The global nonlinear stability of Minkowski space for self-gravitating massive fields. The Wave-Klein-Gordon Model

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ABSTRACT. The Hyperboloidal Foliation Method (introduced by the authors in 2014) is extended here and applied to the Einstein equations of general relativity. Specifically, we establish the nonlinear stability of Minkowski spacetime for self-gravitating massive scalar fields, while existing methods only apply to massless scalar fields. First of all, by analyzing the structure of the Einstein equations in wave coordinates, we exhibit a nonlinear wave-Klein-Gordon model defined on a curved background, which is the focus of the present paper. For this model, we prove here the existence of global-in-time solutions to the Cauchy problem, when the initial data have sufficiently small Sobolev norms. A major difficulty comes from the fact that the class of conformal Killing fields of Minkowski space is significantly reduced in presence of a massive scalar field, since the scaling vector field is not conformal Killing for the Klein-Gordon operator. Our method relies on the foliation (of the interior of the light cone) of Minkowski spacetime by hyperboloidal hypersurfaces and uses Lorentz-invariant energy norms. We introduce a frame of vector fields adapted to the hyperboloidal foliation and we establish several key properties: Sobolev and Hardy-type inequalities on hyperboloids, as well as sup-norm estimates which correspond to the sharp time decay for the wave and the Klein-Gordon equations. These estimates allow us to control interaction terms associated with the curved geometry and the massive field, by distinguishing between two levels of regularity and energy growth and by a successive use of our key estimates in order to close a bootstrap argument.

CONTENTS

1. INTRODUCTION

1.1. **The global existence problem.** In this paper and its companion [23], we study the global-in-time existence problem for small amplitude solutions to nonlinear wave equations, with a two-fold objective:

- First, we provide a significant extension of the Hyperboloidal Foliation Method, recently proposed by the authors [22]. This method is based on a foliation of the interior of the future light cone by hyperboloidal hypersurfaces and on Sobolev and Hardy inequalities adapted to this foliation. This method takes its root in work by Klainerman [19] and, later on, Hörmander [13] concerning the standard Klein-Gordon equation. In comparison

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to our earlier theory in [22], we are now able to encompass a much broader class of coupled wave-Klein-Gordon systems.

- Our second objective is to apply this method to the Einstein equations of general relativity and arrive at a new approach for proving the nonlinear stability of Minkowski spacetime. Our method covers self-gravitating *massive* scalar fields (as will be presented in full details in [23]), while earlier works were restricted to vacuum spacetimes or to spacetimes with massless scalar fields; cf. Christodoulou and Klainerman [7], and Lindblad and Rodnianski [26, 27], as well as Bieri and Zipser [4].

The problem of the global dynamics of self-gravitating massive fields had remained open until now. The presence of a mass term poses a major challenge in order to establish a global existence theory for the Einstein equations (and construct future geodesically complete spacetimes). Namely, the class of conformal Killing fields of Minkowski spacetime is reduced in presence of a massive scalar field, since the so-called scaling vector field is no longer conformal Killing and, therefore, cannot be used in implementing Klainerman's vector field method [18, 19].

In suitably chosen coordinates, the Einstein equations take the form of a coupled system of nonlinear wave-Klein-Gordon equations. More precisely, as in [27], we introduce wave coordinates, also called harmonic or De Donder gauge [1], which allows one to exhibit the (quasi-null, see below) structure of the Einstein equations. The Hyperboloidal Foliation Method [22] was introduced precisely to handle such systems. Yet, due to the presence of metric-related terms in the system under consideration, an important generalization is required before we can tackle the Einstein equations. Proposing such a generalization is our main purpose in the present paper.

By imposing asymptotically flat initial data on a spacelike hypersurface with sufficiently small ADM mass, one can first solve the Cauchy problem for the Einstein equations within a neighborhood of this hypersurface (see [22] for a sketch of the argument[1]) and, next, formulate the Cauchy problem when the initial data are posed on a hyperboloidal hyperspace or, alternatively, on a hyperboloid for the flat Minkowski metric after introducing suitable coordinates. In fact, the hyperboloidal Cauchy problem is, both, geometrically and physically natural. More precisely, let us consider Minkowski spacetime in standard Cartesian coordinates (t, x^1, x^2, x^3) and observe that points on a hypersurface of constant time t cannot be connected by a timelike curve, while points on a hyperboloid can be connected by such curves. Hence, hyperboloidal initial data can be "physically prepared", while data on standard flat hypersurfaces cannot. An alternative standpoint would be to pose the Cauchy problem on a light cone, but while it is physically appealing and the Cauchy problem on a light cone has not been proven to be convenient for global analysis.

We emphasize that hyperboloidal foliations were used by Friedrich [11, 12] in order to establish the stability of Minkowski space in the future of a hyperboloidal hypersurface. Hyperboloidal foliations have also been found to be very efficient in numerical computations [10, 29, 30, 34].

As was demonstrated in [22] for a rather general class of nonlinear wave equations, analyzing the global existence problem is quite natural in the hyperboloidal foliation of Minkowski spacetime and, importantly, lead to *uniform bounds* on the energy of the solutions. Before proceeding with further details, let us summarize the main features of the method we propose:

- **Lorentz invariance.** We rely on the foliation of Minkowski space by hyperboloids (defined as the level sets of constant Lorentzian distance from some origin), so that the fundamental energy of the wave-Klein-Gordon equations remains invariant under Lorentz transformations of Minkowski spacetime. Observe that in our construction, all the hyperboloids are asymptotic to the same limiting cone and approach the same sphere at infinity. (In particular, no energy can escape through null infinity.)
- **Smaller set of Killing fields.** We avoid using the scaling vector field $S := r\partial_r + t\partial_t$, which is the key in order to handle Klein-Gordon equations and cover the Einstein-matter system when the evolution equation for the matter is *not conformally invariant*.

[1]The time of existence of the solution can be made arbitrarily large for compactly supported initial data with sufficiently small norm, so that this neighborhood does contain a hyperboloidal hypersurface.

- **Sharp rate of time decay.** In order to control source-terms related to the curved geometry, we establish sharp pointwise bounds for solutions to wave equations and Klein-Gordon equations with source-terms.

In the rest of this introduction, we explain how to derive, from the Einstein equations, a *model problem* which will be our main focus in the present paper.

1.2. **Einstein equations for massive scalar fields.** We thus consider the Einstein equations for an unknown spacetime (M, g):

$$R_{\alpha\beta} - \frac{R}{2} g_{\alpha\beta} = 8\pi\, T_{\alpha\beta}, \tag{1.1}$$

where $R_{\alpha\beta}$ denotes[2] the Ricci curvature tensor and $R = g^{\alpha\beta} R_{\alpha\beta}$ denotes the scalar curvature. The matter is taken to be a massive scalar field with potential $V = V(\phi)$ and stress-energy tensor

$$T_{\alpha\beta} := \nabla_\alpha \phi \nabla_\beta \phi - \Big(\frac{1}{2} \nabla_\gamma \phi \nabla^\gamma \phi + V(\phi) \Big) g_{\alpha\beta} \tag{1.2}$$

and, specifically,

$$V(\phi) := \frac{c^2}{2} \phi^2, \tag{1.3}$$

where $c^2 > 0$ represents the mass of the scalar field. By applying ∇^α to (1.2) and using the Bianchi identity

$$\nabla^\alpha \big(R_{\alpha\beta} - (R/2) g_{\alpha\beta} \big) = 0,$$

we easily check that the Einstein–scalar field system implies

$$R_{\alpha\beta} = 8\pi \big(\nabla_\alpha \phi \nabla_\beta \phi + V(\phi)\, g_{\alpha\beta} \big), \tag{1.4a}$$

$$\Box_g \phi = V'(\phi) = c^2 \phi. \tag{1.4b}$$

The Cauchy problem for the Einstein-scalar field equations is formulated as follows [5]. An initial data set consists of a Riemannian three-manifold $(\overline{M}, \overline{g})$, a symmetric two-tensor K defined on $\overline{M}$, and two scalar fields $(\overline{\phi}_0, \overline{\phi}_1)$ defined on $\overline{M}$. We then seek for a $(3 + 1)$-dimensional Lorentzian manifold (M, g) satisfying the following properties:

- There exists an embedding $i : \overline{M} \to M$ such that the induced metric $i^*(g)$ coincides with $\overline{g}$, while the second fundamental form of $i(\overline{M}) \subset M$ coincides with the prescribed two-tensor K.
- The restriction of ϕ and $\mathcal{L}_\nu \phi$ to $i(\overline{M})$ coincides with the data ϕ_0 and ϕ_1 respectively, where ν denotes the (future-oriented) unit normal to $i(\overline{M}) \subset M$.
- Moreover, the manifold (M, g) satisfies the Einstein equations (1.4).

More precisely, one seeks for a *globally hyperbolic development* of the given initial data, that is, a Lorentzian manifold such that every time-like geodesic extends toward the past direction in order to meet the initial hypersurface $\overline{M}$. Furthermore, a notion of *maximal development* was defined by Choquet-Bruhat and Geroch [6, 5] and such a development was shown to exist for a large class of matter models. The maximal development need not be future geodesically complete, and a main challenge in the field of mathematical general relativity is the construction of classes of future geodesically complete spacetimes.

Furthermore, it should be emphasized that, in order to fulfill the equations (1.4), the initial data set $(\overline{M}, \overline{g}, K)$ cannot be arbitrary and must satisfy Einstein's constraint equations:

$$\begin{aligned} &\overline{R} + K_{ij}\, K^{ij} - (K^i_i)^2 = 8\pi T_{00}, \\ &\overline{\nabla}^i K_{ij} - \overline{\nabla}_j K^l_l = 8\pi T_{0j}, \end{aligned} \tag{1.5}$$

where $\overline{R}$ is the scalar curvature of the metric $\overline{g}$ and $\overline{\nabla}$ denotes its Levi-Civita connection, and the terms T_{00} and T_{0i} are determined from the data $\overline{\phi}_0$ and $\overline{\phi}_1$.

[2]Throughout, Greek indices α, β, γ take values $0, 1, 2, 3$ and Einstein convention is used.

Minkowski spacetime provides one with a trivial solution to the Einstein equations, which satisfies the Cauchy problem associated with the initial data $(\overline{M}, \overline{g}, K, \overline{\phi}_0, \overline{\phi}_1)$ when $\overline{M} = \mathbb{R}^3$ is endowed with the standard Euclidian metric and $K \equiv 0$, while the matter terms vanish identically $\overline{\phi}_0 = \overline{\phi}_1 \equiv 0$. The question we address in the present paper is whether this solution is dynamically stable under small perturbations of the initial data. More precisely, given an initial data set $(\overline{M}, \overline{g}, K, \overline{\phi}_0, \overline{\phi}_1)$ such that $\overline{M}$ is diffeomorphic to $\mathbb{R}^3$, $\overline{g}$ is close to the flat metric and $K, \overline{\phi}_0, \overline{\phi}_1$ are sufficiently small, does the associated solution (M, g) to the Einstein-massive scalar field system remain close to the flat Minkowski spacetime $\mathbb{R}^{1+3}$?

Clearly, this nonlinear stability problem is of fundamental importance in physics. It is expected that Minkowski spacetime is the ground state state of the theory with the lowest possible energy. As far as massless scalar fields are concerned, the nonlinear stability of Minkowski spacetime was indeed established in Christodoulou and Klainerman's pioneering work [7]. In the present work (including [23]), we solve this question for *massive* scalar fields.

1.3. **Einstein-scalar field equations in wave coordinates.** Our first task is to express the field equations (1.4) in a well-chosen coordinate system and then derive our wave-Klein-Gordon model problem. We follow [5, 26] and work in wave coordinates satisfying, by definition,

$$\Box_g x^\alpha = 0. \tag{1.6}$$

We postpone to [23] the details of the derivation and directly write the formulation of the Einstein-massive scalar field equations in wave coordinates:

$$\widetilde{\Box}_g g_{\alpha\beta} = Q_{\alpha\beta}(g; \partial g, \partial g) + P_{\alpha\beta}(g; \partial g, \partial g) - 16\pi\big(\partial_\alpha\phi\partial_\beta\phi + V(\phi) g_{\alpha\beta}\big), \tag{1.7a}$$

$$\widetilde{\Box}_g \phi - V'(\phi) = 0, \tag{1.7b}$$

where $\widetilde{\Box}_g := g^{\alpha\beta}\partial_\alpha\partial_\beta$ is referred to as the (reduced) wave operator in curved space. In (1.7), we distinguish between several types of nonlinearity:

- **Null terms.** The quadratic terms

$$\begin{aligned} Q_{\alpha\beta} :=& g^{\lambda\lambda'} g^{\delta\delta'} \partial_\delta g_{\alpha\lambda'} \partial_{\delta'} g_{\beta\lambda} - g^{\lambda\lambda'} g^{\delta\delta'} \big(\partial_\delta g_{\alpha\lambda'} \partial_\lambda g_{\beta\delta'} - \partial_\delta g_{\beta\delta'} \partial_\lambda g_{\alpha\lambda'}\big) \\ &+ g^{\lambda\lambda'} g^{\delta\delta'} \big(\partial_\alpha g_{\lambda'\delta'} \partial_\delta g_{\lambda\beta} - \partial_\alpha g_{\lambda\beta} \partial_\delta g_{\lambda'\delta'}\big) + \frac{1}{2} g^{\lambda\lambda'} g^{\delta\delta'} \big(\partial_\alpha g_{\lambda\beta} \partial_{\lambda'} g_{\delta\delta'} - \partial_\alpha g_{\delta\delta'} \partial_{\lambda'} g_{\lambda\beta}\big) \\ &+ g^{\lambda\lambda'} g^{\delta\delta'} \big(\partial_\beta g_{\lambda'\delta'} \partial_\delta g_{\lambda\alpha} - \partial_\beta g_{\lambda\alpha} \partial_\delta g_{\lambda'\delta'}\big) + \frac{1}{2} g^{\lambda\lambda'} g^{\delta\delta'} \big(\partial_\beta g_{\lambda\alpha} \partial_{\lambda'} g_{\delta\delta'} - \partial_\beta g_{\delta\delta'} \partial_{\lambda'} g_{\lambda\alpha}\big) \end{aligned} \tag{1.8}$$

  are standard null forms with cubic corrections. Their treatment in a global existence proof is a now classical matter and, in particular, are already dealt with by standard methods.
- **Quasi-null terms.** The quadratic terms

$$P_{\alpha\beta} := -\frac{1}{2} g^{\lambda\lambda'} g^{\delta\delta'} \partial_\alpha g_{\delta\lambda'} \partial_\beta g_{\lambda\delta'} + \frac{1}{4} g^{\delta\delta'} g^{\lambda\lambda'} \partial_\beta g_{\delta\delta'} \partial_\alpha g_{\lambda\lambda'} \tag{1.9}$$

  are referred to as "weak null" terms in [26], but we prefer to propose the new terminology *"quasi-null terms"*. As first noted in [26], quasi-null terms are found to be analogous to standard null terms, *provided* the tensorial structure of the Einstein equations and the wave coordinate condition are carefully taken into account.
- **Curved metric terms.** Setting now

$$h^{\alpha\beta} := g^{\alpha\beta} - m^{\alpha\beta}, \qquad h_{\alpha\beta} := m_{\alpha\beta} - g_{\alpha\beta} \tag{1.10}$$

  and considering the term $\widetilde{\Box}_g g_{\alpha\beta}$, we see that we must also treat the quasi-linear terms

$$h^{\alpha'\beta'} \partial_{\alpha'} \partial_{\beta'} h_{\alpha\beta}, \qquad h^{\alpha'\beta'} \partial_{\alpha'} \partial_{\beta'} \phi.$$

  We will deal with these metric-related terms by the following two approaches:
  - First, thanks to the wave coordinate condition, we can assume that $h^{\alpha\beta}$ behaves essentially like a null quadratic form and consider, therefore, that $h^{\alpha\beta}$ is null. More precisely, the first term

$$h^{\alpha'\beta'} \partial_{\alpha'} \partial_{\beta'} h_{\alpha\beta}, \qquad h^{\alpha\beta} \text{ being a null form}$$

can be treated by our arguments in [22].
– The second quasi-linear term $h^{\alpha'\beta'}\partial_{\alpha'}\partial_{\beta'}\phi$ (without necessarily imposing the null condition) requires our new technique which is presented in this paper and is based on sharp sup-norm bounds for solutions to wave equations and Klein-Gordon equations.

Our aim is presenting first in a simplifed form several arguments that will be required to cope with the full system of Einstein equations in [23]. In order to derive here a model problem, we proceed by removing (from (1.7)):

- the null terms $Q_{\alpha\beta}$ (which are handled in [22]),
- the quasi-null terms $P_{\alpha\beta}$ (postponed to [23], where the structure of the Einstein equations and the wave coordinate condition will be discussed), and
- the quasi-linear terms $h^{\alpha'\beta'}\partial_{\alpha'}\partial_{\beta'}h_{\alpha\beta}$ (to be treated by the wave coordinate condition and, in turn, the method already presented in [22]).

These formal simplifications, therefore, lead us to the model[3]

$$\Box h_{\alpha\beta} = \partial_\alpha\phi\partial_\beta\phi + m_{\alpha\beta}V(\phi),$$
$$\Box\phi = H^{\alpha\beta}(h)\partial_\alpha\partial_\beta\phi + V'(\phi),$$

with unknowns $h_{\alpha\beta}, \phi$ defined on Minkowski space, where $H^{\alpha\beta}(h)$ can be assumed to depend linearly on $h_{\alpha\beta}$. We are primarily interested in the potential $V(\phi) = \frac{c^2}{2}\phi^2$ and, therefore after changing the notation, we arrive at the following system of two coupled equations:

$$-\Box u = P^{\alpha\beta}\partial_\alpha v\partial_\beta v + Rv^2,$$
$$-\Box v + H^{\alpha\beta}u\partial_\alpha\partial_\beta v + c^2 v = 0,$$

where u, v are two scalar unknowns and $P^{\alpha\beta}, H^{\alpha\beta}, R, c$ are given constants (and only the obvious positivity condition $c^2 > 0$ is relevant).

1.4. **Analysis on the model problem.** As illustrated by the derivation above, in order to deal with the Einstein-massive scalar field equations, we must weaken a key assumption made in [22] and, as we will see, cope with wave equations posed on a curved space for which the Minkowski metric need not represent the underlying geometry in a sufficiently accurate manner. Namely, we recall that, in the notation of [22, Section 1], interaction terms like $u_{\hat{\imath}}\partial\partial v_{\check{\jmath}}$ involving components $u_{\hat{\imath}}$ of wave equations and component $v_{\check{\jmath}}$ of Klein-Gordon equations were not included in our theory. The same restriction was also assumed in a pioneering work by Katayama [16, 17] on wave-Klein-Gordon equations. In the present paper, we overcome this challenging difficulty and extend our earlier analysis (of the system (1.2.1) in [22] by now removing the condition (1.2.4e) therein).

To this end, in the present paper, we derive and take advantage of two pointwise estimates:

- **A sharp sup-norm estimate for solutions to the wave equation** in Minkowski space with source-term, as stated in Theorem 3.1, below. Suitable decay is assumed on the source-term, as is relevant for our analysis, and the proof is based on the solution formula available for the wave equation in flat space.
- **A sharp sup-norm estimate for solutions to the Klein-Gordon equation** in curved space in (3+1)-dimensions (as stated in Theorem 3.3, below). Our estimate is motivated by a pioneering work by Klainerman [19] on the global existence problem for small amplitude solutions to nonlinear Klein-Gordon equations in four spacetime dimensions.

Note that an estimate as above could also be derived in (2+1)-dimensions with different rates [28]. Global existence results for nonlinear Klein-Gordon equations were also established by Shatah in the pioneering work [31]. Klein-Gordon systems have received a lot of attention in the literature and we can, for instance, refer to [2, 3, 8, 9, 13, 19] and the references therein.

[3]Our convention for the wave operator is $\Box := -\partial_t\partial_t + \sum_a \partial_a\partial_a$.

For clarity in the presentation, we do not treat the most general class of systems, but based on our formal derivation from the Einstein equations, we now study the following **wave-Klein-Gordon model**:

$$\begin{aligned} -\Box u &= P^{\alpha\beta}\partial_\alpha v \partial_\beta v + Rv^2, \\ -\Box v + u\, H^{\alpha\beta}\partial_\alpha\partial_\beta v + c^2 v &= 0, \end{aligned} \tag{1.11}$$

with unknowns u, v posed on Minkowski space $\mathbb{R}^{3+1}$ and prescribed initial data[4] u_0, u_1, v_0, v_1 posed on the spacelike hypersurface $t = 2$:

$$\begin{aligned} u|_{t=2} &= u_0, \qquad & \partial_t u|_{t=2} &= u_1, \\ v|_{t=2} &= v_0, \qquad & \partial_t v|_{t=2} &= v_1. \end{aligned} \tag{1.12}$$

Here, $P^{\alpha\beta}, R, H^{\alpha\beta}, c$ are given constants, and the initial data are sufficiently smooth functions that are compactly supported in the unit ball $\{(x_1)^2 + (x_2)^2 + (x_3)^2 < 1\}$ with $x = (x_1, x_2, x_3) \in \mathbb{R}^3$.

We emphasize that, according to our analysis in Section 1.3, (1.11) includes the essential difficulty arising in the Einstein-massive field system. Note in passing that, in (1.11), there is no such term like Ru^2 which would imply finite time blow-up (as first pointed out by John [15]). Our main result in the present paper is as follows.

Theorem 1.1 (Global existence theory for the wave-Klein-Gordon model)**.** *Consider the nonlinear wave-Klein-Gordon system* (1.11) *for some given parameter values $P^{\alpha\beta}, R, H^{\alpha\beta}$ and $c > 0$. Given any integer $N \geqslant 8$, there exists a positive constant $\varepsilon_0 = \varepsilon_0(N) > 0$ such that if the initial data satisfy*

$$\|(u_0, v_0)\|_{H^{N+1}(\mathbb{R}^3)} + \|(u_1, v_1)\|_{H^N(\mathbb{R}^3)} < \varepsilon_0, \tag{1.13}$$

then the Cauchy problem (1.11)-(1.12) *admits a global-in-time solution.*

As done in [23], the Cauchy problem can be reformulated with initial data prescribed on a hyperboloid and the smallness condition (1.13) leads to a similar smallness condition for the hyperboloidal initial data. As already pointed out in [22], the presence of the quasi-linear term $u\, H^{\alpha\beta}\partial_\alpha\partial_\beta v$ may possibly change the asymptotic behavior of solutions for large times. In fact, our proof will only show that the lower-order energy of the wave component remains globally bounded for all times, while the high-order energy of the wave component u and the lower-order energy of the Klein-Gordon component v could in principle grow at the rate t^δ for some (small) $\delta > 0$. On the other hand, the higher-order energy associated with the Klein-Gordon component v may significantly increase at the rate $t^{\delta+1/2}$ for some (small) $\delta > 0$.

The proof of Theorem 1.1 will occupy the rest of this paper which we outline as follows:

- Proceeding with a bootstrap argument, we assume that, within some hyperbolic time interval, the hyperboloidal energy of suitable derivatives of the unknowns (up to a certain order) satisfy a set of bounds.
- Our assumptions use *two levels of regularity* and distinguish between the behavior of lower-order and higher-order energy norms, the low-order derivatives enjoying a much better control in time. Recall that, in [22], we could already prove that the lower-order energy of the wave component is uniformly bounded in time, but the growth rate for the high-order Klein Gordon energy was solely t^δ.
- By Sobolev inequality (on hyperboloids), we can turn these L^2 type inequalities to a set of sup-norm estimates, which we refer to as *basic decay estimates*. These decay estimates are not sharp enough in order to close our bootstrap argument.
- Relying on these basic decay estimates, we establish *refined decay estimates* by relying on two technical sup-norm estimates established below for wave equations and Klein-Gordon equations.
- Equipped with these refined decay estimates, we are able to improve our initial assumptions and close the bootstrap argument.

[4]For convenience in the following proof and without loss of generality, we prescribe data at time $t = 2$.

Before we proceed with the details of the proof (which is rather long), the reader may find it useful to read through the following *heuristic arguments* which rely on notations (only briefly explained here) to be rigorously introduced only later (in the course of the following three sections). Our proof proceeds with a bootstrap argument and considers the largest time interval $[2, s^*]$ (in the 'hyperbolic time' s defined as $s^2 = t^2 - r^2$) within which the following energy bounds hold:

$$\begin{aligned}
&E_m(s, \partial^I L^J u)^{1/2} \leqslant C_1 \varepsilon s^{k\delta}, && |J| = k, \quad |I| + |J| \leqslant N, && \text{wave / high-order},\\
&E_m(s, \partial^I L^J u)^{1/2} \leqslant C_1 \varepsilon, && |I| + |J| \leqslant N - 4, && \text{wave / low-order},\\
&E_{m,c^2}(s, \partial^I L^J v)^{1/2} \leqslant C_1 \varepsilon s^{1/2+k\delta}, && |J| = k, \quad |I| + |J| \leqslant N, && \text{Klein-Gordon / high-order},\\
&E_{m,c^2}(s, \partial^I L^J v)^{1/2} \leqslant C_1 \varepsilon s^{k\delta}, && |J| = k, \quad |I| + |J| \leqslant N - 4 && \text{Klein-Gordon / low-order},
\end{aligned}$$

where ϵ, δ, C_1 are parameters. These bounds concern the energy of the wave component u and the Klein-Gordon component v, and distinguish between low-order and high-order derivatives. We have denoted by E_m the energy associated with the wave equation (for the flat metric m), while ∂^I are partial derivative operators and L^J are combinations of Lorentz boosts (see below for details). The heart of our proof of Theorem 1.1 is proving that, by selecting a sufficiently large constant C_1 and sufficiently small $\varepsilon, \delta > 0$, the above energy bounds in fact imply the following improved energy bounds (obtained by replacing C_1 by $C_1/2$):

$$\begin{aligned}
&E_m(s, \partial^I L^J u)^{1/2} \leqslant \frac{1}{2} C_1 \varepsilon s^{k\delta}, && |J| = k, \quad |I| + |J| \leqslant N, && \text{wave / high-order},\\
&E_m(s, \partial^I L^J u)^{1/2} \leqslant \frac{1}{2} C_1 \varepsilon, && |I| + |J| \leqslant N - 4, && \text{wave / low-order},\\
&E_{m,c^2}(s, \partial^I L^J v)^{1/2} \leqslant \frac{1}{2} C_1 \varepsilon s^{1/2+k\delta}, && |J| = k, \quad |I| + |J| \leqslant N, && \text{Klein-Gordon / high-order},\\
&E_{m,c^2}(s, \partial^I L^J v)^{1/2} \leqslant \frac{1}{2} C_1 \varepsilon s^{k\delta}, && |J| = k, \quad |I| + |J| \leqslant N - 4 && \text{Klein-Gordon / low-order}.
\end{aligned}$$

(Of course, it is then a standard matter to deduce from this property that, in fact, $s^* = +\infty$.)

To derive the improved energy bounds, we differentiate the equations (1.11) with $\partial^I L^J$ with $|I| + |J| \leqslant N$:

$$\begin{aligned}
-\Box \partial^I L^J u &= \partial^I L^J \left(P^{\alpha\beta} \partial_\alpha v \partial_\beta v \right) + \partial^I L^J \left(R v^2 \right),\\
-\Box \partial^I L^J v + u\, H^{\alpha\beta} \partial^I L^J v + c^2 \partial^I L^J v &= -[\partial^I L^J, u\, H^{\alpha\beta} \partial_\alpha \partial_\beta] v.
\end{aligned}$$

For these differentiated equations, we perform energy estimates along the hyperboloidal foliation and we are led to seek for an integrable time decay for the following the three terms[5]:

$$\begin{aligned}
T_1^{I,J}(s) &:= \left\| \partial^I L^J \left(P^{\alpha\beta} \partial_\alpha v \partial_\beta v \right) \right\|_{L^2_f(\mathcal{H}_s)},\\
T_2^{I,J}(s) &:= \left\| \partial^I L^J \left(R v^2 \right) \right\|_{L^2_f(\mathcal{H}_s)},\\
T_3^{I,J}(s) &:= \left\| [\partial^I L^J, u\, H^{\alpha\beta} \partial_\alpha \partial_\beta] v \right\|_{L^2_f(\mathcal{H}_s)}.
\end{aligned} \tag{1.14}$$

For lower-order indices $|I| + |J| \leqslant N - 4$, the terms $T_1^{I,J}(s)$ and $T_2^{I,J}(s)$ are easily controlled, since from the bootstrap assumption and the global Sobolev inequalities on hyperboloids we have (basic) decay estimates which lead to time-integrable bounds:

$$T_1^{I,J}(s) + T_2^{I,J}(s) \lesssim s^{-3/2+(k+2)\delta}, \qquad \text{provided } |I| + |J| \leqslant N - 4 \text{ with } |J| = k. \tag{1.15}$$

On the other hand, for higher-order derivatives these basic decay rates are not sufficient and we can not conclude directly. In addition, for the third term $T_3^{I,J}(s)$ (for arbitrary $|I| + |J|$), we also cannot conclude directly and we need sharper pointwise decay.

To overcome this challenge, we rely on our L^∞–L^∞ sharp decay estimates, established below in Proposition 3.1 (for the wave component) and Proposition 3.3 (for the Klein-Gordon component).

[5]See (2.7) for the notation.

These L^∞–L^∞ bounds allow us to improve the basic pointwise estimates, and we find (for all $|I| + |J| \leqslant N - 4$):

$$
\begin{aligned}
|L^I u| &\lesssim C_1\varepsilon\, t^{-1} s^{k\delta},\\
|\partial^I L^J v| &\lesssim C_1\varepsilon\, (s/t)^{2-7\delta} s^{-3/2+k\delta},\\
|\partial^I L^J \partial_\alpha v| &\lesssim C_1\varepsilon\, (s/t)^{1-7\delta} s^{-3/2+k\delta}.
\end{aligned}
$$

Returning to our bootstrap assumptions, we thus see that for all $|I| + |J| \leqslant N$

$$
\begin{aligned}
\left\|\partial^I L^J \left(\partial_\alpha v \partial_\beta v\right)\right\|_{L^2_f(\mathcal{H}_s)} &\simeq \sum_{\substack{I_1+I_2=I\\ J_1+J_2=J}} \|\partial^{I_1} L^{J_1} \partial_\alpha v\, \partial^{I_2} L^{J_2} \partial_\beta v\|_{L^2_f(\mathcal{H}_s)}\\
&\lesssim C_1\varepsilon s^{-3/2} \|\partial^{I_2} L^{J_2} \partial_\beta v\|_{L^2_f(\mathcal{H}_s)} \lesssim (C_1\varepsilon)^2 s^{-1+k\delta}
\end{aligned}
\tag{1.16}
$$

(by assuming, without loss of generality, $|I_1| + |J_1| \leqslant N - 4$ in the above calculation). Similarly, we also obtain

$$
\|\partial^I L^J \left(v^2\right)\|_{L^2_f(\mathcal{H}_s)} \lesssim (C_1\varepsilon)^2 s^{-1+k\delta}. \tag{1.17}
$$

We thus succeed to uniformly control the terms $T_1^{I,J}(s)$ and $T_2^{I,J}(s)$ (for all relevant I, J), and this is already sufficient to conclude with the desired improved energy bounds for the *wave component*.

Dealing with the last term $T_3^{I,J}(s)$ arising in the equation of the Klein-Gordon component is more delicate. Observe that the commutator is a linear combination of the following three types of terms:

$$
\begin{aligned}
&(\partial^{I_1} L^{J_1} u)\partial^{I_2} L^{L_2} \partial_\alpha\partial_\beta v, && I_1 + I_2 = I, \quad J_1 + J_2 = J, \quad |I_1| \geqslant 1,\\
&(L^{J_1'} u)\partial^I L^{J_2'} \partial_\alpha\partial_\beta v, && J_1' + J_2' = J, \quad J_1' \geqslant 1,\\
&u\partial_\alpha\partial_\beta\partial^I L^{J'} v, && J' \leqslant J - 1.
\end{aligned}
\tag{1.18}
$$

The first expression above is directly controled thanks to the available sharp decay estimate, while for the second and third ones and due to the presence of the term $L^J u$, a refined decay estimates and a Hardy-type inequality (for the hyperboloidal foliation) must be used, as we now explain.

Let us begin by discussing derivatives of higher-order and consider (for instance) the second type of terms in (1.18): for all $|J_1'| \leqslant N-4$, we use the sharp decay bound $|L^I u| \lesssim C_1\varepsilon t^{-1} s^{k\delta}$ combined with the energy bound on $\partial_\alpha\partial_\beta v$ (implied by our bootstrap assumption). When $|I| + |J_2'| \leqslant N-4$, the sharp bound $|\partial^I L^J \partial_\alpha v| \lesssim C_1\varepsilon\, (s/t)^{1-7\delta} s^{-3/2+k\delta}$ and Hardy's inequality are used. We thus find

$$
\left\|[\partial^I L^J, H^{\alpha\beta}\partial_\alpha\partial_\beta] v\right\|_{L^2_f(\mathcal{H}_s)} \lesssim (C_1\varepsilon)^2 s^{-1/2+k\delta}. \tag{1.19}
$$

Dealing with lower-order derivatives is easier and, again, we take the second type of terms in (1.18) as an example: for $|J_1'| \leqslant |I| + |J| \leqslant N - 4$, we apply directly the sharp bound $|L^I u| \lesssim C_1\varepsilon\, t^{-1} s^{k\delta}$ and the energy bound given by our bootstrap assumption. This leads us to the stronger decay

$$
\left\|[\partial^I L^J, H^{\alpha\beta}\partial_\alpha\partial_\beta] v\right\|_{L^2_f(\mathcal{H}_s)} \lesssim (C_1\varepsilon)^2 s^{-1+k\delta}. \tag{1.20}
$$

In conclusion, in view of (1.15)–(1.20), we can gain enough time decay for all of the terms arising in the evolution of our energy expressions and, therefore, the energy estimate on the hyperboloidal foliation leads us to the desired improved energy bounds.

1.5. **A general class of wave-Klein-Gordon systems.** The technique presented here applies immediately to a much broader class of systems. Indeed, it applies to the following system of wave–Klein-Gordon equations

$$
\begin{cases}
\Box u_i + B^{j\alpha\beta} u_j \partial_\alpha\partial_\beta u_i = F_i(u, \partial u, v, \partial v) = P_i^{jk\alpha\beta} \partial_\alpha u_j \partial_\beta u_k + R_i v^2 + S_i^{\alpha\beta} \partial_\alpha v \partial_\beta v,\\
\Box v + B^{j\alpha\beta} u_j \partial_\alpha\partial_\beta v - c^2 v^2 = 0,\\
w_i|_{t=2} = w_{i0}, \qquad v|_{t=2} = v_0,\\
\partial_t w_i|_{t=2} = w_{i1}, \qquad \partial_t v|_{t=2} = v_1,
\end{cases}
\tag{1.21}
$$

with unknowns $u = (u_i)$ $(1 \leqslant i \leqslant n)$ and v defined on Minkowski space $\mathbb{R}^{3+1}$, while w_{i0}, v_0, w_{i1}, v_1 are prescribed initial data and $c > 0$ is a constant. As usual, we assume the symmetry conditions

$$B^{j\alpha\beta} = B^{j\beta\alpha} \tag{1.22}$$

and our main assumption is the null condition for the wave components w_i:

$$B^{j\alpha\beta}\xi_\alpha\xi_\beta = P_i^{jk\alpha\beta}\xi_\alpha\xi_\beta = 0 \qquad \text{for all } (\xi_0)^2 - \sum_a (\xi_a)^2 = 0. \tag{1.23}$$

In the earlier work [22], the nonlinear terms $B^{j\alpha\beta}w_j\partial_\alpha\partial_\beta v$ (actually denoted $B_{\check{i}}^{\hat{j}\check{k}\alpha\beta}w_{\hat{j}}\partial_\alpha\partial_\beta w_{\check{k}}$ therein) were assumed to be vanishing, and in fact this was our only genuine restriction since, with such terms, solutions may not have the same time decay and asymptotics of solutions as the ones of the homogeneous linear wave-Klein-Gordon equations in Minkowski space. With the new technique in the present paper, the hyperboloidal foliation method does extend to encompass these terms (provided $B^{j\alpha\beta}$ is a null quadratic form).

Let us consider the initial value problem (1.21) with sufficiently smooth initial data posed on the spacelike hypersurface $\{t = 2\}$ of constant time and compactly supported in the ball $\{t = 2;\, |x| \leqslant 1\}$. Under the conditions (1.22)–(1.23), there exists a real $\epsilon_0 > 0$ such that, for all initial data $w_{i0}, w_{i1}, v_0, v_1 : \mathbb{R}^3 \to \mathbb{R}$ satisfying the smallness condition

$$\sum_i \|(w_{i0}, v_0)\|_{\mathbf{H}^{N+1}(\mathbb{R}^3)} + \|(w_{i1}, v_1)\|_{\mathbf{H}^{N}(\mathbb{R}^3)} < \epsilon_0, \tag{1.24}$$

the Cauchy problem (1.21) admits a unique, smooth global-in-time solution. In addition, the lower-order energy of the wave components remains globally bounded in time.

2. The hyperboloidal foliation method

2.1. **The semi-hyperboloidal frame.** We begin with basic notions and consider the $(3 + 1)$-dimensional Minkowski space with signature $(-, +, +, +)$. In canonical Cartesian coordinates, we write $(t, x) = (x^0, x^1, x^2, x^3)$ and $r^2 := |x|^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$. In addition to the partial derivative fields $\partial_t = \partial_0$ and ∂_a, we will also use the *Lorentz boosts* (for $a = 1, 2, 3$):

$$L_a := x^a\partial_t + t\partial_a = x_a\partial_0 - x_0\partial_a, \tag{2.1}$$

where we raise and lower indices with the Minkowski metric.

More precisely, throughout, we analyze solutions defined in the interior of the future light cone

$$\mathcal{K} := \{(t, x) \,/\, r < t - 1\}$$

with vertex $(1, 0, 0, 0)$, and we introduce the following foliation of the interior of the cone $\big\{(t, x) \,/\, |x| < t\big\}$ by hyperboloidal hypersurfaces with hyperbolic radius s:

$$\mathcal{H}_s := \big\{(t, x) \,/\, t^2 - r^2 = s^2; \quad t > 0\big\}.$$

The sub-domain of $\mathcal{K}$ limited by two hyperboloids (with $s_0 < s_1$) is denoted by

$$\mathcal{K}_{[s_0,s_1]} := \big\{(t, x) \,/\, s_0^2 \leqslant t^2 - r^2 \leqslant s_1^2; \quad r < t - 1\big\} \subset \mathcal{K}.$$

Observe that the hyperboloids eventually "exit" the region $\mathcal{K}$ and are asymptote to the same light cone $\big\{t - r = 0\big\}$.

The semi-hyperboloidal frame, as we call it, is defined by rescaling the Lorentz boosts:

$$\underline{\partial}_0 := \partial_t, \qquad \underline{\partial}_a := \frac{x_a}{t}\partial_t + \partial_a \qquad (a = 1, 2, 3). \tag{2.2}$$

Observe that the vectors $\underline{\partial}_a$ generates the tangent space to the hyperboloids. Furthermore, we also introduce the vector field $\underline{\partial}_\perp := \partial_t + \frac{x^a}{t}\partial_a$, which is orthogonal to the hyperboloids for the Minkowski metric. (This vector field also coincides, up to an essential factor $1/t$, with the scaling vector field S.)

To make explicit the change of frame formulas $\underline{\partial}_\alpha = \Phi_\alpha^\beta \partial_\beta$ and $\partial_\alpha = \Psi_\alpha^\beta \underline{\partial}_\beta$, we need the following matrices

$$\big(\Phi_\alpha^\beta\big) = \big(\Phi_\alpha{}^\beta\big) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ x^1/t & 1 & 0 & 0 \\ x^2/t & 0 & 1 & 0 \\ x^3/t & 0 & 0 & 1 \end{pmatrix}, \qquad \big(\Psi_\alpha^\beta\big) = \big(\Psi_\alpha{}^\beta\big) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -x^1/t & 1 & 0 & 0 \\ -x^2/t & 0 & 1 & 0 \\ -x^3/t & 0 & 0 & 1 \end{pmatrix}.$$

Any tensor can be expressed in either the Cartesian natural frame $\{\partial_\alpha\}$ or the semi-hyperboloidal frame $\{\underline{\partial}_\alpha\}$. We use standard letters for components in the Cartesian frame and we use underlined letters for components in the semi-hyperboloidal frame, so that, for example, $T^{\alpha\beta}\partial_\alpha \otimes \partial_\beta = \underline{T}^{\alpha\beta}\underline{\partial}_\alpha \otimes \underline{\partial}_\beta$, and the relations between $T^{\alpha\beta}$ and $\underline{T}^{\alpha\beta}$ are

$$\underline{T}^{\alpha\beta} = \Psi^\alpha_{\alpha'}\Psi^\beta_{\beta'}T^{\alpha'\beta'}, \qquad T^{\alpha\beta} = \Phi^\alpha_{\alpha'}\Phi^\beta_{\beta'}\underline{T}^{\alpha'\beta'}.$$

Associated with the semi-hyperboloidal frame, we have the dual semi-hyperboloidal frame

$$\underline{\theta}^0 := dt - \frac{x^a}{t}dx^a, \qquad \underline{\theta}^a := dx^a, \tag{2.3}$$

and the relations between the semi-hyperboloidal dual frame and the standard dual frame are $\underline{\theta}^\alpha = \Psi^\alpha_{\alpha'}dx^{\alpha'}$, $dx^\alpha = \Phi^\alpha_{\alpha'}\underline{\theta}^{\alpha'}$. Hence, for any two-tensor $T_{\alpha\beta}dx^\alpha \otimes dx^\beta = \underline{T}_{\alpha\beta}\underline{\theta}^\alpha \otimes \underline{\theta}^\beta$, we have the change of basis formulas

$$\underline{T}_{\alpha\beta} = T_{\alpha'\beta'}\Phi^{\alpha'}_\alpha\Phi^{\beta'}_\beta, \qquad T_{\alpha\beta} = \underline{T}_{\alpha'\beta'}\Psi^{\alpha'}_\alpha\Psi^{\beta'}_\beta.$$

With the above notation, in the semi-hyperboloidal frame we can express the Minkowski metric and its inverse as

$$\underline{m}_{\alpha\beta} = \begin{pmatrix} -1 & -x^1/t & -x^2/t & -x^3/t \\ -x^1/t & 1-(x^1/t)^2 & -x^1x^2/t^2 & -x^1x^3/t^2 \\ -x^2/t & -x^2x^1/t^2 & 1-(x^2/t)^2 & -x^2x^3/t^2 \\ -x^3/t & -x^3x^1/t^2 & -x^3x^2/t^2 & 1-(x^3/t)^2 \end{pmatrix},$$

$$\underline{m}^{\alpha\beta} = \begin{pmatrix} -s^2/t^2 & -x^1/t & -x^2/t & -x^3/t \\ -x^1/t & 1 & 0 & 0 \\ -x^2/t & 0 & 1 & 0 \\ -x^3/t & 0 & 0 & 1 \end{pmatrix}.$$

Furthermore, given any multi-index $I = (\alpha_n, \alpha_{n-1}, \dots, \alpha_1)$ (where the order is chosen for convenience), we denote by $\partial^I := \partial_{\alpha_n}\partial_{\alpha_{n-1}} \dots \partial_{\alpha_1}$ the product of $n = |I|$ partial derivatives (with $0 \leqslant \alpha_i \leqslant 3$) and, similarly, by $L^J = L_{a_n}L_{a_{n-1}} \dots L_{a_1}$ the product of $n = |J|$ Lorentz boosts (with $1 \leqslant a_i \leqslant 3$).

2.2. **The hyperbolic variables and the hyperboloidal frame.** Within the future cone $\mathcal{K}$, we introduce the change of variables

$$\overline{x}^0 = s := \sqrt{t^2 - r^2}, \qquad \overline{x}^a = x^a, \tag{2.4}$$

together with the corresponding natural frame

$$\begin{aligned} \overline{\partial}_0 &:= \partial_s = \frac{s}{t}\partial_t = \frac{\sqrt{t^2-r^2}}{t}\partial_t, \\ \overline{\partial}_a &:= \partial_{\overline{x}^a} = \frac{\overline{x}^a}{t}\partial_t + \partial_a = \frac{x^a}{t}\partial_t + \partial_a, \end{aligned} \tag{2.5}$$

which we refer to as the *hyperboloidal frame*. The transition matrices between the hyperboloidal frame and the Cartesian frame are

$$\big(\overline{\Phi}_\alpha^\beta\big) = \big(\overline{\Phi}^\beta{}_\alpha\big) = \begin{pmatrix} s/t & 0 & 0 & 0 \\ x^1/t & 1 & 0 & 0 \\ x^2/t & 0 & 1 & 0 \\ x^3/t & 0 & 0 & 1 \end{pmatrix},$$

$$\big(\overline{\Phi}^{\beta}_{\alpha}\big)^{-1} = \big(\overline{\Psi}^{\beta}_{\alpha}\big) = \big(\overline{\Psi}^{\beta}{}_{\alpha}\big) = \begin{pmatrix} t/s & 0 & 0 & 0 \\ -x^1/s & 1 & 0 & 0 \\ -x^2/s & 0 & 1 & 0 \\ -x^3/s & 0 & 0 & 1 \end{pmatrix},$$

so that $\overline{\partial}_\alpha = \overline{\Phi}^{\beta}_{\alpha}\partial_\beta$ and $\partial_\alpha = \overline{\Psi}^{\beta}_{\alpha}\overline{\partial}_\beta$.

The dual hyperboloidal frame then reads $d\overline{x}^0 := ds = \frac{t}{s}dt - \frac{x^a}{s}dx^a$ and $d\overline{x}^a := dx^a$. The Minkowski metric in the hyperboloidal frame reads[6]

$$\overline{m}^{\alpha\beta} = \begin{pmatrix} -1 & -x^1/s & -x^2/s & -x^3/s \\ -x^1/s & 1 & 0 & 0 \\ -x^2/s & 0 & 1 & 0 \\ -x^3/s & 0 & 0 & 1 \end{pmatrix}.$$

In summary, an arbitrary tensor can be expressed in three different frames: the standard frame $\{\partial_\alpha\}$, the semi-hyperboloidal frame $\{\underline{\partial}_\alpha\}$, or the hyperboloidal frame $\{\overline{\partial}_\alpha\}$. We use symbols, underlined symbols, and overlined symbols for tensor components in these frames, respectively. For example, a tensor $T^{\alpha\beta}\partial_\alpha \otimes \partial_\beta$ is written as

$$T^{\alpha\beta}\partial_\alpha \otimes \partial_\beta = \underline{T}^{\alpha\beta}\underline{\partial}_\alpha \otimes \underline{\partial}_\beta = \overline{T}^{\alpha\beta}\overline{\partial}_\alpha \otimes \overline{\partial}_\beta,$$

where $\overline{T}^{\alpha\beta} = \overline{\Psi}^{\alpha}_{\alpha'}\overline{\Psi}^{\beta}_{\beta'}T^{\alpha'\beta'}$ and, moreover, by setting $C := \max_{\alpha\beta}|T^{\alpha\beta}|$, we have in the hyperboloidal frame

$$|\overline{T}^{00}| \leqslant C(t/s)^2, \qquad |\overline{T}^{a0}| \leqslant C\,(t/s), \qquad |\overline{T}^{ab}| \leqslant C. \tag{2.6}$$

2.3. **Energy estimate on hyperboloids.** Throughout this paper, for any function $u = u(t,x)$ defined in $\mathbb{R}^{3+1}$ (or a subset of it), we consider the integral of on the hyperboloids $\mathcal{H}_s$ defined as follows:

$$\|u\|_{L^1_f(\mathcal{H}_s)} := \int_{\mathcal{H}_s} u\,dx = \int_{\mathbb{R}^3} u\big(\sqrt{s^2+r^2},x\big)dx \tag{2.7}$$

The subscript refer the fact that we are endowing $\mathcal{H}_s$ with the *flat* Euclidian metric. We emphasize that this is *not* an integration with respect to the induced Riemannian metric and volume form which should be $(s/t)\,dx$. This notation will be more convenient for our analysis in this paper but, of course, it is completely straightforward to re-state all of our estimates by including the weight s/t.

Consider the hyperboloidal foliation of a region $\mathcal{K}_{[2,s_1]} = \bigcup_{2\leqslant s\leqslant s_1}\mathcal{H}_s$, together with the *hyperboloidal energy* (associated to the Minkowski metric) at some hyperbolic time $s \in [2,s_1]$

$$\begin{aligned} E_{m,c}(s,u) &:= \int_{\mathcal{H}_s}\Big((\partial_t u)^2 + \sum_a(\partial_a u)^2 + 2(x^a/t)\partial_t u\partial_a u + c^2u^2\Big)dx \\ &= \int_{\mathcal{H}_s}\Big(\big((s/t)\partial_t u\big)^2 + \sum_a(\underline{\partial}_a u)^2 + c^2u^2\Big)\,dx \\ &= \int_{\mathcal{H}_s}\left((\underline{\partial}_\perp u)^2 + \sum_a\big((s/t)\partial_a u\big)^2 + \sum_{a<b}\big(t^{-1}\Omega_{ab}u\big)^2 + c^2u\right)dx, \end{aligned} \tag{2.8}$$

where we have also introduced the rotational vector fields $\Omega_{ab} := x^a\partial_b - x^b\partial_a$ (not directly used here). When $c=0$, we also write $E_m(s,u) := E_{m,0}(s,u)$ for short.

We will also need the *hyperboloidal energy for the curved metric* $g^{\alpha\beta} = m^{\alpha\beta} + h^{\alpha\beta}$:

$$E_{g,c}(s,u) := E_{m,c}(s,u) + \int_{\mathcal{H}_s}\Big(2h^{\alpha\beta}\partial_t v\partial_\beta v X_\alpha - h^{\alpha\beta}\partial_\alpha v\partial_\beta v\Big)\,dx, \tag{2.9}$$

where we have used the notation $X^0 = 1$ and $X^a = -x^a/t$.

All of our estimates concern the interior of the light cone $\mathcal{K}\cap\{t\geqslant 2\}$ away from the origin. From here onwards, we assume that all the functions under consideration are *spatially compact*

[6]Our sign convention is opposite to the one in our monograph [22], since the metric here has signature $(-,+,+,+)$.

and, in particular, vanish identically in a neighborhood of the light cone $\{t-1=|x|=r\}$. More precisely, we assume that the initial data on the slice $t=2$ are supported in the ball $|x|\leqslant M$ for some $M\in(0,1)$, and we construct solutions supported in the larger domain $|x|\leqslant M+t$. In short, we will say that the functions under consideration are **spatially compactly supported in** $\mathcal{K}$ or, in short, spatially compactly supported.

We easily adapt the energy estimate in [22, Proposition 2.3.1] to the equation (1.11), as follows.

Proposition 2.1 (Energy estimate for the hyperboloidal foliation)**.** *1. For every function u which is defined in the region $\mathcal{K}_{[2,s]}$ and spatially supported in $\mathcal{K}$, one has for all $s\geqslant 2$*

$$E_m(s,u)^{1/2}\leqslant E_m(2,u)^{1/2}+\int_2^s \|\Box u\|_{L^2_{\bar f}(\mathcal{H}_{\bar s})}d\bar s. \tag{2.10}$$

2. Let v be a solution to the Klein-Gordon equation on a curved space (with the definition of $\widetilde{\Box}$ given earlier)

$$-\widetilde{\Box}_g v+c^2v=f, \tag{2.11}$$

defined the region $\mathcal{K}_{[2,s]}$ and spatially supported in $\mathcal{K}$. Suppose that $h^{\alpha\beta}=g^{\alpha\beta}-m^{\alpha\beta}$ satisfies the following two conditions (for some constant $\kappa\geqslant 1$ and some function M):

$$\kappa^{-2}E_{g,c}(s,v)\leqslant E_{m,c}(s,v)\leqslant \kappa^2 E_{g,c}(s,v), \tag{2.12a}$$

$$\left|\int_{\mathcal{H}_s}(s/t)\Big(\partial_\alpha h^{\alpha\beta}\partial_t v\partial_\beta v-\frac12\partial_t h^{\alpha\beta}\partial_\alpha v\partial_\beta v\Big)\,dx\right|\leqslant M(s)E_{m,c}(s,v)^{1/2}. \tag{2.12b}$$

Then, the evolution of the hyperboloidal energy is controlled (for all $s\geqslant 2$) by

$$E_{m,c}(s,v)^{1/2}\leqslant \kappa^2 E_m(2,v)^{1/2}+\kappa^2\int_2^s\Big(\|f\|_{L^2_{\bar f}(\mathcal{H}_{\bar s})}+M(s)\Big)d\bar s. \tag{2.13}$$

Proof. We apply the multiplier $\partial_t u$ to $\Box u$ and, by a standard calculation,

$$\frac12\partial_t\Big((\partial_t u)^2+\sum_a(\partial_a u)^2\Big)-\sum_a\partial_a(\partial_t u\partial_a u)=\partial_t u\Box u.$$

We integrate this identity in $\mathcal{K}_{[2,s]}$ and apply Stokes' formula by observing that, by assumption, the functions under consideration are spatially supported in $\mathcal{K}$, so that there is no "boundary" contribution, and we find (see [22, Sec. 2.3])

$$\frac12 E_m(s,u)-\frac12 E_m(2,u)=\int_2^s\int_{\mathcal{H}_{\bar s}}(\bar s/t)\partial_t u\Box u\,dxd\bar s.$$

We differentiate this identity with respect to s and apply Cauchy-Schwarz inequality, as follows:

$$E_m(s,u)^{1/2}\frac{d}{ds}E_m(s,u)^{1/2}=\int_{\mathcal{H}_s}(\bar s/t)\partial_t v\Box u\,dx\leqslant\|\Box u\|_{L^2_{\bar f}(\mathcal{H}_s)}\|(s/t)\partial_t u\|_{L^2_{\bar f}(\mathcal{H}_s)}.$$

Next, by recalling (2.8), we find $\frac{d}{ds}E_m(s,u)^{1/2}\leqslant\|\Box u\|_{L^2_{\bar f}(\mathcal{H}_s)}$ and, by integration over $[2,s]$, the inequality (2.10) is established.

Next, for the derivation of (2.13), we rely on the multiplier $\partial_t v$ and, by a standard calculation, we get

$$\begin{aligned}&\frac12\partial_t\big((\partial_t v)^2+\sum_a(\partial_a v)^2+c^2v^2\big)+\sum_a\partial_a\big(-\partial_a v\partial_t v\big)\\&+\partial_\alpha\big(-h^{\alpha\beta}\partial_\beta v\partial_t v\big)+\frac12\partial_t\big(h^{\alpha\beta}\partial_\alpha v\partial_\beta v\big)\\&=\partial_t v f-\partial_\alpha g^{\alpha\beta}\partial_\beta v\partial_t v+\frac12\partial_t g^{\alpha\beta}\partial_\alpha v\partial_\beta v.\end{aligned}$$

As was done in the derivation of (2.10), we combine this identity with (2.12a)-(2.12b) and obtain

$$E_{g,c}(s,v)^{1/2}\frac{d}{ds}E_{g,c}(s,v)^{1/2}\leqslant\kappa\big(\|f\|_{L^2_{\bar f}(\mathcal{H}_s)}+M(s)\big)E_{g,c}(s,v)^{1/2}.$$

We integrate this inequality on $[2,s]$ and find

$$E_{g,c}(s,v)^{1/2} \leqslant E_{g,c}(2,v)^{1/2} + \kappa \int_2^s \big(\|f\|_{L^2_f(\mathcal{H}_{\bar s})} + M(\bar s)\big) ds.$$

Finally, we again apply (2.12a), and (2.13) is proven. □

2.4. **Sobolev inequality on hyperboloids.** In order to turn an L^2 energy estimate into an L^∞ estimate, we will rely on the following version of the Sobolev inequality (Klainerman [19], Hörmander [13, Lemma 7.6.1], LeFloch and Ma [22, Section 5]).

Proposition 2.2 (Sobolev-type estimate on hyperboloids)**.** *For any sufficiently smooth function $u = u(s,x)$ which is defined in $\mathcal{K}_{[2,+\infty)}$ and is spatially supported in $\mathcal{K}$, one has*

$$\sup_{(s,x)\in\mathcal{H}_s} (s+|x|)^{3/2}\,|u(s,x)| \lesssim \sum_{|I|\leqslant 2} \|L^I u(s,\cdot)\|_{L^2_f(\mathcal{H}_s)}, \qquad s \geqslant 2, \tag{2.14}$$

where the implied constant is independent of s and u.

Proof. Recalling that $t = \sqrt{s^2+|x|^2}$ on $\mathcal{H}_s$, we consider the restriction to the hyperboloid

$$w_s(x) := u(\sqrt{s^2+|x|^2}, x).$$

Fixing s_0 and a point $(t_0,x_0)\in\mathcal{H}_{s_0}$ (with $t_0 = \sqrt{s_0^2+|x_0|^2}$), we observe that

$$\partial_a w_{s_0}(x) = \underline{\partial}_a u\big(\sqrt{s_0^2+|x|^2}, x\big) = \underline{\partial}_a u(t,x), \tag{2.15}$$

with $t = \sqrt{s_0^2+|x|^2}$ and

$$t\partial_a w_{s_0}(x) = t\underline{\partial}_a u\big(\sqrt{s_0^2+|x|^2}, t\big) = L_a u(t,x). \tag{2.16}$$

We introduce the function $g_{s_0,t_0}(y) := w_{s_0}(x_0 + t_0\, y)$ and write

$$g_{s_0,t_0}(0) = w_{s_0}(x_0) = u\big(\sqrt{s_0^2+|x_0|^2}, x_0\big) = u(t_0,x_0).$$

By applying the standard Sobolev inequality to the function g_{s_0,t_0}, we find

$$\big|g_{s_0,t_0}(0)\big|^2 \leqslant C \sum_{|I|\leqslant 2} \int_{B(0,1/3)} |\partial^I g_{s_0,t_0}(y)|^2\, dy,$$

where $B(0,1/3)\subset\mathbb{R}^3$ is the ball centered at the origin and with radius $1/3$.

Next, taking into account the identity (with $x = x_0 + t_0 y$)

$$\begin{aligned}\partial_a g_{s_0,t_0}(y) &= t_0 \partial_a w_{s_0}(x_0 + t_0 y)\\ &= t_0 \partial_a w_{s_0}(x) = t_0 \underline{\partial}_a u\big(t,x\big)\end{aligned}$$

and, in view of (2.15), we find (for all I) $\partial^I g_{s_0,t_0}(y) = (t_0\underline{\partial})^I u(t,x)$ and, thus,

$$\begin{aligned}\big|g_{s_0,t_0}(0)\big|^2 \leqslant& C \sum_{|I|\leqslant 2} \int_{B(0,1/3)} \big|(t_0\underline{\partial})^I u\big(t,x)\big)\big|^2 dy\\ =& C t_0^{-3} \sum_{|I|\leqslant 2} \int_{B((t_0,x_0),t_0/3)\cap\mathcal{H}_{s_0}} \big|(t_0\underline{\partial})^I u\big(t,x)\big)\big|^2 dx.\end{aligned}$$

We note that

$$\begin{aligned}(t_0\underline{\partial}_a(t_0\underline{\partial}_b w_{s_0})) &= t_0^2 \underline{\partial}_a\underline{\partial}_b w_{s_0}\\ &= (t_0/t)^2 (t\underline{\partial}_a)(t\underline{\partial}_b) w_{s_0} - (t_0/t)^2 (x^a/t) L_b w_{s_0}\end{aligned}$$

and that $x^a/t = x_0^a/t + y t_0/t = (x_0^a/t_0 + y)(t_0/t)$. Consequently, in the region $y\in B(0,1/3)$ of interest, the factor $|x^a/t|$ is bounded by $C(t_0/t)$ and we conclude that (for $|I|\leqslant 2$)

$$|(t_0\underline{\partial})^I u| \leqslant \sum_{|J|\leqslant |I|} |L^J u| (t_0/t)^2.$$

On the other hand, in the region $|x_0| \leqslant t_0/2$, we have $t_0 \leqslant \frac{2}{\sqrt{3}} s_0$ and thus

$$t_0 \leqslant C s_0 \leqslant C\sqrt{|x|^2 + s_0^2} = Ct$$

for some fixed constant $C > 0$. When $|x_0| \geqslant t_0/2$ then in the region $B((t_0, x_0), t_0/3) \cap \mathcal{H}_{s_0}$ we have $t_0 \leqslant C|x| \leqslant C\sqrt{|x|^2 + s_0^2} = Ct$ and, consequently,

$$|(t_0 \underline{\partial})^I u| \leqslant C \sum_{|J| \leqslant |I|} |L^J u|$$

and

$$\begin{aligned} \big|g_{s_0,t_0}(y_0)\big|^2 \leqslant & C t_0^{-3} \sum_{|I| \leqslant 2} \int_{B(x_0,t_0/3) \cap \mathcal{H}_{s_0}} \big|(t\underline{\partial})^I u\big(t,x\big)\big|^2 \, dx \\ \leqslant & C t_0^{-3} \sum_{|I| \leqslant 2} \int_{\mathcal{H}_{s_0}} \big|L^I u(t,x)\big|^2 \, dx. \end{aligned}$$

□

2.5. **Hardy-type estimate along the hyperboloidal foliation.** The following variant of Hardy's inequality was established in [22, Section 5]. This inequality plays an essential role in order to estimate the L^2 norm of the wave component itself (but not only its gradient).

Proposition 2.3 (Hardy-type estimate on the hyperboloidal foliation)**.** *For any sufficiently smooth function which is defined in the future region $\mathcal{K}_{[2,s]}$ and is spatially supported in $\mathcal{K}$, one has for $s \geqslant 2$*

$$\begin{aligned} \|s^{-1}u\|_{L^2_f(\mathcal{H}_s)} \lesssim & \|u\|_{L^2(\mathcal{H}_2)} + \sum_a \|\underline{\partial}_a u\|_{L^2_f(\mathcal{H}_s)} \\ & + \sum_a \int_2^s \bar{s}^{-1} \Big(\|\underline{\partial}_a u\|_{L^2_f(\mathcal{H}_{\bar{s}})} + \|(\bar{s}/t)\partial_a u\|_{L^2_f(\mathcal{H}_{\bar{s}})} \Big) \, d\bar{s}, \end{aligned} \tag{2.17}$$

where the implied constant is independent of s and u.

The proof uses a version of the classical Hardy inequality on hyperboloids, as well as a vector field that will be introduced in the proof of the proposition, below.

Lemma 2.4. *For any sufficiently smooth function which is defined in the future region $\mathcal{K}_{[2,s]}$ and is spatially supported in $\mathcal{K}$, one has for all $s \geqslant 2$*

$$\|r^{-1}u\|_{L^2_f(\mathcal{H}_s)} \lesssim \sum_a \|\underline{\partial}_a u\|_{L^2_f(\mathcal{H}_s)},$$

where the implied constant is independent of s and u.

Proof. As in the proof of Proposition 2.2, we consider the function $w_s(x) := u\big(\sqrt{s^2 + |x|^2}, x\big)$, which satisfies $\partial_a w_s(x) = \underline{\partial}_a u\big(\sqrt{s^2 + |x|^2}, x\big)$, and we apply the classical Hardy inequality to w_s. It follows that

$$\begin{aligned} \int_{\mathbb{R}^3} |r^{-1} w_s(x)|^2 dx \lesssim \int_{\mathbb{R}^3} |\nabla w_s(x)|^2 dx = C \sum_a \int_{\mathbb{R}^3} \big|\underline{\partial}_a u(\sqrt{s^2 + r^2}, x)\big|^2 dx \\ \lesssim \sum_a \int_{\mathcal{H}_s} \big|\underline{\partial}_a u(t,x)\big|^2 dx. \end{aligned}$$

□

Proof of Proposition 2.3. Let χ be a a smooth cut-off function satisfying

$$\chi(r) = \begin{cases} 0, & 0 \leqslant r \leqslant 1/3 \\ 1, & 2/3 \leqslant r, \end{cases}$$

and let us distinguish between the region "near" and "away" from the light cone. We consider the decomposition

$$\|s^{-1}u\|_{L^2_f(\mathcal{H}_s)} \leqslant \|\chi(r/t) s^{-1} u\|_{L^2_f(\mathcal{H}_s)} + \|(1 - \chi(r/t)) s^{-1} u\|_{L^2_f(\mathcal{H}_s)}.$$

Our estimate of $\|(1-\chi(r/t))s^{-1}u\|_{L^2_f(\mathcal{H}_s)}$ is based on the inequality $\big(1-\chi(r/t)\big)s^{-1}\leqslant Ct^{-1}$ so that, by Lemma 2.4,

$$\begin{aligned}\|(1-\chi(r/t))us^{-1}\|_{L^2_f(\mathcal{H}_s)}&\leqslant\|t^{-1}u\|_{L^2_f(\mathcal{H}_s)}\\&\leqslant\|r^{-1}u\|_{L^2_f(\mathcal{H}_s)}\leqslant C\sum_a\|\underline{\partial}_a u\|_{L^2_f(\mathcal{H}_s)}.\end{aligned}\tag{2.18}$$

The estimate near the light cone is more delicate and we first observe that, in the region $\mathcal{K}_{[2,s]}$ of interest, $\chi(r/t)\lesssim\frac{\chi(r/t)r}{(1+r^2)^{1/2}}$ and, thus,

$$\|\chi(r/t)s^{-1}u\|_{L^2_f(\mathcal{H}_s)}\leqslant C\|r(1+r^2)^{-1/2}\chi(r/t)s^{-1}u\|_{L^2_f(\mathcal{H}_s)},$$

and the right-hand side of this inequality is controlled as follows. We introduce the vector field $W=\big(0,-x^a\frac{t(u\chi(r/t))^2}{(1+r^2)s^2}\big)$ and compute its divergence

$$\begin{aligned}\operatorname{div}W=&-2s^{-1}\partial_a u\frac{r\chi(r/t)u}{(1+r^2)^{1/2}s}\cdot\frac{x^a t\chi(r/t)}{r(1+r^2)^{1/2}}-2s^{-1}\frac{u}{r}\frac{r\chi(r/t)u}{s(1+r^2)^{1/2}}\cdot\frac{\chi'(r/t)r}{(1+r^2)^{1/2}}\\&-\left(\frac{r^2t+3t}{(1+r^2)^2s^2}+2\frac{r^2t}{(1+r^2)s^4}\right)\big(u\chi(r/t)\big)^2.\end{aligned}\tag{2.19}$$

By applying Stokes' theorem in the region $\mathcal{K}_{[2,s_1]}$, we find

$$\begin{aligned}\int_{\mathcal{K}_{[2,s_1]}}\operatorname{div}W\,dxdt&=\int_{\mathcal{H}_s}W\cdot n\,d\sigma+\int_{\mathcal{H}_2}W\cdot n\,d\sigma\\&=\int_{\mathcal{H}_s}\frac{r^2}{1+r^2}\big|u\chi(r/t)s^{-1}\big|^2dx-\int_{\mathcal{H}_2}\frac{r^2}{1+r^2}\big|u\chi(r/t)s^{-1}\big|^2dx.\end{aligned}$$

Differentiating this identity with respect to s leads us to

$$\begin{aligned}\frac{d}{ds}\left(\int_{\mathcal{K}_{[2,s_1]}}\operatorname{div}W\,dxdt\right)&=\frac{d}{ds}\left(\int_{\mathcal{H}_s}\frac{r^2}{1+r^2}\big|u\chi(r/t)s^{-1}\big|^2dx\right)\\&=2\left\|\frac{ru\chi(r/t)}{s(1+r^2)^{1/2}}\right\|_{L^2(\mathcal{H}_s)}\frac{d}{ds}\left\|\frac{ru\chi(r/t)}{s(1+r^2)^{1/2}}\right\|_{L^2(\mathcal{H}_s)}.\end{aligned}\tag{2.20}$$

We then integrate (2.19) in the region $\mathcal{K}_{[2,s_1]}\subset\mathcal{K}\cap\{2\leqslant\sqrt{t^2-r^2}\leqslant s_1\}$:

$$\begin{aligned}\int_{\mathcal{K}_{[2,s_1]}}\operatorname{div}W\,dxdt=&-2\int_{\mathcal{K}_{[2,s_1]}}s^{-1}\left(\partial_a u\frac{r\chi(r/t)u}{(1+r^2)^{1/2}s}\frac{x^a t\chi(r/t)}{r(1+r^2)^{1/2}}\right)dxdt\\&-2\int_{\mathcal{K}_{[2,s_1]}}s^{-1}\frac{u}{r}\frac{r\chi(r/t)u}{s(1+r^2)^{1/2}}\frac{\chi'(r/t)r}{(1+r^2)^{1/2}}dxdt\\&-\int_{\mathcal{K}_{[2,s_1]}}\left(\frac{r^2t+3t}{(1+r^2)^2s^2}+2\frac{r^2t}{(1+r^2)s^4}\right)\big(u\chi(r/t)\big)^2dxdt,\end{aligned}$$

which yields the identity

$$\begin{aligned}\int_{\mathcal{K}_{[2,s_1]}}\operatorname{div}W\,dxdt=&-2\int_2^{s_1}\int_{\mathcal{H}_s}(s/t)s^{-1}\left(\partial_a u\frac{r\chi(r/t)u}{(1+r^2)^{1/2}s}\frac{x^a t\chi(r/t)}{r(1+r^2)^{1/2}}\right)dxds\\&-2\int_2^{s_1}\int_{\mathcal{H}_s}(s/t)s^{-1}\frac{u}{r}\frac{r\chi(r/t)u}{s(1+r^2)^{1/2}}\frac{\chi'(r/t)r}{(1+r^2)^{1/2}}dxds\\&-\int_2^{s_1}\int_{\mathcal{H}_s}(s/t)\left(\frac{r^2t+3t}{(1+r^2)^2s^2}+2\frac{r^2t}{(1+r^2)s^4}\right)\big(u\chi(r/t)\big)^2dxds\\=:&\int_2^{s_1}\big(T_1+T_2+T_3\big)\,ds.\end{aligned}$$

Here, we have $T_3 \leqslant 0$, while

$$\begin{aligned}
T_1 =& -2s^{-1}\int_{\mathcal{H}_s}(s/t)\bigg(\partial_a u\frac{r\chi(r/t)u}{(1+r^2)^{1/2}s}\frac{x^a t\chi(r/t)}{r(1+r^2)^{1/2}}\bigg)\,dx\\
\leqslant& 2s^{-1}\bigg\|\frac{ru\chi(r/t)}{s(1+r^2)^{1/2}}\bigg\|_{L^2_f(\mathcal{H}_s)}\\
&\qquad\cdot\sum_a\|(s/t)\partial_a u\|_{L^2_f(\mathcal{H}_s)}\big\|\chi(r/t)x^a t r^{-1}(1+r^2)^{-1/2}\big\|_{L^\infty(\mathcal{H}_s)}\\
\leqslant& Cs^{-1}\bigg\|\frac{ru\chi(r/t)}{s(1+r^2)^{1/2}}\bigg\|_{L^2_f(\mathcal{H}_s)}\sum_a\|(s/t)\partial_a u\|_{L^2_f(\mathcal{H}_s)}
\end{aligned}$$

and

$$\begin{aligned}
T_2 =& -2s^{-1}\int_{\mathcal{H}_s}(s/t)\frac{u}{r}\frac{r\chi(r/t)u}{s(1+r^2)^{1/2}}\frac{\chi'(r/t)r}{(1+r^2)^{1/2}}dx\\
\leqslant& Cs^{-1}\bigg\|\frac{ru\chi(r/t)}{s(1+r^2)^{1/2}}\bigg\|_{L^2_f(\mathcal{H}_s)}\|ur^{-1}\|_{L^2_f(\mathcal{H}_s)}\big\|r\chi'(r/t)(1+r^2)^{-1/2}\big\|_{L^\infty(\mathcal{H}_s)}\\
\leqslant& Cs^{-1}\bigg\|\frac{ru\chi(r/t)}{s(1+r^2)^{1/2}}\bigg\|_{L^2_f(\mathcal{H}_s)}\sum_a\|\underline{\partial}_a u\|_{L^2_f(\mathcal{H}_s)},
\end{aligned}$$

where Lemma 2.4 was used.

We write our identity in the form $\frac{d}{ds}\Big(\int_{\mathcal{K}_{[2,s]}}\operatorname{div} W\,dxdt\Big) = T_1+T_2+T_3$ and obtain

$$\begin{aligned}
&\frac{d}{ds}\Big(\int_{\mathcal{K}_{[2,s_1]}}\operatorname{div} W\,dxdt\Big)\\
&\lesssim s^{-1}\bigg\|\frac{ru\chi(r/t)}{s(1+r^2)^{1/2}}\bigg\|_{L^2_f(\mathcal{H}_s)}\sum_a\Big(\|(s/t)\partial_a\|_{L^2_f(\mathcal{H}_s)}+\|\underline{\partial}_a u\|_{L^2_f(\mathcal{H}_s)}\Big).
\end{aligned}\tag{2.21}$$

Finally, combining (2.21) and (2.20) yields us

$$\frac{d}{ds}\bigg\|\frac{ru\chi(r/t)}{s(1+r^2)^{1/2}}\bigg\|_{L^2(\mathcal{H}_s)}\lesssim s^{-1}\sum_a\big(\|\frac{s}{t}\partial_a u\|_{L^2(\mathcal{H}_s)}+\|\underline{\partial}_a u\|_{L^2(\mathcal{H}_s)}\big)$$

and, by integration over $[2,s]$,

$$\begin{aligned}
&\big\|r(1+r^2)^{-1/2}\chi(r/t)s^{-1}u\big\|_{L^2_f(\mathcal{H}_s)}\\
&\leqslant\big\|r(1+r^2)^{-1/2}\chi(r/t)2^{-1}u\big\|_{L^2(\mathcal{H}_2)}+\sum_a\int_2^s\bar{s}^{-1}\big(\|\frac{\bar{s}}{t}\partial_a u\|_{L^2_f(\mathcal{H}_{\bar{s}})}+\|\underline{\partial}_a u\|_{L^2_f(\mathcal{H}_{\bar{s}})}\big)d\bar{s}.
\end{aligned}\tag{2.22}$$

From Lemma 2.4, we then deduce that

$$\begin{aligned}
\|\chi(r/t)s^{-1}u\|_{L^2_f(\mathcal{H}_s)}&\lesssim\|r(1+r^2)^{-1/2}\chi(r/t)s^{-1}u\|_{L^2_f(\mathcal{H}_s)}\\
&\lesssim\|2^{-1}u\|_{L^2(\mathcal{H}_2)}+\sum_a\int_2^s\bar{s}^{-1}\big(\|\frac{\bar{s}}{t}\partial_a u\|_{L^2_f(\mathcal{H}_{\bar{s}})}+\|\underline{\partial}_a u\|_{L^2_f(\mathcal{H}_{\bar{s}})}\big)d\bar{s},
\end{aligned}\tag{2.23}$$

and remains to combine (2.18) with (2.23). □

3. Sup-norm estimates for the wave and Klein-Gordon equations

3.1. **Statement for the wave equation.**

Proposition 3.1 (A sup-norm estimate for the wave equation with source)**.** *Let u be a spatially compactly supported to the wave equation*

$$\begin{aligned}
&-\Box u = f,\\
&u|_{t=2}=0,\qquad \partial_t u|_{t=2}=0,
\end{aligned}\tag{3.1}$$

where the source f is spatially compactly supported in $\mathcal{K}$ and satisfies the estimate

$$|f| \leqslant C_f t^{-2-\nu}(t-r)^{-1+\mu}$$

for some constants $C_f > 0$, $0 < \mu \leqslant 1/2$, and $0 < |\nu| \leqslant 1/2$. Then, the following estimate holds:

$$|u(t,x)| \lesssim \begin{cases} \frac{C_f}{\nu\mu}(t-r)^{\mu-\nu}t^{-1}, & 0 < \nu \leqslant 1/2, \\ \frac{C_f}{|\nu|\mu}(t-r)^{\mu}t^{-1-\nu}, & -1/2 \leqslant \nu < 0. \end{cases} \tag{3.2}$$

We recall that the energy estimate on wave equation does not control the solution itself but only its gradient. So when we apply the Sobolev inequality and obtain a sup-norm estimate (cf. for example [22]), there is no immediate estimate on the sup-norm of the solution itself. The estimate above yields a (sharp) sup-norm estimate on the solution itself and will play an essential role for the control of the quasi-linear term $u\partial_t\partial_t v$ in our model problem. We emphasize that the range $-1/2 \leqslant \nu < 0$ will only be used in the second part [23].

3.2. **Proof of the sup-norm estimate for the wave equation.** We now state a technical lemma and give the proof of Proposition 3.1, but postpone the proof of the lemma to the end of this section. Let $d\sigma$ be the Lebesgue measure on the sphere $\{|y| = 1-\lambda\}$ and $x \in \mathbb{R}^3$ with $r = |x|$. We are interested in controling the integral

$$I(\lambda) = I(\lambda, t, x/t) := \int_{|y|=1-\lambda, |\frac{x}{t}-y| \leqslant \lambda - t^{-1}} \frac{d\sigma(y)}{\left(\lambda - \left|\frac{x}{t} - y\right|\right)^{1-\mu}}.$$

Our bounds below are consistent with the obvious estimate where $x = 0$:

$$I(\lambda, t, 0) = 4\pi(2\lambda - 1)^{-1+\mu}(1-\lambda)^2. \tag{3.3}$$

Clearly, when $0 < \lambda \leqslant \frac{t-r+1}{2t}$, one has $I(\lambda) = 0$.

Lemma 3.2. *When $\frac{t-r+1}{2t} \leqslant \lambda \leqslant 1$, the following estimate holds:*

$$I(\lambda) \lesssim \begin{cases} \dfrac{\lambda t(1-\lambda)}{\mu r}\left(\dfrac{t-r}{t}\right)^{\mu}, & \dfrac{t-r+1}{2t} \leqslant \lambda \leqslant \dfrac{t+r+1}{2t}, \\ (1-\lambda)\left(\dfrac{t+r}{t} - \lambda\right)\left(2\lambda - \dfrac{t+r}{t}\right)^{-1+\mu}, & \dfrac{t+r+1}{2t} \leqslant \lambda \leqslant \dfrac{t-r}{t}, \\ & \text{provided } \dfrac{t+r+1}{2t} \leqslant \dfrac{t-r}{t}, \\ \dfrac{(1-\lambda)t}{\mu r}\left(\dfrac{t-r}{t}\right)^{\mu}, & \max\left(\dfrac{t-r}{t}, \dfrac{t+r+1}{2t}\right) \leqslant \lambda \leqslant 1. \end{cases}$$

Proof of Proposition 3.1. From the explicit expression

$$u(t,x) = \frac{1}{4\pi}\int_2^t \frac{1}{t-\bar{s}} \int_{|y|=t-\bar{s}} f(\bar{s}, x-y)\, d\sigma d\bar{s}, \tag{3.4}$$

in which the integration is made on the intersection of the cone $\{(\bar{s}, y)\,/\,|y-x| = t-\bar{s}, 2 \leqslant \bar{s} \leqslant t\}$ and $\{(t,x)\,/\,r < t-1, t^2 - r^2 \leqslant s^2, t \geqslant 2\}$, we obtain

$$\begin{aligned} |u(t,x)| &\leqslant \frac{C_f}{4\pi}\int_2^t \int_{|y|=t-\bar{s}, |x-y| \leqslant \bar{s}-1} \frac{\bar{s}^{-2-\nu}(\bar{s} - |x-y|)^{-1+\mu}}{t-\bar{s}} d\sigma d\bar{s} \\ &= \frac{C_f}{4\pi t^{1+\nu-\mu}}\int_{\frac{2}{t}}^1 \int_{|y'|=1-\lambda, |\frac{x}{t}-y'| \leqslant \lambda - t^{-1}} \frac{(1-\lambda)^{-1}\lambda^{-2-\nu} d\sigma d\lambda}{\left(\lambda - \left|\frac{x}{t} - y'\right|\right)^{1-\mu}} \quad (\lambda := \bar{s}/t, \quad y' := y/t) \\ &= \frac{C_f}{4\pi t^{1+\nu-\mu}}\int_{\frac{2}{t}}^1 (1-\lambda)^{-1}\lambda^{-2-\nu} \int_{|y'|=1-\lambda, |\frac{x}{t}-y'| \leqslant \lambda - t^{-1}} \frac{d\sigma}{\left(\lambda - \left|\frac{x}{t} - y'\right|\right)^{1-\mu}}\, d\lambda. \end{aligned}$$

When $|\frac{x}{t} - y'| \leqslant \lambda - t^{-1}$ holds, we obtain $\frac{t-r+1}{2t} \leqslant \lambda \leqslant 1$. For convenience in the notation, in the following calculation we replace y' by y. We first assume that $r > 0$ and we distinguish between two main cases:

Case 1: $\frac{t-r}{t} > \frac{t+r+1}{2t} \Leftrightarrow r \leqslant \frac{t-1}{3}$. In Lemma 3.2, all three cases are possible:

$$\begin{aligned}
|u(t,x)| \leqslant & \frac{C_f}{4\pi t^{1+\nu-\mu}} \int_{\frac{t-r+1}{2t}}^1 (1-\lambda)^{-1}\lambda^{-2-\nu} \int_{|y|=1-\lambda, |\frac{x}{t}-y|\leqslant \lambda - t^{-1}} \frac{d\sigma}{\left(\lambda - \left|\frac{x}{t}-y\right|\right)^{1-\mu}}\, d\lambda \\
\lesssim & \frac{C_f}{\mu t^{1+\nu-\mu}} \int_{\frac{t-r+1}{2t}}^{\frac{t+r+1}{2t}} (1-\lambda)^{-1}\lambda^{-2-\nu} \frac{\lambda t(1-\lambda)}{r} \left(\frac{t-r}{t}\right)^{\mu} d\lambda \\
& + \frac{C_f}{t^{1+\nu-\mu}} \int_{\frac{t+r+1}{2t}}^{\frac{t-r}{t}} (1-\lambda)^{-1}\lambda^{-2-\nu}(1-\lambda) \left(\frac{t+r}{t}-\lambda\right) \left(2\lambda - \frac{t+r}{t}\right)^{-1+\mu} d\lambda \\
& + \frac{C_f}{\mu t^{1+\nu-\mu}} \int_{\frac{t-r}{t}}^1 (1-\lambda)^{-1}\lambda^{-2-\nu} \frac{(1-\lambda)t}{r} \left(\frac{t-r}{t}\right)^{\mu} d\lambda,
\end{aligned}$$

thus

$$\begin{aligned}
|u(t,x)| \lesssim & \frac{C_f}{\mu t^{1+\nu-\mu}} \frac{t}{r} \left(\frac{t-r}{t}\right)^{\mu} \int_{\frac{t-r+1}{2t}}^{\frac{t+r+1}{2t}} \lambda^{-1-\nu}\, d\lambda \\
& + \frac{C_f}{t^{1+\nu-\mu}} \int_{\frac{t+r+1}{2t}}^{\frac{t-r}{t}} \lambda^{-2-\nu} \left(\frac{t+r}{t}-\lambda\right) \left(2\lambda - \frac{t+r}{t}\right)^{-1+\mu} d\lambda \\
& + \frac{C_f}{\mu t^{1+\nu-\mu}} \frac{t}{r} \left(\frac{t-r}{t}\right)^{\mu} \int_{\frac{t-r}{t}}^1 \lambda^{-2-\nu}\, d\lambda.
\end{aligned}$$

For the first integral, we recall that $r \leqslant \frac{t-1}{3}$ and that $0 < |\nu| \leqslant 1/2$, and write

$$\frac{t}{r} \int_{\frac{t-r+1}{2t}}^{\frac{t+r+1}{2t}} \lambda^{-1-\nu}\, d\lambda \lesssim \left(\frac{t}{t-r}\right)^{1+\nu} \lesssim 1,$$

so that

$$\left| \frac{C_f}{\mu t^{1+\nu-\mu}} \frac{t}{r} \left(\frac{t-r}{t}\right)^{\mu} \int_{\frac{t-r+1}{2t}}^{\frac{t+r+1}{2t}} \lambda^{-1-\nu}\, d\lambda \right| \lesssim C_f \mu^{-1} (t-r)^{\mu} t^{-1-\nu}.$$

Next, for the second integral in the right-hand-side, we just remark that

$$\begin{aligned}
& \int_{\frac{t+r+1}{2t}}^{\frac{t-r}{t}} \lambda^{-2-\nu} \left(\frac{t+r}{t}-\lambda\right) \left(2\lambda - \frac{t+r}{t}\right)^{-1+\mu} d\lambda \\
& \lesssim \int_{\frac{t+r+1}{2t}}^{\frac{t-r}{t}} \left(2\lambda - \frac{t+r}{t}\right)^{-1+\mu} d\lambda = \frac{1}{\mu} \left(2\lambda - \frac{t+r}{t}\right)^{\mu} \Big|_{\frac{t+r+1}{2t}}^{\frac{t-r}{t}} \lesssim \frac{1}{\mu}.
\end{aligned}$$

This leads to

$$\frac{C_f}{t^{1+\nu-\mu}} \int_{\frac{t+r+1}{2t}}^{\frac{t-r}{t}} \lambda^{-2-\nu} \left(\frac{t+r}{t}-\lambda\right) \left(2\lambda - \frac{t+r}{t}\right)^{-1+\mu} d\lambda \lesssim \frac{C_f}{\mu t^{1+\nu-\mu}}.$$

For the third term, in view of $\frac{t-r}{t} \geqslant \frac{t+r+t1}{2t} \geqslant \frac{1}{2}$, we obtain

$$\begin{aligned}
\frac{C_f}{\mu t^{1+\nu-\mu}} \frac{t}{r} \left(\frac{t-r}{t}\right)^{\mu} \int_{\frac{t-r}{t}}^1 \lambda^{-2-\nu}\, d\lambda \lesssim & \frac{C_f}{\mu t^{1+\nu-\mu}} \frac{t}{r} \left(\frac{t-r}{t}\right)^{\mu} \int_{\frac{t-r}{t}}^1 2^{2+\mu}\, d\lambda \\
\lesssim & C_f \mu^{-1} (t-r)^{\mu} t^{-1-\nu}.
\end{aligned}$$

So we conclude that in the case $0 < r \leqslant \frac{t-1}{3}$, $|u(t,x)| \lesssim C_f \mu^{-1}(t-r)^{\mu} t^{-1-\nu}$.

Case 2: $\frac{t+r+1}{2t} \geqslant \frac{t-r}{t} \Leftrightarrow r \geqslant \frac{t-1}{3}$. The second case in Lemma 3.2 is not possible, and we have

$$|u(t,x)| \lesssim \frac{C_f}{\mu t^{1+\nu-\mu}} \left(\frac{t-r}{t}\right)^{\mu} \left(\int_{\frac{t-r+1}{2t}}^{\frac{t+r+1}{2t}} \lambda^{-1-\nu}\, d\lambda + \int_{\frac{t+r+1}{2t}}^1 \lambda^{-2-\nu}\, d\lambda \right).$$

Since $\frac{t+r+1}{2t} \geqslant 1/2$, the second integral is bounded by a constant C. For the first integral, we see that when $\nu > 0$,

$$\int_{\frac{t-r+1}{2t}}^{\frac{t+r+1}{2t}} \lambda^{-1-\nu}\, d\lambda \lesssim \frac{1}{\nu}\left(\frac{t-r+1}{t}\right)^{-\nu}.$$

So in this case when $\nu > 0$, we obtain $|u(t,x)| \lesssim C_f(\mu\nu)^{-1}(t-r)^{\mu-\nu}t^{-1}$.

When $\nu < 0$, we write

$$\int_{\frac{t-r+1}{2t}}^{\frac{t+r+1}{2t}} \lambda^{-1-\nu}\, d\lambda \lesssim \frac{1}{|\nu|}\left(\frac{t+r+1}{t}\right)^{-\nu} \lesssim \frac{1}{|\nu|}$$

and, therefore, we obtain $|u(t,x)| \lesssim C_f(\mu|\nu|)^{-1}(t-r)^{\mu}t^{-1-\nu}$.

When $r = 0$, we make the following direct calculation, remark that in this case, $\frac{t+1}{2t} \leqslant \lambda \leqslant 1$, by (3.3):

$$\begin{aligned}
|u(t,x)| &\leqslant \frac{C_f}{4\pi t^{1+\nu-\mu}} \int_{\frac{t-r+1}{2t}}^{1} (1-\lambda)^{-1}\lambda^{-2-\nu}\, d\lambda \int_{|y|=1-\lambda,\,|\frac{x}{t}-y|\leqslant\lambda-t^{-1}} \frac{d\sigma}{\left(\lambda-\left|\frac{x}{t}-y\right|\right)^{1-\mu}} \\
&\lesssim \frac{C_f}{t^{1+\nu-\mu}} \int_{\frac{t+1}{2t}}^{1} (1-\lambda)^{-1}\lambda^{-2-\nu}(2-\lambda)^{-1+\mu}(1-\lambda)^2 d\lambda \\
&\lesssim \frac{C_f}{\mu} t^{-1-\nu+\mu} = \frac{C_f}{\mu}(t-r)^{\mu}t^{-1-\nu} \qquad (\text{since } r=0),
\end{aligned}$$

which completes the proof. □

Proof of Lemma 3.2. When $r = 0$, the estimate is trivial. When $r > 0$, without loss of generality, let $x = (r,0,0)$. The surface $S_\lambda := \{|y| = 1-\lambda\} \cap \{\left|\frac{x}{t} - y\right| \leqslant \lambda - t^{-1}\}$ is parameterized as follows:

- θ: angle from $(1,0,0)$ to y with $0 \leqslant \theta \leqslant \pi$,
- ϕ: angle from the plane determined by $(1,0,0)$ and $(0,1,0)$ to the plane determined by y and $(1,0,0)$, with $0 \leqslant \phi \leqslant 2\pi$.

Then, we have $y = (1-\lambda)\big(\cos\theta, \sin\theta\cos\phi, \sin\theta\sin\phi\big)$ and we distinguish between two cases, as follows.

Case 1. When $\frac{t-r+1}{2t} \leqslant \lambda \leqslant \frac{t+r+1}{2t}$, we only have a part of the sphere $\{|y| = 1-\lambda\}$ contained in the ball $\{\left|\frac{x}{t} - y\right| \leqslant \lambda - t^{-1}\}$ where $\cos(\theta) \geqslant \frac{(r/t)^2+(1-\lambda)^2-\left(\lambda-t^{-1}\right)^2}{(2r/t)(1-\lambda)}$. So we set $\theta_0 := \arccos\left(\frac{(r/t)^2+(1-\lambda)^2-\left(\lambda-t^{-1}\right)^2}{(2r/t)(1-\lambda)}\right)$ and see that

$$\lambda - \left|\frac{x}{t} - y\right| = \lambda - \sqrt{\frac{r^2}{t^2} + (1-\lambda)^2 - 2\frac{r}{t}(1-\lambda)\cos\theta}$$

and $d\sigma = (1-\lambda)^2\sin(\theta)d\theta d\phi$. The integral is estimated as follows:

$$\begin{aligned}
&\int_{|y|=1-\lambda,\,|\frac{x}{t}-y|\leqslant\lambda-t^{-1}} \frac{d\sigma}{\left(\lambda-\left|\frac{x}{t}-y\right|\right)^{1-\mu}} \\
&= \int_0^{2\pi} d\phi \int_0^{\theta_0} (1-\lambda)^2\sin\theta\left(\lambda - \sqrt{\frac{r^2}{t^2} + (1-\lambda)^2 - 2\frac{r}{t}(1-\lambda)\cos\theta}\right)^{-1+\mu} d\theta \\
&= 2\pi \int_0^{\theta_0} (1-\lambda)^2\sin\theta\left(\lambda - \sqrt{\frac{r^2}{t^2} + (1-\lambda)^2 - 2\frac{r}{t}(1-\lambda)\cos\theta}\right)^{-1+\mu} d\theta \\
&= -2\pi(1-\lambda)^2 \int_0^{\theta_0} \left(\lambda - \sqrt{\frac{r^2}{t^2} + (1-\lambda)^2 - 2\frac{r}{t}(1-\lambda)\cos\theta}\right)^{-1+\mu} d\cos\theta
\end{aligned}$$

thus by setting $\omega = \cos\theta$

$$\begin{aligned}
&\int_{|y|=1-\lambda, |\frac{x}{t}-y|\leqslant\lambda-t^{-1}} \frac{d\sigma}{\left(\lambda-\left|\frac{x}{t}-y\right|\right)^{1-\mu}} \\
&=2\pi(1-\lambda)^2\int_{\cos\theta_0}^1 \left(\lambda-\sqrt{\frac{r^2}{t^2}+(1-\lambda)^2-2\frac{r}{t}(1-\lambda)\omega}\right)^{-1+\mu} d\omega \\
&=\frac{\pi t(1-\lambda)}{r}\int_{|\frac{r}{t}-(1-\lambda)|^2}^{(\lambda-t^{-1})^2}\left(\lambda-\sqrt{\gamma}\right)^{-1+\mu}d\gamma = 2\frac{\pi t(1-\lambda)}{r}\int_{t^{-1}}^{\lambda-|\frac{r}{t}-(1-\lambda)|}\zeta^{-1+\mu}(\lambda-\zeta)\,d\zeta,
\end{aligned}$$

where we have used $\gamma = \frac{r^2}{t^2}+(1-\lambda)^2-2\frac{r}{t}(1-\lambda)\omega$ and $\zeta := \lambda-\sqrt{\gamma}$. Then, we distinguish between the following two sub-cases.

Case 1.1: $\frac{r}{t}\leqslant 1-\lambda$ or, equivalently, $\lambda\leqslant\frac{t-r}{t}$. We now find

$$\begin{aligned}
&2\frac{\pi t(1-\lambda)}{r}\int_{t^{-1}}^{\lambda-|\frac{r}{t}-(1-\lambda)|}\zeta^{-1+\mu}(\lambda-\zeta)\,d\zeta \\
&= 2\frac{\pi t(1-\lambda)}{r}\int_{t^{-1}}^{2(\lambda-\frac{t-r}{2t})}\zeta^{-1+\mu}(\lambda-\zeta)\,d\zeta \lesssim \frac{\lambda t(1-\lambda)}{\mu r}\frac{(t-r)^\mu}{t^\mu}.
\end{aligned}$$

Case 1.2: $1-\lambda<\frac{r}{t}$ or, equivalently, $\lambda>\frac{t-r}{t}$. We find

$$\begin{aligned}
&2\frac{\pi t(1-\lambda)}{r}\int_{t^{-1}}^{\lambda-|\frac{r}{t}-(1-\lambda)|}\zeta^{-1+\mu}(\lambda-\zeta)\,d\zeta \\
&= 2\frac{\pi t(1-\lambda)}{r}\int_{t^{-1}}^{\frac{t-r}{t}}\zeta^{-1+\mu}(\lambda-\zeta)\,d\zeta \lesssim \frac{\lambda t(1-\lambda)}{\mu r}\frac{(t-r)^\mu}{t^\mu}.
\end{aligned}$$

Case 2. When $\frac{t+r+1}{2t}\leqslant\lambda\leqslant 1$, the sphere $\{|y|=1-\lambda\}$ is entirely contained in $\{|(x/t)-y|\leqslant\lambda-t^{-1}\}$:

$$\begin{aligned}
&\int_{|y|=1-\lambda, |\frac{x}{t}-y|\leqslant\lambda-t^{-1}} \frac{d\sigma}{\left(\lambda-\left|\frac{x}{t}-y\right|\right)^{1-\mu}} = \int_{|y|=1-\lambda} \frac{d\sigma}{\left(\lambda-\left|\frac{x}{t}-y\right|\right)^{1-\mu}} \\
&= 2\pi\int_0^\pi (1-\lambda)^2\sin\theta\left(\lambda-\sqrt{\frac{r^2}{t^2}+(1-\lambda)^2-2\frac{r}{t}(1-\lambda)\cos\theta}\right)^{-1+\mu}d\theta \\
&= 2\pi(1-\lambda)^2\int_{-1}^1\left(\lambda-\sqrt{\frac{r^2}{t^2}+(1-\lambda)^2-2\frac{r}{t}(1-\lambda)\omega}\right)^{-1+\mu}d\omega
\end{aligned}$$

and thus

$$\begin{aligned}
\int_{|y|=1-\lambda, |\frac{x}{t}-y|\leqslant\lambda-t^{-1}} \frac{d\sigma}{\left(\lambda-\left|\frac{x}{t}-y\right|\right)^{1-\mu}} &= 2\frac{\pi t(1-\lambda)}{r}\int_{\lambda-(\frac{r}{t}+(1-\lambda))}^{\lambda-|\frac{r}{t}-(1-\lambda)|}\zeta^{-1+\mu}(\lambda-\zeta)\,d\zeta \\
&= 2\frac{\pi t(1-\lambda)}{r}\int_{2\lambda-\frac{t+r}{t}}^{\lambda-|\frac{r}{t}-(1-\lambda)|}\zeta^{-1+\mu}(\lambda-\zeta)\,d\zeta.
\end{aligned}$$

We now distinguish between two sub-cases.

Case 2.1: When $\frac{r}{t}\leqslant 1-\lambda$ or, equivalently, $\lambda\leqslant\frac{t-r}{t}$, we find

$$\begin{aligned}
&2\frac{\pi t(1-\lambda)}{r}\int_{2\lambda-\frac{t+r}{t}}^{\lambda-|\frac{r}{t}-(1-\lambda)|}\zeta^{-1+\mu}(\lambda-\zeta)\,d\zeta \\
&= 2\frac{\pi t(1-\lambda)}{r}\int_{2\lambda-\frac{t+r}{t}}^{2\lambda-\frac{t-r}{t}}\zeta^{-1+\mu}(\lambda-\zeta)\,d\zeta \leqslant C(1-\lambda)\left(\frac{t+r}{t}-\lambda\right)\left(2\lambda-\frac{t+r}{t}\right)^{-1+\mu},
\end{aligned}$$

where we have observed that in the integral the function $\zeta^{-1+\mu}(\lambda-\zeta)$ is decreasing and we can bound this integral by the value of the function taken at the inferior boundary (which is $2\lambda-\frac{t+r}{t}$) times the length of the interval which is $2r/t$.

Case 2.2: When $1-\lambda<\frac{r}{t}$ or, equivalently, $\lambda>\frac{t-r}{t}$, we have

$$\begin{aligned}
&2\frac{\pi t(1-\lambda)}{r}\int_{2\lambda-\frac{t+r}{t}}^{\lambda-|\frac{r}{t}-(1-\lambda)|}\zeta^{-1+\mu}(\lambda-\zeta)\,d\zeta\\
&=2\frac{\pi t(1-\lambda)}{r}\int_{2\lambda-\frac{t+r}{t}}^{\frac{t-r}{t}}\zeta^{-1+\mu}(\lambda-\zeta)\,d\zeta\leqslant C(1-\lambda)\frac{t}{r}\int_{2\lambda-\frac{t+r}{t}}^{\frac{t-r}{t}}\zeta^{-1+\mu}d\zeta\\
&\leqslant\frac{C(1-\lambda)t}{\mu r}\zeta^{\mu}\Big|_0^{\frac{t-r}{r}}=\frac{C(1-\lambda)t}{\mu r}\left(\frac{t-r}{t}\right)^{\mu}.
\end{aligned}$$

When $\frac{t+r+1}{2t}\leqslant\frac{t-r}{t}$, both case above may occur, while only Case 2.2 is possible if the opposite inequality holds true. □

3.3. **Statement for the Klein-Gordon equation.** Consider (sufficiently smooth and spatially compactly supported) solutions to a Klein-Gordon equation on a curved space and, specifically,

$$\begin{aligned}
&-\widetilde{\Box}_g v+c^2v=f,\\
&v|_{\mathcal{H}_2}=v_0,\qquad \partial_t v|_{\mathcal{H}_2}=v_1,
\end{aligned}\tag{3.5}$$

with initial data v_0, v_1 given on $\mathcal{H}_2$ and compactly supported in $\mathcal{H}_2\cap\mathcal{K}$, and the metric has the form $g^{\alpha\beta}=m^{\alpha\beta}-h^{\alpha\beta}$ with $h^{\alpha\beta}$ is spatially compactly supported in $\mathcal{K}$ with $\sup|\overline{h}^{00}|\leqslant 1/3$.

Before we can state our estimate, we need some notation. Given a constant $C>0$ and using the notation $s=\sqrt{t^2-r^2}$, we consider the function

$$h_{t,x}(\lambda):=\overline{h}^{00}(\lambda t/s,\lambda x/s),$$

and, by denoting by $h'_{t,x}(\lambda)$ for the derivative with respect to λ,

$$\begin{aligned}
h'_{t,x}(\lambda)&=\frac{t}{s}\partial_t\overline{h}^{00}(\lambda t/s,\lambda x/s)+\frac{x^a}{s}\partial_a\overline{h}^{00}(\lambda t/s,\lambda x/s)\\
&=\frac{t}{s}\underline{\partial}_{\perp}\overline{h}^{00}(\lambda t/s,\lambda x/s).
\end{aligned}$$

We set

$$s_0:=\begin{cases}2, & 0\leqslant r/t\leqslant 3/5,\\ \sqrt{\dfrac{t+r}{t-r}}, & 3/5\leqslant r/t\leqslant 1,\end{cases}\tag{3.6}$$

and introduce the following function V which is defined by distinguishing between the regions "near" and "far" from the light cone:

$$V:=\begin{cases}\big(\|v_0\|_{L^\infty(\mathcal{H}_2)}+\|v_1\|_{L^\infty(\mathcal{H}_2)}\big)\Big(1+\displaystyle\int_2^s|h'_{t,x}(\bar s)|e^{C\int_{\bar s}^s|h'_{t,x}(\lambda)|d\lambda}\,d\bar s\Big)\\ \qquad\qquad+F(s)+\displaystyle\int_2^s F(\bar s)|h'_{t,x}(\lambda)|e^{C\int_{\bar s}^s|h'_{t,x}(\lambda)|d\lambda}\,d\bar s, & 0\leqslant r/t\leqslant 3/5,\\ F(s)+\displaystyle\int_{s_0}^s F(\bar s)|h'_{t,x}(\bar s)|e^{C\int_{\bar s}^s|h'_{t,x}(\lambda)|d\lambda}\,d\bar s, & 3/5<r/t<1,\end{cases}$$

with

$$F(\bar s):=\int_{s_0}^{\bar s}\Big(\big(R_1[v]+R_2[v]+R_3[v]\big)(\lambda t/s,\lambda x/s)+\lambda^{3/2}f(\lambda t/s,\lambda x/s)\Big)d\lambda$$

and

$$R_1[v] := s^{3/2}\sum_a \overline{\partial}_a\overline{\partial}_a v + \frac{x^a x^b}{s^{1/2}}\overline{\partial}_a\overline{\partial}_b v + \frac{3}{4s^{1/2}}v + \sum_a \frac{3x^a}{s^{1/2}}\overline{\partial}_a v,$$

$$R_2[v] := \overline{h}^{00}\left(\frac{3v}{4s^{1/2}} + 3s^{1/2}\overline{\partial}_0 v\right) - s^{3/2}\big(2\overline{h}^{0b}\overline{\partial}_0\overline{\partial}_b v + \overline{h}^{ab}\overline{\partial}_a\overline{\partial}_b v + h^{\alpha\beta}\partial_\alpha\overline{\Psi}_\beta^{\beta'}\,\overline{\partial}_{\beta'}v\big),$$

$$R_3[v] := \overline{h}^{00}\left(2x^a s^{1/2}\overline{\partial}_0\overline{\partial}_a v + \frac{2x^a}{s^{1/2}}\overline{\partial}_a v + \frac{x^a x^b}{s^{1/2}}\overline{\partial}_a\overline{\partial}_b v\right).$$

With these notations, our result is as follows.

Proposition 3.3 (A sup-norm estimate for the Klein-Gordon equation with source)**.** *Considering the Klein-Gordon problem* (3.5) *for a every sufficiently smooth and spatially compactly supported solution v defined the future region $\mathcal{K}_{[2,+\infty)}$, one has (for all relevant (t,x))*

$$s^{3/2}|v(t,x)| + (s/t)^{-1}s^{3/2}|\underline{\partial}_\perp v(t,x)| \lesssim V(t,x), \tag{3.7}$$

This result is motivated by a pioneering work by Klainerman [19] and the decomposition in Lemma 3.4 below. An analogue statement in two spatial dimensions and flat Minkowski spacetime is discussed in [28]; see also the earlier work [9].

3.4. **Proof of the sup-norm estimate for the Klein-Gordon equation.** We begin with two technical results.

Lemma 3.4 (A decomposition identity)**.** *For every sufficiently smooth solution v to* (3.5)*, the function*

$$w_{t,x}(\lambda) := \lambda^{3/2}v(\lambda t/s, \lambda x/s), \qquad (t,x)\in\mathcal{K},$$

satisfies the following second-order ODE in λ

$$\begin{aligned}&\frac{d^2}{d\lambda^2}w_{t,x}(\lambda) + \frac{c^2}{1+\overline{h}^{00}(\lambda t/s,\lambda x/s)}w_{t,x}(\lambda)\\ &= \big(1+\overline{h}^{00}(\lambda t/s,\lambda x/s)\big)^{-1}\big(R_1[v]+R_2[v]+R_3[v]+s^{3/2}f\big)(\lambda t/s,\lambda x/s).\end{aligned}$$

Lemma 3.5 (Technical ODE estimate)**.** *Let G be a function defined on an interval $[s_0,s_1]$ and satisfying* $\sup|G| \leqslant 1/3$*. and k be an integrable function defined on $[s_0,s_1]$. Then, the solution z to the ordinary differential equation*

$$\begin{aligned}&z''(\lambda) + \frac{c^2}{1+G(\lambda)}z(\lambda) = k(\lambda),\\ &z(s_0) = z_0, \qquad z'(s_0) = z_1,\end{aligned} \tag{3.8}$$

with prescribed initial data z_0, z_1 satisfies the uniform bound

$$|z(s)| + |z'(s)| \lesssim \big(|z_0|+|z_1|+K(s)\big) + \int_{s_0}^{s}\Big(|z_0|+|z_1|+K(\bar s)\Big)\,|G'(\bar s)|e^{C\int_{\bar s}^{s}|G'(\lambda)|d\lambda}\,d\bar s \tag{3.9}$$

for all $s\in[s_0,s_1]$ and with $K(s) := \int_{s_0}^{s}|k(\bar s)|\,d\bar s$ and a suitable constant $C>0$.

Proof of Lemma 3.4. **1. Decomposition of the flat wave operator.** By recalling $s=\sqrt{t^2-r^2}$ and $r=|x|$, an elementary calculation shows that the flat wave operator $\Box$ in the hyperboloidal frame reads

$$-\Box = \overline{\partial}_0\overline{\partial}_0 - \sum_a \overline{\partial}_a\overline{\partial}_a + 2\sum_a\frac{x^a}{s}\overline{\partial}_0\overline{\partial}_a + \frac{3}{s}\overline{\partial}_0. \tag{3.10}$$

Given a function v, we can set

$$w(t,x) = s^{3/2}v(t,x) = (t^2-|x|^2)^{3/4}v(t,x),$$

and obtain

$$-s^{3/2}\Box v = \overline{\partial}_0\overline{\partial}_0 w - \sum_a\overline{\partial}_a\overline{\partial}_a w + 2\sum_a\frac{x^a}{s}\overline{\partial}_0\overline{\partial}_a w - \frac{3w}{4s^2} - \sum_a\frac{3x^a\overline{\partial}_a w}{s^2}. \tag{3.11}$$

Again, we define a function of a single variable by

$$w_{t,x}(\lambda) := w(\lambda t/s, \lambda x/s) = \lambda^{3/2} v(\lambda t/s, \lambda x/s).$$

We see that

$$\frac{d}{d\lambda} w_{t,x}(\lambda) = \big(\overline{\partial}_0 + s^{-1} x^a \overline{\partial}_a\big) w(\lambda t/s, \lambda x/s) = \frac{t}{s} \underline{\partial}_\perp w\,(\lambda t/s, \lambda x/s)$$

and

$$\frac{d^2}{d\lambda^2} w_{t,x}(\lambda) = \left(\overline{\partial}_0 \overline{\partial}_0 + 2\frac{x^a}{s} \overline{\partial}_0 \overline{\partial}_a + \frac{x^a x^b}{s^2} \overline{\partial}_a \overline{\partial}_b\right) w(\lambda t/s, \lambda x/s). \tag{3.12}$$

Combining this with (3.11) and recalling that $w(t,x) = s^{3/2} v(t,x)$, we get

$$\begin{aligned} &\left(\overline{\partial}_0 \overline{\partial}_0 + 2\frac{x^a}{s} \overline{\partial}_0 \overline{\partial}_a + \frac{x^a x^b}{s^2} \overline{\partial}_a \overline{\partial}_b\right) w \\ &= -s^{3/2} \Box v + \sum_a \overline{\partial}_a \overline{\partial}_a w + \frac{x^a x^b}{s^2} \overline{\partial}_a \overline{\partial}_b w + \frac{3}{4s^2} w + \sum_a \frac{3x^a}{s^2} \overline{\partial}_a w = -s^{3/2} \Box v + R_1[v]. \end{aligned} \tag{3.13}$$

2. Decomposition of the curved wave operator. We write

$$-\Box v = h^{\alpha\beta} \partial_\alpha \partial_\beta v - c^2 v + f$$

and, by performing a change of frame,

$$\begin{aligned} h^{\alpha\beta} \partial_\alpha \partial_\beta v =& \overline{h}^{\alpha\beta} \overline{\partial}_\alpha \overline{\partial}_\beta v + h^{\alpha\beta} \partial_\alpha \overline{\Psi}_\beta^{\beta'}\, \overline{\partial}_{\beta'} v \\ =& \overline{h}^{00} \overline{\partial}_0 \overline{\partial}_0 v + 2\overline{h}^{0b} \overline{\partial}_0 \overline{\partial}_b v + \overline{h}^{ab} \overline{\partial}_a \overline{\partial}_b v + h^{\alpha\beta} \partial_\alpha \overline{\Psi}_\beta^{\beta'}\, \overline{\partial}_{\beta'} v. \end{aligned}$$

Then, we obtain

$$\begin{aligned} -s^{3/2} \Box v =& - s^{3/2} \overline{h}^{00} \overline{\partial}_0 \overline{\partial}_0 v - s^{3/2} \big(2\overline{h}^{0b} \overline{\partial}_0 \overline{\partial}_b v + \overline{h}^{ab} \overline{\partial}_a \overline{\partial}_b v + h^{\alpha\beta} \partial_\alpha \overline{\Psi}_\beta^{\beta'}\, \overline{\partial}_{\beta'} v\big) - c^2 s^{3/2} v + s^{3/2} f \\ =& - \overline{h}^{00} \overline{\partial}_0 \overline{\partial}_0 \big(s^{3/2} v\big) - c^2 s^{3/2} v \\ &+ \overline{h}^{00} \left(\frac{3v}{4s^{1/2}} + 3s^{1/2} \overline{\partial}_0 v\right) - s^{3/2} \big(2\overline{h}^{0b} \overline{\partial}_0 \overline{\partial}_b v + \overline{h}^{ab} \overline{\partial}_a \overline{\partial}_b v + h^{\alpha\beta} \partial_\alpha \overline{\Psi}_\beta^{\beta'}\, \overline{\partial}_{\beta'} v\big) + s^{3/2} f, \end{aligned}$$

and we conclude with

$$\begin{aligned} -s^{3/2} \Box v =& -\overline{h}^{00} \overline{\partial}_0 \overline{\partial}_0 w - c^2 w + \overline{h}^{00} \left(\frac{3v}{4s^{1/2}} + 3s^{1/2} \overline{\partial}_0 v\right) \\ &- s^{3/2} \big(2\overline{h}^{0b} \overline{\partial}_0 \overline{\partial}_b v + \overline{h}^{ab} \overline{\partial}_a \overline{\partial}_b v + h^{\alpha\beta} \partial_\alpha \overline{\Psi}_\beta^{\beta'}\, \overline{\partial}_{\beta'} v\big) + s^{3/2} f \\ =& -\overline{h}^{00} \overline{\partial}_0 \overline{\partial}_0 w - c^2 w + R_2[v] + s^{3/2} f. \end{aligned} \tag{3.14}$$

We then combine (3.13) with (3.14) and obtain

$$\overline{\partial}_0 \overline{\partial}_0 w + 2\frac{x^a}{s} \overline{\partial}_0 \overline{\partial}_a w + \frac{x^a x^b}{s^2} \overline{\partial}_a \overline{\partial}_b w - \overline{h}^{00} \overline{\partial}_0 \overline{\partial}_0 w + c^2 w = R_1[v] + R_2[v] + s^{3/2} f. \tag{3.15}$$

3. Conclusion. We continue with (3.15) and write

$$\begin{aligned} &\big(1 + \overline{h}^{00}\big) \left(\overline{\partial}_0 \overline{\partial}_0 + 2\frac{x^a}{s} \overline{\partial}_0 \overline{\partial}_a + \frac{x^a x^b}{s^2} \overline{\partial}_a \overline{\partial}_b\right) w + c^2 w \\ &= \overline{h}^{00} \left(2\frac{x^a}{s} \overline{\partial}_0 \overline{\partial}_a + \frac{x^a x^b}{s^2} \overline{\partial}_a \overline{\partial}_b\right) w + R_1[v] + R_2[v] + s^{3/2} f \end{aligned}$$

and, so, we have

$$\begin{aligned} &\left(\overline{\partial}_0 \overline{\partial}_0 + 2\frac{x^a}{s} \overline{\partial}_0 \overline{\partial}_a + \frac{x^a x^b}{s^2} \overline{\partial}_a \overline{\partial}_b\right) w + \frac{c^2 w}{1 + \overline{h}^{00}} \\ &= \big(1 + \overline{h}^{00}\big)^{-1} \big(R_1[v] + R_2[v] + R_3[v] + s^{3/2} f\big). \end{aligned} \tag{3.16}$$

It follows that

$$\frac{d^2}{d\lambda^2} w_{t,x}(\lambda) + \frac{c^2 w_{t,x}(\lambda)}{1+\overline{h}^{00}(\lambda t/s, \lambda x/s)} \\ = \big(1+\overline{h}^{00}(\lambda t/s, \lambda x/s)\big)^{-1} \big(R_1[v] + R_2[v] + R_3[v] + s^{3/2} f\big)(\lambda t/s, \lambda x/s). \tag{3.17}$$

□

Proof of Lemma 3.5. We simply need to integrate out the ODE. We consider the vector field $b(\lambda) = \big(z(\lambda), z'(\lambda)\big)^T$ and the matrix $A(\lambda) := \begin{pmatrix} 0 & 1 \\ -c^2(1+G)^{-1} & 0 \end{pmatrix}$. We write $b' = Ab + \begin{pmatrix} 0 \\ k \end{pmatrix}$. and introduce the diagonalization of $A = PQP^{-1}$ with

$$Q = \begin{pmatrix} ic\big(1+G\big)^{-1/2} & 0 \\ 0 & -ic\big(1+G\big)^{-1/2} \end{pmatrix}$$

and

$$P = \begin{pmatrix} 1 & 1 \\ \frac{}{ic} & -\frac{}{ic} \\ \frac{}{(1+G)^{1/2}} & \frac{}{(1+G)^{1/2}} \end{pmatrix}, \qquad P^{-1} = \begin{pmatrix} 1/2 & \frac{(1+G)^{1/2}}{2ic} \\ 1/2 & -\frac{(1+G)^{1/2}}{2ic} \end{pmatrix}.$$

We thus have $b' = PQP^{-1}b + \begin{pmatrix} 0 \\ k \end{pmatrix}$, which leads us to

$$\big(P^{-1}b\big)' = Q\big(P^{-1}b\big) + \big(P^{-1}\big)'b + P^{-1}\begin{pmatrix} 0 \\ k \end{pmatrix}.$$

We regard the term $\big(P^{-1}\big)'b$ as a source term and, by a standard formula,

$$P^{-1}b(s) = e^{\int_{s_0}^s Q(\bar s)d\bar s} P^{-1}b(s_0) + \int_{s_0}^s e^{\int_\lambda^s Q(\bar s)d\bar s} P^{-1}\begin{pmatrix} 0 \\ k \end{pmatrix} d\lambda \\ + \int_{s_0}^s e^{\int_\lambda^s Q(\bar s)d\bar s} \big(P^{-1}\big)'(\lambda)\, b(\lambda)\, d\lambda.$$

Recall that when $\sup_{\lambda\in[1,s]} |G(\lambda)| \leqslant 1/3$, the norm of $P(\lambda)$ and $P^{-1}(\lambda)$ are bounded for $\lambda \in [s_0, s]$. We also remarks that the norm of $\big(P^{-1}\big)'(\lambda)$ is bounded by $C|G'(\lambda)|$ with C a constant depending only on c, and the norm of Q is also bounded by a constant $C > 0$. Furthermore, we observe that

$$\int_\lambda^s Q(\bar s)d\bar s = \begin{pmatrix} ic\int_\lambda^s (1+G)^{-1/2}(\bar s)d\bar s & 0 \\ 0 & -ic\int_\lambda^s (1+G)^{-1/2}(\bar s)d\bar s \end{pmatrix}$$

and thus

$$e^{\int_\lambda^s Q(\bar s)d\bar s} = \begin{pmatrix} e^{ic\int_\lambda^s (1+G)^{-1/2}(\bar s)d\bar s} & 0 \\ 0 & e^{-ic\int_\lambda^s (1+G)^{-1/2}(\bar s)d\bar s} \end{pmatrix}.$$

The norm of the matrix $e^{\int_\lambda^s Q(\bar s)d\bar s}$ is uniformly bounded by a constant, and the following estimate is now proven:

$$|z(s)| + |z'(s)| \leqslant C(|z(s_0)| + |z'(s_0)|) + C\,K(s) + C\int_{s_0}^s |G'(\lambda)|\big(|z(\lambda)| + |z'(\lambda)|\big)\, d\lambda,$$

and we conclude with Gronwall's lemma. □

Proof of Proposition 3.3. The proof is based on a combination of the bounds (3.9) and (3.17). By recalling the definition of $w_{t,x}(\lambda)$, we have

$$w_{t,x}(\lambda) = \lambda^{3/2} v(\lambda t/s, \lambda x/s), \\ w'_{t,x}(\lambda) = \frac{3}{2}\lambda^{1/2} v(\lambda t/s, \lambda x/s) + \frac{t}{s}\lambda^{3/2} \underline{\partial}_\perp v(\lambda t/s, \lambda x/s).$$

That is, $w_{t,x}$ is the restriction of $w(t,x)=s^{3/2}v(t,x)$ on the line segment $\big\{(\lambda t/s,\lambda x/s),\lambda\in[s_0,s]\big\}$. We then apply (3.9) and (3.17) to this line segment, with

$$s_0=\begin{cases}2, & 0\leqslant r/t\leqslant 3/5,\\ \sqrt{\dfrac{t+r}{t-r}}, & 3/5\leqslant r/t\leqslant 1.\end{cases}$$

This segment is the part of the line $\{(\lambda t/s,\lambda x/s)\}$ between the point (t,x) and the boundary of $\mathcal{K}_{[s_0,+\infty)}$.

Recall that v is supported in $\mathcal{K}$. and the restriction of v on the initial hyperboloid $\mathcal{H}_2$ is supported in $\mathcal{H}_2\cap\mathcal{K}$. We recall that when $3/5\leqslant r/t\leqslant 1$, $w_{t,x}(s_0)=0$ and when $0\leqslant r/t\leqslant 3/5$, $w_{t,x}(s_0)$ is determined by v_0.

When $0\leqslant r/t\leqslant 3/5$, we apply (3.9) with $s_0=2$. When $\lambda=2$, we write $w_{t,x}(2)=w(2t/s,2x/s)=2^{3/2}v(2t/s,2x/s)=2^{3/2}v_0(2x/s)$, and

$$\begin{aligned}w'_{s,x}(2)=&\frac{d}{d\lambda}\big(\lambda^{3/2}v(\lambda t/s,\lambda x/s)\big)\big|_{\lambda=2}\\ =&\frac{3\sqrt{2}}{2}v(2t/s,2x/s)+2^{3/2}(s/t)^{-1}\underline{\partial}_\perp v(2t/s,2x/s)\\ =&\frac{3\sqrt{2}}{2}v(2t/s,2x/s)+2^{3/2}(s/t)^{-1}\partial_t v(2t/s,2x/s)+2^{3/2}(x^a/s)\partial_a v(2t/s,2x/s)\\ =&\frac{3\sqrt{2}}{2}v_0(2x/s)+2^{3/2}(x^a/s)\partial_a v_0(2x/s)+2^{3/2}(s/t)^{-1}v_1(2t/s,2x/s).\end{aligned}$$

Recall that when $0\leqslant r/t\leqslant 3/5$, we have $4/5\leqslant s/t\leqslant 1$. So we see that $|w_{t,x}(s_0)|+|w'_{t,x}(s_0)|\leqslant C(\|v_0\|_{L^\infty(\mathcal{H}_2)}+\|v_1\|_{L^\infty(\mathcal{H}_2)})$. Then by (3.9) and (3.17) we have

$$\begin{aligned}|w_{t,x}(s)|+|w'_{t,x}(s)|\leqslant&C(\|v_0\|_{L^\infty(\mathcal{H}_2)}+\|v_1\|_{L^\infty(\mathcal{H}_2)})+CF(s)\\ &+C(\|v_0\|_{L^\infty(\mathcal{H}_2)}+\|v_1\|_{L^\infty(\mathcal{H}_2)})\int_2^s|h'_{t,x}(\bar s)|e^{C\int_{\bar s}^s|h'_{t,x}(\lambda)|d\lambda}\,d\bar s\\ &+C\int_2^s F(\bar s)|h'_{t,x}(\bar s)|e^{C\int_{\bar s}^s|h'_{t,x}(\lambda)|d\lambda}\,d\bar s.\end{aligned}$$

We recall that when $3/5\leqslant r/t\leqslant 1$, $w_{t,x}(s_0)=w'_{t,x}(s_0)=0$ and so we have

$$|w_{t,x}(s)|+|w'_{t,x}(s)|\leqslant CF(s)+C\int_{s_0}^s F(\bar s)|h'_{t,x}(\bar s)|e^{C\int_{\bar s}^s|h'_{t,x}(\lambda)|d\lambda}\,d\bar s,$$

which leads to $|w_{t,x}(s)|+|w'_{t,x}(s)|\lesssim V(t,x)$. It remains to recall the relation between v and w, that is, $v(t,x)=s^{3/2}w_{t,x}(s)$ and

$$(s/t)^{-1}s^{3/2}\underline{\partial}_\perp v(t,x)=w'_{t,x}(s)-\frac{3}{2}s^{1/2}v(t,x)=w'_{t,x}(s)-\frac{3}{2}s^{-1}w_{t,x}(s),$$

and the desired estimate is established. □

4. Commutator estimates

4.1. **Algebraic decomposition of the commutators.** We consider the commutators $[X,Y]u:=X(Yu)-Y(Xu)$ of operators associated with our vector fields when the function u is defined in the future cone $\mathcal{K}=\{|x|<t-1\}$. Our uniform bounds rely on homogeneity arguments and on the observation that the coefficients of our decompositions are smooth in $\mathcal{K}$.

First of all, the vector fields ∂_α, and L_a are Killing fields for the (flat) wave operator $\Box$, so that the following commutation relations hold:

$$[\partial_\alpha,\Box]=0,\qquad [L_a,\Box]=0. \tag{4.1}$$

By introducing the notation

$$[L_a,\partial_\beta]=:\Theta^\gamma_{a\beta}\partial_\gamma,\qquad [\partial_\alpha,\underline{\partial}_\beta]=:t^{-1}\underline{M}^\gamma_{\alpha\beta}\partial_\gamma,\qquad [L_a,\underline{\partial}_\beta]=:\underline{\Theta}^\gamma_{a\beta}\underline{\partial}_\gamma, \tag{4.2}$$

we find easily that

$$
\begin{aligned}
&\Theta^\gamma_{a0} = -\delta^\gamma_a, && \Theta^\gamma_{ab} = -\delta_{ab}\delta^\gamma_0, && \\
&\underline{M}^\gamma_{0b} = -\frac{x^b}{t}\delta^\gamma_0 = \Psi^0_b\delta^\gamma_0, && \underline{M}^\gamma_{\alpha 0} = 0, && \underline{M}^\gamma_{ab} = \delta_{ab}\delta^\gamma_0, \\
&\underline{\Theta}^\gamma_{a0} = -\delta^\gamma_a + \frac{x^a}{t}\delta^\gamma_0 = -\delta^\gamma_a + \Phi^a_0\delta^\gamma_0, && \underline{\Theta}^\gamma_{ab} = -\frac{x^b}{t}\delta^\gamma_a = \Psi^0_b\delta^\gamma_a, &&
\end{aligned} \tag{4.3}
$$

where Φ and Ψ were defined at the beginning of Section 2. All of these coefficients are smooth in the (open) cone $\mathcal{K}$ and homogeneous of degree 0. Furthermore, we can also check that

$$
\underline{\Theta}^0_{ab} = 0, \quad \text{so that} \quad [L_a, \underline{\partial}_b] = \underline{\Theta}^c_{ab}\underline{\partial}_c, \tag{4.4}
$$

which means that the commutator of a "good" derivative $\underline{\partial}_b$ with L_a is again a "good" derivative. (That is, these derivatives enjoy better decay compared to the gradient itself.)

Lemma 4.1 (Algebraic decomposition of commutators. I)**.** *There exist constants λ^I_{aJ} such that*

$$
[\partial^I, L_a] = \sum_{|J|\leqslant|I|} \lambda^I_{aJ}\partial^J. \tag{4.5}
$$

Proof. We proceed by induction on $|I|$. For $|I| = 1$, the result is guaranteed by (4.2). Suppose that (4.5) holds for all $|I_1| \leqslant m$, we will prove that it is still valid for $|I| \leqslant m+1$. Let $I = (\alpha, \alpha_m, \alpha_{m-1}, \dots, \alpha_1)$ and $I_1 = (\alpha_m, \alpha_{m-1}, \dots, \alpha_1)$, so that $\partial^I = \partial_\alpha\partial^{I_1}$. Then we have

$$
\begin{aligned}
[\partial^I, L_a] &= [\partial_\alpha\partial^{I_1}, L_a] = \partial_\alpha\big([\partial^{I_1}, L_a]\big) + [\partial_\alpha, L_a]\partial^{I_1} = \partial_\alpha\bigg(\sum_{|J|\leqslant|I_1|}\lambda^{I_1}_{aJ}\partial^J\bigg) - \Theta^\gamma_{a\alpha}\partial_\gamma\partial^{I_1} \\
&= \sum_{|J|\leqslant|I_1|}\lambda^{I_1}_{aJ}\partial_\alpha\partial^J - \Theta^\gamma_{a\alpha}\partial_\gamma\partial^{I_1},
\end{aligned}
$$

which yields the desired statement for $|I| = m+1$. □

Lemma 4.2 (Algebraic decomposition of commutators. II)**.** *There exist constants $\theta^{I\gamma}_{\alpha J}$ such that*

$$
[L^I, \partial_\alpha] = \sum_{|J|\leqslant|I|-1,\gamma} \theta^{I\gamma}_{\alpha J}\partial_\gamma L^J. \tag{4.6}
$$

Proof. We proceed by induction and observe that the case $|I| = 1$ is already covered by (4.2). We assume that (4.6) is valid for $|I| \leqslant m$ and we will prove that it is still valid when $|I| = m+1$. For this purpose, we take $L^I = L_aL^{I_1}$ with $|I_1| = m$, and we have

$$
\begin{aligned}
[L^I, \partial_\alpha] =& [L_aL^{I_1}, \partial_\alpha] = L_a\big([L^{I_1}, \partial_\alpha]\big) + [L_a, \partial_\alpha]L^{I_1} \\
=& L_a\bigg(\sum_{|J|\leqslant|I_1|-1,\gamma}\theta^{I_1\gamma}_{\alpha J}\partial_\gamma L^J\bigg) + \sum_\gamma \Theta^\gamma_{a\alpha}\partial_\gamma L^{I_1} \\
=& \sum_{|J|\leqslant|I_1|-1,\gamma}\theta^{I_1\gamma}_{\alpha J}L_a\partial_\gamma L^J + \sum_\gamma \Theta^\gamma_{a\alpha}\partial_\gamma L^{I_1}
\end{aligned}
$$

and, therefore,

$$
\begin{aligned}
[L^I, \partial_\alpha] &= \sum_{|J|\leqslant|I_1|-1,\gamma}\theta^{I_1\gamma}_{\alpha J}\partial_\gamma L_aJ^J + \sum_{|J|\leqslant|I_1|-1,\gamma}\theta^{I_1\gamma}_{\alpha J}[L_a, \partial_\gamma]J^J + \sum_\gamma \Theta^\gamma_{a\alpha}\partial_\gamma L^{I_1} \\
&= \sum_{|J|\leqslant|I_1|-1,\gamma}\theta^{I_1\gamma}_{\alpha J}\partial_\gamma L_aJ^J + \sum_{|J|\leqslant|I_1|-1,\gamma}\theta^{I_1\gamma}_{\alpha J}\Theta^{\gamma'}_{a\gamma}\partial_{\gamma'}L^J + \sum_\gamma \Theta^\gamma_{a\alpha}\partial_\gamma L^{I_1}.
\end{aligned}
$$

□

An immediate consequence of (4.6) is

$$
[\partial^I L^J, \partial_\alpha]u = \sum_{|J'|<|J|,\gamma}\theta^{J\gamma}_{\alpha J'}\partial_\gamma\partial^I L^{J'}u. \tag{4.7}
$$

Lemma 4.3 (Algebraic decomposition of commutators. III)**.** *In the future cone $\mathcal{K}$, the following identity holds:*

$$[\partial^I L^J, \underline{\partial}_\beta] = \sum_{\substack{|J'|\leqslant|J|,|I'|\leqslant|I|\\ |I'|+|J'|<|I|+|J|}} \underline{\theta}^{IJ\gamma}_{\beta I'J'}\partial_\gamma \partial^{I'}L^{J'}, \tag{4.8}$$

where the coefficients $\underline{\theta}^{IJ\gamma}_{\beta I'J'}$ are smooth functions and satisfy (in $\mathcal{K}$)

$$\big|\partial^{I_1}L^{J_1}\underline{\theta}^{IJ\gamma}_{\beta I'J'}\big| \leqslant \begin{cases} C\big(|I|,|J|,|I_1|,|J_1|\big)\, t^{-|I_1|} & \text{when } |J'|<|J|,\\ C\big(|I|,|J|,|I_1|,|J_1|\big)\, t^{-|I_1|-1} & \text{when } |I'|<|I|.\end{cases} \tag{4.9}$$

Proof. Consider the identity

$$[\partial^I L^J, \underline{\partial}_\beta] = [\partial^I L^J, \Phi^\gamma_\beta \partial_\gamma] = \Phi^\gamma_\beta [\partial^I L^J, \partial_\gamma] + \sum_{\substack{I_1+I_2=I, J_1+J_2=J\\ |I_1|+|J_1|<|I|+|J|}} \partial^{I_1}L^{J_1}\Phi^\gamma_\beta \partial^{I_2}L^{J_2}\partial_\gamma.$$

In the first sum, we commute $\partial^{I_2}L^{J_2}$ and ∂_γ and obtain

$$\begin{aligned}
[\partial^I L^J, \underline{\partial}_\beta] =& \Phi^\gamma_\beta [\partial^I L^J, \partial_\gamma]\\
&+ \sum_{\substack{I_1+I_2=I, J_1+J_2=J\\ |I_1|+|J_1|<|I|+|J|}} \partial^{I_1}L^{J_1}\Phi^\gamma_\beta \partial_\gamma \partial^{I_2}L^{J_2} + \sum_{\substack{I_1+I_2=I, J_1+J_2=J\\ |I_1|+|J_1|<|I|+|J|}} \partial^{I_1}L^{J_1}\Phi^\gamma_\beta [\partial^{I_2}L^{J_2}, \partial_\gamma]\\
=& \sum_{\substack{I_1+I_2=I, J_1+J_2=J\\ |I_1|+|J_1|<|I|+|J|}} \partial^{I_1}L^{J_1}\Phi^\gamma_\beta \partial_\gamma \partial^{I_2}L^{J_2} + \sum_{\substack{I_1+I_2=I\\ J_1+J_2=J}} \partial^{I_1}L^{J_1}\Phi^\gamma_\beta [\partial^{I_2}L^{J_2}, \partial_\gamma]\\
=& \sum_{\substack{I_1+I_2=I, J_1+J_2=J\\ |I_1|+|J_1|<|I|+|J|}} \partial^{I_1}L^{J_1}\Phi^\gamma_\beta \partial_\gamma \partial^{I_2}L^{J_2} + \sum_{\substack{I_1+I_2=I\\ J_1+J_2=J}} \sum_{|J_2'|<|J_2|} \big(\partial^{I_1}L^{J_1}\Phi^\gamma_\beta\big)\, \theta^{J_2\delta}_{\gamma J_2'}\partial_\delta \partial^{I_2}L^{J_2'}.
\end{aligned}$$

Hence, $\underline{\theta}^{IJ\alpha}_{\gamma I'J'}$ are linear combinations of $\partial^{I_1}L^{J_1}\Phi^\gamma_\beta$ and $\big(\partial^{I_1}L^{J_1}\Phi^\gamma_\beta\big)\theta^{J_2\delta}_{\gamma J_2'}$ and $J_1+J_2=J$, which yields (4.8). Note that $\theta^{J_2\delta}_{\gamma J_2'}$ are constants, so that

$$\partial^{I_3}L^{J_3}\big(\partial^{I_1}L^{J_1}\Phi^\gamma_\beta \theta^{J_2\delta}_{\gamma J_2'}\big) = \theta^{J_2\delta}_{\gamma J_2'}\partial^{I_3}L^{J_3}\partial^{I_1}L^{J_1}\Phi^\gamma_\beta.$$

By definition, Φ^γ_β is a homogeneous function of degree zero, so that $\partial^{I_1}L^{J_1}\Phi^\gamma_\beta$ is again homogeneous but with degree $\leqslant 0$. We thus arrive at (4.9). □

Lemma 4.4 (Algebraic decomposition of commutators. IV)**.** *Within the future cone $\mathcal{K}$, the following identity holds*

$$[L^I, \underline{\partial}_c] = \sum_{|J|<|I|} \sigma^{Ia}_{cJ}\underline{\partial}_a L^J, \tag{4.10}$$

where the coefficients σ^{Ia}_{cJ} are smooth functions and satisfy (in $\mathcal{K}$)

$$\big|\partial^{I_1}L^{J_1}\sigma^{Ia}_{cJ}\big| \leqslant C(|I|,|J|,|I_1|,|J_1|)t^{-|I_1|}. \tag{4.11}$$

Proof. This is also by induction. Again, when $|I|=1$, (4.10) together with (4.11) are guaranteed by (4.4). Assume that (4.10) and (4.11) hold for $|I|\leqslant m$, we will prove that they are valid for $|I|=m+1$. We take $L^I = L_a L^J$ with $|J|=m$, and obtain

$$\begin{aligned}
[L^I, \underline{\partial}_c] =& [L_a L^J, \underline{\partial}_c] = L_a\big([L^J, \underline{\partial}_c]\big) + [L_a, \underline{\partial}_c]L^J\\
=& L_a\bigg(\sum_{|J'|<|J|} \sigma^{Ja}_{cJ'}\underline{\partial}_a L^{J'}\bigg) + \underline{\Theta}^b_{ac}\underline{\partial}_b L^J\\
=& \sum_{|J'|<|J|} L_a\sigma^{Jb}_{cJ'}\underline{\partial}_b L^{J'} + \sum_{|J'|<|J|} \sigma^{Jb}_{cJ'}L_a\underline{\partial}_b L^{J'} + \underline{\Theta}^b_{ac}\underline{\partial}_b L^J,
\end{aligned}$$

so that

$$[L^I, \underline{\partial}_c] = \sum_{|J'|<|J|} L_a \sigma^{Jb}_{cJ'} \underline{\partial}_b L^{J'} + \sum_{|J'|<|J|} \sigma^{Jb}_{cJ'} \underline{\partial}_b L_a L^{J'} + \sum_{|J'|<|J|} \sigma^{Jb}_{cJ'} [L_a, \underline{\partial}_b] L^{J'} + \underline{\Theta}^b_{ac} \underline{\partial}_b L^J$$
$$= \sum_{|J'|<|J|} L_a \sigma^{Jb}_{cJ'} \underline{\partial}_b L^{J'} + \sum_{|J'|<|J|} \sigma^{Jb}_{cJ'} \underline{\partial}_b L_a L^{J'} + \sum_{|J'|<|J|} \sigma^{Jb}_{cJ'} \underline{\Theta}^d_{ab} \underline{\partial}_d L^{J'} + \underline{\Theta}^b_{ac} \underline{\partial}_b L^J.$$

In each term the coefficients are homogeneous of degree 0 (by applying (4.11)), and the desired result is proven. □

The following result is also checked by induction along the same lines as above, and so its proof is omitted.

Lemma 4.5 (Algebraic decomposition of commutators. V)**.** *Within the future cone $\mathcal{K}$, the following identity holds:*

$$[\partial^I, \underline{\partial}_c] = t^{-1} \sum_{|J| \leqslant |I|} \rho^I_{cJ} \partial^J, \tag{4.12}$$

where the coefficients ρ^I_{cJ} are smooth functions and satisfy (in $\mathcal{K}$)

$$\left|\partial^{I_1} L^{J_1} \rho^I_{cJ}\right| \leqslant C(|I|, |J|, |I_1|, |J_1|) t^{-|I_1|}. \tag{4.13}$$

4.2. **Estimates for the commutators.** The following statements are now immediate in view of (4.5), (4.6), and (4.10), and (4.12).

Proposition 4.6 (Estimates on commutators. I)**.** *For all sufficiently regular functions u defined in the future cone $\mathcal{K}$, the following estimates hold:*

$$\left|[\partial^I L^J, \partial_\alpha] u\right| \leqslant C(|I|, |J|) \sum_{|J'|<|J|, \beta} |\partial_\beta \partial^I L^{J'} u|, \tag{4.14}$$

$$\left|[\partial^I L^J, \underline{\partial}_c] u\right| \leqslant C(|I|, |J|) \left(\sum_{\substack{|J'|<|J|, a \\ |I'|\leqslant |I|}} |\underline{\partial}_a \partial^{I'} L^{J'} u| + t^{-1} \sum_{\substack{|I|\leqslant |I'| \\ |J|\leqslant |J'|}} |\partial^{I'} L^{J'} u| \right), \tag{4.15}$$

$$\left|[\partial^I L^J, \underline{\partial}_\alpha u]\right| \leqslant C(|I|, |J|) t^{-1} \sum_{\substack{\beta, |I'|<|I| \\ |J'|\leqslant |J|}} \left|\partial_\beta \partial^{I'} L^{J'} u\right| + C(|I|, |J|) \sum_{\substack{\beta, |I'|\leqslant |I| \\ |J'|<|J|}} \left|\partial_\beta \partial^{I'} L^{J'} u\right|, \tag{4.16}$$

$$\left|[\partial^I L^J, \partial_\alpha \partial_\beta] u\right| \leqslant C(|I|, |J|) \sum_{\substack{\gamma, \gamma' \\ |I|\leqslant |I'|, |J'|<|I|}} \left|\partial_\gamma \partial_{\gamma'} \partial^{I'} L^{J'} u\right|, \tag{4.17}$$

$$\begin{aligned} &\left|[\partial^I L^J, \underline{\partial}_a \underline{\partial}_\beta] u\right| + \left|[\partial^I L^J, \underline{\partial}_\alpha \underline{\partial}_b] u\right| \\ &\leqslant C(|I|, |J|) \left(\sum_{\substack{c, \gamma, |I'|\leqslant |I| \\ |J'|<|J|}} \left|\underline{\partial}_c \underline{\partial}_\gamma \partial^{I'} L^{J'} u\right| + t^{-1} \sum_{\substack{c, \gamma, |I'|<|I| \\ |J'|\leqslant |J|}} \left|\underline{\partial}_c \underline{\partial}_\gamma \partial^{I'} L^{J'} u\right| + t^{-1} \sum_{\substack{\gamma, |I'|\leqslant |I| \\ |J'|\leqslant |J|}} \left|\partial_\gamma \partial^{I'} L^{J'} u\right| \right). \end{aligned} \tag{4.18}$$

Further estimates will be also needed, as now stated.

Proposition 4.7 ([Estimates on commutators. II)**.** *For all sufficiently regular functions u defined in the future cone $\mathcal{K}$, the following estimate holds (for all I, J, α)*

$$\left|\partial^I L^J \big((s/t) \partial_\alpha u\big)\right| \leqslant \left|(s/t) \partial_\alpha \partial^I L^J u\right| + C(|I|, |J|) \sum_{\substack{\beta, |I'|\leqslant |I| \\ |J'|\leqslant |J|}} \left|(s/t) \partial_\beta \partial^{I'} L^{J'} u\right|. \tag{4.19}$$

Recall that the proof of the above result (given in [22]) relies on the following technical observation, concerning products of first-order linear operators with homogeneous coefficients of order 0 or 1.

Lemma 4.8. *For all multi-indices I, the function*

$$\Xi^{I,J} := (t/s)\partial^I L^J(s/t),$$

defined in the closed cone $\overline{\mathcal{K}} = \{|x| \leqslant t-1\}$, is smooth and all of its derivatives (of any order) are bounded in $\overline{\mathcal{K}}$. Furthermore, it is homogeneous of degree η with $\eta \leqslant 0$.

5. Initialization of the bootstrap argument

From this section onwards, we begin the proof of Theorem 1.1, which is a rather involved bootstrap argument along the lines of the method presented in [22]. We fix some integer $N \geqslant 8$ throughout, and first summarize our strategy, as follows.

Let (u, v) be the local-in-time solution to the Cauchy problem associated with the system (1.11). From a standard local existence result (cf., for instance, [22, Section 11]), we can construct a local-time solution from the data given on the initial hypersurface and, consequently, guarantee that on the *initial hyperboloid* and for all $|I| + |J| \leqslant N$,

$$E_m(2, \partial^I L^J u)^{1/2} \leqslant C_0\varepsilon, \qquad E_m(2, \partial^I L^J v)^{1/2} \leqslant C_0\varepsilon$$

for some uniform constant $C_0 > 0$. On some (hyperbolic) time interval $[2, s_1]$, we can thus assume the following energy conditions for some constants $C_1, \varepsilon, \delta > 0$ (yet to be determined):

(5.1)

$$\begin{aligned}
&E_m(s, \partial^I L^J u)^{1/2} \leqslant C_1\varepsilon s^{k\delta}, && |J| = k, && |I| + |J| \leqslant N, && \text{wave / high-order},\\
&E_m(s, \partial^I L^J u)^{1/2} \leqslant C_1\varepsilon, && && |I| + |J| \leqslant N-4, && \text{wave / low-order},\\
&E_m(s, \partial^I L^J v)^{1/2} \leqslant C_1\varepsilon s^{1/2+k\delta}, && |J| = k, && |I| + |J| \leqslant N, && \text{Klein-Gordon / high-order},\\
&E_m(s, \partial^I L^J v)^{1/2} \leqslant C_1\varepsilon s^{k\delta}, && |J| = k, && |I| + |J| \leqslant N-4 && \text{Klein-Gordon / low-order}.
\end{aligned}$$

We will prove that on the same interval the following *improved energy bounds* are valid when ε is sufficiently small and $C_1 > C_0$ with $\frac{1}{10N} \leqslant \delta \leqslant \frac{1}{5N}$ (fixed once for all):

(5.2)

$$\begin{aligned}
&E_m(s, \partial^I L^J u)^{1/2} \leqslant \frac{1}{2}C_1\varepsilon s^{k\delta}, && |J| = k, && |I| + |J| \leqslant N, && \text{wave / high-order},\\
&E_m(s, \partial^I L^J u)^{1/2} \leqslant \frac{1}{2}C_1\varepsilon, && && |I| + |J| \leqslant N-4, && \text{wave / low-order},\\
&E_m(s, \partial^I L^J v)^{1/2} \leqslant \frac{1}{2}C_1\varepsilon s^{1/2+k\delta}, && |J| = k, && |I| + |J| \leqslant N, && \text{Klein-Gordon / high-order},\\
&E_m(s, \partial^I L^J v)^{1/2} \leqslant \frac{1}{2}C_1\varepsilon s^{k\delta}, && |J| = k, && |I| + |J| \leqslant N-4 && \text{Klein-Gordon / low-order}.
\end{aligned}$$

Once this property is proven, we set

$$s_1 := \sup\Big\{s \,/\, (5.1) \text{ holds on } [2, s]\Big\}$$

and we can deduce that $s_1 = +\infty$. Indeed, by a continuity argument, $C_1 > C_0$ implies $s_1 > 2$. Again by a continuity argument, we deduce that when $s = s_1$, at least one of the inequalities (5.1) must be an equality. But, when (5.2) holds, none of them can become an equality. This means that $s_1 = +\infty$ and the rest of our work consists of proving (5.2).

Proposition 5.1 (Formulation of the bootstrap argument)**.** *Given any integer $N \geqslant 8$ and $\frac{1}{10N} < \delta < \frac{1}{5N}$, there exist constants $C_1, \varepsilon > 0$ satisfying $\varepsilon C_1 < 1$ such that any local-in-time solution (u, v) to (1.11), defined in the time interval $[2, s_1]$ and satisfying the energy conditions (5.1) for some $\varepsilon \in (0, \varepsilon_0]$, also satisfies the improved energy bounds (5.2).*

The remaining text is devoted to the proof of this proposition, which we decompose into three parts. First, we derive a series of L^2 and sup-norm estimates directly from (5.1), and from the Sobolev inequality on hyperboloids (2.14) and the commutator estimates (i.e. Propositions 4.6 and 4.7). Second, we improve the sup-norm estimates by using (3.2) and (3.7). Finally, we combine the improved sup-norm estimates and the L^2 estimates established in the first part and we get the improved energy estimates (5.2).

From now on, we consider a **solution satisfying the energy bound** (5.1) and throughout we set $|J| = k$.

6. Basic estimates

6.1. **Basic L^2 estimates of the first generation.** Throughout, we always assume that $2 \leqslant s \leqslant s_1$. We begin by stating the L^2 type estimates provided to us by the energy assumption (5.1). For all $|I| + |J| \leqslant N$ with $|J| = k$, we have the high-order bounds

$$
\begin{aligned}
\|(s/t)\partial_\alpha \partial^I L^J u\|_{L^2_f(\mathcal{H}_s)} + \|(s/t)\underline{\partial}_\alpha \partial^I L^J u\|_{L^2_f(\mathcal{H}_s)} &\lesssim C_1 \varepsilon s^{k\delta},\\
\|\underline{\partial}_a \partial^I L^J u\|_{L^2_f(\mathcal{H}_s)} + \|\underline{\partial}_\perp \partial^I L^J u\|_{L^2_f(\mathcal{H}_s)} &\lesssim C_1 \varepsilon s^{k\delta},\\
\|(s/t)\partial_\alpha \partial^I L^J v\|_{L^2_f(\mathcal{H}_s)} + \|(s/t)\underline{\partial}_\alpha \partial^I L^J v\|_{L^2_f(\mathcal{H}_s)} &\lesssim C_1 \varepsilon s^{1/2+k\delta},\\
\|\underline{\partial}_a \partial^I L^J v\|_{L^2_f(\mathcal{H}_s)} + \|\underline{\partial}_\perp \partial^I L^J v\|_{L^2_f(\mathcal{H}_s)} &\lesssim C_1 \varepsilon s^{1/2+k\delta},\\
\|\partial^I L^J v\|_{L^2_f(\mathcal{H}_s)} &\lesssim C_1 \varepsilon s^{1/2+k\delta},
\end{aligned}
\tag{6.1}
$$

where the last estimate implies, for all $|I| + |J| \leqslant N - 1$ and $|J| = k$, the following estimate

$$
\|\partial_\alpha \partial^I L^J v\|_{L^2_f(\mathcal{H}_s)} \lesssim C_1 \varepsilon s^{1/2+k\delta},
\tag{6.2}
$$

as well as, for $|I| + |J| \leqslant N - 4$ with $|J| = k$, the low-order energy bounds imply:

$$
\begin{aligned}
\|(s/t)\partial_\alpha \partial^I L^J u\|_{L^2_f(\mathcal{H}_s)} + \|(s/t)\underline{\partial}_\alpha \partial^I L^J u\|_{L^2_f(\mathcal{H}_s)} &\lesssim C_1 \varepsilon,\\
\|\underline{\partial}_a \partial^I L^J u\|_{L^2_f(\mathcal{H}_s)} + \|\underline{\partial}_\perp \partial^I L^J u\|_{L^2_f(\mathcal{H}_s)} &\lesssim C_1 \varepsilon,\\
\|(s/t)\partial_\alpha \partial^I L^J v\|_{L^2_f(\mathcal{H}_s)} + \|(s/t)\underline{\partial}_\alpha \partial^I L^J v\|_{L^2_f(\mathcal{H}_s)} &\lesssim C_1 \varepsilon s^{k\delta},\\
\|\underline{\partial}_\perp \partial^I L^J v\|_{L^2_f(\mathcal{H}_s)} + \|\underline{\partial}_a \partial^I L^J v\|_{L^2_f(\mathcal{H}_s)} &\lesssim C_1 \varepsilon s^{k\delta},\\
\|\partial^I L^J v\|_{L^2_f(\mathcal{H}_s)} &\lesssim C_1 \varepsilon s^{k\delta}.
\end{aligned}
\tag{6.3}
$$

In addition, they also imply, for all $|I| + |J| \leqslant N - 5$ with $|J| = k$,

$$
\|\partial_\alpha \partial^I L^J v\|_{L^2_f(\mathcal{H}_s)} \lesssim C_1 s^{k\delta}.
\tag{6.4}
$$

6.2. **Basic L^2 estimates of the second generation.** The following estimates are obtained by applying the above energy estimate combined with the commutator estimates presented in Proposition 4.6. For all $|I| + |J| \leqslant N$ with $|J| = k$, we have the high-order bounds

$$
\begin{aligned}
\|(s/t)\partial^I L^J \partial_\alpha u\|_{L^2_f(\mathcal{H}_s)} + \|(s/t)\partial^I L^J \underline{\partial}_\alpha u\|_{L^2_f(\mathcal{H}_s)} &\lesssim C_1 \varepsilon s^{k\delta},\\
\|\partial^I L^J \underline{\partial}_a u\|_{L^2_f(\mathcal{H}_s)} + \|\partial^I L^J \underline{\partial}_\perp u\|_{L^2_f(\mathcal{H}_s)} &\lesssim C_1 \varepsilon s^{k\delta},\\
\|(s/t)\partial^I L^J \partial_\alpha v\|_{L^2_f(\mathcal{H}_s)} + \|(s/t)\partial^I L^J \underline{\partial}_\alpha v\|_{L^2_f(\mathcal{H}_s)} &\lesssim C_1 \varepsilon s^{1/2+k\delta},\\
\|\partial^I L^J \underline{\partial}_\perp v\|_{L^2_f(\mathcal{H}_s)} + \|\partial^I L^J \underline{\partial}_a v\|_{L^2_f(\mathcal{H}_s)} &\lesssim C_1 \varepsilon s^{1/2+k\delta},\\
\|\partial^I L^J v\|_{L^2_f(\mathcal{H}_s)} &\lesssim C_1 \varepsilon s^{1/2+k\delta},
\end{aligned}
\tag{6.5}
$$

which, for $|I| + |J| \leqslant N - 1$ with $|J| = k$, imply the low-order bounds (for instance, by expressing $t\underline{\partial}_a = L_a$ in the first inequality):

$$
\begin{aligned}
\|t\underline{\partial}_a \partial^I L^J v\|_{L^2_f(\mathcal{H}_s)} &\lesssim C_1 \varepsilon s^{1/2+(k+1)\delta},\\
\|t\partial^I L^J \underline{\partial}_a v\|_{L^2_f(\mathcal{H}_s)} &\lesssim C_1 \varepsilon s^{1/2+(k+1)\delta},\\
\|\partial^I L^J \partial_\alpha v\|_{L^2_f(\mathcal{H}_s)} + \|\partial^I L^J \underline{\partial}_\alpha v\|_{L^2_f(\mathcal{H}_s)} &\lesssim C_1 \varepsilon s^{1/2+k\delta},\\
\|(s/t)\partial^I L^J \partial_\alpha \partial_\beta v\|_{L^2_f(\mathcal{H}_s)} + \|(s/t)\partial^I L^J \underline{\partial}_\alpha \underline{\partial}_\beta v\|_{L^2_f(\mathcal{H}_s)} &\lesssim C_1 \varepsilon s^{1/2+k\delta},\\
\|s\partial^I L^J \underline{\partial}_\alpha \underline{\partial}_b v\|_{L^2_f(\mathcal{H}_s)} + \|s\partial^I L^J \underline{\partial}_a \underline{\partial}_\alpha v\|_{L^2_f(\mathcal{H}_s)} &\lesssim C_1 \varepsilon s^{1/2+(k+1)\delta}.
\end{aligned}
\tag{6.6}
$$

Observe especially that, for derivation of the last term, we write

$$\partial^I L^J \underline{\partial}_a \underline{\partial}_\alpha v = \partial^I L^J \big(t^{-1} L_a \underline{\partial}_\alpha v\big) = \sum_{\substack{I_1+I_2=I\\ J_1+J_2=J}} \partial^{I_1} L^{J_1} L_a \underline{\partial}_\alpha v \partial^{I_2} L^{J_2}(t^{-1})$$

and then we proceed by homogeneity (by noting that t^{-1} is homogeneous of degree -1 and its derivatives are homogeneous of degree $\leqslant -1$):

$$\begin{aligned}\|s\partial^I L^J \underline{\partial}_a \underline{\partial}_\alpha v\|_{L^2_f(\mathcal{H}_s)} &\lesssim \sum_{\substack{|I_1|\leqslant |I|\\ |J_1|\leqslant |J|}} \|st^{-1}\partial^{I_1} L^{J_1} L_a \underline{\partial}_\alpha v\|_{L^2_f(\mathcal{H}_s)} \lesssim \sum_{\substack{|I_1|\leqslant |I|\\ |J_1|\leqslant |J|}} \|(s/t)\partial^{I_1} L^{J_1} L_a \underline{\partial}_\alpha v\|_{L^2_f(\mathcal{H}_s)}\\ &\lesssim C_1\varepsilon s^{1/2+(|J_1|+1)\delta} \lesssim C_1\varepsilon s^{1/2+(k+1)\delta}.\end{aligned}$$

We also have, for $|I|+|J|\leqslant N-2$,

$$\|\partial^I L^J \partial_\alpha \partial_\beta v\|_{L^2_f(\mathcal{H}_s)} \lesssim C_1\varepsilon s^{1/2+k\delta}. \tag{6.7}$$

For $|I|+|J|\leqslant N-4$ with $|J|=k$, we have

$$\begin{aligned}\|(s/t)\partial^I L^J \partial_\alpha u\|_{L^2_f(\mathcal{H}_s)} + \|(s/t)\partial^I L^J \underline{\partial}_\alpha u\|_{L^2_f(\mathcal{H}_s)} &\lesssim C_1\varepsilon,\\ \|\partial^I L^J \underline{\partial}_a u\|_{L^2_f(\mathcal{H}_s)} + \|\partial^I L^J \underline{\partial}_\perp u\|_{L^2_f(\mathcal{H}_s)} &\lesssim C_1\varepsilon,\\ \|(s/t)\partial^I L^J \partial_\alpha v\|_{L^2_f(\mathcal{H}_s)} + \|(s/t)\partial^I L^J \underline{\partial}_\alpha v\|_{L^2_f(\mathcal{H}_s)} &\lesssim C_1\varepsilon s^{k\delta},\\ \|\partial^I L^J \underline{\partial}_\perp v\|_{L^2_f(\mathcal{H}_s)} + \|\partial^I L^J \underline{\partial}_a v\|_{L^2_f(\mathcal{H}_s)} &\lesssim C_1\varepsilon s^{k\delta},\\ \|\partial^I L^J v\|_{L^2_f(\mathcal{H}_s)} &\lesssim C_1\varepsilon s^{k\delta}.\end{aligned} \tag{6.8}$$

For $|I|+|J|\leqslant N-5$, $|J|=k$, we have

$$\begin{aligned}\|t\underline{\partial}_a \partial^I L^J v\|_{L^2_f(\mathcal{H}_s)} + \|t\partial^I L^J \underline{\partial}_a v\|_{L^2_f(\mathcal{H}_s)} &\lesssim C_1\varepsilon s^{(k+1)\delta},\\ \|\partial^I L^J \partial_\alpha v\|_{L^2_f(\mathcal{H}_s)} + \|\partial^I L^J \underline{\partial}_\alpha v\|_{L^2_f(\mathcal{H}_s)} &\lesssim C_1\varepsilon s^{k\delta},\\ \|(s/t)\partial^I L^J \partial_\alpha \partial_\beta v\|_{L^2_f(\mathcal{H}_s)} + \|(s/t)\partial^I L^J \underline{\partial}_\alpha \underline{\partial}_\beta v\|_{L^2_f(\mathcal{H}_s)} &\lesssim C_1\varepsilon s^{k\delta},\\ \|s\partial^I L^J \underline{\partial}_\alpha \underline{\partial}_b v\|_{L^2_f(\mathcal{H}_s)} + \|s\partial^I L^J \underline{\partial}_a \underline{\partial}_\alpha v\|_{L^2_f(\mathcal{H}_s)} &\lesssim C_1\varepsilon s^{(k+1)\delta}.\end{aligned} \tag{6.9}$$

For $|I|+|J|\leqslant N-6$, we have

$$\begin{aligned}\|\partial^I L^J \partial_\alpha \partial_\beta v\|_{L^2_f(\mathcal{H}_s)} + \|\partial^I L^J \underline{\partial}_\alpha \underline{\partial}_\beta v\|_{L^2_f(\mathcal{H}_s)} &\lesssim C_1\varepsilon s^{k\delta},\\ \|t\partial^I L^J \underline{\partial}_a \underline{\partial}_\beta v\|_{L^2_f(\mathcal{H}_s)} + \|t\partial^I L^J \underline{\partial}_\beta \underline{\partial}_a v\|_{L^2_f(\mathcal{H}_s)} &\lesssim C_1\varepsilon s^{(k+1)\delta}.\end{aligned} \tag{6.10}$$

6.3. **Basic sup-norm estimates of the first generation.** We combine the Sobolev inequality on hyperboloids (2.14) with our L^2 estimates. In view of the high-order L^2 bounds, for $|I|+|J|\leqslant N-2$ with $|J|=k$ we have

$$\begin{aligned}\sup_{\mathcal{H}_s}\big(t^{1/2}s|\partial_\alpha \partial^I L^J u|\big) + \sup_{\mathcal{H}_s}\big(t^{1/2}s|\underline{\partial}_\alpha \partial^I L^J u|\big) &\lesssim C_1\varepsilon s^{(k+2)\delta},\\ \sup_{\mathcal{H}_s}\big(t^{3/2}|\underline{\partial}_a \partial^I L^J u|\big) + \sup_{\mathcal{H}_s}\big(t^{3/2}|\underline{\partial}_\perp \partial^I L^J u|\big) &\lesssim C_1\varepsilon s^{(k+2)\delta},\\ \sup_{\mathcal{H}_s}\big(t^{1/2}s|\partial_\alpha \partial^I L^J v|\big) + \sup_{\mathcal{H}_s}\big(t^{1/2}s|\underline{\partial}_\alpha \partial^I L^J v|\big) &\lesssim C_1\varepsilon s^{1/2+(k+2)\delta},\\ \sup_{\mathcal{H}_s}\big(t^{3/2}|\underline{\partial}_\perp \partial^I L^J v|\big) + \sup_{\mathcal{H}_s}\big(t^{3/2}|\underline{\partial}_a \partial^I L^J v|\big) &\lesssim C_1\varepsilon s^{1/2+(k+2)\delta},\\ \sup_{\mathcal{H}_s}\big(t^{3/2}|\partial^I L^J v|\big) &\lesssim C_1\varepsilon s^{1/2+(k+2)\delta}.\end{aligned} \tag{6.11}$$

For $|I|+|J|\leqslant N-3$ with $|J|=k$, we have

$$\sup_{\mathcal{H}_s}\big(t^{3/2}|\partial_\alpha \partial^I L^J v|\big) \lesssim C_1\varepsilon s^{1/2+(k+2)\delta}. \tag{6.12}$$

From the low-order L^2 bounds, for $|I| + |J| \leqslant N - 6$ with $|J| = k$ we have

$$
\begin{aligned}
\sup_{\mathcal{H}_s} \big(t^{1/2} s |\partial_\alpha \partial^I L^J u|\big) + \sup_{\mathcal{H}_s} \big(t^{1/2} s |\underline{\partial}_\alpha \partial^I L^J u|\big) &\lesssim C_1 \varepsilon, \\
\sup_{\mathcal{H}_s} \big(t^{3/2} |\underline{\partial}_a \partial^I L^J u|\big) + \sup_{\mathcal{H}_s} \big(t^{3/2} |\underline{\partial}_\perp \partial^I L^J u|\big) &\lesssim C_1 \varepsilon, \\
\sup_{\mathcal{H}_s} \big(t^{1/2} s |\partial_\alpha \partial^I L^J v|\big) + \sup_{\mathcal{H}_s} \big(t^{1/2} s |\underline{\partial}_\alpha \partial^I L^J v|\big) &\lesssim C_1 \varepsilon s^{(k+2)\delta}, \\
\sup_{\mathcal{H}_s} \big(t^{3/2} |\underline{\partial}_\perp \partial^I L^J v|\big) + \sup_{\mathcal{H}_s} \big(t^{3/2} |\underline{\partial}_a \partial^I L^J v|\big) &\lesssim C_1 \varepsilon s^{(k+2)\delta}, \\
\sup_{\mathcal{H}_s} \big(t^{3/2} |\partial^I L^J v|\big) &\lesssim C_1 \varepsilon s^{(k+2)\delta}.
\end{aligned}
\tag{6.13}
$$

For $|I| + |J| \leqslant N - 7$ with $|J| = k$, we have

$$
\sup_{\mathcal{H}_s} \big(t^{3/2} |\partial_\alpha \partial^I L^J v|\big) \lesssim C_1 \varepsilon s^{(k+2)\delta}.
\tag{6.14}
$$

6.4. **Basic sup-norm estimates of the second generation.** For $|I| + |J| \leqslant N - 2$ with $|J| = k$, we have the high-order bounds

$$
\begin{aligned}
\sup_{\mathcal{H}_s} \big(t^{1/2} s |\partial^I L^J \partial_\alpha u|\big) + \sup_{\mathcal{H}_s} \big(t^{1/2} s |\partial^I L^J \underline{\partial}_\alpha u|\big) &\lesssim C_1 \varepsilon s^{(k+2)\delta}, \\
\sup_{\mathcal{H}_s} \big(t^{3/2} |\partial^I L^J \underline{\partial}_a u|\big) + \sup_{\mathcal{H}_s} \big(t^{3/2} |\partial^I L^J \underline{\partial}_\perp u|\big) &\lesssim C_1 \varepsilon s^{(k+2)\delta}, \\
\sup_{\mathcal{H}_s} \big(t^{1/2} s |\partial^I L^J \partial_\alpha v|\big) + \sup_{\mathcal{H}_s} \big(t^{1/2} s |\partial^I L^J \underline{\partial}_\alpha v|\big) &\lesssim C_1 \varepsilon s^{1/2+(k+2)\delta}, \\
\sup_{\mathcal{H}_s} \big(t^{3/2} |\partial^I L^J \underline{\partial}_a v|\big) &\lesssim C_1 \varepsilon s^{1/2+(k+2)\delta}, \\
\sup_{\mathcal{H}_s} \big(t^{3/2} |\partial^I L^J v|\big) &\lesssim C_1 \varepsilon s^{1/2+(k+2)\delta}
\end{aligned}
\tag{6.15}
$$

and, for $|I| + |J| \leqslant N - 3$ with $|J| = k$,

$$
\begin{aligned}
\sup_{\mathcal{H}_s} \big(t^{5/2} |\underline{\partial}_a \partial^I L^J v|\big) &\lesssim C_1 \varepsilon s^{1/2+(k+3)\delta}, \\
\sup_{\mathcal{H}_s} \big(t^{5/2} |\partial^I L^J \underline{\partial}_a v|\big) &\lesssim C_1 \varepsilon s^{1/2+(k+3)\delta}, \\
\sup_{\mathcal{H}_s} \big(t^{3/2} |\partial^I L^J \partial_\alpha v|\big) + \sup_{\mathcal{H}_s} \big(t^{3/2} |\partial^I L^J \underline{\partial}_\alpha v|\big) &\lesssim C_1 \varepsilon s^{1/2+(k+2)\delta}, \\
\sup_{\mathcal{H}_s} \big(t^{1/2} s |\partial^I L^J \partial_\alpha \partial_\beta v|\big) + \sup_{\mathcal{H}_s} \big(t^{1/2} s |\partial^I L^J \underline{\partial}_\alpha \underline{\partial}_\beta v|\big) &\lesssim C_1 s^{1/2+(k+2)\delta}, \\
\sup_{\mathcal{H}_s} \big(t^{3/2} s |\partial^I L^J \underline{\partial}_\alpha \underline{\partial}_b v|\big) + \sup_{\mathcal{H}_s} \big(t^{3/2} s |\partial^I L^J \underline{\partial}_a \underline{\partial}_\beta v|\big) &\lesssim C_1 s^{1/2+(k+3)\delta}.
\end{aligned}
\tag{6.16}
$$

For $|I| + |J| \leqslant N - 6$ with $|J| = k$, we have

$$
\begin{aligned}
\sup_{\mathcal{H}_s} \big(t^{1/2} s |\partial^I L^J \partial_\alpha u|\big) + \sup_{\mathcal{H}_s} \big(t^{1/2} s |\partial^I L^J \underline{\partial}_\alpha u|\big) &\lesssim C_1 \varepsilon, \\
\sup_{\mathcal{H}_s} \big(t^{3/2} |\partial^I L^J \underline{\partial}_a u|\big) + \sup_{\mathcal{H}_s} \big(t^{3/2} |\partial^I L^J \underline{\partial}_\perp u|\big) &\lesssim C_1 \varepsilon, \\
\sup_{\mathcal{H}_s} \big(t^{1/2} s |\partial^I L^J \partial_\alpha v|\big) + \sup_{\mathcal{H}_s} \big(t^{1/2} s |\partial^I L^J \underline{\partial}_\alpha v|\big) &\lesssim C_1 \varepsilon s^{(k+2)\delta}, \\
\sup_{\mathcal{H}_s} \big(t^{3/2} |\partial^I L^J v|\big) &\lesssim C_1 \varepsilon s^{(k+2)\delta}.
\end{aligned}
\tag{6.17}
$$

In addition, for $|I|+|J|\leqslant N-7$ with $|J|=k$, we have

$$\begin{aligned}
\sup_{\mathcal{H}_s}\big(t^{5/2}|\underline{\partial}_a\partial^I L^J v|\big)&\lesssim C_1\varepsilon s^{(k+3)\delta},\\
\sup_{\mathcal{H}_s}\big(t^{5/2}|\partial^I L^J\underline{\partial}_a v|\big)&\lesssim C_1\varepsilon s^{(k+3)\delta},\\
\sup_{\mathcal{H}_s}\big(t^{3/2}|\partial^I L^J\partial_\alpha v|\big)+\sup_{\mathcal{H}_s}\big(t^{3/2}|\partial^I L^J\underline{\partial}_\alpha v|\big)&\lesssim C_1\varepsilon s^{(k+2)\delta},\\
\sup_{\mathcal{H}_s}\big(t^{1/2}s|\partial^I L^J\partial_\alpha\partial_\beta v|\big)+\sup_{\mathcal{H}_s}\big(t^{1/2}s|\partial^I L^J\underline{\partial}_\alpha\underline{\partial}_\beta v|\big)&\lesssim C_1 s^{(k+2)\delta},\\
\sup_{\mathcal{H}_s}\big(t^{3/2}s|\partial^I L^J\underline{\partial}_\alpha\underline{\partial}_b v|\big)+\sup_{\mathcal{H}_s}\big(t^{3/2}s|\partial^I L^J\underline{\partial}_a\underline{\partial}_\beta v|\big)&\lesssim C_1 s^{(k+3)\delta}.
\end{aligned}\tag{6.18}$$

Moreover, $|I|+|J|\leqslant N-8$ with $|J|=k$, we have

$$\begin{aligned}
\sup_{\mathcal{H}_s}\big(t^{3/2}|\partial^I L^J\partial_\alpha\partial_\beta v|\big)+\sup_{\mathcal{H}_s}\big(t^{3/2}|\partial^I L^J\underline{\partial}_\alpha\underline{\partial}_\beta v|\big)&\lesssim C_1 s^{(k+2)\delta},\\
\sup_{\mathcal{H}_s}\big(t^{3/2}|\partial_\alpha\partial_\beta\partial^I L^J v|\big)+\sup_{\mathcal{H}_s}\big(t^{3/2}|\underline{\partial}_\alpha\underline{\partial}_\beta\partial^I L^J v|\big)&\lesssim C_1 s^{(k+2)\delta}.
\end{aligned}\tag{6.19}$$

6.5. **Estimates based on Hardy's inequality on hyperboloids.** We now substitute the basic L^2 estimates in Hardy's inequality (2.17) and find

$$\|s^{-1}L^J u\|_{L^2_f(\mathcal{H}_s)}\lesssim C_0\varepsilon+C_1\varepsilon s^{k\delta},\qquad |L|\leqslant N,\tag{6.20a}$$

as well as the inequality (which will not be used in the following)

$$\|s^{-1}L^J u\|_{L^2_f(\mathcal{H}_s)}\lesssim (C_0+C_1)\varepsilon+C_1\varepsilon\ln s,\qquad |L|\leqslant N-4.\tag{6.20b}$$

6.6. **Estimate based on integration along radial rays.** By the first estimate in (6.13) and since $t^{-1/2}s^{-1}\lesssim t^{-1}(t-r)^{-1/2}$ (in the domain of interest), we obtain

$$|\partial_r\partial^I L^J u(t,x)|\lesssim C_1\varepsilon t^{-1}(t-r)^{-1/2},\qquad |I|+|J|\leqslant N-6.$$

Then we integrate this inequality in space along the rays $(t,\lambda x)|_{0\leqslant\lambda\leqslant t-1}$ for any $x\in\mathbb{S}^3$:

$$|\partial^I L^J u(t,x)|\lesssim C_1\varepsilon t^{-1}(t-r)^{1/2}\simeq C_1\varepsilon t^{-3/2}s,\qquad |I|+|J|\leqslant N-6.\tag{6.21}$$

7. Refined sup-norm estimates

7.1. **Overview of the analysis in this section.** We now proceed by using the structure of the nonlinear wave system under consideration, and relying now on sharp sup-norm estimates. In the following sections we are going to establish the following estimates: For $|I|\leqslant N-4$, we have

$$\sup_{\mathcal{K}_{[s_0,s_1]}} t|u|\lesssim C_1\varepsilon,\tag{7.1a}$$

$$\sup_{\mathcal{H}_s}\big((s/t)^{-3/2+4\delta}t^{3/2}|\underline{\partial}_\perp\partial^I v|\big)+\sup_{\mathcal{H}_s}\big((s/t)^{-1/2+4\delta}t^{3/2}|\partial^I v|\big)\lesssim C_1\varepsilon,\tag{7.1b}$$

and, more generally, for $|I|+|J|\leqslant N-4$ with $|J|=k$,

$$\sup_{\mathcal{H}_s} t|L^J u|\lesssim C_1\varepsilon s^{k\delta},\tag{7.2a}$$

$$\sup_{\mathcal{H}_s}\big((s/t)^{-3/2+4\delta}t^{3/2}|\underline{\partial}_\perp\partial^I L^J v|\big)+\sup_{\mathcal{H}_s}\big((s/t)^{-1/2+4\delta}t^{3/2}|\partial^I L^J v|\big)\lesssim C_1\varepsilon s^{k\delta}.\tag{7.2b}$$

The property (7.1) is essentially a special case of (7.2): we will establish it first and it will next serve in the proof of (7.2), done by induction on k.

The sup-norm estimate for the Klein-Gordon component (3.7) and the sup-norm estimate for the wave equation (3.2) will now be used. We proceed with the following calculation:

$$-\Box\big(\partial^I L^J u\big)=P^{\alpha\beta}\partial^I L^J\big(\partial_\alpha v\partial_\beta v\big)+R\partial^I L^J(v^2),\tag{7.3}$$

$$-\Box\big(\partial^I L^J v\big)+H^{\alpha\beta}u\partial_\alpha\partial_\beta\partial^I L^J v+c^2\partial^I L^J v=[H^{\alpha\beta}u\partial_\alpha\partial_\beta,\partial^I L^J]v.\tag{7.4}$$

We also recall by (3.7), the R_i terms in this context (with $h^{\alpha\beta} = H^{\alpha\beta}u$) read as follows:

$$
\begin{aligned}
R_1[\partial^I L^J v] &= \Big(s^{3/2} \sum_a \overline{\partial}_a \overline{\partial}_a + \frac{x^a x^b}{s^{1/2}} \overline{\partial}_a \overline{\partial}_b + \frac{3}{4s^{1/2}} + \sum_a \frac{3x^a}{s^{1/2}} \overline{\partial}_a \Big) \partial^I L^J v, \\
R_2[\partial^I L^J v] &= \overline{h}^{00} \Big(\frac{3\partial^I L^J v}{4s^{1/2}} + 3s^{1/2} \overline{\partial}_0 \partial^I L^J v \Big) \\
&\qquad - s^{3/2} \big(2\overline{h}^{0b} \overline{\partial}_0 \overline{\partial}_b \partial^I L^J v + \overline{h}^{ab} \overline{\partial}_a \overline{\partial}_b \partial^I L^J v + h^{\alpha\beta} \partial_\alpha \overline{\Psi}_\beta^{\beta'} \overline{\partial}_{\beta'} \partial^I L^J v \big), \\
R_3[\partial^I L^J v] &= \overline{h}^{00} \Big(2x^a s^{1/2} \overline{\partial}_0 \overline{\partial}_a + \frac{2x^a}{s^{1/2}} \overline{\partial}_a + \frac{x^a x^b}{s^{1/2}} \overline{\partial}_a \overline{\partial}_b \Big) \partial^I L^J v.
\end{aligned}
$$

Hence, the following four terms must be controlled:

$$
\partial^I L^J \big(\partial_\alpha v \partial_\beta v \big), \quad \partial^I L^J (v^2), \quad R_i[\partial^I L^J v], \quad [H^{\alpha\beta} u \partial_\alpha \partial_\beta, \partial^I L^J] v. \tag{7.5}
$$

7.2. **First improvement of the sup-norm of the wave component.** We now present estimates which use only the basic sup-norm estimates already established in Sections 6.3 and 6.4. We first estimate the terms $\partial^I L^J \big(\partial_\alpha v \partial_\beta v \big)$ and $\partial^I L^J (v^2)$.

Lemma 7.1. *If the energy bounds* (5.1) *hold, then for all $|I| + |J| \leqslant N - 7$ with $|J| = k$, the following estimate holds in the region $\mathcal{K}_{[2,s_1]}$:*

$$
\big| \partial^I L^J \big(\partial_\alpha v \partial_\beta v \big) \big| + \big| \partial^I L^J \big(v^2 \big) \big| \leqslant C (C_1 \varepsilon)^2 t^{-3} s^{(k+4)\delta}. \tag{7.6}
$$

Proof. We have

$$
\partial^I L^J \big(\partial_\alpha v \partial_\beta v \big) = \sum_{\substack{I_1 + I_2 = I \\ J_1 + J_2 = J}} \partial^{I_1} L^{J_1} \partial_\alpha v \partial^{I_2} L^{J_2} v, \tag{7.7}
$$

where, in the right-hand side, each term satisfies $|I_1| + |I_2| = |I|$ and $|J_1| + |J_2| = |J|$. Then we obtain

$$
\big| \partial^{I_1} L^{J_1} \partial_\alpha v \partial^{I_2} L^{J_2} v \big| \leqslant C (C_1 \varepsilon)^2 s^{(|J_1|+2)\delta} s^{(|J_2|+2)\delta} t^{-3} = C (C_1 \varepsilon)^2 t^{-3} s^{(k+4)\delta},
$$

where we have used the third inequality in (6.18) for each term. The estimate of $\partial^I L^J \big(v^2 \big)$ is derived similarly. □

We improve the bound on u, as follows.

Proposition 7.2 (First improvement of the sup-norm of the wave component)**.** *For $|I| + |J| \leqslant N - 7$ one has*

$$
|\partial^I L^J u(t,x)| \lesssim C_0 \varepsilon t^{-3/2} + (C_1 \varepsilon)^2 (s/t)^{(k+4)\delta} t^{-1} s^{(k+4)\delta}. \tag{7.8}
$$

Proof. The proof is a direct application of the sup-norm estimate for the wave equation (3.2). First of all, $\partial^I L^J u$ solves the Cauchy problem

$$
\begin{aligned}
&\Box \partial^I L^J u = \partial^I L^J \big(P^{\alpha\beta} \partial_\alpha v \partial_\beta v \big) + \partial^I L^J \big(R v^2 \big), \\
&\partial^I L^J u(2,x) = U_0(I,J,x), \quad \partial_t \partial^I L^J u(2,x) = U_1(I,J,x),
\end{aligned}
$$

where $U_0(I,J,x)$ and $U_1(I,J,x)$ are restrictions of $\partial^I L^J u$ and $\partial^I L^J u$ on the initial hyperplane $\{t = 2\}$. We remark that they are linear combinations of $\partial_x^{I'} u$ and $\partial_t \partial_x^{I'} u$ with $|I'| \leqslant |I| + |J|$. Hence, u is decomposed as follows:

$$
u(t,x) = w_1(t,x) + w_2(t,x)
$$

with

$$
\begin{aligned}
&\Box L^J w_2 = L^J \big(P^{\alpha\beta} \partial_\alpha v \partial_\beta v \big) + L^J \big(R v^2 \big), \\
&w_2(2,x) = \partial_t w_2(2,x) = 0,
\end{aligned}
$$

while

$$
\begin{aligned}
&\Box w_1 = 0, \\
&w_1(2,x) = U_1(I,J,x), \quad \partial_t w_1(2,x) = U_2(I,J,x).
\end{aligned}
$$

The sup-norm bound for w_1 comes directly from the explicit expression of the solutions (cf., for instance, [32]) while for w_2 we apply (3.2). Observe that for the terms in the right-hand side:

$$\begin{aligned}\big|\partial^I L^J\big(P^{\alpha\beta}\partial_\alpha v\partial_\beta v\big)\big| + \big|\partial^I L^J\big(Rv^2\big)\big| \lesssim &(C_1\varepsilon)^2 t^{-3}s^{(k+4)\delta}\\ \lesssim &(C_1\varepsilon)^2 t^{-2-(1-(2+k/2)\delta)}(t-r)^{-1+(1+(2+k/2)\delta)}\end{aligned} \tag{7.9}$$

in $\mathcal{K}_{[2,s]}\subset\mathcal{K}_{[2,s_1]}$. Recall that this estimate also holds in $\{(t,x)|r<t-1, t^2-r^2\leqslant s_1^2, t\geqslant 2\}$. Then by (3.2), the desired result is proven. □

We next estimate the terms $R_i[\partial^I L^J v]$.

Lemma 7.3. *For $|I|+|J|\leqslant N-4$ with $|J|=k$, the following estimates hold in $\mathcal{K}_{[2,s_1]}$:*

$$\big|R_1[\partial^I L^J v]\big| \lesssim C_1\varepsilon(s/t)^{3/2}s^{-3/2+(k+4)\delta}, \tag{7.10a}$$

$$\big|R_2[\partial^I L^J v]\big| \lesssim C_1\varepsilon|tu|(s/t)^{3/2}s^{-3/2+(k+3)\delta} + (C_1\varepsilon)^2(s/t)^{3/2}s^{-3/2+(k+3)\delta}, \tag{7.10b}$$

$$\big|R_3[\partial^I L^J v]\big| \lesssim (C_1\varepsilon)^2(s/t)s^{-2+(k+4)\delta} + C_1\varepsilon|tu|(s/t)^{3/2}s^{-3/2+(k+3)\delta}. \tag{7.10c}$$

Proof. The proof is a substitution of the basic sup-norm estimates into the corresponding expression. We begin with R_1 and focus first on $\overline{\partial}_a\overline{\partial}_b\partial^I L^J v$:

$$\begin{aligned}\overline{\partial}_a\overline{\partial}_b\partial^I L^J v =& t^{-1}L_a\big(t^{-1}L_b\partial^I L^J v\big) = t^{-1}L_a\big(t^{-1}\partial^I L_b L^J v + t^{-1}[L_b,\partial^I]L^J v\big)\\ =& t^{-1}L_a\big(t^{-1}\big)\partial^I L_b L^J v + t^{-2}L_a\partial^I L_b L^J v + t^{-1}L_a\big(t^{-1}\big)[L_b,\partial^I]L^J v\\ &+ t^{-2}L_a[L_b,\partial^I]L^J v\\ =& t^{-1}L_a\big(t^{-1}\big)\partial^I L_b L^J v + t^{-2}\partial^I L_a L_b L^J v + t^{-2}[L_a,\partial^I]L_b L^J v\\ &+ t^{-1}L_a\big(t^{-1}\big)[L_b,\partial^I]L^J v + t^{-2}L_a[L_b,\partial^I]L^J v.\end{aligned} \tag{7.11}$$

For the last term, we apply (4.5) as follows:

$$\begin{aligned}t^{-2}L_a[L_b,\partial^I]L^J v =& -t^{-2}\sum_{|I'|\leqslant|I|}\lambda^I_{bI'}L_a\partial^{I'}L^J v\\ =& -t^{-2}\sum_{|I'|\leqslant|I|}\lambda^I_{bI'}\partial^{I'}L_a L^J v - t^{-2}\sum_{|I'|\leqslant|I|}\lambda^I_{bI'}[L_a,\partial^{I'}]L^J v\\ =& -t^{-2}\sum_{|I'|\leqslant|I|}\lambda^I_{bI'}\partial^{I'}L_a L^J v + t^{-2}\sum_{|I'|\leqslant|I|}\lambda^I_{bI'}\sum_{|I''|\leqslant|I'|}\lambda^{I'}_{aI''}\partial^{I''}L^J v,\end{aligned}$$

and the term $[L_a,\partial^I]L_b L^J v$ is bounded in the same manner. Then we conclude that

$$\big|t^{-2}L_a[L_b,\partial^I]L^J v\big| \leqslant Ct^{-2}\sum_{\substack{|I'|\leqslant|I|\\ |J'|\leqslant|J|+1}}\big|\partial^{I'}L^{J'}v\big|.$$

In view of the inequality $\big|L_a\big(t^{-1}\big)\big|\leqslant Ct^{-1}$ (in $\mathcal{K}$), the terms in the right-hand side of (7.11) are bounded by $Ct^{-2}\sum_{\substack{|I'|\leqslant|I|\\ |J'|\leqslant|J|+2}}\big|\partial^{I'}L^{J'}v\big|$. Then, by the last equation in (6.11), we have

$$\big|s^{3/2}\overline{\partial}_a\overline{\partial}_b\partial^I L^J v\big| \lesssim C_1\varepsilon(s/t)^{7/2}s^{-3/2+(k+4)\delta}$$

and, similarly,

$$\begin{aligned}\big|x^a s^{-1/2}\overline{\partial}_a\partial^I L^J v\big| &\lesssim C_1\varepsilon(s/t)^{3/2}s^{-3/2+(k+3)\delta},\\ \big|x^a x^b s^{-1/2}\overline{\partial}_a\overline{\partial}_b\partial^I L^J v\big| &\lesssim C_1\varepsilon(s/t)^{3/2}s^{-3/2+(k+4)\delta},\\ \big|s^{-1/2}\partial^I L^J v\big| &\lesssim C_1\varepsilon(s/t)^{3/2}s^{-3/2+(k+2)\delta}.\end{aligned}$$

So we conclude that

$$\big|R_1[\partial^I L^J v]\big| \lesssim C_1\varepsilon t^{-3/2}s^{(k+4)\delta} \lesssim C_1\varepsilon(s/t)^{3/2}s^{-3/2+(k+4)\delta}.$$

For the derivation in the paragraph above, let us provide some more details by observing that, for the first term,

$$|\overline{\partial}_a\overline{\partial}_b\partial^I L^J v| = |t^{-1}L_a\big(t^{-1}L_b\partial^I L^J v| \lesssim t^{-2}|L_a L_b\partial^I L^J v| + t^{-1}|L_a(t^{-1})|\cdot|L_b\partial^I L^J v|.$$

Here, we remark that

$$s^{3/2}t^{-2}|L_aL_b\partial^IL^Jv|\lesssim t^{-2}C_1\varepsilon s^{3/2}t^{-2}s^{1/2+(|I|+2+2)\delta}t^{-3/2}\lesssim (s/t)^{7/2}s^{-3/2+(k+4)\delta}.$$

In the first inequality, we have used the fact that

$$|L_aL_b\partial^IL^Jv|\lesssim |\partial^IL_aL_bL^Jv|+|[L_aL_b,\partial^I]L^Jv|.$$

For the first term in the right-hand-side of the above inequality, we get the upper bound $CC_1\varepsilon t^{-3/2}s^{1/2+(|J|+2+2)\delta}t^{-3/2}$ (for $|I|+|J|+2\leqslant N-2$) and for the second we recall the estimate on commutator (using (4.5) twice), and see that

$$|[L_aL_b,\partial^I]L^Jv|\lesssim \sum_{\substack{|I'|\leqslant |I|\\ |J'|\leqslant |J+1|}}\partial^{I'}|L^{J'}\partial^{I'}L^{J'}v|\lesssim C_1\varepsilon t^{-3/2}s^{1/2+(|J|+1+2)\delta}.$$

The estimates of R_2 and R_3 are quite similar. We just need to observe that, by (6.21) and by recalling that $|\overline{H}^{00}|\leqslant C(t/s)^2$, $|\overline{H}^{a0}|\leqslant C(t/s)$, and $|\overline{H}^{ab}|\leqslant C$, we obtain

$$|\overline{h}^{00}|=|\overline{H}^{00}u|\lesssim C_1\varepsilon t^{1/2}s^{-1},\qquad |\overline{h}^{a0}|\lesssim C_1\varepsilon t^{-1/2},\qquad |\overline{h}^{ab}u|\lesssim C_1\varepsilon t^{-3/2}s \tag{7.12}$$

and

$$\begin{aligned}\big|\overline{\partial}_0\overline{\partial}_a\partial^IL^Jv\big|=&(s/t)\big|\partial_t\overline{\partial}_a\partial^IL^Jv\big|\\ \leqslant&(s/t)t^{-2}\big|L_a\partial^IL^Jv\big|+t^{-1}(s/t)\big|\partial_tL_a\partial^IL^Jv\big|.\end{aligned}$$

As was done in (7.11) and by applying (4.5) and the fifth equation in (6.15) we find

$$\big|\overline{\partial}_0\overline{\partial}_a\partial^IL^Jv\big|\lesssim C_1\varepsilon(s/t)^{7/2}s^{-2+(k+3)\delta}. \tag{7.13}$$

Equipped with (7.12) and (7.13), we see that in $R_2[\partial^IL^Jv]$,

$$\begin{aligned}\big|s^{-1/2}\overline{h}^{00}\partial^IL^Jv\big|&\lesssim (C_1\varepsilon)^2(s/t)s^{-2+(k+2)\delta}\lesssim (C_1\varepsilon)^2(s/t)^{3/2}s^{-3/2+(k+2)\delta},\\ \big|s^{1/2}\overline{h}^{00}\overline{\partial}_0\partial^IL^Jv\big|&\lesssim C_1\varepsilon|tu|(s/t)^{3/2}s^{-3/2+(k+2)\delta},\\ s^{3/2}\big|\overline{h}^{0b}\overline{\partial}_0\overline{\partial}_b\partial^IL^Jv\big|&\lesssim C_1\varepsilon|tu|(s/t)^{5/2}s^{-3/2+(k+3)\delta},\\ s^{3/2}\big|\overline{h}^{ab}\overline{\partial}_a\overline{\partial}_b\partial^IL^Jv\big|&\lesssim (C_1\varepsilon)^2(s/t)^5s^{-2+(k+4)\delta},\\ s^{3/2}\big|h^{\alpha\beta}\partial_\alpha\overline{\Psi}^0_\beta\,\overline{\partial}_0\partial^IL^Jv\big|&\lesssim C_1\varepsilon|tu|(s/t)^{3/2}s^{-3/2+(k+2)\delta},\\ s^{3/2}\big|h^{\alpha\beta}\partial_\alpha\overline{\Psi}^b_\beta\,\overline{\partial}_b\partial^IL^Jv\big|&=0,\end{aligned}$$

while, in the expression $R_3[\partial^IL^J]v$,

$$\begin{aligned}\big|\overline{h}^{00}x^as^{1/2}\overline{\partial}_0\overline{\partial}_a\partial^IL^Jv\big|&\lesssim C_1\varepsilon|tu|(s/t)^{3/2}s^{-3/2+(k+3)\delta},\\ \Big|\overline{h}^{00}x^as^{-1/2}\overline{\partial}_a\partial^IL^Jv\Big|&\lesssim (C_1\varepsilon)^2(s/t)^{3/2}s^{-3/2+(k+3)\delta},\\ s^{-1/2}\big|\overline{h}^{00}x^ax^b\overline{\partial}_a\overline{\partial}_b\partial^IL^Jv\big|&\lesssim (C_1\varepsilon)^2(s/t)s^{-2+(k+4)\delta}.\end{aligned}$$

□

7.3. **Second improvement on the wave component and first improvement on the Klein-Gordon component.** We now establish (7.1a)-(7.1b) and, for latter use, we first derive the following improved estimates on the terms R_i.

Lemma 7.4. *For $|I|+|J|\leqslant N-4, |J|=k$, the following estimates hold in $\mathcal{K}_{[2,s_1]}$:*

$$\sum_{i=1}^3\big|R_i[\partial^IL^Jv]\big|\lesssim C_1\varepsilon(s/t)^{3/2}s^{-3/2+(k+7)\delta}. \tag{7.14}$$

Proof. This is a combination of Lemma 7.3 and (7.8) (take $k=0$ then considering the condition $C_1\varepsilon\leqslant 1$) and the fact that in $\mathcal{K}, t^{1/2}\leqslant s\leqslant t$. □

Then we need to estimate the term $|h'_{t,x}(\lambda)|$ in Proposition 3.3.

Lemma 7.5. *The following estimate holds for $(t,x) \in \mathcal{K}_{[2,s_1]}$:*

$$\int_{s_0}^{s} |h'_{t,x}(\lambda)| d\lambda \lesssim C_1\varepsilon, \tag{7.15}$$

where $h_{t,x}(\lambda) := \overline{h}^{00}(\lambda t/s, \lambda x/s) = \overline{H}^{00}(\lambda t/s, \lambda x/s) u(\lambda t/s, \lambda x/s)$.

Proof. With the notation of Proposition 3.3, we have $\overline{h}^{00} = \overline{H}^{00} u$, and we observe that

$$\overline{H}^{00} = H^{\alpha\beta}\overline{\Psi}^0_\alpha \overline{\Psi}^0_\beta = H^{00}(t/s)^2 - 2\sum_a H^{0a}(x^a t/s) + \sum_{a,b} H^{ab}(x^a x^b/s^2).$$

Note that $\overline{H}^{00}(\lambda t/s, \lambda x/s) = \overline{H}^{00}(t,x)$, so that $\overline{H}^{00}$ is constant along the segment $(\lambda t/s, \lambda x/s)$, $s_0 \leqslant \lambda \leqslant s$. So we find

$$h'_{t,x}(\lambda) = \overline{H}^{00}(t,x)(t/s)\underline{\partial}_\perp u(\lambda t/s, \lambda x/s),$$

and we conclude that

$$|h'_{t,x}(\lambda)| \leqslant C(t/s)^3 |\underline{\partial}_\perp u(\lambda t/s, \lambda x/s)|.$$

Next, we observe the identity

$$\underline{\partial}_\perp u = \frac{s^2}{t^2}\partial_t u + \frac{x^a}{t}\underline{\partial}_a u = \frac{s^2}{t^2}\partial_t u + \frac{x^a}{t^2} L_a u \tag{7.16}$$

and, by the first inequality in (6.17) and (7.8) with $\partial^I L^J = L_a$,

$$|\underline{\partial}_\perp u| \lesssim C_1\varepsilon(s/t)t^{-3/2} + C_1\varepsilon(s/t)^{5\delta} t^{-2} s^{5\delta}.$$

Therefore, we obtain

$$|h'_{t,x}(\lambda)| \lesssim C_1\varepsilon(s/t)^{-1/2}\lambda^{-3/2} + C_1\varepsilon(s/t)^{-1+5\delta}\lambda^{-2+5\delta}.$$

Then, to apply the sup-norm estimate for the Klein-Gordon equation (3.7), we proceed as follows. In the range $0 \leqslant r/t \leqslant 3/5$, we have $4/5 \leqslant s/t \leqslant 1$, and

$$\int_{s_0}^{s} |h'_{t,x}(\lambda)| d\lambda \lesssim C_1\varepsilon \int_2^s \lambda^{-3/2}\, d\lambda \lesssim C_1\varepsilon.$$

In the range $3/5 < r/t < 1$, we obtain

$$\begin{aligned}\int_{s_0}^{s} |h'_{t,x}(\lambda)| d\lambda &\lesssim C_1\varepsilon(s/t)^{-1/2}\int_{s_0}^{s}\lambda^{-3/2}\, d\lambda + C_1\varepsilon(s/t)^{-1+5\delta}\int_{s_0}^{s}\lambda^{-2+5\delta}\, d\lambda \\ &\lesssim C_1\varepsilon(s/t)^{-1/2}s_0^{-1/2} + C_1\varepsilon(s/t)^{-1+5\delta}s_0^{-1+5\delta}.\end{aligned}$$

We recall that, when $3/5 < r/t < 1$, $s_0 = \sqrt{\frac{t+r}{t-r}} \geqslant t/s$, so that $\int_{s_0}^{s} |h'_{t,x}(\lambda)| d\lambda \lesssim C_1\varepsilon$, and the desired result is established. □

Now we give a second application of the sup-norm estimate for the Klein-Gordon component (3.7).

Proposition 7.6 (Second improvement on the wave component and first improvement on the Klein-Gordon component)**.** *The following estimate also holds in $\mathcal{K}_{[2,s_1]}$:*

$$|v(t,x)| + \frac{t}{s}|\underline{\partial}_\perp v(t,x)| \lesssim C_1\varepsilon(s/t)^{2-7\delta}s^{-3/2}, \tag{7.17a}$$

$$|u(t,x)| \lesssim C_1\varepsilon t^{-1}. \tag{7.17b}$$

Proof. We rely here on (7.7) and the sup-norm estimate for the wave equation (3.2), and we first establish (7.17a). In view of (7.15), we have

$$\begin{aligned}\int_{s_0}^{s} |h'_{t,x}(\bar{s})| e^{\int_{\bar{s}}^{s} |h'_{t,x}(\lambda)| d\lambda}\, d\bar{s} &\leqslant \int_{s_0}^{s} |h'_{t,x}(\bar{s})| e^{\int_{s_0}^{s} |h'_{t,x}(\lambda)| d\lambda}\, d\bar{s} \\ &\lesssim \int_{s_0}^{s} |h'_{t,x}(\bar{s})| e^{CC_1\varepsilon}\, d\bar{s} \lesssim C_1\varepsilon.\end{aligned}$$

On the other hand, to estimate $F(\bar{s})$, we write

$$\begin{aligned} F(s) &= \int_{s_0}^{s} \big(R_1[v] + R_2[v] + R_3[v]\big)(\lambda t/s, \lambda x/s)\, d\lambda \\ &\lesssim C_1\varepsilon \int_{s_0}^{s} (s/t)^{3/2}\lambda^{-3/2+7\delta} d\lambda \lesssim C_1\varepsilon (s/t)^{3/2} s_0^{-1/2+7\delta}. \end{aligned}$$

Now for $0 \leqslant r/t \leqslant 3/5$ we see that $4/5 \leqslant s/t \leqslant 1$ and $s_0 = 2$, and we have

$$F(s) \lesssim C_1\varepsilon (s/t)^{3/2} s_0^{-1/2+7\delta} \lesssim C_1\varepsilon (s/t)^{2-7\delta}.$$

For $3/5 < r/t < 1$, we see that $s_0 = \sqrt{\frac{t+r}{t-r}} \leqslant t/s$, so that

$$F(s) \lesssim C_1\varepsilon (s/t)^{2-7\delta}. \tag{7.18}$$

Then, by combining (7.15), (7.18) and (3.7), we conclude that (7.17a) holds. On the other hand, (7.17b) follows directly from substituting (7.17a) into (3.2).

Let us explain in more detail the above argument. In the equation

$$\Box u = P_{\alpha\beta}\partial_\alpha v \partial_\beta v + Rv^2,$$

we need to estimate $\big|P^{\alpha\beta}\partial_\alpha v\partial_\beta v\big|$ and $\big|Rv^2\big|$. First we rewrite the expresison $P^{\alpha\beta}\partial_\alpha v\partial_\beta v$ in the semi-hyperboloidal frame:

$$\begin{aligned} P^{\alpha\beta}\partial_\alpha v\partial_\beta v =& \underline{P}^{\alpha\beta}\underline{\partial}_\alpha v\underline{\partial}_\beta v \\ =& \underline{P}^{00}\partial_t v\partial_t v + \underline{P}^{a0}\underline{\partial}_a v\partial_t v + \underline{P}^{0b}\partial_t v\underline{\partial}_b v + \underline{P}^{ab}\underline{\partial}_a v\underline{\partial}_b v. \end{aligned}$$

The last three terms in the right-hand-side can be controlled by applying the first and the third inequalities in (6.18)

$$\big|P^{a0\beta}\underline{\partial}_a v\underline{\partial}_\beta v\big| + \big|P^{\alpha b}\underline{\partial}_\alpha\underline{\partial}_b v\big| \leqslant C(C_1\varepsilon)^2 t^{-4}s^{5\delta}.$$

For the first term, we see that

$$\partial_t v = \frac{t^2}{s^2}\left(\underline{\partial}_\perp v - \frac{x^a}{t}\underline{\partial}_a v\right).$$

Then by (7.17a) and the third inequality in (6.18), we obtain

$$\begin{aligned} |\partial_t v| \leqslant& CC_1\varepsilon (s/t)^{1-7\delta}s^{-3/2} + CC_1\varepsilon t^{-5/2}s^{3\delta} \\ \leqslant& CC_1\varepsilon (s/t)^{1-7\delta}s^{-3/2}. \end{aligned}$$

This leads to

$$\big|\underline{P}^{00}\partial_t v\partial_t v\big| \leqslant C(C_1\varepsilon)^2 (t-r)^{-1+(1/2-7\delta/2)} t^{-2-(1/2-7\delta/2)}.$$

The term $\big|Rv^2\big|$ is bounded directly by (7.17a), and we have

$$\big|Rv^2\big| \leqslant C(C_1\varepsilon)^2 (s/t)^{4-14\delta}s^{-3} \leqslant CC_1\varepsilon t^{-3}.$$

Then by applying (3.2), the desired bound on u is guaranteed. □

With (7.17b), we can improve again the estimate on R_i. Namely, the proof of the following estimate is immediate by substituting (7.17b) into (7.7), and using (3.2).

Lemma 7.7. *The following estimates hold:*

$$\sum_{i=1}^{3} R_i[\partial^I L^J v] \lesssim C_1\varepsilon (s/t)^{3/2} s^{-3/2+(k+4)\delta}. \tag{7.19}$$

7.4. **Second improvement on the Klein-Gordon component.** We now establish (7.1b) and, to do so, our first task is to estimate the commutator $[H^{\alpha\beta}u\partial_\alpha\partial_\beta, \partial^I L^J]$. First of all, from the following identities

$$\partial_t = \frac{t^2}{s^2}\big(\underline{\partial}_\perp - (x^a/t)\underline{\partial}_a\big), \qquad \partial_a = -\frac{tx^a}{s^2}\underline{\partial}_\perp + \frac{x^a x^b}{t^2}\underline{\partial}_b + \underline{\partial}_a, \tag{7.20}$$

the following estimates are immediate:

$$\begin{aligned}
\big|\partial_t\partial^I L^J v\big| &\leqslant (t/s)^2\big|\underline{\partial}_\perp\partial^I L^J v\big| + (t/s)^2\sum_a\big|\underline{\partial}_a\partial^I L^J v\big|,\\
\big|\partial_a\partial^I L^J v\big| &\leqslant (t/s)^2\big|\underline{\partial}_\perp\partial^I L^J v\big| + C(t/s)^2\sum_a\big|\underline{\partial}_a\partial^I L^J v\big|.
\end{aligned} \tag{7.21}$$

Based on this result, we estimate the commutator $[H^{\alpha\beta}u\partial_\alpha\partial_\beta, \partial^I]$. (In the statement below, as usual, a sum over the empty set vanishes.)

Lemma 7.8. *The following estimates are valid in $\mathcal{K}$ for $|I|+|J|\leqslant N-4$ and $|J|=k$:*

$$\begin{aligned}
&\big|[H^{\alpha\beta}u\partial_\alpha\partial_\beta, \partial^I L^J]v\big|\\
&\lesssim C_1\varepsilon t^{-1}(s/t)^{-2}\sum_{\substack{|I_2|\leqslant|I|,\beta\\ |J_2|\leqslant|J|-1}}|\underline{\partial}_\perp\partial_\beta\partial^{I_2}L^{J_2}v| + \sum_{\substack{J_1+J_2=J\\ |J_1|\geqslant 1}}|L^{J_1}u|\,|\partial^I L^{J_2}\partial_\alpha\partial_\beta v|\\
&\quad + C_1\varepsilon t^{-3/2}(s/t)^{-3}\sum_{\substack{|I_2|+|J_2|\leqslant|I|+|J|-1,\beta\\ N-7\leqslant|I_2|+|J_2|\leqslant N-5}}|\underline{\partial}_\perp\partial_\beta\partial^{I_2}L^{J_2}v|\\
&\quad + (C_1\varepsilon)^2(s/t)^{3/2}s^{-3+(k+4)\delta}.
\end{aligned} \tag{7.22}$$

Proof. We write the decomposition

$$\begin{aligned}
[H^{\alpha\beta}u\partial_\alpha\partial_\beta, \partial^I L^J]v &= \sum_{\substack{I_1+I_2=I\\ J_1+J_2=J\\ |I_1|+|J_1|\geqslant 1}} H^{\alpha\beta}\partial^{I_1}\partial^{J_1}u\partial^{I_2}L^{J_2}\partial_\alpha\partial_\beta v + H^{\alpha\beta}u\partial^I([L^J,\partial_\alpha\partial_\beta]v)\\
&=: T_0 + T_7.
\end{aligned}$$

We recall that by (4.17) and (7.17b), T_7 is bounded as follows:

$$\begin{aligned}
|T_7| &\lesssim C_1\varepsilon t^{-1}\sum_{\substack{\alpha,\beta\\ |J_2|\leqslant|J|-1}}|\partial^I\partial_\alpha\partial_\beta L^{J_2}v|\\
&\lesssim C_1\varepsilon t^{-1}(t/s)^2\sum_{\substack{\beta\\ |J_2|\leqslant|J|-1}}|\underline{\partial}_\perp\partial_\beta\partial^I L^{J_2}v| + C_1\varepsilon t^{-1}(t/s)^2\sum_{\substack{a,\beta\\ |J_2|\leqslant|J|-1}}|\underline{\partial}_a\partial_\beta\partial^I L^{J_2}v|.
\end{aligned}$$

The second term in the right-hand side is bounded as follows:

$$\begin{aligned}
C_1\varepsilon t^{-1}(t/s)^2|\underline{\partial}_a\partial_\beta\partial^I L^{J_2}v| &\lesssim C_1\varepsilon t^{-1}(t/s)^2 C_1\varepsilon t^{-5/2}s^{1/2+(k+3)\delta}\\
&\lesssim (C_1\varepsilon)^2(s/t)^{3/2}s^{-3+(k+3)\delta}.
\end{aligned}$$

We then write

$$\begin{aligned}
\big|T_0\big| &\leqslant \sum_{\substack{I_1+I_2=I,|I_1|\geqslant 1\\ J_1+J_2=J,\alpha,\beta,\gamma}}|\partial^{I_1}L^{J_1}u|\,|\partial^{I_2}L^{J_2}\partial_\alpha\partial_\beta v| + \sum_{\substack{J_1+J_2=J\\ |J_1|\geqslant 1,\alpha,\beta}}|L^{J_1}u|\,|\partial^I L^{J_2}\partial_\alpha\partial_\beta v|\\
&=: T_1 + T_2.
\end{aligned}$$

Then we see that T_1 is again decomposed as follows:

$$\begin{aligned}
T_1 &= \sum_{\substack{I_1+I_2=I,|I_1|\geqslant 1\\ J_1+J_2=J,\alpha,\beta,\gamma\\ |I_2|+|J_2|\leqslant N-8}}|\partial^{I_1}L^{J_1}u|\,|\partial^{I_2}L^{J_2}\partial_\alpha\partial_\beta v| + \sum_{\substack{I_1+I_2=I,|I_1|\geqslant 1\\ J_1+J_2=J,\alpha,\beta,\gamma\\ N-7\leqslant|I_2|+|J_2|\leqslant N-5}}|\partial^{I_1}L^{J_1}u|\,|\partial^{I_2}L^{J_2}\partial_\alpha\partial_\beta v|\\
&=: T_3 + T_4.
\end{aligned}$$

We have

$$T_3 \lesssim C_1\varepsilon t^{-1/2}s^{-1+(|J_1|+2)\delta}C_1\varepsilon t^{-3/2}s^{(|J_2|+2)\delta} \lesssim (C_1\varepsilon)^2(s/t)^2 s^{-3+(k+4)\delta},$$

where we applied (6.19) and the fourth estimate in (6.18). Then by applying (7.21) and in view of (4.17), the term T_4 is bounded by

$$\begin{aligned}T_4 \leqslant & \sum_{\substack{I_1+I_2=I, |I_1|\geqslant 1\\ |J_1|+|J_2|\leqslant |J|, \alpha,\beta,\gamma\\ N-7\leqslant |I_2|+|J_2|\leqslant N-5}} |\partial^{I_1}L^{J_1}u|\,|\partial_\alpha\partial_\beta\partial^{I_2}L^{J_2}v|\\ \leqslant & (t/s)^2 \sum_{\substack{I_1+I_2=I, |I_1|\geqslant 1\\ |J_1|+|J_2|\leqslant |J|\alpha,\beta,\gamma\\ N-7\leqslant |I_2|+|J_2|\leqslant N-5}} |\partial^{I_1}L^{J_1}u|\,|\underline{\partial}_\perp\partial_\beta\partial^{I_2}L^{J_2}v|\\ & + (t/s)^2 \sum_{\substack{I_1+I_2=I, |I_1|\geqslant 1\\ |J_1|+|J_2|\leqslant |J|, a,\beta,\gamma\\ N-7\leqslant |I_2|+|J_2|\leqslant N-5}} |\partial^{I_1}L^{J_1}u|\,|\underline{\partial}_a\partial_\beta\partial^{I_2}L^{J_2}v| =: \quad T_5+T_6.\end{aligned}$$

Then, in the expression T_5, $N-7\leqslant |I_2|+|J_2|\leqslant N-5$ implies $|I_1|+|J_1|\leqslant 3\leqslant N-6$ and recall $|I_1|\geqslant 1$, so we see $|\partial^{I_1}L^{J_1}u|\leqslant C\sum_\gamma \partial_\gamma\partial^{I_1'}L^{J_1}u$ were $|I_1'|+|J_1|\leqslant 2\leqslant N-6$. Then by the first estimate in (6.13), we find

$$T_5 \lesssim C_1\varepsilon t^{-3/2}(s/t)^{-3} \sum_{\substack{|I_2|+|J_2|\leqslant |I|+|J|-1\\ N-7\leqslant |I_2|+|J_2|\leqslant N-5}} |\underline{\partial}_\perp\partial_\beta\partial^{I_2}L^{J_2}v|.$$

Furthermore, we have

$$\begin{aligned}T_6 \lesssim & (t/s)^2 C_1\varepsilon t^{-1/2}s^{-1}\, C_1\varepsilon t^{-5/2}s^{1/2+(|J_2|+3)\delta} \lesssim (C_1\varepsilon)^2(s/t)s^{-7/2+(k+2)\delta}\\ \lesssim & (C_1\varepsilon)^2(s/t)^{3/2}s^{-3+(k+3)\delta}\end{aligned}$$

and the desired estimate is established. □

We are now in a position to establish the desired bound (7.1b).

Proposition 7.9 (Second improvement on the Klein-Gordon component)**.** *The following estimate holds in $\mathcal{K}_{[2,s_1]}$ for $|I|\leqslant N-4$:*

$$|\underline{\partial}_\perp\partial^I v(t,x)| \lesssim C_1\varepsilon(s/t)^{3/2-4\delta}t^{-3/2}, \tag{7.23a}$$

$$|\partial^I v(t,x)| \lesssim C_1\varepsilon(s/t)^{1/2-4\delta}t^{-3/2}. \tag{7.23b}$$

Proof. We first discuss the case where $|I|-1\geqslant N-7$ and, in this case, using (7.22)

$$\big|[H^{\alpha\beta}u\partial_\alpha\partial_\beta,\partial^I]v\big| \lesssim C_1\varepsilon t^{-3/2}(s/t)^{-3} \sum_{\substack{|I_2|\leqslant |I|-1,\beta\\ N-7\leqslant |I_2|\leqslant N-5}} |\underline{\partial}_\perp\partial_\beta\partial^{I_2}v| + (C_1\varepsilon)^2(s/t)^{3/2}s^{-3+4\delta}.$$

For all $\bar s\in[s_0,s]$, using (7.19) and the above estimate we have

$$\begin{aligned}F(\bar s) \leqslant & \sum_{i=1}^3 \int_{s_0}^{\bar s} |R_i[\partial^I v](\lambda t/s,\lambda x/s)|d\lambda + \int_{s_0}^{\bar s}\lambda^{3/2}\big|[H^{\alpha\beta}u\partial_\alpha\partial_\beta,\partial^I]v\big|d\lambda\\ \lesssim & C_1\varepsilon(s/t)^{3/2}\int_{s_0}^{\bar s}\lambda^{-3/2+3\delta}\,d\lambda + (C_1\varepsilon)^2(s/t)^{3/2}\int_{s_0}^{\bar s}\lambda^{-3/2+4\delta}d\lambda\\ & + C_1\varepsilon(s/t)^{-3/2} \sum_{\substack{|I_2|\leqslant |I|-1,\beta\\ N-7\leqslant |I_2|\leqslant N-5}} \int_{s_0}^{\bar s} |\underline{\partial}_\perp\partial_\beta\partial^{I_2}v(\lambda t/s,\lambda x/s)|d\lambda\\ \lesssim & C_1\varepsilon(s/t)^{-3/2} \sum_{\substack{|I_2|\leqslant |I|-1,\beta\\ N-7\leqslant |I_2|\leqslant N-5}} \int_{s_0}^{\bar s} |\underline{\partial}_\perp\partial_\beta\partial^{I_2}v(\lambda t/s,\lambda x/s)|d\lambda + C_1\varepsilon(s/t)^{3/2}s_0^{-1/2+4\delta}.\end{aligned}$$

Case I: $3/5<r/t<1$. In this case, $s_0=\sqrt{\frac{t+r}{t-r}}\geqslant t/s$ and we have

$$F(\bar s) \lesssim C_1\varepsilon(s/t)^{2-4\delta} + C_1\varepsilon(s/t)^{-3/2} \sum_{\substack{|I_2|\leqslant |I|-1,\beta\\ N-7\leqslant |I_2|\leqslant N-5}} \int_{s_0}^{\bar s} |\underline{\partial}_\perp\partial_\beta\partial^{I_2}v(\lambda t/s,\lambda x/s)|d\lambda.$$

We define

$$V_{t,x}(\lambda) := (\lambda t/s)^{3/2} \sum_{\substack{|I_2|\leqslant |I|-1,\beta \\ N-7\leqslant |I_2|\leqslant N-5}} |\underline{\partial}_\perp \partial_\beta \partial^{I_2} v(\lambda t/s, \lambda x/s)|$$

and find

$$F(\bar{s}) \lesssim C_1\varepsilon(s/t)^{2-4\delta} + C_1\varepsilon \int_{s_0}^{\bar{s}} \lambda^{-3/2} V_{t,x}(\lambda)\, d\lambda, \quad s_0 \leqslant \bar{s} \leqslant s. \tag{7.24}$$

Recalling the sup-norm estimate for the Klein-Gordon component (3.7) in the case $1 > r/t > 3/5$, we obtain

$$|\underline{\partial}_\perp \partial^I v(t,x)| \leqslant Cs^{-1/2}t^{-1}\Big(F(s) + \int_{s_0}^{s} F(\bar{s})|h'_{t,x}(\bar{s})| e^{\int_{\bar{s}}^{s} |h'_{t,x}(\theta)|d\theta}\, d\bar{s}\Big).$$

We replace (t,x) by $(\lambda t/s, \lambda x/s)$ with $s_0 \leqslant \lambda \leqslant s$, we see that $(\lambda t/s, \lambda x/s)$ is again contained in $\mathcal{K}_{[2,s_1]}$. Then (3.7) still holds, and so

$$|\underline{\partial}_\perp \partial^I v(\lambda t/s, \lambda x/s)| \leqslant C(s/t)\lambda^{-3/2}\Big(F(\lambda) + \int_{s_0}^{\lambda} F(\bar{s})|h'_{t,x}(\bar{s})| e^{\int_{\bar{s}}^{\lambda} |h'_{t,x}(\theta)|d\theta}\, d\bar{s}\Big),$$

which implies

$$(\lambda t/s)^{3/2}|\underline{\partial}_\perp \partial^I v(\lambda t/s, \lambda x/s)| \lesssim C_1\varepsilon(s/t)^{-1/2}\Big(F(\lambda) + \int_{s_0}^{\lambda} F(\bar{s})|h'_{t,x}(\bar{s})| e^{\int_{\bar{s}}^{\lambda} |h'_{t,x}(\theta)|d\theta}\, d\bar{s}\Big).$$

Recall that (7.15) holds for $1 > r/t > 3/5$ and that F is increasing, then

$$\int_{s_0}^{\lambda} F(\bar{s})|h'_{t,x}(\bar{s})| e^{\int_{\bar{s}}^{\lambda} |h'_{t,x}(\theta)|d\theta}\, d\bar{s} \leqslant F(\lambda)\int_{s_0}^{\lambda} |h'_{t,x}(\bar{s})| e^{\int_{\bar{s}}^{\lambda} |h'_{t,x}(\theta)|d\theta}\, d\bar{s} \lesssim C_1\varepsilon F(\lambda).$$

So we see that

$$(\lambda t/s)^{3/2}|\underline{\partial}_\perp \partial^I v(\lambda t/s, \lambda x/s)| \lesssim C_1\varepsilon(s/t)^{-1/2}F(\lambda)$$

and, in combination with (7.24),

$$(\lambda t/s)^{3/2}|\underline{\partial}_\perp \partial^I v(\lambda t/s, \lambda x/s)| \lesssim C_1\varepsilon(s/t)^{3/2-4\delta} + C_1\varepsilon(s/t)^{-1/2}\int_{s_0}^{\lambda} \bar{s}^{-3/2} V_{t,x}(\bar{s}) d\bar{s},$$

which implies (by taking sum over $N-6 \leqslant |I| \leqslant N-4$):

$$V_{t,x}(\lambda) \lesssim C_1\varepsilon(s/t)^{3/2-4\delta} + C_1\varepsilon(s/t)^{-1/2}\int_{s_0}^{\lambda} \bar{s}^{-3/2} V_{t,x}(\bar{s}) d\bar{s}. \tag{7.25}$$

Then, by Gronwall lemma, we see that

$$\begin{aligned}
\int_{s_0}^{\lambda} \bar{s}^{-3/2} V_{t,x}(\bar{s}) d\bar{s} &\lesssim C_1\varepsilon(s/t)^{3/2-4\delta}\int_{s_0}^{\lambda} \bar{s}^{-3/2} e^{CC_1\varepsilon(s/t)^{-1/2}\int_{s_0}^{\bar{s}} \theta^{-3/2}d\theta}\, d\bar{s} \\
&\lesssim C_1\varepsilon(s/t)^{3/2-4\delta}\int_{s_0}^{\lambda} \bar{s}^{-3/2} e^{CC_1\varepsilon(s/t)^{-1/2}s_0^{-1/2}}\, d\bar{s} \\
&\lesssim C_1\varepsilon(s/t)^{3/2-4\delta} s_0^{-1/2} e^{CC_1\varepsilon(s/t)^{-1/2}s_0^{-1/2}}
\end{aligned}$$

Here we recall that $s_0 = \sqrt{\frac{t+r}{t-r}} \geqslant t/s$, then

$$V_{t,x}(\lambda) \lesssim C_1\varepsilon(s/t)^{3/2-4\delta}. \tag{7.26}$$

Now we substitute (7.26) into (7.24), and obtain

$$F(\bar{s}) \lesssim C_1\varepsilon(s/t)^{2-4\delta}, \quad s_0 \leqslant \bar{s} \leqslant s. \tag{7.27}$$

Then we apply the sup-norm estimate (3.7) in the case $1 > r/t > 3/5$ and considering (7.15),

$$|\underline{\partial}_\perp \partial^I v(t,x)| \lesssim C_1\varepsilon(s/t)^{3-4\delta}s^{-3/2}, \qquad |\partial^I v(t,x)| \lesssim C_1\varepsilon(s/t)^{2-4\delta}s^{-3/2}. \tag{7.28}$$

Case II: $0 \leqslant r/t \leqslant 3/5$. In this case, $4/5 \leqslant s/t \leqslant 1$ and $s_0 = 2$, so the discussion is simpler. We just remark that as in the former case,

$$F(\bar{s}) \lesssim C_1\varepsilon(s/t)^{3/2}s_0^{-1/2+6\delta} + C_1\varepsilon(s/t)^{-3/2} \sum_{\substack{|I_2|\leqslant |I|-1,\beta \\ N-7\leqslant |I_2|\leqslant N-5}} \int_{s_0}^{\bar{s}} |\underline{\partial}_\perp \partial_\beta \partial^{I_2} v(\lambda t/s, \lambda x/s)| d\lambda$$

$$\lesssim C_1\varepsilon + C_1\varepsilon \int_2^{\bar{s}} \lambda^{-3/2} V_{t,x}(\lambda)\, d\lambda.$$

Then by the sup-norm estimate (3.7) (with $0 \leqslant r/t \leqslant 3/5$),

$$|\underline{\partial}_\perp \partial^I v(t,x)| \lesssim C_0\varepsilon t^{-3/2}\Big(1 + \int_2^s |h'_{t,x}(\bar{s}) e^{C\int_{\bar{s}}^s |h'_{t,x}(\theta) d\theta|}|\, d\bar{s}\Big)$$
$$+ t^{-3/2}\Big(F(s) + \int_2^s F(\bar{s}) |h'_{t,x}(\bar{s})| e^{C\int_{\bar{s}}^s \theta |h'_{t,x}(\theta)| d\theta}\, d\bar{s}\Big).$$

Then similar to the former case, we get (recall (7.15))

$$|(\lambda t/s)^{3/2} \underline{\partial}_\perp \partial^I v(t,x)| \lesssim (C_0 + C_1)\varepsilon + C_1\varepsilon \int_2^\lambda \bar{s}^{-3/2} V_{t,x}(\bar{s}) d\bar{s} \lesssim C_1\varepsilon + C_1\varepsilon \int_2^\lambda \bar{s}^{-3/2} V_{t,x}(\bar{s}) d\bar{s},$$

provided by $C_1 \geqslant C_0$, which implies

$$V_{t,x}(\lambda) \lesssim C_1\varepsilon + C_1\varepsilon \int_2^\lambda \bar{s}^{-3/2} V_{t,x}(\bar{s}) d\bar{s}$$

Then, Gronwall lemma implies $V_{t,x}(\lambda) \lesssim C_1\varepsilon$ and, therefore,

$$|\underline{\partial}_\perp \partial^I v(t,x)| \lesssim C_1\varepsilon t^{-3/2} \lesssim C_1 (s/t)^{3/2-4\delta} t^{-3/2}.$$

And again, as in the former case, we see that $|\partial^I v(t,x)| \lesssim C_1(s/t)^{1/2-4\delta} t^{-3/2}$.

When $|I| - 1 < N - 7$, we see that in this case

$$\big|[H^{\alpha\beta} u \partial_\alpha \partial_\beta, \partial^I] v\big| \lesssim (C_1\varepsilon)^2 (s/t)^{3/2} s^{-3+4\delta}.$$

A direct application of the sup-norm estimate (3.7) combined with (7.19) will give the estimate on $\partial^I v$ and $\partial_\alpha \partial^I v$. Finally, combining these two cases, we see that the desired estimates are established. □

7.5. **Third improvement on the wave and Klein-Gordon components.** We now establish (7.2), by combining the sup-norm estimate for the Klein-Gordon equation (3.7) and the sup-norm estimate for the wave equation (3.2), together with an additional bootstrap argument.

Proposition 7.10 (Third improvement on the wave and Klein-Gordon components)**.** *There exist constants $C, \varepsilon_2 > 0$ (depending only on $N \geqslant 8$ and the structure of the model system* (1.11)*) such that if the bootstrap assumption* (5.1) *holds for $\varepsilon \leqslant \varepsilon_2$ and $C_1\varepsilon \leqslant 1$, then the following estimates also hold for all $s \in [2, s_1]$ and $|I| + |J| \leqslant N - 4$, $|J| = k$:*

$$\sup_{\mathcal{H}_s} \big(t|L^J u|\big) \lesssim C_1\varepsilon s^{k\delta}, \tag{7.29a}$$

$$\sup_{\mathcal{H}_s} \big((s/t)^{-3+7\delta} s^{3/2} |\underline{\partial}_\perp \partial^I L^J v|\big) + \sup_{\mathcal{H}_s} \big((s/t)^{-2+7\delta} s^{3/2} |\partial^I L^J v|\big) \lesssim C_1\varepsilon s^{k\delta}, \tag{7.29b}$$

$$\sup_{\mathcal{H}_s} \big((s/t)^{-1+7\delta} s^{3/2} |\partial_\alpha \partial^I L^J v|\big) \lesssim C_1\varepsilon s^{k\delta}. \tag{7.29c}$$

Furthermore, we see that by the commutator estimates in Proposition 4.6, the following refined decay estimates are a direct consequence of (7.29c):

$$\sup_{\mathcal{H}_s} \big((s/t)^{1/2+7\delta} t^{3/2} |\partial^I L^J \partial_\alpha v|\big) \lesssim C_1\varepsilon s^{k\delta}, \qquad |I| + |J| \leqslant N - 4, |J| = k, \tag{7.30}$$

$$\sup_{\mathcal{H}_s} \big((s/t)^{1/2+7\delta} t^{3/2} |\partial^I L^J \partial_\alpha \partial_\beta v|\big) \lesssim C_1\varepsilon s^{k\delta}, \qquad |I| + |J| \leqslant N - 5, |J| = k, \tag{7.31}$$

$$\sup_{\mathcal{H}_s}\big((s/t)^{-3/2+7\delta}t^{3/2}|\partial^I L^J\partial_\alpha v|\big)\lesssim C_1\varepsilon s^{k\delta},\qquad |I|+|J|\leqslant N-5, |J|=k, \tag{7.32}$$

$$\sup_{\mathcal{H}_s}\big((s/t)^{-3/2+7\delta}t^{3/2}|\partial^I L^J\partial_\alpha\partial_\beta v|\big)\lesssim C_1\varepsilon s^{k\delta},\qquad |I|+|J|\leqslant N-6, |J|=k. \tag{7.33}$$

Proof. We proceed by induction on $|J|$ and introduce the notation

$$V_{k,0}(\lambda):=\sup_{\substack{2\leqslant s\leqslant\lambda,|J|\leqslant k\\ |I|+|J|\leqslant N-4}}\sup_{\mathcal{H}_s}\big((s/t)^{-2+7\delta}s^{3/2}|\partial^I L^J v|\big),$$
$$V_{k,1}(\lambda):=\sup_{\substack{2\leqslant s\leqslant\lambda,|J|\leqslant k\\ |I|+|J|\leqslant N-4}}\sup_{\mathcal{H}_s}\big((s/t)^{-3+7\delta}s^{3/2}|\underline{\partial}_\perp\partial^I L^J v|\big),$$

and, with $|J|\leqslant k$, $U_k(\lambda):=\sup_{\substack{2\leqslant s\leqslant\lambda\\ |J|\leqslant k}}\sup_{\mathcal{H}_s}\big(t|L^J u|\big)$. To begin with, we observe that by (7.23) and (7.17b), there exists a positive constant C determined by the structure of the system (1.11) such that

$$V_{0,0}(\lambda)\lesssim C_1\varepsilon,\qquad V_{0,1}(\lambda)\lesssim C_1\varepsilon,\qquad U_0(\lambda)\lesssim C_1\varepsilon,$$

That is, (7.29) is proved in the case where $k=0$.

Then we suppose that for all $0\leqslant j\leqslant k-1\leqslant N-5$, there exists a (sufficient large) constant C_{k-1} depending only on the structure of the system (1.11) and a positive constant ε'_{k-1} such that for all $\varepsilon\leqslant\varepsilon'_{k-1}$,

$$V_{j,0}(s)\leqslant C_{k-1}C_1\varepsilon s^{j\delta},\qquad V_{j,1}(s)\leqslant C_{k-1}C_1\varepsilon s^{j\delta},\qquad U_j\leqslant C_{k-1}C_1\varepsilon s^{j\delta} \tag{7.34}$$

hold on $[2,s_1]$ with C_{k-1} depending only on k and the structure of (1.11). Then we will prove that there exists a pair of positive constant (C_k,ε'_k) depending only on N and the structure of the model system (1.11) such that if (5.1) holds with $\varepsilon\leqslant\varepsilon'_k$ and $C_1\varepsilon\leqslant 1$, then

$$V_{k,0}(s)\leqslant C_kC_1\varepsilon s^{k\delta},\qquad V_{k,1}(s)\leqslant C_kC_1\varepsilon s^{k\delta},\qquad U_k\leqslant C_kC_1\varepsilon s^{k\delta}. \tag{7.35}$$

We rely on a bootstrap argument. First, we observe that on the initial hyperboloid $\mathcal{H}_2$, there exists a positive constant $C_0>0$ such that

$$\begin{aligned}\max_{\substack{|I|+|J|\leqslant N-4\\ |J|\leqslant k}}\sup_{\mathcal{H}_2}\big((2/t)^{-1/2+7\delta}t^{3/2}|\partial^I L^J v|\big)&\leqslant C_{0,k}C_1\varepsilon,\\ \max_{\substack{|I|+|J|\leqslant N-4\\ |J|\leqslant k}}\sup_{\mathcal{H}_2}\big((2/t)^{-3/2+7\delta}t^{3/2}|\underline{\partial}_\perp\partial^I L^J v|\big)&\leqslant C_{0,k}C_1\varepsilon,\\ \max_{|J|\leqslant k}\sup_{\mathcal{H}_2}\big(t|L^J u|\big)&\leqslant C_{0,k}C_1\varepsilon.\end{aligned} \tag{7.36}$$

We choose $C_k>C_{0,k}$ and set

$$\begin{aligned}s_{2,k}:=\sup\Big\{s\in[2,s_1]\ \ &V_{k,0}(s)\leqslant C_kC_1\varepsilon s^{k\delta},\\ &V_{k,1}(s)\leqslant C_kC_1\varepsilon s^{k\delta},\qquad U_k\leqslant C_kC_1\varepsilon s^{k\delta}\Big\}.\end{aligned}$$

By continuity, we have $s_{2,k}>2$. We will prove that for all sufficiently large constant $C_k\geqslant\max\{C_{0,k},C_{k-1},1\}$ the following bounds hold on $[2,s_{2,k}]$:

$$V_{k,0}(s)\leqslant\frac{1}{2}C_kC_1\varepsilon s^{k\delta},\qquad V_{k,1}(s)\leqslant\frac{1}{2}C_kC_1\varepsilon s^{k\delta},\qquad U_k\leqslant\frac{1}{2}C_kC_1\varepsilon s^{k\delta} \tag{7.37}$$

for sufficiently small ε. Once this is proven, we conclude that $s_{2,k}=s_1$. Namely, proceeding by contradiction, we see that in the opposite case at $s_{2,k}<s_1$, at least one of the following conditions must hold:

$$V_{k,0}(s)=C_kC_1\varepsilon s^{k\delta},\qquad V_{k,1}(s)=C_kC_1\varepsilon s^{k\delta},\qquad U_k=C_kC_1\varepsilon s^{k\delta},$$

which contradicts the improved estimates (7.37).

It remains to establish (7.37) and we derive first the following estimate for $|I|+|J|\leqslant N-4$, $|J|=j\leqslant k$ (again provided $2\leqslant s\leqslant s_{2,k}$)

$$\big|\partial^I L^J v\big|\lesssim C_kC_1\varepsilon(s/t)^{2-7\delta}s^{-3/2+j\delta}, \tag{7.38a}$$

$$\big|\partial_\alpha \partial^I L^J v\big| \lesssim C_k C_1 \varepsilon (s/t)^{1-7\delta} s^{-3/2+j\delta}, \tag{7.38b}$$

$$\big|L^J u\big| \lesssim C_k C_1 \varepsilon t^{-1} s^{j\delta}. \tag{7.38c}$$

The derivation of (7.38a) is direct from the decay assumption (7.34) and the induction assumption (7.34), while (7.38b) follows directly from (7.21), the decay assumption (7.35) or the induction assumption (7.34):

$$\begin{aligned}\big|\partial_\alpha \partial^I L^J v\big| \lesssim\, & (s/t)^{-2}\big|\underline{\partial}_\perp \partial^I L^J v\big| + (s/t)^{-2}\sum_a \big|\underline{\partial}_a \partial^I L^J v\big| \\ \lesssim\, & C_k C_1 \varepsilon (s/t)^{-2}(s/t)^{3-7\delta} s^{-3/2+j\delta} + C_1\varepsilon (s/t)^{-2} t^{-5/2} s^{1/2+(j+3)\delta} \\ \lesssim\, & C_k C_1 \varepsilon (s/t)^{1-7\delta} s^{-3/2+j\delta} + C_1\varepsilon (s/t)^{1/2} s^{-2+(j+3)\delta} \\ \lesssim\, & C_k C_1 \varepsilon (s/t)^{1-7\delta} s^{-3/2+j\delta},\end{aligned}$$

where the first equation in (6.16) was used. On the other hand, (7.38c) is also direct from (7.34) and (7.35).

Then we need the following two estimates for $|I| + |J| \leqslant N-4$, $|J| = k$:

$$\big|\partial^I L^J\big(\partial_\alpha v \partial_\beta v\big)\big| + \big|\partial^I L^J\big(v^2\big)\big| \lesssim (C_k C_1\varepsilon)^2 t^{-2-(1/2-7\delta+k\delta/2)}(t-r)^{-1+(1/2-7\delta+k\delta/2)}, \tag{7.39}$$

$$\big|[H^{\alpha\beta} u\partial_\alpha\partial_\beta, \partial^I L^J]v\big| \lesssim (C_k C_1\varepsilon)^2 (s/t)^{2-7\delta} s^{-5/2+k\delta}. \tag{7.40}$$

The estimate (7.39) follows directly from (7.38). We see that

$$\begin{aligned}\big|\partial^I L^J(v^2)\big| \leqslant & \sum_{\substack{I_1+I_2=I\\ J_1+J_2=J}} \big|\partial^{I_1}\partial^{J_1} v \partial^{I_2} L^{J_2} v\big| \\ \lesssim\, & C_k C_1\varepsilon (s/t)^{2-7\delta} s^{-3/2+|J_1|\delta} C_k C_1\varepsilon (s/t)^{2-7\delta} s^{-3/2+|J_2|\delta} \\ \simeq\, & (C_k C_1\varepsilon)^2 (s-r)^{-1+(1/2-7\delta+k\delta/2)} t^{-2-(1/2-7\delta-k\delta/2)}\end{aligned}$$

and

$$\begin{aligned}\big|\partial^I L^J\big(\partial_\alpha v\partial_\beta v\big)\big| \leqslant & \sum_{\substack{I_1+I_2=I\\ J_1+J_2=J}} \big|\partial^{I_1} L^{J_1}\partial_\alpha v \partial^{I_2} L^{J_2}\partial_\beta v\big| \lesssim (C_k C_1\varepsilon)^2 (s/t)^{2-14\delta} s^{-3+k\delta} \\ \simeq\, & (C_k C_1\varepsilon)^2 t^{-2-(1/2-7\delta-k\delta/2)}(t-r)^{-1+(1/2-7\delta+k\delta/2)}.\end{aligned}$$

The estimate of (7.40) is also direct by substituting (7.38). We recall (7.22) an write

$$\begin{aligned}&\big|[H^{\alpha\beta} u\partial_\alpha\partial_\beta, \partial^I L^J]v\big| \\ &\lesssim C_1\varepsilon (s/t)^{-2} t^{-1} C_k\varepsilon (s/t)^{3-7\delta} s^{-3/2+k\delta} + (C_k\varepsilon)^2 \sum_{|J_1|+|J_2|\leqslant |J|} t^{-1} s^{|J_1|\delta} (s/t)^{1-7\delta} s^{-3/2+|J_2|\delta} \\ &\quad + C_1\varepsilon t^{-3/2}(s/t)^{-3} C_k\varepsilon (s/t)^{3-7\delta} s^{-3/2+k\delta} + (C_1\varepsilon)^2 (s/t)^{3/2} s^{-3+(k+4)\delta} \\ &\lesssim (C_k C_1\varepsilon)^2 (s/t)^{2-7\delta} s^{-5/2+k\delta},\end{aligned}$$

where we have assumed that $C_k \geqslant C_1$.

Now we substitute (7.39) into (3.2) and find that (similar to the proof of Proposition 7.2)

$$\big|\partial^I L^J u\big| \lesssim C_{0,k} C_1\varepsilon t^{-3/2} + (C_k C_1\varepsilon)^2 (s/t)^{k\delta} t^{-1} s^{k\delta}, \tag{7.41}$$

which is

$$U_k(s) \lesssim C_{0,k} C_1\varepsilon + (C_k C_1\varepsilon)^2 s^{k\delta}. \tag{7.42}$$

On the other hand, the estimate on $|\partial^I L^J v|$ and $\big|\partial_\alpha \partial^I L^J v\big|$ is a bit more difficult. We see that

$$F(\bar s) \leqslant \int_{s_0}^{\bar s} \sum_{i=1}^3 R_i[\partial^I L^J v](\lambda t/s, \lambda x/s)\, d\lambda + \int_{s_0}^{\bar s} \lambda^{3/2} \big|[H^{\alpha\beta} u\partial_\alpha\partial_\beta, \partial^I L^J]v\big|(\lambda t/s, \lambda x/s)\, d\lambda.$$

By the sup-norm estimate (3.7), in the region $\mathcal{K} \cap \{3/5 < r/t < 1\}$, recall that $s_0 \geqslant Ct/s$, we can calculate each term in the right-hand side of the above inequality with that aid of (7.40) and (7.14) and find that

$$\begin{aligned}\big|F(s)\big| &\lesssim C_1\varepsilon(s/t)^{2-7\delta-k\delta} + (C_kC_1\varepsilon)^2(s/t)^{2-7\delta}s^{k\delta}\\ &\lesssim C_1\varepsilon(s/t)^{2-7\delta}s^{k\delta} + (C_kC_1\varepsilon)^2(s/t)^{2-7\delta}s^{k\delta} \simeq \big(C_1\varepsilon + (C_kC_1\varepsilon)^2\big)(s/t)^{2-7\delta}s^{k\delta}.\end{aligned}$$

Then, we apply the sup-norm estimate (3.7) with (7.15) and by the same procedure in the proof of Proposition (7.6), we conclude that when $3/5 < r/t < 1$,

$$|\underline{\partial}_\perp\partial^I L^J v(t,x)| \lesssim \big(C_1\varepsilon + (C_kC_1\varepsilon)^2\big)(s/t)^{3-7\delta}s^{-3/2+k\delta}, \tag{7.43a}$$

$$|\partial^I L^J v(t,x)| \lesssim \big(C_1\varepsilon + (C_kC_1\varepsilon)^2\big)(s/t)^{2-7\delta}s^{-3/2+k\delta}. \tag{7.43b}$$

When $0 \leqslant r/t \leqslant 3/5$, we see that $4/5 \leqslant s/t \leqslant 1$, then

$$F(s) \lesssim \big(C_1\varepsilon + (C_kC_1\varepsilon)^2\big)t^{-3/2}s^{k\delta}.$$

Then, also by the sup-norm estimate (3.7) and (7.15), we find that

$$|\underline{\partial}_\perp\partial^I L^J v| \lesssim (C_{0,k}+1)C_1\varepsilon(s/t)^{3-7\delta}s^{-3/2+k\delta} + (C_kC_1\varepsilon)^2(s/t)^{3-7\delta}s^{-3/2+k\delta}, \tag{7.44a}$$

$$|\partial^I L^J v| \lesssim (C_{0,k}+1)C_1\varepsilon(s/t)^{2-7\delta}s^{-3/2+k\delta} + (C_kC_1\varepsilon)^2(s/t)^{2-7\delta}s^{-3/2+k\delta}, \tag{7.44b}$$

where we recall that $C_1 > C_0$

Then we conclude that there exists a positive constant $\overline{C}$ determined only by the structure of the system (1.11) such that

$$V_{k,1}(s) \leqslant \overline{C}(C_{0,k}+1)C_1\varepsilon s^{k\delta} + \overline{C}(C_kC_1\varepsilon)^2 s^{k\delta}, \tag{7.45a}$$

$$V_{k,0}(s) \leqslant \overline{C}(C_{0,k}+1)C_1\varepsilon s^{k\delta} + \overline{C}(C_kC_1\varepsilon)^2 s^{k\delta}. \tag{7.45b}$$

Now we consider together (7.42) and (7.45) and see that if $C_k > 2C(C_{0,k}+1)$, then we can take $\varepsilon_k' := \frac{C_k - 2\overline{C}(C_{0,k}+1)}{2\overline{C}C_k^2C_1}$. Then we find that

$$V_{0,k}(s) \leqslant \frac{1}{2}C_kC_1\varepsilon s^{k\delta}, \qquad V_{1,k}(s) \leqslant \frac{1}{2}C_kC_1\varepsilon s^{k\delta}, \qquad U_k(s) \leqslant \frac{1}{2}C_kC_1\varepsilon s^{k\delta},$$

for all $\varepsilon \leqslant \varepsilon_k'$. This conclude that $s_{2,k} = s_2$ so the case $|J| = k$ is proven. Then by induction we see that for all $k \leqslant N-4$, (7.34) is established. Then taking $\varepsilon_2 := \min_{k\leqslant N-4}\{\varepsilon_k'\}$ and $C_\infty := \max_{k\leqslant N-4} C_k$, we see that (7.29a) and (7.29b) are established for all $k \leqslant N-4$ and, more precisely,

$$\sup_{\mathcal{H}_s}\big(t|L^J u|\big) \leqslant C_\infty C_1\varepsilon s^{k\delta}, \tag{7.46a}$$

$$\sup_{\mathcal{H}_s}\big((s/t)^{-3+7\delta}s^{3/2}|\underline{\partial}_\perp\partial^I L^J v|\big) + \sup_{\mathcal{H}_s}\big((s/t)^{-2+7\delta}s^{3/2}|\partial^I L^J v|\big) \leqslant C_\infty C_1\varepsilon s^{k\delta}. \tag{7.46b}$$

$$\sup_{\mathcal{H}_s}\big((s/t)^{-1+7\delta}s^{3/2}|\partial_\alpha\partial^I L^J v|\big) \leqslant C_\infty C_1\varepsilon s^{k\delta}. \tag{7.46c}$$

From its definition, we see that C_∞ is determined only from the structure of the system and therefore, we have proven (7.29). □

8. Refined energy estimate and completion of the bootstrap argument

8.1. **Overview.** In this section, we derive the improved energy estimates (5.2) which concludes the main result. The improved estimates are classified in two categories. The first refers to the energy estimates of order higher than or equal to $N-3$, the second refers to those of order lower that or equal to $N-4$.

First, we apply $\partial^I L^J$ (with $|I|+|J|\leqslant N$) to our system of equations

$$-\Box\partial^I L^J u = P^{\alpha\beta}\partial^I L^J\big(\partial_\alpha v\partial_\beta v\big) + R\partial^I L^J(v^2), \tag{8.1}$$

$$-\Box\partial^I L^J v + c^2\partial^I L^J v + H^{\alpha\beta}u\partial_\alpha\partial_\beta\partial^I L^J v = [H^{\alpha\beta}u\partial_\alpha\partial_\beta, \partial^I L^J]v. \tag{8.2}$$

To be able to apply the energy estimate (Proposition 2.1), we need first to check (2.12a) and (2.12b).

Lemma 8.1. *There exists a positive constant ε_0 such that if the energy assumption* (5.1) *is valid with $C_1\varepsilon\leqslant 1$ and $\varepsilon\leqslant\varepsilon_0$, then the following estimates hold:*

$$\frac{1}{2}E_{m,c}\leqslant E_{g,c}\leqslant 2E_{m,c}, \tag{8.3}$$

$$\begin{aligned}&\int_{\mathcal{H}_s}(s/t)|\partial_\alpha h^{\alpha\beta}\partial_t\partial^I L^J v\partial_\beta\partial^I L^J v|dx + \int_{\mathcal{H}_s}(s/t)|\partial_t h^{\alpha\beta}\partial_\alpha\partial^I L^J v\partial_\beta\partial^I L^J v|dx\\ &\lesssim M(s)E(s,\partial^I L^J v)^{1/2}\end{aligned} \tag{8.4}$$

with

$$M(s)\leqslant\begin{cases} C_1\varepsilon s^{-1/2+k\delta}, & N-3\leqslant|I|+|J|\leqslant N,\\ C_1\varepsilon s^{-1+k\delta}, & |I|+|J|\leqslant N-4.\end{cases}$$

Proof. The proof of (8.3) follows directly from (7.17b). We remark that

$$|h^{\alpha\beta}| = |H^{\alpha\beta}u|\lesssim C_1\varepsilon t^{-1}\lesssim C_1\varepsilon(s/t)^2$$

where we have observed that $t^{1/2}\leqslant s\leqslant t$ in $\mathcal{K}$. We get

$$\int_{\mathcal{H}_s}|h^{\alpha\beta}\partial_t\partial^I L^J v\partial_\beta\partial^I L^J v|dx\lesssim C_1\varepsilon\int_{\mathcal{H}_s}|(s/t)^2\partial_\alpha\partial^I L^J v\partial_\beta\partial^I L^J v|dx\lesssim C_1\varepsilon E_{g,c}(s,\partial^I L^J v),$$

$$\int_{\mathcal{H}_s}|h^{\alpha\beta}\partial_\alpha\partial^I L^J v\partial_\beta\partial^I L^J v|\lesssim C_1\varepsilon\int_{\mathcal{H}_s}|(s/t)^2\partial_\alpha\partial^I L^J v\partial_\beta\partial^I L^J v|dx\lesssim C_1\varepsilon E_{g,c}(s,\partial^I L^J v),$$

where we have used $\int_{\mathcal{H}_s}|(s/t)\partial_\alpha\partial^I L^J v|^2dx\leqslant E_{g,c}(s,\partial^I L^J v)$.

So for some $C'>0$ we have

$$\big|E_{g,c}(s,\partial^I L^J v)-E_{m,c}(s,\partial^I L^J v)\big|\leqslant C'C_1\varepsilon E_{g,c}(s,\partial^I L^J v),$$

and we choose $\varepsilon_0\leqslant\frac{1}{2C'C_1}$. Then, for $\varepsilon\leqslant\varepsilon_0$, it holds

$$\big|E_{g,c}(s,\partial^I L^J v)-E_{m,c}(s,\partial^I L^J v)\big|\lesssim C_1\varepsilon E_{g,c}(s,\partial^I L^J v)\leqslant\frac{1}{2}E_{g,c}(s,\partial^I L^J v),$$

which yields (8.3).

To derive (8.4), we just need to observe that

$$\begin{aligned}&\int_{\mathcal{H}_s}\big|\partial_\gamma h^{\alpha\beta}\partial_\alpha\partial^I L^J v\big|^2dx\\ &\lesssim C_1\varepsilon\int_{\mathcal{H}_s}t^{-1}s^{-2}(t/s)^2\big|(s/t)\partial_\alpha\partial^I L^J v\big|^2dx\simeq C_1\varepsilon\int_{\mathcal{H}_s}ts^{-4}\big|(s/t)\partial_\alpha\partial^I L^J v\big|^2dx\\ &\lesssim C_1\varepsilon s^{-2}E_{g,c}(s,\partial^I L^J v),\end{aligned}$$

and we use the first estimate in (6.13):

$$\int_{\mathcal{H}_s}\big|\partial_\gamma h^{\alpha\beta}\partial_\alpha\partial^I L^J v\big|^2dx\leqslant\begin{cases} C(C_1\varepsilon)^2 s^{-1+2k\delta}, & N-3\leqslant|I|+|J|\leqslant N,\\ C(C_1\varepsilon)^2 s^{-2+2k\delta}, & |I|+|J|\leqslant N-4.\end{cases}$$

So we see that

$$\int_{\mathcal{H}_s}(s/t)|\partial_\alpha h^{\alpha\beta}\partial_t\partial^I L^J v\partial_\beta\partial^I L^J v|dx\leqslant\|\partial_\alpha h^{\alpha\beta}\partial_t\partial^I L^J v\|_{L^2_f(\mathcal{H}_s)}\|(s/t)\partial_\beta\partial^I L^J v\|_{L^2_f(\mathcal{H}_s)},$$

which is bounded by the right-hand side of (8.4). The other term in the left-hand side is bounded in the same manner and we thus omit the details. □

8.2. **Lower-order L^2 estimates.** We remark that in lower order case where $|I|+|J|\leqslant N-4$, we have $M(s)\lesssim C_1\varepsilon s^{-1+k\delta}$ and we need again the estimate on the source term $\partial^I L^J\big(P^{\alpha\beta}\partial_\alpha v\partial_\beta v+Rv^2\big)$ and $[H^{\alpha\beta}u\partial_\alpha\partial_\beta,\partial^I L^J]v$.

Lemma 8.2. *Under the assumption of* (5.1)*, the following estimates hold for $|I|+|J|\leqslant N-4$ with $|J|=k$:*

$$\big\|\partial^I L^J\big(P^{\alpha\beta}\partial_\alpha v\partial_\beta v\big)\big\|_{L^2_f(\mathcal{H}_s)}+\big\|R\partial^I L^J v^2\big\|_{L^2_f(\mathcal{H}_s)}\lesssim(C_1\varepsilon)^2 s^{-3/2+k\delta},\tag{8.5}$$

$$\|[H^{\alpha\beta}u\partial_\alpha\partial_\beta,\partial^I L^J]v\|_{L^2_f(\mathcal{H}_s)}\lesssim(C_1\varepsilon)^2 s^{-1+k\delta}.\tag{8.6}$$

Proof. The estimates of these terms relies on the basic L^2 and refined sup-norm estimates. We remark that

$$\begin{aligned}\big\|\partial^I L^J\big(\partial_\alpha v\partial_\beta v\big)\big\|_{L^2_f(\mathcal{H}_s)}&\leqslant\sum_{\substack{I_1+I_2=I\\J_1+J_2=J}}\big\|\partial^{I_1}L^{J_1}\partial_\alpha v\partial^{I_2}L^{J_2}\partial_\beta v\big\|_{L^2_f(\mathcal{H}_s)}\\&\leqslant\sum_{\substack{1\leqslant|I_1|+J_1|\leqslant N-4\\|I_2|+|J_2|\leqslant N-5}}\big\|\partial^{I_1}L^{J_1}\partial_\alpha v\big\|_{L^\infty(\mathcal{H}_s)}\big\|\partial^{I_2}L^{J_2}\partial_\beta v\big\|_{L^2_f(\mathcal{H}_s)}\\&\quad+\big\|(t/s)\partial_\alpha v\big\|_{L^\infty(\mathcal{H}_s)}\big\|(s/t)\partial^I L^J\partial_\beta v\big\|_{L^2_f(\mathcal{H}_s)}=:T_1+T_2.\end{aligned}$$

For T_2, we apply (7.23b) with $|I|=1$ and (6.8) and we conclude that

$$T_2\lesssim C_1\varepsilon(s/t)^{-1/2-7\delta}t^{-3/2}C_1\varepsilon s^{|J|\delta}\varepsilon\lesssim(C_1\varepsilon)^2 s^{-3/2+k\delta}.$$

For T_1, we apply (7.30) and (6.9) and we conclude that

$$T_1\lesssim C_1\varepsilon(s/t)^{-1/2-7\delta}t^{-3/2}s^{|J_1|\delta}C_1\varepsilon s^{|J_2|\delta}\varepsilon\lesssim(C_1\varepsilon)^2 s^{-3/2+k\delta}.$$

The estimate on the term $\partial^I L^J\big(v^2\big)$ is similar by apply (7.29b) (6.8) and we omit the details.

To see the estimate on $[H^{\alpha\beta}u\partial_\alpha\partial_\beta,\partial^I L^J]v$ is quite similar, we just need to remark that it is a linear combination of the following terms:

$$L^{J_1'}u\partial^I L^{J_2'}\partial_\alpha\partial_\beta v,\quad\partial^{I_1}L^{J_1}u\partial^{I_2}L^{J_2}\partial_\alpha\partial_\beta v,\quad u\partial_\alpha\partial_\beta\partial^I L^{J_2''}v$$

where $I_1+I_2=I, J_1+J_2=J, J_1'+J_2'=J$ with $|J_1'|\geqslant 1$, $|I_1|\geqslant 1$ and $|J_2''|\leqslant|J|-1$. For the last term we apply (7.17b) and (6.3):

$$\begin{aligned}\big\|u\partial_\alpha\partial_\beta\partial^I L^{J_2''}v\big\|_{L^2_f(\mathcal{H}_s)}&\leqslant\big\|(t/s)u\big\|_{L^\infty(\mathcal{H}_s)}\big\|(s/t)\partial_\alpha\partial_\beta\partial^I L^{J_2''}v\big\|_{L^2_f(\mathcal{H}_s)}\\&\lesssim C_1\varepsilon s^{-1}C_1\varepsilon s^{k\delta}\lesssim(C_1\varepsilon)^2 s^{-1+k\delta}.\end{aligned}$$

For the first term, we see that $|J_1'|\leqslant N-4$, then we apply (7.29a) and (6.3):

$$\begin{aligned}\big\|L^{J_1'}u\partial^I L^{J_2'}v\big\|_{L^2_f(\mathcal{H}_s)}&\leqslant\big\|(t/s)L^{J_1'}u\big\|_{L^\infty(\mathcal{H}_s)}\big\|(s/t)\partial^I L^{J_2'}v\big\|_{L^2_f(\mathcal{H}_s)}\\&\lesssim C_1\varepsilon s^{-1+|J_1'|\delta}C_1\varepsilon s^{|J_2'|\delta}\lesssim(C_1\varepsilon)^2 s^{-1+k\delta}.\end{aligned}$$

For the second term, we see that when $|I_1|=1$ and $J_1=0$,

$$\begin{aligned}\|\partial_\gamma u\partial^{I_2}L^J\partial_\alpha\partial_\beta v\|_{L^2_f(\mathcal{H}_s)}\leqslant&\|(t/s)\partial_\gamma u\|_{L^\infty(\mathcal{H}_s)}\|(s/t)\partial^I L^J\partial_\alpha\partial_\beta v\|_{L^2_f(\mathcal{H}_s)}\\\lesssim&C_1\varepsilon\|(t/s)t^{-1/2}s^{-1}\|_{L^\infty(\mathcal{H}_s)}C_1\varepsilon s^{k\delta}\simeq(C_1\varepsilon)^2 s^{-1+k\delta}.\end{aligned}$$

When $|I_1|+|J_1|\geqslant 2$, we see that $|I_2|+|J_2|\leqslant N-6$. Then by the first inequality in (6.11) and (6.9), we find

$$\begin{aligned}\big\|\partial^{I_1}L^{J_1}u\partial^{I_2}L^{J_2}\partial_\alpha\partial_\beta v\big\|_{L^2_f(\mathcal{H}_s)}\leqslant&\big\|\partial^{I_1}L^{J_1}u\big\|_{L^\infty(\mathcal{H}_s)}\big\|\partial^{I_2}L^{J_2}\partial_\alpha\partial_\beta v\big\|_{L^2_f(\mathcal{H}_s)}\\ \lesssim& C_1\varepsilon s^{-3/2+(|J_1|+2)\delta}C_1\varepsilon s^{|J_2|\delta}\lesssim (C_1\varepsilon)^2 s^{-3/2+(k+2)\delta}\end{aligned}$$

and we conclude with (8.6). □

8.3. **Higher-order L^2 estimates.** When $N-3\leqslant |I|+|J|\leqslant N$, the energy estimate is more complicated. For the source terms we have the following estimates.

Lemma 8.3. *Under the energy assumption* (5.1) *the following estimates hold for $N-4\leqslant|I|+|J|\leqslant N$ and $|J|=k$:*

$$\big\|\partial^I L^J\big(P^{\alpha\beta}\partial_\alpha v\partial_\beta v\big)\big\|_{L^2_f(\mathcal{H}_s)}+\big\|\partial^I L^J\big(Rv^2\big)\big\|_{L^2_f(\mathcal{H}_s)}\lesssim (C_1\varepsilon)^2 s^{-1+k\delta},\tag{8.7}$$

$$\big\|[H^{\alpha\beta}u\partial_\alpha\partial_\beta,\partial^I L^J]v\big\|_{L^2_f(\mathcal{H}_s)}\lesssim (C_1\varepsilon)^2 s^{-1/2+k\delta}.\tag{8.8}$$

Proof. The proof relies on the refined decay estimate (7.29) and the basic L^2 estimates. We begin with (8.7). We remark that $\partial^I L^J\big(\partial_\alpha v\partial_\beta v\big)$ is a linear combination of the following terms

$$\partial^{I_1}L^{J_1}\partial_\alpha v\partial^{I_2}L^{J_2}\partial_\beta v$$

with $I_1+I_2=I$, $J_1+J_2=J$. We see that when $|I_1|+|J_1|=0$, we apply (7.23b) on $\partial_\alpha v$ (with $1\leqslant N-4$) and (6.5)

$$\begin{aligned}\big\|\partial^{I_1}L^{J_1}\partial_\alpha v\partial^{I_2}L^{J_2}\partial_\beta v\big\|_{L^2_f(\mathcal{H}_s)}=&\big\|(t/s)\partial_\alpha v(s/t)\partial^I L^J\partial_\beta v\big\|_{L^2_f(\mathcal{H}_s)}\\ \lesssim& C_1\varepsilon\|(t/s)(s/t)^{1/2-7\delta}t^{-3/2}\|_{L^\infty(\mathcal{H}_s)}CC_1\varepsilon s^{1/2+k\delta}\\ \lesssim&(C_1\varepsilon)^2 s^{-1+k\delta}.\end{aligned}$$

When $1\leqslant|I_1|+|J_1|\leqslant N-4$, we see that $4\leqslant|I_2|+|J_2|\leqslant N-1$. Then we apply (7.29c) and the third inequality in (6.6):

$$\begin{aligned}\big\|\partial^{I_1}L^{J_1}\partial_\alpha v\partial^{I_2}L^{J_2}\partial_\beta v\big\|_{L^2_f(\mathcal{H}_s)}\leqslant&\big\|\partial^{I_1}L^{J_1}\partial_\alpha v\big\|_{L^\infty(\mathcal{H}_s)}\big\|\partial^{I_2}L^{J_2}\partial_\beta v\big\|_{L^2_f(\mathcal{H}_s)}\\ \lesssim& C_k C_1\varepsilon\|(s/t)^{-1/2-7\delta}t^{-3/2+|J_1|\delta}\|_{L^\infty(\mathcal{H}_s)}C_1\varepsilon s^{1/2+|J_2|\delta}\\ \lesssim& C_k(C_1\varepsilon)^2 s^{-1+k\delta}.\end{aligned}$$

When $N-3\leqslant|I_1|+|J_1|\leqslant N-1$, we see that $1\leqslant|I_2|+|J_2|\leqslant 3\leqslant N-4$. Then we apply the third inequality in (6.6) and (7.29c). Similar to the former case,

$$\big\|\partial^{I_1}L^{J_1}\partial_\alpha v\partial^{I_2}L^{J_2}\partial_\beta v\big\|_{\mathcal{H}_s}\lesssim (C_1\varepsilon)^2 s^{-1+k\delta}.$$

When $|I_1|+|J_1|=N$ and $|I_2|+|J_2|=0$, the estimate is derived similarly as in the first case by exchanging the role of $\partial_\alpha v$ and $\partial_\beta v$. The we conclude that

$$\big\|\partial^{I_1}L^{J_1}\partial_\alpha v\partial^{I_2}L^{J_2}\partial_\beta v\big\|_{L^2_f(\mathcal{H}_s)}\lesssim (C_1\varepsilon)^2 s^{-1+k\delta}.$$

The estimate on $\partial^I L^J\big(v^2\big)$ is quite similar by applying (6.5) and (7.29b), we omit the detail.

The estimate on $[H^{\alpha\beta}u\partial_\alpha\partial_\beta,\partial^I L^J]v$ is as follows: we observe that this term is a linear combination of the following terms

$$L^{J_1'}u\partial^I L^{J_2'}\partial_\alpha\partial_\beta v,\quad \partial^{I_1}L^{J_1}u\partial^{I_2}L^{J_2}\partial_\alpha\partial_\beta v,\quad u\partial_\alpha\partial_\beta\partial^{I_2}L^{J_2''}v$$

where $I_1+I_2=I$, $J_1+J_2=J$, $J_1'+J_2'=J$ with $|J_1'|\geqslant 1$ $|I_1|\geqslant 1$ and $|J_2''|\leqslant|J|-1$. The last term is bounded by applying (7.17) and (6.5):

$$\begin{aligned}\big\|u\partial_\alpha\partial_\beta\partial^{I_2}L^{J_2''}v\big\|_{L^2_f(\mathcal{H}_s)}\leqslant&\|(t/s)u\|_{L^\infty(\mathcal{H}_s)}\big\|(s/t)\partial_\alpha\partial_\beta\partial^{I_2}L^{J_2''}v\big\|_{L^2_f(\mathcal{H}_s)}\\ \lesssim& C_1\varepsilon s^{-1}C_1\varepsilon s^{1/2+|J_2''|\delta}\lesssim(C_1\varepsilon)^2 s^{-1/2+k\delta}.\end{aligned}$$

For the first term, we make the following observation. When $1 \leqslant |J_1'| \leqslant N-4$, we have $4 \leqslant |I|+|J_2| \leqslant N-1$. Then we apply (7.29a) and (6.7):

$$\begin{aligned}\big\|L^{J_1'}u\partial^I L^{J_2'}\partial_\alpha\partial_\beta v\big\|_{L(\mathcal{H}_s)} &\leqslant \big\|(t/s)L^{J_1'}u\big\|_{L^\infty(\mathcal{H}_s)}\big\|(s/t)\partial^I L^{J_2'}\partial_\alpha\partial_\beta v\big\|_{L(\mathcal{H}_s)}\\ &\lesssim C_1\varepsilon s^{-1}s^{|J_1'|\delta}C_1\varepsilon s^{1/2+|J_2'|\delta} \lesssim (C_1\varepsilon)^2 s^{-1/2+k\delta}.\end{aligned}$$

When $N-3 \leqslant |J_1'| \leqslant N$, we see that $|I|+|J_2| \leqslant 3 \leqslant N-5$. Then we apply the Hardy inequality in the form (6.20a) as well as (7.31). So we see that

$$\begin{aligned}\big\|L^{J_1'}u\partial^I L^{J_2'}\partial_\alpha\partial_\beta v\big\|_{L^2_f(\mathcal{H}_s)} &\leqslant \big\|s^{-1}L^{J_1'}u\big\|_{L^2_f(\mathcal{H}_s)}\big\|s\partial^I L^{J_2'}\partial_\alpha\partial_\beta v\big\|_{L^\infty(\mathcal{H}_s)}\\ &\lesssim C_1\varepsilon s^{|J_1'|\delta}C_1\varepsilon s^{-1/2+|J_2'|\delta} \lesssim (C_1\varepsilon)^2 s^{-1/2+k\delta}.\end{aligned}$$

The second term is easier, since the factor $\partial^{I_1}L^{J_1}u$ has better decay when $|I_1| \geqslant 1$. Then we see that when $|I_1|=1$ and $|J_2|=0$,

$$\begin{aligned}\|\partial^{I_1}u\partial^{I_2}L^J\partial_\alpha\partial_\beta v\|_{L^2_f(\mathcal{H}_s)} &\leqslant \|(t/s)\partial^{I_1}u\,(s/t)\partial^{I_2}L^J\partial_\alpha\partial_\beta v\big\|_{L^2_f(\mathcal{H}_s)}\\ &\lesssim (C_1\varepsilon)\|t^{1/2}s^{-2}\|_{L^\infty(\mathcal{H}_s)}C_1\varepsilon s^{1/2+k\delta} \simeq (C_1\varepsilon)^2 s^{-1/2+k\delta}\end{aligned}$$

when $2 \leqslant |I_1|+|J_1| \leqslant N-2$, $|I_2|+|J_2| \leqslant N-2$. Then we apply the third inequality in (6.6) and we see that

$$\begin{aligned}\|\partial^{I_1}L^{J_1}u\partial^{I_2}L^{J_2}\partial_\alpha\partial_\beta v\|_{L^2_f(\mathcal{H}_s)} &\leqslant \|\partial^{I_1}L^{J_1}u\|_{L^\infty(\mathcal{H}_s)}\big\|\partial^{I_2}L^{J_2}\partial_\alpha\partial_\beta v\big\|_{L^2_f(\mathcal{H}_s)}\\ &\lesssim (C_1\varepsilon)^2 s^{-1+(k+2)\delta} \leqslant C(C_1\varepsilon)^2 s^{-1/2+k\delta}.\end{aligned}$$

When $N-1 \leqslant |I_1|+|J_1| \leqslant N$, $|I_2|+|J_2| \leqslant 1 \leqslant N-7$ then we apply (6.5) and (7.33). Then, we obtain

$$\begin{aligned}\|\partial^{I_1}L^{J_1}u\partial^{I_2}L^{J_2}\partial_\alpha\partial_\beta v\|_{L^2_f(\mathcal{H}_s)} &\leqslant \|(s/t)\partial^{I_1}L^{J_1}u\|_{L^2_f(\mathcal{H}_s)}\|(t/s)\partial^{I_2}L^{J_2}\partial_\alpha\partial_\beta v\|_{L^\infty(\mathcal{H}_s)}\\ &\lesssim C_1\varepsilon s^{|J_1|\delta}CC_1\varepsilon s^{-3/2+|J_2|\delta} \lesssim (C_1\varepsilon)^2 s^{-1/2+k\delta},\end{aligned}$$

which completes the argument. □

8.4. **Proof of Proposition 5.1.** Our aim is to establish the improved energy estimate (5.2) and to conclude the proof of Theorem 1.1, that is, we now establish Proposition 5.1. The strategy is to apply the energy estimate 2.1 with (8.3), (8.4), (8.5), (8.6), (8.7), and (8.8).

We need to specify the constants and we denote by $\overline{C}$ a sufficiently large constant determined only by the structure of the system such that all of the above estimates hold true. We derive the wave equation of (1.11) by $\partial^I L^J$:

$$-\Box\partial^I L^J u = \partial^I L^J\big(P^{\alpha\beta}\partial_\alpha v\partial_\beta v\big) + \partial^I L^J\big(v^2\big).$$

Recall the energy estimate (2.10)

$$E_m(s,\partial^I L^J u)^{1/2} \leqslant E_m(2,\partial^I L^J u)^{1/2} + \int_2^s \|\Box u\|_{L^2_f(\mathcal{H}_{\bar s})}\,d\bar s$$

with $\|\Box u\|_{L^2_f(\mathcal{H}_{\bar s})} \leqslant \big\|\partial^I L^J\big(P^{\alpha\beta}\partial_\alpha v\partial_\beta v\big)\big\|_{L^2_f(\mathcal{H}_s)} + \big\|\partial^I L^J\big(Rv^2\big)\big\|_{L^2_f(\mathcal{H}_s)}$. Then by (8.5), when $|I|+|J| \leqslant N-4$, we have $\|\Box u\|_{L^2_f(\mathcal{H}_{\bar s})} \leqslant \overline{C}(C_1\varepsilon)^2 s^{-3/2+k\delta}$, and we conclude that

$$E_m(s,\partial^I L^J u)^{1/2} \leqslant \overline{C}C_0\varepsilon + \overline{C}(C_1\varepsilon)^2. \tag{8.9}$$

When $N-3 \leqslant |I|+|J| \leqslant N$ and $|J|=k$, by (8.7)

$$\begin{aligned}E_m(s,\partial^I L^J u)^{1/2} &\leqslant \overline{C}C_0\varepsilon + \overline{C}(C_1\varepsilon)^2\int_2^s \bar s^{-1+k\delta}\,d\bar s\\ &\leqslant \overline{C}C_0\varepsilon + \overline{C}(C_1\varepsilon)^2 s^{k\delta}.\end{aligned} \tag{8.10}$$

For the energy estimates on v, we apply $\partial^I L^J$ to the Klein-Gordon equation in (1.11) and obtain

$$-\Box\partial^I L^J v + H^{\alpha\beta}u\partial_\alpha\partial_\beta\partial^I L^J v + c^2\partial^I L^J v = [H^{\alpha\beta}u\partial_\alpha\partial_\beta, \partial^I L^J]v.$$

Then by (2.13), and (8.3) (with $\kappa = 1/2$), we find

$$E_{m,c}(s, \partial^I L^J v)^{1/2} \leqslant \kappa^2 E_{m,c}(2, \partial^I L^J v)^{1/2} + \kappa^2 \int_2^s \|f\|_{L^2_f(\mathcal{H}_{\bar{s}})}\, d\bar{s} + \kappa^2 \int_2^s \|M(\bar{s})\|_{L^2_f(\mathcal{H}_s)}\, d\bar{s}.$$

When $|I| + |J| \leqslant N - 4$, we rely (8.6) and (8.4) and observe that

$$\begin{aligned} E_{m,c}(s, \partial^I L^J v)^{1/2} &\leqslant \overline{C} C_0 \varepsilon + \overline{C} (C_1 \varepsilon)^2 \int_2^s \bar{s}^{-1+k\delta}\, d\bar{s} \\ &\leqslant \overline{C} C_0 \varepsilon + \overline{C} (C_1 \varepsilon)^2 s^{k\delta}. \end{aligned} \tag{8.11}$$

When $N - 3 \leqslant |I| + |J| \leqslant N$, we apply (8.8) and (8.4) and observe that

$$\begin{aligned} E_{m,c}(s, \partial^I L^J v)^{1/2} &\leqslant \overline{C} C_0 \varepsilon + \overline{C} (C_1 \varepsilon)^2 \int_2^s \bar{s}^{-1/2+k\delta}\, d\bar{s} \\ &\leqslant \overline{C} C_0 \varepsilon + \overline{C} (C_1 \varepsilon)^2 s^{1/2+k\delta}. \end{aligned} \tag{8.12}$$

Finally, by choosing $C_1 \geqslant 4\overline{C} C_0$ and $\varepsilon \leqslant (4\overline{C} C_1)^{-1}$, (8.9)–(8.12) lead to (5.2).

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