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Moutard transform approach to generalized analytic functions with contour poles ^{*}

P.G. Grinevich [†] R.G. Novikov [‡]

Abstract

We continue studies of Moutard-type transforms for the generalized analytic functions started in [13], [14]. In particular, we show that generalized analytic functions with the simplest contour poles can be Moutard transformed to the regular ones, at least, locally. In addition, the later Moutard-type transforms are locally invertible.

1 Introduction

We study the equations

$$\partial_{\bar{z}}\psi = u\bar{\psi} \text{ in } D, \quad (1)$$

$$\partial_{\bar{z}}\psi^+ = -\bar{u}\bar{\psi}^+ \text{ in } D, \quad (2)$$

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where D is an open domain in \mathbb{C} , $u = u(z)$ is a given function in D , $\partial_{\bar{z}} = \partial/\partial\bar{z}$. The functions $\psi = \psi(z)$ satisfying equation (1) are known as generalized analytic functions in D , equation (2) is known as the conjugate equation to (1); see [30]. In the present article the notation $f = f(z)$ does not mean that f is holomorphic.

The classical theory of generalized analytic functions is presented in [3], [30]. In addition, very recently in [13], [14] it was shown that a new progress in this theory is possible by involving ideas of Moutard-type transforms going back to [22]. Actually, ideas of Moutard-type transform were developed and successfully used in the soliton theory in dimension 2+1, in the spectral theory in dimension 2 and in the differential geometry; see [5], [21], [23], [26]-[29] and references therein.

We recall that in our case the Moutard-type transforms assign in quadratures to a given coefficient u and fixed solutions $f_j, f_j^+, j = 1, \dots, N$, of equations (1),(2), and all formal solutions ψ, ψ^+ of (1), (2) a new coefficient \tilde{u} and new related formal solutions $\tilde{\psi}, \tilde{\psi}^+$ for these generalized analytic function equations; see [13], [14]. In turn, the construction of [13] was stimulated by recent articles by I.A. Taimanov [27], [28] on the Moutard-type transforms for the Dirac operators in the framework of the soliton theory in dimension $2 + 1$.

In the classical theory of generalized analytic functions it is usually assumed that

$$u \in L_p(D), \quad p > 2, \quad \text{if } D \text{ is bounded,} \quad (3)$$

$$u \in L_{p,2}(\mathbb{C}), \quad p > 2, \quad \text{if } D = \mathbb{C}, \quad (4)$$

where

$L_{p,\nu}(\mathbb{C})$ denotes complex-valued functions u such that

$$u \in L_p(D_1), \quad u_\nu \in L_p(D_1), \quad \text{where } u_\nu(z) = \frac{1}{|z|^\nu} u\left(\frac{1}{z}\right), \quad (5)$$

$$D_1 = \{z \in \mathbb{C} : |z| \leq 1\};$$

see [30].

On the other hand, one of the most important applications of the generalized analytic functions theory is associated with the inverse scattering in two dimensions, see [2], [8]–[12], [15], [18], [20], [24] and references therein. In addition, already in [15] it was shown that, for the case of the two-dimensional

Schrödinger equation, not only regular generalized analytic functions, where u satisfies (3) or (4), but generalized analytic functions with contour poles, also naturally arise. However, in the latter case, assumptions (3), (4) are not valid at all. It is quite likely, that the classical methods of generalized analytic functions do not admit appropriate generalizations for this case. In particular, the problem of proper solving the generalized analytic function equation (1) for u with contour poles was remaining well-known, but open.

It is in order to precise that in the framework of inverse scattering for the Schrödinger equation in two dimensions the generalized analytic function equation (1) with contour poles arises for the case when

$$u = \frac{u_{-1}}{\Delta}, \quad \psi = \frac{\psi_{-1}}{\Delta}, \quad (6)$$

where Δ is real-valued and u_{-1} , ψ_{-1} , Δ are quite regular (e.g. real-analytic) functions on D , and the pole contours are the zeroes of Δ . In this setting $\psi = \psi(x, z, E)$ are the Faddeev exponentially growing solutions of the Schrödinger equation in $x \in \mathbb{R}^2$ at fixed energy $E \in \mathbb{R}$, u is a particular case of the Faddeev generalized scattering data, Δ is the modified Fredholm determinant for the Lipman-Schwinger-Faddeev integral equation for ψ , and z is a fixed-energy spectral parameter; see [7], [19], [15] and references therein.

Proceeding from the aforementioned inverse scattering motivation, we consider the generalized analytic function equation (1) with contour poles for the case when u is of the form as in (6), and when this equation has sufficiently many (more or less as in the regular case) local solutions ψ of the form as in (6). Adopting the terminology of [17] we say that in this case equation (1) is of meromorphic class. Note also that equation (1) may be of meromorphic class only if principal terms of u near pole contours satisfy solvability conditions; for simple contour poles these conditions were found in [15].

Actually, our recent works [13], [14] were motivated considerably by the aforementioned open problem of proper solving the generalized analytic function equation (1) with contour poles for the meromorphic case. In particular, in [13] we give examples of coefficients u when equation (1) is of meromorphic class and can be efficiently studied using Moutard-type transforms. In addition, we were stimulated by results of [1], [4], [16], [25], [29], [27] on efficient applications of Darboux-Crum and Moutard-type transforms to studies of some important linear ODE's and PDS's with singular coefficients.

The results of the present work can be summarized as follows:

- We give composition and inversion formulas for the simple Moutard-type transforms for the equations (1), (2); see Theorems 1, 2 in Subsection 2.2;
- We show that any equation (1) of meromorphic class with a simple contour pole can be transformed to a regular one in a neighborhood of the pole contour via an appropriate simple Moutard-type transform; see Theorem 3 in Subsection 3.4.

2 Moutard-type transforms for generalized analytic functions

2.1 Basic construction

Let

$$f_j = f_j(z) \quad \text{and} \quad f_j^+ = f_j^+(z), \quad j = 1, \dots, N, \quad (7)$$

denote a set of fixed solutions of equations (1) and (2), respectively.

Let ψ, ψ^+ be arbitrary solutions of (1), (2). We define an imaginary-valued potential $\omega_{\psi, \psi^+} = \omega_{\psi, \psi^+}(z)$ such that

$$\partial_z \omega_{\psi, \psi^+} = \psi \psi^+, \quad \partial_{\bar{z}} \omega_{\psi, \psi^+} = -\overline{\psi \psi^+} \quad \text{in } D. \quad (8)$$

We recall that this definition is self-consistent, at least, under the assumption that D is simply-connected. The integration constant is imaginary-valued and may depend on concrete situation.

As in [13], [14] we consider imaginary-valued potentials ω_{f_j, f_k^+} , $j, k = 1, \dots, N$, and we set

$$\Omega = \begin{bmatrix} \omega_{f_1, f_1^+} & \omega_{f_2, f_1^+} & \dots & \omega_{f_N, f_1^+} \\ \omega_{f_1, f_2^+} & \omega_{f_2, f_2^+} & \dots & \omega_{f_N, f_2^+} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{f_1, f_N^+} & \omega_{f_2, f_N^+} & \dots & \omega_{f_N, f_N^+} \end{bmatrix}. \quad (9)$$

Following [13], for equations (1), (2) we consider the Moutard-type transform

$$u \xrightarrow{\mathcal{M}} \tilde{u}, \quad (10)$$

$$\{\psi, \psi^+\} \xrightarrow{\mathcal{M}} \{\tilde{\psi}, \tilde{\psi}^+\}, \quad (11)$$

$$\mathcal{M} = \mathcal{M}_{u, f, f^+} = \mathcal{M}_{u, f_1, \dots, f_N, f_1^+, \dots, f_N^+} \quad (12)$$

defined as follows:

$$\tilde{u} = u + [f_1 \ \dots \ f_N] \Omega^{-1} \begin{bmatrix} f_1^+ \\ \vdots \\ f_N^+ \end{bmatrix}, \quad (13)$$

$$\tilde{\psi} = \psi - [f_1 \ \dots \ f_N] \Omega^{-1} \begin{bmatrix} \omega_{\psi, f_1^+} \\ \vdots \\ \omega_{\psi, f_N^+} \end{bmatrix}, \quad (14)$$

$$\tilde{\psi}^+ = \psi^+ - [f_1^+ \ \dots \ f_N^+] (\Omega^{-1})^t \begin{bmatrix} \omega_{f_1, \psi^+} \\ \vdots \\ \omega_{f_N, \psi^+} \end{bmatrix}, \quad (15)$$

where ψ , ψ^+ are formal solutions to equations (1), (2), respectively, ω_{ψ, f_j^+} and ω_{f_j, ψ^+} are defined as in (8); t in (15) stands for the matrix transposition.

Due to results of [13], the transformed functions $\tilde{\psi}$, $\tilde{\psi}^+$ solve the transformed generalized-analytic function equations:

$$\partial_{\bar{z}} \tilde{\psi} = \tilde{u} \overline{\tilde{\psi}} \quad \text{in } D, \quad (16)$$

$$\partial_{\bar{z}} \tilde{\psi}^+ = -\tilde{u} \overline{\tilde{\psi}^+} \quad \text{in } D. \quad (17)$$

2.2 Composition and inversion of simple Moutard transforms

We say that the Moutard-type transforms (10)- (15) are simple if $N = 1$. In this case:

$$\tilde{\psi} = \psi - f_1 \frac{\omega_{\psi, f_1^+}}{\omega_{f_1, f_1^+}}, \quad (18)$$

$$\tilde{\psi}^+ = \psi^+ - f_1^+ \frac{\omega_{f_1, \psi^+}}{\omega_{f_1, f_1^+}}, \quad (19)$$

$$\tilde{u} = u + \frac{f_1 \overline{f_1^+}}{\omega_{f_1, f_1^+}}. \quad (20)$$

Proposition 1 For a simple Moutard transform (18)- (20) the following formula holds:

$$\omega_{\tilde{\psi}, \tilde{\psi}^+} = \frac{\omega_{\psi, \psi^+} \omega_{f_1, f_1^+} - \omega_{\psi, f_1^+} \omega_{f_1, \psi^+}}{\omega_{f_1, f_1^+}} + c_{\tilde{\psi}, \tilde{\psi}^+}, \quad (21)$$

where $\omega_{\tilde{\psi}, \tilde{\psi}^+}$ is defined according to (8), and $c_{\tilde{\psi}, \tilde{\psi}^+}$ is an imaginary-valued integration constant.

Proposition 1 was announced in [14] and is proved in Section 4 of the present work.

Let f_1, f_2 and f_1^+, f_2^+ be some fixed solutions of equations (1) and (2), respectively, with given u . Let

$$u \xrightarrow{\mathcal{M}} \tilde{u}, \quad (22)$$

$$\{\psi, \psi^+\} \xrightarrow{\mathcal{M}} \{\tilde{\psi}, \tilde{\psi}^+\}, \quad (23)$$

$$\mathcal{M} = \mathcal{M}_2 \circ \mathcal{M}_1, \quad (24)$$

where

1. \mathcal{M}_1 is the simple Moutard transform for equations (1), (2) with coefficient u , generated by f_1, f_1^+ and given by formulas (18)-(20);
2. \mathcal{M}_2 is the simple Moutard transform for equations (16), (17) with coefficient $\tilde{u} = \mathcal{M}_1 u$, given by:

$$\tilde{\psi} = \tilde{\psi} - \tilde{f}_2 \frac{\omega_{\tilde{\psi}, \tilde{f}_2^+}}{\omega_{\tilde{f}_2, \tilde{f}_2^+}}, \quad (25)$$

$$\tilde{\psi}^+ = \tilde{\psi}^+ - \tilde{f}_2^+ \frac{\omega_{\tilde{f}_2, \tilde{\psi}^+}}{\omega_{\tilde{f}_2, \tilde{f}_2^+}}, \quad (26)$$

$$\tilde{u} = \tilde{u} + \frac{\tilde{f}_2 \overline{\tilde{f}_2^+}}{\omega_{\tilde{f}_2, \tilde{f}_2^+}}, \quad (27)$$

where we assume that:

- $\tilde{\psi} = \mathcal{M}_1 \psi, \tilde{\psi}^+ = \mathcal{M}_1 \psi^+, \tilde{f}_2 = \mathcal{M}_1 f_2, \tilde{f}_2^+ = \mathcal{M}_1 f_2^+$;

- $\omega_{\tilde{\psi}, \tilde{f}_2^+}$, $\omega_{\tilde{f}_2, \tilde{\psi}^+}$ and $\omega_{\tilde{f}_2, \tilde{f}_2^+}$ are given by (21) with $c_{\tilde{\psi}, \tilde{f}_2^+}$, $c_{\tilde{f}_2, \tilde{\psi}^+}$ and $c_{\tilde{f}_2, \tilde{f}_2^+}$ equal to zero.

3. We assume that $\omega_{f_1, f_1^+} \neq 0$, $\omega_{f_2, f_2^+} \neq 0$.

Theorem 1 *Let f_1 , f_2 and f_1^+ , f_2^+ be some fixed solutions of equations (1) and (2), respectively, with given u , and let \mathcal{M} be defined as in (22)-(24). Then \mathcal{M} coincides with the Moutard transform given by formulas (10)-(15) for $N = 2$ and generated by the initial functions f_1 , f_2 , f_1^+ , f_2^+ .*

Schematically, the result of Theorem 1 can be also presented as follows:

$$\begin{array}{ccc} \psi & \xrightarrow[\mathcal{M}_1]{\{f_1, f_1^+\}} & \tilde{\psi} & \xrightarrow[\mathcal{M}_2]{\{\tilde{f}_2, \tilde{f}_2^+\}} & \tilde{\tilde{\psi}} \\ & \searrow & & \nearrow & \\ & & \{f_1, f_2, f_1^+, f_2^+\} & & \end{array} \quad (28)$$

Theorem 1 is proved in Section 4.

Next, in order to inverse simple Moutard transforms, we consider:

$$f_2 \equiv 0, \quad f_2^+ \equiv 0, \quad \omega_{f_2, f_1^+} = i, \quad \omega_{f_1, f_2^+} = i; \quad (29)$$

$$\tilde{f}_2 = \mathcal{M}_1 f_2 = \hat{f} = -i \frac{f_1}{\omega_{f_1, f_1^+}}, \quad \tilde{f}_2^+ = \mathcal{M}_1 f_2^+ = \hat{f}^+ = -i \frac{f_1^+}{\omega_{f_1, f_1^+}}. \quad (30)$$

In particular, \hat{f} and \hat{f}^+ defined in (30) satisfy equations (16) and (17), respectively, with the coefficient \tilde{u} given by (20).

Theorem 2 *Let \mathcal{M}_1 be a simple Moutard transform defined as in (18)-(20), where $\omega_{f_1, f_1^+} \neq 0$. Let \mathcal{M}_2 be the simple Moutard transform for equations (16), (17) with coefficient $\tilde{u} = \mathcal{M}_1 u$, given by (25)-(27), where $\tilde{\psi} = \mathcal{M}_1 \psi$, $\tilde{\psi}^+ = \mathcal{M}_1 \psi^+$, $\tilde{f}_2 = \hat{f}$, $\tilde{f}_2^+ = \hat{f}^+$, where \hat{f} and \hat{f}^+ are defined in (30). Then:*

1. The potentials $\omega_{\tilde{\psi}, \tilde{f}_2^+}$, $\omega_{\tilde{f}_2, \tilde{\psi}^+}$, $\omega_{\tilde{f}_2, \tilde{f}_2^+}$ can be chosen as follows:

$$\omega_{\tilde{\psi}, \tilde{f}_2^+} = -i \frac{\omega_{\psi, f_1^+}}{\omega_{f_1, f_1^+}}, \quad \omega_{\tilde{f}_2, \tilde{\psi}^+} = -i \frac{\omega_{f_1, \psi^+}}{\omega_{f_1, f_1^+}}, \quad \omega_{\tilde{f}_2, \tilde{f}_2^+} = \frac{1}{\omega_{f_1, f_1^+}}; \quad (31)$$

2. Under assumptions (31), the composition $\mathcal{M} = \mathcal{M}_2 \circ \mathcal{M}_1$ is the identical transformation.

Theorem 2 is proved in Section 4.

3 Removing the simplest contour pole singularity

3.1 Real analytic pole contour

We consider a real analytic curve $\Gamma \subset D$:

$$\Gamma = \{z(\tau) : \tau \in]it_1, it_2[\}, \quad t_1, t_2 \in \mathbb{R}, \quad (32)$$

where:

- $z(\tau) \in D$ for $\tau \in]it_1, it_2[$,
- z is (complex-valued) real-analytic on $]it_1, it_2[$,
- $z(\tau'_1) \neq z(\tau'_2)$ for $\tau'_1 \neq \tau'_2$,
- $dz(it)/dt \neq 0$ for $t \in]t_1, t_2[$.

Here D is the domain in (1), (2).

Let

$$\mathcal{T} = \mathcal{T}_{a,b,\varepsilon} = \{\tau \in \mathbb{C} : a < \text{Im } \tau < b, |\text{Re } \tau| < \varepsilon\}, \quad (33)$$

where $a, b, \varepsilon \in \mathbb{R}$, $\varepsilon > 0$.

As a corollary of our assumptions, z in (32) could be continued to a holomorphic bijection Z :

$$\begin{aligned} Z : \mathcal{T}_{a,b,\varepsilon} &\rightarrow D_{a,b,\varepsilon}, & \tau &\rightarrow z(\tau), \\ Z^{-1} : D_{a,b,\varepsilon} &\rightarrow \mathcal{T}_{a,b,\varepsilon}, & z &\rightarrow \tau(z), \end{aligned} \quad (34)$$

for some a, b, ε such that $t_1 < a < b < t_2$, $\varepsilon > 0$, where $D_{a,b,\varepsilon} \subset D$.

Actually, we consider Γ of (32) as a pole contour for equations (1), (2).

3.2 Holomorphic change of variables

In the domain $D_{a,b,\varepsilon}$ of (34) we rewrite equations (1), (2) in variable $\tau \in \mathcal{T}_{a,b,\varepsilon}$. In this connection, following [14] we consider

$$u_*(\tau) = u(z(\tau)) \left| \frac{\partial z(\tau)}{\partial \tau} \right|, \quad (35)$$

$$\psi_*(\tau) = \psi(z(\tau)) \sqrt{\frac{\partial z(\tau)}{\partial \tau}}, \quad \psi_*^+(\tau) = \psi^+(z(\tau)) \sqrt{\frac{\partial z(\tau)}{\partial \tau}}, \quad (36)$$

where $u(z)$, $\psi(z)$, $\psi^+(z)$ are the same that in equations (1), (2), $z(\tau)$ is the same that in (34), $\tau \in \mathcal{T}_{a,b,\varepsilon}$. Then (see [14]):

$$\partial_{\bar{\tau}}\psi_* = u_*\bar{\psi}_* \text{ in } \mathcal{T}_{a,b,\varepsilon}, \quad (37)$$

$$\partial_{\bar{\tau}}\psi_*^+ = -\bar{u}_*\bar{\psi}_*^+ \text{ in } \mathcal{T}_{a,b,\varepsilon}. \quad (38)$$

In addition, we have (see [14]):

$$\mathcal{M}_{u_*,f_*,f_*^+} \circ Z^{-1} = Z^{-1} \circ \mathcal{M}_{u,f,f^+}, \quad (39)$$

where:

- Z^{-1} is considered as a map of the conjugate pairs of equations (1), (2) in $D_{a,b,\varepsilon}$ into the conjugate pairs of equations (37), (38) in $\mathcal{T}_{a,b,\varepsilon}$ and is defined according to (35), (36);
- \mathcal{M}_{u,f,f^+} for (1), (2) in $D_{a,b,\varepsilon}$ and $\mathcal{M}_{u_*,f_*,f_*^+}$ for (37), (38) in $\mathcal{T}_{a,b,\varepsilon}$ are defined according to formulas (10)-(15), where u_* , $f_* = \{f_{1,*}, \dots, f_{N,*}\}$, $f_*^+ = \{f_{1,*}^+, \dots, f_{N,*}^+\}$ are defined according to (35), (36), and

$$\omega_{\psi_*,\psi_*^+}(\tau) = \omega_{\psi,\psi^+}(z(\tau)), \quad (40)$$

for all involved potentials.

In the framework of the Moutard transform approach, using the commutativity relation (39) we reduce local studies of equations (1), (2) with contour pole at Γ in (32) to the case of contour pole at the straight line

$$\Gamma_* = \{\tau \in \mathcal{T}_{a,b,\varepsilon} : \operatorname{Re} \tau = 0\}, \quad (41)$$

where $\mathcal{T}_{a,b,\varepsilon}$ is defined as in (33), (34).

Remark 1 *We recall that, in view of formulas (35), (36), the generalized analytic functions ψ , ψ^+ of (1), (2) can be treated as spinors, i.e. differential forms of type $(\frac{1}{2}, 0)$, and u can be treated as differential form of type $(\frac{1}{2}, \frac{1}{2})$; see [14]. These forms can be written as:*

$$u = u(z)\sqrt{dzd\bar{z}}, \quad \psi = \psi(z)\sqrt{dz}, \quad \psi^+ = \psi^+(z)\sqrt{dz}. \quad (42)$$

3.3 Constraints on the meromorphic class coefficients at singularity

We consider equations (37), (38) in $\mathcal{T} = \mathcal{T}_{a,b,\varepsilon}$ defined by (33) for the case of simplest pole at Γ_* defined by (41). We write $\tau = x + iy$, $\bar{\tau} = x - iy$.

We assume that

$$u_*(\tau) = e^{2i\phi(y)} \sum_{j=-n}^{+\infty} r_j(y) x^j \quad \text{in } \mathcal{T}, \quad (43)$$

where ϕ, r_j are quite regular functions on the interval $]a, b[\approx \Gamma_*$, and ϕ, r_{-n} are real-valued, $n \in \mathbb{N}$. For this case we consider solutions ψ_*, ψ_*^+ of (37), (38) near Γ_* of the following form:

$$\psi_* = \sum_{j=-n'}^{+\infty} \alpha_j(y) x^j, \quad \psi_*^+ = \sum_{j=-n''}^{+\infty} \alpha_j^+(y) x^j, \quad n', n'' \in \mathbb{N}, \quad (44)$$

where α_j, α_j^+ are quite regular on $]a, b[$ and $\alpha_{-n'}, \alpha_{-n''}^+$ are not-zero almost everywhere at $]a, b[$.

Note that in this section we consider u_*, ψ_*, ψ_*^+ in formulas (43), (44) as formal power series in variable x .

Lemma 1 *Assume that equation (37) with coefficient u_* as in (43) has, at least, one solution ψ_* of the form as in (44). Then $n = 1$ and $|r_{-1}(y)| \equiv n'/2$ in (43).*

Lemma 1 follows from formal substitutions of (43), (44) into (37), (38).

In the present article we restrict ourselves to the simplest (but generic) case when $n' = n'' = 1$. In this case, without loss of generality, we can assume that $r_{-1} = -1/2$, adding $\pi/2$ to the phase ϕ , if necessary.

Then, as a corollary of results of [15], equation (37) is of meromorphic class, at least formally, if and only if:

$$\operatorname{Re} r_0(y) \equiv 0, \quad y \in]a, b[, \quad (45)$$

$$\operatorname{Im} r_1(y) = \frac{1}{2} \frac{d^2 \phi(y)}{dy^2}, \quad y \in]a, b[. \quad (46)$$

For completeness of exposition, this result is proved in Section 5.

Here, belonging of equation (37) to meromorphic class means that equation (37) has local solutions ψ_* near $\Gamma_* \subset \mathcal{T}$ of the form (44) with $n' = 1$ parametrised by two real-valued functions (one complex-valued function) on Γ_* , i.e., roughly speaking, equation (37) has as many local solutions ψ_* near Γ_* as in the regular case.

Actually, equation (37) and formulas (43), (44) for u_* , ψ_* with $n = 1$, $n' = 1$, $r_{-1} = -1/2$ imply that

$$\operatorname{Im} e^{-i\phi(y)} \alpha_{-1}(y) = 0, \quad y \in]a, b[. \quad (47)$$

In addition, under conditions (45), (46), the solutions ψ_* of (37), (44) are parametrised by $\beta_{-1}(y)$ and $\operatorname{Im} \beta_1(y)$ on $]a, b[$, where $\alpha_j(y) = e^{i\phi(y)} \beta_j(y)$; see Section 5.

Finally, one can see that the meromorphic class conditions (45), (46) for equation (37) imply the related meromorphic class conditions for the conjugate equation (38).

3.4 Moutard transform to the regular case

We consider equations (37), (38) for the case when

$$u_* = e^{2i\phi(y)} \left(-\frac{1}{2x} + r_0(y) + r_1(y)x + O(x^2) \right) \quad \text{in } \mathcal{T} \cup \partial\mathcal{T}, \quad (48)$$

where

$$\begin{aligned} \phi & \text{ is real-valued, } \phi \in C^2([a, b]), \\ u_* + \frac{e^{2i\phi(y)}}{2x} & \in C^1(\mathcal{T} \cup \partial\mathcal{T}), \\ r_0, r_1 & \text{ satisfy (45), (46),} \end{aligned} \quad (49)$$

\mathcal{T} is defined by (33).

We assume that equations (37), (38) have some solutions f_* , f_*^+ such that

$$\begin{aligned} f_* & = e^{i\phi(y)} \left(\frac{\beta_{-1}(y)}{x} + \beta_0(y) + O(x) \right), \\ \beta_{-1} & \text{ is real-valued, } \beta_{-1} > 0, \quad \beta_{-1} \in C^1([a, b]), \\ f_* - \frac{e^{i\phi(y)} \beta_{-1}(y)}{x} & \in C^1(\mathcal{T} \cup \partial\mathcal{T}), \end{aligned} \quad (50)$$

$$\begin{aligned}
f_*^+ &= e^{-i(\phi(y)+\pi/2)} \left(\frac{\beta_{-1}^+(y)}{x} + \beta_0^+(y) + O(x) \right), \\
\beta_{-1}^+ &\text{ is real-valued, } \beta_{-1}^+ > 0, \beta_{-1}^+ \in C^1([a, b]), \\
f_*^+ - \frac{e^{-i(\phi(y)+\pi/2)}\beta_{-1}^+(y)}{x} &\in C^1(\mathcal{T} \cup \partial\mathcal{T}),
\end{aligned} \tag{51}$$

where $\partial_x O(x) = O(1)$, $\partial_y O(x) = O(x)$.

Note that from point of view of formal considerations of Subsection 3.3 such solutions f_* , f_*^+ always exist.

Theorem 3 *Let u_* satisfy (48), (49) and equations (37), (38) have some solutions f_* , f_*^+ satisfying (50), (51). Let ω_{f_*, f_*^+} be some potential defined according to (8). Then*

$$\tilde{u}_* = \mathcal{M}_{u_*, f_*, f_*^+} u_* = O(1) \text{ in } \mathcal{T}_{a, b, \delta} \cup \partial\mathcal{T}_{a, b, \delta} \tag{52}$$

for some $\delta \in]0, \varepsilon[$, where $\mathcal{M}_{u_*, f_*, f_*^+}$ is defined according to formulas (10)-(15) for $N = 1$.

Theorem 3 is proved in Section 6.

The point is that that the results of Theorems 2, 3 and the commutativity formula (39) reduce local studies of equations (1), (2) near the simplest contour pole singularity to the regular case.

4 Proofs of Proposition 1 and Theorems 1-2

4.1 Proof of Proposition 1

Let $\omega_{\tilde{\psi}, \tilde{\psi}^+}$ be given by (21). Then it is sufficient to show that

$$\partial_z \omega_{\tilde{\psi}, \tilde{\psi}^+} = \tilde{\psi} \tilde{\psi}^+, \quad \partial_{\bar{z}} \omega_{\tilde{\psi}, \tilde{\psi}^+} = -\overline{\tilde{\psi} \tilde{\psi}^+}. \tag{53}$$

Using the definitions of ω_{ψ, ψ^+} , ω_{ψ, f_1^+} , ω_{f_1, ψ^+} , ω_{f_1, f_1^+} , and $\tilde{\psi}$, $\tilde{\psi}^+$ we have:

$$\begin{aligned}
\partial_z \omega_{\tilde{\psi}, \tilde{\psi}^+} &= \partial_z \left(\frac{\omega_{\psi, \psi^+} \omega_{f_1, f_1^+} - \omega_{\psi, f_1^+} \omega_{f_1, \psi^+}}{\omega_{f_1, f_1^+}} + c_{\tilde{\psi}, \tilde{\psi}^+} \right) = \\
&= \partial_z \omega_{\psi, \psi^+} - \partial_z \omega_{\psi, f_1^+} \frac{\omega_{f_1, \psi^+}}{\omega_{f_1, f_1^+}} - \partial_z \omega_{f_1, \psi^+} \frac{\omega_{\psi, f_1^+}}{\omega_{f_1, f_1^+}} + \partial_z \omega_{f_1, f_1^+} \frac{\omega_{f_1, \psi^+}}{\omega_{f_1, f_1^+}} \frac{\omega_{\psi, f_1^+}}{\omega_{f_1, f_1^+}} =
\end{aligned}$$

$$\begin{aligned}
&= \psi\psi^+ - \psi f_1^+ \frac{\omega_{f_1, \psi^+}}{\omega_{f_1, f_1^+}} - f_1 \psi^+ \frac{\omega_{\psi, f_1^+}}{\omega_{f_1, f_1^+}} + \psi\psi^+ \frac{\omega_{f_1, \psi^+}}{\omega_{f_1, f_1^+}} \frac{\omega_{\psi, f_1^+}}{\omega_{f_1, f_1^+}} = \\
&= \left(\psi - f_1 \frac{\omega_{\psi, f_1^+}}{\omega_{f_1, f_1^+}} \right) \left(\psi^+ - f_1^+ \frac{\omega_{f_1, \psi^+}}{\omega_{f_1, f_1^+}} \right) = \tilde{\psi} \tilde{\psi}^+; \\
\partial_{\bar{z}} \omega_{\tilde{\psi}, \tilde{\psi}^+} &= \partial_{\bar{z}} \left(\frac{\omega_{\psi, \psi^+} \omega_{f_1, f_1^+} - \omega_{\psi, f_1^+} \omega_{f_1, \psi^+}}{\omega_{f_1, f_1^+}} + c_{\tilde{\psi}, \tilde{\psi}^+} \right) = \\
&= \partial_{\bar{z}} \omega_{\psi, \psi^+} - \partial_{\bar{z}} \omega_{\psi, f_1^+} \frac{\omega_{f_1, \psi^+}}{\omega_{f_1, f_1^+}} - \partial_{\bar{z}} \omega_{f_1, \psi^+} \frac{\omega_{\psi, f_1^+}}{\omega_{f_1, f_1^+}} + \partial_z \omega_{f_1, f_1^+} \frac{\omega_{f_1, \psi^+}}{\omega_{f_1, f_1^+}} \frac{\omega_{\psi, f_1^+}}{\omega_{f_1, f_1^+}} = \\
&= -\overline{\psi\psi^+} + \overline{\psi} f_1^+ \frac{\omega_{f_1, \psi^+}}{\omega_{f_1, f_1^+}} + f_1 \overline{\psi^+} \frac{\omega_{\psi, f_1^+}}{\omega_{f_1, f_1^+}} - \overline{\psi\psi^+} \frac{\omega_{f_1, \psi^+}}{\omega_{f_1, f_1^+}} \frac{\omega_{\psi, f_1^+}}{\omega_{f_1, f_1^+}} = \\
&= - \left(\overline{\psi} - \overline{f_1} \frac{\omega_{\psi, f_1^+}}{\omega_{f_1, f_1^+}} \right) \left(\overline{\psi^+} - \overline{f_1^+} \frac{\omega_{f_1, \psi^+}}{\omega_{f_1, f_1^+}} \right) = -\overline{\tilde{\psi} \tilde{\psi}^+}.
\end{aligned}$$

Thus, the proof of Proposition 1 is completed.

4.2 Proof of Theorem 1

Due to formula (21) for $\omega_{\tilde{f}_2, \tilde{f}_2^+}$ with $c_{\tilde{f}_2, \tilde{f}_2^+} = 0$ and due to the assumptions that $\omega_{f_1, f_1^+} \neq 0$, $\omega_{\tilde{f}_2, \tilde{f}_2^+} \neq 0$, we have:

$$\det \Omega = \omega_{f_1, f_1^+} \omega_{\tilde{f}_2, \tilde{f}_2^+} \neq 0, \quad (54)$$

where Ω is defined according to (9) for $N = 2$.

Due to the definition of \mathcal{M} according to (22)-(24), using (21) for $\omega_{\tilde{\psi}, \tilde{\psi}^+} = \omega_{\tilde{f}_2, \tilde{f}_2^+}$, $\omega_{\tilde{\psi}, \tilde{f}_2^+}$ with $c_{\tilde{\psi}, \tilde{\psi}^+} = 0$ and using (54), we obtain:

$$\begin{aligned}
\tilde{\psi} &= \tilde{\psi} - \tilde{f}_2 \frac{\omega_{\tilde{\psi}, \tilde{f}_2^+}}{\omega_{\tilde{f}_2, \tilde{f}_2^+}} = \psi - f_1 \frac{\omega_{\psi, f_1^+}}{\omega_{f_1, f_1^+}} - \\
&- \left(f_2 - f_1 \frac{\omega_{f_2, f_1^+}}{\omega_{f_1, f_1^+}} \right) \left(\frac{\omega_{\psi, f_2^+} \omega_{f_1, f_1^+} - \omega_{\psi, f_1^+} \omega_{f_1, f_2^+}}{\omega_{f_1, f_1^+}} \right) \left(\frac{\omega_{f_2, f_2^+} \omega_{f_1, f_1^+} - \omega_{f_2, f_1^+} \omega_{f_1, f_2^+}}{\omega_{f_1, f_1^+}} \right)^{-1} = \\
&= \psi - f_1 \frac{1}{\omega_{f_1, f_1^+} \omega_{f_2, f_2^+} - \omega_{f_2, f_1^+} \omega_{f_1, f_2^+}} \times
\end{aligned}$$

$$\begin{aligned}
& \times \left(\omega_{\psi, f_1^+ \omega_{f_2, f_2^+}} - \frac{\omega_{\psi, f_1^+ \omega_{f_2, f_1^+} \omega_{f_1, f_2^+}}}{\omega_{f_1, f_1^+}} - \omega_{\psi, f_2^+ \omega_{f_2, f_1^+}} + \frac{\omega_{f_2, f_1^+ \omega_{\psi, f_1^+} \omega_{f_1, f_2^+}}}{\omega_{f_1, f_1^+}} \right) - \\
& - f_2 \frac{1}{\omega_{f_1, f_1^+ \omega_{f_2, f_2^+}} - \omega_{f_2, f_1^+ \omega_{f_1, f_2^+}}} \left(\omega_{\psi, f_2^+ \omega_{f_1, f_1^+}} - \omega_{\psi, f_1^+ \omega_{f_1, f_2^+}} \right) = \\
& = \psi - [f_1 \quad f_2] \frac{1}{\omega_{f_1, f_1^+ \omega_{f_2, f_2^+}} - \omega_{f_2, f_1^+ \omega_{f_1, f_2^+}}} \begin{bmatrix} \omega_{\psi, f_1^+ \omega_{f_2, f_2^+}} - \omega_{\psi, f_2^+ \omega_{f_2, f_1^+}} \\ \omega_{\psi, f_2^+ \omega_{f_1, f_1^+}} - \omega_{\psi, f_1^+ \omega_{f_1, f_2^+}} \end{bmatrix}. \tag{55}
\end{aligned}$$

Taking into account that

$$\begin{aligned}
\Omega &= \begin{bmatrix} \omega_{f_1, f_1^+} & \omega_{f_2, f_1^+} \\ \omega_{f_1, f_2^+} & \omega_{f_2, f_2^+} \end{bmatrix}, \quad \Omega^{-1} = \frac{1}{\omega_{f_1, f_1^+ \omega_{f_2, f_2^+}} - \omega_{f_2, f_1^+ \omega_{f_1, f_2^+}}} \begin{bmatrix} \omega_{f_2, f_2^+} & -\omega_{f_2, f_1^+} \\ -\omega_{f_1, f_2^+} & \omega_{f_1, f_1^+} \end{bmatrix}, \\
\Omega^{-1} \begin{bmatrix} \omega_{\psi, f_1^+} \\ \omega_{\psi, f_2^+} \end{bmatrix} &= \frac{1}{\omega_{f_1, f_1^+ \omega_{f_2, f_2^+}} - \omega_{f_2, f_1^+ \omega_{f_1, f_2^+}}} \begin{bmatrix} \omega_{\psi, f_1^+ \omega_{f_2, f_2^+}} - \omega_{\psi, f_2^+ \omega_{f_2, f_1^+}} \\ \omega_{\psi, f_2^+ \omega_{f_1, f_1^+}} - \omega_{\psi, f_1^+ \omega_{f_1, f_2^+}} \end{bmatrix},
\end{aligned}$$

formula (55) can be rewritten as:

$$\tilde{\psi} = \psi - [f_1 \quad f_2] \Omega^{-1} \begin{bmatrix} \omega_{\psi, f_1^+} \\ \omega_{\psi, f_2^+} \end{bmatrix}. \tag{56}$$

One can see that (56) coincides with formula (14) for $N = 2$.

The computations for $\tilde{\psi}^+$ are similar.

In additions, due to the definition of \mathcal{M} according to (22)-(24), using (21) for $\omega_{\tilde{f}_2, \tilde{f}_2^+}$ with $c_{\tilde{f}_2, \tilde{f}_2^+} = 0$ and using (54), we obtain:

$$\begin{aligned}
\tilde{u} &= \tilde{u} + \frac{\tilde{f}_2 \overline{\tilde{f}_2^+}}{\omega_{\tilde{f}_2, \tilde{f}_2^+}} = u + \frac{f_1 \overline{f_1^+}}{\omega_{f_1, f_1^+}} + \frac{\tilde{f}_2 \overline{\tilde{f}_2^+}}{\omega_{\tilde{f}_2, \tilde{f}_2^+}} = \\
&= u + \frac{f_1 \overline{f_1^+}}{\omega_{f_1, f_1^+}} + \left(f_2 - f_1 \frac{\omega_{f_2, f_1^+}}{\omega_{f_1, f_1^+}} \right) \left(\frac{\overline{f_2^+}}{f_2^+} - \overline{f_1^+} \frac{\omega_{f_1, f_2^+}}{\omega_{f_1, f_1^+}} \right) \frac{\omega_{f_1, f_1^+}}{\omega_{f_1, f_1^+ \omega_{f_2, f_2^+}} - \omega_{f_2, f_1^+ \omega_{f_1, f_2^+}}} = \\
&= u + \frac{f_1 \overline{f_1^+} \omega_{f_2, f_2^+} - f_2 \overline{f_1^+} \omega_{f_1, f_2^+} - f_1 \overline{f_2^+} \omega_{f_2, f_1^+} + f_2 \overline{f_2^+} \omega_{f_1, f_1^+}}{\omega_{f_1, f_1^+ \omega_{f_2, f_2^+}} - \omega_{f_2, f_1^+ \omega_{f_1, f_2^+}}} = \\
&= u + \frac{1}{\omega_{f_1, f_1^+ \omega_{f_2, f_2^+}} - \omega_{f_2, f_1^+ \omega_{f_1, f_2^+}}} [f_1 \quad f_2] \begin{bmatrix} \omega_{f_2, f_2^+} & -\omega_{f_2, f_1^+} \\ -\omega_{f_1, f_2^+} & \omega_{f_1, f_1^+} \end{bmatrix} \begin{bmatrix} \overline{f_1^+} \\ \overline{f_2^+} \end{bmatrix}. \tag{57}
\end{aligned}$$

One can see that (57) coincides with formula (13) for $N = 2$.

This completes the proof of Theorem 1.

4.3 Proof of Theorem 2

First, we check that $\omega_{\tilde{f}_2, \tilde{\psi}^+}$, $\omega_{\tilde{\psi}, \tilde{f}_2^+}$, $\omega_{\tilde{f}_2, \tilde{f}_2^+}$ defined in (31) are the potentials for the pairs $\{\hat{f}, \tilde{\psi}^+\}$, $\{\tilde{\psi}, \hat{f}^+\}$, $\{\hat{f}, \hat{f}^+\}$:

$$\begin{aligned} \partial_z \omega_{\tilde{\psi}, \tilde{f}_2^+} &= -i \partial_z \frac{\omega_{\psi, f_1^+}}{\omega_{f_1, f_1^+}} = -i \frac{\psi f_1^+}{\omega_{f_1, f_1^+}} + i \frac{\omega_{\psi, f_1^+}}{\omega_{f_1, f_1^+}^2} f_1 f_1^+ = \\ &= \left(\psi - f_1 \frac{\omega_{\psi, f_1^+}}{\omega_{f_1, f_1^+}} \right) \left(\frac{-i f_1^+}{\omega_{f_1, f_1^+}} \right) = \tilde{\psi} \hat{f}^+, \\ \partial_{\bar{z}} \omega_{\tilde{\psi}, \tilde{f}_2^+} &= -i \partial_{\bar{z}} \frac{\omega_{\psi, f_1^+}}{\omega_{f_1, f_1^+}} = i \frac{\overline{\psi} f_1^+}{\omega_{f_1, f_1^+}} - i \frac{\omega_{\psi, f_1^+}}{\omega_{f_1, f_1^+}^2} \overline{f_1 f_1^+} = \\ &= - \left(\overline{\psi} - \overline{f_1} \frac{\omega_{\psi, f_1^+}}{\omega_{f_1, f_1^+}} \right) \left(\frac{-i \overline{f_1^+}}{\omega_{f_1, f_1^+}} \right) = -\overline{\tilde{\psi}} \overline{\hat{f}^+}; \end{aligned} \quad (58)$$

the calculations for $\omega_{\tilde{f}_2, \tilde{\psi}^+}$ are analogous to the calculations for $\omega_{\tilde{\psi}, \tilde{f}_2^+}$;

$$\begin{aligned} \partial_z \omega_{\tilde{f}_2, \tilde{f}_2^+} &= \partial_z \frac{1}{\omega_{f_1, f_1^+}} = - \frac{\partial_z \omega_{f_1, f_1^+}}{\omega_{f_1, f_1^+}^2} = - \frac{f_1 f_1^+}{\omega_{f_1, f_1^+}^2} = \hat{f} \hat{f}^+, \\ \partial_{\bar{z}} \omega_{\tilde{f}_2, \tilde{f}_2^+} &= \partial_{\bar{z}} \frac{1}{\omega_{f_1, f_1^+}} = - \frac{\partial_{\bar{z}} \omega_{f_1, f_1^+}}{\omega_{f_1, f_1^+}^2} = \frac{\overline{f_1} \overline{f_1^+}}{\omega_{f_1, f_1^+}^2} = -\overline{\hat{f}} \overline{\hat{f}^+}. \end{aligned} \quad (59)$$

Here, we used also that all potentials ω_{ψ, ψ^+} are pure imaginary.

Second, we calculate the transform $\mathcal{M}_2 \circ \mathcal{M}_1$:

$$\begin{aligned} \tilde{\psi} &= \tilde{\psi} - \hat{f} \frac{\omega_{\tilde{\psi}, \tilde{f}_2^+}}{\omega_{\tilde{f}_2, \tilde{f}_2^+}} = \psi - f_1 \frac{\omega_{\psi, f_1^+}}{\omega_{f_1, f_1^+}} - \left(-i \frac{f_1}{\omega_{f_1, f_1^+}} \right) \frac{-i \frac{\omega_{\psi, f_1^+}}{\omega_{f_1, f_1^+}}}{\frac{1}{\omega_{f_1, f_1^+}}} = \\ &= \psi - f_1 \frac{\omega_{\psi, f_1^+}}{\omega_{f_1, f_1^+}} + f_1 \frac{\omega_{\psi, f_1^+}}{\omega_{f_1, f_1^+}} = \psi; \end{aligned}$$

the calculations for $\tilde{\psi}^+$ are analogous to the calculations for $\tilde{\psi}$;

$$\tilde{u} = \tilde{u} + \frac{\hat{f} \hat{f}^+}{\omega_{\tilde{f}_2, \tilde{f}_2^+}} = u + \frac{f_1 \overline{f_1^+}}{\omega_{f_1, f_1^+}} + \frac{\hat{f} \hat{f}^+}{\omega_{\tilde{f}_2, \tilde{f}_2^+}} = u + \frac{f_1 \overline{f_1^+}}{\omega_{f_1, f_1^+}} + \frac{-i \frac{f_1}{\omega_{f_1, f_1^+}} \left[-i \frac{f_1^+}{\omega_{f_1, f_1^+}} \right]}{\frac{1}{\omega_{f_1, f_1^+}}} =$$

$$= u + \frac{f_1 \overline{f_1^+}}{\omega_{f_1, f_1^+}} - \frac{f_1 \overline{f_1^+}}{\omega_{f_1, f_1^+}} = u. \quad (60)$$

This completes the proof of Theorem 2.

5 Proof of meromorphic class conditions

We consider equation (37) and formulas (43), (44) for u_* , ψ_* , where $n = 1$, $r_{-1} = -1/2$, $n' = 1$. In this case formulas (43), (44) can be written as:

$$u_* = e^{2i\phi(y)} \left(-\frac{1}{2x} + r_0(y) + r_1(y)x + r_2(y)x^2 + \dots \right), \quad (61)$$

$$\psi_* = e^{i\phi(y)} \left(\frac{\beta_{-1}(y)}{x} + \beta_0(y) + \beta_1(y)x + \beta_2(y)x^2 + \dots \right). \quad (62)$$

We substitute (61), (62) into (37), and we use that

$$2\partial_{\bar{\tau}}\psi_* = e^{i\phi(y)} \left(-\frac{\beta_{-1}(y)}{x^2} + \frac{i\beta'_{-1}(y) - \phi'(y)\beta_{-1}(y)}{x} + \sum_{k=0}^{+\infty} [i\beta'_k(y) - \phi'(y)\beta_k(y) + (k+1)\beta_{k+1}(y)] x^k \right), \quad (63)$$

$$2u_*\psi_* = e^{i\phi(y)} \left(-\frac{\overline{\beta_{-1}(y)}}{x^2} + \frac{-\overline{\beta_0(y)} + 2r_0(y)\overline{\beta_{-1}(y)}}{x} + \sum_{k=0}^{+\infty} \left[-\overline{\beta_{k+1}(y)} + 2r_{k+1}(y)\overline{\beta_{-1}(y)} + 2\sum_{l=0}^k r_l(y)\overline{\beta_{k-l}(y)} \right] x^k \right). \quad (64)$$

From this point and till the end of the proof ' denotes ∂_y .

Collecting the terms at x^k , $k = -2, -1, 0, 1, 2, \dots$, we obtain:

$$\overline{\beta_{-1}(y)} = \beta_{-1}(y) \quad \text{for } k = -2, \quad (65)$$

$$\overline{\beta_0(y)} = -i\beta'_{-1}(y) + \phi'(y)\beta_{-1}(y) + 2r_0(y)\overline{\beta_{-1}(y)} \quad \text{for } k = -1, \quad (66)$$

$$\begin{aligned} \overline{\beta_1(y)} + \beta_1(y) = & -i\beta'_0(y) + \phi'(y)\beta_0(y) + 2r_1(y)\overline{\beta_{-1}(y)} + \\ & + 2r_0(y)\overline{\beta_0(y)} \quad \text{for } k = 0, \end{aligned} \quad (67)$$

$$\begin{aligned} \overline{\beta_{k+1}(y)} + (k+1)\beta_{k+1}(y) = & -i\beta'_k(y) + \phi'(y)\beta_k(y) + \\ & + 2r_{k+1}(y)\overline{\beta_{-1}(y)} + 2\sum_{l=0}^k r_l(y)\overline{\beta_{k-l}(y)} \quad \text{for } k \geq 1. \end{aligned} \quad (68)$$

One can see that: relation (65) coincides with (47); relation (66) can be considered as a formula for finding β_0 ; relations (68) can be considered as recursion relations for finding β_j , $j \geq 2$. In addition, the real part of (67) can be considered as a formula for finding $\text{Re } \beta_1$, whereas the imaginary part of (67) can be rewritten as

$$\text{Im} \left[-i\beta'_0(y) + \phi'(y)\beta_0(y) + 2r_1(y)\overline{\beta_{-1}(y)} + 2r_0(y)\overline{\beta_0(y)} \right] = 0. \quad (69)$$

Actually, relations (65), (66), (69) yield the solvability constraints on ϕ , r_0 , r_1 . Substituting (65), (66) into (69) we obtain:

$$\begin{aligned} \text{Im} \left[i\phi'(y)\beta'_{-1}(y) + ((\phi'(y))^2 + 2\phi'(y)\overline{r_0'(y)})\beta_{-1}(y) + 2r_1(y)\beta_{-1}(y) - \right. \\ \left. - 2ir_0(y)\beta'_{-1}(y) + (2r_0(y)\phi'(y) + 4(r_0(y))^2)\beta_{-1}(y) + \beta''_{-1}(y) - \right. \\ \left. - i(\phi'(y) + 2\overline{r_0'(y)})\beta'_{-1}(y) - i(\phi''(y) + 2\overline{r_0''(y)})\beta_{-1}(y) \right] = 0. \end{aligned} \quad (70)$$

In turn, (70) can be rewritten as:

$$\begin{aligned} \text{Im} \left[\beta''_{-1}(y) - 2i[r_0(y) + \overline{r_0'(y)}]\beta'_{-1}(y) + [(\phi'(y))^2 + \right. \\ \left. + 2\phi'(y)(\overline{r_0'(y)} + r_0(y)) + 4(r_0(y))^2 + 2r_1(y) - i\phi''(y) - 2i\overline{r_0''(y)}]\beta_{-1}(y) \right] = 0. \end{aligned} \quad (71)$$

In addition, taking into account that ϕ , β_{-1} are real-valued, we simplify (71) as follows:

$$\text{Im} \left[-2i[r_0(y) + \overline{r_0'(y)}]\beta'_{-1}(y) + [4(r_0(y))^2 + 2r_1(y) - i\phi''(y) - 2i\overline{r_0''(y)}]\beta_{-1}(y) \right] = 0. \quad (72)$$

One can see that (72) is fulfilled for all sufficiently regular real-valued β_{-1} if and only if (45), (46) are fulfilled.

Finally, using (65)-(68) under conditions (45), (46), one can see that all ψ_* of (62) satisfying (37) with u_* of 61) are parametrised by β_{-1} and $\text{Im } \beta_1$.

This completes the proof.

6 Proof of Theorem 3

Substituting (48), (50), (51) into (37), (38) we obtain:

$$\beta_0(y) = i(\beta_{-1}(y))' + (\phi'(y) - 2r_0(y))\beta_{-1}(y), \quad y \in [a, b], \quad (73)$$

$$\beta_0^+(y) = i(\beta_{-1}^+(y))' + (-\phi'(y) + 2r_0(y))\beta_{-1}^+(y), \quad y \in [a, b]. \quad (74)$$

As in Section 5 we assume that ' denotes ∂_y .

Using (50), (51) and (73), (74) we obtain:

$$\begin{aligned} f_* f_*^+ &= e^{-i\pi/2} \left(\frac{\beta_{-1}(y)\beta_{-1}^+(y)}{x^2} + \frac{\beta_0(y)\beta_{-1}^+(y) + \beta_{-1}(y)\beta_0^+(y)}{x} + O(1) \right) = \\ &= -i \frac{\beta_{-1}(y)\beta_{-1}^+(y)}{x^2} + \frac{1}{x} \left(\beta_{-1}(y)\beta_{-1}^+(y) \right)' + O(1) \quad \text{in } \mathcal{T} \cup \partial\mathcal{T}. \end{aligned} \quad (75)$$

Next, equation (8) can be rewritten as:

$$\partial_x \omega_{\psi, \psi^+} = 2i \text{Im} \left(\psi \psi^+ \right), \quad \partial_y \omega_{\psi, \psi^+} = 2i \text{Re} \left(\psi \psi^+ \right). \quad (76)$$

As a corollary of (75), (76) and the property that β_1, β_1^+ are real-valued, we have:

$$\partial_x \omega_{f_*, f_*^+} = \frac{-2i\beta_{-1}(y)\beta_{-1}^+(y)}{x^2} + O(1), \quad \partial_y \omega_{f_*, f_*^+} = \frac{2i(\beta_{-1}(y)\beta_{-1}^+(y))'}{x} + O(1); \quad (77)$$

$$\omega_{f_*, f_*^+} = \frac{2i}{x} \beta_{-1}(y)\beta_{-1}^+(y) + O(1) \quad \text{in } \mathcal{T} \cup \partial\mathcal{T}. \quad (78)$$

Remark 2 Note that

$$\text{res} \Big|_{x=0} \partial_x (\omega_{f_*, f_*^+}) = 0.$$

for any fixed $y = y_0$.

Using also the strict positivity of $\beta_{-1}\beta_{-1}^+$ we obtain:

$$\frac{1}{\omega_{f_*, f_*^+}} = -\frac{ix}{2\beta_{-1}(y)\beta_{-1}^+(y)} + O(x^2) \quad \text{in } \mathcal{T}_{a,b,\delta} \cup \partial\mathcal{T}_{a,b,\delta}, \quad (79)$$

for some $\delta \in]0, \varepsilon[$.

Note also that

$$f_* \overline{f_*^+} = ie^{2i\phi(y)} \frac{\beta_{-1}(y)\beta_{-1}^+(y)}{x^2} + O\left(\frac{1}{x}\right) \quad \text{in } \mathcal{T} \cup \partial\mathcal{T}. \quad (80)$$

Finally, due to (20), (48), (79), (80):

$$\begin{aligned} \tilde{u}_* &= u_* + \frac{f_* \overline{f_*^+}}{\omega_{f_*, f_*^+}} = e^{2i\phi(y)} \left(-\frac{1}{2x} + r_0(y) + O(x) \right) + e^{2i\phi(y)} \frac{1}{2x} + O(1) = \\ &= O(1) \quad \text{in } \mathcal{T}_{a,b,\delta} \cup \partial\mathcal{T}_{a,b,\delta}. \end{aligned} \quad (81)$$

Theorem 3 is proved.

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