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Abstract

In this work we consider the numerical resolution of the bilateral obstacle optimal control problem given in Bergounioux et al. Where the main feature of this problem is that the control and the obstacle are the same.

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1 Introduction

Variational inequalities and related optimal control problems have been recognized as suitable mathematical models for dealing with many problems arising in different fields, such as shape optimization theory, image processing and mechanics, (see for example [4], [5], [10], [18], [22]).

Optimal control problem governed by variational inequalities has been studied extensively during the last years by many authors, such as [2], [20], [21]. These authors have studied optimal control problems for obstacle problems (or variational inequalities) where the obstacle is a known function, and the control variables appear as variational inequalities. In other words, controls do not change the obstacle and, on the other hand , in [1], [7], [11] the authors have studied another class of problems where the obstacle functions are unknowns and are considered as control functions.

In this paper, we investigate optimal control problems governed by variational inequalities of obstacle type. This kind of problem is very important and it can lead to the shape optimization problem governed by variational inequality, it may concern the optimal shape of dam [9], for which the obstacle gives the shape to be designed such that the pressure of the fluid inside the dam is close to a desired value. Besides, if we want to design a membrane having an expected shape, we need to choose a suitable obstacle. In this case, the obstacle can be considered as a control, and the membrane as the state (see for example [15]).

It should be pointed out that, in the optimal control problem of a variational inequality, the main difficulty comes from the fact that the mapping $\mathcal{T}$ between the control and the state (control-to-state operator) is not Gateaux differentiable as pointed it out in [21], [20] where one can only define a conical derivative for $\mathcal{T}$ but only Lipschitz-continuous and so it is not easy to get optimality conditions that can be numerically exploitable.

To overcome this difficulty, different authors (see for example, Kunisch et al.[17] V. Barbu [2] and the references therein) consider a Moreau-Yosida approximation technique to reformulate the governing variational inequality problem into a problem governed by a variational equation. Our approach is based on the penalty method and Barbu's treatment as a penalty parameter approaching zero. We then obtain a system of optimality for suitable approximations of the original problem which can be easily used from the numerical point of view.

Nevertheless, the optimal control of variational inequalities of obstacle type is still a very active field of research especially for their numerical treatment which are given in the recent publication [13].

The problem that we are going to study can be set in a wider class of problems, which can be formally described as follows

$$\min\left\{J\left(y,\chi\right),y=\mathcal{T}\left(\chi\right),\chi\in\mathcal{U}_{ad}\subset\mathcal{U}\right\}$$

where $\mathcal{T}$ is an operator which associates y to χ, when y is a solution to

$$\forall y\in\mathcal{K}\left(y,\chi\right),\left\langle A\left(y,\chi\right),y-v\right\rangle\geq 0, \qquad (obs)$$

where K is a multiplication from $\chi\times\mathcal{U}$ to 2^{χ} when χ is a Banach space and A is a differential operator from Y to the dual Y'. Let h be an application from $\mathbb{R}\times\mathbb{R}$ to $\mathbb{R}$, then the variational inequality that relates the control χ to the state y can be written as

$$\left\langle A\left(y,\chi\right),y-v\right\rangle_{Y,Y'}+h\left(\chi,v\right)-h\left(\chi,y\right)\geq\left(\chi,v-y\right),\forall y\in\mathcal{Y},$$

where this formulation gives the obstacle problems where the obstacle is the control.

Following the previous ideas, we may apply a smoothed penalization approach to our problem. More precisely, the idea is to approximates the obstacle problem by introducing an approximating parameter δ, where the approximating method is based on the penalization method and it consists in replacing the obstacle problem ((obs) by a family of semilinear equations. In [7], Bergounioux et al. considered the following bilateral optimal control obstacle problem

$$\min\{J\left(\varphi,\psi\right)=\frac{1}{2}\int_{\Omega}\left(\mathcal{T}\left(\varphi,\psi\right)-z\right)^{2}dx+\frac{\nu}{2}\int_{\Omega}\left(\left(\Delta\varphi\right)^{2}+\left(\Delta\psi\right)^{2}\right)dx,\quad\left(\varphi,\psi\right)\in\mathcal{U}_{ad}\times\mathcal{U}_{ad}\}\quad(1)$$

where ν is a given positive constant and z belongs to $L^{2}\left(\Omega\right)$ as a target profile, such that $y=\mathcal{T}\left(\varphi\right)$ is a solution of the bilateral obstacle problem given by

$$\left\langle Ay,v-y\right\rangle\geq\left(f,v-y\right),\text{ for all }v\text{ in }\mathcal{K}\left(\varphi,\psi\right),$$

where $\mathcal{K}\left(\varphi,\psi\right)$ is given by

$$\mathcal{K}(\varphi,\psi)=\left\{y\in H_0^1(\Omega),\psi\geq y\geq\varphi\right\},$$

and the set of admissible controls $\mathcal{U}_{ad}$ is defined as follows

$$\mathcal{U}_{ad}=\left\{(\varphi,\psi)\in\mathcal{U}\times\mathcal{U}\mid\varphi\leq\psi\right\},$$

where $\mathcal{U}=H^2(\Omega)\times H_0^1(\Omega)$. As we need H^2-priori estimate, we could assume that $\mathcal{U}_{ad}$ is H^2 bounded. For example, we can suppose that $\mathcal{U}_{ad}$ is $\mathcal{B}_{H^2}(0,R)$ i.e. a ball of center 0 and radius R, where R is a large enough positive real number, but according to [13], this choice can lead to technical difficulties to get a numerical solution of the optimality system.

In [13], Ghanem et al., have solved numerically the unilateral optimal control of obstacle problem given by

$$\min\left\{J(\varphi)=\frac{1}{2}\int_\Omega(\mathcal{T}(\varphi)-z)^2dx+\frac{\nu}{2}\int_\Omega(\Delta\varphi)^2dx,\varphi\in\mathcal{U}\right\}\tag{2}$$

instead of the one defined by

$$\min\left\{J(\varphi)=\frac{1}{2}\int_\Omega(\mathcal{T}(\varphi)-z)^2dx+\frac{\nu}{2}\int_\Omega(\nabla\varphi)^2dx,\varphi\in\mathcal{U}_{ad}\right\}$$

where

$$\mathcal{U}_{ad}=\left\{\varphi\in H^2(\Omega),\varphi\in\mathcal{B}_{H^2}(0,R)\right\}\tag{3}$$

such that $y=\mathcal{T}(\varphi)$ is a solution of the unilateral obstacle problem given by

$$\langle Ay,v-y\rangle\geq(f,v-y),\text{ for all }v\text{ in }\mathcal{K}(\varphi)$$

where $\mathcal{K}(\varphi)$ is defined by

$$\mathcal{K}(\varphi)=\left\{y\in H_0^1(\Omega),y\geq\varphi\right\}.$$

According to the result given in [12] the authors point out that, in spite of the elimination of the inequality constraint given by (3), we still get a local convergence property implied by the constraint $\|\varphi_n\|_{H^2(\Omega)}\leq R$. Hence, we are again confronted to the inequality constraint (3).

So we note that it is not necessary to suppress the constraint (3), because it is going to appear again to get the local convergence of the algorithm used for the numerical solution of the problem give by (2).

For the numerical solution of optimal control problem, it is usual to use two kinds of numerical approaches: direct and indirect methods. Direct methods consist in discretizing the cost function, the state and the control and thus reduce the problem to a nonlinear optimization problem with constraints. Indirect methods consist of solving numerically the optimality system given by the state, the adjoint and the projection equations.

The aim of this paper is the numerical solution of the optimal control problem given in [7] by using the indirect approach (after optimisation) based on the same idea and techniques given in [13], where the optimality system is characterized by

$$\begin{cases} Ay^\delta + (\beta_\delta(y^\delta - \varphi^\delta) - \beta_\delta(\psi^\delta - y^\delta)) = f \text{ in } \Omega \text{ and } y^\delta = 0 \text{ on } \partial\Omega \\ \\ A^* p^\delta + \mu_1^\delta + \mu_2^\delta = y^\delta - z \text{ in } \Omega \text{ and } p^\delta = 0 \text{ on } \partial\Omega \\ \\ \left(\mu_1 + \varphi^\delta - \varphi^*, \varphi - \varphi^\delta\right) + \left(\mu_2 + \psi^\delta - \psi^*, \psi - \psi^\delta\right) + \\ +\nu\left(\Delta\varphi^\delta, \Delta\left(\varphi - \varphi^\delta\right)\right) + \nu\left(\Delta\psi^\delta, \Delta\left(\psi - \psi^\delta\right)\right) = 0, \text{ for all } \varphi \text{ in } \mathcal{U}_{ad} \end{cases}$$

For the numerical solution, we first begin by discretizing the optimality system by using finite differences schemes and then by proposing an iterative algorithm based on Gauss-Seidel method that is a combination of damped-Newton-Raphson and a direct method.

The main difficulties of this work compared to the one considered in [13], is to get an optimality system numerically exploitable by the proposed algorithm.

In the sequel, we denote by $\mathcal{B}_V(0, r)$ the V-ball around o of radius r and by C generic positive constants.

The rest of paper is organized as follows: in section 2 we give precise assumptions and some well-known results. In section 3, we introduce the iterative algorithm and give convergence results to solve the optimality system. Section 4 is devoted to numerical examples that illustrate the theoretical findings and in section 5 we present some remarks and a conclusion.

2 Preliminaries and known results

We consider the bilinear form $\sigma(\cdot,\cdot)$ defined in $H^1(\Omega) \times H^1(\Omega)$, where we assume that the following conditions are fulfilled

$\mathbf{H_1}$. Continuity

$$\exists\, C > 0, \forall u, v \in H^1(\Omega),\ |\sigma(u,v)| \leq C \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}$$

$\mathbf{H_2}$. Coercivity

$$\exists\, c > 0, \forall u \in H^1(\Omega),\ \sigma(u,u) \geq c \|u\|^2_{H^1(\Omega)}$$

We call A in $\mathcal{L}(H^1(\Omega), H^{-1}(\Omega))$ the linear self-adjoint elliptic operator (see [19]) associated to σ such that $\langle Au, v\rangle = \sigma(u,v)$, and assume that the adjoint form $\sigma^*(\cdot,\cdot)$ satisfies the conditions H_1 and H_2.

For any φ and ψ in $H_0^1(\Omega)$, we define

$$\mathcal{K}(\varphi, \psi) = \left\{ y \in H_0^1(\Omega) \mid \psi \geq y \geq \varphi \ \text{ in } \ \Omega \right\}, \tag{4}$$

and consider the following variational inequality

$$\sigma(y,\, v - y) \geq (f,\, v - y), \quad \text{for all } v \text{ in } \mathcal{K}(\varphi, \psi), \tag{5}$$

where f belongs to $L^2(\Omega)$ is a source term. From now on, we define the operator $\mathcal{T}$ (control-to-state operator) from $\mathcal{U} \times \mathcal{U}$ to $\mathcal{U}$, such that $y = \mathcal{T}(\varphi, \psi)$ is the unique solution

to the obstacle problem given by (4) and (5) (see [18]), where $\mathcal{U} = H^2(\Omega) \cap H_0^1(\Omega)$.
Let $\mathcal{U}_{ad}$ be the set of admissible controls which is assumed to be $H^2(\Omega)$-bounded subset of $H^2(\Omega) \cap H_0^1(\Omega)$, convex and closed in $H^2(\Omega)$. We may choose, for example,

$$\mathcal{U}_{ad} = \mathcal{B}_{H^2}(0, R) = \{v \text{ in } H^2(\Omega) \cap H_0^1(\Omega) | \|v\|_{H^2} \leq R\} \tag{6}$$

where R is a large enough positive real number. This boundedness assumption for $\mathcal{U}_{ad}$ is crucial: it gives a priori H^2 - estimates on the control functions and leads to the existence of a solution. Now, we consider the optimal control problem (P) defined as follows

$$\min\{J(\varphi, \psi) = \tfrac{1}{2} \int_\Omega (\mathcal{T}(\varphi, \psi) - z)^2 \, dx + \tfrac{\nu}{2} \left(\int_\Omega \left((\nabla\varphi)^2 + (\nabla\psi)^2 \right) dx \right), \quad \text{for all } \varphi, \psi \in \mathcal{U}_{ad}\}, \tag{P}$$

where ν is a strictly given positive constant, z in $L^2(\Omega)$. We seek the obstacles (optimals controls) $(\bar{\varphi}, \bar{\psi})$ in $\mathcal{U}_{ad}^2$, such that the corresponding state is close to a target profile z.

To derive necessary conditions for an optimal control, we would like to differentiate the map $(\varphi, \psi) \mapsto \mathcal{T}(\varphi, \psi)$. Since the map $(\varphi, \psi) \mapsto \mathcal{T}(\varphi, \psi)$ is not directly differentiable (see [21]), the idea here consists in approximating the map $\mathcal{T}(\varphi, \psi)$ by a family of maps $\mathcal{T}^\delta(\varphi, \psi)$ and replacing the obstacle problem (5) and (4) by the following smooth semilinear equation (see [20], [8]):

$$Ay + (\beta_\delta(y - \varphi) - \beta_\delta(\psi - y)) = f \quad \text{in } \Omega, \text{ and } y = 0 \text{ on } \partial\Omega.$$

Then, the approximation map $(\varphi, \psi) \mapsto \mathcal{T}^\delta(\varphi, \psi)$ will then be differentiable and approximate necessary conditions will be derived, such that

$$\beta_\delta(r) = \tfrac{1}{\delta} \begin{cases} 0 & \text{if } r \geq 0 \\ -r^2 & \text{if } r \in \left[-\frac{1}{2}, 0\right] \\ r + \frac{1}{4} & \text{if } r \leq -\frac{1}{2} \end{cases}$$

where $\beta(\cdot)$ is negative and belongs to $\mathscr{C}^1(\mathbb{R})$, such that δ is strictly positive and goes to 0. Then $\beta'_\delta(\cdot)$ is given by

$$\beta'_\delta(r) = \tfrac{1}{\delta} \begin{cases} 0 & \text{if } r \geq 0 \\ -2r & \text{if } r \in \left[-\frac{1}{2}, 0\right] \\ 1 & \text{if } r \leq -\frac{1}{2} \end{cases}$$

As $\beta_\delta(\cdot - \varphi) - \beta_\delta(\psi - \cdot)$ is nondecreasing, it is well known (see [14]), that boundary value problem (2) admits a unique solution y^δ in $H^2(\Omega) \cap H_0^1(\Omega)$ for a fixed φ and ψ in $H^2(\Omega) \cap H_0^1(\Omega)$ and f in $L^2(\Omega)$. In the sequel, we set $y^\delta = \mathcal{T}^\delta(\varphi, \psi)$ and in addition, c or C denotes a general positive constant independent of any approximation parameter. So for any $\delta > 0$, we define

$$J_\delta(\varphi, \psi) = \tfrac{1}{2}[\int_\Omega \left(\mathcal{T}^\delta(\varphi, \psi) - z \right)^2 dx + \nu \int_\Omega \left((\nabla\varphi)^2 + (\nabla\psi)^2 \right) dx]. \tag{P^δ}$$

Then, the approximate optimal control problem is given by

$$min\{J_\delta(\varphi,\psi),\varphi,\psi \text{ in } \mathcal{U}_{ad}\times\mathcal{U}_{ad}\}. \tag{7}$$

and by using the same techniques given in [2] and [7], the problem (7) has, at least, one solution denoted by $(y^\delta,p^\delta,\varphi^\delta,\psi^\delta)$ and characterized by the following Theorem

Theorem 2.1. *Since $(\varphi^\delta,\psi^\delta)$ is an optimal solution to $(\mathcal{P}^\delta)$, and $y^\delta=\mathcal{T}^\delta(\varphi^\delta,\psi^\delta)$. Then there exist p^δ in $\mathcal{U}$, $\mu_1^\delta=\beta_\delta'(y^\delta-\varphi^\delta)p^\delta$ and $\mu_2^\delta=\beta_\delta'(\psi^\delta-y^\delta)p^\delta$ in $L^2(\Omega)$ such that the following optimality system (S^δ) is satisfied*

$$\begin{cases} Ay^\delta+(\beta_\delta(y^\delta-\varphi^\delta)-\beta_\delta(\psi^\delta-y^\delta))=f \text{ in } \Omega \\ A^*p^\delta+\mu_1^\delta+\mu_2^\delta=y^\delta-z \text{ in } \Omega \\ \nu\Delta\varphi^\delta+\nu\Delta\psi^\delta+\beta_\delta{}'(y^\delta-\varphi^\delta)p^\delta+\beta_\delta{}'(\psi^\delta-y^\delta)p^\delta=0 \\ y^\delta=p^\delta=\varphi^\delta=\psi^\delta=0 \text{ on } \partial\Omega \end{cases}$$

Now, we give some important results relevant for the sequel of this paper.

Lemma 2.1. *From the definition of $\beta(\cdot)$ and since p_n^δ belongs to $\mathcal{B}_{H_0^1(\Omega)}(0,\tilde{\rho}_3)$, where $\tilde{\rho}_3$ is a positive constant, and for $(y_i^\delta,\varphi_i^\delta,p_i^\delta)$ in $\tilde{\mathcal{U}}$ where $\tilde{\mathcal{U}}$=$H_0^1(\Omega)\times H_0^2(\Omega)\times H_0^1(\Omega)$ and $i=1,2$, we get*

$$\| \beta_\delta'\left(y_2^\delta-\varphi_2^\delta\right)p_2^\delta-\beta_\delta'\left(y_1^\delta-\varphi_1^\delta\right)p_1^\delta \|_{L^2(\Omega)}\leq \frac{C}{\delta}\| p_2^\delta-p_1^\delta\|_{H^1(\Omega)}+ \frac{C\tilde{\rho}_3}{\delta}\| y_2^\delta-y_1^\delta\|_{L^2(\Omega)}+\frac{C\tilde{\rho}_3}{\delta}\| \varphi_2^\delta-\varphi_1^\delta\|_{L^2(\Omega)}$$

Proof. By the definition of $\beta'(\cdot)$ we get

$$(\beta_\delta'\left(y_2^\delta-\varphi_2^\delta\right)p_2^\delta-\beta_\delta'\left(y_1^\delta-\varphi_1^\delta\right)p_1^\delta)=\beta_\delta'\left(y_2^\delta-\varphi_2^\delta\right)(p_2^\delta-p_1^\delta)+ (\beta_\delta'\left(y_2^\delta-\varphi_2^\delta\right)-\beta_\delta'\left(y_1^\delta-\varphi_1^\delta\right))p_1^\delta.$$

Then, by Cauchy-Schwarz inequality and since p_1^δ belongs to $\mathcal{B}_{H^1(\Omega)}(0,\rho_3)$ and by the Mean-Value Theorem applied in the interval of sides $\{(y_2^\delta-\varphi_2^\delta),(y_1^\delta-\varphi_1^\delta)\}$, we can deduce

$$\| \beta_\delta'\left(y_2^\delta-\varphi_2^\delta\right)p_2^\delta-\beta_\delta'\left(y_1^\delta-\varphi_1^\delta\right)p_1^\delta \|_{L^2(\Omega)}\leq \frac{C}{\delta}\| p_2^\delta-p_1^\delta\|_{H^1(\Omega)}+ \frac{C\tilde{\rho}_3}{\delta}\| \left(y_2^\delta-y_1^\delta\right)-\left(\varphi_2^\delta-\varphi_1^\delta\right)\|_{L^2(\Omega)}$$

□

Lemma 2.2. *Let* $\left(y_i^\delta, \varphi_i^\delta, p_i^\delta\right)$ *belong to* $\tilde{\mathcal{U}}$ *where* $i = 1, 2$ *and by the properties of* $\beta_\delta(\cdot)$*, we get*

$$\| (\beta_\delta(y_2^\delta - \varphi_2^\delta) - \beta_\delta(y_1^\delta - \varphi_1^\delta)) - (\beta_\delta(\psi_2^\delta - y_2^\delta) - \beta_\delta(\psi_1^\delta - y_1^\delta)) \|_{L^2(\Omega)} \leq \frac{C}{\delta} \| y_2^\delta - y_1^\delta \|_{L^2(\Omega)} + \frac{C}{\delta} \| \varphi_2^\delta - \varphi_1^\delta \|_{L^2(\Omega)} + \frac{C}{\delta} \| \psi_2^\delta - \psi_1^\delta \|_{L^2(\Omega)} .$$

Proof. It is easy to see that

$$\| (\beta_\delta(y_2^\delta - \varphi_2^\delta) - \beta_\delta(y_1^\delta - \varphi_1^\delta)) - (\beta_\delta(\psi_2^\delta - y_2^\delta) - \beta_\delta(\psi_1^\delta - y_1^\delta)) \|_{L^2(\Omega)} \leq \| \beta_\delta(y_2^\delta - \varphi_2^\delta) - \beta_\delta(y_1^\delta - \varphi_1^\delta) \|_{L^2(\Omega)} + \| \beta_\delta(\psi_2^\delta - y_2^\delta) - \beta_\delta(\psi_1^\delta - y_1^\delta) \|_{L^2(\Omega)} .$$

By the Mean-Value Theorem applied in the interval of sides $\{(y_2^\delta - \varphi_2^\delta), (y_1^\delta - \varphi_1^\delta)\}$ and $\{(\psi_2^\delta - y_2^\delta), (\psi_1^\delta - y_1^\delta)\}$, we get

$$\| (\beta_\delta(y_2^\delta - \varphi_2^\delta) - \beta_\delta(y_1^\delta - \varphi_1^\delta) - (\beta_\delta(\psi_2^\delta - y_2^\delta) - \beta_\delta(\psi_1^\delta - y_1^\delta)) \|_{L^2(\Omega)} \leq \frac{C}{\delta} \| y_2^\delta - y_1^\delta \|_{L^2(\Omega)} + \frac{C}{\delta} \| \varphi_2^\delta - \varphi_1^\delta \|_{L^2(\Omega)} + \frac{C}{\delta} \| \psi_2^\delta - \psi_1^\delta \|_{L^2(\Omega)} .$$

□

Theorem 2.2. *For any triplet* $\left(y_i^\delta, \varphi_i^\delta, \psi_i^\delta\right)$ *in* $\tilde{\mathcal{U}}$ *that satisfies the optimality system* $\left(S^\delta\right)$ *where* $i = 1, 2$*, and since* $\delta \leq C$*, we get*

$$\| y_2^\delta - y_1^\delta \|_{H^1(\Omega)} \leq l_1(\| \varphi_2^\delta - \varphi_1^\delta \|_{L^2(\Omega)} + \| \psi_2^\delta - \psi_1^\delta \|_{L^2(\Omega)}).$$

where $l_1 := \dfrac{C}{\delta}$*. This means that the mapping* $y^\delta := \mathcal{T}^\delta\left(\varphi^\delta, \psi^\delta\right)$*, is Lipschitizian, with a Lipschitz constant* l_1*.*

Proof. From the state equation of the optimality system $\left(S^\delta\right)$ and by subtraction of the two previous equations and, if we take $v = y_2^\delta - y_1^\delta$, we get

$$\sigma(y_2^\delta - y_1^\delta, y_2^\delta - y_1^\delta) + (\beta_\delta(y_2^\delta - \varphi_2^\delta) - \beta_\delta(y_1^\delta - \varphi_1^\delta) - (\beta_\delta(\psi_2^\delta - y_2^\delta) - \beta_\delta(\psi_1^\delta - y_1^\delta)), y_2^\delta - y_1^\delta) = 0.$$

By the proprieties of $\beta_\delta(\cdot)$, we deduce

$$\begin{aligned} \sigma(y_2^\delta - y_1^\delta, y_2^\delta - y_1^\delta) \quad \leq \quad & -(\beta_\delta(y_2^\delta - \varphi_2^\delta) - \beta_\delta(y_1^\delta - \varphi_1^\delta), \ \left(\varphi_2^\delta - \varphi_1^\delta\right)) \\ & -(\beta_\delta(\psi_2^\delta - y_2^\delta) - \beta_\delta(\psi_1^\delta - y_1^\delta), \left(\psi_1^\delta - \psi_2^\delta\right)). \end{aligned}$$

Then, by using the Mean-Value Theorem and the coercivity condition H_2 of $\sigma(\cdot, \cdot)$, we get

$$c \| y_2^\delta - y_1^\delta \|_{H^1(\Omega)}^2 \leq \frac{c}{\delta}(c \| y_2^\delta - y_1^\delta \|_{H^1(\Omega)} + \| \varphi_2^\delta - \varphi_1^\delta \|_{L^2(\Omega)}) \| \varphi_2^\delta - \varphi_1^\delta \|_{L^2(\Omega)} + \frac{c}{\delta}(c \| y_2^\delta - y_1^\delta \|_{H^1(\Omega)} + \| \psi_1^\delta - \psi_2^\delta \|_{L^2(\Omega)}) \| \psi_1^\delta - \psi_2^\delta \|_{L^2(\Omega)} .$$

From above, and since

1. If $c \parallel y_2^\delta - y_1^\delta \parallel_{H^1(\Omega)} \leq \parallel \varphi_2^\delta - \varphi_1^\delta \parallel_{L^2(\Omega)}$ and $c \parallel y_2^\delta - y_1^\delta \parallel_{H^1(\Omega)} \leq \parallel \psi_1^\delta - \psi_2^\delta \parallel_{L^2(\Omega)}$ then

$$\parallel y_2^\delta - y_1^\delta \parallel_{L^2(\Omega)} \leq \frac{1}{c}(\parallel \varphi_2^\delta - \varphi_1^\delta \parallel_{L^2(\Omega)} + \parallel \psi_1^\delta - \psi_2^\delta \parallel_{L^2(\Omega)}).$$

2. If $c \parallel y_2^\delta - y_1^\delta \parallel_{H^1(\Omega)} \geq \parallel \varphi_2^\delta - \varphi_1^\delta \parallel_{L^2(\Omega)}$ and $c \parallel y_2^\delta - y_1^\delta) \parallel_{H^1(\Omega)} \geq \parallel \psi_1^\delta - \psi_2^\delta \parallel_{L^2(\Omega)}$ we get

$$\parallel y_2^\delta - y_1^\delta \parallel_{H^1(\Omega)} \leq \frac{c}{\delta}(\parallel \varphi_2^\delta - \varphi_1^\delta \parallel_{L^2(\Omega)} + \parallel \psi_1^\delta - \psi_2^\delta \parallel_{L^2(\Omega)}).$$

3. If $c \parallel y_2^\delta - y_1^\delta \parallel_{H^1(\Omega)} \geq \parallel \varphi_2^\delta - \varphi_1^\delta \parallel_{L^2(\Omega)}$ and $c \parallel y_2^\delta - y_1^\delta \parallel_{H^1(\Omega)} \leq \parallel \psi_1^\delta - \psi_2^\delta \parallel_{L^2(\Omega)}$, we get

$$\parallel y_2^\delta - y_1^\delta \parallel_{H^1(\Omega)} \leq \frac{c}{\delta}\left(\parallel \varphi_2^\delta - \varphi_1^\delta \parallel_{L^2(\Omega)} + \parallel \psi_1^\delta - \psi_2^\delta \parallel_{L^2(\Omega)}\right).$$

4. If $c \parallel y_2^\delta - y_1^\delta \parallel_{H^1(\Omega)} \leq \parallel \varphi_2^\delta - \varphi_1^\delta \parallel_{L^2(\Omega)}$ and $c \parallel y_2^\delta - y_1^\delta \parallel_{H^1(\Omega)} \geq \parallel \psi_1^\delta - \psi_2^\delta \parallel_{L^2(\Omega)}$, we get

$$c \parallel y_2^\delta - y_1^\delta \parallel^2_{H^1(\Omega)} \leq \frac{c}{\delta} \parallel \varphi_2^\delta - \varphi_1^\delta \parallel^2_{L^2(\Omega)} + \frac{c}{\delta} \parallel y_2^\delta - y_1^\delta \parallel_{H^1(\Omega)} \parallel \psi_1^\delta - \psi_2^\delta \parallel_{L^2(\Omega)},$$

and since the hypothesis of the Theorem are satisfied, we deduce that in all the previous cases we have

$$\parallel y_2^\delta - y_1^\delta \parallel_{H^1(\Omega)} \leq \frac{c}{\delta}\left(\parallel \varphi_2^\delta - \varphi_1^\delta \parallel_{L^2(\Omega)} + \parallel \psi_1^\delta - \psi_2^\delta \parallel_{L^2(\Omega)}\right).$$

□

Lemma 2.3. *For any triplet $\left(y^\delta, \varphi^\delta, \psi^\delta\right)$ in $\tilde{\mathcal{U}}$, satisfying the optimality system (S^δ), we have*

$$\parallel y^\delta \parallel_{H^1(\Omega)} \leq \max\left(C \parallel \varphi^\delta \parallel_{H^1(\Omega)}, C \parallel \psi^\delta \parallel_{H^1(\Omega)}\right),$$

and moreover when φ^δ and ψ^δ belong to $B_{H^2(\Omega)}(0, \rho_1) \cap \mathcal{W}$, we deduce that

$$\parallel y^\delta \parallel_{H^1(\Omega)} \leq \rho_2.$$

This means that y^δ belongs to $B_{H^1(\Omega)}(0, \rho_2) \cap \mathcal{U}$, where $\rho_2 := C\rho_1$.

Proof. Let v in $\mathcal{K}(\varphi^\delta, \psi^\delta)$, where $\mathcal{K}(\varphi^\delta, \psi^\delta)$ is given by

$$\mathcal{K}(\varphi^\delta, \psi^\delta) = \left\{y \in H_0^1(\Omega) \mid \psi^\delta \geq y \geq \varphi^\delta \text{ in } \Omega\right\}.$$

- If $y^\delta - \varphi^\delta \geq 0$ and $\psi^\delta - y^\delta \geq 0$, we get $\beta(y^\delta - \varphi^\delta) = \beta(\psi^\delta - y^\delta) = 0$.
- If $y^\delta - \varphi^\delta \geq 0$ and $\psi^\delta - y^\delta < 0$, then

$$\beta(\psi^\delta - y^\delta)(\psi^\delta - y^\delta) \geq 0 \text{ and } \beta(y^\delta - \varphi^\delta) = 0.$$

For all v in $\mathcal{K}(\varphi^\delta, \psi^\delta)$, we have $v - y^\delta < v - \psi^\delta < 0$, then

$$-\beta(\psi^\delta - y^\delta)(v - y^\delta) \leq 0.$$

Then, by the following equation

$$\sigma(y^\delta, v - y^\delta) + (\beta_\delta(y^\delta - \varphi^\delta) - \beta_\delta(\psi^\delta - y^\delta), v - y^\delta) = \\ = \left(f, v - y^\delta\right), \text{ for all } v \text{ in } \mathcal{K}(\varphi^\delta, \psi^\delta),$$

and, if we take $v = \psi^\delta$, we get

$$\sigma(y^\delta, y^\delta) \leq \sigma(y^\delta, \psi^\delta) + \left(f, y^\delta - \psi^\delta\right).$$

By the coercivity and continuity conditions of $\sigma(\cdot, \cdot)$ given respectively by H_1 and H_2, we get

$$c\|y^\delta\|^2_{H^1(\Omega)} \leq C\|y^\delta\|_{H^1(\Omega)}\|\psi^\delta\|_{H^1(\Omega)} + \|f\|_{L^2(\Omega)}\left(C\|y^\delta\|_{H^1(\Omega)} + \|\psi^\delta\|_{L^2(\Omega)}\right).$$

From the above inequality, we can deduce that we have two cases

a. If $\|\psi^\delta\|_{L^2(\Omega)} \leq C\|y^\delta\|_{H^1(\Omega)}$
Then, we get

$$\|y^\delta\|_{H^1(\Omega)} \leq C\|\psi^\delta\|_{H^1(\Omega)}.$$

b. If $C\|y^\delta\|_{H^1(\Omega)} \leq \|\psi^\delta\|_{L^2(\Omega)}$, then we obtain

$$\|y^\delta\|_{H^1(\Omega)} \quad \leq \quad C\|\psi^\delta\|_{H^1(\Omega)}.$$

• If $y^\delta - \varphi^\delta < 0$ and $\psi^\delta - y^\delta \geq 0$, we deduce

$$\beta(y^\delta - \varphi^\delta)(y^\delta - \varphi^\delta) \geq 0 \text{ and } \beta(\psi^\delta - y^\delta) = 0.$$

For all v in $\mathcal{K}(\varphi^\delta, \psi^\delta)$, we have $v - y^\delta > v - \varphi^\delta \geq 0$, then

$$\beta(y^\delta - \varphi^\delta)(v - y^\delta) \leq 0.$$

The following equation

$$\sigma(y^\delta, v - y^\delta) + (\beta_\delta(y^\delta - \varphi^\delta) - \beta_\delta(\psi^\delta - y^\delta), v - y^\delta) = \left(f, v - y^\delta\right), \text{ for all } v \text{ in } \mathcal{K}(\varphi^\delta, \psi^\delta),$$

gives

$$\sigma(y^\delta, y^\delta - v) \leq \left(f, y^\delta - v\right), \text{ for all } v \text{ in } \mathcal{K}(\varphi^\delta, \psi^\delta).$$

If we take $v = \varphi^\delta$, and by the coercivity and continuity conditions of $\sigma(\cdot, \cdot)$ given respectively by H_1 and H_2, we get

$$c\|y^\delta\|^2_{H^1(\Omega)} \leq C\|y^\delta\|_{H^1(\Omega)}\|\varphi^\delta\|_{H^1(\Omega)} + \|f\|_{L^2(\Omega)}\left(C\|y^\delta\|_{H^1(\Omega)} + \|\varphi^\delta\|_{L^2(\Omega)}\right).$$

From the above inequality, we can deduce the following two cases

a. If $\|\varphi^\delta\|_{L^2(\Omega)} \leq C\|y^\delta\|_{H^1(\Omega)}$, we get

$$\|y^\delta\|_{H^1(\Omega)} \leq C\|\varphi^\delta\|_{H^1(\Omega)}.$$

b. If $C\|y^\delta\|_{H^1(\Omega)} \leq \|\varphi^\delta\|_{L^2(\Omega)}$, we have

$$\|y^\delta\|_{H^1(\Omega)} \leq C\|\varphi^\delta\|_{H^1(\Omega)}.$$

Therefore, in all cases we get

$$\|y^\delta\|_{H^1(\Omega)} \quad \leq \quad C\|\varphi^\delta\|_{H^1(\Omega)}.$$

• The case when $y^\delta - \varphi^\delta \leq 0$ and $\psi^\delta - y^\delta \leq 0$, is similar to case 3.

□

Lemma 2.4. *For any pair* $\left(p^\delta, y^\delta\right)$ *in* $\mathcal{U} \times (\mathcal{U} \cap B_{H^1}(0, \rho_2))$, *satisfying the optimality system* (S^δ), *we have*

$$\| p^\delta \|_{H^1} \leq C \| y^\delta \|_{H^1(\Omega)},$$

and when y^δ *belongs to* $B_{H^1}(0, \rho_2) \cap \mathcal{U}$, *we deduce that*

$$\| p^\delta \|_{H^1(\Omega)} \leq \rho_3.$$

This means that p^δ *belongs to* $B_{H^1}(0, \rho_3) \cap \mathcal{U}$, *where* $\rho_3 := C\rho_2$.

Proof. From the adjoint equation of optimality system (S^δ), we have

$$\sigma^*\left(p^\delta, v\right) + \left(\mu_1^\delta + \mu_2^\delta, v\right) = \left(y^\delta - z, v\right), \text{ for all } v \text{ in } H_0^1(\Omega)$$

if we take $v = p^\delta$, and by the coercivity condition H_2 of $\sigma^*(\cdot, \cdot)$, we obtain

$$\| p^\delta \|_{H^1(\Omega)} \leq C \| y^\delta \|_{H^1(\Omega)}.$$

□

3 Convergence study of an iterative algorithm

In this section, we give an algorithm to solve problem (P^δ). Roughly speaking, we propose an implicit algorithm to solve the necessary optimality system (S^δ). The proposed algorithm is based on the Gauss-Seidel method and is given below.

This algorithm can be seen as a successive approximation method to compute the five points of the function F that we are going to define. From the different steps of the above algorithm, we define the following functions F_i, for $i = 1, 2, 3, 4$ as

Algorithm 1 Gauss-Seidel algorithm (Continuous version)

Input : $\left\{y_0^\delta, p_0^\delta, \varphi_0^\delta, \psi_0^\delta, \lambda_0^\delta, \delta, \nu, \varepsilon\right\}$ choose $\varphi_0^\delta, \psi_0^\delta$ in $\mathcal{U}, \varepsilon$ and δ in R_+^*;
Begin:
Solve $Ay_n^\delta + \frac{1}{\delta}\left(\beta\left(y_n^\delta - \varphi_{n-1}^\delta\right) - \beta\left(\psi_{n-1}^\delta - y_n^\delta\right)\right) = f$ on y_n^δ
Solve $\left(A + \left(\beta_\delta'\left(y_n^\delta - \varphi_{n-1}^\delta\right) + \beta_\delta'\left(\psi_{n-1}^\delta - y_n^\delta\right)\right)\right) p_n^\delta = y_n^\delta - z$ on p_n^δ.
Calculate $\lambda_n^\delta = \nu\Delta\varphi_{n-1}^\delta + \beta_\delta'\left(y_n^\delta - \varphi_{n-1}^\delta\right) p_n^\delta$
Solve $\nu\Delta\psi_n^\delta + \beta_\delta'\left(\psi_n^\delta - y_n^\delta\right) p_n^\delta = -\lambda_n^\delta$ on ψ_n^δ.
Solve $-\lambda_n^\delta + \nu\Delta\varphi_n^\delta + \beta_\delta'\left(y_n^\delta - \varphi_n^\delta\right) p_n^\delta = 0$ on φ_n^δ.
If the stop criteria is fulfilled **Stop.**
Ensure : $s_n^\delta := \left(y_n^\delta, \varphi_n^\delta, \psi_n^\delta, p_n^\delta\right)$ **is a solution**
Else; $n \leftarrow n+1$, **Go to Begin.**
End if
End algorithm.

- From step 1, we define $F_1 : \mathcal{U} \times \mathcal{U} \to \mathcal{U}$, such that

$$y_n^\delta := F_1\left(\varphi_{n-1}^\delta, \psi_{n-1}^\delta\right),$$

we see that F_1 depends on $\varphi_{n-1}^\delta, \psi_{n-1}^\delta$, and gives y_n^δ as the solution of the following state equation

$$Ay_n^\delta + \beta_\delta\left(y_n^\delta - \varphi_{n-1}^\delta\right) - \beta_\delta\left(\psi_{n-1}^\delta - y_n^\delta\right) = f \text{ in } \Omega, \text{ and } y_n^\delta = 0 \text{ on } \partial\Omega. \tag{8}$$

- From step 2, we define $F_2 : \mathcal{U} \times \mathcal{U} \times \mathcal{U} \to \mathcal{U}$, such that

$$p_n^\delta := F_2\left(y_n^\delta, \varphi_{n-1}^\delta, \psi_{n-1}^\delta\right), \tag{9}$$

we see that F_2 depends on $\varphi_{n-1}^\delta, \psi_{n-1}^\delta$ and y_n^δ, and gives p_n^δ as the solution of the following adjoint state equation

$$Ap_n^\delta + \beta_\delta'\left(y_n^\delta - \varphi_{n-1}^\delta\right) p_n^\delta + \beta_\delta'\left(\psi_{n-1}^\delta - y_n^\delta\right) p_n^\delta = y_n^\delta - z \text{ in } \Omega, \text{ and } p_n^\delta = 0 \text{ on } \partial\Omega. \tag{10}$$

- From step 3, we define $F_3 : \mathcal{U} \times \mathcal{U} \to \mathcal{U}$, such that

$$\psi_n^\delta := F_3\left(y_n^\delta, p_n^\delta\right),$$

we see that F_3 depends on p_n^δ and y_n^δ, and gives ψ_n^δ as the solution of the following equation

$$-\lambda_n^\delta = \nu\Delta\psi_n^\delta + \beta_\delta'\left(\psi_n^\delta - y_n^\delta\right) p_n^\delta \text{ in } \Omega, \text{ and } \psi_n^\delta = 0 \text{ on } \partial\Omega. \tag{11}$$

- From step 4, we define $F_4 : \mathcal{U} \times \mathcal{U} \to \mathcal{U}$, such that

$$\varphi_n^\delta := F_4\left(y_n^\delta, p_n^\delta\right),$$

we see that F_4 depends on p_n^δ, and y_n^δ, and gives φ_n^δ as the solution of the optimality condition equation

$$-\lambda_n^\delta + \nu\Delta\varphi_n^\delta + \beta_\delta'\left(y_n^\delta - \varphi_n^\delta\right)p_n^\delta = 0 \text{ in } \Omega, \text{ and } \varphi_n^\delta = 0 \text{ on } \partial\Omega. \tag{12}$$

Remark 3.1. *We note that the equation given by* (11) *is only used to solve the equation given by* (12)*.*

Then according the above definitions of F_i, where $i = 1, 2, 3, 4$, let us define the map $F : \mathcal{U} \times \mathcal{U} \to \mathcal{U} \times \mathcal{U}$, as

$$\left(\varphi_n^\delta, \psi_n^\delta\right) := F\left(\varphi_{n-1}^\delta, \psi_{n-1}^\delta\right),$$

where

$$\varphi_n^\delta := \tilde{F}_1\left(\varphi_{n-1}^\delta, \psi_{n-1}^\delta\right) := F_4\left(F_1\left(\varphi_{n-1}^\delta, \psi_{n-1}^\delta\right), F_2\left(F_1\left(\varphi_{n-1}^\delta, \psi_{n-1}^\delta\right), \varphi_{n-1}^\delta, \psi_{n-1}^\delta\right)\right),$$

and

$$\psi_n^\delta := \tilde{F}_2\left(\varphi_{n-1}^\delta, \psi_{n-1}^\delta\right) := F_3\left(F_1\left(\varphi_{n-1}^\delta, \psi_{n-1}^\delta\right), F_2\left(F_1\left(\varphi_{n-1}^\delta, \psi_{n-1}^\delta\right), \varphi_{n-1}^\delta, \psi_{n-1}^\delta\right)\right)$$

such that

$$F\left(\varphi_{n-1}^\delta, \psi_{n-1}^\delta\right) = \left(\tilde{F}_1\left(\varphi_{n-1}^\delta, \psi_{n-1}^\delta\right), \tilde{F}_2\left(\varphi_{n-1}^\delta, \psi_{n-1}^\delta\right)\right)$$

Proposition 3.1. *Let* φ_{n-1}^δ *and* ψ_{n-1}^δ *belong to* $\mathcal{U}$ *and* $\left(y_n^\delta, \varphi_n^\delta, \psi_n^\delta, p_n^\delta\right)$ *satisfies equations* (8), (10), (11) *and* (12) *given respectively by* F_1, F_2, F_3 *and* F_4 *such that* $\delta \leq C$*, then we get*

$$\| y_n^\delta \|_{H^1(\Omega)} \leq \frac{c}{\delta} \| \varphi_{n-1}^\delta \|_{H^2(\Omega)} + \frac{c}{\delta} \| \psi_{n-1}^\delta \|_{H^2(\Omega)} + c \| f \|_{L^2(\Omega)} \tag{13}$$

$$\| p_n^\delta \|_{H^1(\Omega)} \leq C + C \| y_n^\delta \|_{H^1(\Omega)} \tag{14}$$

$$\| \varphi_n^\delta \|_{H^2(\Omega)} \leq \frac{C}{\delta\nu} \| p_n^\delta \|_{H^1(\Omega)} + C \tag{15}$$

and

$$\| \psi_n^\delta \|_{H^2(\Omega)} \leq \frac{C}{\delta\nu} \| p_n^\delta \|_{H^1(\Omega)} + C \tag{16}$$

Proof. From the state equation (8), we write

$$\sigma\left(y_n^\delta, y_n^\delta\right) + \left(\beta_\delta\left(y_n^\delta - \varphi_{n-1}^\delta\right) - \beta_\delta\left(\psi_{n-1}^\delta - y_n^\delta\right), y_n^\delta\right) = \left(f, y_n^\delta\right).$$

Then, by the definition of $\beta_\delta(\cdot)$, and by the coercivity condition H_2 of the bilinear form $\sigma(\cdot,\cdot)$ and thanks to the Mean-Value Theorem applied in the interval of sides $\{0,(y_n^\delta - \varphi_{n-1}^\delta)\}$ and $\{0,(\psi_{n-1}^\delta - y_n^\delta)\}$, we obtain

$$\begin{aligned} c \parallel y_n^\delta \parallel^2_{H^1(\Omega)} \leq \frac{c}{\delta}\left(c \parallel y_n^\delta \parallel_{H^1(\Omega)} + \parallel \varphi_{n-1}^\delta \parallel_{L^2(\Omega)}\right) \parallel \varphi_{n-1}^\delta \parallel_{L^2(\Omega)} + \\ + \frac{c}{\delta}\left(\parallel \psi_{n-1}^\delta \parallel_{L^2(\Omega)} + c \parallel y_n^\delta \parallel_{H^1(\Omega)}\right) \parallel \psi_{n-1}^\delta \parallel_{L^2(\Omega)} + c \parallel f \parallel_{L^2(\Omega)} \parallel y_n^\delta \parallel_{H^1(\Omega)} \end{aligned} \quad (17)$$

Then from the above inequality (17), we deduce that we have four cases:

1. If $\parallel \varphi_{n-1}^\delta \parallel_{L^2(\Omega)} \leq c \parallel y_n^\delta \parallel_{H^1(\Omega)}$ and $\parallel \psi_{n-1}^\delta \parallel_{L^2(\Omega)} \leq c \parallel y_n^\delta \parallel_{H^1(\Omega)}$, then

$$\parallel y_n^\delta \parallel_{H^1(\Omega)} \leq \frac{c}{\delta} \parallel \varphi_{n-1}^\delta \parallel_{L^2(\Omega)} + \frac{c}{\delta} \parallel \psi_{n-1}^\delta \parallel_{L^2(\Omega)} + c \parallel f \parallel_{L^2(\Omega)}.$$

2. If $\parallel \varphi_{n-1}^\delta \parallel_{L^2(\Omega)} \leq c \parallel y_n^\delta \parallel_{H^1(\Omega)}$ and $\parallel \psi_{n-1}^\delta \parallel_{L^2(\Omega)} \geq c \parallel y_n^\delta \parallel_{H^1(\Omega)}$, then, we get

$$\parallel y_n^\delta \parallel_{H^1(\Omega)} \leq \left(\frac{c}{\delta} \parallel \varphi_{n-1}^\delta \parallel_{L^2(\Omega)} + c \parallel f \parallel_{L^2(\Omega)}\right) + \frac{c}{\sqrt{\delta}} \parallel \psi_{n-1}^\delta \parallel_{L^2(\Omega)}.$$

3. If $\parallel \varphi_{n-1}^\delta \parallel_{L^2(\Omega)} \geq c \parallel y_n^\delta \parallel_{H^1(\Omega)}$ and $\parallel \psi_{n-1}^\delta \parallel_{L^2(\Omega)} \geq c \parallel y_n^\delta \parallel_{H^1(\Omega)}$, thus

$$\parallel y_n^\delta \parallel_{H^1(\Omega)} \leq c \parallel f \parallel_{L^2(\Omega)} + \frac{c}{\sqrt{\delta}} \parallel \varphi_{n-1}^\delta \parallel_{L^2(\Omega)} + \frac{c}{\sqrt{\delta}} \parallel \psi_{n-1}^\delta \parallel_{L^2(\Omega)}.$$

4. If $\parallel \varphi_{n-1}^\delta \parallel_{L^2(\Omega)} \geq c \parallel y_n^\delta \parallel_{H^1(\Omega)}$ and $\parallel \psi_{n-1}^\delta \parallel_{L^2(\Omega)} \leq c \parallel y_n^\delta \parallel_{H^1(\Omega)}$, therefore

$$\parallel y_n^\delta \parallel_{H^1(\Omega)} \leq \frac{c}{\delta} \parallel \psi_{n-1}^\delta \parallel_{L^2(\Omega)} + c \parallel f \parallel_{L^2(\Omega)} + \frac{c}{\sqrt{\delta}} \parallel \varphi_{n-1}^\delta \parallel_{L^2(\Omega)}.$$

Finally, in any cases we obtain

$$\parallel y_n^\delta \parallel_{H^1(\Omega)} \leq \left(\frac{c}{\delta} \parallel \varphi_{n-1}^\delta \parallel_{L^2(\Omega)} + \frac{c}{\delta} \parallel \psi_{n-1}^\delta \parallel_{L^2(\Omega)} + c \parallel f \parallel_{L^2(\Omega)}\right).$$

Now, from the adjoint state equation (10), we get

$$\langle A p_n^\delta, p_n^\delta \rangle + \left(\beta'_\delta\left(y_n^\delta - \varphi_{n-1}^\delta\right) p_n^\delta + \beta'_\delta\left(\psi_{n-1}^\delta - y_n^\delta\right) p_n^\delta, p_n^\delta\right) = \left(y_n^\delta - z, p_n^\delta\right) \text{ in } \Omega.$$

By the coercivity condition of $\sigma^*(\cdot,\cdot)$ given by H_2, we obtain that

$$\parallel p_n^\delta \parallel_{H^1(\Omega)} \leq C \parallel y_n^\delta \parallel_{H^1(\Omega)},$$

by using the following equation

$$\lambda_n^\delta = \nu \Delta \varphi_{n-1}^\delta + \beta'_\delta\left(y_n^\delta - \varphi_{n-1}^\delta\right) p_n^\delta \text{ and } \lambda_n^\delta = 0 \text{ on } \partial\Omega$$

we deduce that

$$\| \lambda_n^\delta \|_{H^1(\Omega)} \leq \frac{C}{\delta} \| p_n^\delta \|_{H^1(\Omega)} + C\nu \| \varphi_{n-1}^\delta \|_{H^2(\Omega)} .$$

From equation (11), and by the coercivity condition H_2 of $\sigma(\cdot,\cdot)$, and the definition of $\beta'_\delta(\cdot)$, we obtain

$$\begin{aligned} \| \psi_n^\delta \|_{H^2(\Omega)} &\leq \frac{C}{\delta\nu} \| p_n^\delta \|_{H^1(\Omega)} + \frac{C}{\nu} \| \lambda_n^\delta \|_{H^1(\Omega)} \\ &\leq \frac{C}{\delta\nu} \| p_n^\delta \|_{H^1(\Omega)} + C \| \varphi_{n-1}^\delta \|_{H^2(\Omega)} . \end{aligned}$$

Using equation (12), and by the coercivity condition H_2 of $\sigma(.,.)$, and the definition of $\beta'_\delta(\cdot)$, we get

$$\begin{aligned} \| \varphi_n^\delta \|_{H^2(\Omega)} &\leq \frac{C}{\delta\nu} \| p_n^\delta \|_{H^1(\Omega)} + \frac{C}{\nu} \| \lambda_n^\delta \|_{H^1(\Omega)} \\ &\leq \frac{C}{\delta\nu} \| p_n^\delta \|_{H^1(\Omega)} + C \| \varphi_{n-1}^\delta \|_{H^2(\Omega)} . \end{aligned}$$

□

Corollary 3.1. *Since φ_{n-1}^δ and ψ_{n-1}^δ belong to $B_{H^2}(0,\tilde{\rho}_1) \cap \mathcal{U}$, and letting $\left(y_n^\delta, \varphi_{n-1}^\delta, \psi_{n-1}^\delta\right)$ belong to $\tilde{\mathcal{U}}$ to satisfy the conditions* (8), (11) *and* (12) *given respectively by F_1, F_3 and F_4 such that $\delta \leq C$, then we get*

$$\| y_n^\delta \|_{H^1(\Omega)} \leq \tilde{\rho}_2$$

This means that y_n^δ belongs to $B_{H^1}(0,\tilde{\rho}_2) \cap \mathcal{U}$, where $\tilde{\rho}_2 := \left(C + \frac{C}{\delta}\tilde{\rho}_1\right)$.

Proof. By using inequality (13), we obtain

$$\| y_n^\delta \|_{H^1(\Omega)} \leq \frac{c}{\delta} \| \varphi_{n-1}^\delta \|_{H^2(\Omega)} + \frac{c}{\delta} \| \psi_{n-1}^\delta \|_{H^2(\Omega)} + c \| f \|_{L^2(\Omega)} \tag{18}$$

Therefore

$$\| y_n^\delta \|_{H^1(\Omega)} \leq \tilde{\rho}_2 \tag{19}$$

where $\tilde{\rho}_2 := \left(C + \frac{C}{\delta}\tilde{\rho}_1\right)$. □

Corollary 3.2. *Since the hypotheses of Corollary 3.1 are fulfilled, and by letting $(y_n^\delta, p_n^\delta) \in H^1(\Omega) \times H^1(\Omega)$ to satisfy the conditions* (8), (10) *given respectively by F_1, F_2, we get*

$$\| p_n^\delta \|_{H^1(\Omega)} \leq \tilde{\rho}_3$$

This means that p_n^δ belongs to $B_{H^1}(0,\tilde{\rho}_3) \cap \mathcal{U}$, where $\tilde{\rho}_3 := C\tilde{\rho}_2$.

Proof. From inequalities (14) and (19), we obtain

$$\| p_n^\delta \|_{H^1(\Omega)} \leq C\tilde{\rho}_2.$$

Then

$$\| p_n^\delta \|_{H^1(\Omega)} \leq \tilde{\rho}_3,$$

where $\tilde{\rho}_3 = C\tilde{\rho}_2$. □

Corollary 3.3. *Since the hypotheses of corollary* 3.1 *are fulfilled, and by letting* $\left(y_n^\delta, \varphi_n^\delta, \psi_n^\delta\right)$ *in* $\tilde{\mathcal{U}}$ *to satisfy the conditions* (8), (11) *and* (12) *given respectively by* F_1, F_3 *and* F_4, *we get*

$$\| \varphi_n^\delta \|_{H^2(\Omega)} \leq \tilde{\rho}_4.$$

and

$$\| \psi_n^\delta \|_{H^2(\Omega)} \leq \tilde{\rho}_4.$$

This means that φ_n^δ *and* ψ_n^δ *belong respectively to* $\mathcal{B}_{H^2}(0, \tilde{\rho}_4) \cap \mathcal{U}$, *and* $\tilde{\rho}_4 := \dfrac{C}{\delta\nu}\tilde{\rho}_3 + C\tilde{\rho}_1$, *where* $\tilde{\rho}_3$, $\tilde{\rho}_2$ *are given respectively by corollaries* 3.2 *and 3.1.*

Proof. From inequalities (15), (16) and (3.2), we obtain

$$\| \varphi_n^\delta \|_{H^2(\Omega)} \leq \tilde{\rho}_4 \text{ and } \| \psi_n^\delta \|_{H^2(\Omega)} \leq \tilde{\rho}_4,$$

where $\tilde{\rho}_4 := \dfrac{C}{\delta\nu}\tilde{\rho}_3 + C\tilde{\rho}_1$. □

Let us give the following theorem to show that the mapping F is locally Lipschitz.

Theorem 3.1. *If* $\delta \leq C$, *then the mapping* F *is locally Lipschitz from* $\mathcal{B}_{H^2}(0, \tilde{\rho}_1) \cap \mathcal{U} \times \mathcal{B}_{H^2}(0, \tilde{\rho}_1) \cap \mathcal{U}$ *to* $\mathcal{B}_{H^2}(0, \tilde{\rho}_4) \cap \mathcal{U} \times \mathcal{B}_{H^2}(0, \tilde{\rho}_4) \cap \mathcal{U}$, *with the Lipschitz constant* $l := l_1(l_3+l_4)+l_2(l_3+l_4)+l_1l_2(l_3+l_4)$, *where* $\tilde{\rho}_4 = \dfrac{C}{\delta\nu}+(\dfrac{C}{\delta^2\nu}+C)\tilde{\rho}_1$, $l_1 := \dfrac{C}{\delta}$, $l_2 := \left(C + \dfrac{C\tilde{\rho}_3}{\delta}\right)$, $l_3 = l_4 := \dfrac{C}{\delta\nu C - C\tilde{\rho}_3}$, *and* $\tilde{\rho}_3$ *is given by Corollary 3.2.*

To prove the previous theorem, we need the followings Lemmas.

Lemma 3.1. *The function* F_1 *defined by* (3) *is Lipschitz continuous from* $\mathcal{U}$ *to* $\mathcal{U}$, *with a Lipschitz constant* $l_1 := \dfrac{C}{\delta}$.

Proof. Let $y_n^\delta = F_1\left(\varphi_{n-1}^{\delta,1}, \psi_{n-1}^{\delta,1}\right)$ and $z_n^\delta = F_1\left(\varphi_{n-1}^{\delta,2}, \psi_{n-1}^{\delta,2}\right)$, where $\left(y_n^\delta, \varphi_{n-1}^{\delta,1}, \psi_{n-1}^{\delta,1}\right)$ and $\left(z_n^\delta, \varphi_{n-1}^{\delta,2}, \psi_{n-1}^{\delta,2}\right)$ belong to $\mathcal{U} \times \mathcal{U} \times \mathcal{U}$. By the equation given by (8), we get

$$\sigma\left(y_n^\delta - z_n^\delta, y_n^\delta - z_n^\delta\right) + (\left(\beta_\delta\left(y_n^\delta - \varphi_{n-1}^{\delta,1}\right) - \beta_\delta\left(z_n^\delta - \varphi_{n-1}^{\delta,2}\right)\right) - \left(\beta_\delta\left(\psi_{n-1}^{\delta,1} - y_n^\delta\right) - \beta_\delta\left(\psi_{n-1}^{\delta,2} - z_n^\delta\right)\right), y_n^\delta - z_n^\delta) = 0.$$

By the coercivity condition H_2 of $\sigma(\cdot,\cdot)$ and Lemma 2.2, we get

$$\| y_n^\delta - z_n^\delta \|_{H^1(\Omega)}^2 \leq \frac{C}{\delta}\left(\| y_n^\delta - z_n^\delta \|_{L^2(\Omega)} + \| \varphi_{n-1}^{\delta,1} - \varphi_{n-1}^{\delta,2} \|_{L^2(\Omega)}\right) \| \varphi_{n-1}^{\delta,1} - \varphi_{n-1}^{\delta,2} \|_{L^2(\Omega)} + \frac{C}{\delta}\left(\| y_n^\delta - z_n^\delta \|_{L^2(\Omega)} + \| \psi_{n-1}^{\delta,1} - \psi_{n-1}^{\delta,2} \|_{L^2(\Omega)}\right) \| \psi_{n-1}^{\delta,1} - \psi_{n-1}^{\delta,2} \|_{L^2(\Omega)}.$$

From the above inequalities, we have four cases

1. If $\| y_n^\delta - z_n^\delta \|_{L^2(\Omega)} \leq \| \varphi_{n-1}^{\delta,1} - \varphi_{n-1}^{\delta,2} \|_{L^2(\Omega)}$ and $\| y_n^\delta - z_n^\delta \|_{L^2(\Omega)} \leq \| \psi_{n-1}^{\delta,1} - \psi_{n-1}^{\delta,2} \|_{L^2(\Omega)}$, then
$$\| y_n^\delta - z_n^\delta \|_{H^1(\Omega)} \leq \left(\| \varphi_{n-1}^{\delta,1} - \varphi_{n-1}^{\delta,2} \|_{L^2(\Omega)} + \| \psi_{n-1}^{\delta,1} - \psi_{n-1}^{\delta,2} \|_{L^2(\Omega)} \right).$$
2. If $\| y_n^\delta - z_n^\delta \|_{L^2(\Omega)} \geq \| \varphi_{n-1}^{\delta,1} - \varphi_{n-1}^{\delta,2} \|_{L^2(\Omega)}$ and $\| y_n^\delta - z_n^\delta \|_{L^2(\Omega)} \geq \| \psi_{n-1}^{\delta,1} - \psi_{n-1}^{\delta,2} \|_{L^2(\Omega)}$, we get
$$\| y_n^\delta - z_n^\delta \|_{H^1(\Omega)} \leq \frac{C}{\delta} \left(\| \varphi_{n-1}^{\delta,1} - \varphi_{n-1}^{\delta,2} \|_{L^2(\Omega)} + \| \psi_{n-1}^{\delta,1} - \psi_{n-1}^{\delta,2} \|_{L^2(\Omega)} \right).$$
3. If $\| y_n^\delta - z_n^\delta \|_{L^2(\Omega)} \geq \| \varphi_{n-1}^{\delta,1} - \varphi_{n-1}^{\delta,2} \|_{L^2(\Omega)}$ and $\| y_n^\delta - z_n^\delta \|_{L^2(\Omega)} \leq \| \psi_{n-1}^{\delta,1} - \psi_{n-1}^{\delta,2} \|_{L^2(\Omega)}$, then we get
$$\| y_n^\delta - z_n^\delta \|_{H^1(\Omega)} \leq \frac{C}{\delta} \left(\| \varphi_{n-1}^{\delta,1} - \varphi_{n-1}^{\delta,2} \|_{L^2(\Omega)} + \| \psi_{n-1}^{\delta,1} - \psi_{n-1}^{\delta,2} \|_{L^2(\Omega)} \right).$$
4. If $\| y_n^\delta - z_n^\delta \|_{L^2(\Omega)} \geq \| \varphi_{n-1}^{\delta,1} - \varphi_{n-1}^{\delta,2} \|_{L^2(\Omega)}$ and $\| y_n^\delta - z_n^\delta \|_{L^2(\Omega)} \leq \| \psi_{n-1}^{\delta,1} - \psi_{n-1}^{\delta,2} \|_{L^2(\Omega)}$, then we obtain
$$\| y_n^\delta - z_n^\delta \|_{H^1(\Omega)} \leq \frac{C}{\delta} \left(\| \varphi_{n-1}^{\delta,1} - \varphi_{n-1}^{\delta,2} \|_{L^2(\Omega)} + \| \psi_{n-1}^{\delta,1} - \psi_{n-1}^{\delta,2} \|_{L^2(\Omega)} \right).$$

□

Lemma 3.2. *The function F_2 defined by (9), is locally Lipschitz from $(\mathcal{B}_{H^1}(0,\tilde{\rho}_2) \cap \mathcal{U}) \times (\mathcal{B}_{H^2}(0,\tilde{\rho}_1) \cap \mathcal{U}) \times (\mathcal{B}_{H^2}(0,\tilde{\rho}_1) \cap \mathcal{U})$ to $\mathcal{B}_{H^1}(0,\tilde{\rho}_3) \cap \mathcal{U}$, with the Lipschitz constant $l_2 := \left(C + \frac{C\tilde{\rho}_3}{\delta} \right)$, where $\tilde{\rho}_3$ is given by Corollary 3.2.*

Proof. Let $p_n^{\delta,1} = F_2\left(y_n^{\delta,1}, \varphi_{n-1}^{\delta,1}, \psi_{n-1}^{\delta,1}\right)$ and $p_n^{\delta,2} = F_2\left(y_n^{\delta,2}, \varphi_{n-1}^{\delta,2}, \psi_{n-1}^{\delta,2}\right)$ where $\left(y_n^{\delta,1}, \varphi_{n-1}^{\delta,1}, \psi_{n-1}^{\delta,1}\right)$ and $\left(y_n^{\delta,2}, \varphi_{n-1}^{\delta,2}, \psi_{n-1}^{\delta,2}\right)$ belong to $(\mathcal{B}_{H^1}(0,\tilde{\rho}_2) \cap \mathcal{U}) \times (\mathcal{B}_{H^2}(0,\tilde{\rho}_1) \cap \mathcal{W}) \times (\mathcal{B}_{H^2}(0,\tilde{\rho}_1) \cap \mathcal{W})$. Then by the adjoint state equation (10), we get

$$\begin{aligned} C \| p_n^{\delta,2} - p_n^{\delta,1} \|_{H^1(\Omega)}^2 \leq & \\ & \frac{C}{\delta} \| \beta'\left(y_n^{\delta,2} - \varphi_{n-1}^{\delta,2}\right) - \beta'\left(y_n^{\delta,1} - \varphi_{n-1}^{\delta,1}\right) \|_{L^2(\Omega)} \| p_n^{\delta,2} \|_{L^2(\Omega)} \| p_n^{\delta,2} - p_n^{\delta,1} \|_{H^1(\Omega)} + \\ & \frac{C}{\delta} \| \beta'\left(\psi_{n-1}^{\delta,2} - y_n^{\delta,2}\right) - \beta'\left(\psi_{n-1}^{\delta,1} - y_n^{\delta,1}\right) \|_{L^2(\Omega)} \| p_n^{\delta,2} \|_{L^2(\Omega)} \| p_n^{\delta,2} - p_n^{\delta,1} \|_{H^1(\Omega)} + \\ & \| y_n^{\delta,2} - y_n^{\delta,1} \|_{L^2(\Omega)} \| p_n^{\delta,2} - p_n^{\delta,1} \|_{L^2(\Omega)}. \end{aligned}$$

By the definition of $\beta(\cdot)$ and by the Mean-Value Theorem applied in the intervals of sides $\{\left(y_n^{\delta,2} - \varphi_{n-1}^{\delta,2}\right), \left(y_n^{\delta,1} - \varphi_{n-1}^{\delta,1}\right)\}$ and $\{\left(\psi_{n-1}^{\delta,2} - y_n^{\delta,2}\right), \left(\psi_{n-1}^{\delta,1} - y_n^{\delta,1}\right)\}$, we get

$$\begin{aligned} \| p_n^{\delta,2} - p_n^{\delta,1} \|_{H^1(\Omega)} \leq \left(C + \frac{C\,\tilde{\rho}_3}{\delta} \right) & (\| y_n^{\delta,2} - y_n^{\delta,1} \|_{L^2(\Omega)} + \\ & \| \psi_{n-1}^{\delta,2} - \psi_{n-1}^{\delta,1} \|_{L^2(\Omega)} + \| \varphi_{n-1}^{\delta,2} - \varphi_{n-1}^{\delta,1} \|_{L^2(\Omega)}). \end{aligned}$$

□

Lemma 3.3. *Since the following condition,*

$$\tilde{\rho}_3 \leq \delta\nu C,$$

is fulfilled, the function F_3 is locally Lipschitiz from $(\mathcal{B}_{H^1}(0,\tilde{\rho}_2)\cap\mathcal{U})\times(\mathcal{B}_{H^1}(0,\tilde{\rho}_3)\cap\mathcal{U})\times(\mathcal{B}_{H^1}(0,\tilde{\rho}_4)\cap\mathcal{U})$ to $(\mathcal{B}_{H^2}(0,\tilde{\rho}_4)\cap\mathcal{W})$, with Lipschitz constant

$$l_3 := \frac{C}{\delta\nu C - C\tilde{\rho}_3}.$$

Proof. From equation (11), we obtain

$$\nu\sigma\left(\varphi_n^{\delta,1}-\varphi_n^{\delta,2},\varphi_n^{\delta,1}-\varphi_n^{\delta,2}\right)=\left(\beta'_\delta(y_n^{\delta,1}-\varphi_n^{\delta,1})p_n^{\delta,1}-\beta'_\delta(y_n^{\delta,2}-\varphi_n^{\delta,2})p_n^{\delta,2},\varphi_n^{\delta,1}-\varphi_n^{\delta,2}\right)+ \\ +\left(\lambda_n^{\delta,1}-\lambda_n^{\delta,2},\varphi_n^{\delta,1}-\varphi_n^{\delta,2}\right).$$

By the coercivity condition H_2 of $\sigma(.,.)$, and by Lemma 2.1, we obtain

$$\| \varphi_n^{\delta,1}-\varphi_n^{\delta,2} \|_{H^2(\Omega)} \leq \frac{C\ \tilde{\rho}_3}{\nu\delta} \| (y_n^{\delta,1}-y_n^{\delta,2})-(\varphi_n^{\delta,1}-\varphi_n^{\delta,2}) \|_{L_2(\Omega)} + \frac{C}{\nu\delta} \| p_n^{\delta,1}-p_n^{\delta,2} \|_{L_2(\Omega)} + \\ \frac{C}{\nu} \| \lambda_n^{\delta,1}-\lambda_n^{\delta,2} \|_{L_2(\Omega)} \quad (20)$$

For the previous inequality to have a meaning, we must have

$$\tilde{\rho}_3 \leq C\nu\delta.$$

Then, we get

$$\| \varphi_n^{\delta,1}-\varphi_n^{\delta,2} \|_{H^2(\Omega)} \leq \frac{C\tilde{\rho}_3}{\nu\delta - C\tilde{\rho}_3} \| y_n^{\delta,1}-y_n^{\delta,2} \|_{L_2(\Omega)} + \frac{C}{\nu\delta - C\tilde{\rho}_3} \| p_n^{\delta,1}-p_n^{\delta,2} \|_{L_2(\Omega)} + \\ + \frac{\delta C}{\nu\delta - C\tilde{\rho}_3} \| \lambda_n^{\delta,1}-\lambda_n^{\delta,2} \|_{L_2(\Omega)}.$$

□

Lemma 3.4. *Since the following condition,*

$$\tilde{\rho}_3 \leq \delta\nu C,$$

is fulfilled, then, the function F_4 given by (3) is locally Lipschitiz from $(\mathcal{B}_{H^1}(0,\tilde{\rho}_2)\cap\mathcal{U})\times(\mathcal{B}_{H^1}(0,\tilde{\rho}_3)\cap\mathcal{U})\times(\mathcal{B}_{H^1}(0,\tilde{\rho}_4)\cap\mathcal{U})$ to $(\mathcal{B}_{H^2}(0,\tilde{\rho}_4)\cap\mathcal{W})$, with Lipschitz constant

$$l_4 := \frac{C}{\delta\nu C - C\tilde{\rho}_3}.$$

Proof. From equation (11), we obtain

$$\nu\sigma\left(\psi_n^{\delta,1}-\psi_n^{\delta,2},\psi_n^{\delta,1}-\psi_n^{\delta,2}\right)=\left(\beta'_\delta(\psi_n^{\delta,1}-y_n^{\delta,1})p_n^{\delta,1}-\beta'_\delta(\psi_n^{\delta,2}-y_n^{\delta,2})p_n^{\delta,2},\psi_n^{\delta,1}-\psi_n^{\delta,2}\right)+ \\ +\left(\lambda_n^{\delta,1}-\lambda_n^{\delta,2},\psi_n^{\delta,1}-\psi_n^{\delta,2}\right).$$

By the coercivity condition H_2 of $\sigma(\cdot,\cdot)$ and by Lemma 2.2, we get

$$\| \psi_n^{\delta,1} - \psi_n^{\delta,2} \|_{H^2(\Omega)} \leq \frac{C\ \tilde{\rho}_3}{\nu\delta} \| (y_n^{\delta,1} - y_n^{\delta,2}) - (\psi_n^{\delta,1} - \psi_n^{\delta,2}) \|_{L_2(\Omega)} + \frac{C}{\nu\delta} \| p_n^{\delta,1} - p_n^{\delta,2} \|_{L_2(\Omega)} + \\ + \frac{C}{\nu} \| \lambda_n^{\delta,1} - \lambda_n^{\delta,2} \|_{L_2(\Omega)}.$$

For the previous inequality to have a sense, we must have

$$\tilde{\rho}_3 \leq C\nu\delta.$$

Then, we get

$$\| \psi_n^{\delta,1} - \psi_n^{\delta,2} \|_{H^2(\Omega)} \leq \frac{C\tilde{\rho}_3}{\nu\delta - C\tilde{\rho}_3} \| y_n^{\delta,1} - y_n^{\delta,2} \|_{L_2(\Omega)} + \frac{C}{\nu\delta - C\tilde{\rho}_3} \| p_n^{\delta,1} - p_n^{\delta,2} \|_{L_2(\Omega)} + \\ + \frac{\delta C}{\nu\delta - C\tilde{\rho}_3} \| \lambda_n^{\delta,1} - \lambda_n^{\delta,2} \|_{L_2(\Omega)}.$$

□

Now, we give the proof of Theorem 3.1.

Proof. Let

$$(\varphi_n^{\delta,1}, \psi_n^{\delta,1}) := (F_3\left(p_n^{\delta,1}, y_n^{\delta,1}\right), F_4\left(p_n^{\delta,1}, y_n^{\delta,1}\right)),$$

and

$$(\varphi_n^{\delta,2}, \psi_n^{\delta,2}) := (F_3\left(p_n^{\delta,2}, y_n^{\delta,2}\right), F_4\left(p_n^{\delta,2}, y_n^{\delta,2}\right)).$$

Then, we get

$$\| (\varphi_n^{\delta,1}, \psi_n^{\delta,1}) - (\varphi_n^{\delta,2}, \psi_n^{\delta,2}) \|_{H^2(\Omega)} = \| F_3\left(p_n^{\delta,1}, y_n^{\delta,1}\right) - F_3\left(p_n^{\delta,2}, y_n^{\delta,2}\right) \|_{H^2(\Omega)} + \\ + \| F_4\left(p_n^{\delta,1}, y_n^{\delta,1}\right) - F_4\left(p_n^{\delta,2}, y_n^{\delta,2}\right) \|_{H^2(\Omega)}.$$

Thanks to the Lemmas 3.3 and 3.4, we get

$$\| (\varphi_n^{\delta,1}, \psi_n^{\delta,1}) - (\varphi_n^{\delta,2}, \psi_n^{\delta,2}) \|_{H^2(\Omega)} \leq (l_3 + l_4)\left(\| y_n^{\delta,1} - y_n^{\delta,2} \|_{H^1(\Omega)} + \| p_n^{\delta,1} - p_n^{\delta,2} \|_{H^1(\Omega)}\right),$$

where $p_n^{\delta,1} := F_2\left(\varphi_{n-1}^{\delta,1}, y_n^{\delta,1}, \psi_{n-1}^{\delta,1}\right)$, $p_n^{\delta,2} := F_2\left(\varphi_{n-1}^{\delta,1}, y_n^{\delta,2}, \psi_{n-1}^{\delta,1}\right)$, $y_n^{\delta,1} := F_1(\varphi_{n-1}^{\delta,1}, \psi_{n-1}^{\delta,1})$, and $y_n^{\delta,2} := F_1(\varphi_{n-1}^{\delta,2}, \psi_{n-1}^{\delta,2})$, and by Lemmas 3.1 and 3.2, we obtain

$$\| (\varphi_n^{\delta,1}, \psi_n^{\delta,1}) - (\varphi_n^{\delta,2}, \psi_n^{\delta,2}) \|_{H^2(\Omega)} \leq l\left(\| \varphi_{n-1}^{\delta,1} - \varphi_{n-1}^{\delta,2} \|_{H^1(\Omega)} + \| \psi_{n-1}^{\delta,1} - \psi_{n-1}^{\delta,2} \|_{H^1(\Omega)}\right),$$

where $l := l_1 l_2(l_4 + l_3) + l_2(l_4 + l_3) + l_1(l_4 + l_3)$ is the Lipschitz constant of the function F. □

Remark 3.2. *From above, we have proven that the function F is locally Lipschitz, and we can see that it is very difficult to get a sharp estimate of the Lipschitz constant l of F. But we are convinced that appropriate choices of $\tilde{\rho}_1$ and δ (small enough) could make this constant strictly less than* 1*, so that F is contractive.*

In the sequel, we illustrate how the combined direct and dumped Newton method can be used most effectively for solving the optimality system (S_δ). The main idea is to linearize equations given by (8), (3) and (12), for the numerical solution of the set equation (8), (3) and (12). We use the iterative relaxed Newton's method (see [13]) on each mapping F_1,F_3 and F_4, and prove the convergence of the proposed algorithm.

Theorem 3.2. *Since $(\bar{\varphi}^\delta, \bar{\psi}^\delta)$ belongs to $\mathcal{U} \times \mathcal{U}$ is solution of the following equation*

$$(\bar{\varphi}^\delta, \bar{\psi}^\delta) - F\left(\bar{\varphi}^\delta, \bar{\psi}^\delta\right) = 0$$

Then $\left(\bar{y}^\delta, \bar{p}^\delta, \bar{\varphi}^\delta, \bar{\psi}^\delta\right)$ belonging to $\mathcal{U} \times \mathcal{U} \times \mathcal{W} \times \mathcal{W}$ satisfies the optimality system (S^δ), where, in the sequel, we put $\bar{s}^\delta := \left(\bar{y}^\delta, \bar{p}^\delta, \bar{\varphi}^\delta, \bar{\psi}^\delta\right)$.

Proof. Since $\left(\bar{\varphi}^\delta, \bar{\psi}^\delta\right)$ belonging to $\mathcal{W} \times \mathcal{W}$ satisfies equation (3.2), where $\left(\bar{\varphi}^\delta, \bar{\psi}^\delta\right)$ is given by

$$\left(\bar{\varphi}^\delta, \bar{\psi}^\delta\right) := (F_3\left(\bar{y}^\delta, \bar{p}^\delta\right), F_4\left(\bar{y}^\delta, \bar{p}^\delta\right)),$$

where $\bar{y}^\delta$ and $\bar{p}^\delta$ belong to $\mathcal{U}$ can be respectively defined by

$$\bar{y}^\delta := F_1\left(\bar{\varphi}^\delta, \bar{\psi}^\delta\right), \tag{21}$$

and

$$\bar{p}^\delta := F_3\left(\bar{y}^\delta, \bar{\varphi}^\delta, \bar{\psi}^\delta\right). \tag{22}$$

Then, by the definitions of the mappings F_1, F_2, F_3 and F_4, the relations (3), (21) and (22) are respectively written as

$$A\bar{y}^\delta + \beta_\delta\left(\bar{y}^\delta - \bar{\varphi}^\delta\right) - \beta_\delta\left(\bar{\psi}^\delta - \bar{y}^\delta\right) = f, \text{ in } \Omega, \text{ and } \bar{y}^\delta = 0 \text{ on } \partial\Omega \tag{23}$$

$$A\bar{p}^\delta + \beta'_\delta\left(\bar{y}^\delta - \bar{\varphi}^\delta\right)\bar{p}^\delta + \beta'_\delta\left(\bar{\psi}^\delta - \bar{y}^\delta\right)\bar{p}^\delta = \bar{y}^\delta - z, \text{ in } \Omega, \text{ and } \bar{p}^\delta = 0 \text{ on } \partial\Omega \tag{24}$$

$$\nu\Delta\bar{\varphi}^\delta + \beta'_\delta\left(\bar{y}^\delta - \bar{\varphi}^\delta\right)\bar{p}^\delta = -\bar{\lambda}^\delta, \text{ in } \Omega, \text{ and } \bar{\varphi}^\delta = 0 \text{ on } \partial\Omega \tag{25}$$

and

$$\nu\Delta\bar{\psi}^\delta + \beta'_\delta\left(\bar{\psi}^\delta - \bar{y}^\delta\right)\bar{p}^\delta - \bar{\lambda}^\delta = 0, \text{ in } \Omega, \text{ and } \bar{\psi}^\delta = 0 \text{ on } \partial\Omega \tag{26}$$

Hence, we remark that the set of equations (23), (24), (25) and (26) is the same set of the equations of the optimality system (S_δ) when $\left(y^\delta, \varphi^\delta, \psi^\delta, p^\delta\right)$ is replaced by $\left(\bar{y}^\delta, \bar{\varphi}^\delta, \bar{\psi}^\delta, \bar{p}^\delta\right)$. □

The equations (8), (11) and (12) of the optimality system (S^δ) are respectively nonlinear according to y^δ, φ^δ and ψ^δ. Therefore for the solution of the system $(\mathcal{S}^\delta)$, we propose the following iterative algorithm.

Algorithm 2 Newton dumped-Gauss-Seidel algorithm (Continuous version)

Input : $\left\{y_0^\delta, p_0^\delta, \varphi_0^\delta, \lambda_0^\delta, \psi_0^\delta, \delta, \nu, \omega_y, \omega_\varphi, \omega_\psi, \varepsilon\right\}$ choose $\varphi_0^\delta, \psi_0^\delta \in \mathcal{W}, \varepsilon$ and δ in $\mathbb{R}_+^*$;
Begin:
Calculate $J_{n-1} \leftarrow J_{n-1}\left(y_{n-1}^\delta, \varphi_{n-1}^\delta, \psi_{n-1}^\delta\right)$
Step 1
If $\left(A + \beta_\delta'\left(y_{n-1}^\delta - \varphi_{n-1}^\delta\right) + \beta_\delta'\left(\psi_{n-1}^\delta - y_{n-1}^\delta\right)\right)$ is singular **Stop.**
Else
Solve $\left(A + \beta_\delta'\left(y_{n-1}^\delta - \varphi_{n-1}^\delta\right) + \beta_\delta'\left(\psi_{n-1}^\delta - y_{n-1}^\delta\right)\right).r_n^\delta = -\omega_y\left(Ay_{n-1}^\delta + \beta_\delta\left(y_{n-1}^\delta - \varphi_{n-1}^\delta\right) - \beta_\delta\left(\psi_{n-1}^\delta - y_{n-1}^\delta\right) - f\right)$ on r_n^δ,
Then $y_n^\delta = y_{n-1}^\delta + r_n^\delta$.
End if
Step 2
If $\left(A + \beta_\delta'\left(y_n^\delta - \varphi_{n-1}^\delta\right) + \beta_\delta'\left(\psi_{n-1}^\delta - y_n^\delta\right)\right)$ is singular **Stop.**
Else
Solve $\left(A + \beta_\delta'\left(y_n^\delta - \varphi_{n-1}^\delta\right) + \beta_\delta'\left(\psi_{n-1}^\delta - y_n^\delta\right)\right)p_n^\delta = y_n^\delta - z$ on p_n^δ.
End if
Step 3
Calculate $\lambda_n^\delta = \nu\Delta\varphi_{n-1}^\delta + \beta_\delta'\left(y_n^\delta - \varphi_{n-1}^\delta\right)p_n^\delta$
Step 4
If $\left(\nu\Delta + \beta_\delta''\left(\psi_{n-1}^\delta - y_n^\delta\right)p_n^\delta\right)$ is not invertible **Stop.**
Else
Solve $\left(\nu\Delta + \beta_\delta''\left(\psi_{n-1}^\delta - y_n^\delta\right)p_n^\delta\right).r_n^\delta = -\omega_\psi\left(\nu A_h^d\psi_{n-1}^\delta + \beta_\delta'\left(\psi_{n-1}^\delta - y_n^\delta\right)p_n^\delta + \lambda_n^\delta\right)$ on r_n^δ.
Then $\psi_n^\delta = \psi_{n-1}^\delta + r_n^\delta$
Step 5
If $\left(\nu\Delta - \beta_\delta''\left(y_n^\delta - \varphi_{n-1}^\delta\right)p_n^\delta\right)$ is not invertible **Stop.**
Else
Solve $\left(\nu\Delta - \beta_\delta''\left(y_n^\delta - \varphi_{n-1}^\delta\right)p_n^\delta\right).r_n^\delta = -\omega_\varphi\left(\nu A_h^d\varphi_{n-1}^\delta + \beta_\delta'\left(y_n^\delta - \varphi_{n-1}^\delta\right)p_n^\delta - \lambda_n^\delta\right)$ on r_n^δ.
Then $\varphi_n^\delta = \varphi_{n-1}^\delta + r_n^\delta$
Calculate $J_n \leftarrow J_{n-1}\left(y_n^\delta, \varphi_n^\delta, \psi_n^\delta\right)$
End if
If $|J_n - J_{n-1}| \leq \varepsilon$ **Stop.**
Ensure : $s_n^\delta := \left(y_n^\delta, \varphi_n^\delta, \psi_n^\delta, p_n^\delta\right)$ **is a solution**
Else; $n \leftarrow n+1$, **Go to Begin.**
End if
End algorithm.

3.1 Convergence results

In this subsection, we give some conditions on δ and ω to have the convergence of the above algorithm. We denote by $\bar{y}^\delta$, $\bar{p}^\delta$, $\bar{\varphi}^\delta$ and $\bar{\psi}^\delta$ the solutions of the equations (23), (24), (25) and (26) respectively, and let y_n^δ, λ_n^δ, p_n^δ, ψ_n^δ and φ_n^δ be given respectively by step 1, step 2, step 3, step 4, step 5 respectively of the latter algorithm.

Remark 3.3. *From Lemma 3.2, if we replace $y_n^{\delta,2}, \varphi_{n-1}^{\delta,2}, \psi_{n-1}^{\delta,2}$ and $p_n^{\delta,2}$ respectively by $\bar{y}^\delta, \bar{\varphi}^\delta, \bar{\psi}^\delta$ and $\bar{p}^\delta$, we get*

$$\| p_n^{\delta,1} - \bar{p}^\delta \|_{H^1(\Omega)} \leq l_2 \left(\| y_n^{\delta,1} - \bar{y}^\delta \|_{L^2(\Omega)} + \| \varphi_{n-1}^{\delta,1} - \bar{\varphi}^\delta \|_{L^2(\Omega)} + \| \psi_{n-1}^{\delta,1} - \bar{\psi}^\delta \|_{L^2(\Omega)} \right),$$

where $l_2 = C + \frac{C\tilde{\rho}_3}{\delta}$.

Lemma 3.5. *Let $\bar{\lambda}^\delta$ in $\mathcal{U}$ be the solution of the following equation*

$$\bar{\lambda}^\delta = \nu \Delta \bar{\varphi}^\delta + \beta'_\delta \left(\bar{y}^\delta - \bar{\varphi}^\delta \right) \bar{p}^\delta,$$

since

$$\| \bar{p}^\delta \|_{H^1(\Omega)} \leq \tilde{\rho}_3$$

we obtain

$$\| \lambda_n^\delta - \bar{\lambda}^\delta \|_{L^2(\Omega)} \leq k_\lambda \left(\| y_n^\delta - \bar{y}^\delta \|_{L^2(\Omega)} + \| \varphi_{n-1}^\delta - \bar{\varphi}^\delta \|_{L^2(\Omega)} + \| p_n^\delta - \bar{p}^\delta \|_{L^2(\Omega)} \right)$$

where $k_\lambda := \dfrac{C}{\delta}$ and $\tilde{\rho}_3 \leq C\delta\nu$.

Proof. From step 3 of the continuous version of the algorithm 2, and by Lemma 2.1, we get

$$\| \lambda_n^\delta - \bar{\lambda}^\delta \|_{L^2(\Omega)} \leq (\nu + \frac{C\tilde{\rho}_3}{\delta}) \| \varphi_{n-1}^\delta - \bar{\varphi}^\delta \|_{H^2(\Omega)} + \frac{C}{\delta} \| p_n^\delta - \bar{p}^\delta \|_{H^1(\Omega)} + \\ + \| y_n^\delta - \bar{y}^\delta \|_{L^2(\Omega)},$$

then, we get

$$\| \lambda_n^\delta - \bar{\lambda}^\delta \|_{L^2(\Omega)} \leq k_\lambda (\| \varphi_{n-1}^\delta - \bar{\varphi}^\delta \|_{H^2(\Omega)} + \| p_n^\delta - \bar{p}^\delta \|_{H^1(\Omega)} + \\ \frac{C\tilde{\rho}_3}{\delta} \| y_n^\delta - \bar{y}^\delta \|_{L^2(\Omega)}),$$

where

$$k_\lambda := max\{(\nu + \frac{C\tilde{\rho}_3}{\delta}), \frac{C}{\delta}, \frac{C\tilde{\rho}_3}{\delta}\} = \frac{C}{\delta}.$$

□

Lemma 3.6. *Let $\bar{\varphi}^{\delta}$ in $\mathcal{U}$ be the solution of* (25)*, since*

$$\| \bar{p}^{\delta} \|_{H^1(\Omega)} \leq \tilde{\rho}_3,$$

where ω_{φ} is strictly positive, such that

$$\frac{\delta\nu C + C\tilde{\rho}_3}{(C + \delta\nu C - C\tilde{\rho}_3)} \leq \omega_{\varphi} \leq 1,$$

and

$$\omega_{\varphi} < \frac{\delta^2\nu C - C\delta\tilde{\rho}_3}{C\delta + C\tilde{\rho}_3},$$

we obtain

$$\| \varphi_n^{\delta} - \bar{\varphi}^{\delta} \|_{H^2(\Omega)} \leq k_3(\| y_n^{\delta} - \bar{y}^{\delta} \|_{H^1(\Omega)} + \| \varphi_{n-1}^{\delta} - \bar{\varphi}^{\delta} \|_{H^2(\Omega)} + \\ + \| p_n^{\delta} - \bar{p}^{\delta} \|_{H^1(\Omega)} + \| \lambda_n^{\delta} - \bar{\lambda}^{\delta} \|_{H^1(\Omega)}), \quad (27)$$

where $k_3 := \omega_{\varphi} \dfrac{C}{\delta\nu C - C\tilde{\rho}_3}$ and $\tilde{\rho}_3 \leq C\delta\nu$.

Proof. From step 5 of the continuous version of the algorithm 2, we obtain

$$\begin{aligned}
\nu\sigma\left(\varphi_n^{\delta} - \bar{\varphi}^{\delta}, \varphi_n^{\delta} - \bar{\varphi}^{\delta}\right) = &-\left(\beta_{\delta}''\left(y_n^{\delta} - \varphi_{n-1}^{\delta}\right) p_n^{\delta}\left(\varphi_n^{\delta} - \bar{\varphi}^{\delta}\right), \left(\varphi_n^{\delta} - \bar{\varphi}^{\delta}\right)\right) \\
&+ \nu\left(1 - \omega_{\varphi}\right)\sigma\left(\varphi_{n-1}^{\delta} - \bar{\varphi}^{\delta}, \varphi_n^{\delta} - \bar{\varphi}^{\delta}\right) + \\
&+ \left(\beta_{\delta}''\left(y_n^{\delta} - \varphi_{n-1}^{\delta}\right) p_n^{\delta}\left(\varphi_{n-1}^{\delta} - \bar{\varphi}^{\delta}\right), \left(\varphi_n^{\delta} - \bar{\varphi}^{\delta}\right)\right) + \\
+ \omega_{\varphi}\left(\left(\beta_{\delta}'\left(y_n^{\delta} - \varphi_{n-1}^{\delta}\right) - \beta_{\delta}'\left(\bar{y}^{\delta} - \bar{\varphi}^{\delta}\right)\right) p_n^{\delta} + \beta_{\delta}'\left(\bar{y}^{\delta} - \bar{\varphi}^{\delta}\right)\left(p_n^{\delta} - \bar{p}^{\delta}\right), \left(\varphi_n^{\delta} - \bar{\varphi}^{\delta}\right)\right) + \\
&+ \omega_{\varphi}\left(\lambda_n^{\delta} - \bar{\lambda}^{\delta}, \left(\varphi_n^{\delta} - \bar{\varphi}^{\delta}\right)\right).
\end{aligned}$$

By the continuity and coercivity conditions H_1 and H_2 of $\sigma(\cdot,\cdot)$, and the Mean-Value Theorem applied in the interval of sides $\{(y_n^{\delta} - \varphi_{n-1}^{\delta}), (\bar{y}^{\delta} - \bar{\varphi}^{\delta})\}$, where $r^{\delta}(\theta) = \theta\left(y_n^{\delta} - \varphi_{n-1}^{\delta}\right) + (1-\theta)\left(\bar{y}^{\delta} - \bar{\varphi}^{\delta}\right)$ such that $0 \leq \theta \leq 1$, we obtain

$$\begin{aligned}
\left(\tfrac{\delta\nu C - C\tilde{\rho}_3}{\delta}\right) \| \varphi_n^{\delta} - \bar{\varphi}^{\delta} \|_{H^2(\Omega)} \leq &\left(\tfrac{(1-\omega_{\varphi})\delta\nu C + (1+\omega_{\varphi})C\tilde{\rho}_3}{\delta}\right) \| \varphi_{n-1}^{\delta} - \bar{\varphi}^{\delta} \|_{H^2(\Omega)} + \\
&+ \omega_{\varphi}\tfrac{C\tilde{\rho}_3}{\delta} \| y_n^{\delta} - \bar{y}^{\delta} \|_{H^1(\Omega)} + \omega_{\varphi}\tfrac{C}{\delta} \| p_n^{\delta} + \bar{p}^{\delta} \|_{H^1(\Omega)} + \\
&+ \omega_{\varphi} C \| \lambda_n^{\delta} - \bar{\lambda}^{\delta} \|_{H^1(\Omega)}.
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
\| \varphi_n^{\delta} - \bar{\varphi}^{\delta} \|_{H^2(\Omega)} \leq &\left(\frac{(1-\omega_{\varphi})\delta\nu C + (1+\omega_{\varphi})C\tilde{\rho}_3}{\delta\nu C - C\tilde{\rho}_3}\right) \| \varphi_{n-1}^{\delta} - \bar{\varphi}^{\delta} \|_{H^2(\Omega)} + \\
&+ \omega_{\varphi}\frac{C\tilde{\rho}_3}{\delta\nu C - C\tilde{\rho}_3} \| y_n^{\delta} - \bar{y}^{\delta} \|_{H^1(\Omega)} + \omega_{\varphi}\frac{C}{\delta\nu C - C\tilde{\rho}_3} \| p_n^{\delta} - \bar{p}^{\delta} \|_{H^1(\Omega)} + \\
&+ \left(\tfrac{\delta\omega_{\varphi} C}{\delta\nu C - C\tilde{\rho}_3}\right) \| \lambda_n^{\delta} - \bar{\lambda}^{\delta} \|_{H^1(\Omega)}.
\end{aligned}$$

□

Lemma 3.7. *Let $\bar{\psi}^{\delta}$ in $\mathcal{U}$ be the solution of (26), since*

$$\| \bar{p}^{\delta} \|_{H^1(\Omega)} \leq \tilde{\rho}_3$$

where ω_{ψ} is strictly positive, such that

$$\frac{\delta\nu C + C\tilde{\rho}_3}{(C + \delta\nu C - C\tilde{\rho}_3)} \leq \omega_{\psi} \leq 1$$

and

$$\omega_{\psi} < \frac{\delta^2\nu C - C\delta\tilde{\rho}_3}{C\delta + C\tilde{\rho}_3}.$$

Then, we obtain

$$\| \psi_n^{\delta} - \bar{\psi}^{\delta} \|_{H^2(\Omega)} \leq k_2(\| y_n^{\delta} - \bar{y}^{\delta} \|_{H^1(\Omega)} + \| \psi_{n-1}^{\delta} - \bar{\psi}^{\delta} \|_{H^2(\Omega)} + \| p_n^{\delta} - \bar{p}^{\delta} \|_{H^1(\Omega)} + \\ + \| \lambda_n^{\delta} - \bar{\lambda}^{\delta} \|_{H^1(\Omega)})$$

where $\tilde{\rho}_3 \leq C\delta\nu$ and $k_2 := \omega_{\psi}\dfrac{C}{\delta\nu C - C\tilde{\rho}_3}$.

Proof. From step 4 of the continuous version of the algorithm 2, we obtain

$$\begin{aligned}
\nu\sigma\left(\psi_n^{\delta} - \bar{\psi}^{\delta}, \psi_n^{\delta} - \bar{\psi}^{\delta}\right) &= \left(\beta_{\delta}''\left(\psi_{n-1}^{\delta} - y_n^{\delta}\right) p_n^{\delta}\left(\psi_n^{\delta} - \bar{\psi}^{\delta}\right), \left(\psi_n^{\delta} - \bar{\psi}^{\delta}\right)\right) + \\
&+ \nu\left(1 - \omega_{\psi}\right)\sigma\left(\psi_{n-1}^{\delta} - \bar{\psi}^{\delta}, \psi_n^{\delta} - \bar{\psi}^{\delta}\right) + \\
&- \left(\beta_{\delta}''\left(\psi_{n-1}^{\delta} - y_n^{\delta}\right) p_n^{\delta}\left(\psi_{n-1}^{\delta} - \bar{\psi}^{\delta}\right), \left(\psi_n^{\delta} - \bar{\psi}^{\delta}\right)\right) + \\
+ \omega_{\psi}\left(\left(\beta_{\delta}'\left(\psi_{n-1}^{\delta} - y_n^{\delta}\right) - \beta_{\delta}'\left(\bar{\psi}^{\delta} - \bar{y}^{\delta}\right)\right) p_n^{\delta} + \beta_{\delta}'\left(\bar{\psi}^{\delta} - \bar{y}^{\delta}\right)\left(p_n^{\delta} - \bar{p}^{\delta}\right), \left(\psi_n^{\delta} - \bar{\psi}^{\delta}\right)\right) + \\
&+ \omega_{\psi}\left(\lambda_n^{\delta} - \bar{\lambda}^{\delta}, \left(\psi_n^{\delta} - \bar{\psi}^{\delta}\right)\right).
\end{aligned}$$

By the continuity and coercivity conditions H_1 and H_2 of $\sigma(\cdot,\cdot)$, and the Mean-Value Theorem applied in the interval of sides $\{(\psi_{n-1}^{\delta} - y_n^{\delta}),\ (\bar{\psi}^{\delta} - \bar{y}^{\delta})\}$, where $r^{\delta}(\theta) = \theta\left(\psi_{n-1}^{\delta} - y_n^{\delta}\right) + (1-\theta)\left(\bar{\psi}^{\delta} - \bar{y}^{\delta}\right)$ such that $0 \leq \theta \leq 1$, we get

$$\begin{aligned}
\left(\tfrac{\delta\nu C - C\tilde{\rho}_3}{\delta}\right) \| \psi_n^{\delta} - \bar{\psi}^{\delta} \|_{H^2(\Omega)} &\leq \left(\tfrac{\left(1-\omega_{\psi}\right)\delta\nu C + \left(1+\omega_{\psi}\right)C\tilde{\rho}_3}{\delta}\right) \| \psi_{n-1}^{\delta} - \bar{\psi}^{\delta} \|_{H^2(\Omega)} + \\
&+ \omega_{\psi}\tfrac{C\tilde{\rho}_3}{\delta} \| y_n^{\delta} - \bar{y}^{\delta} \|_{H^1(\Omega)} + \omega_{\psi}\tfrac{C}{\delta} \| p_n^{\delta} - \bar{p}^{\delta} \|_{H^1(\Omega)} + \\
&+ \omega_{\psi} C \| \lambda_n^{\delta} - \bar{\lambda}^{\delta} \|_{H^1(\Omega)}.
\end{aligned}$$

Finally, we obtain

$$\| \psi_n^\delta - \bar{\psi}^\delta \|_{H^2(\Omega)} \leq \left(\frac{(1-\omega_\psi)\,\delta\nu C + (1+\omega_\psi)\,C\tilde{\rho}_3}{\delta\nu C - C\tilde{\rho}_3} \right) \| \psi_{n-1}^\delta - \bar{\psi}^\delta \|_{H^2(\Omega)} +$$
$$+ \omega_\psi \frac{C\tilde{\rho}_3}{\delta\nu C - C\tilde{\rho}_3} \| y_n^\delta - \bar{y}^\delta \|_{H^1(\Omega)} + \omega_\psi \frac{C}{\delta\nu C - C\tilde{\rho}_3} \| p_n^\delta - \bar{p}^\delta \|_{H^1(\Omega)} +$$
$$+ \left(\frac{\omega_\psi C\delta}{\delta\nu C - C\tilde{\rho}_3} \right) \| \lambda_n^\delta - \bar{\lambda}^\delta \|_{H^1(\Omega)} .$$

□

Lemma 3.8. *Let y_n^δ in $\mathcal{U}$ be the solution of (23), since the condition (3.7) of previous Lemma 3.7 is fulfilled, where*

$$\left(\frac{\delta C + C - \delta}{\delta C + C} \right) < \omega_y \leq \frac{(\delta C + C)}{(\delta C + 2C)} \leq 1,$$

we get

$$\| y_n^\delta - \bar{y}^\delta \|_{H^1(\Omega)} \leq k_1 \left\{ \| e_{n-1}^\delta - \bar{e}^\delta \|_{\mathcal{V}}^2 + \| e_{n-1}^\delta - \bar{e}^\delta \|_{\mathcal{V}} \right\}$$

where $k_1 := (1-\omega_y)\left(C + \dfrac{C}{\delta} \right)$, $e_{n-1}^\delta := \left(y_{n-1}^\delta, \varphi_{n-1}^\delta, \psi_{n-1}^\delta \right)$, $\bar{e}^\delta := \left(\bar{y}^\delta, \bar{\varphi}^\delta, \bar{\psi}^\delta \right)$ *and* $\mathcal{V} := H^1(\Omega) \times H^2(\Omega) \times H^2(\Omega)$.

Proof. From step 1 of the algorithm 2, and since $-\left((\beta_\delta' \left(y_{n-1}^\delta - \varphi_{n-1}^\delta \right) + \beta_\delta' \left(\psi_{n-1}^\delta - y_{n-1}^\delta \right)) \left(y_n^\delta - \bar{y}^\delta \right), \left(y_n^\delta - \bar{y}^\delta \right) \right) \leq 0$, and by the Mean-Value Theorem applied in the interval of sides $\{(y_{n-1}^\delta - \varphi_{n-1}^\delta),\ (\bar{y}^\delta - \bar{\varphi}^\delta)\}$ and $\{(\psi_{n-1}^\delta - y_{n-1}^\delta),\ (\bar{\psi}^\delta - \bar{y}^\delta)\}$, we get

$$\begin{aligned}
\sigma \left(y_n^\delta - \bar{y}^\delta, y_n^\delta - \bar{y}^\delta \right) &\leq (1-\omega_y)\,\sigma \left(y_{n-1}^\delta - \bar{y}^\delta, y_n^\delta - \bar{y}^\delta \right) \\
&+ \left((\beta_\delta' \left(y_{n-1}^\delta - \varphi_{n-1}^\delta \right) + \beta_\delta' \left(\psi_{n-1}^\delta - y_{n-1}^\delta \right)) \left(y_{n-1}^\delta - \bar{y}^\delta \right), \left(y_n^\delta - \bar{y}^\delta \right) \right) \\
&- \omega_y \left(\left(\beta_\delta' \left(r_1^\delta(\theta) \right) - \beta_\delta' \left(y_{n-1}^\delta - \varphi_{n-1}^\delta \right) + \beta_\delta' \left(y_{n-1}^\delta - \varphi_{n-1}^\delta \right) \right) \left(y_{n-1}^\delta - \bar{y}^\delta \right), \left(y_n^\delta - \bar{y}^\delta \right) \right) \\
&+ \omega_y \left(\beta_\delta' \left(r_1^\delta(\theta) \right) \left(\varphi_{n-1}^\delta - \bar{\varphi}^\delta \right), \left(y_n^\delta - \bar{y}^\delta \right) \right) \\
&- \omega_y \left(\left(\beta_\delta' \left(r_2^\delta(\theta) \right) - \beta_\delta' \left(\psi_{n-1}^\delta - y_{n-1}^\delta \right) + \beta_\delta' \left(\psi_{n-1}^\delta - y_{n-1}^\delta \right) \right) \left(y_{n-1}^\delta - \bar{y}^\delta \right), \left(y_n^\delta - \bar{y}^\delta \right) \right) \\
&+ \omega_y \left(\beta_\delta' \left(r_2^\delta(\theta) \right) \left(\psi_{n-1}^\delta - \bar{\psi}^\delta \right), \left(y_n^\delta - \bar{y}^\delta \right) \right),
\end{aligned}$$

where

$$r_1^\delta(\theta) = \theta \left(y_{n-1}^\delta - \varphi_{n-1}^\delta \right) + (1-\theta) \left(\bar{y}^\delta - \bar{\varphi}^\delta \right),$$

and

$$r_2^\delta(\theta) = \theta \left(\psi_{n-1}^\delta - y_{n-1}^\delta \right) + (1-\theta) \left(\bar{\psi}^\delta - \bar{y}^\delta \right) \text{ such that } 0 \leq \theta \leq 1.$$

Again by the Mean-Value Theorem applied respectively in the interval of sides $\{r_1^\delta(\theta), (y_{n-1}^\delta - \varphi_{n-1}^\delta)\}$ and $\{r_2^\delta(\theta), (\psi_{n-1}^\delta - y_{n-1}^\delta)\}$, we obtain

$$\begin{aligned}
&\sigma\left(y_n^\delta - \bar{y}^\delta, y_n^\delta - \bar{y}^\delta\right) \leq (1-\omega_y)\,\sigma\left(y_{n-1}^\delta - \bar{y}^\delta, y_n^\delta - \bar{y}^\delta\right) + \\
&+ (1-\omega_y)\left(\beta_\delta'\left(y_{n-1}^\delta - \varphi_{n-1}^\delta\right)\left(y_{n-1}^\delta - \bar{y}^\delta\right), \left(y_n^\delta - \bar{y}^\delta\right)\right) + \\
&+ (1-\omega_y)\left(\beta_\delta'\left(\psi_{n-1}^\delta - y_{n-1}^\delta\right)\left(y_{n-1}^\delta - \bar{y}^\delta\right), \left(y_n^\delta - \bar{y}^\delta\right)\right) + \\
&+ \omega_y\left(\left[\beta_\delta''\left(s_1^\delta(\theta)\right)\left[(1-\theta)\left(y_{n-1}^\delta - \varphi_{n-1}^\delta\right) - (1-\theta)\left(\bar{y}^\delta - \bar{\varphi}^\delta\right)\right]\right]\left(y_{n-1}^\delta - \bar{y}^\delta\right), \left(y_n^\delta - \bar{y}^\delta\right)\right) + \\
&+ \omega_y\left(\beta_\delta'\left(r_1^\delta(\theta)\right)\left(\varphi_{n-1}^\delta - \bar{\varphi}^\delta\right), \left(y_n^\delta - \bar{y}^\delta\right)\right) + \\
&+ \omega_y\left(\left[\beta_\delta''\left(s_2^\delta(\theta)\right)\left[(1-\theta)\left(\psi_{n-1}^\delta - y_{n-1}^\delta\right) - (1-\theta)\left(\bar{\psi}^\delta - \bar{y}^\delta\right)\right]\right]\left(y_{n-1}^\delta - \bar{y}^\delta\right), \left(y_n^\delta - \bar{y}^\delta\right)\right) + \\
&+ \omega_y\left(\beta_\delta'\left(r_2^\delta(\theta)\right)\left(\psi_{n-1}^\delta - \bar{\psi}^\delta\right), \left(y_n^\delta - \bar{y}^\delta\right)\right).
\end{aligned}$$

where, $s_1^\delta(\theta) := \theta\left(y_{n-1}^\delta - \varphi_{n-1}^\delta\right) + (1-\theta)\, r_1^\delta(\theta)$ and $s_2^\delta(\theta) := \theta\left(\psi_{n-1}^\delta - y_{n-1}^\delta\right) + (1-\theta)\, r_2^\delta(\theta)$. By the coercivity and continuity conditions H_1 and H_2 of $\sigma(\cdot,\cdot)$, we obtain

$$\begin{aligned}
C \parallel y_n^\delta - \bar{y}^\delta \parallel_{H^1(\Omega)}^2 \leq\ & (1-\omega_y)\, C \parallel y_{n-1}^\delta - \bar{y}^\delta \parallel_{H^1(\Omega)} \parallel y_n^\delta - \bar{y}^\delta \parallel_{H^1(\Omega)} + \\
&+ \frac{C(1-\omega_y)}{\delta} \parallel y_{n-1}^\delta - \bar{y}^\delta \parallel_{H^1(\Omega)} \parallel y_n^\delta - \bar{y}^\delta \parallel_{H^1(\Omega)} + \\
&+ \frac{C\omega_y(1-\theta)}{\delta} \parallel \left(y_{n-1}^\delta - \bar{y}^\delta\right) - \left(\varphi_{n-1}^\delta - \bar{\varphi}^\delta\right) \parallel_{H^1(\Omega)} \parallel y_{n-1}^\delta - \bar{y}^\delta \parallel_{H^1(\Omega)} \parallel y_n^\delta - \bar{y}^\delta \parallel_{H^1(\Omega)} + \\
&+ \frac{C\omega_y}{\delta} \parallel \varphi_{n-1}^\delta - \bar{\varphi}^\delta \parallel_{H^2(\Omega)} \parallel y_n^\delta - \bar{y}^\delta \parallel_{H^1(\Omega)} + \\
&+ \frac{C\omega_y(1-\theta)}{\delta} \parallel \left(y_{n-1}^\delta - \bar{y}^\delta\right) - \left(\psi_{n-1}^\delta - \bar{\psi}^\delta\right) \parallel_{H^1(\Omega)} \parallel y_{n-1}^\delta - \bar{y}^\delta \parallel_{H^1(\Omega)} \parallel y_n^\delta - \bar{y}^\delta \parallel_{H^1(\Omega)} + \\
&+ \frac{C\omega_y}{\delta} \parallel \psi_{n-1}^\delta - \bar{\psi}^\delta \parallel_{H^2(\Omega)} \parallel y_n^\delta - \bar{y}^\delta \parallel_{H^1(\Omega)}.
\end{aligned}$$

Finally, we have

$$\begin{aligned}
\parallel y_n^\delta - \bar{y}^\delta \parallel_{H^1(\Omega)} \leq k_1 \Big\{ & \parallel y_{n-1}^\delta - \bar{y}^\delta \parallel_{H^1(\Omega)}^2 + \parallel \varphi_{n-1}^\delta - \bar{\varphi}^\delta \parallel_{H^2(\Omega)}^2 + \parallel \psi_{n-1}^\delta - \bar{\psi}^\delta \parallel_{H^2(\Omega)}^2 \\
& + \parallel y_{n-1}^\delta - \bar{y}^\delta \parallel_{H^1(\Omega)} + \parallel \varphi_{n-1}^\delta - \bar{\varphi}^\delta \parallel_{H^2(\Omega)} + \parallel \psi_{n-1}^\delta - \bar{\psi}^\delta \parallel_{H^2(\Omega)} \Big\},
\end{aligned}$$

where

$$k_1 := \max\left\{(1-\omega_y)\left(C + \frac{1}{\delta}C\right), \omega_y \frac{1}{\delta} C(1-\theta), \omega_y \frac{1}{\delta} C\right\} = (1-\omega_y)\left(C + \frac{1}{\delta}C\right).$$

□

Theorem 3.3. *Let* $e_n^\delta := \left(y_n^\delta, \varphi_n^\delta, \psi_n^\delta\right)$, $\bar{e}^\delta := \left(\bar{y}^\delta, \bar{\varphi}^\delta, \bar{\psi}^\delta\right)$ *and* $\mathcal{V} := H^1(\Omega) \times H^2(\Omega) \times H^2(\Omega)$, *then we get*

$$\parallel e_n^\delta - \bar{e}^\delta \parallel_{\mathcal{V}} \leq k \max\left\{ \parallel e_{n-1}^\delta - \bar{e}^\delta \parallel_{\mathcal{V}}^2, \parallel e_{n-1}^\delta - \bar{e}^\delta \parallel_{\mathcal{V}} \right\},$$

where

$$k := 2\left(k_1 + \tilde{k}_3\right),\ \tilde{k}_3 := k_3\left(k_1 + \tilde{l}_2 + 1\right)\ \textit{and}\ \tilde{l}_2 = l_2\left(Ck_1 + C\right).$$

Proof. From equations (3.8) and (3.3), we get

$$\| y_n^\delta - \bar{y}^\delta \|_\nu \leq k_1 \left\{ \| e_{n-1}^\delta - \bar{e}^\delta \|_\nu^2 + \| e_{n-1}^\delta - \bar{e}^\delta \|_\nu \right\},$$

and

$$\| p_n^\delta - \bar{p}^\delta \|_{H^1(\Omega)} \leq l_2(C \| y_n^\delta - \bar{y}^\delta \|_{H^1(\Omega)} + C \| \varphi_{n-1}^\delta - \bar{\varphi}^\delta \|_{H^2(\Omega)} + C \| \psi_{n-1}^\delta - \bar{\psi}^\delta \|_{H^2(\Omega)}),$$

then we obtain

$$\| p_n^\delta - \bar{p}^\delta \|_{H^1(\Omega)} \leq \tilde{l}_2 \left\{ \| e_{n-1}^\delta - \bar{e}^\delta \|_\nu^2 + \| e_{n-1}^\delta - \bar{e}^\delta \|_\nu \right\},$$

where

$$\tilde{l}_2 := l_2\left(Ck_1 + C\right).$$

And by equation (27), we get

$$\| \lambda_n^\delta - \bar{\lambda}^\delta \|_{L^2(\Omega)} \leq k_\lambda(\| \varphi_{n-1}^\delta - \bar{\varphi}^\delta \|_{H^2(\Omega)} + \tilde{l}_2 \left\{ \| e_{n-1}^\delta - \bar{e}^\delta \|_\nu^2 + \| e_{n-1}^\delta - \bar{e}^\delta \|_\nu \right\} + \frac{C\tilde{\rho}_3}{\delta} k_1 \left\{ \| e_{n-1}^\delta - \bar{e}^\delta \|_\nu^2 + \| e_{n-1}^\delta - \bar{e}^\delta \|_\nu \right\}) \quad (28)$$

then, we obtain

$$\| \varphi_n^\delta - \bar{\varphi}^\delta \|_{H^2(\Omega)} \leq \tilde{k}_3 \left\{ \| e_{n-1}^\delta - \bar{e}^\delta \|_\nu^2 + \| e_{n-1}^\delta - \bar{e}^\delta \|_\nu \right\}$$

where

$$\tilde{k}_3 := k_3\left(k_1 + \tilde{l}_2 + 1 + \tilde{k}_\lambda\right),$$

and by equation (27), we get

$$\| \psi_n^\delta - \bar{\psi}^\delta \|_{H^2(\Omega)} \leq k_2(\| y_n^\delta - \bar{y}^\delta \|_{H^1(\Omega)} + \| \psi_{n-1}^\delta - \bar{\psi}^\delta \|_{H^2(\Omega)} + \| p_n^\delta - \bar{p}^\delta \|_{H^1(\Omega)} + \| \lambda_n^\delta - \bar{\lambda}^\delta \|_{H^1(\Omega)}),$$

then, we obtain

$$\| \psi_n^\delta - \bar{\psi}^\delta \|_{H^2(\Omega)} \leq \tilde{k}_2 \left\{ \| e_{n-1}^\delta - \bar{e}^\delta \|_\nu^2 + \| e_{n-1}^\delta - \bar{e}^\delta \|_\nu \right\} \quad (29)$$

where

$$\tilde{k}_2 := k_2\left(k_1 + \tilde{l}_2 + 1 + \tilde{k}_\lambda\right).$$

From equations (3.8), (3.1) and (29), we get

$$\| e_n^\delta - \bar{e}^\delta \|_{\mathcal{V}} \leq 2\left(k_1 + \tilde{k}_3 + \tilde{k}_2\right) \max\left\{\| e_{n-1}^\delta - \bar{e}^\delta \|_{\mathcal{V}}^2, \| e_{n-1}^\delta - \bar{e}^\delta \|_{\mathcal{V}}\right\}.$$

Finally, we get

$$\| e_n^\delta - \bar{e}^\delta \|_{\mathcal{V}} \leq k \max\left\{\| e_{n-1}^\delta - \bar{e}^\delta \|_{\mathcal{V}}^2, \| e_{n-1}^\delta - \bar{e}^\delta \|_{\mathcal{V}}\right\} \tag{30}$$

where

$$k := 2\left(k_1 + \tilde{k}_3 + \tilde{k}_2\right).$$

□

Remark 3.4. *As seen above, it is very difficult to give a sharp estimate of the constant k and to prove that this constant is less than 1 to get the convergence of the latter algorithm. However, we believe that with suitable choices of δ and ω, we can make this constant less than 1.*

Remark 3.5. *From Theorem 3.3, we deduce that y_n^δ converges strongly to $\bar{y}^\delta$ in $H_0^1(\Omega)$ and φ_n^δ converges strongly to $\bar{\varphi}^\delta$ in $H^2(\Omega)$ and ψ_n^δ converges strongly to $\bar{\psi}^\delta$ in $H^2(\Omega)$.*

Corollary 3.4. *By the assumptions of Theorem 3.3, we deduce that*

$$\mid J\left(y_n^\delta, \varphi_n^\delta, \psi_n^\delta\right) - J\left(y_{n-1}^\delta, \varphi_{n-1}^\delta, \psi_{n-1}^\delta\right) \mid \text{ goes to } 0.$$

Proof. From the cost functional defined in $(\mathcal{P}^\delta)$, we can write

$$\begin{aligned}\mid J\left(y_n^\delta, \varphi_n^\delta, \psi_n^\delta\right) - J\left(y_{n-1}^\delta, \varphi_{n-1}^\delta, \psi_{n-1}^\delta\right) \mid = \frac{1}{2} \mid \int_\Omega \left(y_n^\delta - z\right)^2 dx + \\ \nu\left(\int_\Omega \left(\nabla\varphi_n^\delta\right)^2 + \left(\nabla\psi_n^\delta\right)^2 dx\right) - \\ \left(\int_\Omega \left(y_{n-1}^\delta - z\right)^2 dx + \nu\left(\int_\Omega \left(\nabla\varphi_{n-1}^\delta\right)^2 + \left(\nabla\psi_{n-1}^\delta\right)^2 dx\right)\right) \mid .\end{aligned}$$

From Corollary 3.1, we have $\| y_{n-1}^\delta \|_{L^2(\Omega)} \leq \tilde{\rho}_2$, $\| \varphi_{n-1}^\delta \|_{H^2(\Omega)} \leq \tilde{\rho}_1$ and $\| \psi_{n-1}^\delta \|_{H^2(\Omega)} \leq \tilde{\rho}_1$, then, we deduce that

$$\begin{aligned}&\mid J\left(y_n^\delta, \varphi_n^\delta, \psi_n^\delta\right) - J\left(y_{n-1}^\delta, \varphi_{n-1}^\delta, \psi_{n-1}^\delta\right) \mid \leq \\ &\leq \frac{1}{2}((\| y_n^\delta - y_{n-1}^\delta \|_{L^2(\Omega)})(\| y_n^\delta - y_{n-1}^\delta \|_{L^2(\Omega)} + 2 \| u_{n-1}^\delta \|_{L^2(\Omega)} + 2 \| z \|_{L^2(\Omega)}) \\ &\nu\left((\| \nabla\varphi_n^\delta - \nabla\varphi_{n-1}^\delta \|_{L^2(\Omega)})(\| \nabla\varphi_n^\delta - \nabla\varphi_{n-1}^\delta \|_{L^2(\Omega)} + 2 \| \nabla\varphi_{n-1}^\delta \|_{L^2(\Omega)})\right) \\ &\nu\left((\| \nabla\psi_n^\delta - \nabla\psi_{n-1}^\delta \|_{L^2(\Omega)})(\| \nabla\psi_n^\delta - \nabla\psi_{n-1}^\delta \|_{L^2(\Omega)} + 2 \| \nabla\psi_{n-1}^\delta \|_{L^2(\Omega)})\right))\end{aligned}$$

and

$$| J\left(y_n^\delta, \varphi_n^\delta, \psi_n^\delta\right) - J\left(y_{n-1}^\delta, \varphi_{n-1}^\delta, \psi_{n-1}^\delta\right) | \leq \frac{1}{2}(\| y_n^\delta - y_{n-1}^\delta \|_{L^2(\Omega)}^2 + (2\tilde{\rho}_2 + C) \| y_n^\delta - y_{n-1}^\delta \|_{L^2(\Omega)}$$
$$+ \nu \left(\| \nabla\varphi_n^\delta - \nabla\varphi_{n-1}^\delta \|_{L^2(\Omega)}^2 + (2\tilde{\rho}_1) \| \nabla\varphi_n^\delta - \nabla\varphi_{n-1}^\delta \|_{L^2(\Omega)}\right)$$
$$+ \nu \left(\| \nabla\psi_n^\delta - \nabla\psi_{n-1}^\delta \|_{L^2(\Omega)}^2 + (2\tilde{\rho}_1) \| \nabla\psi_n^\delta - \nabla\psi_{n-1}^\delta \|_{L^2(\Omega)}\right)).$$

Finally, we deduce that $| J\left(y_n^\delta, \varphi_n^\delta, \psi_n^\delta\right) - J\left(y_{n-1}^\delta, \varphi_{n-1}^\delta, \psi_{n-1}^\delta\right) |$ strongly converges to 0. □

4 Numerical implementation and computational aspects

Numerical experiments are carried out for one and two dimensional problems. We will attempt to compute a grid function consisting of values $y^{\delta,h} := \left(y_0^\delta, y_1^\delta, ..., y_{N+1}^\delta\right)$, $\varphi^{\delta,h} := \left(\varphi_0^\delta, \varphi_1^\delta, ..., \varphi_{N+1}^\delta\right)$, $\psi^{\delta,h} := \left(\psi_0^\delta, \psi_1^\delta, ..., \psi_{N+1}^\delta\right)$ and $p^{\delta,h} := \left(p_0^\delta, p_1^\delta, ..., p_{N+1}^\delta\right)$, where $y^{\delta,h}, \varphi^{\delta,h}, \psi^{\delta,h}$ and $p^{\delta,h}$ are the vectors values of the discrete solutions of the optimality system $(\mathcal{S}^\delta)$ such that $y_i^\delta := y^\delta(x_i), \varphi_i^\delta := \varphi^\delta(x_i), \psi_i^\delta := \psi^\delta(x_i)$ and $p_i^\delta := p^\delta(x_i)$ for $0 \leq i \leq N+1$, finite-differences approximations involving the three, respectively five, point approximation of the Laplacian in one dimensional space, respectively two dimensional space. Here $x_i = ih$ for $0 \leq i \leq N+1$ and $h := \dfrac{1}{N+1}$ is the distance between two successive grid points. From the boundary conditions $y_0^\delta = y_{N+1}^\delta = 0, p_0^\delta = p_{N+1}^\delta = 0$, $\varphi_0^\delta = \varphi_{N+1}^\delta = 0$, and $\psi_0^\delta = \psi_{N+1}^\delta = 0$, so we have $4N$ unknown values to compute in one dimensional space. Then, for example, if we replace $y^{(2)}(x)$ (respectively $\Delta y(x)$) by the centered difference approximation, we get

$$-y^{(2)}(x) := \tfrac{1}{h^2}(-y_{i+1} + 2y_i - y_{i-1}), \text{ where } 0 \leq i \leq N+1, \tag{31}$$

and respectively

$$-(\Delta y)_{ij} := \tfrac{1}{h^2}\left(-y_{i+1,j} + 4y_{i,j} - y_{i-1,j} - y_{i,j+1} - y_{i,j-1}\right), \text{ where } 0 \leq i,j \leq N+1. \tag{32}$$

Then, we can write the previous systems under the matrix form, as

$$\begin{cases} A_h^d y_n^{\delta,h} + \beta_\delta(y_n^{\delta,h} - \varphi_{n-1}^{\delta,h}) - \beta_\delta(\psi_{n-1}^{\delta,h} - y_n^{\delta,h}) = f^h, \\ (A_h^d + \beta'_\delta(y_n^{\delta,h} - \varphi_{n-1}^{\delta,h}) + \beta'_\delta(\psi_{n-1}^{\delta,h} - y_n^{\delta,h}))p_n^{\delta,h} = y_n^{\delta,h} - z^h, \\ \lambda_n^{\delta,h} = \nu A_h^d \varphi_{n-1}^{\delta,h} + \beta'_\delta\left(y_n^{\delta,h} - \varphi_{n-1}^{\delta,h}\right)p_n^{\delta,h}, \\ \nu A_h^d \psi_n^{\delta,h} + \beta'_\delta\left(\psi_n^{\delta,h} - y_n^{\delta,h}\right)p_n^{\delta,h} = -\lambda_n^{\delta,h}, \\ \nu A_h^d \varphi_n^{\delta,h} + \beta'_\delta\left(y_n^{\delta,h} - \varphi_n^{\delta,h}\right)p_n^{\delta,h} - \lambda_n^{\delta,h} = 0, \end{cases}$$

where $d = 1, 2,\ f^h := (f_0, f_1, ..., f_{N+1})$, $z^h := (z_0, z_1, ..., z_{N+1})$, and such that for one dimensional problem, A_h^1 is $(N+2) \times (N+2)$ symmetric positive definite matrix, where A_h^1 is given in (31) and for two dimensional problem A_h^2 is $(N+2)^2 \times (N+2)^2$ symmetric matrix, where A_h^2 is given in (32). Below, we give the discrete algorithm of the continuous algorithm as

Algorithm 3 Newton dumped-Gauss-Seidel algorithm (Discrete version)

Input : $\left\{ y_0^{\delta,h}, p_0^{\delta,h}, \varphi_0^{\delta,h}, \lambda_0^{\delta,h}, \psi_0^{\delta,h}, \delta, \nu, \omega_y, \omega_\varphi, \omega_\psi, \varepsilon \right\}$ choose $\varphi_0^{\delta,h}, \psi_0^{\delta,h} \in \mathcal{W}, \varepsilon$ and δ in $\mathbb{R}_+^*$;

Begin:

Calculate $J_{n-1} \leftarrow J_{n-1}\left(y_{n-1}^{\delta,h}, \varphi_{n-1}^{\delta,h}, \psi_{n-1}^{\delta,h}\right)$

If $\left(A_h^d + \mathbf{diag}(\beta_\delta'\left(y_{n-1}^{\delta,h} - \varphi_{n-1}^{\delta,h}\right)) + \mathbf{diag}(\beta_\delta'\left(\psi_{n-1}^{\delta,h} - y_{n-1}^{\delta,h}\right))\right)$ is singular **Stop.**

Else

Solve $\left(A_h^d + \mathbf{diag}(\beta_\delta'\left(y_{n-1}^{\delta,h} - \varphi_{n-1}^{\delta,h}\right)) + \mathbf{diag}(\beta_\delta'\left(\psi_{n-1}^{\delta,h} - y_{n-1}^{\delta,h}\right))\right).r_n^\delta = -\omega_y\left(A_h^d y_{n-1}^{\delta,h} + \mathbf{diag}(\beta_\delta\left(y_{n-1}^{\delta,h} - \varphi_{n-1}^{\delta,h}\right)) - \mathbf{diag}(\beta_\delta\left(\psi_{n-1}^{\delta,h} - y_{n-1}^{\delta,h}\right)) - f\right)$ on r_n^δ,

Then $y_n^{\delta,h} = y_{n-1}^{\delta,h} + r_n^\delta$.

End if

If $\left(A_h^d + \mathbf{diag}(\beta_\delta'\left(y_n^{\delta,h} - \varphi_{n-1}^{\delta,h}\right)) + \mathbf{diag}(\beta_\delta'\left(\psi_{n-1}^{\delta,h} - y_n^{\delta,h}\right))\right)$ is singular **Stop.**

Else

Solve $\left(A_h^d + \mathbf{diag}(\beta_\delta'\left(y_n^{\delta,h} - \varphi_{n-1}^{\delta,h}\right)) + \mathbf{diag}(\beta_\delta'\left(\psi_{n-1}^{\delta,h} - y_n^{\delta,h}\right))\right) p_n^{\delta,h} = y_n^{\delta,h} - z$ on $p_n^{\delta,h}$.

End if

Calculate $\lambda_n^{\delta,h} = \nu A_h^d \varphi_{n-1}^{\delta,h} + \mathbf{diag}(\beta_\delta'\left(y_n^{\delta,h} - \varphi_{n-1}^{\delta,h}\right)) p_n^{\delta,h}$

If $\left(\nu A_h^d + \mathbf{diag}(\beta_\delta''\left(\psi_{n-1}^{\delta,h} - y_n^{\delta,h}\right)) p_n^{\delta,h}\right)$ is not invertible **Stop.**

Else

Solve $\left(\nu A_h^d + \left(\mathbf{diag}\left(\beta_\delta''\left(\psi_{n-1}^{\delta,h} - y_n^{\delta,h}\right)\right)\right) p_n^{\delta,h}\right).r_n^\delta = -\omega_\psi\left(\nu A_h^d \psi_{n-1}^{\delta,h} + \mathbf{diag}(\beta_\delta'\left(\psi_{n-1}^{\delta,h} - y_n^{\delta,h}\right)) p_n^{\delta,h} + \lambda_n^{\delta,h}\right)$ on r_n^δ.

Then $\psi_n^{\delta,h} = \psi_{n-1}^{\delta,h} + r_n^\delta$

If $\left(\nu A_h^d - \mathbf{diag}(\beta_\delta''\left(y_n^{\delta,h} - \varphi_{n-1}^{\delta,h}\right)) p_n^{\delta,h}\right)$ is not invertible **Stop.**

Else

Solve $\left(\nu A_h^d - \mathbf{diag}(\beta_\delta''\left(y_n^{\delta,h} - \varphi_{n-1}^{\delta,h}\right)) p_n^{\delta,h}\right).\ r_n^\delta = -\omega_\varphi\left(\nu A_h^d \varphi_{n-1}^{\delta,h} + \mathbf{diag}(\beta_\delta'\left(y_n^{\delta,h} - \varphi_{n-1}^{\delta,h}\right)) p_n^{\delta,h} - \lambda_n^{\delta,h}\right)$ on r_n^δ.

Then $\varphi_n^{\delta,h} = \varphi_{n-1}^{\delta,h} + r_n^\delta$

Calculate $J_n \leftarrow J_{n-1}\left(y_n^{\delta,h}, \varphi_n^{\delta,h}, \psi_n^{\delta,h}\right)$

End if

If $|J_n - J_{n-1}| \leq \varepsilon$ **Stop.**

Ensure : $s_n^\delta := \left(y_n^\delta, \varphi_n^\delta, \psi_n^\delta, p_n^\delta\right)$ **is a solution**

Else; $n \leftarrow n+1$, **Go to Begin.**

End if

End algorithm.

Remark 4.1. *Theorem 3.3 is given for the continuous problem and it is clear that for the discrete form of the proposed algorithm, we must introduce the discretisation parameter h. But for this discrete form of the algorithm* 2*, it is very difficult to give a sharp estimate of the Lipschitz constant k given by Theorem 3.3.*

4.1 Numerical examples in one dimensional space

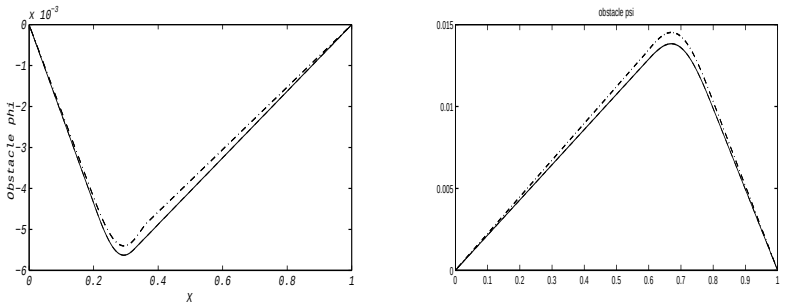
In this section, we take $\Omega = [0, 1]$ and we describe some numerical experiments in one dimensional space based on the previous algorithm. We also give some numerical tests when in each test we vary one of the parameters ω, δ, N and ν, where $f(x) = 100xcos(3\pi x)$, $z(x) = cos(4\pi x^2)$ and $\nu > 0$ are given. In the sequel, we note by ϵ_n the quantity $\max\{\| y_n - y_{n-1} \|_\infty, \| \varphi_n - \varphi_{n-1} \|_\infty\}$.

4.1.1 Test 1: Study of the dependence on the parameter ω with $\delta = h^2$, $\nu = 1$ and $N = 200$

Numerical results are displayed in Table 1 according to the variation of ω. In Figure 1, we give the curves corresponding to the controls φ and ψ. Curves given in Figure 2 show the contact region $I(y)$ between the state and the control functions. Finally, Figure 3 gives graphical variations in a log-log scale of ϵ_n and J_n for each iteration n.

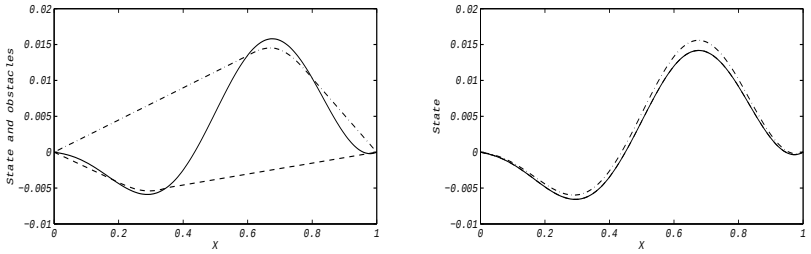
ω	$\sharp$ Iteration	J	$\mid J_n - J_{n-1} \mid$	$\|y - z\|_\infty$	ϵ_n
0.25	143	49.039593172814833	9.663381e-013	0.998935	6.954742e-004
0.5	66	49.039593172804153	9.947598e-013	0.998935	2.859977e-004
0.75	31	49.039593172799911	9.094947e-013	0.998935	2.096388e-005
1	6	49.039593172798838	7.815970e-014	0.998935	6.224867e-004

Table 1: Numerical results for one dimensional space while varying ω



(A) Obstacle φ (B) Obstacle ψ

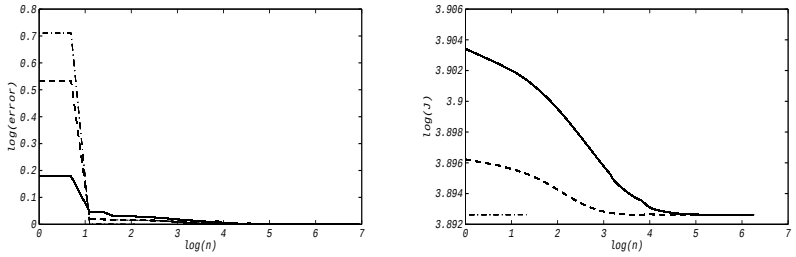
Figure 1: Left (obstacle φ), right (obstacle ψ) continuous line; $\omega = 0.25$, dash line; $\omega = 0.75$, dash-dot line; $\omega = 1$



(A) State and obstacles ($\omega = 0,25$)

(B) State

Figure 2: Left(state and obstacles ($\omega = 0,25$)), right(state) continuous line; $\omega = 0.25$, dash line; $\omega = 0.75$, dash-dot line; $\omega = 1$



(A) Error ϵ_n

(B) Cost functional J_n

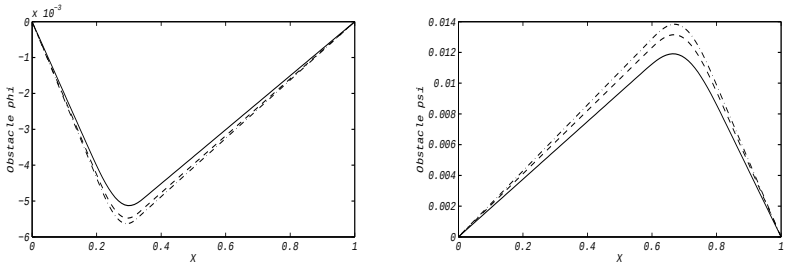
Figure 3: Left(error ϵ_n), right(cost functional J_n) continuous line; $\omega = 0.25$, dash line; $\omega = 0.75$, dash-dot line; $\omega = 1$

4.1.2 Test 2: Study of the dependence on the parameter N with $\delta = h^2$, $\omega = 0,75$ and $\nu = 1$

Numerical results are displayed in Table 2 according to the variation of N. In Figure 4, curves corresponding to the controls φ and ψ are shown. Curves given inFigure 5 show the contact region $I(y)$ between the state and the control functions. Finally, Figure 6 gives graphical variations in a log-log scale of ϵ_n and J_n for each iteration n.

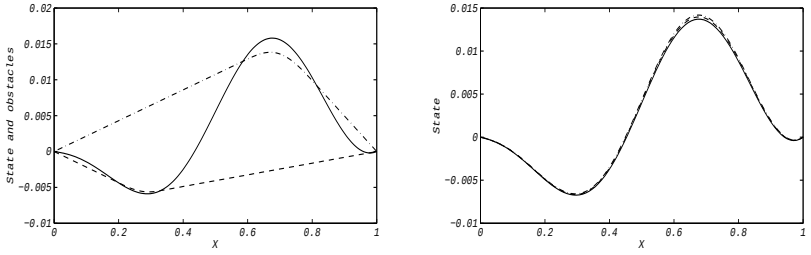
N	♯ Iteration	J	$\mid J_n - J_{n-1} \mid$	$\|y - z\|_\infty$	ϵ_n
100	26	24.392851	9.627854e-013	0.995719	2.621471e-005
150	27	36.716242	9.023892e-013	0.998098	8.192995e-005
200	31	49.039593	9.094947e-013	0.998935	2.096388e-005
250	30	61.362928	9.023892e-013	0.999322	6.492990e-005

Table 2: Numerical results for one dimensional space while varying N



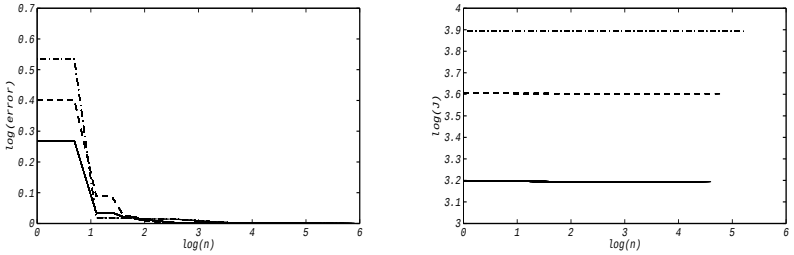
(A) Obstacle φ (B) Obstacle ψ

Figure 4: Left(obstacle φ), right(obstacle ψ) continuous line; $N = 100$, dash line; $N = 150$, dash-dot line; $N = 200$



(A) State and obstacles ($N = 100$) (B) State

Figure 5: Left(state and obstacles ($N = 100$)), right(state) continuous line; $N = 100$, dash line; $N = 150$, dash-dot line; $N = 200$



(A) Error ϵ_n (B) Cost functional J_n

Figure 6: Left(error ϵ_n), right(cost functional J_n), continuous line; $N = 100$, dash line; $N = 150$, dash-dot line; $N = 200$

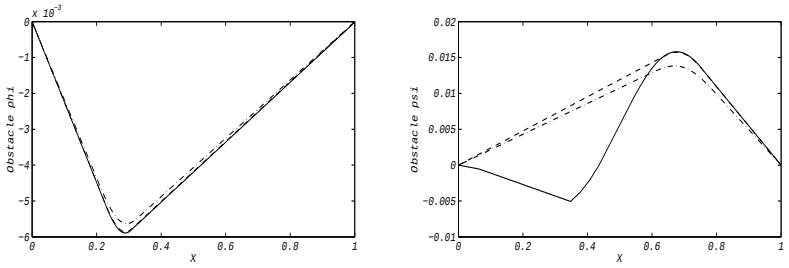
4.1.3 Test 3: Study of the dependence on the parameter ν with $\delta = h^2$, $\omega = 0,75$ and $N = 200$

Numerical results are displayed in Table 3 according to the variation of ν. In Figure 7, curves corresponding to the controls φ and ψ are shown. Curves given in Figure 8 show

the contact region $I(y)$ between the state and the control functions. Finally, Figure 9 gives graphical variations in a log-log scale of ϵ_n and J_n for each iteration n.

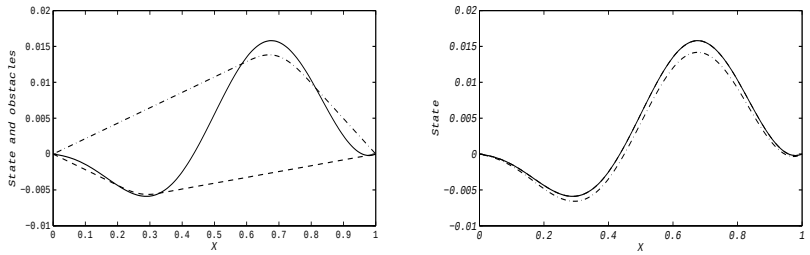
ν	♯ Iteration	J	$\mid J_n - J_{n-1} \mid$	$\|y - z\|_\infty$	ϵ_n
0.0001	28	49.039593172727052	3.907985e-013	0.998935	2.041225e-004
0.001	28	49.039593172727159	3.055333e-013	0.998935	7.349400e-005
0.01	36	49.039593172728459	7.389644e-013	0.998935	3.428793e-004
0.1	41	49.039593172735735	4.334310e-013	0.998935	1.033546e-014
1	223	49.039593172799911	9.094947e-013	0.998935	2.646852e-015

Table 3: Numerical results for one dimensional space while varying ν



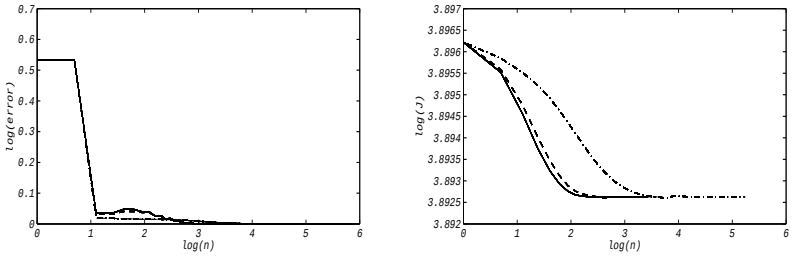
(A) Obstacle φ (B) Obstacle ψ

Figure 7: Left(obstacle φ), right(obstacle ψ) continuous line; $\nu = 0.01$, dash line; $\nu = 0.1$, dash-dot line; $\nu = 1$



(A) State and obstacles ($\nu = 0.01$) (B) State

Figure 8: Left(state and obstacles ($\nu = 0.01$)), right(state) continuous line; $\nu = 0.01$, dash line; $\nu = 0.1$, dash-dot line; $\nu = 1$



(A) Error ϵ_n (B) Cost functional J_n

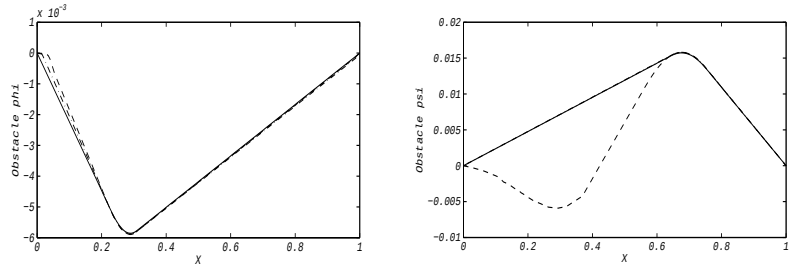
Figure 9: Left(error ϵ_n), right(cost functional J_n), continuous line; $\nu = 0.01$, dash line; $\nu = 0.1$, dash-dot line; $\nu = 1$

4.1.4 Test 4: Study of the dependence on the parameter δ with $N = 200$, $\omega = 0,75$ and $\nu = 0.1$

Numerical results are displayed in Table 4 according to the variation of δ. In Figure 10, curves corresponding to the controls φ and ψ are shown. Curves given in Figure 11 show the contact region $I(y)$ between the state and the control functions. Finally, Figure 12 gives graphical variations in a log-log scale of ϵ_n and J_n for each iteration n.

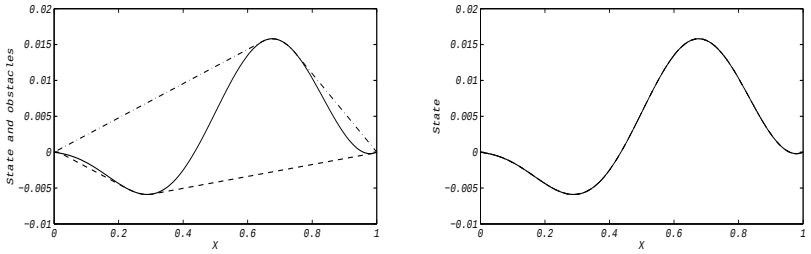
δ	♯ Iteration	J	$\mid J_n - J_{n-1} \mid$	$\|y - z\|_\infty$	ϵ_n
h^2	41	49.039593172735735	4.334310e-013	0.998935	1.033546e-014
$h^{2.5}$	112	49.039593172742727	6.679101e-013	0.998935	1.389559e-004
h^3	155	49.039593172736232	9.308109e-013	0.998935	1.243146e-008
$h^{3.5}$	383	49.039593172738968	6.465938e-013	0.998935	9.571397e-010

Table 4: Numerical results for one dimensional space while varying δ



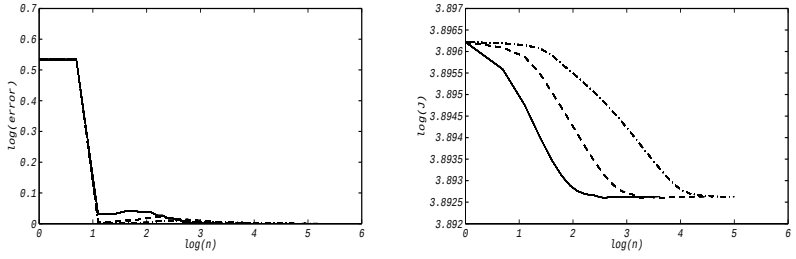
(A) Obstacle φ (B) Obstacle ψ

Figure 10: Left(obstacle φ), right(obstacle ψ), continuous line; $\delta = h^2$, dash line; $\delta = h^{2.5}$, dash-dot line; $\delta = h^3$



(A) State and obstacles ($\delta = h^2$) (B) State

Figure 11: Left(state and obstacle ($\delta = h^2$)), right(state) continuous line; $\delta = h^2$, dash line; $\delta = h^{2.5}$, dash-dot line; $\delta = h^3$



(A) Error ϵ_n (B) Cost functional J_n

Figure 12: Left(error ϵ_n), right(cost functional J_n) continuous line; $\delta = h^2$, dash line; $\delta = h^{2.5}$, dash-dot line; $\delta = h^3$

4.2 Numerical examples in two dimensional space

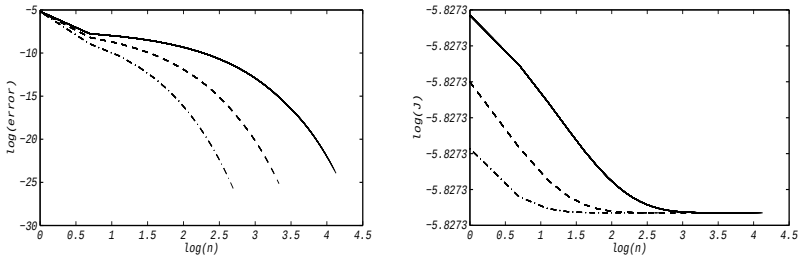
In this section, we describe some numerical experiments in two dimensional space based on the previous algorithm. We also give some numerical tests when in each test we vary one of the parameters ω, δ, N and ν, where $\Omega = [0,1]\times[0,1]$, $f(x,y) = x^3 sin(2\pi x^2) y cos(2\pi y^2)$ and $z(x,y) = sin(2\pi x^2) cos(2\pi y^2))$ and $\omega_y = \omega_\varphi = \omega_\psi = \omega$.

4.2.1 Test 1: Study of the dependence on the parameter ω with $\delta = h^4$, $\nu = 1$ and $N = 40$

Numerical results are displayed in Table 5 according to the variation of ω. Figure 13 gives graphical variations in a log-log scale of ϵ_n and J_n for each iteration n. Curves given in Figure 14 and Figure 15 corresponding to the controls and state functions are shown.

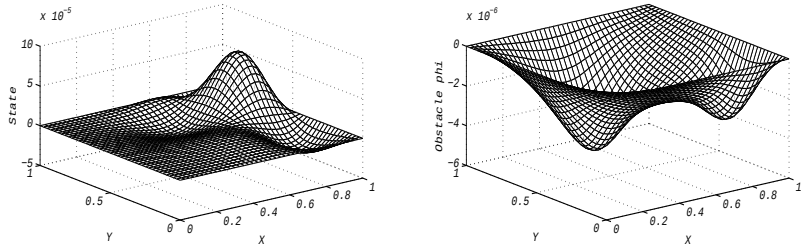
ω	$\sharp$ Iteration	J	$\mid J_n - J_{n-1} \mid$	$\|y - z\|_\infty$	ϵ_n
0.25	62	0.002946055145308	9.601694e-016	0.999991	3.982865e-011
0.5	28	0.002946055145306	6.392456e-016	0.999991	1.180507e-011
0.75	15	0.002946055145305	5.212844e-016	0.999991	5.253112e-012
1	5	0.002946055145305	6.878178e-016	0.999991	9.642406e-010

Table 5: Numerical results for two dimensional space while varying ω



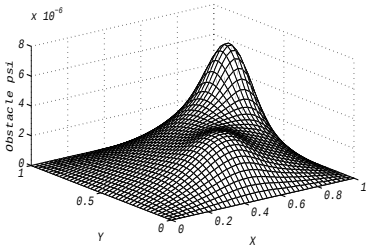
(A) Error ϵ_n (B) Cost functional J_n

Figure 13: Left(error ϵ_n), right(cost functional J_n) continuous line; $\omega = 0.25$, dash line; $\omega = 0.5$, dash-dot line; $\omega = 0.75$



(A) State function y (B) Obstacle function φ

Figure 14: Left(state function y), right(obstacle function φ)



(A) Obstacle function ψ

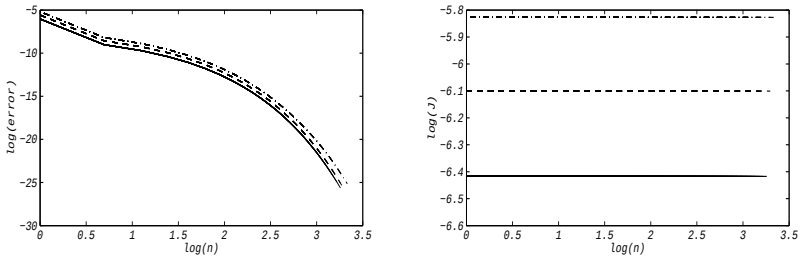
Figure 15: Obstacle function ψ

4.2.2 Test 2: Study of the dependence on the parameter N with $\delta = h^4$, $\omega = 0,5$ and $\nu = 1$

Numerical results are displayed in Table 6 according to the variation of N. Figure 16 gives graphical variations in a log-log scale of ϵ_n and J_n for each iteration n. Curves given in Figure 17 Figure 18 corresponding to the controls and state functions are shown.

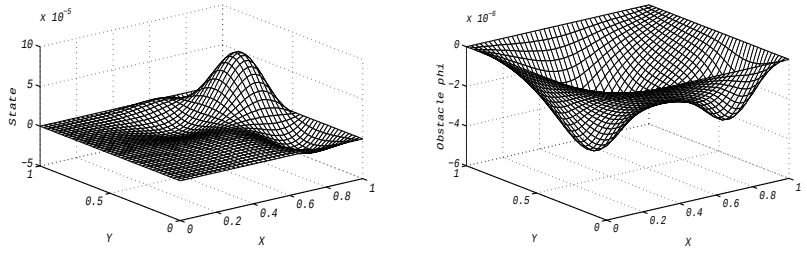
N	♯ Iteration	J	$\mid J_n - J_{n-1} \mid$	$\|y - z\|_\infty$	ϵ_n
30	26	0.001632	5.193328e-016	0.999974	7.355250e-012
35	27	0.002241	5.165139e-016	0.996074	6.513361e-012
40	28	0.002946	6.392456e-016	0.999991	1.180507e-011
45	30	0.003746	7.350890e-016	0.999966	2.389886e-011

Table 6: Numerical results for two dimensional space while varying N



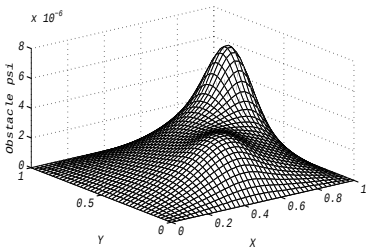
(A) Error ϵ_n (B) Cost functional J_n

Figure 16: Left(error ϵ_n), right(cost functional J_n) continuous line; $N = 30$, dash line; $N = 35$, dash-dot line; $N = 40$



(A) State function y (B) Obstacle function φ

Figure 17: Left(state function y), right(obstacle function φ)



(A) Obstacle function ψ

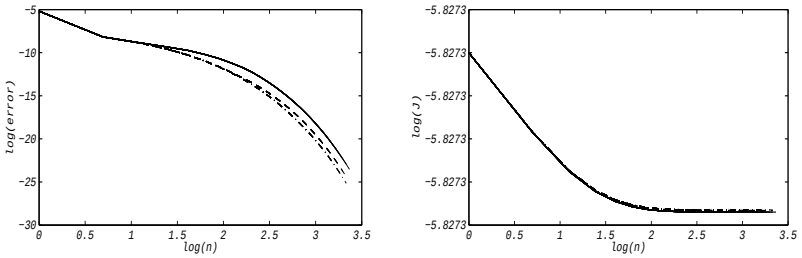
Figure 18: Obstacle function ψ

4.2.3 Test 3: Study of the dependence on the parameter ν with $\delta = h^4$, $\omega = 0,5$ and $N = 40$

Numerical results are displayed in Table 7 according to the variation of ν. Figure 19 gives graphical variations in a log-log scale of ϵ_n and J_n for each iteration n. Curves given in Figure 20 and Figure 21 corresponding to the controls and state functions are shown.

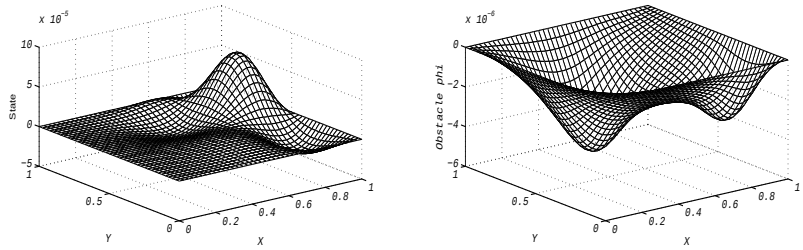
ν	♯ Iteration	J	$\mid J_n - J_{n-1} \mid$	$\|y - z\|_\infty$	ϵ_n
0.001	300	0.002946054157953	1.227802e-013	0.999991	3.695397e-005
0.01	29	0.002946054121818	5.676882e-016	0.999991	5.966039e-011
0.1	29	0.002946054416021	5.676882e-016	0.999991	5.966039e-011
0.5	28	0.002946054948036	7.650130e-016	0.999991	2.529592e-011
1	28	0.002946055145306	6.392456e-016	0.999991	1.180507e-011

Table 7: Numerical results for two dimensional space while varying ν



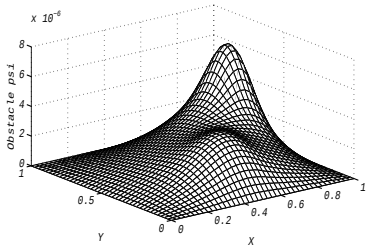
(A) Error ϵ_n (B) Cost functional J_n

Figure 19: Left (error ϵ_n), right(cost functional J_n) continuous line $\nu = 0.1$; dash line $\nu = 0.5$; dash-dot line $\nu = 1$



(A) State function y (B) Obstacle function φ

Figure 20: Left(state function y), right(obstacle function φ)



(A) Obstacle function ψ

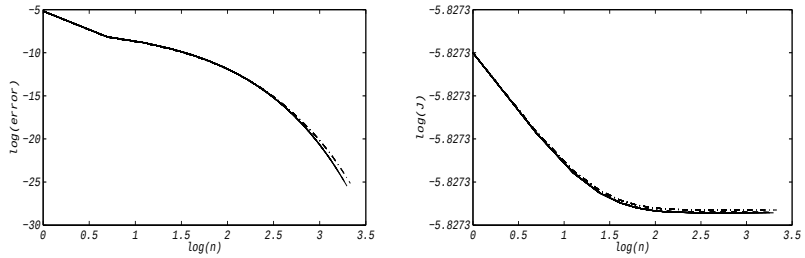
Figure 21: Obstacle function ψ

4.2.4 Test 4: Study of the dependence on the parameter δ with $N = 40$, $\omega = 0,5$ and $\nu = 1$

Numerical results are displayed in Table 8 according to the variation of δ. Figure 22 gives graphical variations in a log-log scale of ϵ_n and J_n for each iteration n. Curves given in Figure 23 and Figure 24 corresponding to the controls and state functions are shown.

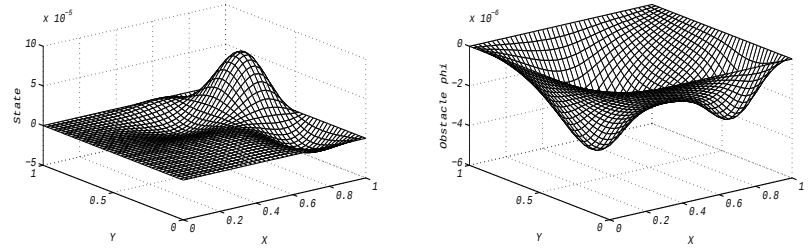
δ	♯ Iteration	J	$\mid J_n - J_{n-1} \mid$	$\|y - z\|_\infty$	ϵ_n
h^3	27	0.002946054193747	8.135853e-016	0.999991	8.627484e-012
$h^{3.5}$	27	0.002946054365294	8.270294e-016	0.999991	8.809409e-012
h^4	28	0.002946055145306	6.392456e-016	0.999991	1.180507e-011
$h^{4.5}$	33	0.002946056696808	9.358833e-016	0.999991	1.016003e-010

Table 8: Numerical results for two dimensional space while varying δ



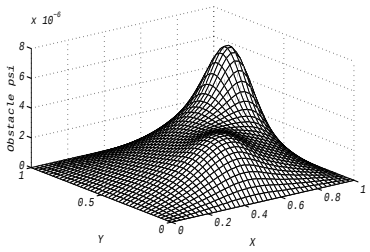
(A) Error ϵ_n (B) Cost functional J_n

Figure 22: Left(error ϵ_n), right(cost functional J_n) continuous line; $\delta = h^2$, dash line; $\delta = h^3$, dash-dot line; $\delta = h^4$



(A) State function y (B) Obstacle function φ

Figure 23: Left(state function y), right(obstacle function φ)



(A) Obstacle function ψ

Figure 24: Obstacle function ψ

5 Conclusion and remarks

We notice that techniques used in the paper of Ghanem et al. [13] can be easily applied to the numerical resolution of the problem considered in this work. The given numerical results are acceptable although the convergence of the algorithm is not fast. They also consolidate our perception given in Remarks 3.5 and 4.1 about the Lipschitz constants. In order to improve the speed of convergence, we can either apply other algorithms of resolution or should improve the used algorithm by optimizing the choice of the parameter (by the line search method, for example).

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