the level of interest rates matters for debt sustainability to the extent it induces repayment incentives. We have a different interpretation. First, repayment incentives are actually guaranteed by the roll-over property of debt limits, independently of the level of interest

¹⁰The transversality condition is satisfied because the equilibrium is Markovian stationary. Formally, we have

$$\sum_{s^t \in S^t} \beta^t \pi(s^t) u'(c^{R_k}(s^t)) a^{R_k}(s^t) \leq \beta^t u'(\underline{c}) \delta \xrightarrow{t \rightarrow \infty} 0.$$

rates. Second, our example shows that interest rates are important from the lenders' perspective. Indeed, the presence of bubbles in debt limits is compatible with the supply of credit only if there is aggregate lending at infinite, which requires sufficiently low interest rates.

Some additional observations about this example deserve further discussion. This is done in the following remarks.

As already stressed, the poor country sustains a level of debt higher than its natural ability to repay (represented by the natural debt limits). Formally, the poor country infinitely rolls over its debt since $a^P(s^t) = -\delta$ for each event s^t . Market clearing then implies that the supply side is lending (in present value terms) at infinite:¹¹

$$(3.4) \quad \sum_{s^t \in S^t} p(s^t)[a^{R1}(s^t) + a^{R2}(s^t)] = \delta, \quad \text{for any period } t.$$

REMARK 3.2 (The need of more than one lender) The presence of (at least) two lenders is fundamental for sustaining debt based on this financial intermediation mechanism. If there was only one agent R acting as a lender, then the Euler equations and the transversality conditions would be inconsistent with lending at infinite. Indeed, the debt constraints $\delta = a^R(s^t) \geq -D^R(s^t)$ would never bind and Euler equations would imply that bond prices coincide with the lender's marginal rates of substitution. This, in turn, would imply that

$$p(s^t) = \beta^t \pi(s^t) \frac{u'(c^R(s^t))}{u'(c^R(s^0))}$$

and, due to equation (3.4),

$$\sum_{s^t \in S^t} \beta^t \pi(s^t) u'(c^R(s^t)) [a^R(s^t) + D^R(s^t)] \geq u'(c^R(s^0)) \delta.$$

This is incompatible with the individual transversality condition (2.4).

REMARK 3.3 (The role of limited commitment) Limited commitment is also indispensable for debt sustainability. Even with two lenders, assuming full commitment implies that the debt constraints would be always non-binding by construction (they would be imposed only to prevent Ponzi schemes) and the marginal rates of substitution of each lender would coincide with bond prices. We would then get

$$p(s^t) = \beta^t \pi(s^t) \frac{u'(c^{Rk}(s^t))}{u'(c^{Rk}(s^0))}, \quad k = \{1, 2\}$$

¹¹Since $p(s^t) = \sum_{s^{t+1} \succ s^t} p(s^{t+1})$ at every event s^t , we have $\sum_{s^t \in S^t} p(s^t) = p(s^0) = 1$ at any period t .

and

$$\begin{aligned} \sum_{s^t \in S^t} \beta^t \pi(s^t) \{u'(c^{R_1}(s^t))[a^{R_1}(s^t) + D^{R_1}(s^t)] + u'(c^{R_2}(s^t))[a^{R_2}(s^t) + D^{R_2}(s^t)]\} \\ \geq \min\{u'(c^{R_1}(s^0)), u'(c^{R_2}(s^0))\} \delta, \end{aligned}$$

which, as before, is incompatible with the lenders' transversality condition (2.4).

REMARK 3.4 (The role of interest rates) The level of interest rates has a preeminent role. If we had two lenders, lack of commitment, but interest rates were such that each country's wealth is finite, then the poor country would not be able to sustain positive levels of debt (actually no agent could sustain positive levels of debt). The reason is not the same as in the full commitment environment. Indeed, because debt limits may bind at equilibrium (this is where the lack of commitment enters the picture), marginal rates of substitution need not coincide with bond prices. Therefore, we may not obtain a contradiction through the violation of individual transversality condition as in the full commitment environment. However—as shown in Lemma A.1—when lenders' wealth is finite, the following “market transversality condition” should hold true for each lender¹²

$$0 = \lim_{t \rightarrow \infty} \sum_{s^t \in S^t} p(s^t)[a^{R_k}(s^t) + D^{R_k}(s^t)] \geq \lim_{t \rightarrow \infty} \sum_{s^t \in S^t} p(s^t) a^{R_k}(s^t).$$

Given equation (3.4), this is inconsistent with the market clearing condition $a^{R_1}(s^t) + a^{R_2}(s^t) = \delta$.

In contrast, when interest rates are such that the lenders' wealth is infinite, then Lemma A.1 does not apply, i.e., the market transversality condition need not be satisfied. Observe that the failure of the market transversality condition is not incompatible with the satisfaction of the standard individual transversality condition (2.4). Indeed, if for each lender, the debt constraints bind infinitely many times, then marginal rates of substitution need not coincide with bond prices (i.e., $p(s^t)$ may not coincide with $\beta^t \pi(s^t) u'(c^{R_k}(s^t)) / u'(c^{R_k}(s^0))$).¹³ Therefore, there is space for “lending at infinite”.

Actually, this is a general result: to sustain positive levels of debt, interest rates must be low enough such that the wealth of some agents (but not necessarily the wealth of the debtors) is infinite. Formally, we obtain the following result.

¹²Observe that assuming the wealth of each agent to be finite together with the market clearing imply that equilibrium consumption allocations must have finite present value.

¹³This is exactly what happens in our example. If $s_t = z_k$ and $s_{t-1} \neq z_k$, then we have

$$a^{R_k}(s^t) = 0 = -D^{R_k}(s^t) \quad \text{and} \quad \beta \pi \frac{u'(c^{R_k}(s^t))}{u'(c^{R_k}(s^{t-1}))} = \beta \pi \frac{u'(\bar{c})}{u'(\underline{c})} < \beta \pi \frac{u'(\underline{c})}{u'(\bar{c})} = q^c.$$

PROPOSITION 3.2 *Debt cannot be self-enforced if interest rates are such that the aggregate wealth of the economy is finite. Formally, if $(q, (c^i, a^i, D^i)_{i \in I})$ is a competitive equilibrium with self-enforcing debt such that $W^i(s^0)$ is finite for each agent i , then $D^i = 0$ and there is no trade.*

We provide the detailed proof of this result because this helps to clarify the role of interest rates to sustain debt at equilibrium. In particular, we do not assume a priori that debt limits are tighter than the natural debt limits since our example above illustrates that self-enforced debt limits may exceed the natural debt limits at equilibrium. We instead show that this is a necessary condition when the wealth of each agent is finite. To prove this, we exploit the fact that the “market transversality condition” (see Lemma A.1) is always satisfied when the optimal consumption of an agent has finite present value.

PROOF OF PROPOSITION 3.2: Let $(q, (c^i, a^i, D^i)_{i \in I})$ be a competitive equilibrium with self-enforcing debt. Assume that for each agent i the wealth $W^i(s^0)$ is finite (this implies that $W^i(s^t)$ is finite at each event s^t). Since consumption markets clear, the present value of each agent’s consumption is finite: $PV(c^i|s^0) < \infty$ for each i . Applying Lemma A.1 we get the following market transversality condition

$$\lim_{t \rightarrow \infty} \sum_{s^t \in S^t} p(s^t) [a^i(s^t) + D^i(s^t)] = 0.$$

Market clearing of bond markets then implies

$$\lim_{t \rightarrow \infty} \sum_{s^t \in S^t} p(s^t) \sum_{i \in I} D^i(s^t) = 0.$$

We know from the characterization result proved by Hellwig and Lorenzoni (2009) (Theorem 2.2) that D^i satisfies exact roll-over. Therefore, we have

$$\sum_{i \in I} D^i(s^0) = \lim_{t \rightarrow \infty} \sum_{s^t \in S^t} p(s^t) \sum_{i \in I} D^i(s^t) = 0.$$

Non-negativity of each $D^i(s^0)$ implies that $D^i(s^0) = 0$ for every i . Since D^i is non-negative and allows for exact roll-over, we have the desired result: $D^i = 0$ for every i . *Q.E.D.*

REMARK 3.5 (Non-binding debt limits) In the equilibrium described in the proof of Proposition 3.1, the debt limits of the borrowing country always bind. It is possible to slightly modify the primitives in order to exhibit a competitive equilibrium where the debt limits of the borrowing country never bind. Indeed, let $(\delta_t)_{t \geq 0}$ be a strictly increasing sequence of positive numbers $\delta_t > 0$ converging to δ such that the sequence $(\varepsilon_t)_{t \geq 0}$ is

strictly decreasing where $\varepsilon_t := \delta_{t+1} - \delta_t$. We also assume that $\varepsilon_t \leq \gamma^t \bar{y}$ for every $t \geq 0$. The endowments of the rich countries R_1 and R_2 are modified as follows:

$$\tilde{y}^{R_k}(s^t) := \begin{cases} \bar{c} + q^c \delta_{t+1}, & \text{if } s_t = z_k \\ \underline{c} - q^c \delta_{t+1} + \varepsilon_t, & \text{otherwise.} \end{cases}$$

Country P's endowment is defined as follows: $\tilde{y}^P(s^t) := y^P(s^t) - \varepsilon_t$ for every $s^t \succ s^0$.

Consider the same consumption allocations (c^P, c^{R_1}, c^{R_2}) , the same price process q and the same debt limits: $(D^P, D^{R_1}, D^{R_2}) := (\delta, 0, 0)$ as in Proposition 3.1. We only modify the asset holdings. Let $\tilde{a}^P(s^t) := -\delta_t$ for every s^t and \tilde{a}^{R_1} and \tilde{a}^{R_2} be defined as follows:

$$\tilde{a}^{R_k}(s^{t+1}) := \begin{cases} \delta_{t+1}, & \text{if } s_{t+1} \neq z_k \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward to see that $(q, (c^i, \tilde{a}^i, D^i)_{i \in I})$ is a competitive equilibrium with self-enforcing debt.¹⁴ In the equilibrium described above, the debt limits of country P never bind: the equilibrium continuation utility of repaying debt is strictly larger than the default option.

REMARK 3.6 (Real indeterminacy) An additional observation regarding our example is that, for the same bond prices, there is a continuum of competitive equilibria with self-enforcing debt where the gains from trade between the rich countries are only partially intermediated by the poor country P. More formally, for any $\alpha \in [0, 1]$, let $D^P(s^t) := \alpha\delta$, $D^{R_1}(s^t) := 0$ and $D^{R_2}(s^t) := (1 - \alpha)\delta$. Countries P and R_2 have both some degree of creditworthiness. Let (c^P, a^P) be defined as follows: $c^P(s^0) := \bar{c} - (1 - \alpha)\delta$, $c^P(s^t) := y^P(s^t)$ for every $s^t \succ s^0$ and $a^P(s^t) := -\alpha\delta$ for every $s^t \succ s^0$. The plan (c^{R_1}, a^{R_1}) is defined as above. We let (c^{R_2}, a^{R_2}) be the plan defined by $c^{R_2}(s^0) := \underline{c} - \alpha\delta$ and for every $s^t \succ s^0$,

$$c^{R_2}(s^t) := \begin{cases} \bar{c} & \text{if } s_t = z_2 \\ \underline{c} & \text{otherwise} \end{cases} \quad \text{and} \quad a^{R_2}(s^{t+1}) := \begin{cases} \alpha\delta & \text{if } s_{t+1} = z_1 \\ -(1 - \alpha)\delta & \text{otherwise.} \end{cases}$$

As before, we can check that we get a competitive equilibrium with self-enforcing debt. This observation illustrates that, when the aggregate wealth of the economy is infinite, there is indeterminacy of creditworthiness or “good reputation” which has real effects.

REMARK 3.7 (Transition to the steady-state) In the description of the equilibrium in Proposition 3.1, we have chosen specific initial asset holdings to exhibit a stationary Markovian equilibrium. Our result remains valid if agents start with zero asset holdings. Indeed, suppose now that the economy begins at date 0 in state $s^0 = s_0 = z_1$ and the

¹⁴Observe that $\tilde{y}^P \leq y^P$, implying that country P's wealth is finite.

initial asset positions are $\tilde{a}^P(s^0) = \tilde{a}^{R1}(s^0) = \tilde{a}^{R2}(s^0) = 0$. Let $(\tilde{c}^i, \tilde{a}^i)_{i \in I}$ be the allocation defined by $(\tilde{c}^i(s^t), \tilde{a}^i(s^t)) := (c^i(s^t), a^i(s^t))$ for every event $s^t \neq (z_1, \dots, z_1)$ along which the state has switched at least once, and for every event $\sigma^t = (z_1, \dots, z_1)$ prior to the first time the state switches from z_1 to z_2 , we pose

$$\tilde{c}^P(\sigma^t) := \bar{y} + q^c \delta, \quad \tilde{c}^{R1}(\sigma^t) := \bar{c} \quad \text{and} \quad \tilde{c}^{R2}(\sigma^t) = \underline{y}$$

together with

$$\tilde{a}^P(\sigma^t, s_{t+1}) := \begin{cases} -\delta, & \text{if } s_{t+1} = z_2 \\ 0, & \text{otherwise,} \end{cases} \quad \tilde{a}^{R1}(\sigma^t, s_{t+1}) := \begin{cases} \delta, & \text{if } s_{t+1} = z_2 \\ 0, & \text{otherwise} \end{cases}$$

and $\tilde{a}^{R2}(\sigma^t, s_{t+1}) := 0$. If the Bernoulli utility u coincides with the logarithmic function in the interval $(0, \bar{y} + q^c \delta]$ and if δ is small enough, then $(q, (\tilde{c}^i, \tilde{a}^i, D^i)_{i \in I})$ is a competitive equilibrium with self-enforcing debt. Indeed, the only delicate step is to check the validity of the Euler equations corresponding to the poor agent's decisions along the path $\sigma^t = (z_1, \dots, z_1)$. For this purpose, it is sufficient to prove that

$$\beta\pi \frac{u'(\underline{c})}{u'(\bar{c})} = q^c \geq \beta\pi \frac{u'(\gamma\bar{y})}{\bar{y} + q^c \delta}.$$

When $u(x) = \ln(x)$ on the interval $(0, \bar{y} + q^c \delta]$, the above inequality reduces to

$$\gamma \frac{\bar{c}}{\underline{c}} \geq 1 + \frac{q^c \delta}{\bar{y}}.$$

Since γ can be chosen in the interval $[\underline{c}/\bar{c}, 1)$, choosing δ small enough, we get the desired result.

3.2. Deterministic Economy with Positive Riskless Interest Rate

The original Impossibility Theorem of Bulow and Rogoff (1989) is stated under the assumption that the asset pricing kernel is risk-neutral, i.e., $q(s^{t+1}) = \beta\pi(s^{t+1}|s^t)$.¹⁵ In particular, the risk-less interest rate $r := \beta^{-1} - 1$ is assumed to be strictly positive. This makes the comparison of Bulow and Rogoff (1989)'s result with the equilibrium described in Proposition 3.1 less transparent since in our economy the equilibrium risk-less interest rate is equal to zero. To convince the reader that the implications of our analysis do not rely on this feature, we provide below a second example where the equilibrium risk-less interest rate is strictly positive but a country sustains debt levels in excess of its natural ability to repay.

¹⁵In that respect, the statement of Theorem 2.1 is slightly more general than the original result proved in Bulow and Rogoff (1989)

EXAMPLE 3.2 Fix arbitrary non-negative numbers δ , ω^{R_1} and ω^{R_2} with $\delta > 0$. Consider a deterministic economy with three countries (P, R₁ and R₂) where the endowment sequences are specified as follows:

Country P's endowments are defined by

$$y_0^P := 0 \quad \text{and} \quad y_t^P := \delta, \quad \text{for all } t \geq 1.$$

Country R₁'s endowments are defined by

$$\forall t \geq 0, \quad y_{2t+1}^{R_1} := \omega^{R_1} \quad \text{and} \quad y_{2t}^{R_1} := \omega^{R_1} + \frac{\delta}{\beta^{2t}} + \frac{\delta}{\beta^{2t+1}}.$$

In other words,

$$y_0^{R_1} = \omega^{R_1} + \delta + \frac{\delta}{\beta}, \quad y_1^{R_1} = \omega^{R_1}, \quad y_2^{R_1} = \omega^{R_1} + \frac{\delta}{\beta^2} + \frac{\delta}{\beta^3}, \quad y_3^{R_1} = \omega^{R_1}, \quad \dots$$

Those of country R₂ are defined by

$$\forall t \geq 0, \quad y_{2t+1}^{R_2} := \omega^{R_2} + \frac{\delta}{\beta^{2t+1}} + \frac{\delta}{\beta^{2t+2}} \quad \text{and} \quad y_{2t}^{R_2} := \omega^{R_2}.$$

In other words,

$$y_0^{R_2} = \omega^{R_2}, \quad y_1^{R_2} = \omega^{R_2} + \frac{\delta}{\beta} + \frac{\delta}{\beta^2}, \quad y_2^{R_2} = \omega^{R_2}, \quad y_3^{R_2} = \omega^{R_2} + \frac{\delta}{\beta^3} + \frac{\delta}{\beta^4}, \quad \dots$$

The choice of ω^{R_1} and ω^{R_2} is irrelevant. To fix ideas, we can set them to be equal to zero.

PROPOSITION 3.3 *The economy of Example 3.2 admits a competitive equilibrium with self-enforcing debt in which country P faces positive not-too-tight debt limits $D_t^P = \delta/(\beta^t)$ although its natural debt limits W_t^P are finite at equilibrium. Moreover, for t large enough, the debt limit is strictly larger than the country's natural debt limit. More specifically, we have*

$$\lim_{t \rightarrow \infty} D_t^P = \infty > \delta/(1 - \beta) = \lim_{t \rightarrow \infty} W_t^P.$$

PROOF: We first describe the equilibrium prices, debt limits and allocations.

Let the price sequence $q = (q_t)_{t \geq 1}$ be defined by $q_t := \beta$ for every $t \geq 1$ (positive and constant interest rate r defined $1 + r = \beta^{-1}$).

Consider the following debt limits: $D_t^{R_1} = D_t^{R_2} = 0$ and $D_t^P := \delta/(\beta^t)$. These debt limits are not-too-tight under the price sequence q since they allow for exact roll-over.

Let (c^P, a^P) defined as follows: $c_0^P := \delta$, $c_t^P := y_t^P = \delta$ for every $t \geq 1$ and $a_t^P := -\delta/(\beta^t)$ for every $t \geq 1$. The poor country borrows and consumes the amount δ at the initial period and then, instead of repaying, it rolls over this debt at infinite.

Let (c^{R_1}, a^{R_1}) be defined as follows:

$$c_t^{R_1} := \begin{cases} y_t^{R_1} - \delta/(\beta^t) & \text{if } t \text{ is even} \\ y_t^{R_1} + \delta/(\beta^t) & \text{if } t \text{ is odd} \end{cases} \quad \text{and} \quad a_t^{R_1} := \begin{cases} \delta/(\beta^t) & \text{if } t \text{ is odd} \\ 0 & \text{if } t \text{ is even.} \end{cases}$$

We let (c^{R_2}, a^{R_2}) be the plan defined by $c_0^{R_2} = y_0^{R_2}$ and for every $t \geq 1$,

$$c_t^{R_2} := \begin{cases} y_t^{R_2} - \delta/(\beta^t) & \text{if } t \text{ is odd} \\ y_t^{R_2} + \delta/(\beta^t) & \text{if } t \text{ is even} \end{cases} \quad \text{and} \quad a_t^{R_2} := \begin{cases} \delta/(\beta^t) & \text{if } t \text{ is even} \\ 0 & \text{if } t \text{ is odd.} \end{cases}$$

At every even date $2t$, the rich country R_1 optimally saves the amount $\beta a_{2t+1}^{R_1} = \delta/(\beta^{2t})$ in order to trade the time-varying endowments

$$(y_{2t}^{R_1}, y_{2t+1}^{R_1}) = (\omega^{R_1} + \delta/(\beta^{2t}) + \delta/(\beta^{2t+1}), \omega^{R_1})$$

in exchange of the constant consumption

$$(c_{2t}^{R_1}, c_{2t+1}^{R_1}) = (\omega^{R_1} + \delta/(\beta^{2t+1}), \omega^{R_1} + \delta/(\beta^{2t+1})).$$

The rich country R_2 follows the same strategy at odd dates $2t+1$: it saves the amount $\beta a_{2t+2}^{R_2} = \delta/(\beta^{2t+1})$ to trade the time-varying endowments

$$(y_{2t+1}^{R_2}, y_{2t+2}^{R_2}) = (\omega^{R_2} + \delta/(\beta^{2t+1}) + \delta/(\beta^{2t+2}), \omega^{R_2})$$

in exchange of the constant consumption

$$(c_{2t+1}^{R_2}, c_{2t+2}^{R_2}) = (\omega^{R_2} + \delta/(\beta^{2t+2}), \omega^{R_2} + \delta/(\beta^{2t+2})).$$

We next show that equilibrium allocations are indeed optimal.

Observe that (c^P, a^P) is optimal since it is budget feasible (with equality), it satisfies the Euler equations (this follows from the fact that consumption is constant) and the transversality condition (debt limits bind infinitely often). Moreover, country P 's wealth (and therefore its natural debt limit) is finite at any period since the interest rate is strictly positive and endowments are bounded from above. Formally, we have

$$W_t^P = \delta + \beta\delta + \beta^2\delta + \dots = \frac{\delta}{1-\beta}, \quad \text{for all } t \geq 1.$$

The plan (c^{R_k}, a^{R_k}) is also optimal since it is budget feasible (with equality), the transversality condition is satisfied (debt constraints bind infinitely many times) and Euler equations are satisfied. Indeed, if $a_{t+1}^i > 0$ (i.e., agent i saves at date t), then agent i is financially unconstrained and we have $c_t^i = c_{t+1}^i$. If $a_{t+1}^i = 0$, then agent i is financially constrained and we have $c_t^i \leq c_{t+1}^i$.

Finally, all markets clear by construction.

Q.E.D.

4. CONCLUSION

In models without commitment, the creditworthiness of an agent is not necessarily limited by his ability to repay out of his future resources. Indeed, we show, by means of two examples, that an agent can sustain positive levels of debt by acting as a financial intermediary that alleviates the incentive compatibility constraints of some other agents. Since this financial service is not related to the agent's wealth, the borrowing capacity can exceed an agent's natural ability to repay represented by the present value of his future endowments. This is in contrast with the standard results of the full commitment literature. Moreover, our examples show that the Impossibility Theorem in Bulow and Rogoff (1989) hinges on the restrictive assumption that debt should be bounded by the natural debt limits. They also clarify that the level of interest rates is important from the lenders' perspective: they should be low enough to provide lending incentives. Indeed, repayment incentives are guaranteed by the bubble property of debt limits, independently of the level of interest rates. However, the bubble property of debt limits is consistent with the supply of credit only if interest rates are lower than some lenders' endowments growth rates.

A. APPENDIX

In this appendix we show that the "market transversality condition" is satisfied when the present value of an agent's optimal consumption is finite. Since we are exclusively concerned with the single-agent problem, we simplify notation by dropping the superscript i .

If c is a strictly positive consumption sequence (in the sense that $c(s^t) > 0$ for every event s^t), then the agent's marginal rate of substitution at event s^t is denoted by $\text{MRS}(c|s^t) := \beta\pi(s^t|s^{t-1})u'(c(s^t))/u'(c(s^{t-1}))$.

LEMMA A.1 Let b denote an initial claim and (c, a) be optimal in $B(D, b|s^\tau)$ where D is a process of not-too-tight debt limits. If c has finite present value, i.e., $\text{PV}(c|s^\tau) < \infty$, then the following market transversality condition is satisfied,

$$(A.1) \quad \lim_{t \rightarrow \infty} \sum_{s^t \in S^t} p(s^t)[a(s^t) + D(s^t)] = 0.$$

PROOF: It suffices to show that for every $s^t \succ s^\tau$, we have $a(s^t) + D(s^t) \leq \text{PV}(c|s^t)$. Assume, by way of contradiction, that there exists $s^t \succ s^\tau$ such that

$$(A.2) \quad a(s^t) + D(s^t) > \text{PV}(c|s^t).$$

Let $\theta(s^r) := \text{PV}(c|s^r)$ for every $s^r \succeq s^t$. By construction we have

$$(A.3) \quad c(s^r) + \sum_{s^{r+1} \succ s^r} q(s^{r+1})\theta(s^{r+1}) = \theta(s^r), \quad \text{for all } s^r \succeq s^t.$$

Moreover, it is easy to see that¹⁶

$$(A.4) \quad D(s^r) \leq y(s^r) + \sum_{s^{r+1} \succ s^r} q(s^{r+1})D(s^{r+1}), \quad \text{for all } s^r \succeq s^t.$$

Posing $\tilde{a} := \theta - D$, it follows that

$$(A.5) \quad c(s^r) + \sum_{s^{r+1} \succ s^r} q(s^{r+1})\tilde{a}(s^{r+1}) = \tilde{a}(s^r), \quad \text{for all } s^r \succeq s^t.$$

Since $\tilde{a}(s^r) \geq -D(s^r)$, we get that $(c, \tilde{a}) \in B(D, \tilde{a}(s^t)|s^t)$. The bond holdings \tilde{a} finance the consumption c when the initial claim is $\tilde{a}(s^t)$. Following Equation (A.2) we have $a(s^t) > \tilde{a}(s^t)$. This contradicts the optimality of a . Indeed, we can increase the consumption at event $\sigma(s^t) = s^{t-1}$ by replacing $(a(s^r))_{s^r \in \Sigma(s^t)}$ with $(\tilde{a}(s^r))_{s^r \in \Sigma(s^t)}$. *Q.E.D.*

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¹⁶Indeed, Let $b = -y(s^r) - \sum_{s^{r+1} \succ s^r} q(s^{r+1})D(s^{r+1})$ and let (c, a) be optimal in $B(D, b|s^r)$. It is straightforward to see that we must have $c(s^r) = 0$ and $a(s^{r+1}) = -D(s^{r+1})$ implying that $U(c|s^r) = u(0) + \beta V(s^{r+1})$. We know that $V(s^r) = J(0, 0|s^r) = U(\hat{c}|s^r)$ for a consumption process \hat{c} satisfying participation constraints at all successor events, i.e., $U(\hat{c}(s^{r+1})) \geq V(s^{r+1})$. In particular, we have $V(s^r) \geq u(\hat{c}(s^r)) + \beta V(s^{r+1}) \geq U(c|s^r)$ which implies that $b \leq -D(s^r)$.

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