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CHARACTERIZATIONS OF FREENESS FOR COHEN-MACAULAY SPACES

DELPHINE POL

Abstract. The purpose of this paper is to develop an analogue of freeness for reduced Cohen-Macaulay spaces embedded in a smooth manifold S which generalizes the notion of Saito free divisors. The main result of this paper is a characterization of freeness in terms of the projective dimension of the module of multi-logarithmic k-forms, where k is the codimension of X in S. We then focus on quasi-homogeneous complete intersection curves, for which we compute explicitly a minimal free resolution of the module of multi-residues and the module of multi-logarithmic forms.

1. Introduction

Let D be the germ of a reduced hypersurface in a smooth analytic variety S. A meromorphic form with simple poles along D is called logarithmic if its differential has also simple poles along D. When the module of logarithmic 1-forms is free, the hypersurface D is called a *free divisor*. For example, normal crossing divisors, plane curves and discriminants of isolated hypersurface singularities are families of free divisors (see [Sai80]).

Several characterizations of freeness are known. Let us mention some of them. Thanks to the duality between the modules of logarithmic vector fields and logarithmic differential 1-forms proved in [Sai80], a hypersurface is free if and only if the module of logarithmic vector fields is free. Another characterization of freeness is the following: a hypersurface is free if and only if the Jacobian ideal is Cohen-Macaulay (see [Ale88]).

A theory of multi-logarithmic differential forms and multi-residues for a reduced complete intersection is developed in [AT01], [AT08], [Ale12] and studied in [GS12]. A generalization of this theory to Cohen-Macaulay spaces is developed in [Ale14], which is inspired by the interpretation of regular meromorphic forms with residue symbols developed in [Ker84], which is also used in [Sch16].

Let X be a reduced Cohen-Macaulay space of codimension k contained in a smooth complex manifold S, and $\mathcal{I}_X$ the defining ideal of X. There exists a reduced complete intersection C defined by a regular sequence $\underline{f} = (f_1, \dots, f_k)$ such that $X \subseteq C$. We set $\mathcal{I}_C = \sum_{i=1}^k f_i \mathcal{O}_S$ and $f = f_1 \cdots f_k$. The module of multi-logarithmic differential q-forms with respect to the pair (X, C) is defined by:

$$\Omega^q(\log X/C, \underline{f}) = \left\{ \omega \in \frac{1}{f}\Omega^q_S \ ; \ \mathcal{I}_X \cdot \omega \subseteq \frac{1}{f}\mathcal{I}_C \Omega^q_S \text{ and } \mathrm{d}(\mathcal{I}_X) \wedge \omega \subseteq \frac{1}{f}\mathcal{I}_C \Omega^{q+1}_S \right\}. \tag{1}$$

Following the definition of freeness for complete intersection suggested in [GS12], we say that X is *free* if the restriction to X of the Jacobian ideal of C is Cohen-Macaulay. We prove in proposition 4.4 that this notion of freeness is independent from the choice of the complete intersection C.

The natural problem which then arises is to characterize the freeness of a Cohen-Macaulay space thanks to the module of multi-logarithmic forms. It leads to the main theorem of this paper (see theorem 4.6):

Theorem. *A reduced Cohen-Macaulay space X contained in a reduced complete intersection C of the same dimension is free if and only if the projective dimension of $\Omega^k(\log X/C, \underline{h})$ is lower than or equal to $k-1$, which is equivalent to the fact that the projective dimension of $\Omega^k(\log X/C)$ is equal to $k-1$.*

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For a free reduced Cohen-Macaulay space of codimension at least 2, the module $\Omega^k(\log X/C)$ is no more free, however, it is of minimal projective dimension.

We then focus on a particular family of free singularities, namely the curves. The computation of multi-logarithmic forms is in general difficult. In this paper, we determine explicitly a free resolution of the module of multi-logarithmic forms for quasi-homogeneous complete intersection curves.

Let us describe more precisely the content of this paper.

We first consider in subsection 2.1 the complete intersection case, which is less technical and is used for the Cohen-Macaulay case.

Let C be a reduced complete intersection defined by a regular sequence $(h_1, \dots, h_k)$. We set $h = h_1 \cdots h_k$, and $\Omega^q(\log C, \underline{h}) = \Omega^q(\log C/C, \underline{h})$.

In subsection 2.1.1 we first recall several definitions and properties of multi-logarithmic forms and multi-residues from [AT01], [AT08] and [Ale12]. We set $\mathcal{R}^{q-k}_{C,\underline{h}}$ for the module of multi-residues of multi-logarithmic q-forms, and $\mathcal{R}_{C,\underline{h}} = \mathcal{R}^0_{C,\underline{h}}$ (see definition 2.7).

In subsections 2.1.1 and 2.1.2 we make explicit the dependence of the modules $\Omega^q(\log C, \underline{h})$ and $\mathcal{R}^{q-k}_{C,\underline{h}}$ on the choice of the equations. We prove that the modules of numerators $h_1 \cdots h_k \cdot \Omega^q(\log C, \underline{h})$ do not depend on the choice of the equations (see proposition 2.3). In addition, the modules $\mathcal{R}^{q-k}_{C,\underline{h}}$ depends only on C (see proposition 2.9). We then fix a regular sequence $(h_1, \dots, h_k)$ and we denote $\Omega^q(\log C) = \Omega^q(\log C, \underline{h})$ and $\mathcal{R}^q_C = \mathcal{R}^q_{C,\underline{h}}$.

Let $\widetilde{C}$ be the normalization of C. A consequence of [Sch16, Proposition 2.1] is that $\mathcal{O}_{\widetilde{C}} \subseteq \mathcal{R}_C$. We give here another proof of this result which is similar to the proof of [Sai80, Lemma 2.8] (see proposition 2.13).

Following [GS12], we define the module of multi-logarithmic k-vector fields as the kernel of the map: $\bigwedge^k \Theta_S \to \mathcal{J}_C, \delta \mapsto \mathrm{d}h_1 \wedge \cdots \wedge \mathrm{d}h_k(\delta)$, where $\mathcal{J}_C$ is the Jacobian ideal of C.

In subsection 2.1.4, we show that the fact that C is reduced imposes a strong condition on the module of logarithmic 1-forms of the divisor D defined by $h_1 \cdots h_k$. We denote by D_i the hypersurface defined by h_i. Then we have (see proposition 2.23):

$$\Omega^1(\log D) = \Omega^1(\log D_1) + \cdots + \Omega^1(\log D_k).$$

In section 2.2, we consider first a reduced equidimensional space X of dimension n. Let C be a reduced complete intersection of dimension n containing X defined by a regular sequence $(f_1, \dots, f_k)$. We first recall definitions and properties from [Ale14]. For all $q \in \mathbb{N}$, the module $f_1 \cdots f_k \Omega^q(\log X/C, \underline{f})$ depends only on X and C, so that we set $\Omega^q(\log X/C) = \Omega^q(\log X/C, \underline{f})$. We have the inclusion $\Omega^q(\log X/C) \subseteq \Omega^q(\log C)$. The multi-residue of a form $\omega \in \Omega^q(\log X/C)$ is defined as the restriction to X of the multi-residue of ω seen as a multi-logarithmic form on C.

Let c_X be the fundamental class of X (see notation 2.36). We prove the following characterization of multi-logarithmic differential forms with respect to the pair (X, C), which generalizes [Ale12, §3, theorem 1] and [Sai80, (1.1)] (see proposition 2.38): a meromorphic q-form $\omega \in \frac{1}{f}\Omega^q_S$ is multi-logarithmic if and only if there exist $g \in \mathcal{O}_S$ which induces a non zero divisor in $\mathcal{O}_C$, $\xi \in \Omega^{q-k}_S$ and $\eta \in \widetilde{\Omega}^q_{\underline{f}}$ such that $g\omega = \dfrac{c_X}{f} + \eta$.

We suggest the following definition of multi-logarithmic k-vector fields: a holomorphic k-vector field δ is multi-logarithmic if $\delta(c_X) \in \mathcal{I}_X$. We then have the following perfect pairing which generalizes the duality of the hypersurface case (see proposition 3.3):

$$\mathrm{Der}^k(-\log X/C) \times \Omega^k(\log X/C) \to \frac{1}{f}\mathcal{I}_C.$$

We denote by $\mathcal{J}_{X/C}$ the restriction to X of the Jacobian ideal of C. In subsection 3.2, using an approach which is similar to the proof of [GS14, Proposition 3.4], and which uses the previous perfect pairing, we prove that $\mathrm{Hom}_{\mathcal{O}_C}(\mathcal{J}_{X/C}, \mathcal{O}_C) = \mathcal{R}_X$, where $\mathcal{R}_X$ is the module of multi-residues of multi-logarithmic k-forms (see proposition 3.11).

Section 4 is devoted to the main results of this paper, namely, the characterizations of freeness for Cohen-Macaulay spaces. Since a hypersurface is free if the module of logarithmic vector fields is free, or equivalently, the module of logarithmic 1-forms is free, it is then natural to investigate conditions on the projective dimensions of $\Omega^k(\log X/C)$ and $\mathrm{Der}^k(-\log X/C)$ to characterize the freeness of a Cohen-Macaulay space.

We prove that a reduced Cohen-Macaulay space is free if and only if the projective dimension of $\mathrm{Der}^k(-\log X/C)$ is $k-1$ (see proposition 4.5). Contrary to the hypersurface case, passing from $\mathrm{Der}^k(-\log X/C)$ to $\Omega^k(\log X/C)$ requires much more work. The proof of our main theorem 4.6 is developed in section 4, and uses Koszul complexes and change of rings spectral sequences.

We also give other characterizations of freeness involving the module of multi-residues in corollary 4.19.

In section 5 we study a particular family of free complete intersections: the complete intersection curves. For quasi-homogeneous curves, we determine a minimal free $\mathcal{O}_S$-resolution of $\mathcal{R}_C$ and $\Omega^{m-1}(\log C)$ (see theorems 5.9 and 5.11). We also give a characterization of plane homogeneous curves using multi-residues (see proposition 5.17). Several properties of complete intersection curves are studied in [Pol15], and are used in this section.

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2. Multi-logarithmic differential forms

2.1. Complete intersection case.

2.1.1. *Definitions.* We give here the definitions and several properties of multi-logarithmic differential forms and their residues along a complete intersection, which are a natural generalization of the theory of logarithmic differential forms developed in [Sai80].

Let S be a smooth analytic complex variety of dimension $m \geqslant 1$. We denote by $\mathcal{O}_S$ the sheaf of holomorphic functions on S. Let $x \in S$, and $(h_1, \dots, h_k)$ be a regular $\mathcal{O}_{S,x}$-sequence such that the ideal $\mathcal{I}_C = \sum_{i=1}^k h_i \mathcal{O}_{S,x}$ is radical. For the statements concerning sheaves, we will consider an open neighbourhood of p such that the equations $(h_1, \dots, h_k)$ converge and define a reduced complete intersection.

Notation 2.1. For all $i \in \{1, \dots, k\}$, we denote by D_i the hypersurface of S defined by h_i, and $D = D_1 \cup \dots \cup D_k$ the hypersurface defined by $h := h_1 \cdots h_k$. For $j \in \{1, \dots, k\}$, we set $\widehat{h_j} = \prod_{i \neq j} h_i$ and $\widehat{D_j}$ the hypersurface of S defined by $\widehat{h_j}$.

We denote by $C = D_1 \cap \dots \cap D_k$ the reduced complete intersection defined by the regular sequence $(h_1, \dots, h_k)$. We set $\mathcal{O}_C := \mathcal{O}_S / \mathcal{I}_C$. We denote $n = \dim(C) = m - k$.

For $q \in \mathbb{N}$, we denote by Ω^q_S, or Ω^q for short, the sheaf of holomorphic differential forms on S. For $I \subseteq \{1, \dots, m\}$, we set $\mathrm{d}x_I = \bigwedge_{i \in I} \mathrm{d}x_i$, and $|I|$ the cardinality of I.

We first recall some definitions and results from [Ale12]. The following definition generalizes [Sai80, (1.1)] to the complete intersection case.

Definition 2.2 ([Ale12]). *Let $x \in S$ and $\omega \in \dfrac{1}{h}\Omega^q_{S,x}$, with $q \in \mathbb{N}$. We call ω a* multi-logarithmic q-form *along the complete intersection C for the equations $(h_1, \dots, h_k)$ if*

$$\forall i \in \{1, \dots, k\}, \mathrm{d}h_i \wedge \omega \in \sum_{j=1}^k \frac{1}{\widehat{h_j}} \Omega^{q+1}_{S,x} = \frac{1}{h} \mathcal{I}_C \Omega^{q+1}_{S,x}.$$

We denote by $\Omega^q_S(\log C, \underline{h})$ or $\Omega^q(\log C, \underline{h})$ the sheaf of germs of multi-logarithmic q-forms along C with respect to the equations $(h_1, \dots, h_k)$.

From the definition, one can see that $h \cdot \Omega^q(\log C, \underline{h})$ is the kernel of the map of coherent sheaves $\varphi : \Omega^q \to \left(\frac{\Omega^{q+1}}{\mathcal{I}_C \Omega^{q+1}}\right)^k$ given by $\varphi(\omega) = (\mathrm{d}h_1 \wedge \omega, \dots, \mathrm{d}h_k \wedge \omega)$. It implies the following result:

Proposition 2.3. *The sheaves $\Omega^q(\log C, \underline{h})$ are coherent $\mathcal{O}_S$-modules and the modules $h \cdot \Omega^q(\log C, \underline{h})$ do not depend on the choice of the defining equations $(h_1, \dots, h_k)$.*

Notation 2.4. To simplify the notations, we set $\widetilde{\Omega}^q_{\underline{h}} := \sum_{j=1}^{k} \frac{1}{\widehat{h_j}} \Omega^q_S = \frac{1}{h} \mathcal{I}_C \Omega^q_S$.

The following characterization of multi-logarithmic forms generalizes [Sai80, (1.1)]:

Theorem 2.5 ([Ale12, §3, theorem 1])**.** *Let $\omega \in \frac{1}{h}\Omega^q_{S,x}$, with $q \in \mathbb{N}$ and $x \in S$. Then ω is multi-logarithmic if and only if there exist a holomorphic function $g \in \mathcal{O}_{S,x}$ which induces a non zero divisor in $\mathcal{O}_{C,x}$, a holomorphic differential form $\xi \in \Omega^{q-k}_{S,x}$ and a meromorphic q-form $\eta \in \widetilde{\Omega}^q_{\underline{h},x}$ such that:*

$$g\omega = \frac{\mathrm{d}h_1 \wedge \cdots \wedge \mathrm{d}h_k}{h} \wedge \xi + \eta. \tag{2}$$

Remark 2.6. For $q < k$, we have the equality $\Omega^q_S(\log C, \underline{h}) = \widetilde{\Omega}^q_{\underline{h}}$.

2.1.2. *Multi-residues.* Let $q \in \mathbb{N}$. The module of Kähler differentials of order q is:

$$\Omega^q_C = \left.\frac{\Omega^q_S}{(h_1, \dots, h_k)\Omega^q_S + \mathrm{d}h_1 \wedge \Omega^{q-1}_S + \cdots + \mathrm{d}h_k \wedge \Omega^{q-1}_S}\right|_C = \left.\frac{\Omega^q_S}{\mathrm{d}\mathcal{I}_C \wedge \Omega^{q-1}_S}\right|_C.$$

Definition 2.7 ([Ale12, §4, Definition 1])**.** *Let $x \in S$ and $\omega \in \Omega^q_{S,x}(\log C, \underline{h}), q \geqslant k$. Let us assume that g, ξ, η satisfy the properties of theorem 2.5. Then the* multi-residue *of ω is:*

$$\mathrm{res}_{C,\underline{h}}(\omega) := \left.\frac{\xi}{g}\right|_C \in \mathcal{M}_{C,x} \otimes_{\mathcal{O}_{C,x}} \Omega^{q-k}_{C,x} = \left(\pi_* \left(\mathcal{M}_{\widetilde{C}} \otimes_{\mathcal{O}_{\widetilde{C}}} \Omega^{q-k}_{\widetilde{C}}\right)\right)_x$$

where $\pi : \widetilde{C} \to C$ is the normalization of C and $\mathcal{M}_C$ is the sheaf of meromorphic functions on C.

The notion of multi-residue is well-defined with respect to the choices of g, ξ, η in (2) (see [Ale12, §4 proposition 2]). We denote $\mathcal{R}^{q-k}_{C,\underline{h}} := \mathrm{res}_{C,\underline{h}}(\Omega^q(\log C, \underline{h}))$. If $q = k$, we set $\mathcal{R}_{C,\underline{h}} := \mathcal{R}^0_{C,\underline{h}} \subset \mathcal{M}_C$.

Proposition 2.8 ([Ale12, §4, lemma 1])**.** *Let $q \in \mathbb{N}$. We have the following exact sequence of $\mathcal{O}_S$-modules:*

$$0 \longrightarrow \widetilde{\Omega}^q_{\underline{h}} \longrightarrow \Omega^q(\log C, \underline{h}) \xrightarrow{\mathrm{res}_{C,\underline{h}}} \mathcal{R}^{q-k}_{C,\underline{h}} \longrightarrow 0 \tag{3}$$

In particular, the sheaf $\mathcal{R}^{q-k}_{C,\underline{h}}$ is coherent.

From now on, we fix a point $x \in S$, and we consider only germs at X.

The modules of multi-residues do not depend on the choice of the defining equations:

Proposition 2.9. *Let $(h_1, \dots, h_k)$ and $(f_1, \dots, f_k)$ be two regular sequences defining the same reduced germ of complete intersection C of S. We set $f = f_1 \cdots f_k$. Let $A = (a_{ij})_{1 \leqslant i,j \leqslant k} \in \mathcal{M}_k(\mathcal{O}_S)$ be such that $(f_1, \dots, f_k)^t = A(h_1, \dots, h_k)^t$. Then for all $q \in \mathbb{N}$ and for all $\alpha \in f \cdot \Omega^q(\log C, \underline{f})$:*

$$\mathrm{res}_{C,\underline{h}}\left(\frac{\alpha}{h}\right) = \det(A)\mathrm{res}_{C,\underline{f}}\left(\frac{\alpha}{f}\right).$$

In particular, for all $p \geqslant 0$, the module $\mathcal{R}^p_{C,\underline{h}}$ does not depend on the choice of the defining equations.

Proof. Let $\alpha \in f_1 \cdots f_k \cdot \Omega^q(\log C, \underline{f})$. Then there exists $g \in \mathcal{O}_S$ which induces a non zero divisor in $\mathcal{O}_C$, $\xi \in \Omega^{q-k}$ and $\eta \in \widetilde{\Omega}^q_{\underline{f}}$ such that $g\alpha = \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k \wedge \xi + f\eta$.

Since for all $j \in \{1, \ldots, k\}$, $\mathrm{d}f_j = \sum_{i=1}^k (h_i \mathrm{d}a_{ji} + a_{ji}\mathrm{d}h_i)$, there exists $\nu \in \mathcal{I}_C\Omega^k_S$ such that:

$$\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k = \nu + \det(A)\mathrm{d}h_1 \wedge \cdots \wedge \mathrm{d}h_k. \tag{4}$$

Thus, $g\alpha = \mathrm{d}h_1 \wedge \cdots \wedge \mathrm{d}h_k \wedge \big(\det(A)\xi\big) + \nu \wedge \xi + f\eta$.

Since $f\eta \in \mathcal{I}_C\Omega^q_S$ and $\nu \in \mathcal{I}_C\Omega^k_S$, we have $\dfrac{\nu \wedge \xi + f\eta}{h_1 \cdots h_k} \in \widetilde{\Omega}^q_{\underline{h}} = \dfrac{1}{h}\mathcal{I}_C\Omega^q_S$, so that

$$\mathrm{res}_{C,\underline{h}}\left(\frac{\alpha}{h}\right) = \frac{\det(A)\xi}{g} = \det(A)\mathrm{res}_{C,\underline{f}}\left(\frac{\alpha}{f}\right).$$

□

In the rest of the paper, we will use the notation $\Omega^\bullet(\log C)$, $\mathcal{R}^\bullet_C$ and res_C where the set of equations is implicitly $(h_1, \ldots, h_k)$.

The following proposition shows that we can assume that m is the embedding dimension of C.

Proposition 2.10. *Let $\ell \in \{1, \ldots, k\}$. We assume that for all $i \in \{1, \ldots, \ell\}$ we have $h_i = x_i$ and for $i \geqslant \ell+1$, $h_i \in \mathbb{C}\{x_{\ell+1}, \ldots, x_m\} \cap \mathfrak{m}^2$. We set $C' = D_{\ell+1} \cap \ldots \cap D_k \subseteq \mathbb{C}^{m-\ell}$, so that $C = \{0\} \times C'$. Then for all $q \geqslant \ell$:*

$$\Omega^q(\log C) = \frac{1}{x_1 \cdots x_\ell}\Omega^{q-\ell}(\log C') \wedge \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_\ell + \widetilde{\Omega}^q_{(h_1,\ldots,h_k)}.$$

Proof. Let $\omega = \frac{1}{h}\sum a_I \mathrm{d}x_I$. Then by definition $\omega \in \Omega^q(\log C)$ if and only if for all $i \in \{1, \ldots, k\}$, $\mathrm{d}h_i \wedge \omega \in \widetilde{\Omega}^{q+1}_{h_1,\ldots,h_k}$. In particular, for all $i \in \{1, \ldots, \ell\}$, $\mathrm{d}x_i \wedge \omega \in \widetilde{\Omega}^{q+1}_{(h_1,\ldots,h_k)}$. This condition implies that for all I such that $\{1, \ldots, \ell\} \not\subseteq I$, $a_I \in (h_1, \ldots, h_k)$. We can write

$$\omega = \frac{1}{x_1 \cdots x_\ell}\omega' \wedge \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_\ell + \eta$$

with $\eta \in \widetilde{\Omega}^q_{(h_1,\ldots,h_k)}$ and $\omega' = \dfrac{1}{h_{\ell+1} \cdots h_k} \displaystyle\sum_{\substack{I' \subseteq \{\ell+1,\ldots,k\} \\ |I'| = q-\ell}} b_{I'}\mathrm{d}x_{I'}$, with $b_{I'} \in \mathbb{C}\{x_{\ell+1}, \ldots, x_m\}$.

Since $h_{\ell+1}, \ldots, h_k \in \mathbb{C}\{x_{\ell+1}, \ldots, x_m\}$, the conditions $\mathrm{d}h_i \wedge \dfrac{1}{x_1 \cdots x_\ell}\omega' \wedge \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_\ell \in \widetilde{\Omega}^{q+1}_{(h_1,\ldots,h_k)}$ are equivalent to the conditions $\mathrm{d}h_i \wedge \omega' \in \widetilde{\Omega}^{q-\ell+1}_{(h_{\ell+1},\ldots,h_k)}$. Therefore, $\omega \in \Omega^q(\log C)$ if and only if $\omega' \in \Omega^{q-\ell}(\log C')$. Hence the result. □

Corollary 2.11. *With the same notations, for all $q \geqslant k$, the module $\mathcal{R}^{q-k}_C$ is equal to the $\mathcal{O}_C$-module $\mathcal{R}^{q-k}_{C'}$.*

Notation 2.12. We denote by $\mathcal{J}_C$ the Jacobian ideal of C, which is the ideal of $\mathcal{O}_C$ generated by the $k \times k$-minors of the Jacobian matrix of $(h_1, \ldots, h_k)$. The Jacobian ideal does not depend on the choice of the equations $(h_1, \ldots, h_k)$.

The proof of the following result is similar to the proof of [Sai80, lemma 2.8]:

Proposition 2.13 (see [GS12])**.** *Let $\pi : \widetilde{C} \to C$ be the normalization of C and $x \in C$. We have the inclusion $\mathcal{O}_{\widetilde{C}} \subseteq \mathcal{R}_C$.*

Proof. Let $\alpha \in \mathcal{O}_{\widetilde{C}}$. Then, thanks to [GLS10, Lemma 4.1], for every $g \in \mathcal{J}_C$, $g\alpha \in \mathcal{O}_C$.

Thus, for every subset $J \subseteq \{1, \ldots, m\}$ with $|J| = k$, there exists $a_J \in \mathcal{O}_S$ such that its class in $\mathcal{O}_C$ satisfies $\Delta_J\alpha = \overline{a_J} \in \mathcal{O}_C$, where Δ_J is the minor of the Jacobian matrix relative to the set J. Let I, J be two subsets of $\{1, \ldots, m\}$ with k elements. Then from the equality $\Delta_I\Delta_J\alpha - \Delta_J\Delta_I\alpha = 0 \in \mathcal{O}_C$ we deduce:

$$\Delta_I a_J - \Delta_J a_I = h_1 b_1^{IJ} + \cdots + h_k b_k^{IJ} \in \mathcal{O}_S.$$

Let us define $\omega = \frac{\sum_{|J|=k} a_J \mathrm{d}x_J}{h} \in \frac{1}{h}\Omega^k_S$. The previous equality gives:

$$\Delta_I \omega = \frac{\sum_J \Delta_I a_J \mathrm{d}x_J}{h} = a_I \frac{\mathrm{d}h_1 \wedge \cdots \wedge \mathrm{d}h_k}{h} + \eta \text{ with } \eta \in \widetilde{\Omega}^k.$$

Moreover, there exists a linear combination of the Δ_J which does not induce a zero-divisor in $\mathcal{O}_C$ (see [GLS10, Lemma 4.1]). Therefore, $\omega \in \Omega(\log C)$ and $\mathrm{res}_C(\omega) = \alpha \in \mathcal{R}_C$. □

Other proofs of propositions 2.9 and 2.13 are suggested respectively in [Ale12] and in [Sch16], which require the introduction of the modules of regular meromorphic forms and theorem 2.16 which is recalled below.

Definition 2.14. *The $\mathcal{O}_C$-module $\omega_C^{m-k} = \mathrm{Ext}^k_{\mathcal{O}_S}(\mathcal{O}_C, \Omega^m_S)$ is called the* Grothendieck dualizing module *of the germ $(C, 0)$. Since C is a complete intersection, it is a free $\mathcal{O}_C$-module of rank 1.*

Definition 2.15. *The module ω^p_C of* regular meromorphic differential p-forms *is:*

$$\omega^p_C = \mathrm{Hom}_{\mathcal{O}_C}\left(\Omega_C^{m-k-p}, \omega_C^{m-k}\right).$$

Theorem 2.16 ([AT08, Theorem 3.1])**.** *For all $p \geqslant 0$, the module of multi-residues $\mathcal{R}^p_C$ is isomorphic to the module ω^p_C of regular meromorphic differential forms.*

2.1.3. *Multi-logarithmic vector fields.* We introduce here a module of multi-logarithmic k-vector fields. We will prove in proposition 3.3 that there exists a perfect pairing between $\Omega^k(\log C)$ and $\mathrm{Der}^k(-\log C)$ with values in $\frac{1}{h}\mathcal{I}_C$, which generalizes [Sai80, (1.6)].

Let Θ_S denote the $\mathcal{O}_S$-module of holomorphic vector fields and $\Theta^k_S := \bigwedge^k \Theta_S$ be the exterior power of order k of Θ_S.

We can evaluate a k-form $\omega \in \Omega^k_S$ on a k-vector field $\delta \in \Theta^k_S$, which gives a function denoted by $\omega(\delta)$.

Definition 2.17 ([GS12])**.** *A k-vector field $\delta \in \Theta^k_S$ is called* multi-logarithmic *along C if it satisfies:*

$$\mathrm{d}h_1 \wedge \cdots \wedge \mathrm{d}h_k(\delta) \in \mathcal{I}_C.$$

We denote by $\mathrm{Der}^k(-\log C)$ the $\mathcal{O}_S$-module of multi-logarithmic k-vector fields along C.

The following proposition is easy to prove with (4):

Proposition 2.18. *The module $\mathrm{Der}^k(-\log C)$ does not depend on the choice of the equations.*

The following proposition comes from definition 2.17, where $\mathcal{J}_C$ is the Jacobian ideal defined in notation 2.12.

Proposition 2.19 ([GS12, (5.1)])**.** *We have the following exact sequence of $\mathcal{O}_S$-modules:*

$$0 \to \mathrm{Der}^k(-\log C) \to \Theta^k_S \xrightarrow{\mathrm{d}h_1 \wedge \cdots \wedge \mathrm{d}h_k} \mathcal{J}_C \to 0. \tag{5}$$

In particular, $\mathrm{Der}^k(-\log C)$ is a coherent $\mathcal{O}_S$-module.

Remark 2.20. The notion of logarithmic vector field studied in [Sai80] can be extended to spaces of higher codimension (see [HM93]). A holomorphic vector field $\eta \in \Theta_S$ is logarithmic if η is tangent to the complete intersection at its smooth points, which is equivalent to $\eta(\mathcal{I}_C) \subseteq \mathcal{I}_C$. We denote by $\mathrm{Der}(-\log C)$ the module of logarithmic vector fields. The following inclusion is a direct consequence of the definition:

$$\mathrm{Der}(-\log C) \wedge \Theta^{k-1}_S \subseteq \mathrm{Der}^k(-\log C). \tag{6}$$

Using the fact that $(h_1, \ldots, h_k)$ is a regular sequence, it is easy to prove that $\mathrm{Der}(-\log D) \subseteq \mathrm{Der}(-\log C)$. In particular, it implies that $\mathrm{Der}(-\log D) \wedge \Theta^{k-1}_S \subseteq \mathrm{Der}^k(-\log C)$ (see [GS12, (5.3)]).

One can check that the inclusion (6) may be strict, by considering for example the complete intersection curve defined by $h_1 = x^3 - y^2$ and $h_2 = x^2 y - z^2$ (see [Pol16, Exemple 3.2.10]).

2.1.4. *A condition on the divisor associated to a regular sequence.* We prove here that the module of logarithmic 1-forms on the divisor $D = D_1 \cup \cdots \cup D_k$ satisfies a decomposition property when $D_1 \cap \cdots \cap D_k$ is a reduced complete intersection (see proposition 2.23).

We first need the following lemma:

Lemma 2.21. *If C is the germ of a reduced complete intersection defined by a regular sequence $(h_1, \ldots, h_k)$, then the complete intersection defined by $(h_1, \ldots, h_{k-1})$ is also reduced.*

In addition, for all $i \in \{1, \ldots, k-1\}$, the complete intersection defined by $(h_1, \ldots, h_{i-1}, \prod_{j=i}^k h_j)$ is also reduced.

Proof. Let $g \in \sqrt{(h_1, \ldots, h_{k-1})}$. Then, since $\sqrt{(h_1, \ldots, h_{k-1})} \subseteq \sqrt{(h_1, \ldots, h_k)} = (h_1, \ldots, h_k)$, there exists $f_1 \in (h_1, \ldots, h_{k-1})$ and $\alpha_1 \in \mathcal{O}_S$ such that $g = f_1 + \alpha_1 h_k$. We then have $\alpha_1 h_k \in \sqrt{(h_1, \ldots, h_{k-1})}$. Since $(h_1, \ldots, h_k)$ is a regular sequence, it implies that $\alpha_1 \in \sqrt{(h_1, \ldots, h_{k-1})}$. Therefore, there exists $f_2 \in (h_1, \ldots, h_{k-1})$ and $\alpha_2 \in \mathcal{O}_S$ such that $g = f_2 + \alpha_2 h_k^2$. By induction, we prove that for all $i \in \mathbb{N}^*$, there exist $f_i \in (h_1, \ldots, h_{k-1})$ and $\alpha_i \in \mathcal{O}_S$ such that $g = f_i + \alpha_i h_k^i$. Hence, the class $\overline{g}$ of g in $\mathcal{O}_S/(h_1, \ldots, h_{k-1})$ satisfies

$$\overline{g} \in \bigcap_{i \geqslant 1} \overline{(h_k^i)} \subseteq \bigcap_{i \geqslant 1} \overline{(\mathfrak{m}^i)} \subset \mathcal{O}_S/(h_1, \ldots, h_{k-1}).$$

By Krull's intersection theorem, it implies that $\overline{g} = 0$, therefore, $g \in (h_1, \ldots, h_{k-1})$. Hence the first statement.

For the second statement, it is sufficient to prove it for $i = k-1$, the result will then follow by induction.

Let us consider $g \in \sqrt{(h_1, \ldots, h_{k-2}, h_{k-1} \cdot h_k)}$. Then, $g \in (h_1, \ldots, h_k)$, so that for all $j \in \{1, \ldots, k\}$ there exist $\alpha_j \in \mathcal{O}_S$ such that $g = \sum_{j=1}^k \alpha_j h_j$. Therefore,

$$\alpha_{k-1} h_{k-1} + \alpha_k h_k \in \sqrt{(h_1, \ldots, h_{k-2}, h_{k-1} \cdot h_k)}.$$

Let $q \in \mathbb{N}^*$ be such that $(\alpha_{k-1} h_{k-1} + \alpha_k h_k)^q \in (h_1, \ldots, h_{k-2}, h_{k-1} \cdot h_k)$. Since $(h_1, \ldots, h_k)$ is a regular sequence, we then have $\alpha_{k-1}^q \in (h_1, \ldots, h_{k-2}, h_k)$ and $\alpha_k^q \in (h_1, \ldots, h_{k-1})$. Since by the first statement, these ideals are radical, it gives the result. □

Proposition 2.22 (see [Ale12, Proposition 1])**.** *For all $q \in \mathbb{N}$ we have the inclusion:*

$$\Omega^q(\log D) \subseteq \Omega^q(\log C).$$

Therefore, $\mathrm{res}_C(\Omega^q(\log D)) \subseteq \mathrm{res}_C(\Omega^q(\log C))$.

Proposition 2.23. *Let C be a reduced complete intersection. We keep the same notations as before. The module of logarithmic 1-forms satisfies:*

$$\Omega^1(\log D) = \Omega^1(\log D_1) + \cdots + \Omega^1(\log D_k).$$

Proof. We first prove the proposition for $k = 2$. The general case will then follow from this special case.

Let $\omega \in \Omega^1(\log D)$. Then by proposition 2.22 and Remark 2.6, $\omega = \frac{\eta_1}{h_1} + \frac{\eta_2}{h_2}$ with $\eta_i \in \Omega_S^1$. Then:

$$\mathrm{d}h \wedge \omega = \mathrm{d}h_1 \wedge \eta_2 + \mathrm{d}h_2 \wedge \eta_1 + \frac{h_2 \mathrm{d}h_1 \wedge \eta_1}{h_1} + \frac{h_1 \mathrm{d}h_2 \wedge \eta_2}{h_2} \in \Omega_S^2.$$

Therefore, there exists $\theta \in \Omega_S^2$ such that:

$$h_2^2 \mathrm{d}h_1 \wedge \eta_1 + h_1^2 \mathrm{d}h_2 \wedge \eta_2 = h_1 h_2 \theta.$$

Since (h_1, h_2) is a regular sequence, h_2 divides $\mathrm{d}h_2 \wedge \eta_2$ and h_1 divides $\mathrm{d}h_1 \wedge \eta_1$, which means that $\dfrac{\eta_1}{h_1} \in \Omega^1(\log D_1)$ and $\dfrac{\eta_2}{h_2} \in \Omega^1(\log D_2)$.

The general case is obtained by induction from the case $k = 2$. Indeed, by lemma 2.21, for all $i \in \{1, \ldots k-1\}$, $(h_1, \ldots, h_{i-1}, \prod_{j=i}^k h_j)$ is a regular sequence defining a reduced complete intersection, so that $\Omega^1(\log D) = \Omega^1(\log D_1) + \cdots + \Omega^1(\log D_{i-1}) + \Omega^1(\log(D_i \cup \cdots \cup D_k))$. □

Remark 2.24. In particular, it gives a necessary condition on $\Omega^1(\log D)$ which has to be satisfied when C is a reduced complete intersection.

A divisor is called *splayed* if in some coordinate system $(y_1, \dots, y_m)$ it has an equation of the form $g = g_1 \cdot g_2$ with $g_1 \in \mathbb{C}\{y_1, \dots y_\ell\}$ and $g_2 \in \mathbb{C}\{y_{\ell+1}, \dots, y_m\}$, with $\ell \in \{1, \dots, m-1\}$ (see [AF13] and [Sch16]). The following corollary is then a consequence of [AF13, Theorem 2.12] and proposition 2.23:

Corollary 2.25. *Let C be a reduced complete intersection of codimension 2 defined by (h_1, h_2). If the divisor D defined by $h_1 \cdot h_2$ is free, then D is a splayed divisor.*

2.2. **Generalization to reduced equidimensional spaces.** We generalize to reduced equidimensional spaces the notions of the previous part. We treated first the complete intersection case because the reduced equidimensional case rely partially on it.

The reduced equidimensional case uses a description of regular meromorphic differential forms by residue symbols which has been introduced in [Ker84]. This description was already used by M. Schulze in a talk given at Oberwolfach in 2012, and then in [Sch16], in order to generalize logarithmic residues to reduced equidimensional spaces. A definition of multi-logarithmic differential forms is suggested in [Ale14], where the interpretation of regular meromorphic forms with residue symbols is also used.

2.2.1. *Multi-logarithmic differential forms.* Let $X \subseteq S$ be the germ of a reduced equidimensional analytic subset of S of dimension n defined by a radical ideal $\mathcal{I}_X$. We set $k = m - n$.

One can prove that there exists a regular sequence $(f_1, \dots, f_k) \subseteq \mathcal{I}_X$ such that the ideal generated by $f_1, \dots, f_k$ is radical (see [AT08, Remark 4.3] or [Pol16, Proposition 4.2.1] for a detailed proof of this result). We fix such a sequence $(f_1, \dots, f_k)$. We denote by C the complete intersection defined by the ideal $\mathcal{I}_C$ generated by $(f_1, \dots, f_k)$. In particular, $C = X \cup Y$, where Y is of pure dimension n and does not contain any component of X. We set $f = f_1 \cdots f_k$, and $\mathcal{I}_Y$ the radical ideal defining Y.

We recall that $\widetilde{\Omega}^q_{\underline{f}} = \frac{1}{f}\mathcal{I}_C\Omega^q_S$.

Definition 2.26 ([Ale14, Definition 10.1])**.** *Let $q \in \mathbb{N}$. We define the* module of multi-logarithmic q-forms with respect to the pair (X, C) *as :*

$$\Omega^q(\log X/C) = \left\{ \omega \in \frac{1}{f_1 \cdots f_k}\Omega^q_S \ ; \ \mathcal{I}_X\omega \subseteq \widetilde{\Omega^q_{\underline{f}}} \text{ and } (\mathrm{d}\mathcal{I}_X) \wedge \omega \subseteq \widetilde{\Omega}^{q+1}_{\underline{f}} \right\}.$$

Remark 2.27. For all $q \in \mathbb{N}$, we have :

$$\widetilde{\Omega}^q_{\underline{f}} \subseteq \Omega^q(\log X/C) \subseteq \Omega^q(\log C).$$

In addition, if C is a reduced complete intersection, we have $\Omega^q(\log C/C) = \Omega^q(\log C)$.

Proposition 2.28. *Let $q \in \mathbb{N}$. The module $\Omega^q(\log X/C)$ is coherent and the module of numerators $f \cdot \Omega^q(\log X/C)$ depends only on X and C.*

Proof. Let $q \in \mathbb{N}$ and $(h_1, \dots, h_r)$ be a generating family of $\mathcal{I}_X$. The module $f \cdot \Omega^q(\log X/C)$ is the kernel of the map $\beta : \Omega^q_S \to \left(\Omega^q_S/\mathcal{I}_C\Omega^q_S\right)^r \oplus \left(\Omega^{q+1}_S/\mathcal{I}_C\Omega^{q+1}_S\right)^r$ defined for $\omega \in \Omega^q_S$ by $\beta(\omega) = (h_1\omega, \dots, h_r\omega, \mathrm{d}h_1 \wedge \omega, \dots, \mathrm{d}h_r \wedge \omega)$. Hence the result. □

Remark 2.29. The modules $f \cdot \Omega^q(\log X/C)$ depend on the choice of the complete intersection C, since $\mathcal{I}_C \cdot \Omega^q_S \subseteq f \cdot \Omega^q(\log X/C)$.

Definition 2.30 ([Ale14, Proposition 10.1])**.** *The multi-residue map* $\mathrm{res}_{X/C} : \Omega^q(\log X/C) \to \mathcal{M}_X \otimes_{\mathcal{O}_X} \Omega^{q-k}_X$ *is defined as the restriction of the map* res_C.

For $q \in \mathbb{N}$, we set $\mathcal{R}^q_{X/C} := \mathrm{res}_{X/C}(\Omega^{q+k}(\log X/C))$.

Proposition 2.31 ([Ale14, Theorem 10.2])**.** *We have the following exact sequence of $\mathcal{O}_S$-modules:*

$$0 \to \widetilde{\Omega}^q_{\underline{f}} \to \Omega^q(\log X/C) \xrightarrow{\mathrm{res}_{X/C}} \mathcal{R}^{q-k}_{X/C} \to 0. \tag{7}$$

We have the following theorem:

Theorem 2.32 ([Ale14, Theorem 10.2])**.** *Let $q \in \mathbb{N}$. The module $\mathcal{R}^q_{X/C}$ is isomorphic to the module of regular meromorphic forms ω^q_X. In particular, $\mathcal{R}^q_{X/C}$ does not depend on the choice of the complete intersection C.*

The proof of this theorem uses [Kun86, (E.20)] and [Kun86, (E.21)], and the following characterization of regular meromorphic forms with residue symbols:

Proposition 2.33 ([Ker84, (1.2)])**.** *Let $q \in \mathbb{N}$. The module of regular meromorphic forms ω^q_X satisfies:*

$$\omega^q_X = \left\{ \begin{bmatrix} \alpha \\ f_1, \dots, f_k \end{bmatrix}; \alpha \in \Omega^{q+k}_S, (f_1, \dots, f_k) \subseteq \mathcal{I}_X \text{ a regular sequence}, \right.$$
$$\left. \mathcal{I}_X \alpha \subseteq \sum f_i \Omega^{k+q}_S, \mathrm{d}\mathcal{I}_X \wedge \alpha \subseteq \sum f_i \Omega^{q+k+1}_S \right\}$$

where $\begin{bmatrix} \alpha \\ f_1, \dots, f_k \end{bmatrix}$ denotes the residue symbol (see [Ker83] for more details).

We end this section with a characterization of multi-logarithmic q-forms with respect to the pair (X, C), which generalizes theorem 2.5. We first need to introduce the fundamental form of X (see for example [Ker84, (1.3)]). We recall that $C = X \cup Y$ is an irredundant decomposition of C.

Notation 2.34. Let $\beta_f \in \mathcal{O}_{\widetilde{C}}$ be such that $\beta_f|_X = 1$ and $\beta_f|_Y = 0$.

A direct consequence of [GLS10, Lemma 4.1] is the following lemma:

Lemma 2.35. *The form $\overline{\beta_f \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k} \in \Omega^k_S \otimes_{\mathcal{O}_S} \mathcal{M}_C$ satisfies $\overline{\beta_f \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k} \in \Omega^k_S \otimes_{\mathcal{O}_S} \mathcal{O}_C$*

Notation 2.36. We fix $c_X \in \Omega^k_S$ such that $\overline{c_X} = \overline{\beta_f \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k} \in \Omega^k_S \otimes_{\mathcal{O}_S} \mathcal{O}_C$.

The following result is a consequence of the definition of c_X and [Ker84, (1.3)]:

Proposition 2.37 (see [Ker84, (1.3)])**.** *We have $\frac{c_X}{f} \in \Omega^k(\log X/C)$ and $\mathrm{res}_{X/C}\left(\frac{c_X}{f}\right) = 1 \in \mathcal{M}_X$.*

The following proposition gives a characterization of multi-logarithmic forms which generalizes theorem 2.5.

Proposition 2.38. *Let $\omega \in \frac{1}{f}\Omega^q_S$. Then $\omega \in \Omega^q(\log X/C)$ if and only if there exist $g \in \mathcal{O}_S$ which induces a non zero divisor in $\mathcal{O}_C$, $\xi \in \Omega^{q-k}_S$ and $\eta \in \widetilde{\Omega}^q_{\underline{f}}$ such that:*

$$g\omega = \frac{c_X}{f} \wedge \xi + \eta \tag{8}$$

In addition, we then have $\mathrm{res}_{X/C}(\omega) = \frac{\xi}{g}\big|_X$.

Proof. Let $\omega \in \Omega^q(\log X/C)$. Then $\omega \in \Omega^q(\log C)$. Let g, ξ, η satisfying theorem 2.5 such that:

$$g\omega = \frac{\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k}{f} \wedge \xi + \eta.$$

Then $g\mathrm{res}_{X/C}(\omega) = \xi = \mathrm{res}_{X/C}\left(\frac{c_X}{f} \wedge \xi\right)$. By proposition 2.31, there exists $\eta' \in \widetilde{\Omega}^q_{\underline{f}}$ such that

$$g\omega = \frac{c_X}{f} \wedge \xi + \eta'.$$

Conversely, let $\omega \in \frac{1}{f}\Omega^q_S$ be such that $g\omega = \frac{c_X}{f} \wedge \xi + \eta$ with g, ξ, η as in the statement of the proposition. Let $h \in \mathcal{I}_X$. Since $\frac{c_X}{f} \in \Omega^k(\log X/C)$, we deduce that $hg\omega \in \widetilde{\Omega}^q_{\underline{f}}$ and $\mathrm{d}h \wedge g\omega \in \widetilde{\Omega}^{q+1}_{\underline{f}}$. Therefore, since g induces a non zero divisor in $\mathcal{O}_C$, we have $\omega \in \Omega^q(\log X/C)$. □

We deduce from [Sch16, (2.14)] the following equality:

$$\mathcal{R}_X = \{\rho_{|X} \ ; \ \rho \in \mathcal{R}_C \text{ such that } \rho_{|Y} = 0\}. \tag{9}$$

In particular, it implies that $\mathcal{O}_{\widetilde{X}} \subseteq \mathcal{R}_X$.

As a consequence, we have the following proposition:

Proposition 2.39. *Let $\omega \in \Omega^k(\log C)$. Then:*

$$\omega \in \Omega^k(\log X/C) \iff \mathrm{res}_C(\omega)|_Y = 0.$$

Proof. Let $\omega \in \Omega^k(\log C)$, and $\rho = \mathrm{res}_C(\omega)$. Then:

$$\begin{aligned} \rho_{|Y} = 0 &\iff \exists \omega' \in \Omega^k(\log X/C), \rho_{|X} = \mathrm{res}_{X/C}(\omega') \\ &\iff \exists w' \in \Omega^k(\log X/C), \exists \eta \in \widetilde{\Omega}^k_{\underline{f}},\ \omega = \omega' + \eta \in \Omega^k(\log X/C). \end{aligned}$$

Hence the result. □

2.2.2. *Multi-logarithmic vector fields.* We suggest the following definition of multi-logarithmic k-vector fields, which generalizes definition 2.17:

Definition 2.40. *Let $\delta \in \Theta^k_S$. We say that δ is a* multi-logarithmic k-vector field with respect to the pair (X, C) *if $c_X(\delta) \in \mathcal{I}_X$.*

We denote by $\mathrm{Der}^k(-\log X/C)$ the module of multi-logarithmic k-vector fields with respect to (X, C).

Remark 2.41. Since $c_X \in \mathcal{I}_Y \Omega^k_S$, for all $\delta \in \Theta^k_S$, $c_X(\delta) \in \mathcal{I}_Y$. Therefore,

$$\mathrm{Der}^k(-\log X/C) = \left\{\delta \in \Theta^k_S \ ; \ c_X(\delta) \in \mathcal{I}_C\right\}.$$

As a consequence, we have the following proposition:

Proposition 2.42. *We have the following inclusion:*

$$\mathrm{Der}^k(-\log C) \subseteq \mathrm{Der}^k(-\log X/C).$$

Remark 2.43. If C is a reduced complete intersection, we have $\mathrm{Der}^k(-\log C/C) = \mathrm{Der}^k(-\log C)$.

3. Duality results

3.1. **Perfect pairing between multi-logarithmic vector fields and multi-logarithmic forms.** For a reduced hypersurface D, a $\mathcal{O}_S$-duality is satisfied between $\Omega^1(\log D)$ and $\mathrm{Der}(-\log D)$ (see [Sai80, Lemma (1.6)]).

Let us prove that there is a perfect pairing between $\Omega^k(\log X/C)$ and $\mathrm{Der}^k(-\log X/C)$, and in particular, for a complete intersection C, between $\Omega^k(\log C)$ and $\mathrm{Der}^k(-\log C)$. We fix a point $x \in S$ and consider germs at this point, without mentioning it.

We first need the following lemma, which is a direct consequence of proposition 2.38 and the definition of $\mathrm{Der}^k(-\log X/C)$.

To simplify the notations, we set $\Sigma = \frac{1}{f}\mathcal{I}_C$.

Lemma 3.1. *Let $\delta \in \mathrm{Der}^k(-\log X/C)$ and $\omega \in \Omega^k(\log X/C)$. Then $\omega(\delta) \in \Sigma$.*

Thanks to lemma 3.1, we see that we have a natural pairing

$$\mathrm{Der}^k(-\log X/C) \times \Omega^k(\log X/C) \to \Sigma.$$

Let us notice the following lemma:

Lemma 3.2. *We have the following perfect pairings:*

$$\widetilde{\Omega}^k_{\underline{f}} \times \Theta^k_S \to \Sigma, \tag{10}$$

$$\frac{1}{f}\Omega^k_S \times \sum_{i=1}^{k} f_i \Theta^k_S \to \Sigma. \tag{11}$$

Proof. Let us notice that $\widetilde{\Omega}^k_{\underline{f}} = \Omega^k_S \otimes \Sigma$ and $\sum f_i \Theta^k_S = f\Theta^k_S \otimes \Sigma$. Let M be either Ω^k_S or $f\Theta^k_S$. It is easy to prove that $\mathrm{Hom}_{\mathcal{O}_S}(M \otimes \Sigma, \Sigma) = \mathrm{Hom}_{\mathcal{O}_S}(M, \mathcal{O}_S)$ and $\mathrm{Hom}_{\mathcal{O}_S}(M, \Sigma) = \mathrm{Hom}_{\mathcal{O}_S}(M, \mathcal{O}_S) \otimes \Sigma$. Hence the result. □

Proposition 3.3. *We have the following perfect pairing:*

$$\Omega^k(\log X/C) \times \mathrm{Der}^k(-\log X/C) \to \Sigma. \tag{12}$$

Proof. We have the following inclusions:

$$\widetilde{\Omega}^k_{\underline{f}} \subseteq \Omega^k(\log X/C) \subseteq \frac{1}{f}\Omega^k_S \tag{13}$$

$$\Theta^k_S \supseteq \mathrm{Der}^k(-\log X/C) \supseteq \mathrm{Der}^k(-\log X/C). \tag{14}$$

We deduce from lemmas 3.1, 3.2 and the inclusions (13) and (14) the following:

$$\mathrm{Der}^k(-\log X/C) \subseteq \mathrm{Hom}_{\mathcal{O}_S}\left(\Omega^k(\log X/C), \Sigma\right) \subseteq \Theta^k_S,$$

$$\Omega^k(\log X/C) \subseteq \mathrm{Hom}_{\mathcal{O}_S}\left(\mathrm{Der}^k(-\log X/C), \Sigma\right) \subseteq \frac{1}{f}\Omega^k_S.$$

Let us prove that the left-hand-side inclusions are equalities.

Since $\frac{c_X}{f} \in \Omega^k(\log X/C)$, for all $\delta \in \mathrm{Hom}_{\mathcal{O}_S}\left(\Omega^k(\log X/C), \Sigma\right)$ we have $c_X(\delta) \in \mathcal{I}_C$ so that $\delta \in \mathrm{Der}^k(-\log X/C)$. Therefore, $\mathrm{Hom}_{\mathcal{O}_S}\left(\Omega^k(\log X/C), \Sigma\right) = \mathrm{Der}^k(-\log X/C)$.

Let $\omega \in \mathrm{Hom}_{\mathcal{O}_S}\left(\mathrm{Der}^k(-\log X/C), \Sigma\right)$. Let us prove that $\omega \in \Omega^k(\log X/C)$. Let us set $\omega = \sum_{|I|=k} \frac{1}{f}\omega_I \mathrm{d}x_I$. Let $h \in \mathcal{I}_X$. Then for all I, $h\partial_{x_I} \in \mathrm{Der}^k(-\log X/C)$, so that $h\partial_{x_I}(\omega) = h\frac{1}{f}\omega_I \in \Sigma$. Therefore, $h\omega \in \widetilde{\Omega}^k_{\underline{f}}$.

We set for $J \subseteq \{1, \dots, m\}$ with $|J| = k+1$, $\delta_J = \sum_{\ell=1}^{k+1} (-1)^{\ell-1} \frac{\partial h}{\partial x_{j_\ell}} \partial x_{j_1} \wedge \cdots \wedge \widehat{\partial x_{j_\ell}} \wedge \cdots \wedge \partial x_{j_{k+1}}$.

We thus have $\omega(\delta_J) = (\mathrm{d}h \wedge \omega)(\partial_{x_J})$. Let us prove that $\delta_J \in \mathrm{Der}^k(-\log X/C)$. We have:

$$(\beta_f \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k)(\delta_J) = \beta_f \sum_{\ell=1}^{k+1} (-1)^{\ell-1} \frac{\partial h}{\partial x_{j_\ell}} \Delta_{j_1 \dots \widehat{j_\ell} \dots j_{k+1}} = \beta_f \begin{vmatrix} \frac{\partial h}{\partial x_{j_1}} & \cdots & \frac{\partial h}{\partial x_{j_{k+1}}} \\ \frac{\partial f_1}{\partial x_{j_1}} & \cdots & \frac{\partial f_1}{\partial x_{j_{k+1}}} \\ \vdots & & \vdots \\ \frac{\partial f_k}{\partial x_{j_1}} & \cdots & \frac{\partial f_k}{\partial x_{j_{k+1}}} \end{vmatrix}.$$

Since the codimension of X is k and $(h, f_1, \dots, f_k) \subseteq \mathcal{I}_X$, the restriction of the previous determinant to X is zero. Given that $\beta_f|_Y = 0$, we have $\overline{(\beta_f \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k)(\delta_J)} = \overline{0} \in \mathcal{M}_C$. Thus, $\delta_J \in \mathrm{Der}^k(-\log X/C)$. Since $\omega(\delta_J) = (\mathrm{d}h \wedge \omega)(\partial_{x_J})$, we deduce that $\mathrm{d}h \wedge \omega \in \widetilde{\Omega}^{k+1}_{\underline{f}}$ and $\mathrm{Hom}_{\mathcal{O}_S}\left(\mathrm{Der}^k(-\log X/C), \Sigma\right) = \Omega^k(\log X/C)$. □

Remark 3.4. By lemma 3.2 and proposition 3.3, if C is a reduced complete intersection, we have the following perfect pairings between modules which all depend only on C and not on the choice of equations:

$$\mathcal{I}_C \Omega^k_S \times \Theta^k_S \to \mathcal{I}_C$$

$$\Omega^k_S \times \mathcal{I}_C \Theta^k_S \to \mathcal{I}_C$$

$$f \cdot \Omega^k(\log C) \times \mathrm{Der}^k(-\log C) \to \mathcal{I}_C$$

3.2. **Multi-residues and Jacobian ideal.** Let C be a reduced complete intersection. A consequence of [Sch16, Lemma 5.4] and theorem 2.16 is that the dual of the Jacobian ideal $\mathcal{J}_C$ of C is the module of multi-residues $\mathcal{R}_C$.

In this subsection, we give another proof this duality, which does not depend on the isomorphism of theorem 2.16. In addition, we extend it to the case of reduced equidimensional spaces. Our proof uses the perfect pairings of proposition 3.3 and is analogous to the proof of [GS14, Proposition 3.4].

Notation 3.5. Let X be a reduced equidimensional space of codimension k in S and C be a reduced complete intersection of codimension k containing X. We denote by $\mathcal{J}_{X/C} \subseteq \mathcal{O}_X$ the restriction of the Jacobian ideal $\mathcal{J}_C$ of C to the space X.

Remark 3.6. Since $\beta_f|_X = 1$, the ideal $\mathcal{J}_{X/C}$ is given by $\mathcal{J}_{X/C} = \left\{\overline{\delta(c_X)} \in \mathcal{O}_X \ ; \ \delta \in \Theta_S^k\right\}$.

The following result is a consequence of the definitions:

Proposition 3.7. *We have the following exact sequence of $\mathcal{O}_S$-modules:*

$$0 \to \mathrm{Der}^k(-\log X/C) \to \Theta_S^k \to \mathcal{J}_{X/C} \to 0.$$

Remark 3.8. Several notions of Jacobian ideals are associated to a reduced equidimensional space: the ideal $\mathcal{J}_X \subseteq \mathcal{O}_X$ generated by the $k \times k$ minors of the Jacobian matrix of $(h_1, \dots, h_r)$, where $\sum_{i=1}^r h_i \mathcal{O}_S = \mathcal{I}_X$, the ω-Jacobian $\mathcal{J}_X^\omega$ (see for example[Sch16]), and the ideal $\mathcal{J}_{X/C}$ considered above. These three notions may not coincide, as one can check on the example of the non Gorenstein curve of $\mathbb{C}^3$ defined by $h_1 = y^3 - x^2 z$, $h_2 = x^3 y - z^2$ and $h_3 = x^5 - y^2 z$, which is the irreducible curve parametrized by (t^5, t^7, t^{11}) (see [Pol16, Exemple 4.2.35]).

We first need the following lemma:

Lemma 3.9. *Let X be a reduced space of pure dimension $n = m - k$. We assume $k \geqslant 2$. Then $\mathrm{Ext}^1_{\mathcal{O}_S}\left(\mathcal{J}_{X/C}, \mathcal{O}_S\right) = 0$ and $\mathrm{Ext}^1_{\mathcal{O}_S}\left(\mathcal{J}_{X/C}, \Sigma\right) = \mathrm{Hom}_{\mathcal{O}_C}\left(\mathcal{J}_{X/C}, \mathcal{O}_C\right)$.*

Proof. We apply the functor $\mathrm{Hom}_{\mathcal{O}_S}(\mathcal{J}_{X/C}, -)$ to the exact sequence $0 \to \Sigma \xrightarrow{\times f} \mathcal{O}_S \to \mathcal{O}_C \to 0$. It gives:

$$0 \to \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{J}_{X/C}, \mathcal{O}_C) \to \mathrm{Ext}^1_{\mathcal{O}_S}\left(\mathcal{J}_{X/C}, \Sigma\right) \to \mathrm{Ext}^1_{\mathcal{O}_S}\left(\mathcal{J}_{X/C}, \mathcal{O}_S\right) \to \dots$$

The depth of $\mathcal{O}_S$ is m and since $\mathcal{J}_{X/C}$ is a fractional ideal of $\mathcal{O}_X$, the dimension of $\mathcal{J}_{X/C}$ is $m - k = \dim \mathcal{O}_X$. Thus, by Ischebeck's lemma (see [Mat80, 15.E]), we have $\mathrm{Ext}^1_{\mathcal{O}_S}\left(\mathcal{J}_{X/C}, \mathcal{O}_S\right) = 0$. Hence the result. □

Notation 3.10. Let $I \subset \mathcal{M}_C$ be an ideal. We set $I^\vee = \mathrm{Hom}_{\mathcal{O}_C}(I, \mathcal{O}_C)$.

Proposition 3.11. *We have $\mathcal{J}_{X/C}^\vee \simeq \mathcal{R}_X$. In particular, if C is a reduced complete intersection, $\mathcal{J}_C^\vee = \mathcal{R}_C$.*

Proof. We assume $k \geqslant 2$. For $k = 1$, we refer to [GS14, Proposition 3.4].

We consider the double complex $\mathrm{Hom}_{\mathcal{O}_S}\left(\mathrm{Der}^k(-\log X/C) \hookrightarrow \Theta_S^k, f : \Sigma \to \mathcal{O}_S\right)$, which gives almost the same diagram as the dual of (3.8) in [GS14]. By lemma 3.9, $\mathrm{Ext}^1_{\mathcal{O}_S}(\mathcal{J}_{X/C}, \mathcal{O}_S) = 0$, so that we obtain the following commutative diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & \\
 & 0 \arrow[r] & \mathrm{Hom}_{\mathcal{O}_S}\left(\Theta_S^k, \Sigma\right) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{O}_S}\left(\mathrm{Der}^k(-\log X/C), \Sigma\right) \arrow[r] \arrow[d, "\cdot f"] & \mathrm{Ext}^1_{\mathcal{O}_S}\left(\mathcal{J}_{X/C}, \Sigma\right) \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{O}_S}\left(\Theta_S^k, \mathcal{O}_S\right) \arrow[r, "\lambda"] \arrow[d, "\pi"] & \mathrm{Hom}_{\mathcal{O}_S}\left(\mathrm{Der}^k(-\log X/C), \mathcal{O}_S\right) \arrow[r] \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathrm{Hom}_{\mathcal{O}_S}\left(\mathcal{J}_{X/C}, \mathcal{O}_C\right) \arrow[r, "\sigma"] \arrow[d] & \mathrm{Hom}_{\mathcal{O}_S}\left(\Theta_S^k, \mathcal{O}_C\right) \arrow[r, "\mu"] \arrow[d] & \mathrm{Hom}_{\mathcal{O}_S}\left(\mathrm{Der}^k(-\log X/C), \mathcal{O}_C\right) \arrow[r] \arrow[d] & \mathrm{Ext}^1_{\mathcal{O}_S}\left(\mathcal{J}_{X/C}, \mathcal{O}_C\right) \arrow[r] \arrow[d] & 0 \\
 & \mathrm{Ext}^1_{\mathcal{O}_S}\left(\mathcal{J}_{X/C}, \Sigma\right) \arrow[r] \arrow[d] & 0 \arrow[r] & \mathrm{Ext}^1_{\mathcal{O}_S}\left(\mathrm{Der}^k(-\log X/C), \Sigma\right) \arrow[r] & \mathrm{Ext}^2_{\mathcal{O}_S}\left(\mathcal{J}_{X/C}, \Sigma\right) \arrow[r] & 0 \\
 & 0 & & & &
\end{tikzcd}$$

Let $\varphi : \mathrm{Der}^k(-\log X/C) \to \Sigma$. Thanks to the isomorphism β of the proof of proposition 3.3, φ coresponds to $\omega = \frac{1}{f}\sum_I \varphi(f\partial x_I)\mathrm{d}x_I \in \Omega^k(\log X/C)$, and $\varphi(\delta) = \omega(\delta)$.

By a diagram chasing process, we obtain the map:

$$\begin{aligned} \Omega^k(\log X/C) &\to \mathrm{Hom}_{\mathcal{O}_S}\left(\mathcal{J}_{X/C}, \mathcal{O}_C\right) \\ \omega &\mapsto \left(\overline{a} \mapsto \overline{\mathrm{res}_C(\omega)a}\right). \end{aligned}$$

The map $\overline{a} \mapsto \overline{\mathrm{res}_C(\omega)a} \in \mathcal{O}_C$ is well defined since by proposition 2.39, $\mathrm{res}_C(\omega)|_Y = 0$.

Similarly, the same diagram chasing process starting from the lower left $\mathrm{Hom}_{\mathcal{O}_S}\left(\mathcal{J}_{X/C}, \mathcal{O}_C\right)$ to the upper right $\mathrm{Hom}_{\mathcal{O}_S}\left(\mathrm{Der}^k(-\log X/C), \Sigma\right)$, show that the map

$$\begin{aligned} \theta : \mathcal{R}_X &\to \mathcal{J}_{X/C}^{\vee} \\ \rho \mapsto \theta_\rho &: \begin{cases} \mathcal{J}_{X/C} \to \mathcal{O}_C \\ \overline{a} \mapsto \overline{\rho a} \end{cases} \end{aligned}$$

is an isomorphism. □

4. Freeness for Cohen-Macaulay spaces

We prove here our main result, namely, theorem 4.6, which is a characterization of freeness for Cohen-Macaulay spaces by the projective dimension of the module $\Omega^k(\log X/C)$, which generalizes the hypersurface case.

4.1. **Definition and statements.** In [Sai80], K. Saito studies a particular family of hypersurfaces for which the module of logarithmic vector fields is free. This kind of hypersurfaces are called *free divisors*. Among the different characterizations of free divisors, let us mention the following one:

Theorem 4.1 ([Ale88])**.** *The germ of a reduced singular divisor is free if and only if its singular locus is Cohen Macaulay of codimension* 1 *in* D*. Equivalently, a reduced divisor is free if and only if the Jacobian ideal is Cohen-Macaulay.*

Our purpose here is to extend the notion and characterizations of freeness to Cohen-Macaulay spaces. We give the following definition for freeness, which is inspired by theorem 4.1:

Definition 4.2. *A reduced Cohen-Macaulay space* X *contained in a reduced complete intersection* C *of the same dimension is called* free *if* $\mathcal{J}_{X/C}$ *is Cohen-Macaulay.*

Remark 4.3. Definition 4.2 generalizes the notion of freeness for complete intersections defined in [GS12, Definition 5.1].

Proposition 4.4. *If there exists a reduced complete intersection* C *of dimension* n *containing* X *such that* $\mathcal{J}_{X/C}$ *is Cohen-Macaulay, then for all reduced complete intersection* C' *of dimension* n *containing* X*,* $\mathcal{J}_{X/C'}$ *is Cohen-Macaulay. In other words, the notion of freeness does not depend on the choice of* C*.*

Proof. Let C and C' be two reduced complete intersections of dimension n containing X. Let C'' be a reduced complete intersection containing C and C'. Let A be a transition matrix from $(f_1, \dots, f_k)$ to $(f_1'', \dots, f_k'')$, where $(f_1'', \dots, f_k'')$ are defining equations for C''. There exists $\nu \in \mathcal{I}_C\Omega_S^k$ such that:

$$\mathrm{d}f_1'' \wedge \cdots \wedge \mathrm{d}f_k'' = \det(A)\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k + \nu.$$

Therefore, we have: $\mathcal{J}_{X/C''} = \det(A)\mathcal{J}_{X/C} \subseteq \mathcal{O}_X$. In addition, since $X \subseteq C \cap C''$ and X is not included in the singular locus of C or C'', $\det(A)$ is a non zero divisor of $\mathcal{O}_X$. Therefore, $\mathcal{J}_{X/C}$ and $\mathcal{J}_{X/C''}$ are isomorphic. Similarly, $\mathcal{J}_{X/C'}$ and $\mathcal{J}_{X/C''}$ are isomorphic, which gives us the result. □

We now state our main results:

Proposition 4.5. *Let* X *be a reduced Cohen-Macaulay space of codimension* k *in* S*, and* C *be a reduced complete intersection of codimension* k *containing* X*. The following statements are equivalent:*

(1) X is free,
(2) $\mathcal{O}_X/\mathcal{J}_{X/C}$ is Cohen-Macaulay of dimension $m-k-1$ or $\mathcal{O}_X/\mathcal{J}_{X/C}=(0)$,
(3) $\operatorname{projdim}(\operatorname{Der}^k(-\log X/C)) \leqslant k-1$,
(4) $\operatorname{projdim}(\operatorname{Der}^k(-\log X/C)) = k-1$.

The first three equivalences are given in [GS12] for reduced complete intersections.
We add to this list other characterizations, which leads to the following theorem:

Theorem 4.6. *We can extend the list of equivalences of proposition 4.5 with:*
(5) $\operatorname{projdim}\left(\Omega^k(\log X/C)\right) \leqslant k-1$,
(6) $\operatorname{projdim}\left(\Omega^k(\log X/C)\right) = k-1$.

In particular, for $k=1$, we recognize the several characterizations of freeness for divisors we mentioned before. In the hypersurface case, the duality between $\operatorname{Der}(-\log D)$ and $\Omega^1(\log D)$ gives immediately the fact that if one of the two modules is free, the other is also free, whereas for for Cohen-Macaulay spaces of codimension greater than 2, the statement on the projective dimension of $\Omega^k(\log X/C)$ needs much more work, that's why we consider it separately from the others.

4.2. **Proof of proposition 4.5.** We prove the equivalences of proposition 4.5. They are based on the depth lemma as stated in [dJP00, Lemma 6.5.18] and the Auslander-Buchsbaum formula.

The equivalence 1. $\iff$ 2. is proved in [Sch16, Proposition 5.6] for Gorenstein spaces. Our proof is completely similar.

If $\mathcal{J}_{X/C}=\mathcal{O}_X$, the statement is clear. Let us assume that $\mathcal{J}_{X/C}\neq\mathcal{O}_X$.

Let us consider the following exact sequence of $\mathcal{O}_X$-modules:

$$0 \to \mathcal{J}_{X/C} \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{J}_{X/C} \to 0. \tag{15}$$

By assumption, $\mathcal{O}_X$ is Cohen-Macaulay of dimension n. Moreover, since we assume C to be reduced, the singular locus of C is of dimension at most $n-1$, and therefore the depth of $\mathcal{O}_X/\mathcal{J}_{X/C}$ is at most $n-1$. We deduce from the depth lemma that $\operatorname{depth}(\mathcal{J}_{X/C})=n \iff \operatorname{depth}(\mathcal{O}_X/\mathcal{J}_{X/C})=n-1$.

We now prove 1. $\Rightarrow$ 4. We recall the following exact sequence:

$$0 \to \operatorname{Der}^k(-\log X/C) \to \Theta_S^k \to \mathcal{J}_{X/C} \to 0. \tag{16}$$

Then, thanks to the depth lemma, since the depth of $\mathcal{J}_{X/C}$ is $m-k$ and the depth of Θ_S^k is m, we have $\operatorname{depth}(\operatorname{Der}^k(-\log X/C)) = m-k+1$. By the Auslander-Buchsbaum formula, we have $\operatorname{projdim}(\operatorname{Der}^k(-\log X/C)) = k-1$.

The implication 4 $\Rightarrow$ 3. is trivial.

Let us prove 3. $\Rightarrow$ 1. This implication is mentioned in [GS12] for complete intersections. By the Auslander-Buchsbaum formula, $\operatorname{depth}(\operatorname{Der}^k(-\log X/C)) \geqslant m-k+1$. In addition, we have $\operatorname{depth}(\Theta_S^k)=m$, and $\operatorname{depth}(\mathcal{J}_{X/C}) \leqslant m-k$. As a consequence of the exact sequence (16) and of the depth lemma we have $\operatorname{depth}(\operatorname{Der}^k(-\log X/C)) = m-k+1$ and $\operatorname{depth}(\mathcal{J}_{X/C})=m-k$, so that $\mathcal{J}_{X/C}$ is maximal Cohen-Macaulay.

4.3. **Preliminary to the proof of theorem 4.6.** The methods used to prove proposition 4.5 applied to the short exact sequence (17) are not sufficient to prove directly theorem 4.6.

$$0 \to \widetilde{\Omega}^k_{\underline{f}} \to \Omega^k(\log X/C) \to \mathcal{R}_X \to 0. \tag{17}$$

The proof of theorem 4.6 is based on the explicit computation of some modules and morphisms of the long exact sequence obtained by applying the functor $\operatorname{Hom}_{\mathcal{O}_S}(-,\mathcal{O}_S)$ to the short exact sequence (17):

$$0 \to \operatorname{Hom}_{\mathcal{O}_S}(\mathcal{R}_X,\mathcal{O}_S) \to \operatorname{Hom}_{\mathcal{O}_S}(\Omega^k(\log X/C),\mathcal{O}_S) \to \operatorname{Hom}_{\mathcal{O}_S}(\widetilde{\Omega}^k_{\underline{f}},\mathcal{O}_S) \to \operatorname{Ext}^1_{\mathcal{O}_S}(\mathcal{R}_X,\mathcal{O}_S) \to \dots \tag{18}$$

The structure of the proof is the following. Thanks to the Koszul complex, we compute the modules $\operatorname{Ext}^q_{\mathcal{O}_S}\left(\widetilde{\Omega}^k_{\underline{f}},\mathcal{O}_S\right)$. We then determine the modules $\operatorname{Ext}^q_{\mathcal{O}_S}(\mathcal{R}_X,\mathcal{O}_S)$ for $q\leqslant k$ using the change

of rings spectral sequence. The most technical part is the explicit computation of the connecting morphism

$$\alpha' : \mathrm{Ext}^{k-1}_{\mathcal{O}_S}\left(\widetilde{\Omega}^k_{\underline{f}}, \mathcal{O}_S\right) \to \mathrm{Ext}^k_{\mathcal{O}_S}\left(\mathcal{R}_X, \mathcal{O}_S\right).$$

This computation is necessary in order to identify the kernel and the image of α', which are used in the end of the proof.

4.3.1. We first compute the terms $\mathrm{Ext}^q_{\mathcal{O}_S}\left(\widetilde{\Omega}^k_{\underline{f}}, \mathcal{O}_S\right)$ of the long exact sequence (18). We use the Koszul complex associated to the regular sequence $(f_1, \dots, f_k)$.

Notation 4.7. We denote by $K(\underline{f})$ the Koszul complex of $(f_1, \dots, f_k)$ in $\mathcal{O}_S$:

$$K(\underline{f}) \ : \ 0 \to \bigwedge^k \mathcal{O}_S^k \xrightarrow{d_k} \cdots \xrightarrow{d_2} \bigwedge^1 \mathcal{O}_S^k \xrightarrow{d_1} \mathcal{O}_S \to 0. \tag{19}$$

We also set $\widetilde{K}(\underline{f})$ the complex obtained from $K(\underline{f})$ obtained by removing the last $\mathcal{O}_S$.

Lemma 4.8 ([Eis95, Corollary 17.5, proposition 17.15])**.** *Since the sequence $(f_1, \dots, f_k)$ is regular, $K(\underline{f})$ is a free $\mathcal{O}_S$-resolution of $\mathcal{O}_C$.*

The dual complex $\mathrm{Hom}_{\mathcal{O}_S}(K(\underline{f}), \mathcal{O}_S)$ of the Koszul complex is a free resolution of $\mathcal{O}_C$.

Remark 4.9. A consequence of lemma 4.8 is that $\widetilde{K}(\underline{f})$ gives a free $\mathcal{O}_S$-resolution of $\Sigma \simeq \mathcal{I}_C$.

We can therefore use the complex $\widetilde{K}(\underline{f})$ to compute the modules $\mathrm{Ext}^\bullet_{\mathcal{O}_S}\left(\widetilde{\Omega}^k_{\underline{f}}, \mathcal{O}_S\right)$:

Lemma 4.10. *We assume $k \geqslant 2$. The projective dimension of $\widetilde{\Omega}^k_{\underline{f}}$ is $k-1$. Moreover, we have $\mathrm{Hom}_{\mathcal{O}_S}\left(\widetilde{\Omega}^k_{\underline{f}}, \mathcal{O}_S\right) = f\Theta^k_S$, $\mathrm{Ext}^{k-1}_{\mathcal{O}_S}\left(\widetilde{\Omega}^k_{\underline{f}}, \mathcal{O}_S\right) = \Theta^k_S \otimes_{\mathcal{O}_S} \mathcal{O}_C$, and for all $j \notin \{0, k-1\}$, $\mathrm{Ext}^j_{\mathcal{O}_S}\left(\widetilde{\Omega}^k_{\underline{f}}, \mathcal{O}_S\right) = 0$.*

Proof. Since $\widetilde{\Omega}^k_{\underline{f}} = \Omega^k_S \otimes_{\mathcal{O}_S} \Sigma$, we have for all $q \in \mathbb{N}$, $\mathrm{Ext}^q_{\mathcal{O}_S}\left(\widetilde{\Omega}^k_{\underline{f}}, \mathcal{O}_S\right) = \Theta^k_S \otimes_{\mathcal{O}_S} \mathrm{Ext}^q_{\mathcal{O}_S}(\Sigma, \mathcal{O}_S)$.

By remark 4.9, $\mathrm{projdim}(\widetilde{\Omega}^k_{\underline{f}}) = k-1$ and $\mathrm{Ext}^{k-1}_{\mathcal{O}_S}\left(\widetilde{\Omega}^k_{\underline{f}}, \mathcal{O}_S\right) = \Theta^k_S \otimes_{\mathcal{O}_S} \mathcal{O}_C$ and for all $j \notin \{0, k-1\}$, $\mathrm{Ext}^j_{\mathcal{O}_S}\left(\widetilde{\Omega}^k_{\underline{f}}, \mathcal{O}_S\right) = 0$.

In addition, since $\mathrm{Ext}^1_{\mathcal{O}_S}(\mathcal{O}_C, \mathcal{O}_S) = 0$, we deduce from the short exact sequence

$$0 \to \Sigma \xrightarrow{f.} \mathcal{O}_S \to \mathcal{O}_C \to 0$$

that $\mathrm{Hom}_{\mathcal{O}_S}(\mathcal{O}_S, \mathcal{O}_S)$ and $\mathrm{Hom}_{\mathcal{O}_S}(\Sigma, \mathcal{O}_S)$ are isomorphic. More precisely, $\mathrm{Hom}_{\mathcal{O}_S}(\Sigma, \mathcal{O}_S) = f\mathcal{O}_S$. Hence the result. □

4.3.2. We compute the modules $\mathrm{Ext}^q_{\mathcal{O}_S}(\mathcal{R}_X, \mathcal{O}_S)$ for $q \leqslant k$.

To compute the modules involving $\mathcal{R}_X$, we introduce the change of rings spectral sequence (see for example [CE56, Chapter XV and XVI] for details on spectral sequences). The change of rings spectral sequence applied to an $\mathcal{O}_C$-module M and $\mathcal{O}_S$ gives:

$$E_2^{pq} = \mathrm{Ext}^p_{\mathcal{O}_C}\left(M, \mathrm{Ext}^q_{\mathcal{O}_S}(\mathcal{O}_C, \mathcal{O}_S)\right) \Rightarrow \mathrm{Ext}^{p+q}_{\mathcal{O}_S}(M, \mathcal{O}_S). \tag{20}$$

Lemma 4.11. *For all $q < k$, $\mathrm{Ext}^q_{\mathcal{O}_S}(M, \mathcal{O}_S) = 0$ and $\mathrm{Ext}^k_{\mathcal{O}_S}(M, \mathcal{O}_S) = \mathrm{Hom}_{\mathcal{O}_C}(M, \mathcal{O}_C)$.*

Proof. Since $(f_1, \dots, f_k)$ is a regular sequence, we have for all $q \neq k$, $\mathrm{Ext}^q_{\mathcal{O}_S}(\mathcal{O}_C, \mathcal{O}_S) = 0$ and $\mathrm{Ext}^k_{\mathcal{O}_S}(\mathcal{O}_C, \mathcal{O}_S) = \mathcal{O}_C$. Therefore, the only non zero terms of the second sheet of the spectral sequence (20) are the E_2^{pk}, so that the spectral sequence degenerates at rank 2. Hence the result. □

We deduce from the previous propositions the following exact sequence:

Corollary 4.12. *The long exact sequence (18) gives:*

$$\cdots \to 0 \to \mathrm{Ext}^{k-1}_{\mathcal{O}_S}\left(\Omega^k(\log X/C), \mathcal{O}_S\right) \to \Theta^k_S \otimes_{\mathcal{O}_S} \mathcal{O}_C \xrightarrow{\alpha} \mathcal{R}^\vee_X \to \mathrm{Ext}^k_{\mathcal{O}_S}\left(\Omega^k(\log X/C), \mathcal{O}_S\right) \to 0 \to \dots \tag{21}$$

where $\mathcal{R}^\vee_X = \mathrm{Hom}_{\mathcal{O}_C}\left(\mathcal{R}_X, \mathcal{O}_C\right)$.

4.3.3. *Computation of the connecting morphism.* The previous results show that there exist isomorphisms β and β' such that the following diagram is commutative:

$$\begin{tikzcd}
\Theta^k_S \otimes_{\mathcal{O}_S} \mathcal{O}_C \arrow[r, "\alpha"] & \mathcal{R}^\vee_X \\
\mathrm{Ext}^{k-1}_{\mathcal{O}_S}\left(\widetilde{\Omega}^k_{\underline{f}}, \mathcal{O}_S\right) \arrow[u, "\beta'"] \arrow[r, "\alpha'"'] & \mathrm{Ext}^k_{\mathcal{O}_S}\left(\mathcal{R}_X, \mathcal{O}_S\right) \arrow[u, "\beta"']
\end{tikzcd}$$

We recall that c_X is the fundamental form of X (see notation 2.36). In particular, if X is a complete intersection defined by $(h_1, \dots, h_k)$, we have $c_X = \mathrm{d}h_1 \wedge \dots \wedge \mathrm{d}h_k$.

The purpose of this subsection is to prove the following proposition:

Proposition 4.13. *The connecting morphism of the exact sequence of corollary 4.12 is:*

$$\begin{aligned} \alpha : \Theta^k_S \otimes_{\mathcal{O}_S} \mathcal{O}_C &\to \mathcal{R}^\vee_X \\ \delta \otimes \overline{a} &\mapsto a \cdot \delta(c_X) \end{aligned}$$

In particular, the image of α *is* $\mathcal{J}_{X/C}$.

Thanks to this proposition, we are able to compare $\mathcal{J}_{X/C}$ and $\mathcal{R}^\vee_X$, which is used in the end of the proof of theorem 4.6.

The computation of α is quite technical. We determine explicitly the isomorphisms β and β', and the connecting morphism α'.

We fix an injective resolution $(\mathcal{I}^\bullet, \varepsilon_\bullet)$ of $\mathcal{O}_S$.

Lemma 4.14. *Let* M *be a finite type* $\mathcal{O}_C$*-module. The isomorphism of lemma 4.11 is:*

$$\begin{aligned} \beta : \mathrm{Ext}^k_{\mathcal{O}_S}\left(M, \mathcal{O}_S\right) = H^k\left(\mathrm{Hom}_{\mathcal{O}_S}\left(M, \mathcal{I}^\bullet\right)\right) &\to \mathrm{Hom}_{\mathcal{O}_C}\left(M, H^k\left(\mathrm{Hom}_{\mathcal{O}_S}\left(\mathcal{O}_C, \mathcal{I}^\bullet\right)\right)\right) = \mathrm{Hom}_{\mathcal{O}_C}\left(M, \mathcal{O}_C\right) \\ [\psi] &\mapsto \left(\widetilde{\psi} : \rho \mapsto [\widetilde{\psi}_\rho : \overline{a} \mapsto a.\psi(\rho)]\right) \end{aligned}$$

Proof. Let (P_p, δ) be a free $\mathcal{O}_C$-resolution of M. There are two spectral sequences associated to the double complex $A^{pq} = \mathrm{Hom}_{\mathcal{O}_C}\left(P_p, \mathrm{Hom}_{\mathcal{O}_S}\left(\mathcal{O}_C, \mathcal{I}^q\right)\right)$. The announced isomorphism follows from the definitions of the spectral sequences (see [CE56, Chapter XV and XVI]) and the fact that both degenerate at rank two. □

Lemma 4.15. *The following map is the isomorphism of lemma 4.10:*

$$\begin{aligned} \beta' : \underbrace{H^{k-1}\left(\mathrm{Hom}_{\mathcal{O}_S}\left(\widetilde{\Omega}^k_{\underline{f}}, \mathcal{I}^\bullet\right)\right)}_{=\mathrm{Ext}^{k-1}_{\mathcal{O}_S}\left(\widetilde{\Omega}^k_{\underline{f}}, \mathcal{O}_S\right)} &\to \Theta^k_S \otimes_{\mathcal{O}_S} \underbrace{H^{k-1}\left(\mathcal{I}^\bullet / \mathrm{Ann}_{\mathcal{I}^\bullet}(f_1, \dots, f_k)\right)}_{=\mathcal{O}_C} \\ [\varphi] &\mapsto \sum_I \partial x_I \otimes [\overline{m_I}] \end{aligned}$$

where $m_I \in \mathcal{I}^{k-1}$ *satisfies* $f \cdot m_I = \varphi(\mathrm{d}x_I)$.

Proof. For all $j \in \mathbb{N}$, there is an isomorphism $\zeta : \mathrm{Hom}_{\mathcal{O}_S}\left(\widetilde{\Omega}^k_{\underline{f}}, \mathcal{I}^j\right) \to \Theta^k_S \otimes_{\mathcal{O}_S} \mathrm{Hom}_{\mathcal{O}_S}\left(\Sigma, \mathcal{I}^j\right)$ given by $\zeta(\varphi) = \sum_I \partial x_I \otimes (a \mapsto \varphi(a\mathrm{d}x_I))$.

Since $0 \to \Sigma \xrightarrow{f} \mathcal{O}_S \to \mathcal{O}_C \to 0$ is exact and $\mathcal{I}^j$ is injective, the following map is an isomorphism:

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}_S}\left(\mathcal{O}_S, \mathcal{I}^j\right) / \mathrm{Hom}_{\mathcal{O}_S}\left(\mathcal{O}_C, \mathcal{I}^j\right) &\to \mathrm{Hom}_{\mathcal{O}_S}\left(\Sigma, \mathcal{I}^j\right) \\ \left[\varphi : \mathcal{O}_S \to \mathcal{I}^j\right] &\mapsto (a \mapsto \varphi(f \cdot a)) \end{aligned}$$

Moreover, $\mathrm{Hom}_{\mathcal{O}_S}\left(\mathcal{O}_S, \mathcal{I}^j\right) \simeq \mathcal{I}^j$ and $\mathrm{Hom}_{\mathcal{O}_S}\left(\mathcal{O}_C, \mathcal{I}^j\right) \simeq \mathrm{Ann}_{\mathcal{I}^j}(f_1, \ldots, f_k)$, so that we obtain the isomorphism

$$\begin{aligned} \xi : \mathcal{I}^j/\mathrm{Ann}_{\mathcal{I}^j}(f_1, \ldots, f_k) &\to \mathrm{Hom}_{\mathcal{O}_S}\left(\Sigma, \mathcal{I}^j\right) \\ [\overline{m}] &\mapsto (a \mapsto a \cdot fm) \end{aligned}$$

Using the isomorphisms ζ and ξ^{-1} we obtain the isomorphism β' announced in the statement of lemma 4.15. □

As we mention before in lemma 4.10, $H^{k-1}\left(\mathrm{Hom}_{\mathcal{O}_S}(\Sigma, \mathcal{O}_S)\right) = \mathcal{O}_C$. Therefore, there exists an isomorphism $\gamma_1 : H^{k-1}\left(\mathcal{I}^\bullet/\mathrm{Ann}_{\mathcal{I}^\bullet}(f_1, \ldots, f_k)\right) \to \mathcal{O}_C$.

Moreover, since for all $j \in \mathbb{N}$, $\mathrm{Ann}_{\mathcal{I}^j}(f_1, \ldots, f_k)$ is isomorphic to $\mathrm{Hom}_{\mathcal{O}_S}(\mathcal{O}_C, \mathcal{I}^j)$, we obtain an isomorphism $\gamma_2 : H^k\left(\mathrm{Ann}_{\mathcal{I}^\bullet}(f_1, \ldots, f_k)\right) \to \mathcal{O}_C$.

The following lemma gives the isomorphism between the modules $H^{k-1}\left(\mathcal{I}^\bullet/\mathrm{Ann}_{\mathcal{I}^\bullet}(f_1, \ldots, f_k)\right)$ and $H^k\left(\mathrm{Ann}_{\mathcal{I}^\bullet}(f_1, \ldots, f_k)\right)$.

Lemma 4.16. *The following map is an isomorphism:*

$$\begin{aligned} \gamma : H^{k-1}\left(\mathcal{I}^\bullet/\mathrm{Ann}_{\mathcal{I}^\bullet}(f_1, \ldots, f_k)\right) &\to H^k\left(\mathrm{Ann}_{\mathcal{I}^\bullet}(f_1, \ldots, f_k)\right) \\ [\overline{m}] &\mapsto [\varepsilon_{k-1}(m)] \end{aligned}$$

Proof. Let us denote $\overline{\varepsilon_{k-1}} : \mathcal{I}^{k-1}/\mathrm{Ann}_{\mathcal{I}^{k-1}}(f_1, \ldots, f_k) \to \mathcal{I}^k/\mathrm{Ann}_{\mathcal{I}^k}(f_1, \ldots, f_k)$. We first prove that γ is well defined.

If $\overline{m} \in \mathrm{Ker}(\overline{\varepsilon_{k-1}})$ then $\varepsilon_{k-1}(m) \in \mathrm{Ann}_{\mathcal{I}^k}(f_1, \ldots, f_k)$. If $\overline{m} = \overline{\varepsilon_{k-2}}(\overline{m'})$ for an element $\overline{m'} \in \mathcal{I}^{k-2}/\mathrm{Ann}_{\mathcal{I}^{k-2}}(f_1, \ldots, f_k)$, then $[\varepsilon_{k-1}(\varepsilon_{k-2}(m'))] = 0$ so that the map γ is well defined.

Let us assume that $[\varepsilon_{k-1}(m)] = 0$. Then, there exists $m' \in \mathrm{Ann}_{\mathcal{I}^{k-1}}(f_1, \ldots, f_k)$ such that $\varepsilon_{k-1}(m) = \varepsilon_{k-1}(m')$, so that $m - m' \in \mathrm{Ker}(\varepsilon_{k-1}) = \mathrm{Im}(\varepsilon_{k-2})$. Hence $[\overline{m}] = 0$, therefore, the map γ is injective. Let us consider $[m] \in H^k\left(\mathrm{Ann}_{\mathcal{I}^\bullet}(f_1, \ldots, f_k)\right)$. Then $\varepsilon_k(m) = 0$ thus there exists $m' \in \mathcal{I}^{k-1}$ such that $\varepsilon_{k-1}(m') = m$. Then $[m] = \gamma([\overline{m'}])$ so that γ is surjective. □

We now have all the identifications we need to compute α.

Proof of proposition 4.13. Let us construct explicitly the connecting morphism:

$$\alpha' : H^{k-1}\left(\mathrm{Hom}_{\mathcal{O}_S}\left(\widetilde{\Omega}^k_{\underline{f}}, \mathcal{I}^\bullet\right)\right) \to H^k\left(\mathrm{Hom}_{\mathcal{O}_S}\left(\mathcal{R}_X, \mathcal{I}^\bullet\right)\right).$$

We use a diagram chasing process based on the following commutative diagram:

$$\begin{tikzcd}
0 & \mathrm{Hom}_{\mathcal{O}_S}\left(\widetilde{\Omega}^k_{\underline{f}}, \mathcal{I}^{k-1}\right) \arrow[l] \arrow[d, "\varepsilon_{k-1}"] & \mathrm{Hom}_{\mathcal{O}_S}\left(\Omega^k(\log X/C), \mathcal{I}^{k-1}\right) \arrow[l, "i^*"'] \arrow[d, "\varepsilon_{k-1}"] & \mathrm{Hom}_{\mathcal{O}_S}\left(\mathcal{R}_X, \mathcal{I}^{k-1}\right) \arrow[l, "\mathrm{res}^*_{X/C}"'] \arrow[d, "\varepsilon_{k-1}"] & 0 \arrow[l] \\
0 & \mathrm{Hom}_{\mathcal{O}_S}\left(\widetilde{\Omega}^k_{\underline{f}}, \mathcal{I}^{k}\right) \arrow[l] & \mathrm{Hom}_{\mathcal{O}_S}\left(\Omega^k(\log X/C), \mathcal{I}^{k}\right) \arrow[l, "i^*"'] & \mathrm{Hom}_{\mathcal{O}_S}\left(\mathcal{R}_X, \mathcal{I}^{k}\right) \arrow[l, "\mathrm{res}^*_{X/C}"'] & 0 \arrow[l]
\end{tikzcd}$$

Let $\varphi : \widetilde{\Omega}^k_{\underline{f}} \to \mathcal{I}^{k-1}$ be such that $\varepsilon_{k-1}(\varphi) = 0$. Let $\delta \otimes [\overline{m}] \in \Theta^k_S \otimes H^{k-1}\left(\mathcal{I}^\bullet/\mathrm{Ann}_{\mathcal{I}^\bullet}(f_1, \ldots, f_k)\right)$ be the image of $[\varphi] \in \mathrm{Ext}^{k-1}_{\mathcal{O}_S}\left(\widetilde{\Omega}^k_{\underline{f}}, \mathcal{O}_S\right)$ by β'. In particular, it means that for $\eta \in \widetilde{\Omega}^k_{\underline{f}}$, $\varphi(\eta) = \delta(f\eta) \cdot m$.

There exists $\Phi : \Omega^k(\log X/C) \to \mathcal{I}^{k-1}$ such that $\Phi \circ i = \varphi$. Let $\omega \in \Omega^k(\log X/C)$. By proposition 2.38, there exists g, ξ, η such that $g\omega = \xi \frac{c_X}{f} + \eta$. Then

$$g\Phi(\omega) = \xi\Phi\left(\frac{c_X}{f}\right) + \varphi(\eta).$$

Moreover, for all $i \in \{1, \ldots, k\}$, $f_i\Phi\left(\frac{c_X}{f}\right) = \varphi\left(f_i\frac{c_X}{f}\right) = f_i\delta(c_X) \cdot m$. Therefore,

$$\Phi\left(\frac{c_X}{f}\right) = \delta(c_X) \cdot m + m'$$

with $m' \in \mathrm{Ann}_{\mathcal{I}^{k-1}}(f_1, \ldots, f_k)$.

The image by ε_{k-1} of Φ satisfies:

$$g \cdot \varepsilon_{k-1}(\Phi)(\omega) = \xi \left(\delta(c_X) \cdot \varepsilon_{k-1}(m) + \varepsilon_{k-1}(m')\right).$$

Since $i^*\varepsilon_{k-1}(\Phi) = 0$, there exists $\Psi : \mathcal{R}_X \to \mathcal{I}^k$ such that $\varepsilon_{k-1}(\Phi) = \mathrm{res}^*_{X/C}(\Psi)$. In particular, for all $\rho \in \mathcal{R}_X$, we have[1]:

$$g\Psi(\rho) = g\rho \left(\delta(\mathrm{d}h_1 \wedge \cdots \wedge \mathrm{d}h_k) \cdot \varepsilon_{k-1}(m) + \varepsilon_{k-1}(m')\right).$$

We thus obtain the expression of $g\alpha'(\beta'^{-1}(\delta \otimes [\overline{m}])) \in \mathrm{Ext}^k_{\mathcal{O}_S}(\mathcal{R}_X, \mathcal{O}_S)$.

By the isomorphism β of lemma 4.14, and using the identification of $\mathrm{Hom}_{\mathcal{O}_S}(\mathcal{O}_C, \mathcal{I}^\bullet)$ with $\mathrm{Ann}_{\mathcal{I}^\bullet}(f_1, \ldots, f_k)$, the class of $[g\Psi] \in H^k(\mathrm{Hom}_{\mathcal{O}_S}(\mathcal{R}_X, \mathcal{I}^\bullet))$ corresponds to the map:

$$\begin{aligned} \mathcal{R}_X &\to H^k(\mathrm{Ann}_{\mathcal{I}^\bullet}(f_1, \ldots, f_k)) \\ \rho &\mapsto [g\rho \cdot \left(\delta(c_X) \cdot \varepsilon_{k-1}(m) + \varepsilon_{k-1}(m')\right)] \end{aligned}$$

In addition, since $m' \in \mathrm{Ann}_{\mathcal{I}^{k-1}}(f_1, \ldots, f_k)$, we have for all $\rho \in \mathcal{R}_X$:

$$[g\rho \cdot \left(\delta(c_X) \cdot \varepsilon_{k-1}(m) + \varepsilon_{k-1}(m')\right)] = [g\rho \cdot (\delta(c_X) \cdot \varepsilon_{k-1}(m))]$$

Moreover, we have the isomorphisms:

$$\mathcal{O}_C \xleftarrow{\gamma_1} H^{k-1}(\mathcal{I}^\bullet/\mathrm{Ann}_{\mathcal{I}^\bullet}(f_1, \ldots, f_k)) \xrightarrow{\gamma} H^k(\mathrm{Ann}_{\mathcal{I}^\bullet}(f_1, \ldots, f_k)) \xrightarrow{\gamma_2} \mathcal{O}_C$$

Let $\overline{a} = \gamma_1([\overline{m}]) \in \mathcal{O}_C$. Since $\gamma, \gamma_1, \gamma_2$ are isomorphisms, we can assume that $\gamma_2 \circ \gamma \circ \gamma_1^{-1}(\overline{1}) = \overline{1}$, so that $\gamma_2([\varepsilon_{k-1}(m)]) = \overline{a} \in \mathcal{O}_C$.

Consequently, $[g\Psi]$ is identified with the map $\begin{cases} \mathcal{R}_X \to \mathcal{O}_C \\ \rho \mapsto \overline{\rho g \delta(c_X) a} \end{cases}$, and since g is a non zero divisor in $\mathcal{O}_C$, the map $[\Psi]$ is identified with:

$$\begin{aligned} \mathcal{R}_X &\to \mathcal{O}_C \\ \rho &\mapsto \overline{\rho \delta(c_X) a} \end{aligned}$$

Hence the result: let $\delta \otimes \overline{a} \in \Theta_S \otimes_{\mathcal{O}_S} \mathcal{O}_C$, then $\alpha(\delta \otimes \overline{a}) = \overline{a} \cdot \delta(c_X)$. □

4.4. **End of the proof of theorem 4.6.** We also need the following results, which are obtained from the following short exact sequence by using similar methods as the ones used in the proof of proposition 4.5:

$$0 \to \widetilde{\Omega}^k_{\underline{f}} \to \Omega^k(\log X/C) \to \mathcal{R}_X \to 0.$$

We first notice the following property, which is a direct consequence of [Eis95, Theorem 21.21]:

Lemma 4.17. *If X is a free Cohen-Macaulay space, then $\mathcal{R}_X$ is a maximal Cohen-Macaulay module and $\mathcal{R}_X^\vee \simeq \mathcal{J}_{X/C}$.*

Lemma 4.18. *Let X be a reduced Cohen-Macaulay space. If $\mathrm{projdim}(\Omega^k(\log X/C)) \leqslant k-1$, then $\mathcal{R}_X$ is a maximal Cohen-Macaulay module. If X is free, then $\mathrm{projdim}(\Omega^k(\log X/C)) \leqslant k$.*

Proof. Let us consider the exact sequence $0 \to \widetilde{\Omega}^k_{\underline{f}} \to \Omega^k(\log X/C) \to \mathcal{R}_X \to 0$.

If $\mathrm{projdim}(\Omega^k(\log X/C)) \leqslant k-1$, by Auslander-Buchsbaum formula, $\mathrm{depth}(\Omega^k(\log X/C)) \geqslant m-k+1$. Since $\mathrm{depth}(\widetilde{\Omega}^k_{\underline{f}}) = m-k+1$ and $\mathrm{depth}(\mathcal{R}_X) \leqslant m-k$, by the depth lemma, $\mathrm{depth}(\mathcal{R}_X) = m-k = \dim(\mathcal{R}_X)$.

If X is free, by lemma 4.17, we have $\mathrm{depth}(\mathcal{R}_X) = m-k$. By lemma 4.10 and Auslander Buchsbaum Formula, we have $\mathrm{depth}(\widetilde{\Omega}^k_{\underline{f}}) = m-k+1$. Therefore, by the depth lemma, $\mathrm{depth}(\Omega^k(\log X/C)) \geqslant m-k$ and $\mathrm{projdim}(\Omega^k(\log X/C)) \leqslant k$. □

Thanks to the explicit computation of the connecting morphism α of proposition 4.13, we are able to compare $\mathrm{Im}(\alpha) = \mathcal{J}_{X/C}$ and $\mathcal{R}_X^\vee$, so that we can finish the proof of theorem 4.6, using lemma 4.18.

[1] We notice that $\varepsilon_{k-1}(m)$ and $\varepsilon_{k-1}(m')$ are canceled by $(f_1, \ldots, f_k)$, so that multiplying by $g\rho \in \mathcal{O}_C$ makes sense.

End of the proof of theorem 4.6. We start with the implication 1. $\Rightarrow$ 6. By lemma 4.18, we have $\operatorname{projdim}(\Omega^k(\log X/C)) \leqslant k$. Moreover, by lemma 4.17, $\mathcal{R}_X^\vee = \mathcal{J}_{X/C}$ so that the map α of proposition 4.13 is surjective. Therefore, we have $\operatorname{Ext}^k_{\mathcal{O}_S}\left(\Omega^k(\log X/C), \mathcal{O}_S\right) = 0$.

Let

$$0 \to \mathcal{O}_S^{\ell_k} \xrightarrow{d_k} \mathcal{O}_S^{\ell_{k-1}} \to \cdots \to \mathcal{O}_S^{\ell_0} \to \Omega^k(\log X/C) \to 0. \tag{22}$$

be a minimal free resolution of $\Omega^k(\log X/C)$. In particular, it means that the coefficients of d_k belongs to the maximal ideal $\mathfrak{m}$ of $\mathcal{O}_S$. We apply the functor $\operatorname{Hom}_{\mathcal{O}_S}(-, \mathcal{O}_S)$ to this resolution, and we identify $\operatorname{Hom}_{\mathcal{O}_S}\left(\mathcal{O}_S^{\ell_j}, \mathcal{O}_S\right)$ with $\mathcal{O}_S^{\ell_j}$. Then $\operatorname{Ext}^k_{\mathcal{O}_S}\left(\Omega^k(\log X/C), \mathcal{O}_S\right)$ is equal to $\mathcal{O}_S^{\ell_k}/\operatorname{Im}({}^t d_k)$. Since it is zero, by Nakayama lemma, $\mathcal{O}_S^{\ell_k} = 0$ and therefore $\operatorname{projdim}(\Omega^k(\log X/C)) \leqslant k-1$.

In addition, since there are relations between the maximal minors of the Jacobian matrix, the map α has a non zero kernel.

Therefore, $\operatorname{Ext}^{k-1}_{\mathcal{O}_S}\left(\Omega^k(\log X/C), \mathcal{O}_S\right) \neq 0$ and $\operatorname{projdim}(\Omega^k(\log X/C)) = k-1$.

The implication 6. $\Rightarrow$ 5. is trivial.

Let us prove 5. $\Rightarrow$ 1.

We assume $\operatorname{projdim}\left(\Omega^k(\log X/C)\right) \leqslant k-1$. The exact sequence (21) becomes:

$$0 \to \operatorname{Ext}^{k-1}_{\mathcal{O}_S}\left(\Omega^k(\log X/C), \mathcal{O}_S\right) \to \Theta_S^k \otimes_{\mathcal{O}_S} \mathcal{O}_C \xrightarrow{\alpha} \mathcal{R}_X^\vee \to 0.$$

Since by proposition 4.13 the image of α is $\mathcal{J}_{X/C}$, we have $\mathcal{R}_X^\vee = \mathcal{J}_{X/C}$. By lemma 4.18, $\mathcal{R}_X$ is a maximal Cohen-Macaulay $\mathcal{O}_C$-module. Therefore, by [Eis95, Theorem 21.21], $\mathcal{J}_{X/C}$ is also maximal Cohen-Macaulay. □

The following corollary gives other characterizations of freeness using the module of multi-residues:

Corollary 4.19. *Let X be a reduced Cohen-Macaulay space contained in a reduced complete intersection C of the same dimension. The following statements are equivalent:*

(1) X is free,
(2) $\operatorname{projdim}(\mathcal{R}_X) \neq \operatorname{projdim}(\Omega^k(\log X/C))$,
(3) $\mathcal{R}_X$ is Cohen-Macaulay and $\operatorname{Hom}_{\mathcal{O}_C}(\mathcal{R}_X, \mathcal{O}_C) \simeq \mathcal{J}_{X/C}$.

Proof. We start with (1) $\iff$ (2). We consider the exact sequence (17). The depth of $\widetilde{\Omega}^k_{\underline{f}}$ is $m-k+1$. Since $\operatorname{depth}(\mathcal{R}_X) \leqslant m-k$, the depth lemma gives that $\operatorname{depth}(\Omega^k(\log X/C)) \neq \operatorname{depth}(\mathcal{R}_X)$ if and only if $\operatorname{depth}(\mathcal{R}_X) = m-k$ and $\operatorname{depth}(\Omega^k(\log X/C)) \geqslant m-k+1$. This is equivalent to the fact that X is free by theorem 4.6 and the Auslander-Buchsbaum formula.

The implication (1) $\Rightarrow$ (3) is given by lemma 4.17. Conversely, if $\operatorname{Hom}_{\mathcal{O}_C}(\mathcal{R}_X, \mathcal{O}_C) = \mathcal{J}_{X/C}$ and $\mathcal{R}_X$ is Cohen-Macaulay, then, by [Eis95, Theorem 21.21], $\mathcal{J}_{X/C}$ is Cohen-Macaulay so that X is free. □

Remark 4.20. The condition that $\mathcal{R}_X$ is Cohen-Macaulay may not be satisfied (see for example [OT95, Example 5.6]).

4.5. **Consequences of freeness.** We give here a relation between the modules $\Omega^k(\log X/C)$ and $\operatorname{Der}^k(-\log X/C)$ which is satisfied by free Cohen-Macaulay spaces.

Corollary 4.21. *Let X be a Cohen-Macaulay space of codimension at least 2. Then:*

$$\operatorname{Hom}_{\mathcal{O}_S}\left(\Omega^k(\log X/C), \mathcal{O}_S\right) = f\Theta_S^k,$$

$$\operatorname{Ext}^{k-1}_{\mathcal{O}_S}\left(\Omega^k(\log X/C), \mathcal{O}_S\right) = \frac{\operatorname{Der}^k(-\log X/C)}{\left(\sum_{i=1}^k f_i \Theta_S^k\right)},$$

and for all $1 \leqslant q \leqslant k-2$, $\operatorname{Ext}^q_{\mathcal{O}_S}\left(\Omega^k(\log X/C), \mathcal{O}_S\right) = 0$. If moreover X is free, then for all $q \geqslant k$, we have $\operatorname{Ext}^q_{\mathcal{O}_S}\left(\Omega^k(\log X/C), \mathcal{O}_S\right) = 0$.

Proof. The beginning of the long exact sequence (18) gives

$$\mathrm{Hom}_{\mathcal{O}_S}\left(\Omega^k(\log X/C),\mathcal{O}_S\right)\simeq \mathrm{Hom}_{\mathcal{O}_S}\left(\widetilde{\Omega}^k_{\underline{f}},\mathcal{O}_S\right)=f\Theta^k_S.$$

Moreover, by lemma 4.11, for all $q\leqslant k-1$, $\mathrm{Ext}^q_{\mathcal{O}_S}(\mathcal{R}_X,\mathcal{O}_S)=0$ and for all $1\leqslant q\leqslant k-2$, $\mathrm{Ext}^q_{\mathcal{O}_S}\left(\widetilde{\Omega}^k_{\underline{f}},\mathcal{O}_S\right)=0$ so that for all $1\leqslant q\leqslant k-2$, $\mathrm{Ext}^q_{\mathcal{O}_S}\left(\Omega^k(\log X/C),\mathcal{O}_S\right)=0$. Thanks to the long exact sequence (21), we see that $\mathrm{Ext}^{k-1}_{\mathcal{O}_S}\left(\Omega^k(\log X/C),\mathcal{O}_S\right)$ is the kernel of the map $\alpha:\Theta^k_S\otimes_{\mathcal{O}_S}\mathcal{O}_C\to\mathcal{J}_{X/C}$ computed in proposition 4.13. It is easy to see that this kernel is $\frac{\mathrm{Der}^k(-\log X/C)}{\left(\sum_{i=1}^k f_i\Theta^k_S\right)}$. In addition, if X is free, $\mathrm{projdim}(\Omega^k(\log X/C))=k-1$ which implies that for all $q>k-1$, $\mathrm{Ext}^q_{\mathcal{O}_S}\left(\Omega^k(\log X/C),\mathcal{O}_S\right)=0$. □

Remark 4.22. If X is not free, the module $\mathrm{Ext}^k_{\mathcal{O}_S}\left(\Omega^k(\log X/C),\mathcal{O}_S\right)$ is isomorphic to $\mathcal{R}^\vee_X/\mathcal{J}_{X/C}$.

Proposition 4.23. *Let X be a reduced Cohen-Macaulay space of codimension at least two. Then:*

$$\mathrm{Hom}_{\mathcal{O}_S}\left(\mathrm{Der}^k(-\log X/C),\mathcal{O}_S\right)=\Omega^k_S,$$

$$\mathrm{Ext}^{k-1}_{\mathcal{O}_S}\left(\mathrm{Der}^k(-\log X/C),\mathcal{O}_S\right)\simeq\mathcal{R}_X,$$

and for all $1\leqslant q\leqslant k-2$, $\mathrm{Ext}^q_{\mathcal{O}_S}\left(\mathrm{Der}^k(-\log X/C),\mathcal{O}_S\right)=0$. If moreover X is free, then for all $q\geqslant k$, we have $\mathrm{Ext}^q_{\mathcal{O}_S}\left(\mathrm{Der}^k(-\log X/C),\mathcal{O}_S\right)=0$.

Proof. We apply the functor $\mathrm{Hom}_{\mathcal{O}_S}(-,\mathcal{O}_S)$ to the following exact sequence:

$$0\to\mathrm{Der}^k(-\log X/C)\to\Theta^k_S\to\mathcal{J}_{X/C}\to 0$$

Since Θ^k_S is free, for all $q\geqslant 1$, $\mathrm{Ext}^q_{\mathcal{O}_S}(\Theta^k_S,\mathcal{O}_S)=0$. In addition, by lemma 4.11, for all $q<k$, $\mathrm{Ext}^q_{\mathcal{O}_S}(\mathcal{J}_{X/C},\mathcal{O}_S)=0$ and $\mathrm{Ext}^k_{\mathcal{O}_S}(\mathcal{J}_{X/C},\mathcal{O}_S)\simeq\mathrm{Hom}_{\mathcal{O}_S}\left(\mathcal{J}_{X/C},\mathcal{O}_C\right)=\mathcal{R}_X$. It gives us the result.

If X is free, by proposition 4.5, $\mathrm{projdim}(\mathrm{Der}^k(-\log X/C))=k-1$ so that for all $q\geqslant k$, we have $\mathrm{Ext}^q_{\mathcal{O}_S}\left(\mathrm{Der}^k(-\log X/C),\mathcal{O}_S\right)=0$. □

5. Complete intersection curves

Several families of Saito free divisors are known: normal crossing divisors, plane curves, discriminants of deformation of isolated singularities... In that case, the module $\Omega^1(\log D)$ is a free $\mathcal{O}_S$-module of rank m, so that we know exactly how many generators are needed, and it depends only on the dimension of the ambient space. Moreover, there is a criterion, known as Saito criterion (see [Sai80, (1.8)]) which can be used to check if a given family is a basis of $\Omega^1(\log D)$.

For Cohen-Macaulay spaces the situation is more complicated. Indeed, if we have a free Cohen-Macaulay space, we proved in theorem 4.6 that the projective dimension of $\Omega^k(\log X/C)$ is $k-1$, but it does not give any information on the minimal number of generators of $\Omega^k(\log X/C)$. In addition, we do not have a criterion as Saito criterion to determine if a given family of multi-logarithmic differential forms generates the module $\Omega^k(\log C)$.

In this section, we describe explicitly a free $\mathcal{O}_S$-resolution of $\Omega^{m-1}(\log C)$ when C is a quasi-homogeneous complete intersection curve. In particular, it gives a minimal generating family of $\Omega^{m-1}(\log C)$.

Let us notice the following property which is easy to prove from the definition of freeness:

Proposition 5.1. *Reduced curves are free Cohen-Macaulay spaces.*

Complete intersection curves are studied in [Pol15]. In this paper, the author proves that the set of values of the module of multi-residues $\mathcal{R}_C$ satisfies a symmetry with the values of the Jacobian ideal, and gives the relation between the values of $\mathcal{R}_C$ and the values of the Kähler differentials. This result is then generalized in [KTS15] for more general curves by considering the dualizing module.

5.1. **Quasi-homogeneous curves.** We describe here explicitly the module of multi-logarithmic differential forms for a quasi-homogeneous complete intersection curve. We will see in example 5.13 that this result can not be generalized to Cohen-Macaulay curves.

We recall the following notations from [Pol15].

Let $C = C_1 \cup \cdots \cup C_p$ be a reduced complete intersection curve with p irreducible components. The normalization of C satisfies $\mathcal{O}_{\widetilde{C}} = \bigoplus_{i=1}^{p} \mathbb{C}\{t_i\}$. It induces for all $i \in \{1, \ldots, p\}$ a valuation map

$$\mathrm{val}_i : \mathcal{M}_C \ni g \mapsto \mathrm{val}_i(g) \in \mathbb{Z} \cup \{\infty\}.$$

The value of an element $g \in \mathcal{M}_C$ is $\mathrm{val}(g) = (\mathrm{val}_1(g), \ldots, \mathrm{val}_p(g)) \in (\mathbb{Z} \cup \{\infty\})^p$. For a fractional ideal $I \subset \mathcal{M}_C$, we set $\mathrm{val}(I) := \{\mathrm{val}(g)\ ;\ g \in I \text{ non zero divisor}\} \subset \mathbb{Z}^p$.

We consider the product order on $\mathbb{Z}^p$, so that for all $\alpha, \beta \in \mathbb{Z}^p$, $\alpha \leqslant \beta$ means that for all $i \in \{1, \ldots, p\}$, $\alpha_i \leqslant \beta_i$. We set $\underline{1} = (1, \ldots, 1)$.

We denote by $\mathcal{C}_C = \mathcal{O}_{\widetilde{C}}^{\vee}$ the conductor ideal. There exists $\gamma \in \mathbb{N}^p$ such that $\mathcal{C}_C = \underline{t}^{\gamma}\mathcal{O}_{\widetilde{C}}$. In particular, $\gamma = \inf\{\alpha \in \mathbb{N}^p; \alpha + \mathbb{N}^p \subseteq \mathrm{val}(\mathcal{O}_C)\}$.

Let C be a reduced complete intersection curve defined by a regular sequence $(h_1, \ldots, h_{m-1})$. Let us consider the following properties:

Conditions 5.2.

a) There exist $(w_1, \ldots, w_m) \in \mathbb{N}^m$ such that for all $i \in \{1, \ldots, m-1\}$, h_i is quasi-homogeneous of degree d_i with respect to the weight $(w_1, \ldots, w_m)$.
b) m is the embedding dimension. Equivalently, for all $i \in \{1, \ldots, m-1\}$, $h_i \in \mathfrak{m}^2$ where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_S$ (see proposition 2.10).

5.1.1. *Generators of* $\Omega^{m-1}(\log C)$.

Lemma 5.3. *Let C be a reduced complete intersection curve satisfying condition a). Then*

$$\omega_0 = \frac{\sum_{i=1}^{m}(-1)^{i-1} w_i x_i \widehat{\mathrm{d}x_i}}{h} \in \Omega^{m-1}(\log D).$$

Proof. Let $i \in \{1, \ldots, m-1\}$. We have $\sum_{k=1}^{m} w_k x_k \dfrac{\partial h_i}{\partial x_k} = d_i h_i$ so that $\mathrm{d}h_i \wedge \omega_0 = \dfrac{d_i}{\widehat{h_i}} \mathrm{d}\underline{x}$. Since $\mathrm{d}h = \sum_{i=1}^{m-1} \widehat{h_i} \mathrm{d}h_i$, we have $\mathrm{d}h \wedge \omega_0 \in \Omega_S^m$. Thus, $\omega_0 \in \Omega^{m-1}(\log D)$. □

Remark 5.4. By proposition 2.22, we also have $\omega_0 \in \Omega^{m-1}(\log C)$.

For $i \in \{1, \ldots, m\}$, we denote by J_i the $(m-1) \times (m-1)$ minor of the Jacobian matrix obtained by removing the column $\left(\frac{\partial h_j}{\partial x_i}\right)_{1 \leqslant j \leqslant m-1}$.

Lemma 5.5. *For all $i \in \{1, \ldots, m\}$, $J_i \mathrm{res}_C(\omega_0) = (-1)^{i-1} w_i x_i$. Let $c_1, \ldots, c_m \in \mathbb{C}$ be such that $\sum_{i=1}^{m} c_i J_i$ induces a non zero divisor of $\mathcal{O}_C$. We thus have*

$$\mathrm{res}_C(\omega_0) = \frac{\sum_{i=1}^{m}(-1)^{i-1} c_i w_i x_i}{\sum_{i=1}^{m} c_i J_i}.$$

Proof. Let $i_0 \in \{1, \ldots, m\}$. We recall that $\mathrm{d}h_1 \wedge \cdots \wedge \mathrm{d}h_{m-1} = \sum_{i=1}^{m} J_i \widehat{\mathrm{d}x_i}$. We have:

$$J_{i_0}\omega_0 = (-1)^{i_0-1} w_{i_0} x_{i_0} \frac{\mathrm{d}h_1 \wedge \cdots \wedge \mathrm{d}h_{m-1}}{h} + \sum_{\substack{i=1 \\ i \neq i_0}}^{m} \underbrace{\left((-1)^{i-1} w_i x_i J_{i_0} - (-1)^{i_0-1} w_{i_0} x_{i_0} J_i\right)}_{=:\lambda_i} \frac{\widehat{\mathrm{d}x_i}}{h}. \tag{23}$$

Let $i \in \{1, \ldots, m\}$, $i \neq i_0$. We develop J_i with respect to the column i_0, and J_{i_0} with respect to the column i. For $\{i_1, i_2\} \subseteq \{1, \ldots, m\}$ and $j \in \{1, \ldots, m-1\}$, we denote by J_{i_1,i_2}^{j} the minor of the Jacobian matrix obtained by removing the columns i_1, i_2 and the line j. We then obtain:

$$\lambda_i = \mathrm{sgn}(i_0 - i) \sum_{\ell=1}^{m-1} (-1)^{\ell-1} \left(w_i x_i \frac{\partial h_\ell}{\partial x_i} + w_{i_0} x_{i_0} \frac{\partial h_\ell}{\partial x_{i_0}} \right) \cdot J_{i_0,i}^{\ell}$$

By developing a convenient determinant, one can prove that for $p \notin i, i_0$ and $\ell \in \{1, \dots, m-1\}$:

$$\sum_{\ell=1}^{m-1} (-1)^{\ell-1} \frac{\partial h_\ell}{\partial x_p} J^\ell_{i,i_0} = 0. \tag{24}$$

We then have:

$$\lambda_i = \operatorname{sgn}(i_0 - i) \sum_{\ell=1}^{m-1} (-1)^{\ell-1} \left(\sum_{p=1}^{m} w_p x_p \frac{\partial h_\ell}{\partial x_p} \right) J^\ell_{i_0,i} = \operatorname{sgn}(i_0 - i) \sum_{\ell=1}^{m-1} (-1)^{\ell-1} d_\ell h_\ell J^\ell_{i_0,i}.$$

Therefore, there exists $\eta \in \widetilde{\Omega}^{m-1}$ such that

$$J_{i_0}\omega_0 = (-1)^{i_0-1} w_{i_0} x_{i_0} \frac{\mathrm{d}h_1 \wedge \cdots \wedge \mathrm{d}h_k}{h_1 \cdots h_k} + \eta. \tag{25}$$

Hence the result. □

Lemma 5.6. *With the notations of lemma 5.3, we have:*

$$\inf(\operatorname{val}(\mathcal{R}_C)) = \operatorname{val}(\operatorname{res}_C(\omega_0)) = -\gamma + \underline{1}$$

Proof. By [Pol15, Proposition 3.30 and (3.18)], we have $\operatorname{val}(J_i) = \gamma + \operatorname{val}(x_i) - \underline{1}$. By considering for all branch C_i of C an index $j(i)$ such that $x_{i(j)|_{C_i}} \neq 0$, we deduce from lemma 5.5 that $\operatorname{val}(\operatorname{res}_C(\omega_0)) = -\gamma + \underline{1}$.

Let us prove that $\inf(\operatorname{val}(\mathcal{R}_C)) = \operatorname{val}(\operatorname{res}_C(\omega_0))$. By [Pol15, Proposition 3.30], we also have $\gamma + \operatorname{val}(\mathcal{O}_C) - \underline{1} \subseteq \operatorname{val}(\mathcal{J}_C)$. Therefore, $2\gamma - \underline{1} + \mathbb{N}^p \subseteq \operatorname{val}(\mathcal{J}_C)$.

As in [Pol15], we set for $v \in \mathbb{Z}^p$, $\Delta_i(v, \mathcal{J}_C) = \{\alpha \in \operatorname{val}(\mathcal{J}_C); \alpha_i = v_i \text{ and } \forall j \neq i, \alpha_j > v_j\}$ and $\Delta(v, I) = \bigcup_{i=1}^p \Delta_i(v, I)$. We then have $\max\{v \in \mathbb{Z}^p; \Delta(v, \mathcal{J}_C) = \emptyset\} \leqslant 2\gamma - \underline{2}$.

By [Pol15, Theorem 2.4] we have $v \in \operatorname{val}(\mathcal{R}_C) \iff \Delta(\gamma - v - \underline{1}, \mathcal{J}_C) = \emptyset$. It implies that $\inf(\operatorname{val}(\mathcal{R}_C)) \geqslant -\gamma + \underline{1}$. Hence the result: $\inf(\operatorname{val}(\mathcal{R}_C)) = \operatorname{val}(\operatorname{res}_C(\omega_0))$. □

Proposition 5.7. *Let C be a singular complete intersection satisfying condition a). Then $\mathcal{R}_C$ is generated by $\operatorname{res}_C\left(\frac{\mathrm{d}h_1 \wedge \cdots \wedge \mathrm{d}h_k}{h}\right) = 1$ and $\operatorname{res}_C(\omega_0)$, where ω_0 is given in lemma 5.3. In addition, this generating family is minimal.*

Proof. We set $Z = \operatorname{Sing}(C)$ the singular locus of C. By dualizing over $\mathcal{O}_C$ the exact sequence $0 \to \mathcal{J}_C \to \mathcal{O}_C \to \mathcal{O}_Z \to 0$, we obtain

$$0 \to \mathcal{O}_C \to \mathcal{R}_C \to \omega_Z \to 0 \tag{26}$$

where ω_Z is the dualizing module of Z. Moreover, the singular locus of a quasi-homogeneous curve is Gorenstein (see [KW84, Satz 2]), so that $\omega_Z = \mathcal{O}_Z$. The exact sequence (26) implies that $\mathcal{R}_C$ is generated by two elements, the image of $1 \in \mathcal{O}_C$, which is $1 \in \mathcal{R}_C$, and the antecedent of $1 \in \mathcal{O}_Z$. Therefore, there exists $\rho_0 \in \mathcal{R}_C$ such that $(1, \rho_0)$ generates $\mathcal{R}_C$.

It remains to prove that we can take $\rho_0 = \operatorname{res}_C(\omega_0)$. By lemma 5.6, $\operatorname{val}(\operatorname{res}_C(\omega_0)) = -\gamma + \underline{1}$.

We assume first that $-\gamma + \underline{1} \notin \mathbb{N}^p$. For example, $-\gamma_1 + 1 < 0$. There exists $\alpha_0, \alpha_1 \in \mathcal{O}_C$ such that $\operatorname{res}_C(\omega_0) = \alpha_0\rho_0 + \alpha_1$. Since $\operatorname{val}(\alpha_1) \geqslant 0$, and $\inf(\operatorname{val}(\mathcal{R}_C)) = \operatorname{val}(\operatorname{res}_C(\omega_0))$, we have $\operatorname{val}_1(\rho_0) = \operatorname{val}_1(\omega_0)$ therefore $\operatorname{val}_1(\alpha_0) = 0$ which implies that $\operatorname{val}(\alpha_0) = 0$ and α_0 is invertible. Thus, $(\operatorname{res}_C(\omega_0), 1)$ generates $\mathcal{R}_C$.

Let us assume that $-\gamma + \underline{1} \in \mathbb{N}^p$. Since $\gamma \geqslant 0$, we must have $\gamma = \underline{1}$ or $\gamma = \underline{0}$. However, if $\gamma = \underline{0}$, we have $\mathcal{O}_C = \mathcal{O}_{\widetilde{C}}$ so that C is smooth. Therefore, $\gamma = \underline{1}$. By [Pol15, Corollary 3.32], we have $\operatorname{val}(\mathcal{J}_C) = \operatorname{val}(\mathcal{C}_C) = \underline{1} + \mathbb{N}^p$. It implies that $\mathcal{J}_C = \mathcal{C}_C$, so that by duality, $\mathcal{R}_C = \mathcal{O}_{\widetilde{C}}$. By [Sch16, Proposition 4.11], it implies that C is a plane normal crossing curve. By Saito criterion [Sai80, (1.8)], if $h = xy$ defines a plane curve C, then $(\omega_0 = \frac{x\mathrm{d}y - y\mathrm{d}x}{h}, \frac{\mathrm{d}h}{h})$ is a basis of $\Omega^1(\log C)$. Hence the result. □

Since $\left\{\frac{h_i \widehat{\mathrm{d}x_j}}{h}; i \in \{1, \dots, m-1\}, j \in \{1, \dots, m\}\right\}$ generates $\widetilde{\Omega}^{m-1}_{\underline{h}}$, proposition 5.7 gives:

Corollary 5.8. *Let C be a singular complete intersection curve satisfying condition a). Then $\Omega^{m-1}(\log C)$ is generated by the multi-logarithmic form ω_0 of lemma 5.3, $\dfrac{\mathrm{d}h_1 \wedge \cdots \wedge \mathrm{d}h_k}{h}$, and the family $\left\{ \frac{h_i \widehat{\mathrm{d}x_j}}{h}; i \in \{1, \ldots, m-1\}, j \in \{1, \ldots, m\} \right\}$.*

We will see in the next section that this generating family is minimal if the conditions 5.2 are satisfied.

5.1.2. *Free resolution.* Since curves are free, lemma 4.17 yields $\operatorname{depth}(\mathcal{R}_C) = 1$, so that by the Auslander-Buchsbaum Formula, the projective dimension of $\mathcal{R}_C$ as a $\mathcal{O}_S$-module is $m-1$. In addition, by theorem 4.6, the projective dimension of $\Omega^{m-1}(\log C)$ is $m-2$.

In corollary 5.8, we give a generating family of $\Omega^{m-1}(\log C)$. We can go further and compute explicitly a free $\mathcal{O}_S$-resolution of $\mathcal{R}_C$ and $\Omega^{m-1}(\log C)$ for a quasi-homogeneous complete intersection curve.

Theorem 5.9. *Let C be a reduced complete intersection curve satisfying the conditions 5.2. We set for $p \in \{0, \ldots, m-2\}$:*

$$F_p = \bigwedge^p \mathcal{O}_S^{m-1} \oplus \bigwedge^p \mathcal{O}_S^m$$

and $F_{m-1} = \bigwedge^{m-1} \mathcal{O}_S^m$.

There exist differentials $\delta_\bullet : F_\bullet \to F_{\bullet-1}$ such that $(F_\bullet, \delta_\bullet)$ is a minimal free resolution of $\mathcal{R}_C$ as a $\mathcal{O}_S$-module.

In particular, the Betti numbers of $\mathcal{R}_C$ as a $\mathcal{O}_S$-module are:

$$\forall p \in \{0, \ldots, m-2\}, b_j(\mathcal{R}_C) = \binom{m-1}{p} + \binom{m}{p} \text{ and } b_{m-1}(\mathcal{R}_C) = m$$

In order to prove this theorem, we introduce the following exact sequence, where the middle module is isomorphic to $\mathcal{R}_C$ and where a free resolution of the two other modules is given by Koszul complexes.

Lemma 5.10. *We keep the hypothesis of theorem 5.9. Let $c_1, \ldots, c_m \in \mathbb{C}$ be such that $g = \sum_{i=1}^m c_i J_i \in \mathcal{O}_S$ induces a non zero divisor in $\mathcal{O}_C$. Let $y = \sum_{i=1}^m (-1)^{i-1} c_i w_i x_i$. In particular, $\mathcal{R}_C \simeq y\mathcal{O}_C + g\mathcal{O}_C$. We have the following exact sequence:*

$$0 \to \mathcal{O}_C \xrightarrow{y} y\mathcal{O}_C + g\mathcal{O}_C \to \frac{(g\mathcal{O}_S + y\mathcal{O}_S + \mathcal{I}_C)}{y\mathcal{O}_S + \mathcal{I}_C} \to 0 \tag{27}$$

In addition, a free resolution of $\dfrac{(g\mathcal{O}_S + y\mathcal{O}_S + \mathcal{I}_C)}{y\mathcal{O}_S + \mathcal{I}_C}$ is given by the Koszul complex associated with the regular sequence $(w_1x_1, \ldots, w_mx_m)$.

Proof. The exact sequence (27) is just a consequence of the fact that y is a non zero divisor of $\mathcal{O}_C$.

We have:

$$0 \to ((y, h_1, \ldots, h_{m-1}) : g)_{\mathcal{O}_S} \to \mathcal{O}_S \xrightarrow{g} \frac{(g\mathcal{O}_S + y\mathcal{O}_S + \mathcal{I}_C)}{y\mathcal{O}_S + \mathcal{I}_C} \to 0.$$

By (25), for all $i \in \{1, \ldots, m\}$, $J_i\omega_0 = (-1)^{i-1} w_i x_i \dfrac{\mathrm{d}h_1 \wedge \cdots \wedge \mathrm{d}h_{m-1}}{h} + \eta_i$ with $\eta_i \in \widetilde{\Omega}^{m-1}$. We have for all $i, j \in \{1, \ldots, m\}$:

$$(-1)^{i-1} w_i x_i J_j = (-1)^{j-1} w_j x_j J_i \mod (h_1, \ldots, h_{m-1}) \tag{28}$$

We thus have $yJ_j = (-1)^{j-1} w_j x_j g \mod (h_1, \ldots, h_{m-1})$. Therefore,

$$(w_1x_1, \ldots, w_mx_m) \subseteq ((y, h_1, \ldots, h_{m-1}) : g)_{\mathcal{O}_S}.$$

Moreover, by proposition 5.7, $(1, \frac{y}{g})$ is a minimal generating family of $\mathcal{R}_C$, thus, $g \notin (y, h_1, \ldots, h_{m-1})$. We thus have $\mathfrak{m} = (w_1x_1, \ldots, w_mx_m) = ((y, h_1, \ldots, h_{m-2}) : J_1)_{\mathcal{O}_S}$. Hence the result. □

Proof of theorem 5.9. We consider the exact sequence (27). A minimal free resolution of $\mathcal{O}_C$ is given by the Koszul complex associated with the regular sequence $(h_1, \dots, h_{m-1})$ and a minimal free resolution of $\frac{(g\mathcal{O}_S + y\mathcal{O}_S + \mathcal{I}_C)}{y\mathcal{O}_S + \mathcal{I}_C}$ is given by the Koszul complex associated to $(w_1x_1, \dots, w_mx_m)$. We deduce from these two resolutions a free resolution (F'_j, δ'_j) of $(y, g)\mathcal{O}_C$, whose length is m. However, the projective dimension of $\mathcal{R}_C$ is $m-1$, so that the free resolution we obtain is not minimal. Thanks to the explicit computation of the differential δ'_j, one can see that all the coefficients of the differentials belongs to the maximal ideal $\mathfrak{m}$ of $\mathcal{O}_S$, except from δ'_m which has an invertible coefficient. A minimization of these free resolution then gives the announced result. The precise expression of the differential can be computed exactly as in [Pol16, Théorème 6.1.29], where we assume that $g = J_1$ is a non zero divisor in $\mathcal{O}_C$. □

Theorem 5.11. *We keep the notations and hypothesis of theorem 5.9. We set for all $j \in \{0, \dots, m-3\}$,*

$$P_j = \left(\bigwedge^{j+1} \mathcal{O}_S^{m-1} \otimes \Omega_S^{m-1} \right) \oplus F_j$$

and $P_{m-2} = F_{m-2}$. There exist $\alpha_\bullet : P_\bullet \to P_{\bullet-1}$ such that $(P_\bullet, \alpha_\bullet)$ is a minimal free resolution of $\Omega^{m-1}(\log C)$.

In particular, the Betti numbers of $\Omega^{m-1}(\log C)$ are for all $j \in \{0, \dots, m-3\}$, $b_j(\Omega^{m-1}(\log C)) = m\binom{m-1}{j+1} + \binom{m-1}{j} + \binom{m}{j}$ and $b_{m-2}(\Omega^{m-1}(\log C)) = m - 1 + \binom{m}{m-2}$.

Proof. We consider the exact sequence

$$0 \to \widetilde{\Omega}^{m-1} \to \Omega^{m-1}(\log C) \to \mathcal{R}_C \to 0.$$

The free resolutions of $\widetilde{\Omega}^{m-1}$ and $\mathcal{R}_C$ induce a free resolution $(P'_\bullet, \alpha'_\bullet)$ of $\Omega^{m-1}(\log C)$, whose length is $m-1$. By theorem 4.6, the projective dimension of $\Omega^{m-1}(\log C)$ is $m-2$ since C is free. Therefore, the previous free resolution of $\Omega^{m-1}(\log C)$ is not minimal. A minimization gives the announced result. The expression of the differentials can be found in [Pol16, Théorème 6.1.33] □

The following example shows that theorems 5.9 and 5.11 can not be generalized to Cohen-Macaulay quasi-homogeneous curves. Let us notice that for a space quasi-homogeneous complete intersection curve, a free resolution of $\Omega^2(\log C)$ is:

$$0 \to \mathcal{O}_S^5 \to \mathcal{O}_S^8 \to \Omega^2(\log C) \to 0.$$

Remark 5.12. For a complete intersection curve $C \subseteq \mathbb{C}^3$, a free resolution of $\mathrm{Der}^2(-\log C)$ is given in [GS12, Proposition 5.5].

Example 5.13. Let us consider the curve X of $\mathbb{C}^3$ parametrized by (t^3, t^4, t^5). This curve is a quasi-homogeneous curve which is not Gorenstein. The reduced ideal defining X is the ideal generated by $h_1 = xz - y^2$, $h_2 = x^3 - yz$ and $h_3 = x^2y - z^2$. We set C the reduced complete intersection defined by $(xz - y^2, x^3 - yz)$. A computation made with Singular gives the following minimal free resolution of $\Omega^2(\log X/C)$:

$$0 \to \mathcal{O}_S^6 \to \mathcal{O}_S^9 \to \Omega^2(\log X/C) \to 0.$$

In particular, the number of generators of $\mathcal{R}_X$ is 3.

If the curve is not quasi-homogeneous, the number of generators can be strictly greater:

Example 5.14. Let $h_1 = x_1^7 - x_2^5 + x_1^5x_2^3$ and $h_2 = x_1^3x_2 - x_3^2$. The sequence (h_1, h_2) defines a reduced complete intersection curve of $\mathbb{C}^3$, which is not quasi-homogeneous. We use Singular to compute a minimal free resolution of $\Omega^2(\log C)$:

$$0 \to \mathcal{O}_S^6 \to \mathcal{O}_S^9 \to \Omega^2(\log C) \to 0.$$

Remark 5.15. It is then natural to look for analogous results for quasi-homogeneous complete intersections of higher dimension. However, the situation seems to be more complicated than in the curve case. In particular, we compute with Singular examples of homogeneous complete intersection surfaces in $\mathbb{C}^4$ for which the Betti numbers are different (see [Pol16, §A.2.2]).

5.2. **Complements.** The following property shows that for a reduced quasi-homogeneous curve, even the quasi-homogeneity of the equations cannot be dropped in the statement of [Ale12, §6, Theorem 2]. An interesting question would be to determine precisely under which hypothesis on the equations the equality $\mathrm{res}_C(\Omega^{m-1}(\log D)) = \mathrm{res}_C(\Omega^{m-1}(\log C))$ holds.

We denote by $\text{w-deg}(f)$ the weighted degree of a quasi-homogeneous element $f \in \mathcal{O}_C$, with respect to the weight $(w_1, \dots, w_m)$.

Proposition 5.16. *Let C be a reduced complete intersection curve defined by a regular sequence $(h_1, \dots, h_{m-1})$ satisfying conditions 5.2.*

Let $A = (a_{ij})_{1 \leqslant i,j \leqslant m-1} \in \mathcal{M}_{m-1}(\mathbb{C})$ be an invertible matrix with constant coefficients. We set $(f_1, \dots, f_{m-1})^t = A \cdot (h_1, \dots, h_{m-1})^t$. In particular, $(f_1, \dots, f_{m-1})$ also defines the complete intersection C. Let D_f be the hypersurface defined by $f = f_1 \cdots f_{m-1}$.

If there exists $\ell, i, j \in \{1, \dots, m-1\}$ such that $f_\ell = \sum_{q=1}^{m-1} a_{\ell,q} h_q$ with $\text{w-deg}(h_i) \neq \text{w-deg}(h_j)$ and $a_{\ell,i} a_{\ell,j} \neq 0$ then:

$$\mathrm{res}_C(\Omega^{m-1}(\log D_f)) \neq \mathrm{res}_C(\Omega^{m-1}(\log C)).$$

Proof. For the sake of simplicity, we assume that $a_{1,1} \cdot a_{1,2} \neq 0$ and $\text{w-deg}(h_1) \neq \text{w-deg}(h_2)$.

By lemma 5.3 and proposition 2.3, $\omega = \dfrac{\sum_{i=1}^m (-1)^{i-1} w_i x_i \widehat{\mathrm{d}x_i}}{f_1 \dots f_{m-1}} \in \Omega^{m-1}(\log C, \underline{f})$. Let us prove that $\mathrm{res}_C(\omega) \notin \mathrm{res}_C(\Omega^{m-1}(\log D_f))$. We thus have to prove that for all $\eta \in \widetilde{\Omega}^{m-1}_{\underline{f}}$, $\omega + \eta \notin \Omega^{m-1}(\log D_f)$.

Let $\eta \in \widetilde{\Omega}^{m-1}_{\underline{f}}$. Then $\eta = \sum_{j=1}^{m-1} \dfrac{f_j \eta_j}{f}$ with $\eta_i \in \Omega^{m-1}_S$. We have $\mathrm{d}f_i = \sum_{j=1}^{m-1} a_{ij} \mathrm{d}h_j$ and:

$$\mathrm{d}f \wedge (\omega + \eta) = \sum_{i=1}^{m-1} \widehat{f_i} \cdot \left(\sum_{j=1}^{m-1} (a_{ij} d_j h_j) \frac{\mathrm{d}\underline{x}}{f} + \sum_{j=1}^{m-1} \left(\frac{\mathrm{d}f_i \wedge f_j \eta_j}{f} \right) \right).$$

If $\omega + \eta \in \Omega^{m-1}(\log D_f)$ then $\mathrm{d}f \wedge (\omega + \eta)$ is holomorphic. Since the sequence $(f_1, \dots, f_{m-1})$ is regular, it implies that f_1 divides $\sum_{j=1}^{m-1} (a_{1j} d_j h_j + f_j \theta_{1,j})$, where $\theta_{1,j} \mathrm{d}\underline{x} = \mathrm{d}f_1 \wedge \eta_j$. In particular, since for all $i \in \{1, \dots, m-1\}$, $h_i \in \mathfrak{m}^2$, we also have $f_i \in \mathfrak{m}^2$ so that we have $\theta_{1,j} \in \mathfrak{m}$. There exists $q \in \mathcal{O}_S$ such that:

$$\sum_{\ell=1}^{m-1} \left(a_{1\ell} d_\ell + \sum_{j=1}^{m-1} (a_{j,\ell} \theta_{1,j}) - q a_{1,\ell} \right) \cdot h_\ell = 0.$$

Since $(h_1, \dots, h_{m-1})$ is a regular sequence, we have for all $\ell \in \{1, \dots, m-1\}$,

$$a_{1\ell} d_\ell + \sum_{j=1}^{m-1} a_{j,\ell} \theta_{1,j} - q a_{1,\ell} \in (h_1, \dots, h_{m-1}).$$

It implies that for all $\ell \in \{1, \dots, m-1\}$, $a_{1\ell}(d_\ell - q(0, \cdots, 0)) = 0$. In particular, since $a_{1,1} \neq 0$ and $a_{1,2} \neq 0$, we have $d_1 - q(0, \dots, 0) = 0$ and $d_2 - q(0, \dots, 0) = 0$. Since by assumption, $d_1 \neq d_2$, it leads to a contradiction, so that $\mathrm{d}f \wedge (\omega + \eta) \notin \Omega^m_S$ and $\omega + \eta \notin \Omega^{m-1}(\log D_f)$. □

We end this subsection by a characterization of plane homogeneous curves *via* multi-residues.

Proposition 5.17. *Let $(h_1, \dots, h_{m-1})$ be a regular sequence defining a reduced complete intersection curve C such that for all $i \in \{1, \dots, m-1\}$, h_i is homogeneous of degree d_i and m is the embedding dimension. We set $C = C_1 \cup \dots \cup C_p$ the irreducible decomposition of the complete intersection defined by $(h_1, \dots, h_{m-1})$, which is an union of lines. We assume $p \geqslant 2$. Then*

$$\inf(\mathrm{val}(\mathcal{R}_C)) = (-p+2, \dots, -p+2) \iff C \text{ is plane.}$$

Proof. The implication $\Leftarrow$ comes from a direct computation. Let us prove the implication $\Rightarrow$.

Lemma 5.3 gives an element ω_0 in $\Omega^{m-1}(\log C)$ such that $\mathrm{val}(\mathrm{res}_C(\omega_0)) = 1 - \sum_{i=1}^{m-1}(d_i - 1)$. Moreover, this multi-residue satisfies $\inf(\mathrm{val}(\mathcal{R}_C)) = \mathrm{val}(\mathrm{res}_C(\omega_0))$.

Since the curve C is reduced, by Bezout theorem, we have $p = d_1 \cdots d_{m-1}$. Moreover, since m is the embedding dimension, for all $i \in \{1, \dots, p\}$, $d_i \geqslant 2$.

Let us assume that $\inf(\mathrm{val}(\mathcal{R}_C)) = (-p+2, \dots, -p+2)$. It means that

$$1 - \sum_{i=1}^{m-1} (d_i - 1) = -d_1 \cdots d_{m-1} + 2.$$

which is equivalent to $d_1 \cdots d_{m-1} = \left(\sum_{i=1}^{m-1} d_i\right) - m + 2$. We assume that $d_1 = \max(d_i)$. The left-hand-side term is greater than $2^{m-2} d_1$, the right-hand-side term is stricly lower than $(m-1)d_1$ if $m \geqslant 3$. Therefore, equality cannot hold if $m \geqslant 3$. Hence the result. □

Appendix A. Comparison between simple poles and arbitrary poles

The definition 2.2 of multi-logarithmic differential forms we choose here is the definition proposed in [Ale12]. In previous works, another definition is considered where arbitrary poles are allowed (see [AT01], [AT08]).

The purpose of this appendix is to clarify the relation between the two definitions of multi-logarithmic differential forms, and to deduce directly from this relation that $(\mathcal{R}_C^\bullet, \mathrm{d})$ is a complex.

For $q \in \mathbb{N}$, we set $\Omega^q(*D)$ the module of meromorphic q-forms with arbitrary poles along D.

Definition A.1 ([AT01], [AT08])**.** *Let $\omega \in \Omega^q(\star D)$ with $q \in \mathbb{N}$. We say that ω is a multi-logarithmic differential form along C with arbitrary poles along D if the following properties are satisfied:*

(1) for all $j \in \{1, \dots, k\}$, $h_j \omega \in \sum_{i=1}^k \Omega^q\left(\star \widehat{D_i}\right)$,

(2) for all $j \in \{1, \dots, k\}$, $\mathrm{d}h_j \wedge \omega \in \sum_{i=1}^k \Omega^{q+1}\left(\star \widehat{D_i}\right)$.

We denote by $\underline{\Omega^q}(\log C)$ the module of multi-logarithmic differential forms along C with arbitrary poles along D.

To simplify the notations, we set $\widetilde{\underline{\Omega^q}} = \sum_{i=1}^k \Omega^q\left(\star \widehat{D_i}\right)$.

Remark A.2. It is easy to prove that the modules $\underline{\Omega^\bullet}(\log C)$ induce a complex for the de Rham differentiation. This property is not satisfied by $\Omega^\bullet(\log C)$. However, the main drawback of the modules $\underline{\Omega^\bullet}(\log C)$ is that they are not of finite type, whereas the modules $\Omega^\bullet(\log C)$ are.

The relation between the modules $\Omega^q(\log C)$ and $\underline{\Omega^q}(\log C)$ is:

Proposition A.3. *For all $q \in \mathbb{N}$ we have the equality:*

$$\underline{\Omega^q}(\log C) = \Omega^q(\log C) + \widetilde{\underline{\Omega^q}}.$$

We first introduce the following lemma, which can be proved by induction on $j \in \{1, \dots, k\}$, using the fact that $(h_1, \dots, h_k)$ is a regular sequence in a local ring:

Lemma A.4 ([CMNM02, Lemma 2.3])**.** *Let $p = (p_1, \dots, p_k) \in \mathbb{N}^k$ and $n = (n_1, \dots, n_k) \in \mathbb{N}^k$ be such that for all $i \in \{1, \dots, k\}$, $p_i \geqslant n_i \geqslant 1$ and $a \in \mathcal{O}_S$.*

If $h_1^{p_1 - n_1} \cdots h_k^{p_k - n_k} a \in \left(h_1^{p_1}, \dots, h_k^{p_k}\right)$, then $a \in \left(h_1^{n_1}, \dots, h_k^{n_k}\right)$.

Proof of proposition A.3. The inclusion $\Omega^q(\log C) + \widetilde{\underline{\Omega^q}} \subseteq \underline{\Omega^q}(\log C)$ is obvious.

Let $\omega \in \underline{\Omega^q}(\log C)$. We can choose $\eta \in \widetilde{\underline{\Omega^q}}$ and for all $I \subseteq \{1, \dots, m\}$ with $|I| = q$, $a_I \in \mathcal{O}_S$, $n^I \in \mathbb{N}^k$ such that $a_I \notin (h_1^{n_1^I}, \dots, h_k^{n_k^I})$, for all $j \in \{1, \dots, k\}$, $a_I \notin (h_j)$ and

$$\omega = \sum_I \frac{a_I \mathrm{d}x_I}{h_1^{n_1^I} \cdots h_k^{n_k^I}} + \eta.$$

Let us prove that we can take for all I, $n^I = (1, \dots, 1)$.

Since $\omega \in \underline{\Omega^q}(\log C)$, we have $h_1\omega \in \widetilde{\underline{\Omega^q}}$. We fix $I \subseteq \{1,\dots,m\}$ with $|I| = q$. There exist $(b_1,\dots,b_k) \in \mathcal{O}_S^k$ and for all $j \in \{1,\dots,k\}$, $m^j = (m_1^j,\dots,m_k^j) \in \mathbb{N}^k$ with $m_j^j = 0$ such that

$$\frac{a_I}{h_1^{n_1^I-1}h_2^{n_2^I}\cdots h_k^{n_k^I}} = \sum_{j=1}^{k} \frac{b_j}{\underline{h}^{m^j}}$$

where $\underline{h}^{m^j} = h_1^{m_1^j}\cdots h_k^{m_k^j}$. To simplify the notations, we denote for all $j \in \{1,\dots,k\}$ n_j^I by n_j, and $n' = (n_1-1, n_2,\dots,n_k)$. We set for all $i \in \{1,\dots,k\}$, $p_i = \max_{j\in\{1,\dots,k\}} m_i^j$. We have:

$$\frac{a_I}{\underline{h}^{n'}} = \frac{\underline{h}^{p-m^1}b_1 + \cdots + \underline{h}^{p-m^k}b_k}{\underline{h}^p}.$$

Thus, since $\frac{a_I\underline{h}^p}{\underline{n'}}$ is holomorphic and by assumption, for all $i \in \{1,\dots,k\}$, $a_I \notin (h_i)$, we have for all $i \in \{1,\dots,k\}$, $p_i \geqslant n_i'$. Since for all $j \in \{1,\dots,k\}$, $m_j^j = 0$, we have $\underline{h}^{p-n'}a_I \in \left(h_1^{p_1},\dots,h_k^{p_k}\right)$. lemma A.4 gives $a_I \in \left(h_1^{n_1-1}, h_2^{n_2},\dots,h_k^{n_k}\right)$, so that

$$\frac{a_I \mathrm{d}x_I}{\underline{h}^n} = \frac{c_I}{h_1h_2^{n_2}\cdots h_k^{n_k}} + \eta'$$

with $\eta' \in \widetilde{\underline{\Omega^q}}$ and $c_I \in \mathcal{O}_S$. By induction, one can prove that there exists a q-form $\omega' \in \frac{1}{h}\Omega^q$ with simple poles along D and $\eta'' \in \widetilde{\underline{\Omega^q}}$ such that $\omega = \omega' + \eta''$.

For all $j \in \{1,\dots,k\}$, $\mathrm{d}h_j \wedge \omega \in \widetilde{\underline{\Omega^q}}$, thus, $\mathrm{d}h_j \wedge \omega' \in \widetilde{\underline{\Omega^q}}$. lemma A.4 can be used in a similar proof to show that $\mathrm{d}h_j \wedge \omega' \in \widetilde{\Omega}^q$, and therefore, $\omega' \in \Omega^q(\log C)$, which finishes the proof of proposition A.3. □

The following result is then a direct consequence of both proposition A.3 and theorem 2.5:

Corollary A.5 ([AT01, Proposition 2.1])**.** *Let $\omega \in \underline{\Omega^q}(\log C)$. There exist $g \in \mathcal{O}_S$ which induces a non zero divisor in $\mathcal{O}_C$, a holomorphic $(q-k)$-form $\xi \in \Omega_S^{q-k}$, and $\eta \in \widetilde{\underline{\Omega^q}}$ such that*

$$g\omega = \frac{\mathrm{d}h_1 \wedge \cdots \wedge \mathrm{d}h_k}{h_1\cdots h_k} \wedge \xi + \eta.$$

Definition A.6. *The multi-residue of $\omega \in \underline{\Omega^q}(\log C)$ is $\underline{\mathrm{res}_C}(\omega) = \left.\frac{\xi}{g}\right|_C \in \Omega_C^{q-k} \otimes_{\mathcal{O}_C} \mathcal{M}_C$.*

As for res_C, the map $\underline{\mathrm{res}_C}$ is well-defined (see [AT08, Proposition 1.2]). Thanks to proposition A.3, we have the following equality:

$$\underline{\mathrm{res}_C}(\underline{\Omega^q}(\log C)) = \mathrm{res}_C(\Omega^q(\log C)). \tag{29}$$

As a consequence of corollary A.5, we have:

Corollary A.7. *The following diagramm is commutative:*

$$\begin{array}{ccc} \underline{\Omega^q}(\log C) & \xrightarrow{d} & \underline{\Omega^{q+1}}(\log C) \\ \downarrow{\scriptstyle \underline{\mathrm{res}_C}} & & \downarrow{\scriptstyle \underline{\mathrm{res}_C}} \\ \mathcal{R}_C^{q-k} & \xrightarrow{d} & \mathcal{R}_C^{q+1-k} \end{array}$$

In particular, $\mathcal{R}_C^\bullet$ is stable by differentiation.

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