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The field of Omicran-reals

A new approach to nonstandard analysis

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Abstract

From the works of Abraham Robinson, we know that the heuristic idea of infinite and infinitesimal numbers has obtained a formal rigor, he proved that the field of real numbers \mathbb{R} can be considered as a proper subset of a new field, *R , which is called field of hyperreal [1] numbers and contains the infinite and infinitesimal numbers. From the approach of Robinson we can represent every hyperreal by a sequence of $\mathbb{R}^{\mathbb{N}}$ modulo a maximal ideal M , this ideal is defined by using an ultrafilter \mathcal{U} . Unfortunately, the Ultrafilter \mathcal{U} and the order relation defined on *R are unknown, only the existence can be proved by the axiom of choice.

In this paper, we find a new extension of real numbers which contains the infinite and infinitesimal numbers as the field of hyperreals, the new set is called the field of Omicran-reals \mathcal{O} . Moreover, the new approach of construction is simple compared to other methods [1, 7, 8] and very precise, and the field \mathcal{O} is endowed with a well defined total order relation.

1 Introduction

In 1961 Abraham Robinson [1] showed how infinitely large and infinitesimal numbers can be rigorously defined and used to develop the field of non-standard analysis. To understand his theory nonconstructively, it's necessary to use the essential properties given by the model theory and mathematical logic.

After the birth of this new theory, more mathematicians have found the important applications of the nonstandard analysis in physic [2, 5, 6], numerical analysis and variational methods [3, 4].

In 1977 a new axiomatic representation of hyperreals is given by Edward Nelson [7], in the sense to simplify the Robinson's method, he proposed to add three axioms on the set theory and obtained a new theory called internal set theory [7].

From the construction of Robinson we can see every hyperreal as an element of $\mathbb{R}^{\mathbb{N}}$ modulo a maximal ideal M , the ideal M is defined with a non principal ultrafilter \mathcal{U} , the existence of \mathcal{U} is proved by the axiom of choice, by using the ultrafilter \mathcal{U} we define the order relation in the field of hyperreals. Unfortunately, we can't determine exactly this order relation because the ultrafilter is unknown.

Our aims in this article is to give a new field which contains the infinite and infinitesimal numbers, without using the properties of model theory, and without adding the new axioms to ZFC (Zermelo-Frankel+Axiom of choice). The new approach is very simple and we can determine precisely the order relation defined on the new field. The new field is noted \mathcal{O} and called field of Omicran-reals.

In this paper we restart in the section of preliminary results by some notions to define the infinitesimal and infinite numbers. In the section 3, we construct a new unitary ring by using a strict subset of $\mathbb{R}^{\mathbb{N}}$. After we define an ideal \mathcal{I} of the above ring which maximal, and we deduce that the quotient of this ring by \mathcal{I} is a new field. Finally, we construct the new field of **Omicran – reals** which is a totally ordered field and an extension of \mathbb{R} .

2 Preliminary results

Let $H(D(0, \varepsilon))$ the set of the holomorphic functions on the disk $D(0, \varepsilon)$.

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Theorem 1 If h is a holomorphic function on the disk $D(0, \varepsilon)$, and $h(0) = 0$ then : $h(z) = z^k g(z)$ on a neighborhood of 0 , where k is a non-zero integer, and $g \in H(D(0, \varepsilon))$ and $g(0) \neq 0$.

Theorem 2 The zeros of a nonconstant analytic function are isolated.

Definition 1 We define the following assertions :

- A totally ordered set (E, \preceq) is called a ordered \mathbb{R} -extension if

$$\begin{cases} \mathbb{R} \subseteq E & ; \\ x \preceq y \Leftrightarrow x \leq y & \forall (x, y) \in \mathbb{R}^2. \end{cases}$$

- In addition if $(E, +)$ is a commutative group, we consider $|\alpha| = \max(\alpha, -\alpha) = \begin{cases} \alpha, & \text{when } -\alpha \not\preceq \alpha, \\ -\alpha, & \text{when } \alpha \not\preceq -\alpha. \end{cases}$
- We note $x \prec y$ while : $x \preceq y$ and $x \neq y$.
- Let the set I_E be defined as follows $I_E = \{\alpha \in E / 0 \prec |\alpha| \prec \varepsilon \forall \varepsilon \in \mathbb{R}^{+*}\}$.
 I_E is a set of infinitesimals.

Remark 1 If it has not ambiguity we replace the symbol \preceq by \leq , and \prec by $<$.

To construct the new extension of \mathbb{R} which contains the infinitesimals, it's sufficient to prove the following theorem :

Theorem 3 There exist a extension field $(E, +, \cdot)$ of $(\mathbb{R}, +, \cdot)$, and partial order \leq such that :
 (E, \leq) is a ordered \mathbb{R} -extension and $I_E \neq \emptyset$.

Remark 2 An element δ of $I_E \neq \emptyset$ is called infinitesimal.

3 Construction of a new extension of the reals

3.1 The metallic map

Let $D(0, 1)$ (resp. $D'(0, 1)$) the open (resp. closed) disk of radius 1 and centre 0.

Definition 2 Let u be a map from $]0, 1[$ to \mathbb{R} , such that :

- There exists a map \tilde{u} is defined on $D'(0, 1)$, and holomorphic in a neighborhood of 0 .
- There exists $\varepsilon > 0$, such that $\forall x \in]0, \varepsilon[$ we have a : $\tilde{u}(x) = u(x)$.

The u is called a metallic map, and \tilde{u} is a metallic extension of u .

Example 1 If f is defined in an interval $]0, 1[$ as : $f(x) = \begin{cases} 2x + 1 & \text{when } x \in]\varepsilon, 1[\\ 1 - 3x^2 & \text{when } x \in]0, \varepsilon[\end{cases}$
then a metallic extension \tilde{f} is given by : $\tilde{f}(z) = 1 - 3z^2$ in the disk $D'(0, 1)$.

Remark 3 If u is metallic, then the two metallics extension \tilde{u} and \hat{u} of u are identic in a disk $D(0, \varepsilon)$ from the theorem 2.

Definition 3 We note $\Delta_1 = \{ u, u \text{ is a metallic map } \}$, and we have the following definitions :

- $\Delta_1(\frac{1}{n}) = \{ u(\frac{1}{n})_{n \geq 1}, u \text{ is a metallic map } \}$.
- $H_0 =$ the set of maps \tilde{u} defined on the disk $D'(0, 1)$ and holomorphic in a neighborhood of 0 .
- $(H_0, +)$ is a commutative group, let \mathcal{O}_0 be a subgroup of H_0 containing the maps defined on the disk $D'(0, 1)$ and are zero in a neighborhood of 0 .
- Let θ_0 be a map defined as :

$$\theta_0 : \begin{array}{ccc} \Delta_1(\frac{1}{n}) & \longrightarrow & H_0/\mathcal{O}_0 \\ u(\frac{1}{n})_{n \geq 1} & \longrightarrow & C(\tilde{u}) \end{array}$$

which $C(\tilde{u})$ is the equivalence classe of \tilde{u} modulo \mathcal{O}_0 . The map θ_0 is a well-defined from the unicity of $C(\tilde{u})$.

- We consider the surjective map θ_1 defined as :

$$\theta_1 : \begin{array}{l} \Delta_1(\frac{1}{n}) \longrightarrow \theta_0(\Delta_1(\frac{1}{n})) \\ u(\frac{1}{n})_{n \geq 1} \longrightarrow C(\tilde{u}) \end{array}$$

and the set : $\overline{\Delta_1(\frac{1}{n})} = \{ \theta_1^{-1}(C(\tilde{u})) , C(\tilde{u}) \in \theta_0(\Delta_1(\frac{1}{n})) \}$.

- We define on the set $\Delta_1(\frac{1}{n})$ the following equivalence relation \sim :

$$u(\frac{1}{n})_{n \geq 1} \sim v(\frac{1}{n})_{n \geq 1} \iff \exists n_0, \forall n \geq n_0, u(\frac{1}{n}) = v(\frac{1}{n}).$$

- $\overline{u(\frac{1}{n})_{n \geq 1}}$ is the equivalence classe of $u(\frac{1}{n})_{n \geq 1}$ modulo \sim .

Remark 4 1. We can check the equality: $\overline{u(\frac{1}{n})_{n \geq 1}} = \theta_1^{-1}(C(\tilde{u}))$.

Then :

$$\overline{\Delta_1(\frac{1}{n})} = \left\{ \overline{u(\frac{1}{n})_{n \geq 1}}, u \in \Delta_1 \right\}.$$

2. The sets Δ_1 and $\overline{\Delta_1(\frac{1}{n})}$ are commutative groups.

3. The map defined as :

$$\overline{\theta_1} : \begin{array}{l} \overline{\Delta_1(\frac{1}{n})} \longrightarrow E_1 = \theta_0(\Delta_1(\frac{1}{n})) \\ \overline{u(\frac{1}{n})_{n \geq 1}} \longrightarrow C(\tilde{u}) \end{array}$$

is an isomorphism between two groups.

Definition 4 • $\mathcal{A}_2 = \left\{ \frac{1}{u}, u \in \Delta_1 \forall x \in]0, 1[\ u(x) \neq 0 \text{ and } \lim u(\frac{1}{n}) = 0 \right\}$.

- $\Delta_2 = \{ v / v :]0, 1[\longrightarrow \mathbb{R}, \text{ there exists } 1/u \in \mathcal{A}_2 \text{ and } \varepsilon > 0 \text{ such that } v_{/]0, \varepsilon[} = (\frac{1}{u})_{/]0, \varepsilon[} \}$.

- $\Delta_2(\frac{1}{n}) = \left\{ (v(\frac{1}{n}))_{n \geq 1}, v \in \Delta_2 \right\}$.

3.2 Construction of a unitary ring

Lemma 1 Let $\Delta = \Delta_1 \cup \Delta_2$, $(\Delta, +, \cdot)$ is a unitary ring.

Proof.

- The stability of the sum : The set Δ is non-empty set, because $\Delta_1 \neq \emptyset$ and $\mathbb{R} \subseteq \Delta_1$ (we identify the constant functions by the real numbers). Show that : $\forall g \in \Delta$, and $h \in \Delta$ then $g + h \in \Delta$.

First case : if $(g, h) = (u, v) \in \Delta_1^2$, we verify that the function $s = f + g$ is matalic map, and : $\tilde{s} = \tilde{u} + \tilde{v}$.

Second case : if $(g, h) \in \Delta_2^2$, there exists $(u, v) \in \Delta_1^2$ such that $\lim u(\frac{1}{n}) = \lim v(\frac{1}{n}) = 0$ and $u(x)v(x) \neq 0$ for every $x \in]0, \varepsilon[$, for $\varepsilon > 0$ small enough, in addition : $g_{/]0, \varepsilon[} = \frac{1}{u}_{/]0, \varepsilon[}$ and $h_{/]0, \varepsilon[} = \frac{1}{v}_{/]0, \varepsilon[}$.

Since \tilde{u} and \tilde{v} are holomorphic functions on a neighborhood of 0, there exists $(m, n, l) \in \mathbb{N}^3$ such that :

$$\tilde{u}(z) = z^n b_1(z), \tilde{v}(z) = z^m b_2(z) \text{ and } \tilde{u}(z) + \tilde{v}(z) = z^l b_3(z).$$

where b_1, b_2, b_3 are 3 holomorphic functions on a neighborhood of 0, such that : $b_1(0)b_2(0)b_3(0) \neq 0$.

Let :

$$\psi(x) = \frac{u(x)v(x)}{u(x) + v(x)}.$$

The map $g + h$ is defined in the interval $]0, 1[$, for $\varepsilon > 0$ small enough, and $b_1(z)b_2(z)b_3(z) \neq 0$ in the disk $D(0, \varepsilon)$, we have : $g(x) + h(x) = \frac{1}{u(x)} + \frac{1}{v(x)} = \frac{1}{\psi(x)} = x^{l-m-n} \frac{b_3(x)}{b_1(x)b_2(x)}$, for every $x \in]0, \varepsilon[$.

✓ If $l - m - n \geq 0$, The map defined as $\tilde{\phi}(z) = \begin{cases} z^{l-m-n} \frac{b_3(z)}{b_1(z)b_2(z)} & \text{in } D(0, \varepsilon) \\ 1 & \text{if not} \end{cases}$ is a metallic extension of $g + h$, then $g + h$ is a element of Δ_1 .

✓ If $m + n - l < 0$, the map defined as : $\tilde{\psi}(z) = \begin{cases} z^{m+n-l} \frac{b_1(z)b_2(z)}{b_3(z)} & \text{in } D(0, \varepsilon) \\ 1 & \text{if not} \end{cases}$ is a metallic extension of ψ , in addition $\lim \psi(\frac{1}{n}) = 0$, we deduce that the is $g + h$ is a element of Δ_2 .

Third case : If $(g, h) \in \Delta_1 \times \Delta_2$, there exists $(u, v) \in \Delta_1^2$, such that : $g = u$, $h_{/]0, \varepsilon]} = \frac{1}{v}_{/]0, \varepsilon]}$, $\lim v(\frac{1}{n}) = 0$.

Let $k(z) = \frac{\tilde{v}(z)}{\tilde{u}(z)\tilde{v}(z) + 1}$.

Since $\tilde{v}(0) = 0$, we have $k(0) = 0$, and k is a holomorphic function in a disk $D(0, \varepsilon)$, for $\varepsilon > 0$ small enough. The map k is nonzero and holomorphic at $z = 0$, we can choose the ε so that $k(x) \neq 0$ for every $x \in]0, \varepsilon]$. Let ϕ the function defined in $]0, 1]$ as :

$$\phi(x) = \begin{cases} k(x) & \text{if } x \in]0, \varepsilon] \\ 1 & \text{if not} \end{cases} .$$

We verify that : $g + h_{/]0, \varepsilon]} = (\frac{1}{\phi})_{/]0, \varepsilon]}$, $\phi \in \Delta_1$ then $g + h \in \Delta_2 \subset \Delta$.

Finally, we deduce that $(\Delta, +)$ is a commutative group.

• Now, we can show the stability of the law $(.)$ in Δ , for that, we distiguish tree cases :

1. We can easily verify that the product of two metallic functions is a metallic function.
(if $g \in \Delta_1$ and $h \in \Delta_1$ then $gh \in \Delta_1 \subseteq \Delta$)

2. In this case, we assume that $g \in \Delta_1$ and $h \in \Delta_2$, we can show that $gh \in \Delta$, in fact, there exists $(u, v) \in \Delta_1^2$, such that : $g = u$, $h_{/]0, \varepsilon]} = \frac{1}{v}_{/]0, \varepsilon]}$, $\lim v(\frac{1}{n}) = 0$.

- if $\lim u(\frac{1}{n}) \neq 0$, then $\frac{\tilde{v}}{u}$ is holomorphic in the disk $D(0, \varepsilon)$, which implies that : $\frac{u}{v} \in \Delta_2 \subseteq \Delta$.

- if $\lim u(\frac{1}{n}) = 0$, then $\lim \tilde{u}(\frac{1}{n}) = 0$ and $\tilde{u}(0) = 0$, we deduce that: $\tilde{u}(z) = z^k b_1(z)$ in $D(0, \varepsilon)$, and $\tilde{v}(z) = z^{k'} b_2(z)$, where $b_i(z) \in H(D(0, \varepsilon))$, for $i \in \{0, 1\}$ and $b_i(0) \neq 0$.

We get:

$$\frac{\tilde{u}(z)}{\tilde{v}(z)} = z^{k-k'} \frac{b_1(z)}{b_2(z)}$$

-First case : if $k = k'$ then the function $\frac{\tilde{u}}{\tilde{v}}$ is holomorphic in $D(0, \varepsilon)$, which implies that $\frac{u}{v} \in \Delta_1$.

-Second case : if $k > k'$ then $\lim_{z \rightarrow 0} \frac{\tilde{u}}{\tilde{v}}(z) = 0$, and $\frac{\tilde{u}}{\tilde{v}}$ is a holomorphic function in the disk $D(0, \varepsilon)$.

Then: $\frac{u}{v} \in \Delta_1$.

-Third case : if $k < k'$ then $\lim_{z \rightarrow 0} \frac{\tilde{u}}{\tilde{v}}(z) = 0$, which implies that $\frac{u}{v} \in \Delta_2$.

3. In the case of $g \in \Delta_2$ and $h \in \Delta_2$, we verify easily the stability of the law $(.)$.

Finally, we deduce that $(\Delta, +, .)$ is a commutative and unitary ring, where the constant function $\mathbf{1}_\Delta$ is a multiplicative identity of Δ .

3.3 Construction of the new field

Let \mathcal{I}_0 the set defined as :

$$\mathcal{I}_0 = \{ u_{/]0, 1]} \quad / \quad u \in \mathcal{O}_0 \quad \text{and} \quad u(]0, 1]) \subset \mathbb{R} \quad \}$$

it is a set of maps defined in $]0, 1]$, and zero on $]0, \varepsilon]$ (for $0 < \varepsilon \leq 1$).

Thus, we have demonstrated the following proposition.

Proposition 1 \mathcal{I}_0 is a maximal ideal of Δ .

Theorem 4 The ring $(\Delta/\mathcal{I}_0, +, .)$ is a field.

Proof.

From the above proposition, the ideal \mathcal{I}_0 is maximal, so we deduce that the ring $(\Delta/\mathcal{I}_0, +, \cdot)$ is a field.

4 The field of Omicran-reals

We consider the set defined as $\Delta(\frac{1}{n}) = \{ h(\frac{1}{n})_{n \geq 1}, h \in \Delta \}$.

Let \sim the equivalence relation defined on the set $\Delta(\frac{1}{n})$ as :

$$g(\frac{1}{n})_{n \geq 1} \sim h(\frac{1}{n})_{n \geq 1} \Leftrightarrow \exists n_0 / \forall n \geq n_0, h(\frac{1}{n}) = g(\frac{1}{n}).$$

The equivalence class is given by : $\overline{g(\frac{1}{n})_{n \geq 1}} = \{ h(\frac{1}{n})_{n \geq 1}, \text{ for } h \in \Delta \text{ and } \exists n_0 \in \mathbb{N} / \forall n \geq n_0, h(\frac{1}{n}) = g(\frac{1}{n}) \}$.

The map

$$\theta : \frac{(\overline{\Delta(\frac{1}{n})}, +, \cdot)}{g(\frac{1}{n})_{n \geq 1}} \longrightarrow (\Delta/\mathcal{I}_0, +, \cdot)$$

$$g(\frac{1}{n})_{n \geq 1} \longrightarrow C(g) = \overline{g}$$

is well defined, in addition we have :

1. $(\overline{\Delta(\frac{1}{n})}, +, \cdot)$ is a field.
2. θ is an isomorphism.

Theorem 5 *There exists a set \mathcal{O} and a total order \leq such that :*

- $(\mathcal{O}, +, \cdot)$ is an extension field of $(\mathbb{R}, +, \cdot)$.
- (\mathcal{O}, \leq) is a ordered \mathbb{R} -extension.
- $I_{\mathcal{O}} \neq \emptyset$.

Proof.

The set Δ/\mathcal{I}_0 is noted Δ_0 , we replace $C(g) = \overline{g}$ by g (if it has not ambiguity).

From what precedes, we deduce that the map :

$$\theta : \frac{(\overline{\Delta(\frac{1}{n})}, +, \cdot)}{g(\frac{1}{n})_{n \geq 1}} \longrightarrow (\Delta_0, +, \cdot)$$

$$g(\frac{1}{n})_{n \geq 1} \longrightarrow g$$

is a ring isomorphism.

Let δ an indeterminate, and $\mathcal{O} = \Delta_0(\delta) = \{ g(\delta), g \in \Delta_0 \}$.

The map defined as :

$$\delta^* : \Delta_0 \longrightarrow E$$

$$g \longrightarrow g(\delta)$$

is a ring isomorphism.

Then $(\mathcal{O}, +, \cdot)$ is a field, in addition, the map $\varphi = \delta^* \circ \theta$ defined as :

$$\varphi : \frac{(\overline{\Delta(\frac{1}{n})}, +, \cdot)}{(g(\frac{1}{n}))_{n \geq 1}} \longrightarrow (\mathcal{O}, +, \cdot)$$

$$g(\frac{1}{n})_{n \geq 1} \longrightarrow g(\delta)$$

is a isomorphism.

- for $g = cste = l$, we note $\overline{l} = \overline{(g(\frac{1}{n}))_{n \geq 1}}$, we identify l by the image of \overline{l} by φ , and we find $\varphi(\overline{l}) = l$. by using the identification, we deduce that : $\mathbb{R} \subseteq \mathcal{O}$.
- We can define on \mathcal{O} the following relation \leq :
 $g(\delta) \leq h(\delta)$ if and only if there exists an integer n_0 , such that : $g(\frac{1}{n}) \leq h(\frac{1}{n})$ for every $n \geq n_0$.
It is easy to check that \leq is reflexive, transitive and antisymmetric, then it is a partial order.
- To show that set (\mathcal{O}, \leq) is ordered \mathbb{R} -extension, we must show that relation \leq is total.
For that we consider $g, h \in \Delta_0$, we assume that these propositions are true (*not* $g(\delta) \leq h(\delta)$) and (*not* $h(\delta) \leq g(\delta)$).
Next, we need to find a contradiction.
If the above propositions are true, then : $\forall k \in \mathbb{N}, \exists n_k > k$ and $\exists n'_k > k / g(\frac{1}{n_k}) > h(\frac{1}{n_k})$ and $g(\frac{1}{n'_k}) < h(\frac{1}{n'_k})$.

1. We assume that $g, h \in \Delta_1$. From the Intermediate value theorem we deduce that there exists $\beta_k \in]\frac{1}{n_k}, \frac{1}{n_k} |$ such that $(g-h)(\beta_k) = 0$, (we can choose the β_k so that the sequence (β_k) is strictly decreasing), then the holomorphic function $\tilde{g} - \tilde{h}$ has an infinite number of roots in neighborhood of 0, from the theorem2, we deduce that the function $\tilde{g} - \tilde{h}$ is the zero function, then $g = h$, which is absurd.
2. Now, we suppose that $g, h \in \Delta_2$. There exists $(u, v) \in \Delta_1^2$, such that :
 $u_{/]0, \varepsilon]} = (\frac{1}{g})_{/]0, \varepsilon]}$ and $v_{/]0, \varepsilon]} = (\frac{1}{h})_{/]0, \varepsilon]}$.
from the above results, we deduce that : $u(\delta) \leq v(\delta)$ ou $v(\delta) \leq u(\delta)$.
which implies that: $g(\delta)$ and $h(\delta)$ are comparable.
3. In the case of $g \in \Delta_1$ and $h \in \Delta_2$. There exists $h_1 \in \Delta_1$ and $\varepsilon > 0$, such that: $h_{/]0, \varepsilon]} = (\frac{1}{h_1})_{/]0, \varepsilon]}$.
Since h_1 is a metallic function, then the sequence $(h_1(\frac{1}{n}))$ keeps a constant sign from a certain rank, in fact: if it's not the case, then $\forall k \in \mathbb{N}$ there exists $n_k > k$ et $n'_k > k$ such that: $h_1(\frac{1}{n_k}) > 0$ and $h_1(\frac{1}{n'_k}) < 0$.
From the Intermediate value theorem we deduce that there exists $\beta_k \in]\frac{1}{n_k}, \frac{1}{n_k} |$ and $(h_1)(\beta_k) = 0$, (we can choose the β_k such that the sequence (β_k) is strictly decreasing), from the theorem2, h_1 is the zero function on the neighborhood of 0, which absurd(because we suppose $(h_1(\frac{1}{n}))$ does not keep a constant sign from a certain rank)
Finally, we deduce that the sequence $(h_1(\frac{1}{n}))$ keeps a constant sign, since $\lim h_1(\frac{1}{n}) = 0$, then $\lim \frac{1}{h_1(\frac{1}{n})}$ exists, and $\lim \frac{1}{h_1(\frac{1}{n})} = \pm\infty$, which implies $\lim h(\frac{1}{n}) = \pm\infty$.
- if $\lim h(\frac{1}{n}) = +\infty$, then $g(\delta) \leq h(\delta)$ (because $g(\frac{1}{n}) \leq h(\frac{1}{n})$, from a certain rank).
- if $\lim h(\frac{1}{n}) = -\infty$, then $h(\delta) \leq g(\delta)$ (because $h(\frac{1}{n}) \leq g(\frac{1}{n})$, from a certain rank).

- Now, it remains to show that: $I_{\mathcal{O}} \neq \emptyset$.

For that, it is necessary to find a element $\delta \in \mathcal{O}$ which infinitesimal.

for $u : x \rightarrow x$, we have $u \in \Delta$ (more precisely Δ_1) and $\delta = u(\delta) = \overline{u(\frac{1}{n})}_{n \geq 1}$.

In addition, we have : $0 < \delta < \varepsilon$ for every real strictly positive ε , because there exists $p \in \mathbb{N}$ such that: $0 < u(\frac{1}{n}) < \varepsilon$, for every integer $n > p$. Then δ is infinitesimal.

Conclusions 1 Finally, we deduce that:

- $(\mathcal{O}, +, \cdot)$ is a extension field of $(\mathbb{R}, +, \cdot)$.
- (\mathcal{O}, \leq) is a ordered \mathbb{R} -extension, which contains the infinitesimal element δ .

The field $(\mathcal{O}, +, \cdot)$ is called **a field of Omicran-reals**.

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