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# A CHARACTERIZATION OF FREENESS FOR COMPLETE INTERSECTIONS

DELPHINE POL

ABSTRACT. The purpose of this paper is to study the notion of freeness for reduced complete intersections, which is a generalization of the notion of free divisors introduced by K. Saito. We give some properties of multi-logarithmic differential forms and their multi-residues along a reduced complete intersection defined by a regular sequence  $(h_1, \dots, h_k)$ . We first establish a kind of duality between multi-logarithmic differential  $k$ -forms and multi-logarithmic  $k$ -vector fields. We use it to prove a duality between the Jacobian ideal and multi-residues. The main result is a characterization of freeness in terms of the projective dimension of the module of multi-logarithmic forms. We then focus on quasi-homogeneous curves, for which we compute explicitly a minimal free resolution of the module of multi-logarithmic forms.

## 1. INTRODUCTION

In [Sai80], K. Saito introduces the modules of logarithmic differential forms along reduced hypersurfaces and their residues, and the module of logarithmic vector fields. If  $D$  is the germ of a reduced hypersurface in a smooth analytic variety  $S$  and  $h \in \mathcal{O}_S$  is an equation of  $D$ , then for  $q \in \mathbb{N}$  the module of logarithmic  $q$ -forms is:

$$\Omega^q(\log D) = \left\{ \omega \in \frac{1}{h} \Omega_S^q, dh \wedge \omega \in \Omega_S^{q+1} \right\}$$

The dual of the module of logarithmic differential 1-forms is the module of logarithmic vector fields  $\text{Der}(-\log D)$ . A holomorphic vector field is logarithmic if it is tangent to  $D$  at its smooth points.

As a consequence, the freeness of  $\Omega^1(\log D)$  is equivalent to the freeness of  $\text{Der}(-\log D)$ . In that case,  $D$  is called a free divisor. For example, normal crossing divisors, plane curves and discriminants of isolated hypersurface singularities are families of free divisors. We can also characterize free divisors in a more geometric way. Indeed, A.G. Aleksandrov proves in [Ale88] that a singular divisor is free if and only if its singular locus is Cohen-Macaulay of codimension one in  $D$ .

A similar theory is developed by A.G. Aleksandrov and A. Tsikh in [AT01], [AT08] and [Ale12] for reduced complete intersections. Let  $C$  be a reduced complete intersection defined by a regular sequence  $\underline{h} = (h_1, \dots, h_k)$ . We set for  $i \in \{1, \dots, k\}$ ,  $\widehat{h}_i = h_1 \cdots h_{i-1} \cdot h_{i+1} \cdots h_k$  and  $h = h_1 \cdots h_k$ . The multi-logarithmic differential  $q$ -forms along  $C$  are defined by:

$$\Omega_S^q(\log C, \underline{h}) = \left\{ \omega \in \frac{1}{h} \Omega_S^q, \forall i \in \{1, \dots, k\}, dh_i \wedge \omega \in \sum_{j=1}^k \frac{1}{\widehat{h}_j} \Omega_S^{q+1} \right\}$$

In [GS12], the authors suggest a generalization of the notion of freeness using Aleksandrov's characterization of free divisors. A reduced complete intersection is called free if it is smooth or if its singular locus is Cohen-Macaulay of codimension one in  $C$ .

The purpose of this paper is to investigate the module of multi-logarithmic differential forms along complete intersections, giving a particular attention to the case of free complete intersections.

In subsection 2.1 we first recall several definitions and properties of [AT01], [AT08] and [Ale12]. If  $\omega$  is a multi-logarithmic  $q$ -form, there exist  $g \in \mathcal{O}_S$  which does not induce a zero divisor in

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$\mathcal{O}_C = \mathcal{O}_S/(h_1, \dots, h_k)$ , a holomorphic  $(q - k)$ -form  $\xi$  and a meromorphic  $q$ -form  $\eta \in \sum_{j=1}^k \frac{1}{h_j} \Omega_S^q$  such that (see Theorem 2.8):

$$g\omega = \frac{dh_1 \wedge \dots \wedge dh_k}{h_1 \dots h_k} \wedge \xi + \eta$$

The multi-residue of  $\omega$  is  $\frac{\xi}{g}|_C \in \Omega_C^{q-k} \otimes \mathcal{M}_C$ , where  $\Omega_C^p$  is the module of Kähler differentials and  $\mathcal{M}_C$  is the ring of meromorphic functions on  $C$ . We denote by  $\mathcal{R}_C^{q-k}$  the module of multi-residues of multi-logarithmic  $q$ -forms. The module  $\mathcal{R}_C^0$  always contains the ring  $\mathcal{O}_{\tilde{C}}$  of weakly holomorphic functions on  $C$ . We give here a proof of this inclusion which is almost the same as the proof of K. Saito in the hypersurface case.

In subsections 2.1 and 2.2 we make explicit the dependence of the modules  $\Omega_S^q(\log C, \underline{h})$  and  $\mathcal{R}_C^{q-k}$  on the choice of the equations. If  $(f_1, \dots, f_k)$  is another regular sequence defining  $C$ , we have (see Corollary 2.4):

$$(1) \quad \Omega_S^q(\log C, \underline{h}) = \frac{f_1 \dots f_k}{h_1 \dots h_k} \Omega_S^q(\log C, \underline{f})$$

Therefore, the modules  $h_1 \dots h_k \Omega^q(\log C, \underline{h})$  do not depend on the choice of the equations. In addition, modules  $\mathcal{R}_C^{q-k}$  depends only on  $C$  (see Proposition 2.17). We then fix the regular sequence  $(h_1, \dots, h_k)$  and we denote  $\Omega^q(\log C) = \Omega^q(\log C, \underline{h})$ .

We also prove in Proposition 2.20 that we can assume that  $k$  is the embedding dimension of  $C$  without changing the modules of multi-residues.

In subsection 2.3, we prove that the fact that  $C$  is reduced imposes a strong condition on the module of logarithmic 1-forms of the divisor  $D$  defined by  $h_1 \dots h_k$ . We denote by  $D_i$  the hypersurface defined by  $h_i$ . Then (see Proposition 2.24):

**Proposition 1.1.**

$$\Omega^1(\log D) = \Omega^1(\log D_1) + \dots + \Omega^1(\log D_k)$$

In subsections 2.4 and 2.5 we generalize some properties of hypersurfaces to complete intersections. Let  $\mathcal{J}_C$  be the Jacobian ideal of  $C$  (see notation 2.12) and  $\Theta_S$  be the module of holomorphic vector fields. Following [GS12], we define the module of multi-logarithmic  $k$ -vector fields  $\text{Der}^k(-\log C)$  as the kernel of the map:  $\bigwedge^k \Theta_S \ni \delta \mapsto dh_1 \wedge \dots \wedge dh_k(\delta) \in \mathcal{J}_C$ . We link this module to the module  $\Omega^k(\log C)$  in the following way, generalizing K. Saito's duality:

**Proposition 1.2.**

$$\text{Hom}_{\mathcal{O}_S} \left( \text{Der}^k(-\log C), \sum_{i=1}^k \frac{1}{h_i} \mathcal{O}_S \right) = \Omega^k(\log C)$$

Proposition 1.2 remains true if we exchange the roles of  $\Omega^k(\log C)$  and  $\text{Der}^k(-\log C)$ .

Moreover, we also answer a question of the authors of [GS12]. Indeed, using an approach which is similar to their proof, we prove, using Proposition 1.2, that

$$\text{Hom}_{\mathcal{O}_C}(\mathcal{J}_C, \mathcal{O}_C) = \mathcal{R}_C$$

Section 3 is devoted to the main result of this paper, namely, the characterization of freeness by the projective dimension of the module  $\Omega^k(\log C)$ . Together with the results of [GS12] and [Sch13], it leads to the following Theorem:

**THEOREM 1.3** (see Proposition 3.4 and Theorem 3.5). *Let  $C$  be a singular reduced complete intersection. The following are equivalent:*

- (1)  $C$  is free
- (2)  $\mathcal{J}_C$  is Cohen-Macaulay
- (3)  $\text{Der}^k(-\log C)$  has projective dimension  $k - 1$
- (4)  $\Omega^k(\log C)$  has projective dimension  $k - 1$

The equivalences between (1), (2) and (3) are already proved in [GS12] and [Sch13], but a characterization using multi-logarithmic differential forms was missing. Contrary to the hypersurface case, passing from  $\text{Der}^k(-\log C)$  to  $\Omega^k(\log C)$  needs much more work.

In section 4 we study a particular family of free complete intersections: the complete intersection curves. In general, the computation of  $\Omega^{m-1}(\log C)$  is difficult. For quasi-homogeneous curves, we determine a minimal free  $\mathcal{O}_S$ -resolution of  $\mathcal{R}_C$  and  $\Omega^{m-1}(\log C)$  (see Theorem 4.10). We also give a characterization of plane homogeneous curves using multi-residues (see Proposition 4.16). Several properties of complete intersection curves are already studied in [Pol15b], and are used in this section.

We then compare the notion of multi-logarithmic  $k$ -vector fields and the notion of logarithmic vector fields studied by B. Pike in [Pik14]. We end with a list of examples of surfaces and questions.

In Appendix A we suggest several algorithms which can be used with SINGULAR [DGPS15] to study explicit examples. The first algorithm A.1 is used to determine if a given sequence  $(h_1, \dots, h_k)$  defines a free complete intersection. Algorithms A.2.1 and A.3 compute a free resolution of the module  $\Omega^k(\log C)$  for curves or surfaces.

In Appendix B, we clarify the relation between the different definitions of multi-logarithmic differential forms suggested by A. Aleksandrov and A. Tsikh. In particular, it gives a proof of the stability by the de Rham differentiation of the modules of multi-residues, which do not use the isomorphism with regular meromorphic forms.

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## 2. MULTI-LOGARITHMIC DIFFERENTIAL FORMS

**2.1. Definitions.** We give here the definitions and several properties of multi-logarithmic differential forms and their residues, which are a natural generalization of the theory of logarithmic differential forms introduced in [Sai80].

Let  $S$  be a smooth analytic complex variety of dimension  $m \geq 1$ . We denote by  $\mathcal{O}_S$  the sheaf of holomorphic functions on  $S$ . By restricting  $S$  to an open subset of  $S$ , we may assume that there exist local coordinates  $(x_1, \dots, x_m)$  on  $S$ . Let  $h_1, \dots, h_k \in \mathcal{O}_S(S)$  be such that:

- (1) for all  $x \in S$ ,  $(h_1, \dots, h_k)$  is a regular  $\mathcal{O}_{S,x}$ -sequence.
- (2) the ideal sheaf  $(h_1, \dots, h_k)$  is radical.

*Notation 2.1.* For all  $i \in \{1, \dots, k\}$ , we set  $D_i$  the hypersurface defined by  $h_i$ , and  $D = D_1 \cup \dots \cup D_k$ . We denote by  $C = D_1 \cap \dots \cap D_k$  the reduced complete intersection defined by the regular sequence  $(h_1, \dots, h_k)$ . We set  $\mathcal{O}_C := \mathcal{O}_S / (h_1, \dots, h_k)$ .

For  $j \in \{1, \dots, k\}$ , we set  $\widehat{h}_j = \prod_{i \neq j} h_i$  and  $\widehat{D}_j$  the hypersurface of  $S$  defined by  $\widehat{h}_j$ .

For  $q \in \mathbb{N}$ , we denote by  $\Omega_S^q$ , or  $\Omega^q$  for short, the sheaf of holomorphic differential forms on  $S$ . For  $I \subseteq \{1, \dots, m\}$ , we set  $dx_I = \bigwedge_{i \in I} dx_i$ , and  $|I|$  the cardinality of  $I$ .

Let us recall some definitions and results from [Ale12]. The following definition is a generalization to the complete intersection case of the notion of logarithmic differential forms developed in [Sai80, 1.1].

**Definition 2.2.** Let  $x \in S$  and  $\omega \in \frac{1}{h}\Omega_{S,x}^q$ , with  $q \in \mathbb{N}$ . We call  $\omega$  a multi-logarithmic  $q$ -form along the complete intersection  $C$  for the equations  $(h_1, \dots, h_k)$  if

$$\forall i \in \{1, \dots, k\}, dh_i \wedge \omega \in \sum_{j=1}^k \frac{1}{h_j} \Omega_{S,x}^{q+1}$$

We denote by  $\Omega_S^q(\log C, \underline{h})$  the sheaf of germs of multi-logarithmic  $q$ -forms along  $C$  with respect to  $(h_1, \dots, h_k)$ .

**Proposition 2.3.** The sheaves  $\Omega_S^q(\log C, \underline{h})$  are coherent  $\mathcal{O}_S$ -modules.

*Proof.* Let  $q \in \mathbb{N}$ . The sheaf  $h\Omega_S^q(\log C, \underline{h})$  is the kernel of the following map of coherent sheaves:

$$\begin{aligned} \Omega^q &\rightarrow \left( \Omega^{q+1} / \left( (h_1, \dots, h_k) \cdot \Omega^{q+1} \right) \right)^k \\ \omega &\mapsto (dh_1 \wedge \omega, \dots, dh_k \wedge \omega) \end{aligned}$$

□

A direct consequence of the proof of Proposition 2.3 is the following:

**Corollary 2.4.** The modules  $h\Omega^q(\log C)$  do not depend on the choice of the defining equations  $(h_1, \dots, h_k)$ .

*Remark 2.5.* Contrary to the hypersurface case, if  $k \geq 2$ , the family  $(\Omega_S^\bullet(\log C, \underline{h}))$  is not closed for the exterior differentiation<sup>1</sup>. This problem can be solved by considering arbitrary poles instead of simple poles, but it gives no more finite type modules (see Appendix B, where we compare [AT08, Definition 1.1] and Definition 2.2).

*Remark 2.6.* Let us consider a reduced analytic subset  $C \subset S$  such that it is locally defined as a complete intersection, but not globally defined by a given regular sequence. There exists an open covering  $(U_\alpha)_{\alpha \in \Lambda}$  of  $S$  such that for all  $\alpha \in \Lambda$ , there exists a sequence  $(h_1^{(\alpha)}, \dots, h_k^{(\alpha)})$  satisfying the hypothesis (1) and (2) of the beginning of this section. We can therefore define a sheaf  $\Omega_{U_\alpha}^q(\log C)$  on each open subset  $U_\alpha$ , but we can not glue them together into a sheaf over  $S$ .

*Notation 2.7.* To simplify the notations, we set  $\Sigma_{\underline{h}} := \sum_{j=1}^k \frac{1}{h_j} \mathcal{O}_S$  and  $\tilde{\Omega}_{\underline{h}}^q := \sum_{j=1}^k \frac{1}{h_j} \Omega_S^q = \Omega_S^q \otimes_{\mathcal{O}_S} \Sigma_{\underline{h}}$ .

In particular,  $\Sigma_{\underline{h}} = \tilde{\Omega}_{\underline{h}}^0$ .

The following characterization of multi-logarithmic forms should be compared with [Sai80, (1.1)]:

**THEOREM 2.8** ([Ale12, §3, Theorem 1]). Let  $\omega \in \frac{1}{h}\Omega_{S,x}^q$ , with  $q \in \mathbb{N}$  and  $x \in S$ . Then  $\omega$  is multi-logarithmic if and only if there exist a holomorphic function  $g \in \mathcal{O}_{S,x}$  which does not induce a zero-divisor in  $\mathcal{O}_{C,x}$ , a holomorphic differential form  $\xi \in \Omega_{S,x}^{q-k}$  and a meromorphic  $q$ -form  $\eta \in \tilde{\Omega}_{\underline{h},x}^q$  such that:

$$(2) \quad g\omega = \frac{dh_1 \wedge \dots \wedge dh_k}{h} \wedge \xi + \eta$$

*Remark 2.9.* For  $q < k$ , we have the equality  $\Omega_S^q(\log C, \underline{h}) = \tilde{\Omega}_{\underline{h}}^q$ .

Let  $q \in \mathbb{N}$ . The module of Kähler differentials of order  $q$ , which does not depend on the choice of the equations  $(h_1, \dots, h_k)$ , is:

<sup>1</sup>For example,  $\frac{1}{h_1} \in \Omega_S^0(\log C, \underline{h})$  but  $d\left(\frac{1}{h_1}\right) = -\frac{dh_1}{h_1^2}$  has poles of order 2.

$$\Omega_C^q = \frac{\Omega_S^q}{(h_1, \dots, h_k)\Omega_S^q + dh_1 \wedge \Omega_S^{q-1} + \dots + dh_k \wedge \Omega_S^{q-1}} \Big|_C$$

We then define the multi-residue of a multi-logarithmic form (see [Ale12, §4, Definition 1]):

**Definition 2.10.** *Let  $x \in S$  and  $\omega \in \Omega_{S,x}^q(\log C, \underline{h})$ ,  $q \geq k$ . Let us assume that  $g, \xi, \eta$  satisfy the properties of Theorem 2.8. Then the multi-residue of  $\omega$  is:*

$$\text{res}_C(\omega) := \frac{\xi}{g} \Big|_C \in \mathcal{M}_{C,x} \otimes_{\mathcal{O}_{C,x}} \Omega_{C,x}^{q-k} = \left( \pi_* \left( \mathcal{M}_{\tilde{C}} \otimes_{\mathcal{O}_{\tilde{C}}} \Omega_{\tilde{C}}^{q-k} \right) \right)_x$$

where  $\pi : \tilde{C} \rightarrow C$  is the normalization of  $C$  and  $\mathcal{M}_C$  is the sheaf of meromorphic functions on  $C$ .

The notion of multi-residue is well-defined with respect to the choices of  $g, \xi, \eta$  in (2) (see [Ale12, §4 Proposition 2]).

We denote  $\mathcal{R}_{C,\underline{h}}^{q-k} := \text{res}_C(\Omega^q(\log C, \underline{h}))$ . If  $q = k$ , we set  $\mathcal{R}_{C,\underline{h}} := \mathcal{R}_{C,\underline{h}}^0 \subset \mathcal{M}_C$ . We will see in Proposition 2.17 that the modules  $\mathcal{R}_{C,\underline{h}}^{q-k}$  do not depend on the choice of the equations  $(h_1, \dots, h_k)$ .

**Proposition 2.11** ([Ale12, §4, Lemma 1]). *Let  $q \in \mathbb{N}$ . We have the following exact sequence of  $\mathcal{O}_S$ -modules:*

$$(3) \quad 0 \longrightarrow \tilde{\Omega}_{\underline{h}}^q \longrightarrow \Omega^q(\log C, \underline{h}) \longrightarrow \mathcal{R}_{C,\underline{h}}^{q-k} \longrightarrow 0$$

*In particular, the sheaf  $\mathcal{R}_{C,\underline{h}}^{q-k}$  is coherent.*

*Notation 2.12.* Let us fix a point  $x \in S$ . We denote by  $\mathcal{J}_{C,x}$  the Jacobian ideal of  $C$ , which is the ideal of  $\mathcal{O}_{C,x}$  generated by the  $k \times k$ -minors of the Jacobian matrix of  $(h_1, \dots, h_k)$ . The Jacobian ideal does not depend on the choice of the equations  $(h_1, \dots, h_k)$ .

The following result is already given in [GS12] and [Sch13], but we give here a different proof which is almost the same as the proof of [Sai80, Lemma 2.8].

**Proposition 2.13** (see [GS12]). *Let  $\pi : \tilde{C} \rightarrow C$  be the normalization of  $C$ . The ring  $(\pi_* \mathcal{O}_{\tilde{C}})_x$  is included in  $\mathcal{R}_{C,x}$ .*

*Proof.* For the sake of simplicity, we write  $\mathcal{O}_{\tilde{C}}$  instead of  $(\pi_* \mathcal{O}_{\tilde{C}})_x$ ,  $\mathcal{O}_C$  for  $\mathcal{O}_{C,x}$  and so on. Let  $\alpha \in \mathcal{O}_{\tilde{C}}$ . Then, thanks to [GLS10, Lemma 4.1], for every  $g \in \mathcal{J}_C$ ,  $g\alpha \in \mathcal{O}_C$ .

Thus, for every subset  $J \subseteq \{1, \dots, m\}$  with  $|J| = k$ , there exists  $\bar{a}_J \in \mathcal{O}_C$  such that  $\Delta_J \alpha = \bar{a}_J \in \mathcal{O}_C$ , where  $\Delta_J$  is the minor of the Jacobian matrix relative to the set  $J$ . Let  $I, J$  be two subsets of  $\{1, \dots, m\}$  with  $k$  elements. Then from the equality  $\Delta_I \Delta_J \alpha - \Delta_J \Delta_I \alpha = 0 \in \mathcal{O}_C$  we can deduce:

$$\Delta_I a_J - \Delta_J a_I = h_1 b_1^{IJ} + \dots + h_k b_k^{IJ} \in \mathcal{O}_S$$

Let us define  $\omega = \frac{\sum a_J dx_J}{h}$ . Then the previous equality gives:

$$\begin{aligned} \Delta_I \omega &= \frac{\sum_J \Delta_I a_J dx_J}{h} = \frac{a_I}{h} \sum_J \Delta_J dx_J + \eta \text{ with } \eta \in \tilde{\Omega}_{\underline{h}}^k \\ &= a_I \frac{dh_1 \wedge \dots \wedge dh_k}{h} + \eta \end{aligned}$$

Moreover, there exists a linear combination of the  $\Delta_J$  which does not induce a zero-divisor in  $\mathcal{O}_C$  (see [GLS10, Lemma 4.1]). We can then deduce that  $\text{res}_C(\omega) = \alpha \in \mathcal{R}_{C,\underline{h}}$ .  $\square$

The proof of Proposition 2.13 in [Sch13] uses the isomorphism between regular meromorphic differential forms and multi-residues (see [AT01, Theorem 2.4]). Let us recall it. We first give some definitions:

**Definition 2.14.** *The  $\mathcal{O}_C$ -module*

$$\omega_C^{m-k} = \text{Ext}_{\mathcal{O}_S}^k(\mathcal{O}_C, \Omega_S^m)$$

*is called the Grothendieck dualizing module of the germ  $(C, 0)$ . Since  $C$  is a complete intersection, it is a free  $\mathcal{O}_C$ -module of rank 1.*

**Definition 2.15.** *The module  $\omega_C^p$  of regular meromorphic differential  $p$ -forms is:*

$$\omega_C^p = \text{Hom}_{\mathcal{O}_C}(\Omega_C^{m-k-p}, \omega_C^{m-k})$$

**THEOREM 2.16** ([AT01, Theorem 2.4]). *For all  $p \geq 0$ , the module of multi-residues  $\mathcal{R}_C^p$  is isomorphic to the module  $\omega_C^p$  of regular meromorphic differential forms.*

**2.2. Choice of equations.** Corollary 2.4 shows that the modules  $h\Omega^q(\log C)$  do not depend on the choice of the regular sequence  $(h_1, \dots, h_k)$ .

The following proposition describes explicitly the behaviour of the multi-residues when we change the set of equations:

**Proposition 2.17.** *Let  $(h_1, \dots, h_k)$  and  $(f_1, \dots, f_k)$  be two regular sequences defining the same reduced complete intersection  $C$  of  $S$ . Then for all  $q \in \mathbb{N}$  and  $\omega \in f_1 \cdots f_k \Omega^q(\log C, \underline{f})$ :*

$$\text{res}_C\left(\frac{\omega}{h}\right) = \Delta \text{res}_C\left(\frac{\omega}{f}\right)$$

where  $\Delta$  is the determinant of the transition matrix from  $(h_1, \dots, h_k)$  to  $(f_1, \dots, f_k)$ .

*In particular, for all  $p \geq 0$ , the modules  $\mathcal{R}_C^p$  do not depend on the choice of the defining equations.*

*Proof.* Let  $x \in S$ . We write  $\Omega^q(\log C, \underline{h})$  instead of  $\Omega_{S,x}^q(\log C, \underline{h})$ , and so on.

Let  $\omega \in f_1 \cdots f_k \cdot \Omega^q(\log C, \underline{f})$ . Then there exists  $g \in \mathcal{O}_S$  which induces a non zero divisor in  $\mathcal{O}_C$ ,  $\xi \in \Omega^{q-k}$  and  $\eta \in \tilde{\Omega}_{\underline{f}}^q$  such that

$$g\omega = df_1 \wedge \cdots \wedge df_k \wedge \xi + f_1 \cdots f_k \eta$$

There exists an invertible matrix  $A = (a_{ij})_{1 \leq i, j \leq k}$  such that  $A(h_1, \dots, h_k)^t = (f_1, \dots, f_k)^t$ . Therefore there exists  $\nu \in (h_1, \dots, h_k)\Omega^k$  such that:

$$df_1 \wedge \cdots \wedge df_k = \nu + \det(A)dh_1 \wedge \cdots \wedge dh_k$$

Thus:

$$g\omega = dh_1 \wedge \cdots \wedge dh_k \wedge (\det(A)\xi) + \nu \wedge \xi + f_1 \cdots f_k \cdot \eta$$

with  $\frac{\nu \wedge \xi + f_1 \cdots f_k \eta}{h_1 \cdots h_k} \in \tilde{\Omega}_{\underline{h}}^q$ , so that  $\text{res}_C\left(\frac{\omega}{h}\right) = \frac{\Delta \xi}{g} = \delta \text{res}_C\left(\frac{\omega}{f}\right)$ .  $\square$

**Remark 2.18.** We already mention in Remark 2.4 that the modules  $h\Omega^q(\log C)$  do not depend on the choice of the regular sequence defining  $C$ . The proof of Proposition 2.17 gives an alternative proof of this statement.

**Remark 2.19.** The independence of  $\mathcal{R}_C^p$  from the choice of the equations is proved in [Ale12] thanks to the isomorphism of Theorem 2.16.

In the remain of the paper, we will use the notation  $\Omega^\bullet(\log C)$ , and  $\tilde{\Omega}^q$  where the set of equations is implicitly  $(h_1, \dots, h_k)$ .

The following proposition shows that we can assume that  $k$  is the embedding dimension of  $C$ .

**Proposition 2.20.** *Let  $\ell \in \{1, \dots, k\}$ . We assume<sup>2</sup> that for all  $i \in \{1, \dots, \ell\}$  we have  $h_i = x_i$ , and<sup>3</sup> for  $i \geq \ell + 1$ ,  $h_i \in \mathbb{C}\{x_{\ell+1}, \dots, x_m\}$ . We set  $C' = D_{\ell+1} \cap \dots \cap D_k \subseteq \mathbb{C}^{m-\ell}$ , so that  $C = C' \times \{0\}$ . Then for all  $q \geq \ell$ :*

<sup>2</sup>Maybe after an analytic change of coordinates and change of generators

<sup>3</sup>by taking  $h_i(0, \dots, 0, x_{\ell+1}, \dots, x_m)$  instead of  $h_i$

$$\Omega^q(\log C) = \frac{1}{x_1 \cdots x_\ell} \Omega^{q-\ell}(\log C') \wedge dx_1 \wedge \cdots \wedge dx_\ell + \tilde{\Omega}_{(h_1, \dots, h_k)}^q$$

*Proof.* Let  $\omega = \frac{1}{h} \sum a_I dx_I$ . Then by definition  $\omega \in \Omega^q(\log C)$  if and only if for all  $i \in \{1, \dots, k\}$ ,  $dh_i \wedge \omega \in \tilde{\Omega}^{q+1}$ . In particular, for all  $i \in \{1, \dots, \ell\}$ ,  $dx_i \wedge \omega \in \tilde{\Omega}^{q+1}$ . This condition implies that for all  $I$  such that  $\{1, \dots, \ell\} \not\subseteq I$ ,  $a_I \in (h_1, \dots, h_k)$ . We can write

$$\omega = \frac{1}{x_1 \cdots x_\ell} \omega' \wedge dx_1 \wedge \cdots \wedge dx_\ell + \eta$$

with  $\eta \in \tilde{\Omega}^q$  and  $\omega' = \frac{1}{h_{\ell+1} \cdots h_k} \sum_{\substack{I' \subseteq \{\ell+1, \dots, k\} \\ |I'|=q-\ell}} a_{I'} dx_{I'}$ , with  $a_{I'} \in \mathbb{C}\{x_{\ell+1}, \dots, x_m\}$ .

Since  $h_{\ell+1}, \dots, h_k \in \mathbb{C}\{x_{\ell+1}, \dots, x_m\}$ , the conditions  $dh_i \wedge \frac{1}{x_1 \cdots x_k} \omega' \wedge dx_1 \wedge \cdots \wedge dx_\ell \in \tilde{\Omega}^{q+1}$  is equivalent to the condition  $dh_i \wedge \omega' \in \tilde{\Omega}_{(h_{\ell+1}, \dots, h_k)}^{q-\ell+1}$ . Therefore,  $\omega \in \Omega^q(\log C)$  if and only if  $\omega' \in \Omega^{q-\ell}(\log C')$ . Hence the result.  $\square$

**Corollary 2.21.** *With the same notations, for all  $q \geq \ell$ , the module  $\mathcal{R}_C^{q-k}$  is equal to the  $\mathcal{O}_C$ -module  $\mathcal{R}_{C'}^{q-k}$ .*

**2.3. A condition on the divisor.** We prove here that the module of logarithmic 1-forms on the divisor  $D = D_1 \cup \cdots \cup D_k$  satisfies a decomposition property when  $D_1 \cap \cdots \cap D_k$  is a reduced complete intersection (see Proposition 2.24).

**Lemma 2.22.** *If  $C$  is a reduced complete intersection defined by a regular sequence  $(h_1, \dots, h_k)$ , then the complete intersection defined by  $(h_1, \dots, h_{k-1})$  is also reduced.*

*In addition, for all  $i \in \{1, \dots, k-1\}$ , the complete intersection defined by  $(h_1, \dots, h_{i-1}, h_i \cdots h_k)$  is also reduced. In particular,  $D$  is reduced.*

*Proof.* Let  $g \in \sqrt{(h_1, \dots, h_{k-1})}$ . Then, since  $\sqrt{(h_1, \dots, h_{k-1})} \subseteq \sqrt{(h_1, \dots, h_k)} = (h_1, \dots, h_k)$ , there exists  $f_1 \in (h_1, \dots, h_{k-1})$  and  $\alpha_1 \in \mathcal{O}_S$  such that  $g = f_1 + \alpha_1 h_k$ . We then have  $\alpha_1 h_k \in \sqrt{(h_1, \dots, h_{k-1})}$ . Since  $(h_1, \dots, h_k)$  is a regular sequence, it implies that  $\alpha_1 \in \sqrt{(h_1, \dots, h_{k-1})}$ . Therefore, there exists  $f_2 \in (h_1, \dots, h_{k-1})$  and  $\alpha_2 \in \mathcal{O}_S$  such that  $g = f_2 + \alpha_2 h_k^2$ . By induction, we can prove that for all  $i \in \mathbb{N}^*$ , there exist  $f_i \in (h_1, \dots, h_{k-1})$  and  $\alpha_i \in \mathcal{O}_S$  such that  $g = f_i + \alpha_i h_k^i$ . Hence, the class  $\bar{g}$  of  $g$  in  $\mathcal{O}_S/(h_1, \dots, h_{k-1})$  satisfies

$$\bar{g} \in \bigcap_{i \geq 1} \overline{(h_k^i)} \subseteq \bigcap_{i \geq 1} \overline{(\mathfrak{m}^i)} \subset \mathcal{O}_S/(h_1, \dots, h_{k-1})$$

By Krull's intersection Theorem (see [dJP00, Corollary 1.3.5]), it implies that  $\bar{g} = 0$ , therefore,  $g \in (h_1, \dots, h_{k-1})$ . Hence the first statement.

For the second statement, it is sufficient to prove it for  $i = k-1$ , the result will then follow by induction.

Let us consider  $g \in \sqrt{(h_1, \dots, h_{k-2}, h_{k-1} \cdot h_k)}$ . Then,  $g \in (h_1, \dots, h_k)$ , so that for all  $j \in \{1, \dots, k\}$  there exist  $\alpha_j \in \mathcal{O}_S$  such that  $g = \sum_{j=1}^k \alpha_j h_j$ . Therefore,

$$\alpha_{k-1} h_{k-1} + \alpha_k h_k \in \sqrt{(h_1, \dots, h_{k-2}, h_{k-1} \cdot h_k)}$$

Let  $q \in \mathbb{N}^*$  be such that  $(\alpha_{k-1} h_{k-1} + \alpha_k h_k)^q \in (h_1, \dots, h_{k-2}, h_{k-1} \cdot h_k)$ . Since  $(h_1, \dots, h_k)$  is a regular sequence, we then have  $\alpha_{k-1}^q \in (h_1, \dots, h_{k-2}, h_k)$  and  $\alpha_k^q \in (h_1, \dots, h_{k-1})$ . Since by the first statement, these ideals are radical, it gives the result.  $\square$

First, we recall the following result:



**Proposition 2.23** ([Ale12, Proposition 1]). *For all  $q \in \mathbb{N}$  we have the inclusion:*

$$\Omega^q(\log D) \subseteq \Omega^q(\log C)$$

Therefore,  $\text{res}_C(\Omega^q(\log D)) \subseteq \text{res}_C(\Omega^q(\log C))$ .

More precisely, he proves that for all  $i \in \{1, \dots, k\}$ ,  $dh_i \wedge \Omega^q(\log D) \subseteq \Omega^{q+1}(\log \widehat{D}_i)$ .

**Proposition 2.24.** *The module of logarithmic 1-forms satisfies:*

$$\Omega^1(\log D) = \Omega^1(\log D_1) + \dots + \Omega^1(\log D_k)$$

*Proof.* We first prove the proposition for  $k = 2$ . The general case will then follow.

Let  $\omega \in \Omega^1(\log D)$ . Then by Remark 2.9,  $\omega = \frac{\eta_1}{h_1} + \frac{\eta_2}{h_2}$  with  $\eta_i \in \Omega_S^1$ . Then:

$$dh \wedge \omega = dh_1 \wedge \eta_2 + dh_2 \wedge \eta_1 + \frac{h_2 dh_1 \wedge \eta_1}{h_1} + \frac{h_1 dh_2 \wedge \eta_2}{h_2} \in \Omega_S^2$$

Therefore, there exists  $\theta \in \Omega_S^2$  such that:

$$h_2^2 dh_1 \wedge \eta_1 + h_1^2 dh_2 \wedge \eta_2 = h_1 h_2 \theta$$

Since  $(h_1, h_2)$  is a regular sequence,  $h_2$  divides  $dh_2 \wedge \eta_2$  and  $h_1$  divides  $dh_1 \wedge \eta_1$ .

Hence the result:  $\frac{\eta_1}{h_1} \in \Omega^1(\log D_1)$  and  $\frac{\eta_2}{h_2} \in \Omega^1(\log D_2)$ .

The general case is obtained by induction from the case  $k = 2$ . Indeed, by Lemma 2.22, for all  $i \in \{1, \dots, k-1\}$ ,  $(h_1, \dots, h_{i-1}, h_i \cdots h_k)$  is a regular sequence defining a reduced complete intersection, so that  $\Omega^1(\log D) = \Omega^1(\log D_1) + \dots + \Omega^1(\log D_{i-1}) + \Omega^1(\log(D_i \cup \dots \cup D_k))$ .  $\square$

*Remark 2.25.* In particular, it gives a necessary condition on  $\Omega^1(\log D)$  for  $C$  being a reduced complete intersection.

It also implies the following corollary (see [AF13] and [Sch13]). A divisor is called splayed if in some coordinate system  $(y_1, \dots, y_m)$  it has an equation of the form  $g = g_1 \cdot g_2$  with  $g_1 \in \mathbb{C}\{y_1, \dots, y_\ell\}$  and  $g_2 \in \mathbb{C}\{y_{\ell+1}, \dots, y_m\}$ .

**Corollary 2.26.** *Let  $C$  be a reduced complete intersection of codimension 2 defined by  $(h_1, h_2)$ . If the divisor  $D$  defined by  $h_1 \cdot h_2$  is free, then  $D$  is a splayed divisor.*

**2.4. Multi-logarithmic vector fields and multi-logarithmic forms.** In the theory of Saito (see [Sai80]), the modules of logarithmic differential 1-forms and the module of logarithmic vector fields along a reduced hypersurface  $D$  are in duality over  $\mathcal{O}_S$ . Let us prove that an analogous property is satisfied by complete intersections.

We first define the notion of multi-logarithmic  $k$ -vector fields, as suggested in [GS12].

**Definition 2.27.** *Let  $\Theta_S$  denote the  $\mathcal{O}_S$ -module of holomorphic vector fields and  $\Theta_S^k := \bigwedge^k \Theta_S$  be the exterior power of order  $k$  of  $\Theta_S$ . A  $k$ -vector field  $\delta \in \Theta_S^k$  is called multi-logarithmic if it satisfies:*

$$dh_1 \wedge \dots \wedge dh_k(\delta) \in (h_1, \dots, h_k)\mathcal{O}_S$$

We denote by  $\text{Der}^k(-\log C)$  the  $\mathcal{O}_S$ -module of multi-logarithmic  $k$ -vector fields along  $C$ .

**Proposition 2.28.** *The module  $\text{Der}^k(-\log C)$  does not depend on the choice of the equations.*

*Proof.* We keep the same notations as in the statement and proof of Proposition 2.17. Let us assume that  $\delta \in \Theta^k$  satisfies  $dh_1 \wedge \dots \wedge dh_k(\delta) \in (h_1, \dots, h_k)$ . Then

$$df_1 \wedge \dots \wedge df_k(\delta) = \nu(\delta) + \det(A) \cdot dh_1 \wedge \dots \wedge dh_k(\delta) \in (h_1, \dots, h_k) = (f_1, \dots, f_k)$$

Hence the result.  $\square$

The following proposition comes from Definition 2.27, where  $\mathcal{J}_C$  is the Jacobian ideal defined in notation 2.12.

**Proposition 2.29.** *We have the following exact sequence of  $\mathcal{O}_S$ -modules:*

$$(4) \quad 0 \rightarrow \mathrm{Der}^k(-\log C) \rightarrow \Theta_S^k \xrightarrow{dh_1 \wedge \cdots \wedge dh_k} \mathcal{J}_C \rightarrow 0$$

For a hypersurface  $D$ , the evaluation of a logarithmic 1-form on a logarithmic vector field is holomorphic. It is no more true when we consider complete intersections of codimension at least 2:

**Lemma 2.30.** *Let  $\omega \in \Omega^k(\log C)$  and  $\delta \in \mathrm{Der}^k(-\log C)$ . Then  $\omega(\delta) \in \Sigma = \sum_{i=1}^k \frac{\mathcal{O}_S}{h_1 \cdots h_{i-1} h_{i+1} \cdots h_k}$ .*

*Proof.* Let us write thanks to Theorem 2.8  $g\omega = \xi \frac{dh_1 \wedge \cdots \wedge dh_k}{h} + \eta$ . Then, since  $\delta \in \mathrm{Der}^k(-\log C)$ , by definition  $\frac{dh_1 \wedge \cdots \wedge dh_k}{h}(\delta) \in \Sigma$  so that  $g\omega(\delta) \in \Sigma$ . Since  $g$  induces a non zero divisor in  $\mathcal{O}_C$ , we have  $\omega(\delta) \in \Sigma$ .  $\square$

*Remark 2.31.* For  $k = 1$ , one has  $\Sigma = \mathcal{O}_S$ , so that the lemma above is still true.

Thanks to Lemma 2.30, we see that we have a natural pairing

$$\mathrm{Der}^k(-\log C) \times \Omega^k(\log C) \rightarrow \Sigma$$

Therefore, we consider the left-exact functor  $\mathrm{Hom}_{\mathcal{O}_S}(-, \Sigma)$  instead of  $\mathrm{Hom}_{\mathcal{O}_S}(-, \mathcal{O}_S)$ .

**Proposition 2.32.** *We assume  $k \geq 2$ . For  $k = 1$ , we refer to [Sai80, (1.6)]. We consider the following sequences of inclusions:*

$$(5) \quad \Omega_S^k \subseteq \tilde{\Omega}^k \subseteq \Omega^k(\log C) \subseteq \frac{1}{h} \Omega_S^k \subseteq \Sigma \otimes_{\mathcal{O}_S} \frac{\Omega_S^k}{h}$$

$$(6) \quad \Sigma \otimes_{\mathcal{O}_S} \Theta_S^k \supseteq \Theta_S^k \supseteq \mathrm{Der}^k(-\log C) \supseteq \sum h_i \Theta_S^k \supseteq h \Theta_S^k$$

- a) *The functor  $\mathrm{Hom}_{\mathcal{O}_S}(-, \Sigma)$  applied to the sequence of inclusions (5) gives the sequence of inclusions (6).*  
b) *Conversely, the functor  $\mathrm{Hom}_{\mathcal{O}_S}(-, \Sigma)$  applied to the sequence of inclusions (6) gives the sequence of inclusions (5).*

*Proof.* For  $I = \{i_1, \dots, i_k\} \subseteq \{1, \dots, m\}$ , we denote  $\partial x_I = \partial x_{i_1} \wedge \cdots \wedge \partial x_{i_k} \in \Theta_S^k$ . We notice that the module  $\Sigma$  is torsion-free and for any inclusion  $M_1 \subseteq M_2$  with  $M_1, M_2$  both in (5) or both in (6), the module  $M_2/M_1$  is a torsion module, so that we have  $\mathrm{Hom}_{\mathcal{O}_S}(M_1, \Sigma) \supseteq \mathrm{Hom}_{\mathcal{O}_S}(M_2, \Sigma)$ .

- a) Let  $\varphi \in \mathrm{Hom}_{\mathcal{O}_S}(\Omega_S^k, \Sigma)$ . Since  $\Omega_S^k$  is free and  $\mathrm{Hom}_{\mathcal{O}_S}(\Omega_S^k, \mathcal{O}_S) = \Theta_S^k$ , we have the following isomorphism of  $\mathcal{O}_S$ -modules:

$$\begin{aligned} \xi : \mathrm{Hom}_{\mathcal{O}_S}(\Omega_S^k, \Sigma) &\xrightarrow{\sim} \Sigma \otimes_{\mathcal{O}_S} \Theta_S^k \\ \varphi &\mapsto \xi_\varphi = \sum_{|I|=k} \varphi(dx_I) \partial x_I \end{aligned}$$

Thanks to the isomorphism  $\xi$ , we can see each module  $\mathrm{Hom}_{\mathcal{O}_S}(M, \Sigma)$  for  $M$  in (5) as a submodule of  $\widetilde{\Theta}_S^k := \Sigma \otimes_{\mathcal{O}_S} \Theta_S^k$ . Moreover, it gives also:

$$\mathrm{Hom}_{\mathcal{O}_S}\left(\frac{1}{h} \Omega_S^k, \Sigma\right) = h \cdot \mathrm{Hom}_{\mathcal{O}_S}(\Omega_S^k, \Sigma) = h \cdot (\Sigma \otimes_{\mathcal{O}_S} \Theta_S^k) = \sum_{i=1}^k h_i \Theta_S^k$$

Obviously, we have the inclusion  $\Theta_S^k \subseteq \mathrm{Hom}_{\mathcal{O}_S}(\tilde{\Omega}^k, \Sigma) \subseteq \widetilde{\Theta}_S^k$ . Let  $\delta \in \widetilde{\Theta}_S^k$ ,  $\delta = \sum_I \delta_I \partial x_I$  with  $\delta_I \in \Sigma$ . Let us assume<sup>4</sup> that there is  $I$  such that  $\delta_I \notin \mathcal{O}_S$ . For example  $\delta_I$  has a pole at least along  $D_1$ . Then  $\frac{dx_I}{h_1}(\delta) = \frac{\delta_I}{h_1}$  has a pole of order at least 2 along  $D_1$ , and  $\delta \notin \mathrm{Hom}_{\mathcal{O}_S}(\tilde{\Omega}^k, \Sigma)$ . Therefore,  $\mathrm{Hom}_{\mathcal{O}_S}(\tilde{\Omega}^k, \Sigma) \simeq \Theta_S^k$ . Since  $\tilde{\Omega}^k = \Omega_S^k \otimes_{\mathcal{O}_S} \Sigma$ , it gives also  $\mathrm{Hom}_{\mathcal{O}_S}(\frac{1}{h} \Omega_S^k \otimes \Sigma, \Sigma) = h \Theta_S^k$ .

<sup>4</sup>We use here the assumption  $k \geq 2$

We have  $\text{Hom}_{\mathcal{O}_S}(\Omega^k(\log C), \Sigma) \subseteq \Theta_S^k$ . Thanks to Lemma 2.30, we have  $\text{Der}^k(-\log C) \subseteq \text{Hom}_{\mathcal{O}_S}(\Omega^k(\log C), \Sigma)$ . Conversely, let  $\delta \in \Theta_S^k$ . If  $\delta \in \text{Hom}_{\mathcal{O}_S}(\Omega^k(\log C), \Sigma)$ , then in particular  $\frac{dh_1 \wedge \dots \wedge dh_k}{h}(\delta) \in \Sigma$ , so that  $\delta \in \text{Der}^k(-\log C)$ . Hence  $\text{Hom}_{\mathcal{O}_S}(\Omega^k(\log C), \Sigma) = \text{Der}^k(-\log C)$ .

- b) Let  $\psi \in \text{Hom}_{\mathcal{O}_S}(h\Theta_S^k, \Sigma)$ . Since  $h\Theta_S^k$  is free and  $\text{Hom}_{\mathcal{O}_S}(h\Theta_S^k, \mathcal{O}_S) = \frac{1}{h}\Omega_S^k$ , we have the following isomorphism of  $\mathcal{O}_S$ -modules:

$$\begin{aligned} \beta : \text{Hom}_{\mathcal{O}_S}(h\Theta_S^k, \Sigma) &\rightarrow \frac{1}{h}\Omega_S^k \otimes \Sigma \\ \psi &\mapsto \omega_\psi = \frac{1}{h} \sum_{|I|=k} \psi(h\partial x_I) dx_I \end{aligned}$$

We have  $\psi(h\partial x_I) = \omega_\psi(h\partial x_I)$ . Thanks to the isomorphism  $\beta$ , we can see each module  $\text{Hom}_{\mathcal{O}_S}(M, \Sigma)$  for  $M$  in (6) as a submodule of  $\frac{1}{h}\tilde{\Omega}^k$ . It gives also  $\text{Hom}_{\mathcal{O}_S}(\Theta_S^k, \Sigma) = \tilde{\Omega}^k$ .

It is easy to see that  $\frac{1}{h}\Omega_S^k \subseteq \text{Hom}_{\mathcal{O}_S}(\sum h_i\Theta_S^k, \Sigma) \subseteq \frac{1}{h}\tilde{\Omega}^k$ . Let  $\omega \in \text{Hom}_{\mathcal{O}_S}(\sum h_i\Theta_S^k, \Sigma)$ ,  $\omega = \sum \frac{1}{h}\omega_I dx_I$  with  $\omega_I \in \Sigma$ . Let  $I \subset \{1, \dots, m\}$  with  $|I| = k$ . For all  $i \in \{1, \dots, k\}$ , we have  $\omega(h_i\partial x_I) = \frac{1}{h_i}\omega_I \in \Sigma$ . Therefore,  $\omega_I \in \mathcal{O}_S$  and  $\text{Hom}_{\mathcal{O}_S}(\sum h_i\Theta_S^k, \Sigma) = \frac{1}{h}\Omega_S^k$ . Since  $\Theta_S^k \otimes_{\mathcal{O}_S} \Sigma = \frac{1}{h} \sum h_i\Theta_S^k$ , we also have:  $\text{Hom}_{\mathcal{O}_S}(\Theta_S^k \otimes_{\mathcal{O}_S} \Sigma, \Sigma) = \Omega_S^k$ .

Since  $\text{Der}^k(-\log C)$  contains  $\sum h_i\Theta_S^k$ , we have  $\text{Hom}_{\mathcal{O}_S}(\text{Der}^k(-\log C), \Sigma) \subseteq \frac{1}{h}\Omega_S^k$ . Thanks to Lemma 2.30, we have  $\Omega^k(\log C) \subseteq \text{Hom}_{\mathcal{O}_S}(\text{Der}^k(-\log C), \Sigma)$ . In analogy with [Sai80, (1.6)], we introduce a well-chosen  $k$ -vector field  $\delta$  to prove the equality. Let us first notice that if  $\omega = \frac{1}{h}a_I dx_I$  with  $a_I \in \mathcal{O}_S$ , then for all  $i \in \{1, \dots, k\}$  we have:

$$(7) \quad dh_i \wedge \omega = \sum_j \sum_I \frac{1}{h} \frac{\partial h_i}{\partial x_j} a_I dx_j \wedge dx_I = \sum_{|J|=k+1} \sum_{\ell=1}^{k+1} (-1)^{\ell-1} \frac{1}{h} \frac{\partial h_i}{\partial x_{j_\ell}} a_{J \setminus \{j_\ell\}} dx_J$$

We set for all  $J = \{j_1, \dots, j_{k+1}\} \subseteq \{1, \dots, m\}$  and  $i \in \{1, \dots, k\}$ :

$$\delta_J^i = \sum_{\ell=1}^{k+1} (-1)^{\ell-1} \frac{\partial h_i}{\partial x_{j_\ell}} \partial x_{j_1} \wedge \dots \wedge \widehat{\partial x_{j_\ell}} \wedge \dots \wedge \partial x_{j_{k+1}}$$

We first prove that  $\delta_J^i \in \text{Der}^k(-\log C)$ . For  $I \subseteq \{1, \dots, m\}$  with  $|I| = k$  we denote by  $\Delta_I$  the  $k \times k$  minor of the Jacobian matrix of  $(h_1, \dots, h_k)$  relative to the set  $I$ . We then have  $dh_1 \wedge \dots \wedge dh_k = \sum \Delta_I dx_I$ . Thus:

$$dh_1 \wedge \dots \wedge dh_k(\delta_J^i) = \sum_{\ell=1}^{k+1} (-1)^{\ell-1} \frac{\partial h_i}{\partial x_{j_\ell}} \Delta_{j_1 \dots \widehat{j_\ell} \dots j_{k+1}} = \begin{vmatrix} \frac{\partial h_i}{\partial x_{j_1}} & \dots & \frac{\partial h_i}{\partial x_{j_{k+1}}} \\ \frac{\partial h_1}{\partial x_{j_1}} & \dots & \frac{\partial h_1}{\partial x_{j_{k+1}}} \\ \vdots & & \vdots \\ \frac{\partial h_k}{\partial x_{j_1}} & \dots & \frac{\partial h_k}{\partial x_{j_{k+1}}} \end{vmatrix} = 0$$

Therefore,  $\delta_J^i \in \text{Der}^k(-\log C)$ . Let us consider  $\omega = \sum \frac{1}{h}a_I dx_I \in \text{Hom}_{\mathcal{O}_S}(\text{Der}^k(-\log C), \Sigma)$ , with  $a_I \in \mathcal{O}_S$ . Then, for all  $J \subseteq \{1, \dots, m\}$  and  $i \in \{1, \dots, k\}$  we have:

$$\omega(\delta_J^i) = \sum_I \frac{1}{h} a_I dx_I(\delta_J^i) = \sum_{\ell=1}^{k+1} (-1)^{\ell-1} \frac{1}{h} \frac{\partial h_i}{\partial x_{j_\ell}} a_{J \setminus \{j_\ell\}} \in \Sigma$$

Thus, by (7), for all  $i \in \{1, \dots, k\}$ , we have  $dh_i \wedge \omega = \sum_{|J|=k+1} \omega(\delta_J^i) dx_J \in \tilde{\Omega}^{k+1}$ , so that  $\omega \in \Omega^k(\log C)$ . Hence the result:  $\text{Hom}_{\mathcal{O}_S}(\text{Der}^k(-\log C), \Sigma) = \Omega^k(\log C)$ .  $\square$

**2.5. Multi-residues and the Jacobian ideal.** In this subsection, we give another proof of [Sch13, Proposition 4.1], which does not depend on the isomorphism of Theorem 2.16. Our proof uses the duality of Proposition 2.32 and is analogous to the proof of [GS14, Proposition 3.4]. We recall that  $\mathcal{J}_C \subseteq \mathcal{O}_C$  is the Jacobian ideal.

We will need the following lemma:

**Lemma 2.33.** *We assume  $k \geq 2$ . Then  $\text{Ext}_{\mathcal{O}_S}^1(\mathcal{J}_C, \mathcal{O}_S) = 0$  and:*

$$\text{Ext}_{\mathcal{O}_S}^1(\mathcal{J}_C, \Sigma) = \text{Hom}_{\mathcal{O}_C}(\mathcal{J}_C, \mathcal{O}_C)$$

*Proof.* Let us apply the functor  $\text{Hom}_{\mathcal{O}_S}(\mathcal{J}_C, -)$  to the exact sequence:

$$0 \rightarrow \Sigma \xrightarrow{\times h} \mathcal{O}_S \rightarrow \mathcal{O}_C \rightarrow 0$$

It gives:

$$0 \rightarrow \text{Hom}_{\mathcal{O}_S}(\mathcal{J}_C, \mathcal{O}_C) \rightarrow \text{Ext}_{\mathcal{O}_S}^1(\mathcal{J}_C, \Sigma) \rightarrow \text{Ext}_{\mathcal{O}_S}^1(\mathcal{J}_C, \mathcal{O}_S) \rightarrow \dots$$

The depth of  $\mathcal{O}_S$  is  $m$  and since  $\mathcal{J}_C$  is a fractional ideal, the dimension of  $\mathcal{J}_C$  is  $m - k = \dim \mathcal{O}_C$ . Thus, by Ischebeck's Lemma (see [Mat80, 15.E]), we have  $\text{Ext}_{\mathcal{O}_S}^1(\mathcal{J}_C, \mathcal{O}_S) = 0$ . Hence the result.  $\square$

**Definition 2.34.** *Let  $I \subset \mathcal{M}_C$  be a fractional ideal. The dual of  $I$  is  $I^\vee = \text{Hom}_{\mathcal{O}_C}(I, \mathcal{O}_C)$ .*

*Remark 2.35.* Thanks to [dJP00, Proof of Lemma 1.5.14], we can identify  $I^\vee = \{f \in \mathcal{M}_C; f \cdot I \subseteq \mathcal{O}_C\}$ .

**Proposition 2.36.** *The dual of the Jacobian ideal is  $\mathcal{J}_C^\vee = \mathcal{R}_C$ .*

*Proof.* We assume  $k \geq 2$ . For  $k = 1$ , we refer to [GS14, Proposition 3.4].

We consider the double complex  $\text{Hom}_{\mathcal{O}_S}(\text{Der}^k(-\log C) \hookrightarrow \Theta_S^k, h : \Sigma \rightarrow \mathcal{O}_S)$ , which gives almost the same diagram as the dual of (3.8) in [GS14]. By Lemma 2.33,  $\text{Ext}_{\mathcal{O}_S}^1(\mathcal{J}_C, \mathcal{O}_S) = 0$ , so that we obtain the following commutative diagram:

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \text{Hom}_{\mathcal{O}_S}(\Theta_S^k, \Sigma) & \longrightarrow & \text{Hom}_{\mathcal{O}_S}(\text{Der}^k(-\log C), \Sigma) & \longrightarrow & \text{Ext}_{\mathcal{O}_S}^1(\mathcal{J}_C, \Sigma) \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \text{Hom}_{\mathcal{O}_S}(\Theta_S^k, \mathcal{O}_S) & \longrightarrow & \text{Hom}_{\mathcal{O}_S}(\text{Der}^k(-\log C), \mathcal{O}_S) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \text{Hom}_{\mathcal{O}_S}(\mathcal{J}_C, \mathcal{O}_C) & \longrightarrow & \text{Hom}_{\mathcal{O}_S}(\Theta_S^k, \mathcal{O}_C) & \longrightarrow & \text{Hom}_{\mathcal{O}_S}(\text{Der}^k(-\log C), \mathcal{O}_C) \longrightarrow \text{Ext}_{\mathcal{O}_S}^1(\mathcal{J}_C, \mathcal{O}_C) \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & \text{Ext}_{\mathcal{O}_S}^1(\mathcal{J}_C, \Sigma) & \longrightarrow & 0 & \longrightarrow & \text{Ext}_{\mathcal{O}_S}^1(\text{Der}^k(-\log C), \Sigma) \longrightarrow \text{Ext}_{\mathcal{O}_S}^2(\mathcal{J}_C, \Sigma) \longrightarrow 0 \\
& & \downarrow & & & & \downarrow \\
& & 0 & & & & 0
\end{array}$$

Let us prove by a diagram chasing process that  $\mathcal{J}_C^\vee = \mathcal{R}_C$ .

Let  $\varphi : \text{Der}^k(-\log C) \rightarrow \Sigma$ . Thanks to the isomorphism  $\beta$  of the proof of Proposition 2.32,  $\varphi$  corresponds to  $\omega_\varphi = \frac{1}{h} \sum_I \varphi(h \partial x_I) dx_I \in \Omega^k(\log C)$ . Then  $\varphi(\delta) = \omega_\varphi(\delta)$ .

By the vertical map,  $\varphi$  is sent on  $h\varphi : \text{Der}^k(-\log C) \rightarrow \mathcal{O}_S$ ,  $\delta \mapsto h\omega_\varphi(\delta)$ . Since the horizontal map is an isomorphism, there exists  $\Phi : \Theta_S^k \rightarrow \mathcal{O}_S$  such that  $\Phi|_{\text{Der}^k(-\log C)} = h\varphi$ . For all  $i \in \{1, \dots, k\}$ ,  $h_i \Phi(\partial x_I) = h\varphi(h_i \partial x_I) = h_i \varphi(h \partial x_I)$  and therefore,  $\Phi(\partial x_I) = \varphi(h \partial x_I) = h\omega_\varphi(\partial x_I)$ .

Let us denote by  $\bar{\Phi} : \Theta_S^k \rightarrow \mathcal{O}_C$  the image of  $\Phi$  by the vertical map. The image of  $\bar{\Phi}$  by the horizontal map is zero, so that  $\bar{\Phi}$  is the image of an element of  $\text{Hom}_{\mathcal{O}_S}(\mathcal{J}_C, \mathcal{O}_C)$ . Let us compute it explicitly. Since  $\omega_\varphi \in \Omega^k(\log C)$ , we can write thanks to Theorem 2.8,  $g\omega_\varphi = \xi \frac{dh_1 \wedge \dots \wedge dh_k}{h} + \eta$  with  $g \in \mathcal{O}_S$  which does not induce a zero divisor in  $\mathcal{O}_C$ ,  $\xi \in \mathcal{O}_S$  and  $\eta \in \tilde{\Omega}^k$ . Then:

$$g\bar{\Phi}(\partial x_I) = \overline{g\omega_\varphi(\partial x_I)} = \overline{\xi \partial x_I (dh_1 \wedge \dots \wedge dh_k)} \in \mathcal{O}_C$$

Since  $g$  is a non zero divisor in  $\mathcal{O}_C$ , we have:  $\overline{\Phi}(\partial x_I) = \overline{\text{res}_C(\omega_\varphi) \partial x_I (dh_1 \wedge \cdots \wedge dh_k)}$ . Hence, by identifying  $\text{Hom}_{\mathcal{O}_S}(\text{Der}^k(-\log C), \Sigma)$  and  $\Omega^k(\log C)$ , we obtain the map  $\text{res}_C : \Omega^k(\log C) \rightarrow \mathcal{J}_C^\vee$ ,  $\omega \mapsto \text{res}_C(\omega)$ .

In fact, by a similar diagram chasing process starting from the lower left  $\text{Hom}_{\mathcal{O}_S}(\mathcal{J}_C, \mathcal{O}_C)$  to the upper right  $\text{Hom}_{\mathcal{O}_S}(\text{Der}^k(-\log C), \Sigma)$ , one can show that  $\text{res}_C : \Omega^k(\log C) \rightarrow \mathcal{J}_C^\vee$  is surjective, hence,  $\mathcal{J}_C^\vee = \mathcal{R}_C$ .  $\square$

*Remark 2.37.* As in [GS14, (3.8)], we can also consider the double complex associated with

$$\text{Hom}_{\mathcal{O}_S} \left( \tilde{\Omega}^k \hookrightarrow \Omega^k(\log C), h : \Sigma \rightarrow \mathcal{O}_S \right)$$

It yields to the long exact sequence

$$(8) \quad 0 \rightarrow \text{Der}^k(-\log C) \rightarrow \Theta_S^k \xrightarrow{\sigma} \text{Ext}_{\mathcal{O}_S}^1(\mathcal{R}_C, \Sigma) \rightarrow \text{Ext}_{\mathcal{O}_S}^1(\Omega^k(\log C), \Sigma) \rightarrow \text{Ext}_{\mathcal{O}_S}^1(\tilde{\Omega}^k, \Sigma) \rightarrow \cdots$$

A proof similar to the proof of Lemma 2.33 and [GS14, Proposition 3.4] gives  $\text{Ext}_{\mathcal{O}_S}^1(\mathcal{R}_C, \Sigma) = \mathcal{R}_C^\vee$  and  $\sigma(\delta) = dh_1 \wedge \cdots \wedge dh_k(\delta)$ .

### 3. FREE COMPLETE INTERSECTIONS

In [Sai80], K. Saito studies a particular family of hypersurfaces for which the module of logarithmic vector fields is free. This kind of hypersurfaces are called free divisors. Several characterizations of freeness for divisors can be extended to the complete intersection case (see Proposition 3.4 and Theorem 3.5).

We first recall the definition and a characterization of freeness for reduced divisors.

**Definition 3.1.** *Let  $D \subset S$  be the germ of a reduced divisor defined by  $f \in \mathcal{O}_S$ . We call  $D$  a germ of free divisor if  $\Omega^1(\log D)$  is a free  $\mathcal{O}_S$ -module, or equivalently  $\text{Der}(-\log D)$  is a free  $\mathcal{O}_S$ -module.*

**THEOREM 3.2** ([Ale88]). *The germ of a reduced singular divisor is free if and only if its singular locus is Cohen-Macaulay of codimension 1 in  $D$ . It is equivalent to the Jacobian ideal being maximal Cohen-Macaulay.*

Our purpose here is to extend the notion of freeness to the case of complete intersections. As in [GS12] and [Sch13], we give the following definition for freeness, which comes obviously from Theorem 3.2.

**Definition 3.3.** *A reduced complete intersection  $C$  is called free if it is smooth or if its singular locus is Cohen-Macaulay of codimension 1 (i.e.  $\mathcal{O}_C/\mathcal{J}_C$  is a Cohen-Macaulay  $\mathcal{O}_S$ -module of dimension  $m - k - 1$ ).*

They also give the following characterizations of freeness, where  $\text{projdim}(M)$  denotes the minimal length of a projective  $\mathcal{O}_S$ -resolution of a  $\mathcal{O}_S$ -module  $M$ :

**Proposition 3.4.** *Let  $C$  be a singular reduced complete intersection. The following statements are equivalent:*

- (1)  $C$  is free
- (2)  $\mathcal{J}_C$  is a Cohen-Macaulay  $\mathcal{O}_C$ -module
- (3)  $\text{projdim}(\text{Der}^k(-\log C)) \leq k - 1$
- (4)  $\text{projdim}(\text{Der}^k(-\log C)) = k - 1$

We add to this list other characterizations, which leads to our main result:

**THEOREM 3.5.** *We can extend the list of equivalences of Proposition 3.4 with:*

- (5)  $\text{projdim}(\Omega^k(\log C)) \leq k - 1$
- (6)  $\text{projdim}(\Omega^k(\log C)) = k - 1$

In particular, for  $k = 1$ , we recognize the several characterizations of freeness for divisors we have just mentioned. In the hypersurface case, the duality between  $\text{Der}(-\log D)$  and  $\Omega^1(\log D)$  gives immediately the fact that if one of the two modules is free, the other is also free, whereas for complete intersections with  $k \geq 2$ , the statement on the projective dimension of  $\Omega^k(\log C)$  needs much more work.

**3.1. Proof of Proposition 3.4.** We first prove the equivalences of Proposition 3.4. These equivalences are already proved or mentioned in [GS12] or [Sch13]. The proof is based on the Depth Lemma and the Auslander-Buchsbaum Formula, which are recalled below, and can be found for example in [dJP00].

**Proposition 3.6** (Depth Lemma, [dJP00, Lemma 6.5.18]). *Let  $R$  be a Noetherian local ring. Let*

$$0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$$

*be a short exact sequence of  $R$ -modules. Then:*

$$\text{depth}(M_2) \geq \min(\text{depth}(M_1), \text{depth}(M_3))$$

*In case this inequality is strict, we have:  $\text{depth}(M_1) = \text{depth}(M_3) + 1$ .*

**Proposition 3.7** (Auslander-Buchsbaum Formula, [dJP00, Theorem 6.5.20]). *Let  $R$  be a Noetherian local ring and  $M$  be a finite type  $R$ -module of finite projective dimension. Then:*

$$\text{depth}(M) + \text{projdim}(M) = \text{depth}(R)$$

*Remark 3.8.* By Hilbert's Syzygy Theorem (see [dJP00, Theorem 6.5.16]), since the ring  $\mathcal{O}_S$  is a regular local ring, all finite type  $\mathcal{O}_S$ -module has finite projective dimension, so that we can apply Auslander-Buchsbaum Formula.

*Remark 3.9.* The dimension of a fractional ideal of  $\mathcal{O}_C$  (as a  $\mathcal{O}_S$ -module or a  $\mathcal{O}_C$ -module) is  $m - k$ . The depth of a fractional ideal of  $\mathcal{O}_C$  as a  $\mathcal{O}_C$ -module is equal to its depth as a  $\mathcal{O}_S$ -module.

*Proof of Proposition 3.4.* The equivalence 1.  $\iff$  2. is proved in [Sch13], but we recall here the proof.

Let us consider the following exact sequence of  $\mathcal{O}_C$ -modules:

$$(9) \quad 0 \rightarrow \mathcal{J}_C \rightarrow \mathcal{O}_C \rightarrow \mathcal{O}_C/\mathcal{J}_C \rightarrow 0$$

We set  $n = m - k$ , which is the dimension of  $C$ . Since  $C$  is a complete intersection,  $\mathcal{O}_C$  is Cohen-Macaulay of dimension  $n$ . Moreover, since we assume  $C$  to be reduced, the singular locus of  $C$  is of dimension at most  $n - 1$ , and therefore the depth of  $\mathcal{O}_C/\mathcal{J}_C$  is at most  $n - 1$ . Then, by the Depth Lemma 3.6, we have

$$\text{depth}(\mathcal{J}_C) = \text{depth}(\mathcal{O}_C/\mathcal{J}_C) + 1$$

Therefore,  $\text{depth}(\mathcal{J}_C) = n \iff \text{depth}(\mathcal{O}_C/\mathcal{J}_C) = n - 1$ , hence 1.  $\iff$  2.

We now prove 2.  $\implies$  4. We recall the following exact sequence:

$$(10) \quad 0 \rightarrow \text{Der}^k(-\log C) \rightarrow \Theta_S^k \rightarrow \mathcal{J}_C \rightarrow 0$$

Then, thanks to the Depth Lemma 3.6, since the depth of  $\mathcal{J}_C$  is  $m - k$  and the depth of  $\Theta_S^k$  is  $m$ , we have  $\text{depth}(\text{Der}^k(-\log C)) = m - k + 1$ . By the Auslander-Buchsbaum Formula 3.7, we have  $\text{projdim}(\text{Der}^k(-\log C)) = k - 1$ .

The implication 4  $\implies$  3. is trivial.

Let us prove 3.  $\implies$  2. This implication is already mentioned in [GS12] but we give here a more detailed proof. By the Auslander-Buchsbaum Formula 3.7,  $\text{depth}(\text{Der}^k(-\log C)) \geq m - k + 1$ . Moreover,  $\text{depth}(\Theta_S^k) = m$ , and  $\text{depth}(\mathcal{J}_C) \leq m - k$ . As a consequence of the exact sequence (10) and the Depth Lemma 3.6 we have  $\text{depth}(\text{Der}^k(-\log C)) = m - k + 1$  and  $\text{depth}(\mathcal{J}_C) = m - k$ , so that  $\mathcal{J}_C$  is maximal Cohen-Macaulay.  $\square$

**3.2. Preliminary to the proof of Theorem 3.5.** The methods used to prove Proposition 3.4 are not sufficient for Theorem 3.5. We will study the long exact sequence obtained by applying the functor  $\text{Hom}_{\mathcal{O}_S}(-, \mathcal{O}_S)$  to the short exact sequence

$$0 \rightarrow \tilde{\Omega}^k \rightarrow \Omega^k(\log C) \rightarrow \mathcal{R}_C \rightarrow 0$$

We also introduce the Koszul complex associated to a regular sequence and the change of rings spectral sequence, which are used to determine the modules and morphisms involved in the long exact sequence.

*Notation 3.10.* We denote by  $K(\underline{h})$  the Koszul complex of  $(h_1, \dots, h_k)$  in  $\mathcal{O}_S$ :

$$(11) \quad K(\underline{h}) : 0 \rightarrow \bigwedge^k \mathcal{O}_S \xrightarrow{d_k} \dots \xrightarrow{d_2} \bigwedge^1 \mathcal{O}_S \xrightarrow{d_1} \mathcal{O}_S \rightarrow 0$$

where for  $I \subseteq \{1, \dots, k\}$  of cardinal  $p$ ,  $d_i(e_I) = \sum_{\ell=1}^p (-1)^{\ell-1} h_{i_\ell} e_{i_1} \wedge \dots \wedge \widehat{e_{i_\ell}} \wedge \dots \wedge e_{i_p}$ .

We also set  $\tilde{K}(\underline{h})$  the complex obtained from  $K(\underline{h})$  obtained by removing the last  $\mathcal{O}_S$ .

**Lemma 3.11** ([Eis95, Corollary 17.5, Proposition 17.15]). *Since the sequence  $(h_1, \dots, h_k)$  is regular,  $K(\underline{h})$  is a free  $\mathcal{O}_S$ -resolution of  $\mathcal{O}_C$ .*

*The dual complex  $\text{Hom}_{\mathcal{O}_S}(K(\underline{h}), \mathcal{O}_S)$  of the Koszul complex is a free resolution of  $\mathcal{O}_C$ .*

*Remark 3.12.* One can see that a consequence of Lemma 3.11 is that  $\tilde{K}(\underline{h})$  is a free  $\mathcal{O}_S$ -resolution of  $\Sigma \simeq (h_1, \dots, h_k)$ .

The following properties will be useful to prove Theorem 3.5.

**Proposition 3.13.** *If  $C$  is a free complete intersection, then  $\mathcal{R}_C$  is a maximal Cohen-Macaulay module and  $\mathcal{R}_C^\vee = \mathcal{J}_C$ .*

*Proof.* Since  $C$  is free,  $\mathcal{J}_C$  is a maximal Cohen-Macaulay module. Thanks to [Eis95, Theorem 21.21], we can deduce that  $\mathcal{J}_C^\vee$  is also a maximal Cohen-Macaulay module, and  $\mathcal{J}_C^{\vee\vee} = \mathcal{J}_C$ . But from Proposition 2.36,  $\mathcal{J}_C^\vee = \mathcal{R}_C$ . Hence the result.  $\square$

**Lemma 3.14.** *We have:*

$$\text{Hom}_{\mathcal{O}_S}(\tilde{\Omega}^k, \mathcal{O}_S) = h\Theta_S^k$$

$$\text{Ext}_{\mathcal{O}_S}^{k-1}(\tilde{\Omega}^k, \mathcal{O}_S) = \Theta_S^k \otimes_{\mathcal{O}_S} \mathcal{O}_C$$

and for all  $j \neq 0, k-1$ ,  $\text{Ext}_{\mathcal{O}_S}^j(\tilde{\Omega}^k, \mathcal{O}_S) = 0$ .

Moreover,  $\text{projdim}(\tilde{\Omega}^k) = k-1$ .

*Proof.* Since  $\tilde{\Omega}^k = \Omega_S^k \otimes_{\mathcal{O}_S} \Sigma$ , we have  $\text{Hom}_{\mathcal{O}_S}(\tilde{\Omega}^k, \mathcal{O}_S) = \Theta_S^k \otimes_{\mathcal{O}_S} \text{Hom}_{\mathcal{O}_S}(\Sigma, \mathcal{O}_S)$ .

Let  $\psi \in \text{Hom}_{\mathcal{O}_S}(\Sigma, \mathcal{O}_S)$ . Then it is determined by  $\psi(1)$ . Since for all  $i \in \{1, \dots, k\}$ ,  $\psi\left(\frac{1}{h_i}\right) = \frac{1}{h_i}\psi(1) \in \mathcal{O}_S$ , we have  $\psi(1) \in (h)\mathcal{O}_S$ . Hence the first equality.

By Remark 3.12, the complex  $\tilde{K}(\underline{h})$  is a free resolution of  $\Sigma$ . Therefore  $\tilde{K}(\underline{h}) \otimes_{\mathcal{O}_S} \Omega_S^k$  is a free resolution of  $\tilde{\Omega}^k$ . The lemma follows from Lemma 3.11.  $\square$

The same methods as the ones used in the proof of Proposition 3.4 applied to the exact sequence  $0 \rightarrow \tilde{\Omega}^k \rightarrow \Omega^k(\log C) \rightarrow \mathcal{R}_C \rightarrow 0$  give the following result, which will be used in the end of the proof of Theorem 3.5:

**Lemma 3.15.** *If  $C$  is free, then  $\text{projdim}(\Omega^k(\log C)) \leq k$ .*

*Let  $C$  be a reduced complete intersection, not necessarily free. If  $\text{projdim}(\Omega^k(\log C)) \leq k-1$ , then  $\mathcal{R}_C$  is a maximal Cohen-Macaulay module.*

*Proof.* Let us consider the exact sequence  $0 \rightarrow \tilde{\Omega}^k \rightarrow \Omega^k(\log C) \rightarrow \mathcal{R}_C \rightarrow 0$ . Then, if  $C$  is free, by Proposition 3.13, we have  $\text{depth}(\mathcal{R}_C) = m - k$ . By Lemma 3.14 and the Auslander Buchsbaum Formula 3.7, we have  $\text{depth}(\tilde{\Omega}^k) = m - k + 1$ . Therefore, by the Depth Lemma 3.6,  $\text{depth}(\Omega^k(\log C)) \leq m - k$ .

Let us assume now that  $\text{projdim}(\Omega^k(\log C)) \leq k - 1$ . Then,  $\text{depth}(\Omega^k(\log C)) \geq m - k + 1$ . Since  $\text{depth}(\tilde{\Omega}^k) = m - k + 1$  and  $\text{depth}(\mathcal{R}_C) \leq m - k$ , by the Depth Lemma,  $\text{depth}(\mathcal{R}_C) = m - k = \dim(\mathcal{R}_C)$ . Hence the result.  $\square$

We have to use extra tools to prove Theorem 3.5. Let us apply the functor  $\text{Hom}_{\mathcal{O}_S}(-, \mathcal{O}_S)$  to the short exact sequence

$$0 \rightarrow \tilde{\Omega}^k \rightarrow \Omega^k(\log C) \rightarrow \mathcal{R}_C \rightarrow 0$$

It gives the following long exact sequence:

(12)

$$0 \rightarrow \text{Hom}_{\mathcal{O}_S}(\mathcal{R}_C, \mathcal{O}_S) \rightarrow \text{Hom}_{\mathcal{O}_S}(\Omega^k(\log C), \mathcal{O}_S) \rightarrow \text{Hom}_{\mathcal{O}_S}(\tilde{\Omega}^k, \mathcal{O}_S) \rightarrow \text{Ext}_{\mathcal{O}_S}^1(\mathcal{R}_C, \mathcal{O}_S) \rightarrow \dots$$

The purpose of the remaining of this subsection is to prove the following proposition:

**Proposition 3.16.** *The long exact sequence (12) gives:*

(13)

$$\dots \rightarrow 0 \rightarrow \text{Ext}_{\mathcal{O}_S}^{k-1}(\Omega^k(\log C), \mathcal{O}_S) \rightarrow \Theta_S^k \otimes_{\mathcal{O}_S} \mathcal{O}_C \xrightarrow{\alpha} \mathcal{R}_C^\vee \rightarrow \text{Ext}_{\mathcal{O}_S}^k(\Omega^k(\log C), \mathcal{O}_S) \rightarrow 0 \rightarrow \dots$$

Moreover,  $\alpha : \Theta_S^k \otimes_{\mathcal{O}_S} \mathcal{O}_C \ni \delta \otimes a \mapsto a \cdot \delta(dh_1 \wedge \dots \wedge dh_k) \in \mathcal{R}_C^\vee$ , so that the image of  $\alpha$  is  $\mathcal{I}_C$ .

All the modules involving  $\tilde{\Omega}^k$  in (12) are computed in Lemma 3.14. To compute the modules involving  $\mathcal{R}_C$ , we introduce the change of rings spectral sequence (see for example [CE56, Chapter XV and XVI] for details on spectral sequences). The change of rings spectral sequence applied to  $\mathcal{R}_C$  and  $\mathcal{O}_S$  gives:

$$(14) \quad E_2^{pq} = \text{Ext}_{\mathcal{O}_C}^p(\mathcal{R}_C, \text{Ext}_{\mathcal{O}_S}^q(\mathcal{O}_C, \mathcal{O}_S)) \Rightarrow \text{Ext}_{\mathcal{O}_S}^{p+q}(\mathcal{R}_C, \mathcal{O}_S)$$

**Lemma 3.17.** *For all  $q < k$ ,  $\text{Ext}_{\mathcal{O}_S}^q(\mathcal{R}_C, \mathcal{O}_S) = 0$  and  $\text{Ext}_{\mathcal{O}_S}^k(\mathcal{R}_C, \mathcal{O}_S) = \text{Hom}_{\mathcal{O}_C}(\mathcal{R}_C, \mathcal{O}_C)$ . More precisely, if  $\mathcal{I}^\bullet$  is an injective resolution of  $\mathcal{O}_S$ , the change of ring isomorphism is given by:*

$$\begin{aligned} \beta : H^k(\text{Hom}_{\mathcal{O}_S}(\mathcal{R}_C, \mathcal{I}^\bullet)) &\rightarrow \text{Hom}_{\mathcal{O}_C}(\mathcal{R}_C, H^k(\text{Hom}_{\mathcal{O}_S}(\mathcal{O}_C, \mathcal{I}^\bullet))) \\ [\psi] &\mapsto \tilde{\psi} : \rho \mapsto [\tilde{\psi}_\rho : a \mapsto a \cdot \psi(\rho)] \end{aligned}$$

*Proof.* Since  $(h_1, \dots, h_k)$  is a regular sequence, we have for all  $q \neq k$ ,  $\text{Ext}_{\mathcal{O}_S}^q(\mathcal{O}_C, \mathcal{O}_S) = 0$  and  $\text{Ext}_{\mathcal{O}_S}^k(\mathcal{O}_C, \mathcal{O}_S) = \mathcal{O}_C$ . Therefore, the only non zero terms of the second sheet of the spectral sequence (14) are the  $E_2^{pk}$ , so that the spectral sequence degenerates at rank 2. Hence the first claim.

To compute explicitly the isomorphism between  $\text{Ext}_{\mathcal{O}_S}^k(\mathcal{R}_C, \mathcal{O}_S)$  and  $\text{Hom}_{\mathcal{O}_C}(\mathcal{R}_C, \mathcal{O}_C)$ , we consider a free  $\mathcal{O}_C$ -resolution of  $\mathcal{R}_C$  (with differential  $\delta$ ) and an injective  $\mathcal{O}_S$ -resolution of  $\mathcal{O}_S$  (with differential  $\varepsilon$ ). It leads to the double complex  $A^{pq} = \text{Hom}_{\mathcal{O}_C}(P_p, \text{Hom}_{\mathcal{O}_S}(\mathcal{O}_C, \mathcal{I}^q))$ . There are two associated spectral sequences:

$${}^I E_2^{pq} = H^p(\text{Hom}_{\mathcal{O}_C}(P_*, H^q(\text{Hom}_{\mathcal{O}_S}(\mathcal{O}_C, \mathcal{I}^\bullet)))) = \text{Ext}_{\mathcal{O}_C}^p(\mathcal{R}_C, \text{Ext}_{\mathcal{O}_S}^q(\mathcal{O}_C, \mathcal{O}_S))$$

$${}^II E_2^{pq} = H^q(\text{Hom}_{\mathcal{O}_C}(H_p(P_*), \text{Hom}_{\mathcal{O}_S}(\mathcal{O}_C, \mathcal{I}^\bullet)))$$

For the first spectral sequence, the only non zero terms are the  ${}^I E_2^{pk}$  and for the second spectral sequence, the only non zero terms are  ${}^II E_2^{0q}$ . In particular, it means that the two spectral sequences



degenerates at rank 2, and for all  $j \geq 1$ ,  ${}''E_\infty^{j,k-j}$  and  ${}'E_\infty^{j,k-j}$  are both zero. Therefore, one has:

$$\mathrm{Ext}_{\mathcal{O}_S}^k(\mathcal{R}_C, \mathcal{O}_S) = {}'E_2^{0k} = {}'E_\infty^{0k} = {}''E_\infty^{0k} = {}''E_2^{0k}$$

One can then see from the definitions of spectral sequences (see [CE56, Chapter XV and XVI]) that an element in  ${}'E_\infty^{0k}$  can be represented by an element  $\psi \in \mathrm{Hom}_{\mathcal{O}_C}(P_0, \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{O}_C, \mathcal{I}^k))$ , which defines the same class in  ${}''E_\infty^{0k}$ . Moreover, in  ${}'E_2^{0k}$ ,  $\psi$  defines an element  $[\psi] \in H^k(\mathrm{Hom}_{\mathcal{O}_S}(\mathcal{R}_C, \mathcal{I}^\bullet))$ , and in  ${}''E_2^{0k}$ , it defines  $\tilde{\psi} : \rho \mapsto [\tilde{\psi}_\rho : a \mapsto a \cdot \psi(\rho)] \in \mathrm{Hom}_{\mathcal{O}_C}(\mathcal{R}_C, H^k(\mathrm{Hom}_{\mathcal{O}_S}(\mathcal{O}_C, \mathcal{I}^\bullet)))$ , which gives the announced isomorphism.  $\square$

*Proof of Proposition 3.16.* The exactness of the sequence comes from Lemmas 3.14 and 3.17. We only have to compute  $\alpha$ . We consider an injective  $\mathcal{O}_S$  resolution  $\mathcal{I}^\bullet$  of  $\mathcal{O}_S$  with differential  $\varepsilon_j : \mathcal{I}^j \rightarrow \mathcal{I}^{j+1}$ .

We want to construct the connecting homomorphism

$$\alpha : H^{k-1}(\mathrm{Hom}_{\mathcal{O}_S}(\tilde{\Omega}^k, \mathcal{I}^\bullet)) \rightarrow H^k(\mathrm{Hom}_{\mathcal{O}_S}(\mathcal{R}_C, \mathcal{I}^\bullet))$$

For all  $j \in \mathbb{N}$ , we have an isomorphism  $\mathrm{Hom}_{\mathcal{O}_S}(\tilde{\Omega}^k, \mathcal{I}^j) \rightarrow \Theta_S^k \otimes_{\mathcal{O}_S} \mathrm{Hom}_{\mathcal{O}_S}(\Sigma, \mathcal{I}^j)$ .

Since  $0 \rightarrow \Sigma \xrightarrow{h} \mathcal{O}_S \rightarrow \mathcal{O}_C \rightarrow 0$  is exact and  $\mathcal{I}^j$  is injective, the following map is an isomorphism:

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{O}_S, \mathcal{I}^j) / \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{O}_C, \mathcal{I}^j) &\rightarrow \mathrm{Hom}_{\mathcal{O}_S}(\Sigma, \mathcal{I}^j) \\ [\varphi : \mathcal{O}_S \rightarrow \mathcal{I}^j] &\mapsto (a \mapsto \varphi(h \cdot a)) \end{aligned}$$

Moreover,  $\mathrm{Hom}_{\mathcal{O}_S}(\mathcal{O}_S, \mathcal{I}^j) \simeq \mathcal{I}^j$  and  $\mathrm{Hom}_{\mathcal{O}_S}(\mathcal{O}_C, \mathcal{I}^j) \simeq \mathrm{Ann}_{\mathcal{I}^j}(h_1, \dots, h_k)$ , so that we construct the isomorphism

$$\beta_{k-1} : H^{k-1}(\mathrm{Hom}_{\mathcal{O}_S}(\tilde{\Omega}^k, \mathcal{I}^\bullet)) \rightarrow \Theta_S^k \otimes_{\mathcal{O}_S} H^{k-1}(\mathcal{I}^\bullet / \mathrm{Ann}_{\mathcal{I}^\bullet}(h_1, \dots, h_k))$$

which is given by:  $[\varphi] \in H^{k-1}(\mathrm{Hom}_{\mathcal{O}_S}(\tilde{\Omega}^k, \mathcal{I}^\bullet)) \mapsto \sum_I \partial x_I \otimes [\overline{m}_I]$  where  $h \cdot m_I = \varphi(dx_I)$ . The inverse is given by  $\delta \otimes [\overline{m}] \mapsto [\varphi : \omega \mapsto \delta(h\omega) \cdot m]$ .

On the other hand, let us prove that the following homomorphism is well defined and is an isomorphism:

$$\begin{aligned} \gamma : H^{k-1}(\mathcal{I}^\bullet / \mathrm{Ann}_{\mathcal{I}^\bullet}(h_1, \dots, h_k)) &\rightarrow H^k(\mathrm{Ann}_{\mathcal{I}^\bullet}(h_1, \dots, h_k)) \\ [\overline{m}] &\mapsto [\varepsilon_{k-1}(m)] \end{aligned}$$

Let us denote  $\overline{\varepsilon_{k-1}} : \mathcal{I}^{k-1} / \mathrm{Ann}_{\mathcal{I}^{k-1}}(h_1, \dots, h_k) \rightarrow \mathcal{I}^k / \mathrm{Ann}_{\mathcal{I}^k}(h_1, \dots, h_k)$ . If  $\overline{m} \in \mathrm{Ker}(\overline{\varepsilon_{k-1}})$  then  $\varepsilon_{k-1}(m) \in \mathrm{Ann}_{\mathcal{I}^k}(h_1, \dots, h_k)$ . If  $\overline{m} = \overline{\varepsilon_{k-2}(m')}$  for an element  $m' \in \mathcal{I}^{k-2} / \mathrm{Ann}_{\mathcal{I}^{k-2}}(h_1, \dots, h_k)$ , then  $[\varepsilon_{k-1}(\varepsilon_{k-2}(m'))] = 0$  so that the map  $\gamma$  is well defined. Let us assume that  $[\varepsilon_{k-1}(m)] = 0$ . Then, there exists  $m' \in \mathrm{Ann}_{\mathcal{I}^{k-1}}(h_1, \dots, h_k)$  such that  $\varepsilon_{k-1}(m) = \varepsilon_{k-1}(m')$ , so that  $m - m' \in \mathrm{Ker}(\varepsilon_{k-1}) = \mathrm{Im}(\varepsilon_{k-2})$ . Hence  $[\overline{m}] = 0$ , therefore, the map  $\gamma$  is injective. Let us consider  $[m] \in H^k(\mathrm{Ann}_{\mathcal{I}^\bullet}(h_1, \dots, h_k))$ . Then  $\varepsilon_k(m) = 0$  thus there exists  $m' \in \mathcal{I}^{k-1}$  such that  $\varepsilon_{k-1}(m') = m$ . Then  $[m] = \gamma([\overline{m'}])$ .

We now have all the identifications we need to compute  $\alpha$ . This homomorphism can be computed thanks to a diagram chasing process based on the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longleftarrow & \mathrm{Hom}_{\mathcal{O}_S}(\tilde{\Omega}^k, \mathcal{I}^{k-1}) & \xleftarrow{i^*} & \mathrm{Hom}_{\mathcal{O}_S}(\Omega^k(\log C), \mathcal{I}^{k-1}) & \xleftarrow{\mathrm{res}_C^*} & \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{R}_C, \mathcal{I}^{k-1}) \longleftarrow 0 \\ & & \downarrow \varepsilon_{k-1} & & \downarrow \varepsilon_{k-1} & & \downarrow \varepsilon_{k-1} \\ 0 & \longleftarrow & \mathrm{Hom}_{\mathcal{O}_S}(\tilde{\Omega}^k, \mathcal{I}^k) & \xleftarrow{i^*} & \mathrm{Hom}_{\mathcal{O}_S}(\Omega^k(\log C), \mathcal{I}^k) & \xleftarrow{\mathrm{res}_C^*} & \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{R}_C, \mathcal{I}^k) \longleftarrow 0 \end{array}$$

Let  $\delta \otimes [\overline{m}] \in \Theta_S^k \otimes H^{k-1}(\mathcal{I}^\bullet / \mathrm{Ann}_{\mathcal{I}^\bullet}(h_1, \dots, h_k))$ . It corresponds to the class of an element

$$\begin{aligned} \varphi : \tilde{\Omega}^k &\rightarrow \mathcal{I}^{k-1} \\ \eta &\mapsto \delta(h\omega) \cdot m \end{aligned}$$

which satisfies  $\varepsilon_{k-1}(\varphi) = 0$ .

There exists  $\Phi : \Omega^k(\log C) \rightarrow \mathcal{I}^{k-1}$  such that  $\Phi \circ i = \varphi$ . Let  $\omega \in \Omega^k(\log C)$ . By Theorem 2.8, there exists  $g, \xi, \eta$  such that  $g\omega = \xi \frac{dh_1 \wedge \cdots \wedge dh_k}{h_1 \cdots h_k} + \eta$ . Then

$$g\Phi(\omega) = \xi \Phi \left( \frac{dh_1 \wedge \cdots \wedge dh_k}{h_1 \cdots h_k} \right) + \varphi(\eta)$$

Moreover, for all  $i \in \{1, \dots, k\}$ ,  $h_i \Phi \left( \frac{dh_1 \wedge \cdots \wedge dh_k}{h_1 \cdots h_k} \right) = \varphi \left( h_i \frac{dh_1 \wedge \cdots \wedge dh_k}{h_1 \cdots h_k} \right) = h_i \delta(dh_1 \wedge \cdots \wedge dh_k) \cdot m$ . Therefore,

$$\Phi \left( \frac{dh_1 \wedge \cdots \wedge dh_k}{h_1 \cdots h_k} \right) = \delta(dh_1 \wedge \cdots \wedge dh_k) \cdot m + m'$$

with  $m' \in \text{Ann}_{\mathcal{I}^{k-1}}(h_1, \dots, h_k)$ .

The image by  $\varepsilon_{k-1}$  of  $\Phi$  satisfies:

$$g \cdot \varepsilon_{k-1}(\Phi)(\omega) = \xi \left( \delta(dh_1 \wedge \cdots \wedge dh_k) \cdot \varepsilon_{k-1}(m) + \varepsilon_{k-1}(m') \right)$$

Since  $i^* \varepsilon_{k-1}(\Phi) = 0$ , there exists  $\Psi : \mathcal{R}_C \rightarrow \mathcal{I}^k$  such that  $\Phi = \text{res}_C^*(\Psi)$ . We set:

$$\begin{aligned} \Psi : \mathcal{R}_C &\rightarrow \mathcal{I}^k \\ \rho &\mapsto \rho \cdot \left( \delta(dh_1 \wedge \cdots \wedge dh_k) \cdot \varepsilon_{k-1}(m) + \varepsilon_{k-1}(m') \right) \end{aligned}$$

The image of the class  $[\Psi] \in H^k(\text{Hom}_{\mathcal{O}_S}(\mathcal{R}_C, \mathcal{I}^\bullet))$  by the isomorphism  $\beta$  of Lemma 3.17 with the identification  $H^k(\text{Hom}_{\mathcal{O}_S}(\mathcal{O}_C, \mathcal{I}^\bullet)) = H^k(\text{Ann}_{\mathcal{I}^\bullet}(h_1, \dots, h_k))$  is, since  $m' \in \text{Ann}_{\mathcal{I}^{k-1}}(h_1, \dots, h_k)$ :

$$\begin{aligned} \mathcal{R}_C &\rightarrow H^k(\text{Ann}_{\mathcal{I}^\bullet}(h_1, \dots, h_k)) \\ \rho &\mapsto [\rho \cdot \delta(dh_1 \wedge \cdots \wedge dh_k) \cdot \varepsilon_{k-1}(m)] \end{aligned}$$

Moreover, there are isomorphisms:

$$\mathcal{O}_C \xleftarrow{\gamma_1} H^{k-1}(\mathcal{I}^\bullet / \text{Ann}_{\mathcal{I}^\bullet}(h_1, \dots, h_k)) \xrightarrow{\gamma} H^k(\text{Ann}_{\mathcal{I}^\bullet}(h_1, \dots, h_k)) \xrightarrow{\gamma_2} \mathcal{O}_C$$

Let  $\bar{a} = \gamma_1([\bar{m}]) \in \mathcal{O}_C$ . Since  $\gamma, \gamma_1, \gamma_2$  are isomorphisms, we can assume that  $\gamma_2 \circ \gamma \circ \gamma^{-1}(\bar{1}) = \bar{1}$ , so that  $\gamma_2([\varepsilon_{k-1}(m)]) = \bar{a} \in \mathcal{O}_C$ .

Hence the result: let  $\delta \otimes \bar{a} \in \Theta_S \otimes_{\mathcal{O}_S} \mathcal{O}_C$ , then  $\alpha(\delta \otimes \bar{a}) = \bar{a} \cdot \delta(dh_1 \wedge \cdots \wedge dh_k)$ .  $\square$

**3.3. End of the proof of Theorem 3.5.** The explicit computation of the connecting morphism  $\alpha$  in Proposition 3.16 allows us to compare  $\text{Im}(\alpha) = \mathcal{J}_C$  and  $\mathcal{R}_C^\vee$ , which is useful to finish the proof of Theorem 3.5.

*End of the proof of Theorem 3.5.* We start with the implication 2.  $\Rightarrow$  6. By Lemma 3.15, we have  $\text{projdim}(\Omega^k(\log C)) \leq k$ . Moreover, by Proposition 3.13,  $\mathcal{R}_C^\vee = \mathcal{J}_C$  so that the map  $\alpha$  of Proposition 3.16 is surjective. Therefore, we have  $\text{Ext}_{\mathcal{O}_S}^k(\Omega^k(\log C), \mathcal{O}_S) = 0$ .

Let

$$(15) \quad 0 \rightarrow \mathcal{O}_S^{\ell_k} \xrightarrow{d_k} \mathcal{O}_S^{\ell_{k-1}} \rightarrow \cdots \rightarrow \mathcal{O}_S^{\ell_0} \rightarrow \Omega^k(\log C) \rightarrow 0$$

be a minimal free resolution of  $\Omega^k(\log C)$ . In particular, it means that the coefficients of  $d_k$  are in the maximal ideal  $\mathfrak{m}$  of  $\mathcal{O}_S$ . We apply the functor  $\text{Hom}_{\mathcal{O}_S}(-, \mathcal{O}_S)$  to this resolution, and we identify  $\text{Hom}_{\mathcal{O}_S}(\mathcal{O}_S^{\ell_j}, \mathcal{O}_S)$  with  $\mathcal{O}_S^{\ell_j}$ . Then  $\text{Ext}_{\mathcal{O}_S}^k(\Omega^k(\log C), \mathcal{O}_S)$  is equal to  $\mathcal{O}_S^{\ell_k} / \text{Im}(d_k)$ . Since it is zero, by Nakayama lemma,  $\mathcal{O}_S^{\ell_k} = 0$  and therefore  $\text{projdim}(\Omega^k(\log C)) \leq k - 1$ .

In addition, since there are relations between the maximal minors of the Jacobian matrix, the map  $\alpha$  has a non zero kernel.

Therefore,  $\text{Ext}_{\mathcal{O}_S}^{k-1}(\Omega^k(\log C), \mathcal{O}_S) \neq 0$  and  $\text{projdim}(\Omega^k(\log C)) = k - 1$ .

The implication 6.  $\Rightarrow$  5. is trivial.

Let us prove 5.  $\Rightarrow$  2.

We assume  $\text{projdim}(\Omega^k(\log C)) \leq k - 1$ . The exact sequence (13) becomes:

$$0 \rightarrow \text{Ext}_{\mathcal{O}_S}^{k-1}(\Omega^k(\log C), \mathcal{O}_S) \rightarrow \Theta_S^k \otimes_{\mathcal{O}_S} \mathcal{O}_C \xrightarrow{\alpha} \mathcal{R}_C^\vee \rightarrow 0$$

Since the image of  $\alpha$  is  $\mathcal{J}_C$  by proposition 3.16, we have  $\mathcal{R}_C^\vee = \mathcal{J}_C$ . By Lemma 3.15,  $\mathcal{R}_C$  is a maximal Cohen-Macaulay  $\mathcal{O}_C$ -module. Therefore, by [Eis95, Theorem 21.21],  $\mathcal{J}_C$  is also maximal Cohen-Macaulay.  $\square$

#### 4. COMPLETE INTERSECTION CURVES

Several families of Saito free divisors are known: normal crossing divisors, plane curves, discriminants of deformation of isolated singularities... In that case, the module  $\Omega^1(\log D)$  is a free  $\mathcal{O}_S$ -module of rank  $m$ , so that we know exactly how many generators are needed, and it depends only on the dimension of the ambient space. Moreover, there is a criterion, known as Saito criterion (see [Sai80, (1.8)]) which can be used to check if a given family is a basis of  $\Omega^1(\log D)$ .

For complete intersections the situation is more complicated. Indeed, if we have a free complete intersection, we proved in Theorem 3.5 that the projective dimension of  $\Omega^k(\log C)$  is  $k - 1$ . But it does not give any information on the minimal number of generators of  $\Omega^k(\log C)$ . In addition, we do not have a criterion as Saito criterion to determine if a given family of multi-logarithmic differential forms generates the module  $\Omega^k(\log C)$ . In this section, we describe explicitly a free  $\mathcal{O}_S$ -resolution of  $\Omega^{m-1}(\log C)$  when  $C$  is a quasi-homogeneous curve. In particular, it gives a minimal generating family of  $\Omega^{m-1}(\log C)$ . For more general curves, computations can be made using a computer system such as SINGULAR and algorithm A.2.1.

First of all, let us notice the following property which is easy to prove from the definition of freeness:

**Proposition 4.1.** *Reduced complete intersection curves are free complete intersections.*

*Proof.* If the curve is smooth, there is nothing to do. Otherwise, since  $C$  is reduced, the singular locus is of dimension 0 and therefore is Cohen-Macaulay.  $\square$

Complete intersection curves are studied in [Pol15b]. In this paper, the author proves that the set of values of the module of multi-residues  $\mathcal{R}_C$  defined below satisfies a symmetry with the values of the Jacobian ideal, and gives the relation between the values of  $\mathcal{R}_C$  and the values of the Kähler differentials.

**4.1. Quasi-homogeneous curves.** We describe here explicitly the module of multi-logarithmic differential forms for a quasi-homogeneous complete intersection curve. We will need several results and notations from [Pol15b], which are recalled below.

Let  $C = C_1 \cup \dots \cup C_p$  be a reduced complete intersection curve with  $p$  irreducible components. The normalization of  $C$  satisfies  $\mathcal{O}_{\tilde{C}} = \bigoplus_{i=1}^p \mathbb{C}\{t_i\}$ . It induces for all  $i \in \{1, \dots, p\}$  a valuation map

$$\text{val}_i : \mathcal{M}_C \ni g \mapsto \text{val}_i(g) \in \mathbb{Z} \cup \{\infty\}$$

The value of an element  $g \in \mathcal{M}_C$  is  $\text{val}(g) = (\text{val}_1(g), \dots, \text{val}_p(g)) \in (\mathbb{Z} \cup \{\infty\})^p$ . For a fractional ideal  $I \subset \mathcal{M}_C$ , we set

$$\text{val}(I) := \{\text{val}(g); g \in I \text{ non zero divisor}\} \subset \mathbb{Z}^p$$

We consider the product order on  $\mathbb{Z}^p$ , so that for all  $\alpha, \beta \in \mathbb{Z}^p$ ,  $\alpha \leq \beta$  means that for all  $i \in \{1, \dots, p\}$ ,  $\alpha_i \leq \beta_i$ . We set  $\underline{1} = (1, \dots, 1)$ .

We denote by  $\mathcal{C}_C = \mathcal{O}_C^\vee$  the conductor ideal. There exists  $\gamma \in \mathbb{N}^p$  such that  $\mathcal{C}_C = \underline{t}^\gamma \mathcal{O}_{\tilde{C}}$ . In particular,

$$\gamma = \inf \{\alpha \in \mathbb{N}^p; \alpha + \mathbb{N}^p \subseteq \text{val}(\mathcal{O}_C)\}$$

Let  $C$  be a reduced complete intersection curve defined by a regular sequence  $(h_1, \dots, h_{m-1})$ . Let us consider the following properties:

- a) There exist  $(w_1, \dots, w_m) \in \mathbb{N}^m$  such that for all  $i \in \{1, \dots, m-1\}$ ,  $h_i$  is quasi-homogeneous of degree  $d_i$  with respect to the weight  $(w_1, \dots, w_m)$ .
- b)  $m \geq 3$
- c)  $m-1$  is the embedding dimension. Equivalently, for all  $i \in \{1, \dots, m-1\}$ ,  $h_i \in \mathfrak{m}^2$  where  $\mathfrak{m}$  is the maximal ideal of  $\mathcal{O}_S$  (see Proposition 2.20).

**Lemma 4.2.** *Let  $C$  be a complete intersection curve satisfying condition a). Then*

$$\omega_0 = \frac{\sum_{i=1}^m (-1)^{i-1} w_i x_i \widehat{dx}_i}{h} \in \Omega^{m-1}(\log D)$$

*Proof.* Let  $i \in \{1, \dots, m-1\}$ . We have  $\sum_{i=1}^m w_i x_i \frac{\partial h_i}{\partial x_i} = d_i h_i$  so that  $dh_i \wedge \omega_0 = \frac{d_i}{h_i} d\mathbf{x}$ . Since  $dh = \sum_{i=1}^{m-1} \widehat{h}_i dh_i$ , we have  $dh \wedge \omega_0 \in \Omega_S^m$ . Thus,  $\omega_0 \in \Omega^{m-1}(\log D)$ .  $\square$

*Remark 4.3.* By Proposition 2.23, we also have  $\omega_0 \in \Omega^{m-1}(\log C)$ .

Let us prove the following lemma:

**Lemma 4.4.** *With the notations of Lemma 4.2, we have:*

$$\inf(\text{val}(\mathcal{R}_C)) = \text{val}(\text{res}_C(\omega_0)) = -\gamma + \underline{1}$$

*Proof.* By [Pol15b, Proposition 3.30], we have  $\text{val}(\text{res}_C(\omega_0)) = -\gamma + \underline{1}$  and

$$(16) \quad \gamma + \text{val}(\mathcal{O}_C) - \underline{1} \subseteq \text{val}(\mathcal{J}_C)$$

Let us prove that  $\inf(\text{val}(\mathcal{R}_C)) = \text{val}(\text{res}_C(\omega_0))$ .

By the inclusion (16), we have  $2\gamma - \underline{1} + \mathbb{N}^p \subseteq \text{val}(\mathcal{J}_C)$ .

As in [Pol15b], we set for  $v \in \mathbb{Z}^p$ ,  $\Delta_i(v, \mathcal{J}_C) = \{\alpha \in \text{val}(\mathcal{J}_C); \alpha_i = v_i \text{ and } \forall j \neq i, \alpha_j > v_j\}$  and  $\Delta(v, I) = \bigcup_{i=1}^p \Delta_i(v, I)$ .

We then have:

$$\max\{v \in \mathbb{Z}^p; \Delta(v, \mathcal{J}_C) = \emptyset\} \leq 2\gamma - \underline{2}$$

By [Pol15b, Theorem 2.4] we have

$$v \in \text{val}(\mathcal{R}_C) \iff \Delta(\gamma - v - \underline{1}, \mathcal{J}_C) = \emptyset$$

It implies that  $\inf(\text{val}(\mathcal{R}_C)) \geq -\gamma + \underline{1}$ .

Hence the result:  $\inf(\text{val}(\mathcal{R}_C)) = \text{val}(\text{res}_C(\omega_0))$ .  $\square$

**Proposition 4.5.** *Let  $C$  be a singular complete intersection satisfying condition a). Then  $\mathcal{R}_C$  is generated by  $\text{res}_C\left(\frac{dh_1 \wedge \dots \wedge dh_k}{h}\right) = 1$  and  $\text{res}_C(\omega_0)$ , where  $\omega_0$  is given in Lemma 4.2.*

*Proof.* We set  $Z = \text{Sing}(C)$  the singular locus of  $C$ . By dualizing over  $\mathcal{O}_C$ :

$$0 \rightarrow \mathcal{J}_C \rightarrow \mathcal{O}_C \rightarrow \mathcal{O}_Z \rightarrow 0$$

we obtain

$$(17) \quad 0 \rightarrow \mathcal{O}_C \rightarrow \mathcal{R}_C \rightarrow \omega_Z \rightarrow 0$$

where  $\omega_Z$  is the dualizing module of  $Z$ . Moreover, the singular locus of a quasi-homogeneous curve is Gorenstein (see [KW84, Satz 2]), so that  $\omega_Z = \mathcal{O}_Z$ . The exact sequence (17) implies that  $\mathcal{R}_C$  is generated by two elements, the image of  $1 \in \mathcal{O}_C$ , which is  $1 \in \mathcal{R}_C$ , and the antecedent of  $1 \in \mathcal{O}_Z$ . Therefore, there exists  $\rho_0 \in \mathcal{R}_C$  such that  $(1, \rho_0)$  generates  $\mathcal{R}_C$ .

It remains to prove that we can take  $\rho_0 = \text{res}_C(\omega_0)$ .

By Lemma 4.4, we have  $\text{val}(\text{res}_C(\omega_0)) = -\gamma + \underline{1}$ .

We assume first that  $-\gamma + \underline{1} \notin \mathbb{N}^p$ . For example,  $-\gamma_1 + 1 < 0$ . There exists  $\alpha_0, \alpha_1 \in \mathcal{O}_C$  such that  $\omega_0 = \alpha_0 \rho + \alpha_1$ . Since  $\text{val}(\alpha_1) \geq 0$ , and  $\inf(\text{val}(\mathcal{R}_C)) = \text{val}(\text{res}_C(\omega_0))$ , we have  $\text{val}_1(\rho) = \text{val}_1(\omega_0)$  therefore  $\text{val}_1(\alpha_0) = 0$ . By [DdlM88, (1.1.1)], it implies that  $\text{val}(\alpha_0) = 0$  so that  $\alpha_0$  is invertible, and  $(\text{res}_C(\omega_0), 1)$  generates  $\mathcal{R}_C$ .

Let us assume that  $-\gamma + \underline{1} \in \mathbb{N}^p$ . Since  $\gamma \geq 0$ , by [DdlM88, (1.1.1)], we must have  $\gamma = \underline{1}$  or  $\gamma = \underline{0}$ . However, if  $\gamma = \underline{0}$ , we have  $\mathcal{O}_C = \mathcal{O}_{\tilde{C}}$  so that  $C$  is smooth. Therefore,  $\gamma = \underline{1}$ . By [Pol15b, Corollary 3.32], we have

$$\text{val}(\mathcal{J}_C) = \text{val}(\mathcal{C}_C) = \underline{1} + \mathbb{N}^p$$

It implies that  $\mathcal{J}_C = \mathcal{C}_C$ , so that by duality,  $\mathcal{R}_C = \mathcal{O}_{\tilde{C}}$ . By [Sch13, Proposition 3.14], it implies that  $C$  is a plane normal crossing curve. By Saito criterion [Sai80, (1.8)], if  $h = xy$  defines a plane curve  $C$ , then  $(\omega_0 = \frac{xdy-ydx}{h}, \frac{dh}{h})$  is a basis of  $\Omega^1(\log C)$ . Hence the result.  $\square$

Since the family  $\left\{ \frac{h_i \widehat{dx}_j}{h}; i \in \{1, \dots, m-1\}, j \in \{1, \dots, m\} \right\}$  generates  $\widetilde{\Omega}^{m-1}$ , Proposition 4.5 gives:

**Corollary 4.6.** *Let  $C$  be a singular complete intersection curve satisfying condition a). Then  $\Omega^{m-1}(\log C)$  is generated by the multi-logarithmic form  $\omega_0$  of Lemma 4.2,  $\frac{dh_1 \wedge \dots \wedge dh_k}{h}$ , and the family  $\left\{ \frac{h_i \widehat{dx}_j}{h}; i \in \{1, \dots, m-1\}, j \in \{1, \dots, m\} \right\}$ .*

With additional hypothesis, the generating family of Corollary 4.6 is minimal:

**Proposition 4.7.** *Let  $C$  be a singular complete intersection curve satisfying condition a), b), c). The generating family of Corollary 4.6 is a minimal generating family of  $\Omega^{m-1}(\log C)$ .*

*Proof.* Let  $\alpha_0, \alpha_1, (\beta_{ij})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq m-1}}$  be such that:

$$(18) \quad \alpha_0 \omega_0 = \alpha_1 \frac{dh_1 \wedge \dots \wedge dh_{m-1}}{h} + \sum_{i=1}^m \sum_{j=1}^{m-1} \beta_{ij} h_j \frac{\widehat{dx}_i}{h}$$

Since  $(\text{res}_C(\omega_0), 1)$  is a minimal generating family of  $\mathcal{R}_C$ , we have  $\alpha_0 \in \mathfrak{m}$  and  $\alpha_1 \in \mathfrak{m}$ , where  $\mathfrak{m}$  is the maximal ideal of  $\mathcal{O}_S$ .

Let us prove that for all  $i \in \{1, \dots, m\}$  and  $j \in \{1, \dots, m-1\}$ ,  $\beta_{ij} \in \mathfrak{m}$ .

We notice that the class  $\overline{\alpha_0} \in \mathcal{O}_C$  satisfies  $\overline{\alpha_0} \cdot \text{res}_C(\omega_0) = \overline{\alpha_1} \in \mathcal{O}_C$  and  $\overline{\alpha_0} \in \mathcal{O}_C$ . Thus,  $\overline{\alpha_0} \in \mathcal{R}_C^\vee = \mathcal{J}_C$ . We denote by  $J_i$  the maximal minor of the Jacobian matrix of  $(h_1, \dots, h_{m-1})$  obtained by removing the  $i$ th column. Then  $\alpha_0 \in (J_1, \dots, J_m, h_1, \dots, h_{m-1}) \subseteq \mathcal{O}_S$ . We notice that

$$dh_1 \wedge \dots \wedge dh_{m-1} = \sum_{i=1}^m J_i \widehat{dx}_i$$

Let  $i_0 \in \{1, \dots, m\}$ . Let us compute  $J_{i_0} \omega_0$ . We have:

$$(19) \quad J_{i_0} \omega_0 = (-1)^{i_0-1} w_{i_0} x_{i_0} \frac{dh_1 \wedge \dots \wedge dh_{m-1}}{h} + \sum_{\substack{i=1 \\ i \neq i_0}}^m \underbrace{((-1)^{i-1} w_i x_i J_{i_0} - (-1)^{i_0-1} w_{i_0} x_{i_0} J_i)}_{=: \lambda_i} \frac{\widehat{dx}_i}{h}$$

For  $i \neq j \in \{1, \dots, m\}$  and  $\ell \in \{1, \dots, m-1\}$ , we denote by  $\Delta_{i,j}^\ell$  the  $(k-1) \times (k-1)$  minor of the Jacobian matrix obtained by removing the  $i$ th and  $j$ th columns and the  $\ell$ th line.

Let  $i \in \{1, \dots, m\}$ ,  $i \neq i_0$ . We develop  $J_i$  with respect to the  $i_0$ th column, and  $J_{i_0}$  with respect to the  $i$ th column. We denote by  $\varepsilon_{i_0-i}$  the sign of  $i_0 - i$ . We obtain:

$$\begin{aligned} \lambda_i &= (-1)^{i-1} w_i x_i \varepsilon_{i_0-i} (-1)^{i-1} \left( \sum_{\ell=1}^{m-1} (-1)^{\ell-1} \frac{\partial h_\ell}{\partial x_i} \Delta_{i_0,i}^\ell \right) + (-1)^{i_0} w_{i_0} x_{i_0} \varepsilon_{i_0-i} (-1)^{i_0} \left( \sum_{\ell=1}^{m-1} (-1)^{\ell-1} \frac{\partial h_\ell}{\partial x_{i_0}} \Delta_{i_0,i}^\ell \right) \\ &= \varepsilon_{i_0-i} \sum_{\ell=1}^{m-1} (-1)^{\ell-1} \left( w_i x_i \frac{\partial h_\ell}{\partial x_i} + w_{i_0} x_{i_0} \frac{\partial h_\ell}{\partial x_{i_0}} \right) \cdot \Delta_{i_0,i}^\ell \end{aligned}$$

In addition, we notice that for all  $p \in \{1, \dots, m\} \setminus \{i, i_0\}$ :

$$\sum_{\ell=1}^{m-1} (-1)^{\ell-1} \frac{\partial h_\ell}{\partial x_p} \Delta_{i_0, i}^\ell = 0$$

Thus, thanks to condition c), we have:

$$\lambda_i = \varepsilon_{i_0-i} \sum_{\ell=1}^{m-1} (-1)^{\ell-1} \left( \sum_{p=1}^m w_p x_p \frac{\partial h_\ell}{\partial x_p} \right) \Delta_{i_0, i}^\ell = \varepsilon_{i_0-i} \sum_{\ell=1}^{m-1} (-1)^{\ell-1} d_\ell h_\ell \Delta_{i_0, i}^\ell \in \mathfrak{m}(h_1, \dots, h_{m-1})$$

Therefore:

$$J_{i_0} \omega_0 - (-1)^{i_0-1} w_{i_0} x_{i_0} \frac{dh_1 \wedge \dots \wedge dh_{m-1}}{h} = \sum_{\substack{i=1 \\ i \neq i_0}}^m \varepsilon_{i_0-i} \left( \sum_{\ell=1}^{m-1} (-1)^{\ell-1} d_\ell h_\ell \Delta_{i_0, i}^\ell \right) \frac{\widehat{dx}_i}{h} =: \eta_{i_0}$$

In addition, for all  $j \in \{1, \dots, m-1\}$ ,  $h_j \omega_0 \in \mathfrak{m}(h_1, \dots, h_{m-1}) \cdot \frac{1}{h} \Omega^{m-1}$ .

Moreover, if  $J_{i_0} \cdot \omega_0 = \alpha_{1, i_0} \frac{dh_1 \wedge \dots \wedge dh_{m-1}}{h} + \eta'$  with  $\eta' \in \widetilde{\Omega_S^{m-1}}$ , then  $\alpha_{1, i_0} = (-1)^{i_0-1} w_{i_0} x_{i_0} + \alpha'$ , with  $\alpha' \in (h_1, \dots, h_{m-1})$ . Therefore,

$$\eta' = \eta_{i_0} - \alpha' \sum_{i=1}^m J_i \frac{\widehat{dx}_i}{h} \in \mathfrak{m}(h_1, \dots, h_{m-1}) \cdot \frac{1}{h} \Omega_S^{m-1}$$

With the notations of (19),  $\alpha_0$  is a linear combination of  $J_1, \dots, J_m, h_1, \dots, h_{m-1}$ . Therefore, we have:

$$\sum_{i=1}^m \sum_{j=1}^{m-1} \beta_{ij} h_j \frac{\widehat{dx}_i}{h} \in \mathfrak{m}(h_1, \dots, h_{m-1}) \cdot \frac{1}{h} \Omega^{m-1}$$

Since the sequence  $(h_1, \dots, h_{m-1})$  is regular, any relation  $\sum_j \mu_j h_j = 0$  satisfies for all  $j \in \{1, \dots, m-1\}$ ,  $\mu_j \in (h_1, \dots, h_{m-1}) \subseteq \mathfrak{m}$ . Therefore, for all  $i \in \{1, \dots, m\}$  and  $j \in \{1, \dots, m-1\}$ ,  $\beta_{ij} \in \mathfrak{m}$ .  $\square$

*Remark 4.8.* For plane curves, the statement of Proposition 4.7 is not true. Indeed, a plane curve  $D$  defined by a reduced equation  $h$  is always a free divisor so that the module  $\Omega^1(\log D)$  is free of rank 2. A basis of  $\Omega^1(\log D)$  is given by  $\omega_0$  and  $\frac{dh}{h}$ , but we do not need  $dx$  and  $dy$ .

*Remark 4.9.* The statement of Proposition 4.7 is still true if we weaken the condition c) as follows: at least two equations are in  $\mathfrak{m}^2$ .

Since curves are free, by Proposition 3.13, we have  $\text{depth}(\mathcal{R}_C) = m - (m-1) = 1$  so that by the Auslander-Buchsbaum Formula 3.7, the projective dimension of  $\mathcal{R}_C$  as a  $\mathcal{O}_S$ -module is  $m-1$ . In addition, by Theorem 3.5, the projective dimension of  $\Omega^{m-1}(\log C)$  is  $m-2$ .

In Proposition 4.7, we give a minimal generating family of  $\Omega^{m-1}(\log C)$ . By using similar methods, we can go further and compute explicitly a free  $\mathcal{O}_S$ -resolution of  $\mathcal{R}_C$  and  $\Omega^{m-1}(\log C)$  for a quasi-homogeneous complete intersection curve.

**THEOREM 4.10.** *Let  $C$  be a reduced quasi-homogeneous complete intersection curve satisfying conditions a), b) and c). We set for  $i \in \{0, \dots, m-2\}$ ,  $F_i = \bigwedge^i \mathcal{O}_S^{m-1} \oplus \bigwedge^i \mathcal{O}_S^m$  and  $F_{m-1} = \bigwedge^{m-1} \mathcal{O}_S^m$ . For all  $i \in \{1, \dots, m-1\}$ , there exists a map  $\delta_i : F_i \rightarrow F_{i-1}$  such that  $(F_\bullet, \delta_\bullet)$  is a minimal free  $\mathcal{O}_S$ -resolution of  $\mathcal{R}_C$ .*

*In addition, a minimal free  $\mathcal{O}_S$ -resolution  $(P_j, \alpha_j)_{0 \leq j \leq m-2}$  of  $\Omega^{m-1}(\log C)$  is given by*

$$\forall j \in \{0, \dots, m-3\}, P_j = \bigwedge_{j+1}^{j+1} \mathcal{O}_S^{m-1} \otimes \Omega_S^{m-1} \oplus F_j \text{ and } P_{m-2} = F_{m-2}$$

Therefore, the Betti number  $b_j$  of  $\Omega^{m-1}(\log C)$  are:

$$\forall j \in \{0, \dots, m-3\}, b_j = \binom{m-1}{j+1} \binom{m}{m-1} + \binom{m-1}{j} + \binom{m}{j} \text{ and } b_{m-2} = \binom{m-1}{m-2} + \binom{m}{m-2}$$

*Remark 4.11.* In both case, the differentials of the free resolutions can be computed explicitly from the differentials of the Koszul complexes  $K(\underline{h})$  and  $K(w_1x_1, \dots, w_mx_m)$ .

*Proof.* Maybe after a change of coordinates, we assume that  $x_1$  is a non-zero divisor of  $\mathcal{O}_C$ . We denote by  $J_1$  the minor of the Jacobian matrix obtained by removing the first column. By [Pol15b, Proposition 3.30], the condition on  $x_1$  implies that  $J_1$  is also a non zero divisor of  $\mathcal{O}_C$ . Moreover, since  $\text{res}_C(\omega_0) = \frac{w_1x_1}{J_1}$  and  $(\text{res}_C(\omega_0), 1)$  is a minimal generating family of  $\mathcal{R}_C$  by Proposition 4.5,  $J_1 \notin (x_1, h_1, \dots, h_{m-1})$ .

By Proposition 4.5, the module  $\mathcal{R}_C$  is generated by  $\text{res}_C(\omega_0) = \frac{w_1x_1}{J_1}$  and 1. Therefore, since  $J_1$  is a non zero divisor, a free  $\mathcal{O}_S$ -resolution of  $(w_1x_1, J_1)\mathcal{O}_C$  gives a free  $\mathcal{O}_S$ -resolution of  $\mathcal{R}_C$ .

Let us consider the following exact sequence:

$$(20) \quad 0 \rightarrow \mathcal{O}_C \xrightarrow{\times w_1x_1} (w_1x_1, J_1)\mathcal{O}_C \rightarrow (w_1x_1, J_1)\mathcal{O}_C / (w_1x_1)\mathcal{O}_C \rightarrow 0$$

A free  $\mathcal{O}_S$ -resolution of  $\mathcal{O}_C$  is given by the Koszul complex  $K(\underline{h})$  (see Lemma 3.11).

For the right-hand side term, we obviously have

$$(w_1x_1, J_1)\mathcal{O}_C / (w_1x_1)\mathcal{O}_C = J_1 \cdot (\mathcal{O}_S / (x_1, h_1, \dots, h_{m-1}))$$

In addition, for all  $i \in \{1, \dots, m\}$ :

$$(-1)^{i-1} w_i x_i J_1 = w_1 x_1 J_i \text{ mod } (h_1, \dots, h_{m-1})$$

Thus, we have the following exact sequence:

$$0 \rightarrow (w_1x_1, \dots, w_mx_m) \rightarrow \mathcal{O}_S \xrightarrow{\times J_1} J_1 \cdot (\mathcal{O}_S / (x_1, h_1, \dots, h_{m-1})) \rightarrow 0$$

Therefore a free  $\mathcal{O}_S$ -resolution of  $J_1 \cdot (\mathcal{O}_S / (x_1, h_1, \dots, h_{m-1}))$  is given by the Koszul complex  $K(w_1x_1, \dots, w_mx_m)$ . From the resolutions of  $\mathcal{O}_C$  and  $J_1 \cdot (\mathcal{O}_S / (x_1, h_1, \dots, h_{m-1}))$ , we deduce a free resolution  $(P_j, \delta'_j)_{0 \leq j \leq m}$  of  $(w_1x_1, J_1)\mathcal{O}_C$ , with  $P_j = \bigwedge^j \mathcal{O}_S^{m-1} \oplus \bigwedge^j \mathcal{O}_S^m$  for  $0 \leq j \leq m-1$  and  $P_m = \bigwedge^m \mathcal{O}_S^m$ . Since the projective dimension of  $(w_1x_1, J_1)\mathcal{O}_C$  is  $m-1$ , the previous resolution is not minimal.

Thanks to computations analogous to the ones of Proposition 4.7, we can compute explicitly the maps  $\delta'_j$ . In particular, let  $(a, b_1, \dots, b_m)$  be the image of the generator of  $\bigwedge^m \mathcal{O}_S^m$  by  $\delta'_m$ . One can check that  $a \in \mathbb{C} \setminus \{0\}$ , and all the other coefficients of the differentials  $\delta'_j$  are in  $\mathfrak{m}$ . It implies the first statement. For all  $j \in \{0, \dots, m-2\}$ ,  $\delta_j = \delta'_j$  and  $\delta_{m-1}$  is the restriction of  $\delta'_{m-1}$  to  $\bigwedge^{m-1} \mathcal{O}_S^m$ .

For the second statement, we use the exact sequence  $0 \rightarrow \tilde{\Omega}^{m-1} \rightarrow \Omega^{m-1}(\log C) \rightarrow \mathcal{R}_C \rightarrow 0$  together with the free  $\mathcal{O}_S$ -resolutions  $(F_\bullet, \delta_\bullet)$  of  $\mathcal{R}_C$  and  $\tilde{K}(\underline{h}) \otimes \Omega_S^{m-1}$  of  $\tilde{\Omega}^{m-1}$  to compute explicitly a free resolution  $(P'_j, \alpha'_j)_{0 \leq j \leq m-1}$  of  $\Omega^{m-1}(\log C)$ , which is again not minimal. The map  $\alpha'_{m-1}$  composed with the projection on  $\bigwedge^{m-1} \mathcal{O}_S^{m-1} \otimes \Omega_S^{m-1}$  gives a unit multiple of the identity. All the other coefficients of the differentials  $\alpha'_j$  are in  $\mathfrak{m}$ . Hence the second statement.  $\square$

*Remark 4.12.* Let  $C \subseteq \mathbb{C}^3$  be a space quasi-homogeneous complete intersection curve. By Theorem 4.10, a free resolution of  $\Omega^2(\log C)$  is

$$0 \rightarrow \mathcal{O}_S^5 \rightarrow \mathcal{O}_S^8 \rightarrow \Omega^2(\log C) \rightarrow 0$$

A free resolution of  $\text{Der}^2(-\log C)$  is also known (see [GS12, Proposition 5.5]):

$$(21) \quad 0 \rightarrow \mathcal{O}_S^2 \rightarrow \mathcal{O}_S^5 \rightarrow \text{Der}^2(-\log C) \rightarrow 0$$

The authors of [GS12] also give explicitly the maps of the resolution (21).

If the curve is not quasi-homogeneous, the number of generators can be strictly greater. Let us give an example.

*Example 4.13.* Let  $h_1 = x_1^7 - x_2^5 + x_1^5 x_2^3$  and  $h_2 = x_1^3 x_2 - x_3^2$ . The sequence  $(h_1, h_2)$  defines a reduced complete intersection curve of  $\mathbb{C}^3$ . We use SINGULAR with the algorithm of section A.2.1 to compute a minimal free resolution of  $\Omega^2(\log C)$ :

$$0 \rightarrow \mathcal{O}_S^6 \rightarrow \mathcal{O}_S^9 \rightarrow \Omega^2(\log C) \rightarrow 0$$

Thus, the minimal number of generators of  $\Omega^2(\log C)$  is 9. Moreover, among these nine generators, five are needed to generate  $\mathcal{R}_C$ .

We end this subsection with the following property, which gives counterexamples to the surjectivity of the map of [Ale12, §6, Theorem 2].

We denote by  $w\text{-deg}(f)$  the weighted degree of a quasi-homogeneous element  $f \in \mathcal{O}_C$ , with respect to the weight  $(w_1, \dots, w_m)$ .

**Proposition 4.14.** *Let  $C$  be a reduced complete intersection curve defined by a regular sequence  $(h_1, \dots, h_{m-1})$  satisfying conditions a), b), c).*

*Let  $A = (a_{ij})_{1 \leq i, j \leq m-1} \in \mathcal{M}_{m-1}(\mathbb{C})$  be an invertible matrix with constant coefficients. We set  $(f_1, \dots, f_{m-1})^t = A \cdot (h_1, \dots, h_{m-1})^t$ . In particular,  $(f_1, \dots, f_{m-1})$  also defines the complete intersection  $C$ . Let  $D_f$  be the hypersurface defined by  $f = f_1 \cdots f_{m-1}$ .*

*If there exists  $\ell, i, j \in \{1, \dots, m-1\}$  such that  $f_\ell = \sum_{q=1}^{m-1} a_{\ell,q} h_q$  with  $w\text{-deg}(h_i) \neq w\text{-deg}(h_j)$  and  $a_{\ell,i} a_{\ell,j} \neq 0$  then:*

$$\text{res}_C(\Omega^{m-1}(\log D_f)) \neq \text{res}_C(\Omega^{m-1}(\log C))$$

*Proof.* For the sake of simplicity, we assume that there exist  $i_0, j_0 \in \{1, \dots, m-1\}$  such that  $a_{1,i_0} \cdot a_{1,j_0} \neq 0$  and  $w\text{-deg}(h_{i_0}) \neq w\text{-deg}(h_{j_0})$ .

By Lemma 4.2 and Corollary 2.4,  $\omega = \frac{\sum_{i=1}^m (-1)^{i-1} w_i x_i \widehat{dx}_i}{f_1 \cdots f_{m-1}} \in \Omega^{m-1}(\log C, f)$ . Let us prove that  $\text{res}_C(\omega) \notin \text{res}_C(\Omega^{m-1}(\log D_f))$ . Thus, since the kernel of the residue map is  $\widetilde{\Omega}_f^{m-1}$ , we have to prove that for all  $\eta \in \widetilde{\Omega}_f^{m-1}$ ,  $\omega + \eta \notin \Omega^{m-1}(\log D_f)$ .

Let  $\eta \in \widetilde{\Omega}_f^{m-1}$ . Then  $\eta = \sum_{j=1}^{m-1} \frac{f_j \eta_j}{f}$  with  $\eta_j \in \Omega_S^{m-1}$ . We have  $df_i = \sum_{j=1}^{m-1} a_{ij} dh_j$  and:

$$df \wedge (\omega + \eta) = \sum_{i=1}^{m-1} \widehat{f}_i \cdot \left( \sum_{j=1}^{m-1} (a_{ij} d_j h_j) \frac{dx}{f} + \sum_{j=1}^{m-1} (df_i \wedge f_j \eta_j) \right)$$

If  $\omega + \eta \in \Omega^{m-1}(\log D_f)$  then  $df \wedge (\omega + \eta)$  is holomorphic. Since the sequence  $(f_1, \dots, f_{m-1})$  is regular, it implies that  $f_1$  divides  $\sum_{j=1}^{m-1} (a_{1j} d_j h_j + f_j \theta_{1,j})$ , where  $\theta_{1,j} dx = df_1 \wedge \eta_j$ . In particular, since for all  $i \in \{1, \dots, m-1\}$ ,  $h_i \in \mathfrak{m}^2$ , we also have  $f_i \in \mathfrak{m}^2$  so that we have  $\theta_{1,j} \in \mathfrak{m}$ . There exists  $q \in \mathcal{O}_S$  such that:

$$\sum_{\ell=1}^{m-1} \left( a_{1\ell} d_\ell + \sum_{j=1}^{m-1} (a_{j,\ell} \theta_{1,j}) - q a_{1,\ell} \right) \cdot h_\ell = 0$$

Since  $(h_1, \dots, h_{m-1})$  is a regular sequence, we have for all  $j \in \{1, \dots, m-1\}$ ,

$$a_{1\ell} d_\ell + \sum_{j=1}^{m-1} a_{j,\ell} \theta_{1,j} - q a_{1,\ell} \in (h_1, \dots, h_{m-1})$$

It implies that for all  $\ell \in \{1, \dots, m-1\}$ ,  $a_{1\ell}(d_\ell - q(0, \dots, 0)) = 0$ . In particular, for  $i_0$  and  $j_0$ , since  $a_{1,i_0} \neq 0$  and  $a_{1,j_0} \neq 0$ , we have  $d_{i_0} - q(0, \dots, 0) = 0$  and  $d_{j_0} - q(0, \dots, 0) = 0$ . Since by assumption,  $d_{i_0} \neq d_{j_0}$ , it leads to a contradiction, so that  $df \wedge (\omega + \eta) \notin \Omega_S^m$  and  $\omega + \eta \notin \Omega^{m-1}(\log D_f)$ .  $\square$



**4.2. A characterization of plane homogeneous curves *via* multi-residues.** We first describe the set of values of the logarithmic residues of a plane arrangement of lines. Logarithmic residues along plane curves are studied in [Pol15a] and [Pol15b].

**Proposition 4.15.** *Let  $C = C_1 \cup \dots \cup C_p$  be the union of  $p$  lines in  $(\mathbb{C}^2, 0)$  with  $p \geq 2$ . Let  $h = \prod_{i=1}^p (\alpha_i x + \beta_i y)$  be a reduced equation of  $C$ . Then:*

$$\inf(\text{val}(\mathcal{R}_C)) = (-p + 2, \dots, -p + 2)$$

Moreover,  $\text{val}(\mathcal{R}_C) = ((-p + 2, \dots, -p + 2) + \text{val}(\mathcal{O}_C)) \cup \mathbb{N}^p$ .

*Proof.* A basis of  $\Omega^1(\log C)$  is given by  $\omega_1 = \frac{df}{f}$  and  $\omega_2 = \frac{-ydx + xdy}{f}$ . Then,  $\text{res}_C(\omega_1) = 1$  and  $\text{res}_C(\omega_2) = \frac{-ay + bx}{ah'_x + bh'_y}$  with  $a, b \in \mathbb{C}$  such that  $ah'_x + bh'_y$  is a non zero divisor in  $\mathcal{O}_C$ . The denominator is homogeneous of degree  $p - 1$ . Hence the result.  $\square$

The following proposition characterizes plane arrangements of lines.

**Proposition 4.16.** *Let  $(h_1, \dots, h_{m-1})$  be a regular sequence defining a reduced complete intersection curve  $C$  such that for all  $i \in \{1, \dots, m - 1\}$ ,  $h_i$  is homogeneous of degree  $d_i$ . We set  $C = C_1 \cup \dots \cup C_p$  the irreducible decomposition of the complete intersection defined by  $(h_1, \dots, h_{m-1})$ . In particular, each  $C_i$  is a line.*

*We assume  $p \geq 2$ . Then*

$$\inf(\text{val}(\mathcal{R}_C)) = (-p + 2, \dots, -p + 2) \iff C \text{ is plane}$$

*Proof.* The implication  $\Leftarrow$  comes from the previous proposition.

Lemma 4.2 gives an element  $\omega_0$  in  $\Omega^{m-1}(\log C)$  such that  $\text{val}(\text{res}_C(\omega_0)) = 1 - \sum_{i=1}^{m-1} (d_i - 1)$ . Moreover, this multi-residue satisfies  $\inf(\text{val}(\mathcal{R}_C)) = \text{val}(\text{res}_C(\omega_0))$ .

Since the curve  $C$  is reduced, the number of lines is equal to the product of degrees:  $p = d_1 \cdots d_{m-1}$ .

First of all, if there exists  $i \in \{1, \dots, m - 1\}$  such that  $d_i = 1$ , this equation can be removed, and we consider the complete intersection in  $\mathbb{C}^{m-1}$  instead of  $\mathbb{C}^m$  (see Proposition 2.20). We can therefore assume that for all  $i \in \{1, \dots, m - 1\}$ ,  $d_i \geq 2$ .

Let us assume that  $\inf(\text{val}(\mathcal{R}_C)) = (-p + 2, \dots, -p + 2)$ . It means that

$$1 - \sum_{i=1}^{m-1} (d_i - 1) = -d_1 \cdots d_{m-1} + 2$$

which is equivalent to  $d_1 \cdots d_{m-1} = \left( \sum_{i=1}^{m-1} d_i \right) - m + 2$ . We assume that  $d_1 = \max(d_i)$ . The left-hand-side term is greater than  $2^{m-2} d_1$ , the right-hand-side term is strictly lower than  $(m - 1)d_1$ . Therefore, equality cannot hold if  $m \geq 3$ . Hence the result.  $\square$

## 5. OTHER RESULTS, EXAMPLES AND QUESTIONS

**5.1. Relation with logarithmic vector fields.** The notion of logarithmic vector field is defined in [Pik14], for arbitrary reduced analytic germs. We want to compare it with the notion of multi-logarithmic  $k$ -vector fields in case of a complete intersection.

**Definition 5.1** ([Pik14]). *The module of logarithmic vector fields is*

$$\Theta(-\log C) = \{\eta \in \Theta_S, \forall i \in \{1, \dots, k\}, dh_i(\eta) \in (h_1, \dots, h_k)\}$$

A vector field  $\eta \in \Theta_S$  satisfies  $\eta \in \Theta(-\log C)$  if and only if  $\eta$  is tangent to the complete intersection at its smooth points. The following property shows that we can deduce multi-logarithmic  $k$ -vector fields thanks to  $\Theta(-\log C)$ .

**Proposition 5.2.**

$$\Theta(-\log C) \wedge \Theta_S^{k-1} \subseteq \text{Der}^k(-\log C)$$

*Proof.* Let  $\eta \in \Theta(-\log C)$ , and  $\delta_2, \dots, \delta_k \in \Theta_S$ . Then, since for all  $i \in \{1, \dots, k\}$ ,  $dh_i(\eta) \in (h_1, \dots, h_k)$ , we have:

$$dh_1 \wedge \dots \wedge dh_k (\eta \wedge \delta_2 \wedge \dots \wedge \delta_k) \in (h_1, \dots, h_k)$$

□

*Remark 5.3.* Let  $\eta \in \text{Der}(-\log D)$ . Then,  $dh(\eta) = \sum_{i=1}^k \widehat{h}_i dh_i(\eta) \in (h)$ . Since  $(h_1, \dots, h_k)$  is a regular sequence, it implies for all  $i \in \{1, \dots, k\}$ ,  $dh_i(\eta) \in (h_i)$  so that  $\text{Der}(-\log D) \subseteq \Theta(-\log C)$ . In particular, it implies (see [GS12, (5.3)]):

$$\text{Der}(-\log D) \wedge \Theta_S^{k-1} \subseteq \text{Der}^k(-\log C)$$

In the particular case of a complete intersection curve, we have an isomorphism between the module  $\Theta(-\log C)$  and the module  $\Omega^{m-1}(\log C)$  of multi-logarithmic  $(m-1)$ -forms. We denote by  $i_\eta(\omega)$  the interior product of a differential form  $\omega$  by a vector field  $\eta$ .

**Proposition 5.4.** *Let  $C$  be a reduced complete intersection curve. The following map is an isomorphism:*

$$\begin{aligned} \Upsilon : \Theta(-\log C) &\rightarrow \Omega^{m-1}(\log C) \\ \eta = \sum_{i=1}^m \alpha_i \partial_{x_i} &\mapsto \frac{1}{h} i_\eta(dx_1 \wedge \dots \wedge dx_m) = \frac{1}{h} \sum_{i=1}^m (-1)^{i-1} \alpha_i \widehat{dx}_i \end{aligned}$$

*Proof.* The map  $\Theta_S \rightarrow \Omega^{m-1}, \eta \mapsto i_\eta(dx_1 \wedge \dots \wedge dx_m)$  is an isomorphism. Let  $\eta = \sum_{i=1}^m \alpha_i \partial_{x_i}$ . One can easily check that for all  $j \in \{1, \dots, k\}$ :

$$dh_j(\eta) = dh_j \wedge i_\eta(dx_1 \wedge \dots \wedge dx_m)$$

As a consequence, the map  $\Upsilon$  is an isomorphism.

□

**5.2. Examples of surfaces.** In this subsection, we describe several examples computed with the algorithms of Appendix A. The situation with surfaces seems to be more complicated than in the curve case.

*Example 5.5.* Let us consider the complete intersection surface  $C$  in  $\mathbb{C}^4$  given by  $h_1 = t^2 + y^3 + x^3t$  and  $h_2 = x^2 + z^3$ . It can be checked by SINGULAR using algorithm A.1 that this surface is a reduced complete intersection, which is not free. Moreover it is a quasi-homogeneous surface, for the weights  $(3, 6, 2, 9)$ .

A computation by SINGULAR using algorithm A.3 gives 14 generators for the module  $\Omega^2(\log C)$ . It can be seen that a generating family of  $\Omega^2(\log C)$  is given by the 12 obvious generators of  $\widetilde{\Omega}^2$  and two more forms:  $\frac{dh_1 \wedge dh_2}{h_1 h_2}$  and

$$\omega = \frac{6y^2 z dx \wedge dy + 9xz^3 t dx \wedge dz - 2z(2t - xz^3) dx \wedge dt - 9xy^2 dy \wedge dz + 3x(2t - xz^3) dz \wedge dt}{h_1 h_2}$$

A free resolution of  $\Omega^2(\log C)$  is:

$$0 \rightarrow \mathcal{O}_S^2 \rightarrow \mathcal{O}_S^{10} \rightarrow \mathcal{O}_S^{14} \rightarrow \Omega^2(\log C)$$

The module of multi-residues  $\mathcal{R}_C$  is generated over  $\mathcal{O}_C$  by 1 and  $\frac{z}{x}$ .

It is possible to check that in that case  $\text{res}_C(\Omega^2(\log D)) = \text{res}_C(\widetilde{\Omega}^2(\log C))$ .

*Example 5.6.* Let us consider  $h_1 = xyzt$  and  $h_2 = (x + y + z + t)(x - y + z - t)$ . It defines a homogeneous reduced complete intersection which is not free. A free resolution of  $\Omega^2(\log C)$  is:

$$0 \rightarrow \mathcal{O}_S \rightarrow \mathcal{O}_S^{11} \rightarrow \mathcal{O}_S^{16} \rightarrow \Omega^2(\log C) \rightarrow 0$$

The module  $\Omega^2(\log C)$  is generated by the twelve generators of  $\widetilde{\Omega}^2$ , and four other forms, including  $\frac{dh_1 \wedge dh_2}{h_1 h_2}$ .

*Example 5.7.* Let us consider  $h_1 = xz(x+z)$  and  $h_2 = yt$ . It defines a free complete intersection. The hypersurface  $D$  is free and splayed (see Proposition 2.24).

$$0 \rightarrow \mathcal{O}_S^{10} \rightarrow \mathcal{O}_S^{16} \rightarrow \Omega^2(\log C) \rightarrow 0$$

The module  $\Omega^2(\log C)$  is generated by the twelve generators of  $\tilde{\Omega}^2$ , and four other forms, including  $\frac{dh_1 \wedge dh_2}{h_1 h_2}$ .

### 5.3. Questions.

*Question 1.* For curves of codimension at least 2, does the fact that  $\mathcal{R}_C$  is generated by two elements characterize quasi-homogeneous curves? For curves of codimension 1, it is wrong, since  $\mathcal{R}_C$  is always generated by two elements (free divisor).

If a minimal generating family of  $\mathcal{R}_C$  contains 1, is  $C$  quasi-homogeneous?

*Question 2.* Is there again a relation between  $\Omega^k(\log C)$  and  $\text{Tors}(\Omega_C^{m-k})$  as in [Ale05] for hypersurfaces?

*Question 3.* Let  $C = C_1 \cup \dots \cup C_p$  be a reducible reduced complete intersection curve. For a plane curve  $D = D_1 \cup \dots \cup D_p$ , the intersection of  $\text{val}(\mathcal{R}_D)$  with the axis  $j$  for  $j \in \{1, \dots, p\}$  gives the values of the logarithmic residues of the branch  $D_j$ . What can we say about the intersection of  $\text{val}(\mathcal{R}_C)$  with an axis? A component  $C_j$  may not be a complete intersection.

*Question 4.* For a complete intersection  $C$ , is there always or when is there a regular sequence  $(h_1, \dots, h_k)$  defining it so that  $\text{res}_C(\Omega^k(\log D)) = \text{res}_C(\Omega^k(\log C))$ ?

## APPENDIX A. ALGORITHMS FOR SINGULAR

We use SINGULAR for the computation of the examples. Here are given the algorithms used in this paper. We need the following libraries:

```
LIB "sing.lib";
LIB "homolog.lib";
LIB "primdec.lib";
LIB "latex.lib";
LIB "general.lib";
```

**A.1. Freeness.** This first procedure checks if a sequence  $(h_1, \dots, h_k)$  defines a free reduced complete intersection. We denote by  $I$  the ideal generated by  $(h_1, \dots, h_k)$ . We assume that  $I$  defines a singular variety.

```
proc is_freeci (ideal I)
{
  int m=dim(std(0));
  int k=size(I);
  intvec ci=is_ci(I);
  int q=size(ci);
  int qq=ci[q];
  int p=(m-k)-qq;
  if (p<>0)
    {"//not a regular sequence"; };
  if (p==0)
    { ideal pr=radical(I);
      int q=size(pr);
      int pp=0;
      for (int i=1; i<=q; i=i+1)
        { list S(i)=division(pr[i],I);
          poly R(i)=S(i)[2][1];
          if (R(i)<>0)
```

```

        {pp=1;};};
if (pp<>0)
  {"//non reduced complete intersection";};
if (pp==0)
  { matrix Jac=jacob(I);
    ideal mi=minor(Jac,k);
    ideal J=I,mi;
    int u=isCM(J);
    if (u==0)
      {"// It is a reduced complete intersection with a
        non Cohen Macaulay singular locus";};
    if (u==1)
      {int ss=dim(std(J))+1;
        int n=dim(std(I));
        if (ss<>n)
          {"// It is a reduced complete intersection with a Cohen
            Macaulay singular locus, but not of codimension 1";};
        if (ss==n)
          {"// It is a free complete intersection";};}; };};

```

## A.2. Curves.

A.2.1. *Multi-logarithmic forms.* Let  $C$  be a reduced complete intersection curve in  $\mathbb{C}^m$  defined by  $I = (h_1, \dots, h_{m-1})$ . The following procedure returns a list which is a minimal free  $\mathcal{O}_S$  resolution of  $\Omega^{m-1}(\log C)$ .

```

proc logcur (ideal I)
{ int m=dim(std(0));
  int k=size(I);
  int n=m-k;
  if (n<>1)
    {"// It is not a curve...";};
  if (n==1)
    { for (int i=1; i<=k; i=i+1)
      {ideal J(i)=jacob(I[i]);};
    int q=m+k*k;
    matrix A[k][q];
    for (int i=1; i<=k; i=i+1)
      { for (int j=1; j<=m; j=j+1)
        { A[i,j]=(-1)^(j-1)*J(i)[j];};
        for (int j=1; j<=k; j=j+1)
          { A[i,m+k*(i-1)+j]=I[j];};};
    module K=syz(A); int qK=size(K);
    for (int i=1; i<=qK; i=i+1)
      { for (int j=1; j<=m; j=j+1)
        { vector kk(j)=K[i][j]*gen(j);};
        vector v(i)=sum(kk(1..m));};
    module om=v(1..qK);
    list L=mres(om,0);
    return(L); };}

```

A.2.2. *Multi-residues.* The following procedure gives a family of multi-logarithmic forms whose residues generate  $\mathcal{R}_C$ .

```

proc rescur (ideal I)
{list L=logcur(I);
 int m=dim(std(0));
 module omk=L[1];
 int q=size(omk);
 int k=size(I);
 for (int j=1; j<=k; j=j+1)
 { for (int i=1; i<=m; i=i+1)
 { vector v(i)=I[j]*gen(i); };
 module tom(j)=v(1..m);};
 module tom=tom(1..k);
 module omt(0)=tom;
 module M(0);
 for (int i=1; i<=q; i=i+1)
 {list S=division(omk[i], omt(i-1));
 vector q(i)=S[2][1];
 if (q(i)==0)
 { module omt(i)=omt(i-1);
 module M(i)=M(i-1);};
 if (q(i)<>0)
 {module omt(i)=omt(i-1),q(i);
 module M(i)=M(i-1),q(i);};};
 int s=size(M(q))+1;
 module genRc=M(q)[2..s];
 return(genRc);}

```

A.3. **Surfaces.** Let  $C$  be a reduced surface in  $\mathbb{C}^m$  defined by a regular sequence  $I = (h_1, \dots, h_{m-2})$ . The following procedure returns a list which is a minimal free  $\mathcal{O}_S$  resolution of  $\Omega^{m-2}(\log C)$ .

```

proc om2logC (ideal I)
{ int m=dim(std(0));
 int k=size(I);
 int q=m*(m-1) div 2;
 int p=m*(m-1)*(m-2) div 6;
 for (int s=1; s<=k; s=s+1)
{ ideal jac(s)=jacob(I[s]);
 matrix A[k*p][q+k*k*p];
 for (s=1; s<=k; s=s+1)
 { int c=0;
 for (int i=1; i<=m-2; i=i+1)
 { intvec iv(1)=intvec(0..i-1);
 int u(1)=(i-1)*m-sum(iv(1));
 for (int j=i+1; j<=m-1; j=j+1)
 {intvec iv(2)=intvec(0..j-1);
 int u(2)=(j-1)*m-sum(iv(2));
 for (int l=j+1; l<=m; l=l+1)
 { A[p*(s-1)+c+l-j,u(2)+l-j]=jac(s)[i];
 A[p*(s-1)+c+l-j,u(1)+l-i]=-jac(s)[j];
 A[p*(s-1)+c+l-j,u(1)+j-i]=jac(s)[l];
 };
 int d=c; int c=d+m-j;
 };
 };}

```

```

for (int i=1; i<=k*p; i=i+1)
  { for (int s=1; s<=k; s=s+1)
    { A[i,q+(i-1)*k+s]=I[s];};};
module K=syz(A); int qK=size(K);
  for (int i=1; i<=qK; i=i+1)
    {
      for (int j=1; j<=q; j=j+1)
        {
          vector kk(j)=K[i][j]*gen(j);
        };
      vector v(i)=sum(kk(1..q));
    };
module om=v(1..qK);
list L=mres(om,0);
module omk=L[1];
int g=size(omk);
return(L);
}

```

## APPENDIX B. COMPARISON BETWEEN SIMPLE POLES AND ARBITRARY POLES

Since the de Rham differentiation induces a complex  $(\omega_C^\bullet, d)$ , it induces also a complex on  $\mathcal{R}_C^\bullet$  by Theorem 2.16. Nevertheless, we suggest here another proof of the stability of  $\mathcal{R}_C^\bullet$  by differentiation. To do this, we compare the two definitions of multi-logarithmic differential forms suggested by A.G. Aleksandrov and A. Tsikh.

For  $q \in \mathbb{N}$ , we set  $\Omega^q(*D)$  the module of meromorphic  $q$ -forms with arbitrary poles along  $D$ .

**Definition B.1** ([AT01], [AT08]). *Let  $\omega \in \Omega^q(*D)$  with  $q \in \mathbb{N}$ . We say that  $\omega$  is a multi-logarithmic differential form along  $C$  with arbitrary poles along  $D$  if the following properties are satisfied:*

- (1) for all  $j \in \{1, \dots, k\}$ ,  $h_j \omega \in \sum_{i=1}^k \Omega^q(\star \widehat{D}_i)$
- (2) for all  $j \in \{1, \dots, k\}$ ,  $dh_j \wedge \omega \in \sum_{i=1}^k \Omega^{q+1}(\star \widehat{D}_i)$

We denote by  $\underline{\Omega}^q(\log C)$  the module of multi-logarithmic differential forms along  $C$  with arbitrary poles along  $D$ .

To simplify the notations, we set  $\widetilde{\Omega}^q = \sum_{i=1}^k \Omega^q(\star \widehat{D}_i)$ .

*Remark B.2.* As we will see, the modules  $\underline{\Omega}^\bullet(\log C)$  induce a complex for the de Rham differentiation. This property is not satisfied by  $\Omega^\bullet(\log C)$ .

Thus, as regards the stability by  $d$ , the modules  $\underline{\Omega}^\bullet(\log C)$  are better, but their main drawback is that they are not of finite type, whereas the modules  $\Omega^\bullet(\log C)$  are.

The relation between the modules  $\Omega^q(\log C)$  and  $\underline{\Omega}^q(\log C)$  is:

**Proposition B.3.** *For all  $q \in \mathbb{N}$  we have the equality:*

$$\underline{\Omega}^q(\log C) = \Omega^q(\log C) + \widetilde{\Omega}^q$$

*Proof.* The inclusion  $\Omega^q(\log C) + \widetilde{\Omega}^q \subseteq \underline{\Omega}^q(\log C)$  is obvious. For the other inclusion, we need the following lemma, which can be proved by induction on  $j \in \{1, \dots, k\}$ , using the fact that  $(h_1, \dots, h_k)$  is a regular sequence in a local ring:

**Lemma B.4** ([CMNM02, Lemma 2.3]). *Let  $p = (p_1, \dots, p_k) \in \mathbb{N}^k$  and  $n = (n_1, \dots, n_k) \in \mathbb{N}^k$  be such that for all  $i \in \{1, \dots, k\}$ ,  $p_i \geq n_i \geq 1$  and  $a \in \mathcal{O}_S$ .*

*If  $h_1^{p_1-n_1} \dots h_k^{p_k-n_k} a \in (h_1^{p_1}, \dots, h_k^{p_k})$ , then  $a \in (h_1^{n_1}, \dots, h_k^{n_k})$ .*

Let us prove Proposition B.3. Let  $\omega \in \underline{\Omega}^q(\log C)$ . We can choose  $\eta \in \widetilde{\underline{\Omega}}^q$  and for all  $I \subseteq \{1, \dots, m\}$  with  $|I| = q$ ,  $a_I \in \mathcal{O}_S$  and  $n^I \in \mathbb{N}^k$  such that:

$$\omega = \sum_I \frac{a_I dx_I}{h_1^{n_1^I} \dots h_k^{n_k^I}} + \eta$$

with  $a_I \notin (h_1^{n_1^I}, \dots, h_k^{n_k^I})$ , and for all  $j \in \{1, \dots, k\}$ ,  $a_I \notin (h_j)$ .

Let us prove that we can take for all  $I$ ,  $n^I = (1, \dots, 1)$ .

Since  $\omega \in \underline{\Omega}^q(\log C)$ , we have  $h_1 \omega \in \widetilde{\underline{\Omega}}^q$ . We fix  $I \subseteq \{1, \dots, m\}$  with  $|I| = q$ . There exist  $(b_1, \dots, b_k) \in \mathcal{O}_S^k$  and for all  $j \in \{1, \dots, k\}$ ,  $m^j = (m_1^j, \dots, m_k^j) \in \mathbb{N}^k$  with  $m_j^j = 0$  such that

$$\frac{a_I}{h_1^{n_1^I-1} h_2^{n_2^I} \dots h_k^{n_k^I}} = \sum_{j=1}^k \frac{b_j}{\underline{h}^{m^j}}$$

where  $\underline{h}^{m^j} = h_1^{m_1^j} \dots h_k^{m_k^j}$ . To simplify the notations, we denote for all  $j \in \{1, \dots, k\}$   $n_j^I$  by  $n_j$ , and  $n' = (n_1 - 1, n_2, \dots, n_k)$ . We set for all  $i \in \{1, \dots, k\}$ ,  $p_i = \max_{j \in \{1, \dots, k\}} m_i^j$ . We have:

$$\frac{a_I}{\underline{h}^{n'}} = \frac{h^{p-m_1} b_1 + \dots + h^{p-m_k} b_k}{\underline{h}^p}$$

Thus, since  $\frac{a_I h^p}{\underline{h}^{n'}}$  is holomorphic and by assumption, for all  $i \in \{1, \dots, k\}$ ,  $a_I \notin (h_i)$ , we have for all  $i \in \{1, \dots, k\}$ ,  $p_i \geq n'_i$ . Since for all  $j \in \{1, \dots, k\}$ ,  $m_j^j = 0$ , we have  $\underline{h}^{p-n'} a_I \in (h_1^{p_1}, \dots, h_k^{p_k})$ . Lemma B.4 gives  $a_I \in (h_1^{n_1-1}, h_2^{n_2}, \dots, h_k^{n_k})$ , so that

$$\frac{a_I dx_I}{\underline{h}^n} = \frac{c_I}{h_1 h_2 \dots h_k^{n_k}} + \eta'$$

with  $\eta' \in \widetilde{\underline{\Omega}}^q$  and  $c_I \in \mathcal{O}_S$ . By induction, one can prove that there exists a  $q$ -form  $\omega' \in \frac{1}{h} \underline{\Omega}^q$  with simple poles along  $D$  and  $\eta'' \in \widetilde{\underline{\Omega}}^q$  such that

$$\omega = \omega' + \eta''$$

For all  $j \in \{1, \dots, k\}$ ,  $dh_j \wedge \omega \in \widetilde{\underline{\Omega}}^q$ , thus,  $dh_j \wedge \omega' \in \widetilde{\underline{\Omega}}^q$ . Lemma B.4 can be used in a similar proof to show that  $dh_j \wedge \omega' \in \widetilde{\underline{\Omega}}^q$ , and therefore,  $\omega' \in \underline{\Omega}^q(\log C)$ , which finishes the proof of Proposition B.3.  $\square$

The following result is then a direct consequence of both Proposition B.3 and Theorem 2.8:

**Corollary B.5** ([AT01, Proposition 2.1]). *Let  $\omega \in \underline{\Omega}^q(\log C)$ . There exist  $g \in \mathcal{O}_S$  which induces a non zero divisor in  $\mathcal{O}_C$ , a holomorphic  $(q-k)$ -form  $\xi \in \Omega_S^{q-k}$ , and  $\eta \in \widetilde{\underline{\Omega}}^q$  such that*

$$g\omega = \xi \wedge \frac{dh_1 \wedge \dots \wedge dh_k}{h_1 \dots h_k} + \eta$$

**Definition B.6.** *The multi-residue of  $\omega \in \underline{\Omega}^q(\log C)$  is  $\underline{\text{res}}_C(\omega) = \frac{\xi}{g} \Big|_C \in \Omega_C^{q-k} \otimes_{\mathcal{O}_C} \mathcal{M}_C$ .*

As for  $\underline{\text{res}}_C$ , the map  $\underline{\text{res}}_C$  is well-defined (see [AT08, Proposition 1.2]). Thanks to Lemma B.3, we have the following equality:

$$(22) \quad \underline{\text{res}}_C(\underline{\Omega}^q(\log C)) = \underline{\text{res}}_C(\Omega^q(\log C))$$

We can now state the result announced at the beginning of this appendix:

**Corollary B.7.** *The following diagram is commutative:*

$$\begin{array}{ccc} \underline{\Omega}^q(\log C) & \xrightarrow{d} & \underline{\Omega}^{q+1}(\log C) \\ \downarrow \text{res}_C & & \downarrow \text{res}_C \\ \mathcal{R}_C^{q-k} & \xrightarrow{d} & \mathcal{R}_C^{q+1-k} \end{array}$$

In particular,  $\mathcal{R}_C^\bullet$  is stable by differentiation.

*Proof.* It is a consequence of the Corollary B.5. □

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