On Weight-Prioritized Multi-Task Control of Humanoid Robots

Karim Bouyarmane, Abderrahmane Kheddar

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Karim Bouyarmane, *Member, IEEE* and Abderrahmane Kheddar, *Senior Member, IEEE*

***Abstract*—We propose a formal analysis with some theoretical properties of weight-prioritized multi-task inverse-dynamics-like control of humanoid robots, being a case of redundant "manipulators" with a non-actuated free-floating base and multiple unilateral frictional contacts with the environment. The controller builds on a weighted sum scalarization of a multiobjective optimization problem under equality and inequality constraints, which appears as a straightforward solution to account for state and control input viability constraints characteristic of humanoid robots that were usually absent from early existing pseudo-inverse and null-space projection-based prioritized multi-task approaches. We argue that our formulation is indeed well founded and justified from a theoretical standpoint, and we propose an analysis of some stability properties of the approach: Lyapunov stability is demonstrated for the closed-loop dynamical system that we analytically derive in the unconstrained multiobjective optimization case. Stability in terms of solution existence, uniqueness, continuity, and robustness to perturbations, is then formally demonstrated for the constrained quadratic program.**

***Index Terms*—Multiobjective optimization, multi-task control, Lyapunov's indirect method, quadratic-program stability**

I. Introduction

APPLYING early control methods developed for (industrial) manipulators [1], [2] to humanoid robots, e.g. inverse dynamics control, operational or task function space control... raises a number of challenging problems [3]–[6]. Typical such problems include simultaneous resolution of redundancy and underactuation, or actuation through friction-cone-constrained unilateral contact forces. Although each of these problems has already been extensively studied in the context of industrial manipulators or various general cases (e.g. handling redundancy in [7], [8], underactuation in [9], [10], contacts constraints in [11]–[14], bounds on control inputs in [15], and references therein), the specificity of a humanoid robot is that it features and interleaves them all at once, and thus renders the solutions that were proposed for each of these problems taken in a separate setting largely inapplicable in a unified control framework.

We tackle these combined structural problems in a simple formulation in which we make the non-equivocal distinction between the two notions of *constraints* and *tasks*; a distinction that we believe should be made by/in any humanoid control law design at large. *Constraints* are inherent to the well-posedness of the problem, as failing to satisfy them results in a physically or mathematically ill-posed problem. These are the physics laws (Newton-Euler equations or Lagrange equations, Coulomb laws) and the safety and structural limits (torque saturation, joint angle and velocity limits, collision and obstacle avoidance). *Tasks*, on the other hand, allow for more tolerance in their fulfilment and necessitate a certain degree of "compliance" in their execution. Failing to realize them does not result in a mathematical or physical law violation. Since tasks come one way or another from *planning* (off-line or real-time), then it should be the role of the *planner*, not the controller, to ensure that the tasks are consistent and realizable [16].

Another important aspect in which humanoids differ essentially from industrial manipulators is their novel context of applications. An industrial manipulator is confined to a structured, known, and uncertainty-free environment. It is thus conceivable that in that setting tasks are seen as constraints that should be realized perfectly, more so if the manipulator had been specifically designed for the task at hand. Humanoids, even when targeted to manufacturing[1], are neither customized to achieve a particular task nor do they evolve in a structured environment that was exclusively designed for their operations. As such, tasks shall have the flexibility to be set as constraints or as objectives to be realized at best given their actual structural constraints and the uncertain state of their environment.

In this paper we have taken a step back from what we already extensively achieve in experimental humanoid robotics. Firstly, we adapt in an original way, different from the recursive null space projection approach, the inverse dynamics control principles to general multi-task systems and to the "humanoid type of manipulator" in particular accounting for its redundant, underactuated, and constrained nature (e.g. walking stability). Secondly, and this constitutes our novel contribution with respect to existing work, we assess the foundations from a control theoretical perspective of such control schemes.

II. Main Results and Structure of the Paper

In Section III-A we cast the problem of multi-task control as a multiobjective optimization problem. Proposition 1 explores the one-task control case and its exact realization if unconstrained. When there are multiple tasks competing with each other, their exact realization cannot be guaranteed

K. Bouyarmane is with Université de Lorraine, Inria, CNRS, Loria UMR 7503, Larsen team, 615 rue du Jardin Botanique, Vandœuvre-lès-Nancy F-54500 France

A. Kheddar is with CNRS-University of Montpellier LIRMM Interactive Digital Humans group, 161 rue Ada, 34095 Montpellier, France. A. Kheddar is also with CNRS-AIST JRL UMI3218/RL, 1-1-1 Umezono, Tsukuba 305-8568 , Japan

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anymore, but under the conditions in Section III-B, we can approach the realization of a desired task at a given precision. Proposition 2 explores the meaning and consequences of not realizing exactly a desired task but approaching it at the given precision. We show that this results in a uniformly ultimately bounded task error that converges to zero in some circumstances.

In Section III-B we recall for the unfamiliar reader some main results from the multiobjective optimization literature that drive our reasoning (Theorems 1 to 7). We then derive results under which Proposition 2 is applicable, namely Corollary 2: if a task is realizable exactly (when considered alone), then when put in competition with the other tasks there exists a set of weights that makes it realizable at any desired precision.

In Section IV we study the unconstrained dynamical system's ordinary differential equation (ODE) that results from the weighted sum scalarization of the multiobjective optimization as formulated in Section III. The main result is Proposition 4, that characterizes the equilibrium point and gives necessary and sufficient conditions for its exponential stability. The methodology followed in the proof of Proposition 4 is first introduced for the simpler one-task setting in Proposition 3.

In Section V we consider the multi-task control problem of a humanoid robot. Proposition 5 that allows us to position the problem within the context of the framework developed in Section III-B (thanks to results borrowed from Theorems 9 and 10). In this section we consider the full constrained humanoid problem and formulate it as a linearly-constrained quadratic program (QP). The results in Propositions 6 to 9 and Corollary 5 then give us conditions for the well-posedness, robustness to perturbations, and continuity of the solution of that QP.

Note: We label "Theorem" any result that we borrow from the literature and "Proposition", "Corollary" and "Lemma" results that we propose as contribution. We also borrow all the "Definitions" from the literature, as we do not redefine any of the literature terminology.

III. Multi-Task Control as a Multiobjective Optimization Problem

A. General Concepts

Let us recall some concepts of multiobjective optimization (also known as multicriteria optimization, multiple criteria decision making, vector optimization [17], [18]) and demonstrate some useful properties in our context of multi-task control.

Multiobjective optimization studies the problem

$$\text{“}\min_{x\in\mathcal{X}}\text{”}\, f(x) = (f_1(x), \ldots, f_p(x)) \,, \tag{1}$$

where the $\min$ operator is put between quotation marks to emphasize that it is dependent on some specific optimality notion for vector values to be defined. The $f_{1,2,\cdots,p}$ functions are scalar functions and $\mathcal{X}$ is the feasible space (e.g. as defined by a set of constraints on x). A solution $x^* \in \mathcal{X}$ of (1) is called an *efficient* (or *Pareto-optimal*) solution if there is no $x \in \mathcal{X}$ such that $f(x) \leq f(x^*)$. The notation $y^1 \leq y^2$ denotes the *componentwise* order in $\mathbb{R}^p$.

Definition 1 (Componentwise order [17, Definition 2.1 p. 24])**.** *Let y^1 and y^2 be two vectors of $\mathbb{R}^p$. y^2 is said to be dominated*[2] *by y^1, and we denote $y^1 \leq y^2$, if $\forall k \in \{1,\ldots,p\}$ $y^1_k \leq y^2_k$ and $y^1 \neq y^2$, i.e. at least one inequality holds strictly $\exists i \in \{1,\ldots,p\}$ $y^1_i < y^2_i$.*

This notion of componentwise order is to be clearly distinguished from the *weak* and *strict* componentwise orders that we also use in the developments to follow.

Definition 2 (Weak componentwise order [17, Definition 2.24 p. 38])**.** *y^2 is said to be weakly dominated by y^1, and we denote $y^1 \leqq y^2$, if $\forall k \in \{1,\ldots,p\}$ $y^1_k \leq y^2_k$.*

Definition 3 (Strict componentwise order [17, Definition 2.24 p. 38])**.** *y^2 is said to be strictly dominated by y^1, and we denote $y^1 < y^2$, if $\forall k \in \{1,\ldots,p\}$ $y^1_k < y^2_k$.*

Let $\mathcal{Y} = f(\mathcal{X}) \subset \mathbb{R}^p$ denote the image of the feasible set. If x^* is an efficient solution of (1) then its image $y^* = f(x^*)$ is called a *nondominated* point of $\mathcal{Y}$. The set of all efficient solutions of (1) is denoted $\mathcal{X}_E$ and the set of all nondominated points of $\mathcal{Y}$ is denoted $\mathcal{Y}_N$ (sometimes referred to as the *Pareto-optimal front*). We denote

$$y^I = (\min_{x\in\mathcal{X}} f_1(x), \ldots, \min_{x\in\mathcal{X}} f_p(x)) \,, \tag{2}$$

the so-called *ideal point*. In general the ideal point is not realizable, i.e. $y^I \notin \mathcal{Y}$, in that case any point in $\mathcal{Y}_N$ can be seen as a non improvable compromise solution of (1) (note that if however $y^I \in \mathcal{Y}$ then $\mathcal{Y}_N$ reduces to the singleton $\{y^I\}$, i.e. $y^I \in \mathcal{Y} \Leftrightarrow \mathcal{Y}_N = \{y^I\}$).

In a context of multi-task control with p tasks, each task τ_k ($k \in \{1,\ldots,p\}$) is defined through a forward kinematics function $g_k : \mathbb{R}^n \to \mathbb{R}^{n_k}$, mapping the n-dimensional generalized coordinates of the system q to the n_k-dimensional value of the task $\tau_k = g_k(q)$ ($n \geq n_k$). A task is associated with a planned reference trajectory $t \mapsto \tau^r_k(t)$ and an objective *attractor* behavior to realize *exponential* tracking of the reference trajectory. In the case of a humanoid robot system as will be considered in Section V, the tasks τ_k of interest are of vector relative degree 2. This is due to the fact that they explicitly depend only on the configuration variable q (and not on the velocity $\dot{q}$), and that the dynamics model of the robot is of second order, see Section V). Hence, we consider throughout the paper tasks of vector relative degree 2. Denoting the task error $e_k = \tau_k - \tau^r_k$, the attractor behavior takes the form

$$\ddot{e}_k + D_k\dot{e}_k + P_k e_k = 0 \,, \tag{3}$$

where the matrices (P_k, D_k) are so that $\begin{pmatrix} 0 & I_{n_k} \\ -P_k & -D_k \end{pmatrix}$ is *stable* (i.e. has all its eigenvalues with negative real parts).

More generally, denoting the task error state space variable $\eta_k = \begin{pmatrix} e_k \\ \dot{e}_k \end{pmatrix}$, we study tasks for which the reference behavior is of the form $\dot{\eta}_k = A_k\eta_k$ where $A_k \in \mathbb{R}^{2n_k\times 2n_k}$ is any stable matrix. However, some results of the paper are stated under

[2]Note the non intuitive use of the "dominated by" terminology. y^2 is dominated by y^1 in the sense that y^2 is less optimal than y^1, thus dominated by y^1 in the optimality characteristic.

the assumption of the negative definiteness of $A_k + A_k^T$, we recall that this is a sufficient condition for A_k to be stable [19].

For convenience of notation the behavior (3) can also be written in the form

$$\ddot{\tau}_k - \ddot{\tau}_k^d = 0\,, \tag{4}$$

with the desired task acceleration $\ddot{\tau}_k^d = \ddot{\tau}_k^r - D_k\dot{e}_k - P_k e_k$. If the constraints of the robot make it impossible to achieve perfect realization of $\ddot{\tau}_k^d$, then one might want to realize this behavior "at best" in the following sense

$$\min_{x\in\mathcal{X}} ||\ddot{\tau}_k - \ddot{\tau}_k^d||^2\,, \tag{5}$$

where x denotes a control decision variable and $x \in \mathcal{X}$ its constraints. As we will see later (Section V), the particular choice of the square norm $||.||^2$ allows us to formulate the problem as a linearly constrained quadratic program (QP) and use algorithms that are dedicated to this class of optimization problems. Let $J_k = \partial g_k/\partial q \in \mathbb{R}^{n_k\times n}$ denote the Jacobian matrix of the task $\tau_k = g_k(q)$. Here and henceforth we suppose that g_k is continuously differentiable so that J_k exists and is continuous (which is always the case for a large class of robotic systems in practice). In the simplest case where $x = \ddot{q}$ and $\mathcal{X} = \mathbb{R}^n$ we can easily show that:

Proposition 1. *If J_k is full row rank then (5) ⇔ (4).*

Proof. The first order optimality condition for (5) is

$$\frac{\partial||\ddot{\tau}_k - \ddot{\tau}_k^d||^2}{\partial\ddot{q}} = 0\,, \tag{6}$$

$$\Leftrightarrow \frac{\partial(\ddot{\tau}_k - \ddot{\tau}_k^d)}{\partial\ddot{q}}^T(\ddot{\tau}_k - \ddot{\tau}_k^d) + (\ddot{\tau}_k - \ddot{\tau}_k^d)^T\frac{\partial(\ddot{\tau}_k - \ddot{\tau}_k^d)}{\partial\ddot{q}} = 0 \tag{7}$$

$$\Leftrightarrow \qquad 2\,\frac{\partial(\ddot{\tau}_k - \ddot{\tau}_k^d)}{\partial\ddot{q}}^T(\ddot{\tau}_k - \ddot{\tau}_k^d) = 0. \tag{8}$$

Since $\dot{\tau}_k = J_k\dot{q}$ and $\ddot{\tau}_k = J_k\ddot{q} + \dot{J}_k\dot{q}$, we have $\partial\ddot{\tau}_k/\partial\ddot{q} = J_k$ (tasks of vector relative degree two). On the other hand $\partial\ddot{\tau}_k^d/\partial\ddot{q} = 0$. Hence

$$(8) \quad\Leftrightarrow\quad 2\,J_k^T(\ddot{\tau}_k - \ddot{\tau}_k^d) = 0. \tag{9}$$

By the rank-nullity theorem, $\dim\ker J_k^T = n_k - \operatorname{rank} J_k^T = n_k - \operatorname{rank} J_k$; since $\operatorname{rank} J_k = n_k$ then $\dim\ker {J_k}^T = 0$, which means $\ker J_k^T = \{0\}$, the desired equivalence thus follows from (9). □

In the more general case we can state the following:

Definition 4 ([20], [21], [22, Definition 4.6 p. 169])**.** *The solutions of a system $\dot{\chi} = \varphi(\chi, t)$ are said to be uniformly ultimately bounded (UUB) if there exists $b > 0$ and $c > 0$ such that, for every $0 < a < c$, there exists $T(a, b) > 0$ such that*

$$||\chi(0)|| < a \;\Rightarrow\; \forall t \geq T(a, b),\; ||\chi(t)|| < b\,. \tag{10}$$

b is called an ultimate bound of the solutions. If a can be arbitrarily large, i.e. if there exists $b > 0$ such that for every $a > 0$ there exists $T(a, b) > 0$ such that (10) holds, then the solutions are said to be globally uniformly ultimately bounded with ultimate bound b.

Let $\mu(A_k)$ denote the logarithmic norm of A_k associated with the vector norm $||.||$:

Definition 5 ([23])**.** *The logarithmic norm associated with the vector norm $||.||$ in $\mathbb{R}^{2n_k}$ and its subordinate matrix norm $||.||$ in $\mathbb{R}^{2n_k\times 2n_k}$ is defined as*

$$\mu(A_k) = \lim_{h\to 0^+}\frac{||I + hA_k|| - 1}{h}\,. \tag{11}$$

It can be shown [24] that $\mu(A_k) = \lambda_{\max}\left[\frac{1}{2}(A_k + A_k^T)\right]$, the maximum eigenvalue of $\frac{1}{2}(A_k + A_k^T)$.

Proposition 2. *If $A_k + A_k^T$ is negative definite then, for any $\epsilon > 0$, the differential inequality:*

$$||\ddot{\tau}_k - \ddot{\tau}_k^d||^2 < \epsilon\,, \tag{12}$$

results in $\eta_k(t)$ globally uniformly ultimately bounded. Moreover, for any $t \mapsto \varepsilon(t) > 0$ such that $\varepsilon(t) = O\left(e^{2\,\mu(A_k)t}\right)$, the differential inequality

$$||\ddot{\tau}_k - \ddot{\tau}_k^d||^2 < \varepsilon(t)\,, \tag{13}$$

implies, for every initial condition $\eta_k(0)$,

$$\eta_k(t) \xrightarrow[t\to+\infty]{} 0\,. \tag{14}$$

Proof. The inequality (12) can be rewritten as

$$||\dot{\eta}_k - A_k\eta_k|| = \left|\left|\begin{pmatrix} 0 \\ \ddot{\tau}_k - \ddot{\tau}_k^d\end{pmatrix}\right|\right| = ||\ddot{\tau}_k - \ddot{\tau}_k^d|| < \sqrt{\epsilon}\,, \tag{15}$$

which is equivalent to

$$\dot{\eta}_k = A_k\eta_k + \zeta(t)\,, \tag{16}$$

with $||\zeta(t)|| < \sqrt{\epsilon}$. From the properties of the logarithmic norm, it can be shown [23] that (16) implies

$$||\eta_k(t)|| \leq e^{t\mu(A_k)}||\eta_k(0)|| + \int_0^t e^{(t-\theta)\mu(A_k)}||\zeta(\theta)||d\theta\,, \tag{17}$$

$$\leq e^{t\mu(A_k)}||\eta_k(0)|| + \int_0^t e^{(t-\theta)\mu(A_k)}\sqrt{\epsilon}d\theta\,, \tag{18}$$

$$= \left(||\eta_k(0)|| + \frac{\sqrt{\epsilon}}{\mu(A_k)}\right)e^{t\mu(A_k)} - \frac{\sqrt{\epsilon}}{\mu(A_k)}\,. \tag{19}$$

Let $\delta > 0$. We show that $\eta_k(t)$ is globally UUB with ultimate bound $-\frac{\sqrt{\epsilon}}{\mu(A_k)} + \delta$. So let $a > 0$. From (19), $||\eta_k(0)|| < a$ implies that

$$||\eta_k(t)|| < \left(a + \frac{\sqrt{\epsilon}}{\mu(A_k)}\right)e^{t\mu(A_k)} - \frac{\sqrt{\epsilon}}{\mu(A_k)}\,. \tag{20}$$

We also have $\mu(A_k) = \lambda_{\max}\left[\frac{1}{2}(A_k + A_k^T)\right]$. Since $A_k + A_k^T$ is negative definite, $\mu(A_k) < 0$, and hence the right-hand side of (20) goes to $-\frac{\sqrt{\epsilon}}{\mu(A_k)}$ as t goes to $+\infty$. Therefore there exists $T(a, \delta)$ such that $\forall t \geq T(a, \delta) : ||\eta_k(t)|| < -\frac{\sqrt{\epsilon}}{\mu(A_k)} + \delta$, and we can conclude that $\eta_k(t)$ is globally UUB with ultimate bound $-\frac{\sqrt{\epsilon}}{\mu(A_k)} + \delta$.

In the case of (13) with $\varepsilon(t) = O\left(e^{2\,\mu(A_k)t}\right)$, there exists $M > 0$ such that $\varepsilon(t) < Me^{2\,\mu(A_k)t}$, so

$$\int_0^t e^{(t-\theta)\mu(A_k)}\sqrt{\varepsilon(\theta)}d\theta \leq \int_0^t e^{(t-\theta)\mu(A_k)}Me^{\theta\mu(A_k)}d\theta\,, \tag{21}$$

$$= Mte^{t\mu(A_k)}\,, \tag{22}$$

hence (17) implies

$$||\eta_k(t)|| \leq \Big(||\eta_k(0)|| + Mt\Big)e^{t\mu(A_k)}\,. \tag{23}$$

Since $\mu(A_k) < 0$ the right-hand side of (23) goes to 0 as t goes to $+\infty$ and therefore $\lim_{t\to+\infty}\eta_k(t) = 0$. □

Following this train of thought, it appears now that the multi-task problem can indeed be written as a multiobjective optimization problem as introduced earlier in this section

$$\text{“}\min_{x\in\mathcal{X}}\text{”}\, f(x) = (||\ddot{\tau}_1 - \ddot{\tau}_1^d||^2, \ldots, ||\ddot{\tau}_p - \ddot{\tau}_p^d||^2). \tag{24}$$

We thus provide in the following a complete characterization of all the efficient solutions of this problem.

B. Characterization of the Efficient Solutions

It can be shown that, to a certain extent that is precisely defined hereafter, all the efficient solutions of the multiobjective optimization problem (1) can be obtained by solving single objective problems of the form

$$\min_{x\in\mathcal{X}}\sum_{k=1}^{p} w_k f_k(x)\,. \tag{25}$$

The problem (25) is called a weighted sum *scalarization* of the problem (1). Different results on the completeness of the characterization of the solutions of (1) can be obtained depending on whether we consider the non-identically null scalar weights w_k of (25) as only nonnegative or as (strictly) positive (i.e. whether $0 \leq w$ or $0 < w$ using the componentwise order notations of Section III-A). Let us denote the set of optimal points in $\mathcal{Y}$ that are spanned by the problems (25) in these two cases respectively as

$$\mathcal{S}_0(\mathcal{Y}) = \left\{ y^* \in \mathcal{Y} \;\middle|\; \sum_{k=1}^{p} w_k y_k^* = \min_{y\in\mathcal{Y}}\sum_{k=1}^{p} w_k y_k,\ 0 \leq w \right\}\,, \tag{26}$$

$$\mathcal{S}(\mathcal{Y}) = \left\{ y^* \in \mathcal{Y} \;\middle|\; \sum_{k=1}^{p} w_k y_k^* = \min_{y\in\mathcal{Y}}\sum_{k=1}^{p} w_k y_k,\ 0 < w \right\}\,. \tag{27}$$

We need a few more definitions to complete those already introduced in Section III-A. A solution $x^* \in \mathcal{X}$ is said to be a *weakly efficient* solution of (1) if $f(x^*)$ is *weakly nondominated* in $\mathcal{Y}$, that is, if there is no $x \in \mathcal{X}$ such that $f(x) < f(x^*)$. The set of all weakly nondominated points in $\mathcal{Y}$ is then denoted $\mathcal{Y}_{wN}$.

Theorem 1 ([17, Theorem 3.4 p. 69])**.** $\mathcal{S}_0(\mathcal{Y}) \subset \mathcal{Y}_{wN}$.

For the converse inclusion we need the following definition:

Definition 6 ([25, Definition 3.1 p. 329] [17, Definition 3.1 p. 67])**.** *A set $\mathcal{Y}$ is said to be $\mathbb{R}^p_{\geqq}$-convex if $\mathcal{Y}+\mathbb{R}^p_{\geqq}$ is convex. $\mathbb{R}^p_{\geqq} = \{y \in \mathbb{R}^p \mid 0 \leqq y\}$ is the nonnegative orthant.*

Theorem 2 ([17, Theorem 3.5 p. 69])**.** *If $\mathcal{Y}$ is $\mathbb{R}^p_{\geqq}$-convex then $\mathcal{S}_0(\mathcal{Y}) = \mathcal{Y}_{wN}$.*

Thus we can see that under the conditions of Theorem 2 all weakly nondominated solutions of a multiobjective optimization problem can be obtained by weighted sum scalarizations with nonnegative weights. In our coming formulation of multitask control we need the weights to be positive for the sake of stability. Thus we need stronger results, characterizing $\mathcal{S}(\mathcal{Y})$ rather than $\mathcal{S}_0(\mathcal{Y})$.

Theorem 3 ([17, Theorem 3.6 p. 70])**.** $\mathcal{S}(\mathcal{Y}) \subset \mathcal{Y}_N$.

Unfortunately, the inclusion in Theorem 3 is too large, and the converse inclusion does not hold in general. In fact, it can be shown that the positive weights will only yield a set of so-called *properly efficient* solutions.

Definition 7 ([26, Definition p. 618])**.** *A solution $x^* \in \mathcal{X}$ is called properly efficient if it is efficient and $\exists M > 0$ s.t. $\forall x \in \mathcal{X}$, $\forall i \in \{1,\ldots,p\}$: $f_i(x) < f_i(x^*) \Rightarrow \exists j \in \{1,\ldots,p\} \setminus \{i\}$ s.t. $f_j(x^*) < f_j(x)$ and*

$$\frac{f_i(x^*) - f_i(x)}{f_j(x) - f_j(x^*)} \leq M\,.$$

In that case the point $f(x^*)$ is said to be properly nondominated in $\mathcal{Y}$ and the set of all properly nondominated points of $\mathcal{Y}$ is denoted $\mathcal{Y}_{pN}$.

Using Definition 7, a tighter inclusion than that of Theorem 3 can be obtained:

Theorem 4 (Geoffrion (1968) [26, Theorem 1])**.** $\mathcal{S}(\mathcal{Y}) \subset \mathcal{Y}_{pN}$.

The converse inclusion of Theorem 4 holds:

Theorem 5 ([17, Theorem 3.13 p. 74])**.** *If $\mathcal{Y}$ is $\mathbb{R}^p_{\geqq}$-convex then $\mathcal{S}(\mathcal{Y}) = \mathcal{Y}_{pN}$.*

Theorem 5 shows that only the properly efficient solutions of (1) can be attained with positive weights, and that this is the best we can achieve *exactly*. However, the following theorem, due to Hartley (1978), allows us to *approximate* any efficient solution with positive weight scalarization which will prove useful in our application.

Definition 8 ([18, Definition 3.2.4 p. 52])**.** *A set $\mathcal{Y}$ is said to be $\mathbb{R}^p_{\geqq}$-closed if $\mathcal{Y}+\mathbb{R}^p_{\geqq}$ is closed.*

Theorem 6 (Hartley (1978) [27, Theorem 5.5])**.** *If $\mathcal{Y}$ is nonempty, $\mathbb{R}^p_{\geqq}$-convex and $\mathbb{R}^p_{\geqq}$-closed then $\mathcal{Y}_N \subset \mathrm{cl}\left(\mathcal{S}(\mathcal{Y})\right)$.*

Theorem 6 is a powerful tool that allows us to perform our desired approximation. Before applying it we will need the following lemma:

Lemma 1. *There is always at least one efficient solution of problem (1) that exactly realizes a given component of the ideal point y^I (2), i.e. $\forall k \in \{1,\ldots,p\}\ \exists y \in \mathcal{Y}_N$ s.t. $y_k = y_k^I$.*

Proof. Let k be a given index in $\{1,\dots,p\}$. Let $\mathcal{X}'$ denote the set $\mathcal{X}' = \{x \in \mathcal{X} \mid f_k(x) = y_k^I\}$, let $f' : \mathcal{X}' \to \mathbb{R}^{p-1}$ such that $f'(x) = (f_1(x),\dots,f_{k-1}(x), f_{k+1}(x),\dots,f_p(x))$ and let y' be any nondominated point of $\mathcal{Y}' = f'(\mathcal{X}')$. Then it is clear that y such that $y_k = y_k^I$ and $y_i = y'_i$ for $i \neq k$ satisfies the desired result. □

Now, we state the following corollary, supposing in the remainder of this section that the conditions of Theorem 6 are satisfied:

Corollary 1. *For any $\epsilon > 0$ and any index k, there exists a set of positive weights $0 < w$ such that $f_k(x^*) - y_k^I < \epsilon$, where x^* denotes a solution of problem (25).*

Proof. From Lemma 1 there exists $y \in \mathcal{Y}_N$ such that $y_k = y_k^I$. From Theorem 6 we then have $y \in \mathrm{cl}(\mathcal{S}(\mathcal{Y}))$ Since $\mathcal{Y}$ is finite-dimensional all norms are topologically equivalent and thus we can consider the ℓ^∞-norm $||.||_\infty$ for the closure definition $\mathrm{cl}(.)$. Therefore, there exists a sequence of elements $(y^l)_{l\in\mathbb{N}} \in \mathcal{S}(\mathcal{Y})^{\mathbb{N}}$ such that $||y^l - y||_\infty \xrightarrow[l\to+\infty]{} 0$, and as such there exists $l_0 \in \mathbb{N}$ such that $||y^{l_0} - y||_\infty < \epsilon$. Finally we have $y_k^{l_0} - y_k^I = y_k^{l_0} - y_k \leq ||y^{l_0} - y||_\infty < \epsilon$ which shows the desired result. □

Applying Corollary 1 to problem (24) gives us:

Corollary 2. *If a given task τ_k is realizable exactly, i.e. $\exists x \in \mathcal{X}$ s.t. $\ddot{\tau}_k = \ddot{\tau}_k^d$, then it can be reached with weighted-sum scalarization of (24) with positive weights at any given precision, i.e. for any $\epsilon > 0$ there exists $0 < w$ such that*

$$||\ddot{\tau}_k(x^*) - \ddot{\tau}_k^d||^2 < \epsilon\,, \tag{28}$$

where x^* is the solution of the w-weighted sum scalarization of (24):

$$\min_{x\in\mathcal{X}} \sum_{l=1}^{p} w_l\, ||\ddot{\tau}_l(x) - \ddot{\tau}_l^d||^2\,. \tag{29}$$

Proof. Immediate from Corollary 1. □

In redundant manipulator control, one popular optimality notion is what is usually referred to as the *strict priority* ordering of the tasks (or sometimes *strict hierarchy*), which is *de facto* imposed by the nature of the method itself, i.e. the recursive pseudo-inversion of the task "constraint" and the projection in the null space of higher priority constrains [28]. In the context of multiobjective optimization a similar notion is labeled under the term *lexicographic optimization*

$$\operatorname*{lexmin}_{x\in\mathcal{X}}\,(f_1(x),\dots,f_p(x))\,, \tag{30}$$

which consists in finding a point $y^L \in \mathcal{Y}$ called the *lexicographic optimum* such that $\forall y \in \mathcal{Y}\ y^L \leq_{\mathrm{lex}} y$ where $\leq_{\mathrm{lex}}$ denotes the *lexicographic order* (a total order) in $\mathbb{R}^p$.

Theorem 7 ([17, Lemma 5.2 p. 129])**.** *The lexicographic optimum is one particular efficient solution of (1), i.e. $y^L \in \mathcal{Y}_N$.*

Applying again Theorem 6 we get:

Corollary 3. *The lexicographic (strict priority) optimum can be approached at any given precision by positive weighted sum scalarization, i.e., for any $\epsilon > 0$ there exists a set of positive weights $0 < w$ such that $||f(x^*) - y^L|| < \epsilon$, where x^* is the solution of (29).*

Proof. Similar to the proof of Corollary 1 from Theorem 7. □

At this stage, we have characterized the efficient solutions of (24) and justified the use of (29) for solving it. Propositions 1 and 2 give us some stability results in the state-space of the tasks $(\tau_k, \dot{\tau}_k)$, we study in the following the behavior of the system in the state-space of the generalized coordinates of the robot $(q, \dot{q})$.

IV. Stability in the State Space of the Generalized Coordinates

In this section, we restrict ourselves to the case in which $x = \ddot{q}$ and $\mathcal{X} = \mathbb{R}^p$. This would provide us with some insight on the general case that is more complex to study analytically and is out of the scope of this paper. We also consider task function regulation problems in which $t \mapsto \tau_k^r(t)$ are constant in time, and for ease of notation we denote their constant regulation values τ_k^r.

Our aim here is to study the behavior of the system of ordinary differential equations (ODEs) defined by

$$\ddot{q} = \operatorname{argmin} \sum_{k=1}^{p} w_k\, ||\ddot{\tau}_k - \ddot{\tau}_k^d||^2\,, \tag{31}$$

in the state space of $(q, \dot{q})$, where the weights are positive $0 < w$ following our analysis in Section III-B. As for related work concerning this section, see for example [29], [30] that study the stability of the strict priority inverse kinematics control approach, [7], [28] for the stability of strict priority inverse dynamics, [31], [32] for the stability of the weighted approach of a multi-task controller based on control Lyapunov functions (CLF).

We will base our argumentation below on Lyapunov's indirect method. In the Appendix we introduce some general matrix differentiation concepts that we extensively use in the course of its application. This also allows us to introduce along the way the concept of the second derivative of the forward kinematics mapping (the "Jacobian of the Jacobian").

We start with a single task case to illustrate our method in a simple setting, we then generalize the approach to multiple tasks. Note that some of the notations used throughout the rest of the paper are introduced inside the proofs of this section.

Proposition 3. *Suppose $n_k = n$. The system:*

$$\ddot{q} = \operatorname{argmin} ||\ddot{\tau}_k - \ddot{\tau}_k^d||^2\,, \tag{32}$$

has an equilibrium if and only if there exists q^0 such that $g_k(q^0) = \tau_k^r$ and, in that case, if $J_k(q^0)$ is nonsingular then the equilibrium is exponentially stable in the state-space of $(q, \dot{q})$. More generally, the system:

$$\ddot{q} = \operatorname{argmin} ||\dot{\eta}_k - A_k \eta_k||^2\,, \tag{33}$$

where A_k is stable, has an equilibrium if and only if there exists q^0 such that $g_k(q^0) = \tau_k^r$ and, in that case, if $J_k(q^0)$

is nonsingular then the equilibrium is exponentially stable in the state space of $(q, \dot{q})$.

Proof. Let us denote $\xi = (q, \dot{q})$ the state of the system (33). The variable ξ is related to η_k through the nonlinear "forward kinematics" mapping

$$\gamma_k : \xi \mapsto \eta_k = \gamma_k(\xi) = \begin{pmatrix} g_k(q) - \tau_k^r \\ J_k(q)\dot{q} \end{pmatrix}. \tag{34}$$

Let $\mathcal{J}_k(\xi)$ denote the Jacobian matrix of that mapping at ξ. From (34) it appears that $\mathcal{J}_k(\xi)$ is related to $J_k(q)$ through the following relation:

$$\mathcal{J}_k(\xi) = \begin{pmatrix} J_k(q) & 0 \\ \frac{\partial [J(q)\dot{q}]}{\partial q} & J_k(q) \end{pmatrix}. \tag{35}$$

From Proposition 1, the system (33) is equivalent to

$$\dot{\eta}_k = A_k \eta_k \,, \tag{36}$$

which has an equilibrium if and only if there exists q^0 such that $\eta_k = 0$, i.e. such that $g_k(q^0) = \tau_k^r$. In terms of ξ, (36) translates into the nonlinear *descriptor* system

$$\mathcal{J}_k(\xi)\dot{\xi} = A_k \gamma_k(\xi)\,. \tag{37}$$

Let $\xi^0 = (q^0, 0)$. Since $n = n_k$ and $J_k(q^0)$ is nonsingular, we can see from (35) that $\mathcal{J}_k(\xi^0)$ is a square $2n \times 2n$ lower block triangular matrix with $\operatorname{rank} \mathcal{J}_k(\xi^0) = \operatorname{rank} J_k(q^0) + \operatorname{rank} J_k(q^0) = 2n$, therefore $\mathcal{J}_k(\xi^0)$ is also nonsingular. Assuming that the forward kinematics mapping is continuously differentiable, then the mapping $\mathfrak{J} : \xi \mapsto \mathcal{J}_k(\xi)$ is continuous, and as such the inverse image of any open set of $\mathbb{R}^{2n\times 2n}$ under $\mathfrak{J}$ is open. Since the $GL_{2n}(\mathbb{R})$ group is an open subset of $\mathbb{R}^{2n\times 2n}$, $\mathfrak{J}^{-1}\big(GL_{2n}(\mathbb{R})\big)$ is an open set containing ξ^0, therefore there exists a neighborhood V of ξ^0 included in $\mathfrak{J}^{-1}\big(GL_{2n}(\mathbb{R})\big)$. Finally, for any $\xi \in V$, $\mathcal{J}_k(\xi) = \mathfrak{J}(\xi) \in GL_{2n}(\mathbb{R})$, and hence, in that neighborhood V, the descriptor system (37) takes the form of the nonlinear dynamical system:

$$\dot{\xi} = \mathcal{J}_k(\xi)^{-1} A_k \gamma_k(\xi)\,, \tag{38}$$

or, denoting ϕ_k the mapping $\phi_k : \xi \mapsto \mathcal{J}_k(\xi)^{-1} A_k \gamma_k(\xi)$,

$$\dot{\xi} = \phi_k(\xi)\,. \tag{39}$$

Before calculating the Jacobian of ϕ_k at ξ^0 in order to apply Lyapunov's indirect method, we introduce the following matrix:

$$\Gamma_k = D\mathcal{J}_k(\xi) = \frac{\partial \operatorname{vec} \mathcal{J}_k}{\partial \xi}\,. \tag{40}$$

We have (we drop the dependencies on ξ when there is no ambiguity):

$$\begin{aligned} d\phi_k &= d[\mathcal{J}_k(\xi)^{-1} A_k \gamma_k(\xi)]\,, && (41)\\ &= d\mathcal{J}_k(\xi)^{-1} A_k \gamma_k + \mathcal{J}_k^{-1} A_k d\gamma_k(\xi)\,. && (42) \end{aligned}$$

Then

$$\begin{aligned} d\mathcal{J}_k(\xi)^{-1} A_k \gamma_k &= \operatorname{vec}\left[d\mathcal{J}_k(\xi)^{-1} A_k \gamma_k\right]\,, && (43)\\ &= \left(\gamma_k^T A_k^T \otimes I_{2n_k}\right) \operatorname{vec} d\mathcal{J}_k(\xi)^{-1}\,, && (44) \end{aligned}$$

and by (148)

$$\begin{aligned} \operatorname{vec} d\mathcal{J}_k(\xi)^{-1} &= \operatorname{vec}\left[-\mathcal{J}_k^{-1} d\mathcal{J}_k(\xi) \mathcal{J}_k^{-1}\right]\,, && (45)\\ &= -\left(\mathcal{J}_k^{-T} \otimes \mathcal{J}_k^{-1}\right) \operatorname{vec} d\mathcal{J}_k(\xi)\,, && (46)\\ &= -\left(\mathcal{J}_k^{-T} \otimes \mathcal{J}_k^{-1}\right) \Gamma_k d\xi\,. && (47) \end{aligned}$$

We also have

$$d\gamma_k(\xi) = \mathcal{J}_k d\xi. \tag{48}$$

Plugging (44), (47) and (48) into (42) yields

$$\begin{aligned} d\phi_k = \big[-\left(\gamma_k^T A_k^T \otimes I_{2n_k}\right)\left(\mathcal{J}_k^{-T} \otimes \mathcal{J}_k^{-1}\right)\Gamma_k \\ + \mathcal{J}_k^{-1} A_k \mathcal{J}_k\big]\, d\xi\,, \end{aligned} \tag{49}$$

and, therefore, we get the expression of the Jacobian of ϕ_k:

$$\frac{\partial \phi_k}{\partial \xi} = -\left(\gamma_k^T A_k^T \otimes I_{2n_k}\right)\left(\mathcal{J}_k^{-T} \otimes \mathcal{J}_k^{-1}\right)\Gamma_k + \mathcal{J}_k^{-1} A_k \mathcal{J}_k\,. \tag{50}$$

At ξ^0 we have $\gamma_k(\xi^0) = 0$, and (50) simplifies into

$$\left.\frac{\partial \phi_k}{\partial \xi}\right|_{\xi^0} = \mathcal{J}_k(\xi^0)^{-1} A_k \mathcal{J}_k(\xi^0)\,, \tag{51}$$

which has the same eigenvalues as A_k. From Lyapunov's indirect method [19, Theorem 1 p. 246], [22, Corollary 4.3 p. 166] we conclude that (39) is exponentially stable. □

In the multi-task case we also analytically linearize the system in the $(q, \dot{q})$ state space. In what follows, we require that the tasks together span the state space of the system, i.e. more formally that the matrix $B(\xi)$ in (52) below is always positive definite. One practical way to ensure this condition is that at least one of the tasks k_0 is a full-configuration task $\tau_{k_0}(q) = q$, no matter how infinitesimally small its weight w_{k_0} is, as long as it remains positive $w_{k_0} > 0$. This is a non-restrictive assumption following the analysis in Section III-B.

Lemma 2. *If one of the tasks is a full-configuration task then for all* ξ *the matrix*

$$B(\xi) = \sum_{k=1}^{p} w_k \mathcal{J}_k(\xi)^T \mathcal{J}_k(\xi)\,, \tag{52}$$

is nonsingular.

Proof. $B(\xi)$ is clearly a symmetric positive matrix. Since one of the tasks τ_{k_0} is a full-configuration task $\tau_{k_0}(q) = q$, we have $J_{k_0}(q) = I_n$ and from (35) $\mathcal{J}_{k_0}(\xi) = I_{2n}$, therefore

$$B(\xi) = w_{k_0} I_{2n} + \sum_{\substack{k=1\\k\neq k_0}}^{p} w_k \mathcal{J}_k(\xi)^T \mathcal{J}_k(\xi)\,. \tag{53}$$

Since $w_{k_0} > 0$, $B(\xi)$ is positive definite and thus nonsingular. □

Proposition 4. *Let us suppose* $B(\xi) > 0$ *(e.g. under the conditions of Lemma 2). The system*

$$\dot{\xi} = \operatorname{argmin} \sum_{k=1}^{p} w_k ||\dot{\eta}_k - A_k \eta_k||^2\,, \tag{54}$$

has an equilibrium if and only if there exists ξ^0 such that

$$\sum_{k=1}^{p} w_k \mathcal{J}_k(\xi^0)^T A_k \gamma_k(\xi^0) = 0\,. \tag{55}$$

In that case, the equilibrium is exponentially stable if and only if the matrix

$$B^{-1} \sum_{k=1}^{p} w_k \Big(\left(\gamma_k^T A_k^T \otimes I_{2n_k}\right) K_{2n_k 2n} \Gamma_k + \mathcal{J}_k^T A_k \mathcal{J}_k \Big)\,, \tag{56}$$

evaluated at ξ^0 is stable.

Proof. The first order optimality condition for (54) is:

$$\frac{\partial}{\partial \dot{\xi}} \left[\sum_{k=1}^{p} w_k ||\dot{\eta}_k - A_k \eta_k||^2 \right] = 0\,, \tag{57}$$

$$\Leftrightarrow \sum_{k=1}^{p} 2\, w_k \mathcal{J}_k^T (\dot{\eta}_k - A_k \eta_k) = 0\,, \tag{58}$$

$$\Leftrightarrow \left[\sum_{k=1}^{p} w_k \mathcal{J}_k^T \mathcal{J}_k \right] \dot{\xi} = \sum_{k=1}^{p} w_k \mathcal{J}_k^T A_k \eta_k\,. \tag{59}$$

$B(\xi)$ being nonsingular, (59) takes the form of the nonlinear system:

$$\dot{\xi} = B(\xi)^{-1} \sum_{k=1}^{p} w_k \mathcal{J}_k(\xi)^T A_k \gamma_k(\xi)\,, \tag{60}$$

which admits an equilibrium if and only if there exists ξ^0 such that:

$$\sum_{k=1}^{p} w_k \mathcal{J}_k(\xi^0)^T A_k \gamma_k(\xi^0) = 0\,. \tag{61}$$

Let us linearize (60) around such an equilibrium. To do this we calculate the Jacobian of the mapping $\psi : \xi \mapsto B(\xi)^{-1} \sum_{k=1}^{p} w_k \mathcal{J}_k(\xi)^T A_k \gamma_k(\xi)$ using the differential-based treatment introduced in the Appendix. We have (dropping again the dependencies on ξ when appropriate):

$$d\psi = dB(\xi)^{-1} \sum_{k=1}^{p} w_k \mathcal{J}_k^T A_k \gamma_k + B^{-1} \sum_{k=1}^{p} w_k \left[d\mathcal{J}_k(\xi)^T A_k \gamma_k + \mathcal{J}_k^T A_k d\gamma_k(\xi) \right]\,. \tag{62}$$

Let us calculate each term of the right-hand side of (62) separately. To shorten the expressions let C denote the vector $C(\xi) = \sum_{k=1}^{p} w_k \mathcal{J}_k(\xi)^T A_k \gamma_k(\xi)$. We have, by (148),

$$\begin{aligned} dB(\xi)^{-1} C &= -B^{-1} dB(\xi) B^{-1} C\,, && (63)\\ &= \operatorname{vec}\left[-B^{-1} dB(\xi) B^{-1} C \right]\,, && (64)\\ &= -\left(C^T B^{-T} \otimes B^{-1} \right) \operatorname{vec} dB(\xi)\,, && (65) \end{aligned}$$

where

$$\begin{aligned} \operatorname{vec} dB(\xi) &= d \operatorname{vec} B(\xi)\,, && (66)\\ &= \sum_{k=1}^{p} w_k d \operatorname{vec} \mathcal{J}_k(\xi)^T \mathcal{J}_k(\xi)\,, && (67) \end{aligned}$$

and by (150)

$$d \operatorname{vec} \mathcal{J}_k^T \mathcal{J}_k = \Big(I_{4n_k^2} + K_{2n_k 2n_k} \Big) \left(\mathcal{J}_k \otimes I_{2n_k} \right) d \operatorname{vec} \mathcal{J}_k\,, \tag{68}$$

with

$$d \operatorname{vec} \mathcal{J}_k(\xi) = \Gamma_k d\xi\,. \tag{69}$$

This gives us the first term in (62) as

$$dB(\xi)^{-1} C = -\left(C^T B^{-T} \otimes B^{-1} \right) \sum_{k=1}^{p} w_k \Big(I_{4n_k^2} + K_{2n_k 2n_k} \Big) \left(\mathcal{J}_k \otimes I_{2n_k} \right) \Gamma_k d\xi\,. \tag{70}$$

As for the other two terms we write, applying (149) for (73):

$$\begin{aligned} d\mathcal{J}_k(\xi)^T A_k \gamma_k &= \operatorname{vec}\left[d\mathcal{J}_k(\xi)^T A_k \gamma_k \right]\,, && (71)\\ &= \left(\gamma_k^T A_k^T \otimes I_{2n_k} \right) \operatorname{vec} d\mathcal{J}_k(\xi)^T\,, && (72)\\ &= \left(\gamma_k^T A_k^T \otimes I_{2n_k} \right) K_{2n_k 2n} \operatorname{vec} d\mathcal{J}_k(\xi)\,, && (73)\\ &= \left(\gamma_k^T A_k^T \otimes I_{2n_k} \right) K_{2n_k 2n} \Gamma_k d\xi\,, && (74) \end{aligned}$$

and finally the last term

$$\mathcal{J}_k^T A_k d\gamma_k(\xi) = \mathcal{J}_k^T A_k \mathcal{J}_k d\xi\,. \tag{75}$$

Plugging (70), (74) and (75) into (62) gives us

$$d\psi = \left[-\left(C^T B^{-T} \otimes B^{-1} \right) \sum_{k=1}^{p} w_k \Big(I_{4n_k^2} + K_{2n_k 2n_k} \Big) \left(\mathcal{J}_k \otimes I_{2n_k} \right) \Gamma_k + B^{-1} \sum_{k=1}^{p} w_k \Big(\left(\gamma_k^T A_k^T \otimes I_{2n_k} \right) K_{2n_k 2n} \Gamma_k + \mathcal{J}_k^T A_k \mathcal{J}_k \Big) \right] d\xi\,, \tag{76}$$

from which we get the desired analytic expression of the Jacobian of the mapping ψ:

$$\frac{\partial \psi}{\partial \xi} = -\left(C^T B^{-T} \otimes B^{-1} \right) \sum_{k=1}^{p} w_k \Big(I_{4n_k^2} + K_{2n_k 2n_k} \Big) \left(\mathcal{J}_k \otimes I_{2n_k} \right) \Gamma_k + B^{-1} \sum_{k=1}^{p} w_k \Big(\left(\gamma_k^T A_k^T \otimes I_{2n_k} \right) K_{2n_k 2n} \Gamma_k + \mathcal{J}_k^T A_k \mathcal{J}_k \Big)\,. \tag{77}$$

At the equilibrium ξ^0 we have from (61) $C(\xi^0) = 0$, hence (77) simplifies into

$$\left. \frac{\partial \psi}{\partial \xi} \right|_{\xi^0} = B^{-1} \sum_{k=1}^{p} w_k \Big(\left(\gamma_k^T A_k^T \otimes I_{2n_k} \right) K_{2n_k 2n} \Gamma_k + \mathcal{J}_k^T A_k \mathcal{J}_k \Big)\,. \tag{78}$$

Thus, the equilibrium ξ^0 is exponentially stable if and only if this latter matrix is stable. □

Corollary 4. *If the tasks τ_k are ultimately realizable simultaneously, i.e. if there exists ξ^0 such that $\forall k \in \{1, \ldots, p\}$ $\gamma_k(\xi^0) = 0$, then ξ^0 is an equilibrium of (54). In that case, a sufficient condition for ξ^0 to be exponentially stable is that the matrices $A_k + A_k^T$ are negative definite.*

Proof. If $\forall k \in \{1,\ldots,p\}$ $\gamma_k(\xi^0)=0$ then (55) holds, and by Proposition 4, ξ^0 is an equilibrium point of (54). Moreover, in that case, (56) simplifies into

$$\left[\sum_{k=1}^{p} w_k \mathcal{J}_k^T \mathcal{J}_k\right]^{-1} \sum_{k=1}^{p} w_k \mathcal{J}_k^T A_k \mathcal{J}_k = B^{-1}\mathcal{A}\,, \tag{79}$$

where we denoted

$$\mathcal{A} = \sum_{k=1}^{p} w_k \mathcal{J}_k^T A_k \mathcal{J}_k\,. \tag{80}$$

If we additionally suppose that $A_k + A_k^T$ are negative definite, then $\mathcal{A}+\mathcal{A}^T$ is also negative definite since

$$\mathcal{A}+\mathcal{A}^T = w_{k_0}\left(A_{k_0}+A_{k_0}^T\right) + \sum_{\substack{k=1\\k\neq k_0}}^{p} w_k \mathcal{J}_k^T\left(A_k+A_k^T\right)\mathcal{J}_k\,, \tag{81}$$

with $w_{k_0}\left(A_{k_0}+A_{k_0}^T\right)$ negative definite (since $w_{k_0}>0$) and $\forall k\neq k_0$ $w_k\mathcal{J}_k^T\left(A_k+A_k^T\right)\mathcal{J}_k$ negative.

Furthermore, B being positive definite, $\mathcal{B}=B^{-1}$ is also symmetric positive definite. Any matrix congruent to a negative definite matrix is also a negative definite matrix and hence $\mathcal{B}\left(\mathcal{A}^T+\mathcal{A}\right)\mathcal{B}^T$ is negative definite. And given that

$$\begin{aligned}\mathcal{B}\left(\mathcal{A}^T+\mathcal{A}\right)\mathcal{B}^T &= \mathcal{B}\left(\mathcal{A}^T+\mathcal{A}\right)\mathcal{B}\,, \quad (\mathcal{B}\text{ symmetric})\,, && (82)\\ &= \mathcal{B}\mathcal{A}^T\mathcal{B}+\mathcal{B}\mathcal{A}\mathcal{B}\,, && (83)\\ &= \mathcal{B}\left(\mathcal{B}\mathcal{A}\right)^T + \left(\mathcal{B}\mathcal{A}\right)\mathcal{B}\,, && (84)\end{aligned}$$

then the pair of positive definite matrices $\mathcal{Q} = -\mathcal{B}\left(\mathcal{A}^T+\mathcal{A}\right)\mathcal{B}^T$ and $\mathcal{P}=\mathcal{B}$ satisfy Lyapunov equation $\mathcal{P}(\mathcal{B}\mathcal{A})^T + (\mathcal{B}\mathcal{A})\mathcal{P} = -\mathcal{Q}$. Therefore, $\mathcal{B}\mathcal{A} = B^{-1}\mathcal{A}$ is stable. By Proposition 4 we conclude that ξ^0 is exponentially stable. □

Remark 1. *The terms $\left(\gamma_k^T A_k^T \otimes I_{2n_k}\right)K_{2n_k 2n}\Gamma_k$ can all be ignored in the expression of matrix (56) if and only if the tasks are all achievable simultaneously. When the tasks conflict and the equilibrium is a compromise between them, then these terms cannot be ignored and the full expression of (56) has to be considered for evaluating the stability of the system.*

V. Application to Humanoid Multi-Task Control

In this section we determine the nature of the control decision variable x and characterize the constraint set $\mathcal{X}$ in the humanoid control application case. We also cast the problem (29) as a linearly constrained QP inspired by approaches in the literature [33]–[37] (see the discussion at the end of Section V-C) and show some of its stability properties in the sense of existence, uniqueness, continuity, and robustness of its solution (that is, a "stability" sense different from the "Lyapunov stability" sense in Section IV).

A. Physical and Mathematical Constraints

Constraints of the humanoid robot motion include its equation of motion, the non-slipping contact constraints (e.g. at the feet surfaces), the corresponding Coulomb friction constraints, and various bounds on the applicable torques, admissible ranges of joint angles, joint velocities, and collision-avoidance.

The equation of motion of a humanoid robot in a given contact phase is usually written:

$$M(q)\ddot{q} + N(q,\dot{q}) = Su + J^c(q)^T\lambda\,, \tag{85}$$
$$J^c(q)\ddot{q} + \dot{J}^c(q)\dot{q} = 0\,. \tag{86}$$

One additional constraint has to be appended to the system (85)-(86) and yet is often omitted in many existing treatments of the problem, that is the Coulomb friction cone constraint which then results into the following system:

$$M(q)\ddot{q} + N(q,\dot{q}) = Su + J^c(q)^T\lambda\,, \tag{87}$$
$$J^c(q)\ddot{q} + \dot{J}^c(q)\dot{q} = 0\,, \tag{88}$$
$$\lambda \in \mathcal{C}\,, \tag{89}$$

$\mathcal{C}$ denoting a Coulomb friction cone. Note that constraint (88) cannot be derived from any arbitrary holonomic constraint $h(q)=0$ that expresses the fixation of the contact (with $\frac{\partial h}{\partial q} = J^c$) . For example, for any such constraint $h(q)=0$, the constraint $||h(q)||^2=0$ would mathematically express the exact same constraint but would result in a different Jacobian and thus in Lagrange multipliers that would not satisfy the same mathematical relations.

In order for the constraint (89) to physically make sense, λ has to be the actual physical contact forces, not arbitrary constraint forces. For a point contact at a point a belonging to a planar surface $\mathcal{S}$ of the robot with normal $\nu_{\mathcal{S}}$, the physical contact force λ is associated with the constraint $J^a\dot{q}=0$ where J^a is the Jacobian such that $\dot{a}=J^a\dot{q}$. In that case the Coulomb friction cone takes the following form:

$$\mathcal{C}_{\mathcal{S}} = \left\{\lambda\in\mathbb{R}^3 \mid \langle\lambda,\nu_{\mathcal{S}}\rangle>0\,,\ ||\lambda-\langle\lambda,\nu_{\mathcal{S}}\rangle\nu_{\mathcal{S}}|| \leq \mu\langle\lambda,\nu_{\mathcal{S}}\rangle\right\}\,. \tag{90}$$

For distributed surface contact on a surface $\mathcal{S}$ we would have a *continuum* of forces and likewise constraints in a system of the form:

$$M(q)\ddot{q} + N(q,\dot{q}) = Su + \iint_{a\in\mathcal{S}} J^a(q)^T\lambda(a)d\mathcal{S}(a)\,, \tag{91}$$
$$\forall a\in\mathcal{S}\quad J^a(q)\ddot{q} + \dot{J}^a(q)\dot{q} = 0\,, \tag{92}$$
$$\forall a\in\mathcal{S}\quad \lambda(a)\in\mathcal{C}_{\mathcal{S}}\,, \tag{93}$$

This system can however be simplified according to the following theorem

Theorem 8 ([38, Proposition 1])**.** *If $\mathcal{S}$ is a convex polygon*

$$\mathcal{S} = \left\{\sum_{i=1}^{s}\alpha_i a_i \mid \sum_{i=1}^{s}\alpha_i = 1\right\}\,, \tag{94}$$

then we have the following equivalence

$$\begin{aligned}&\forall F\in\mathbb{R}^n:\\ &\quad \exists\lambda:\mathcal{S}\to\mathcal{C}_{\mathcal{S}}\ \textit{s.t.}\ F = \iint_{a\in\mathcal{S}} J^a(q)^T\lambda(a)d\mathcal{S}(a)\\ &\quad\quad\Leftrightarrow\\ &\quad \exists(\lambda_1,\ldots,\lambda_s)\in[\mathcal{C}_{\mathcal{S}}]^s\ \textit{s.t.}\ F = \sum_{i=1}^{s} J^{a_i}(q)^T\lambda_i\,. \end{aligned} \tag{95}$$

Additionally, if we stay under the conditions of Theorem 8, it is clear that

$$(92) \Leftrightarrow \forall i \in \{1,\dots,s\}\ J^{a_i}(q)\ddot{q} + \dot{J}^{a_i}(q)\dot{q} = 0\,, \tag{96}$$
$$\Leftrightarrow \qquad J^{\mathcal{S}}(q)\ddot{q} + \dot{J}^{\mathcal{S}}\dot{q} = 0\,, \tag{97}$$

where $J^{\mathcal{S}}$ denotes the rotational and translational Jacobian of any frame rigidly attached to $\mathcal{S}$. This latter remark together with Theorem 8 allows us to rewrite the continuum system of equations (91) to (93) in the following equivalent finite system form:

$$M(q)\ddot{q} + N(q,\dot{q}) = Su + \sum_{i=1}^{s} J^{a_i}(q)^T\lambda_i\,, \tag{98}$$
$$J^{\mathcal{S}}(q)\ddot{q} + \dot{J}^{\mathcal{S}}\dot{q} = 0 \tag{99}$$
$$\forall i \in \{1,\dots,s\} \quad \lambda_i \in \mathcal{C}_{\mathcal{S}}\,. \tag{100}$$

B. Structural Constraints

We write here the structural constraints using the weak componentwise order notation for vector inequalities as follows

$$u_{\min} \leqq u \leqq u_{\max}\,, \tag{101}$$
$$q_{\min} \leqq q \leqq q_{\max}\,, \tag{102}$$
$$\dot{q}_{\min} \leqq \dot{q} \leqq \dot{q}_{\max}\,, \tag{103}$$

and the collision avoidance between two bodies based on a velocity damper formulation

$$\dot{d} \geq -\kappa\frac{d-\delta_s}{\delta_i - \delta_s}\,, \tag{104}$$

where d denotes the distance between the two bodies and δ_i, δ_s, κ, respectively, an influence distance, a security distance, and a damping constant (see [39] for details on this particular formulation).

C. Casting the problem as a QP

In order to cast the problem as a QP we conservatively approximate the friction cone $\mathcal{C}_{\mathcal{S}}$ with an inscribed polyhedral cone $\hat{\mathcal{C}}_{\mathcal{S}}$ [40]. Let $\mathcal{C}$ denote the matrix of the set of the polyhedral cone generators' coordinates in the world frame, and let c denote the number of generators, $\mathcal{C} \in \mathbb{R}^{3\times c}$, then we have $\lambda \in \hat{\mathcal{C}}_{\mathcal{S}}$ if and only if $\exists\hat{\lambda} \in \mathbb{R}^c_{\geq}$ s.t. $\lambda = \mathcal{C}\hat{\lambda}$. The system (98) to (100) becomes:

$$M(q)\ddot{q} + N(q,\dot{q}) = Su + \sum_{i=1}^{s} J^{a_i}(q)^T\mathcal{C}\hat{\lambda}_i\,, \tag{105}$$
$$J^{\mathcal{S}}(q)\ddot{q} + \dot{J}^{\mathcal{S}}\dot{q} = 0\,, \tag{106}$$
$$\forall i \in \{1,\dots,s\} \quad 0 \leqq \hat{\lambda}_i\,. \tag{107}$$

We also rewrite the constraints (101) to (104) respectively as follows:

$$u_{\min} \leqq u \leqq u_{\max}\,, \tag{108}$$
$$\frac{\dot{q}_{\min}-\dot{q}}{\Delta t} \leqq \ddot{q} \leqq \frac{\dot{q}_{\max}-\dot{q}}{\Delta t}\,, \tag{109}$$
$$\frac{q_{\min}-q-\dot{q}\Delta t}{\frac{1}{2}\Delta t^2} \leqq \ddot{q} \leqq \frac{q_{\max}-q-\dot{q}\Delta t}{\frac{1}{2}\Delta t^2}\,, \tag{110}$$
$$\ddot{d} \geq \frac{1}{\Delta t}\left(-\xi\frac{d-\delta_s}{\delta_i-\delta_s} - \dot{d}\right)\,. \tag{111}$$

where Δt is a fixed parameter (e.g. control time-step). Finally we enforce the compactness of the feasible set by setting an arbitrarily large bound on $\hat{\lambda}$

$$\hat{\lambda} \leqq \hat{\lambda}_{\max}\,. \tag{112}$$

It can now be seen that setting the control decision variable as $x = (\ddot{q}, u, \hat{\lambda}) \in \mathbb{R}^{2n-6+s\cdot c}$, the set of equations and inequalities (105) to (112) defining the feasible set $\mathcal{X} \subset \mathbb{R}^{2n-6+s\cdot c}$ are linear in x, i.e. $\mathcal{X}$ is a an intersection of closed halfspaces. Let $H_e x = b_e$ denote the set of equations (105) and (106) and $H_i x \leqq b_i$ denote the set of inequalities (107) to (112).

$$\mathcal{X} = \left\{x = (\ddot{q}, u, \hat{\lambda}) \in \mathbb{R}^{2n-6+s\cdot c} \mid (105) \text{ to } (112)\right\}, \tag{113}$$
$$= \left\{x \in \mathbb{R}^{2n-6+s\cdot c} \mid H_e x = b_e,\ H_i x \leqq b_i\right\}. \tag{114}$$

Denoting the matrix

$$K(q) = \begin{pmatrix} J^{a_1}(q)^T\mathcal{C} & \cdots & J^{a_s}(q)^T\mathcal{C}\end{pmatrix} \in \mathbb{R}^{n\times s\cdot c}\,, \tag{115}$$

we have, in particular,

$$H_e = \begin{pmatrix} M(q) & -K(q) & -S \\ J^{\mathcal{S}}(q) & 0 & 0\end{pmatrix},\ b_e = \begin{pmatrix} -N(q,\dot{q}) \\ -\dot{J}^{\mathcal{S}}\dot{q}\end{pmatrix} \tag{116}$$

To the set of tasks $\tau_1,\dots,\tau_p$, of which we recall that the task τ_{k_0} is a full-configuration task $\tau_{k_0} = g_{k_0}(q) = q$, we append two additional components in the vector optimization problem (24):

$$\text{“}\min_{x\in\mathcal{X}}\text{”} f(x) = (||\ddot{\tau}_1 - \ddot{\tau}_1^d||^2, \dots, ||\ddot{\tau}_p - \ddot{\tau}_p^d||^2, ||u||^2, ||\hat{\lambda}||^2)\,. \tag{117}$$

We show now that the conditions of Theorem 6 hold. We shall invoke the following two theorems, reusing the notations of Section III:

Theorem 9 ([18, Proposition 2.1.22 p. 15])**.** *A sufficient condition for the $\mathbb{R}^p_{\geqq}$-convexity of $\mathcal{Y} = f(\mathcal{X})$ is that $\mathcal{X}$ is convex and the functions $f_1,\dots,f_p$ are convex.*

Theorem 10 ([18, Lemma 3.2.3 p. 52])**.** *Let Y^+ denote the extended recession cone of a set Y, defined as*

$$Y^+ = \left\{y' \mid \exists(\beta^k) \in \mathbb{R}^{\mathbb{N}},\ \exists(y^k) \in Y^{\mathbb{N}},\ \beta^k > 0, \right.$$
$$\left. s.t.\ \beta^k \xrightarrow[k\to+\infty]{} 0,\ \beta^k y^k \xrightarrow[k\to+\infty]{} y'\right\}. \tag{118}$$

Let Y_1 and Y_2 be two nonempty closed sets. If

$$Y_1^+ \cap (-Y_2^+) = \{0\}\,, \tag{119}$$

then $Y_1 + Y_2$ is closed.

We can now prove the following:

Proposition 5. *if $\mathcal{X}$ is nonempty then the conditions of Theorem 6 hold for the problem (117).*

Proof. We recall that in finite dimension compactness is equivalent to simultaneous closedness and boundedness. Since $\mathcal{X}$ is closed as the intersection of a finite number of closed halfspaces, and $\mathcal{X}$ is bounded by the constraints (107), (108), (109), (112), $\mathcal{X}$ is compact. f in (117) being continuous, $\mathcal{Y} = f(\mathcal{X})$ is therefore compact, which implies that it is closed and bounded.

The extended recession cone of a bounded set is $\{0\}$ by [18, Lemma 3.2.1 p. 52], thus $\mathcal{Y}^+ = \{0\}$, and hence $\mathcal{Y}^+ \cap (-\mathbb{R}^{(p+2)+}_{\geqq}) = \{0\}$. Since $\mathcal{Y}$ and $\mathbb{R}^p_{\geqq}$ are closed, by Theorem 10 $\mathcal{Y} + \mathbb{R}^{p+2}_{\geqq}$ is closed, i.e. $\mathcal{Y}$ is $\mathbb{R}^{p+2}_{\geqq}$-closed.

Moreover, $\mathcal{X}$ is convex as the intersection of a finite number of closed halfspaces which are convex sets, and the functions $f_1, \ldots, f_{p+2}$ in (117) are convex, then by Theorem 9 $\mathcal{Y}$ is $\mathbb{R}^{p+2}_{\geqq}$-convex. □

With Proposition 5 we can now safely consider the weighted-sum scalarization of (117) with strictly positive weights $0 < w$ without sacrificing the completeness of all the achievable task behaviors:

$$\min_{x\in\mathcal{X}} \sum_{k=1}^{p} w_k ||\ddot{\tau}_k - \ddot{\tau}_k^d||^2 + w_{p+1}||u||^2 + w_{p+2}||\hat{\lambda}||^2 \,. \quad (120)$$

Problem (120) is a quadratic program of the form:

$$\begin{aligned} &\min_{x} \ x^T Q x + l^T x, \\ &\text{subject to } H_e x = b_e,\ H_i x \leqq b_i\,, \end{aligned} \quad (121)$$

where, in particular:

$$Q = \begin{pmatrix} \sum_{k=1}^p w_k J_k^T J_k & 0 & 0 \\ 0 & w_{p+1} I_{n-6} & 0 \\ 0 & 0 & w_{p+2} I_{s\cdot c} \end{pmatrix} \,. \quad (122)$$

Different variants of the formulation (120) and (121) were originally derived in the literature, e.g. [33, Eq. (5)], [34, Fig. 4 and Eq. (16)], [36, Eq. (20)], [37, Eq. (20)]. All these formulations can be seen as somewhat equivalent, with the later ones incorporating additional structural constraints and features (e.g. joint and velocity limits) that were absent from earlier ones, hence gradually becoming more complete and physically-consistent. Other differences between the various weighted multi-task QPs in the literature lie in the choice of the particular tasks or objectives, with for example [35, Eq. (13)] incorporating an angular momentum objective to control the center of pressure (CoP) position (although it uses a less accurate, penalty-based rather than constraint-based, contact model). However all these formulations can be fit in the general framework of (120) with, as such, differences in the particular formulation of the $\mathcal{X}$ constraint set and in the choices of the $||\ddot{\tau}_k - \ddot{\tau}_k^d||^2$ tasks.

The humanoid multiobjective QP formulation was later applied (or based upon) in the control architectures of different humanoid robots. [41, QP 5.1] used it in a control architecture for the HRP-2 robot. Many of the Atlas robot teams in the DARPA Robotics Challenge (2015) designed their control architectures based on a multiobjective QP formulation [42]. The WPI-CMU team used an equivalent formulation to the one we presented here [43, Eq. (1) and Sec. 5]. The IHMC team used a reduced and faster formulation with only centroidal dynamics rather than full-body dynamics (at the expense of not accounting for torque limits) and they also wrote the motion objectives as joint acceleration constraints [44, Fig. 1 and Eq. (21)]. Finally, the MIT team used a different formulation, which does not fit in the formulation (120), incorporating LQR-based CoM trajectory optimization directly in the QP, as an additional cost function objective along with the objectives considered here [45, Fig. 6 and QP 1]. However, their formulation was also inspired by the classical framework analyzed here (see the discussion in [45, Sec. 4.4]).

D. Stability of the QP

To conclude this section we study some stability properties of the QP (121). Note that the notion of "stability" we consider here is different from the one in Section IV, as we understand the term "stability" of the QP in the sense of 1) existence and uniqueness of a solution (Propositions 6 and 9), 2) robustness of the solution with respect to problem perturbations (Lemma 3 and Propositions 7 to 9), and 3) continuity of the solution of the QP with respect to its parameters (Corollary 5 to Proposition 9). This is the notion of stability we study here, that is complementary to the one studied in Section IV. Related work for a different control approach can be found for example in [32]. We are interested in the questions of existence, uniqueness and continuity of the solution, as well as robustness to perturbations and modeling uncertainties. We will take as a first assumption the nonemptiness of $\mathcal{X}$ (i.e. the feasibility of the problem) at a given initial state ξ^0. Other assumptions we will make is the full row rank condition of the matrix H_e in (116), i.e. $\operatorname{rank} H_e = n + 6$, and the *regularity* of the system

$$H_e x = b_e,\ \ H_i x \leqq b_i\,. \quad (123)$$

Definition 9 ([46, Definition p. 755] [47, Definition p. 512]). *The system of equations and inequalities (123) is said to be regular if H_e has full row rank and there exists x such that $H_e x = b_e$ and $H_i x < b_i$.*

Lemma 3. *Q is symmetric positive definite. Moreover, for any perturbation resulting from the updating of the state $(q, \dot{q})$ or from uncertainty in the model, the perturbed matrix $Q + \delta Q$ remains positive definite.*

Proof. Isolating the configuration task τ_{k_0} in (122) we get:

$$Q = \begin{pmatrix} w_{k_0} I_n & 0 & 0 \\ 0 & w_{p+1} I_{n-6} & 0 \\ 0 & 0 & w_{p+2} I_{s\cdot c} \end{pmatrix} + \sum_{\substack{k=1\\k\neq k_0}}^{p} \begin{pmatrix} w_k J_k^T J_k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \,. \quad (124)$$

Since $0 < w$, we have in particular $w_{k_0}, w_{p+1}, w_{p+2} > 0$ and therefore Q is symmetric positive definite. The perturbations of the state and the model would affect only J_k for $k \neq k_0$ in the right-hand side of (124), with $(J_k + \delta J_k)^T (J_k + \delta J_k)$ remaining positive, and therefore $Q + \delta Q$ remains positive definite. □

Remark 2. *We can also show that $Q > 0$ from the less strong assumption of $B > 0$. Indeed $B > 0$ implies that $\sum_k w_k J_k^T J_k > 0$ ($\mathcal{J}_k$ being block triangular with both block diagonal terms being equal to J_k). This assumption amounts to the set of tasks spanning the joint space without necessarily requiring that one of the tasks is a full-configuration task.*

Proposition 6. *If $\mathcal{X}$ is nonempty then (121) reaches a minimum at a unique point, i.e. the solution exists and is unique.*

Proof. The set $\mathcal{X}$ being compact and the mapping $\mathfrak{F} : x \mapsto x^T Qx + l^T x$ being continuous, from the extreme value theorem (121) has a minimum. $\mathfrak{F}$ being strictly convex from Q positive definite by Lemma 3, the minimizer is unique. □

Proposition 7. *A sufficient condition for the full row rank condition of H_e is that $\operatorname{rank}\begin{pmatrix}K(q) & S\end{pmatrix} = n$ (i.e. the contact forces completely make up for the underactuation).*

Proof. Let $L(q) = \begin{pmatrix}-K(q) & -S\end{pmatrix}$. We have

$$n + 6 \geq \operatorname{rank} H_e = \operatorname{rank}\begin{pmatrix} M(q) & L(q) \\ J^{\mathbb{S}}(q) & 0 \end{pmatrix}, \tag{125}$$

$$\geq \operatorname{rank} L(q) + \operatorname{rank} J^{\mathbb{S}}(q), \tag{126}$$

$$= \operatorname{rank} L(q) + 6. \tag{127}$$

Therefore $\operatorname{rank} H_e = n + 6$ if $\operatorname{rank} L(q) = n$. □

Proposition 8. *Let x^0 denote the solution of (121) at an initial point ξ^0. If the system (123) is regular, then there exists $\epsilon_1 > 0$ and $\mathcal{K}_1 > 0$ such that, for any update of the state ξ or modeling error (in particular, in $M(q)$, $N(q, \dot{q})$, and the various Jacobians of the robot) the perturbed system*

$$(H_e + \delta H_e)x = b_e + \delta b_e,\ (H_i + \delta H_i)x \leqq b_i + \delta b_i, \tag{128}$$

remains solvable and regular for those perturbations $(\delta H_e, \delta H_i, \delta b_e, \delta b_i)$ such that

$$\left|\left|\begin{pmatrix}\delta H_e \\ \delta H_i\end{pmatrix}\right|\right| + \left|\left|\begin{pmatrix}\delta b_e \\ \delta b_i\end{pmatrix}\right|\right| \leq \epsilon_1, \tag{129}$$

and, denoting x any solution of (128) with $\delta x = x - x^0$, we have

$$||\delta x|| \leq \mathcal{K}_1 \left(\left|\left|\begin{pmatrix}\delta H_e \\ \delta H_i\end{pmatrix}\right|\right| + \left|\left|\begin{pmatrix}\delta b_e \\ \delta b_i\end{pmatrix}\right|\right|\right) \max\{1, ||x^0||\}(1 + ||x^0||). \tag{130}$$

Proof. This is a direct application of [47, Corollary 7] since the conditions of the latter Corollary are all satisfied in the present case. [47, Corollary 7] is itself a direct consequence of the original work of Robinson [46, Theorem 1]. See also the discussion in [48] and in [49]. □

Proposition 9. *Let $p = (\delta Q, \delta l, \delta H_e, \delta H_i, \delta b_e, \delta b_i)$ denote a perturbation of the QP (121). We suppose that H_e and $H_e + \delta H_e$ are both full row rank and that the system (123) is regular at the initial state ξ^0. Then there exists $\epsilon_2 > 0$ and $\mathcal{K}_2 > 0$ such that the solution $x^* = x^0 + \delta x$ of the perturbed QP*

$$\min_x\ x^T (Q + \delta Q)x + (l + \delta l)^T x,$$
$$\text{subject to } (H_e + \delta H_e)x = b_e + \delta b_e,\ (H_i + \delta H_i)x \leqq b_i + \delta b_i, \tag{131}$$

exists and is unique and satisfies, whenever $||p||_\infty < \epsilon_2$

$$||x^* - x^0|| < \mathcal{K}_2 ||p||_\infty. \tag{132}$$

Proof. Our aim here is to apply [50, Theorem 4.4]. We thus shall show that the hypotheses [50, Equations (3.1) to (3.4)] hold. First, we know that the conditions of Proposition 8 hold, thus the first conclusion we can draw from that Proposition is that there exists $\epsilon_1 > 0$ such that the system (128) is regular and solvable whenever (129) hold. Hence both feasible sets of (121) and (131) are nonempty under (129), which constitutes the first of the needed hypotheses. The other hypotheses are already satisfied by our assumptions and therefore we can apply [50, Theorem 4.4], from which we deduce that, under (129), there exist $\epsilon_1' > 0$ and $\mathcal{K}_2$ such that if $||p||_\infty < \epsilon_1'$ and x' is any solution that minimizes (131) we have $||x^0 - x'|| < ||p||_\infty$. From Lemma 3 $Q + \delta Q$ is positive definite and thus x' is unique and we denote it x^*. Take now

$$\epsilon_2 = \min\left\{\frac{\epsilon_1}{4}, \epsilon_1'\right\}. \tag{133}$$

We have

$$||p||_\infty < \epsilon_2 \quad \Rightarrow \quad \text{(129) and } ||p||_\infty < \epsilon_1'. \tag{134}$$

We finally conclude that if $||p||_\infty < \epsilon_2$ then $||x^* - x^0|| < \mathcal{K}_2 ||p||_\infty$. □

Corollary 5. *In the context and with the notations of Proposition 9 the mapping $p \mapsto x^*$ is well defined on a neighborhood of 0 and continuous at 0.*

Proof. Immediate from Proposition 9. □

VI. Experimental Validation

To illustrate our results, we applied the control scheme proposed in Section V to the humanoid robot HRP-4. The robot has to perform a whole-body reaching task with its right hand while keeping balance and sustaining feet contact with the ground, see Fig. 1. The corresponding video and more complex experiments can be found online at the url [51].

Fig. 1. Example experiment with the HRP-4 humanoid robot.

We use the controller formulation (120), or in an equivalent form the QP controller (121). Propositions 6 to 9 and Corollary 5 are hence applicable and as a result the controller outputs a continuous solution, producing a smooth motion as can be read in Figs. 2 to 4. We plot in these figures the feed-forward command sent to the robot with task-level feedback.

The robot has 56 degrees of freedom, including the degrees of freedom of the hand fingers, i.e. $n = 56$ and $q \in \mathbb{R}^{56}$. We define a set of $p = 3$ tasks for the robot: a right hand position task $\tau_{\text{hand}} \in \mathbb{R}^3$ to reach the desired workspace goal, a center-of-mass (COM) task $\tau_{\text{com}} \in \mathbb{R}^3$ to keep equilibrium while performing the task, and a full-configuration task $\tau_q = q \in \mathbb{R}^{56}$ for stability and redundancy resolution as required in Lemmas 2 and 3. For these three tasks we design attractor behaviors (3) with matrices $P_{\text{hand}} = k_{\text{hand}} I_3$, $D_{\text{hand}} = 2\sqrt{k_{\text{hand}}} I_3$, $P_{\text{com}} = k_{\text{com}} I_3$, $D_{\text{com}} = 2\sqrt{k_{\text{com}}} I_3$,

$P_q = k_q I_{56}$, $D_q = 2\sqrt{k_q} I_{56}$, and $(k_{\text{hand}}, k_{\text{com}}, k_q) = (2, 5, 5)$ (standard values we use in most our control scenarios, these do not require any specific fine tuning). These matrices allow us to derive the matrices A_{hand}, A_{com} and A_q respectively. Fig 2 shows the convergence behavior of the tasks along a subset of $3 \times 3 \times 3$ values of the weights as run on the robot HRP-4.

Using the Matlab function `logspace`, we discretize the weight space $(w_{\text{hand}}, w_{\text{com}}, w_q)$ in a $50 \times 50 \times 50$ grid ranging in logarithmic scale from 10^{-1} to 10^5 along each of the three dimensions, i.e. $(w_{\text{hand}}, w_{\text{com}}, w_q) \in \texttt{logspace}(-1, 5, 50)^3$. We then compute the matrices:

$$\begin{aligned}&\mathbb{R}^{112\times 112} \ni \Xi[w_{\text{hand}}, w_{\text{com}}, w_q] \\ &= \left[\sum_{k\in\{\text{hand},\text{com},q\}} w_k \mathcal{J}_k^T \mathcal{J}_k\right]^{-1} \sum_{k\in\{\text{hand},\text{com},q\}} w_k \mathcal{J}_k^T A_k \mathcal{J}_k\,,\end{aligned} \tag{135}$$

which are the forms of the matrix (56) in Proposition 4 when the tasks are achievable (this is the case here as the tasks were planned with a planner and a posture generator). Since at the equilibrium ξ^0 we have $\dot{q} = 0$, then the matrices $\mathcal{J}_k$ take here the forms

$$\mathcal{J}_k(\xi^0) = \begin{pmatrix} J_k(q^0) & 0 \\ 0 & J_k(q^0) \end{pmatrix}, k \in \{\text{hand}, \text{com}, q\}. \tag{136}$$

In order to evaluate the stability of the matrices Ξ, using the Matlab function `eig` we compute the eigenvalues of the matrices $\Xi[w_{\text{hand}}, w_{\text{com}}, w_q]$ that we plot in Fig 6. We then compute the maximum real part of the eigenvalues of each of these matrices, and plot them in Fig. 5. All the matrices are stable since their eigenvalues have all negative real parts. Hence by Proposition 4 the equilibrium point of the system (31), which corresponds to the closed-loop dynamical system resulting from the unconstrained version of the multiobjective optimization, is exponentially stable, giving hints on the Lyapunov stability of the QP (120). Finally, this result is validated by running the controller starting from 10 randomly sampled initial configurations in the upper body of the robot as displayed in Fig. 3.

Note that on the limitations side, we experienced numerical instability issues when the range of weights was extended to a ratio between the smallest and largest weight above 10^7. This is due the real optimization problem running on floating-point hardware and becoming ill-conditioned when that ratio becomes too large, and is an inherent limitation of the non-constructive pure existence proofs of the results in Section III-B, more aimed toward theoretical foundation of the proposed multi-task control approach.

VII. Conclusion

We have demonstrated that the essence of the multi-task control problem can be effectively captured by the multi-objective optimization formal framework. We discussed the pertinence of scalarizing the vector optimization problem as a weighted sum with positive weights and proved that the positive-weight scalarization does indeed satisfy a completeness property with respect to all the efficient solutions, the popular lexicographic solution being one of them. We studied Lyapunov stability of the feedback system resulting from such a weighted-sum scalarization scheme in the unconstrained optimization case and proposed some necessary and/or sufficient conditions for the exponential stability of the equilibrium points of the systems. Finally we applied the study to the particular case of the humanoid robot. We demonstrated that in that case the positive weighted-sum scalarization leads to a linearly-constrained positive definite quadratic problem that is stable (in the robustness and solution-guaranteed sense) and well-behaved under the stated regularity conditions.

Future work is dedicated to translating some of the non-constructive pure existence proofs of this paper, proposed essentially as theoretical foundation layers, into practical weight tuning algorithms, which constitutes an active topic of research. We also plan on extending the Lyapunov stability analysis to the feedback dynamical system resulting from a constrained multiobjective optimization formulation, with both equality and inequality constraints. This is still an open problem, and the contributions of the present paper will be used as the primary building blocks for that follow-up work.

Appendix
Matrix Differentiation Tools for Lyapunov's Indirect Method

We introduce a tool to efficiently differentiate $J_k(q)$ with respect to q, that can somewhat be termed the "Jacobian of the Jacobian" (which is not to be confused with the notion of a *Hessian matrix* that is only defined for scalar functions). Unfortunately the expression

$$\text{“}\,\frac{\partial J_k(q)}{\partial q}\,\text{”}\,, \tag{137}$$

does not make sense and is not properly defined, since it involves the differentiation of a matrix with respect to a vector. Magnus and Neudecker (1985) proposed to use the following quantity that is thoroughly consistent with all the properties of the classical differentiation frameworks (in particular with the chain rule, the notion of the Jacobian, and Cauchy's rule of invariance) [52]:

$$G_k = DJ_k(q) = \frac{\partial\, \mathrm{vec}\, J_k(q)}{\partial q}\,. \tag{138}$$

The vec operator denotes the vectorization operator, which consists for a matrix in stacking its columns as a vector, i.e.

$$\mathrm{vec}\begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix} = \begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \\ \vdots \\ \vdots \\ a_{1m} \\ \vdots \\ a_{nm} \end{pmatrix}. \tag{139}$$

Definition 10 ([53, Definition 3.1 p. 383])**.** *There exists a so-called commutation matrix K_{nm}, that is the $nm \times nm$*

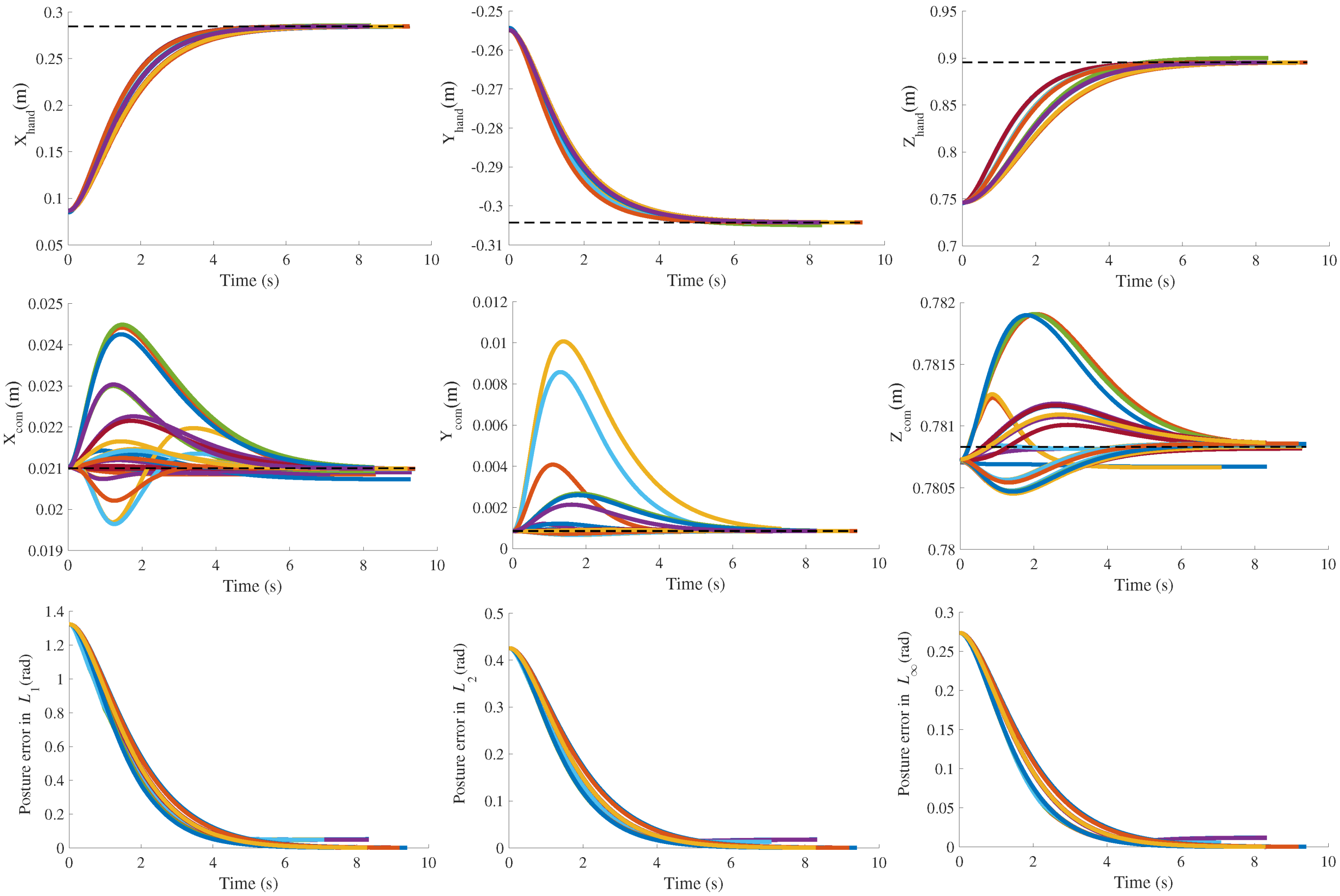

Fig. 2. Tasks and state convergence when varying weights. We plot the trajectories for 27 runs of the hand reaching experiment on the HRP-4 robot with different sets of weights to assess the results from Corollary 2 and Proposition 2 ($3 \times 3 \times 3$ values of $(w_{\text{hand}}, w_{\text{com}}, w_q) \in \{10^{-1}, 10^2, 10^3\}^3$). As predicted by Corollary 2, although the tasks do not necessarily always converge to zero (e.g. 2 runs do not yield a zero-converging Z_{com} and X_{com} error and 2 runs do not yield a zero converging posture error), yet there exists always a set of weights such that the acceleration error of the task is below any given precision ϵ, which in turn ensures the boundedness of the error or the convergence to zero of the error of the task by Proposition 2. The trajectories also illustrate the results from Section V (Propositions 6 to 9 and Corollary 5) as the QP controller outputs a continuous solution.

permutation matrix which transforms $\operatorname{vec} A^T$ *into* $\operatorname{vec} A$ *for any* $n \times m$ *matrix* A*, i.e.* $\forall A \in \mathbb{R}^{n \times m}$ $\operatorname{vec} A^T = K_{mn} \operatorname{vec} A$.

Denoting $\otimes$ the *Kronecker product*:

Theorem 11 ([54, Proposition 7.1.9 p. 401 and Fact 7.4.6 p. 405])**.** *For any vector* X *and matrices* A*,* B *and* C *such that* ABC *is defined we have*

$$\begin{aligned} X &= \operatorname{vec} X\,, && (140)\\ \operatorname{vec}(ABC) &= (C^T \otimes A) \operatorname{vec} B\,, && (141)\\ \operatorname{vec}(AB) &= (I \otimes A) \operatorname{vec} B\,, && (142)\\ \operatorname{vec}(AB) &= (B^T \otimes I) \operatorname{vec} A\,. && (143) \end{aligned}$$

Definition 11 ([52, Definition 5 p. 479])**.** *A matrix function* $F : S \subset \mathbb{R}^{n \times m} \to \mathbb{R}^{p \times q}$ *is differentiable at* $C \in \operatorname{int}(S)$ *if there exists a matrix* $A(C) \in \mathbb{R}^{mn \times pq}$ *such that, for* U *in a neighborhood of* 0 *in* $\mathbb{R}^{n \times m}$*, we have*

$$\operatorname{vec} F(C + U) = \operatorname{vec} F(C) + A(C) \operatorname{vec} U + o(||U||)\,. \quad (144)$$

If $A(C)$ *exists it is unique and the* $p \times q$ *matrix* $dF(C; U)$ *defined by*

$$\operatorname{vec} dF(C; U) = A(C) \operatorname{vec} U\,, \quad (145)$$

is called the differential of F *at* C *with increment* U*.*

Theorem 12 ([55, Theorem 11 p. 108])**.** *If* F *is differentiable at* C *then* $A(C)$ *defined in Definition 6 is the Jacobian of* $\operatorname{vec} F$ *with respect to* $\operatorname{vec} X$ *(*X *denoting the variable of* F*) that we also call the Jacobian of* F *at* X

$$A(C) = DF(C) = \left.\frac{\partial \operatorname{vec} F}{\partial \operatorname{vec} X}\right|_C\,. \quad (146)$$

Theorem 13 (Cauchy's rule of invariance [55, Theorem 13 p. 108])**.** *If* F *is differentiable at* C *and* G *is differentiable at* $B = F(C)$ *then* $H = G \circ F$ *is differentiable at* C *and*

$$dH(C; U) = dG(B; dF(C; U))\,. \quad (147)$$

Example 1 ([55, Theorem 3 p. 71 and Chapter 9 Section 13 pp. 205-208])**.** *The differentials of the mappings* $GL_n(\mathbb{R}) \to \mathbb{R}^n$ $X \mapsto X^{-1}$*;* $\mathbb{R}^{n \times m} \to \mathbb{R}^{m \times n}$*,* $X \mapsto X^T$*; and* $\mathbb{R}^{n \times m} \to \mathbb{R}^{n \times n}$*,* $X \mapsto X^T X$ *can be derived respectively as:*

$$\begin{aligned} d(X^{-1}) &= -X^{-1} dX X^{-1}\,, && (148)\\ d(X^T) &= K_{nm} dX\,, && (149)\\ d(X^T X) &= (K_{mm} + I_{m^2})\left(I_m \otimes X^T\right) dX\,. && (150) \end{aligned}$$

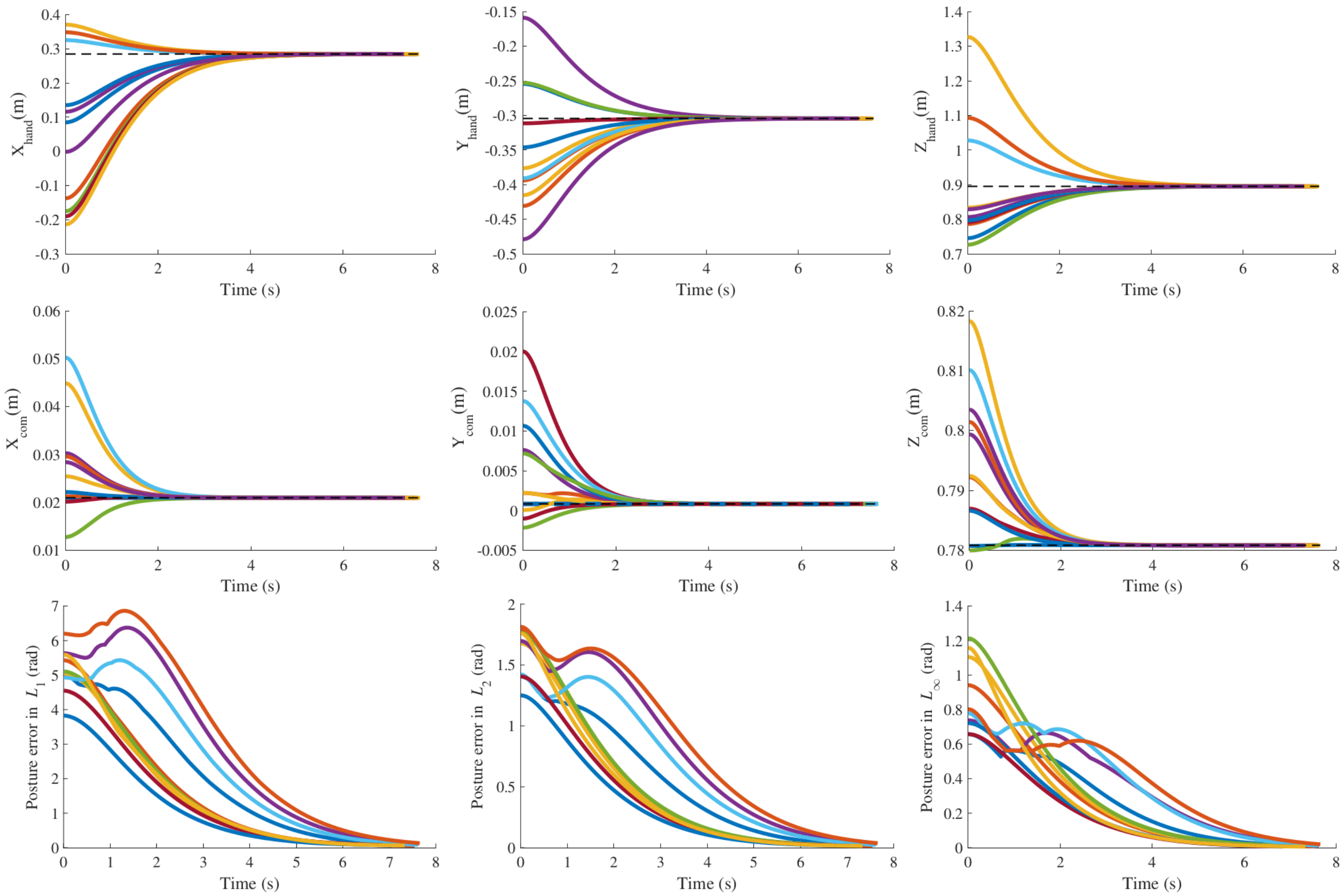

Fig. 3. Tasks and state convergence from random initial states to assess stability of the system (31) by Proposition 4. We plot the trajectories for 10 runs of the hand reaching experiment of the HRP-4 robot starting from 10 randomly sampled initial configurations in the upper-body of the robot (randomly sampled joint angles of the upper-body joints) for a fixed set of weights $(w_{\text{hand}}, w_{\text{com}}, w_q) = (10^3, 10^3, 10^{-1})$. The errors converge to zero from any of these initial random configurations which positively correlates to the stability of the matrices of Proposition 4 as shown in Figs. 5 and 6.

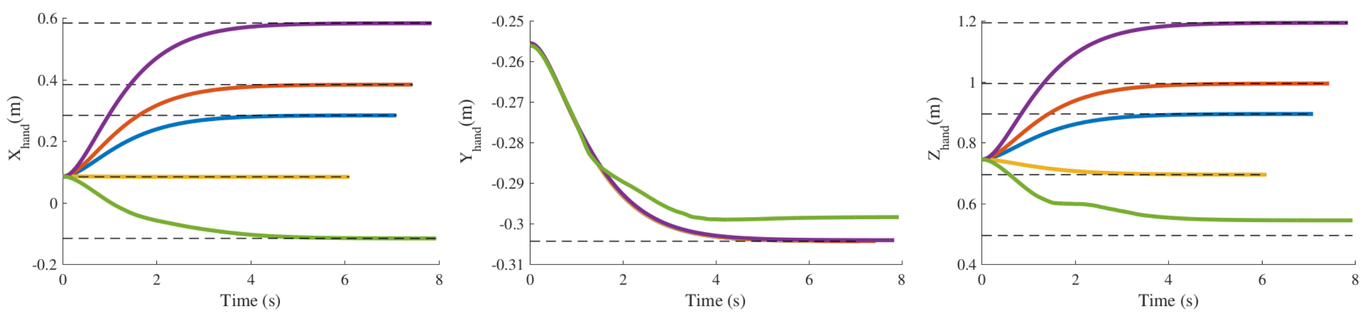

Fig. 4. Changing the task objective for the hand with a fixed set of weights $(w_{\text{hand}}, w_{\text{com}}, w_q) = (10^3, 10^3, 10^{-1})$. One of the positions was unachievable without compromising the equilibrium of the robot, which led to not realizing the task with that set of weights.

Hence by Cauchy's rule of invariance we can write for $J_k(q)$ seen as a function of q:

$$dJ_k(q)^{-1} = -J_k^{-1} dJ_k(q) J_k^{-1} \quad \textit{(J(q) nonsingular)}\,, \tag{151}$$

$$d(J_k(q)^T) = K_{n_k n} dJ_k(q)\,, \tag{152}$$

$$d(J_k(q)^T J_k(q)) = \left(K_{n_k n_k} + I_{n_k^2}\right)\left(I_{n_k} \otimes J_k^T\right) dJ_k(q)\,. \tag{153}$$

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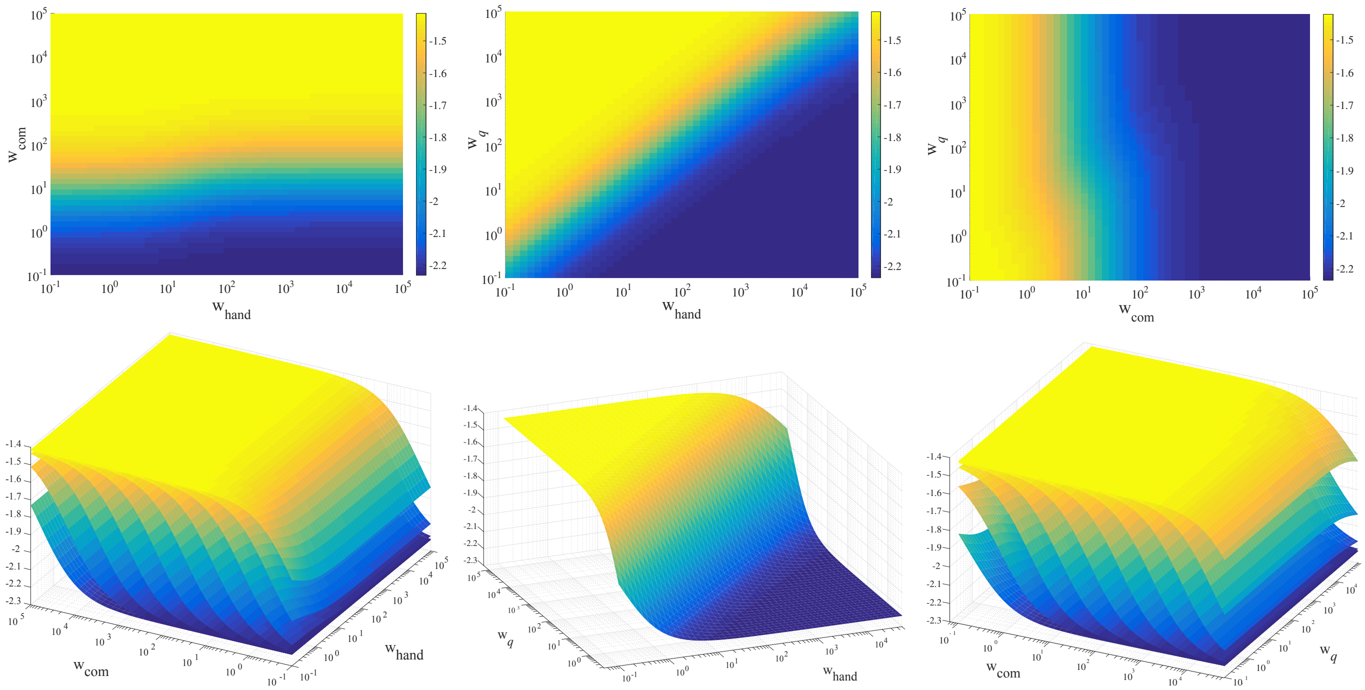

Fig. 5. Matrix stability from Proposition 4. We discretized the $(w_{\text{hand}}, w_{\text{com}}, w_q)$ space in $50 \times 50 \times 50$ grid in logarithmic scale ranging from 10^{-1} to 10^5 along each dimension of the weight vector $(w_{\text{hand}}, w_{\text{com}}, w_q) \in \texttt{logspace}(-1, 5, 50)^3$. We plot in color scale and surface plot the maximum real part of the eigenvalues of the matrix from Proposition 4. We see that this maximum is always negative, hence the matrices are stable. In the top row figures we plot the maximum real part of the eigenvalues along two dimensions of the weight vector for a given value of the third dimension (10^4). In the bottom row figures we plot the same data for 10 values in $\texttt{logspace}(-1, 5, 10)$ along the third dimension while the other two dimensions are in $\texttt{logspace}(-1, 5, 50)^2$ which result in 10 surfaces in each plot (the 10 surfaces are very close to each other in the middle column).

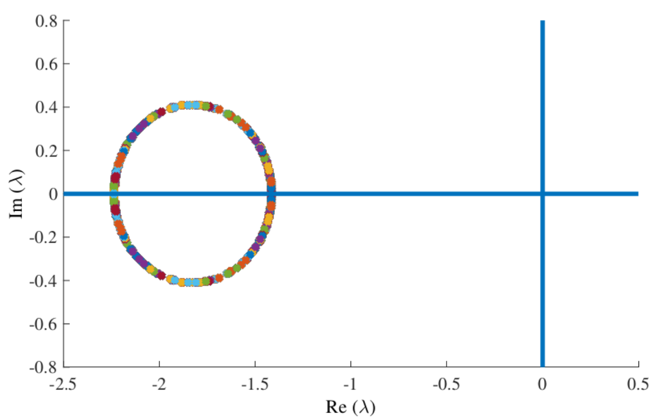

Fig. 6. All 112000 eigenvalues (counting multiplicities) of the 112x112 matrices from Proposition 4 for 1000 set of weights ranging in logarithmic scale from 10^{-1} to 10^5, i.e. $(w_{\text{hand}}, w_{\text{com}}, w_q) \in \texttt{logspace}(-1, 5, 10)^3$. All the eigenvalues are located in the left complex half plane which means they are stable.

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Karim Bouyarmane received the double Ingénieur diploma from the École Polytechnique in Palaiseau in 2007 and from the École Nationale Supérieure des Mines de Paris in 2008, and the PhD degree in Automatic and Microelectronic Systems from the University of Montpellier in 2011 after completing the PhD program for full-time in the Joint Robotics Laboratory (JRL) at the National Institute of Advanced Industrial Science and Technology (AIST) in Tsukuba, Japan. He was a Japan Society for the Promotion of Science (JSPS) post-doctoral fellow at the Advanced Telecommunications Research Institute International (ATR) in Kyoto, in the Computational Neuroscience Laboratories (CNS), department of Brain-Robot Interfaces (BRI). He also held a CNRS fixed-term researcher position at the Laboratory of Informatics, Robotics and Microelectronics of Montpellier (LIRMM). He is presently an Associate Professor (Maître de Conférences) at the Université of Lorraine, in the Lorraine Laboratory of Research in Informatics and Applications (LORIA).

Abderrahmane Kheddar received the BSCS degree from the Institut National d'Informatique (ESI), Algiers, the MSc and PhD degrees in robotics, both from the University of Pierre and Marie Curie, Paris 6. He is presently Directeur de Recherche at CNRS. He is the Director of the CNRS-AIST Joint Robotic Laboratory (JRL), UMI3218/RL, Tsukuba, Japan; and the leader of the Interactive Digital Humans (IDH) team at CNRS-UM LIRMM at Montpellier, France. His research interests include humanoid robotics, haptics, and thought-based control using brain machine interfaces. He is a founding member of the IEEE/RAS chapter on haptics, the co-chair and co-founding member of the IEEE/RAS Technical committee on model-based optimization. He is presently Editor of the IEEE Transactions on Robotics; he is a founding member of the IEEE Transactions on Haptics and served in its editorial board during three years (2007-2010). He is a titular member of the National Academy of Technologies of France (NATF), Knight in the National Order of Merit, and a Senior Member of the IEEE Society.