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***Abstract*—We propose a formal analysis with some theoretical properties of weight-prioritized multi-task inverse-dynamics-like control of humanoid robots, being a case of redundant "manipulators" with a non-actuated free-floating base and multiple unilateral frictional contacts with the environment. The controller builds on a weighted sum scalarization of a multiobjective optimization problem under equality and inequality constraints, which appears as a straightforward solution to account for state and control input viability constraints characteristic of humanoid systems, such as Coulomb friction cone and sustained unilateral contact constraints, torque saturation... that were usually absent from early existing pseudo-inverse and null-space projection-based prioritized multi-task approaches. We argue that our formulation is indeed well founded and justified from a theoretical standpoint and propose an analysis of some stability properties of the resulting closed-loop dynamical system based on the Lyapunov linearization method.**

***Index Terms*—Multiobjective optimization, weighted sum scalarization, multi-task control, matrix differentiation, Lyapunov linearization method, quadratic-program stability**

I. Introduction

APPLYING early control methods developed for (industrial) manipulators [1]–[3] to humanoid robots, e.g. inverse dynamics control, operational or task function space control... raises a number of challenging problems [4]–[9]. Typical such problems include simultaneous resolution of redundancy and underactuation, or actuation through friction-cone-constrained unilateral contact forces. Although each of these problems has already been extensively studied in the context of industrial manipulators or various general cases (see examples of treatments of redundancy in [10], [11], underactuation in [12], [13], constraints though contacts in [14]–[17], bounds on control inputs in [18], and references therein), the specificity of the humanoid robot case is that it features and interleaves them all at once, and thus renders the solutions that were proposed for each of these problems taken in a separate setting largely inapplicable in a unified control framework.

We propose to tackle these combined structural problems in a simple formulation in which we make the non-equivocal distinction between the two notions of *constraints* and *tasks*, a distinction that we believe should be made by/in any humanoid control law design at large. *Constraints* are inherent to the well-posedness of the problem, as failing to satisfy them results in a physically or mathematically ill-posed problem. These are the physics laws (Newton-Euler equations or Lagrange equations, Coulomb laws) and the safety and structural limits (torque saturation, joint angle and velocity limits, collision and obstacle avoidance). *Tasks*, on the other hand, allow for more tolerance in their non-fulfillment and necessitate a certain degree of "compliance" in their execution. Failing to realize them does not result in a mathematical or physical law violation. Since tasks come one way or another from *planning* (off-line or real-time), then it should be the role of the *planner*, not the controller, to ensure that the tasks are consistent and realizable [19].

Another important aspect in which humanoids differ essentially from industrial manipulators is their novel context of applications. An industrial manipulator is confined to a structured, known, and uncertainty-free environment. It is thus conceivable that in that setting tasks are seen as constraints that should be realized perfectly, moreso if the manipulator had been specifically designed for the task at hand. Humanoids, even when targeted to manufacturing[1], are neither customized to achieve a particular task nor do they evolve in a structured environment that was exclusively designed for their operations. As such, tasks shall have the flexibility to be set as constraints or as objectives to be realized at best given their actual structural constraints and the uncertain state of their environment.

In this paper we have taken a step back from what we already extensively achieve in experimental humanoid robotics. Firstly, we adapt in an original way, different from the recursive null space projection approach, the inverse dynamics control principles to general multi-task systems and to the "humanoid type of manipulator" in particular accounting for its redundant, underactuated, and constrained nature (e.g. walking stability). Secondly, and this constitutes our novel contribution with respect to existing work, we assess the foundations from a control theoretical perspective of such control schemes. In Section II we introduce concepts from the area of multiobjective optimization and show that they are suitable to treat the multi-task control problem (Section II-A), we then establish the completeness of the retained solution method to deal with the problem (Section II-B). In Section III we present results on the Lyapunov stability of the solution scheme (Section III-B) based on the matrix differentiation tools that we recall beforehand in Section III-A. Finally in Section IV we cast the problem as a linearly constrained quadratic program in the case of the humanoid robot and study its theoretical stability properties.

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II. Multi-Task Control as a Multiobjective Optimization Problem

A. General Concepts

Let us recall some concepts of multiobjective optimization (also known as multicriteria optimization, multiple criteria decision making, vector optimisation [20], [21]) and demonstrate some useful properties in our context of multi-task control.

Multiobjective optimisation studies the problem

$$\text{“}\min_{x\in\mathcal{X}}\text{”} f(x) = (f_1(x), \dots, f_p(x))\,, \tag{1}$$

where the $\min$ operator is put between quotation marks to emphasize that it is dependent on some specific optimality notion for vector values to be defined. The f_k functions are scalar functions and $\mathcal{X}$ is the feasible space (e.g. as defined by a set of constraints on x). A solution $x^* \in \mathcal{X}$ of (1) is called an *efficient* (or *Pareto-optimal*) solution if there is no $x \in \mathcal{X}$ such that $f(x) \leq f(x^*)$. The notation $y^1 \leq y^2$ denotes the *componentwise* order: $\forall k \in \{1,\dots,p\}\ y^1_k \leq y^2_k$ and $y^1 \neq y^2$, i.e. at least one inequality holds strictly $\exists i \in \{1,\dots,p\}\ y^1_i < y^2_i$. This notion of componentwise order is to be clearly distinguished from the *weak* componentwise order defined as $y^1 \leqq y^2$ if $\forall k \in \{1,\dots,p\}\ y^1_k \leq y^2_k$ and the *strict* componentwise order defined as $y^1 < y^2$ if $\forall k \in \{1,\dots,p\}\ y^1_k < y^2_k$. Let $\mathcal{Y} = f(\mathcal{X}) \subset \mathbb{R}^p$ denote the image of the feasible set. If x^* is an efficient solution of (1) then $y^* = f(x^*)$ is called a *nondominated* point of $\mathcal{Y}$. A point y^1 is said to dominate y^2 if $y^1 \leq y^2$. The set of all efficient solutions of (1) is denoted $\mathcal{X}_E$ and the set of all nondominated points of $\mathcal{Y}$ is denoted $\mathcal{Y}_N$ (sometimes referred to as the *Pareto-optimal front*). We denote

$$y^I = (\min_{x\in\mathcal{X}} f_1(x), \dots, \min_{x\in\mathcal{X}} f_p(x))\,, \tag{2}$$

the so-called *ideal point*. In general the ideal point is not realizable $y^I \notin \mathcal{Y}$ (note that if $y^I \in \mathcal{Y}$ then $\mathcal{Y}_N = \{y^I\}$), in that case any point in $\mathcal{Y}_N$ can be seen as a non improvable compromise solution of (1).

In a context of multi-task control with p tasks, each task τ_k is defined through a forward kinematics function $g_k : \mathbb{R}^n \to \mathbb{R}^{n_k}$, mapping the n-dimensional generalized coordinates of the system q to the n_k-dimensional value of the task $\tau_k = g_k(q)$ ($n \geq n_k$). A task is associated with a planned reference trajectory $t \mapsto \tau^r_k(t)$ and an objective *attractor* behaviour to realize *exponential* tracking of the reference trajectory, denoting $e_k = \tau_k - \tau^r_k$, the attractor behavior takes the form

$$\ddot{e}_k + D_k\dot{e}_k + P_k e_k = 0\,, \tag{3}$$

where the matrices (P_k, D_k) are so that $A_k = \begin{pmatrix} 0 & I_{n_k} \\ -P_k & -D_k \end{pmatrix}$ is *stable* (i.e. has all its eigenvalues with negative real parts). More generally, denoting the task error state space variable $\eta_k = \begin{pmatrix} e_k \\ \dot{e}_k \end{pmatrix}$, the reference behavior is of the form $\dot{\eta}_k = A_k\eta_k$ where $A_k \in \mathbb{R}^{2n_k\times 2n_k}$ is stable. However, some results of the paper will be stated under the assumption of the negative definiteness of $A_k + A_k^T$, we recall the following relation between the two properties:

Theorem 1. *$A_k + A_k^T$ negative definite is a sufficient condition for A_k stable.*

Proof. If $A_k + A_k^T$ is negative definite, then The pair $Q = -(A_k + A_k^T)$, $P = I_{n_k}$ satisfies the Lyapunov equation $A_k^T P + PA_k = -Q$ with P and Q positive definite, therefore A_k is stable. □

For convenience of notation the behavior (3) can also be written in the form

$$\ddot{\tau}_k - \ddot{\tau}^d_k = 0\,, \tag{4}$$

with the desired task acceleration $\ddot{\tau}^d_k = \ddot{\tau}^r_k - D_k\dot{e}_k - P_k e_k$. If the constraints of the robot make it impossible to achieve perfect realization of $\ddot{\tau}^d_k$, then one might want to realize this behavior “at best” in the following sense

$$\min_{x\in\mathcal{X}} ||\ddot{\tau}_k - \ddot{\tau}^d_k||^2\,, \tag{5}$$

where x denotes a control decision variable and $x \in \mathcal{X}$ its constraints. As we will see later (Section IV), the particular choice of the square norm $||.||^2$ allows us to formulate the problem as a linearly constrained quadratic program (QP) and use algorithms that are dedicated to this class of optimization problems. Let $J_k = \partial g_k/\partial q \in \mathbb{R}^{n_k\times n}$ denote the Jacobian matrix of the task $\tau_k = g_k(q)$. Here and henceforth we suppose that g_k is continuously differentiable so that J_k exists and is continuous (which is always the case for a large class of robotic systems in practice). In the simplest case where $x = \ddot{q}$ and $\mathcal{X} = \mathbb{R}^n$ we can easily show that:

Proposition 1. *If J_k is full row rank then (5) $\Leftrightarrow$ (4).*

Proof. Noting that $\dot{\tau}_k = J_k\dot{q}$ and $\ddot{\tau}_k = J_k\ddot{q} + \dot{J}_k\dot{q}$, the first order optimality condition for (5) is

$$\frac{\partial||\ddot{\tau}_k - \ddot{\tau}^d_k||^2}{\partial\ddot{q}} = 2\,J_k^T(\ddot{\tau}_k - \ddot{\tau}^d_k) = 0\,. \tag{6}$$

By the rank-nullity theorem, $\dim\ker J_k^T = n_k - \operatorname{rank} J_k^T = n_k - \operatorname{rank} J_k$; since $\operatorname{rank} J_k = n_k$ then $\dim\ker {J_k}^T = 0$, which means $\ker J_k^T = \{0\}$, the desired equivalence thus follows from (6). □

In the more general case we can state the following, (based on the terminology used in, e.g., [22], [23], [24, Definition 4.6 p. 169]):

Definition 1. *The solutions of a system $\dot{\chi} = \varphi(\chi, t)$ are said to be uniformly ultimately bounded (UUB) if there exists $b > 0$ and $c > 0$ such that, for every $0 < a < c$, there exists $T(a,b) > 0$ such that*

$$||\chi(0)|| < a \;\Rightarrow\; \forall t \geq T(a,b),\ ||\chi(t)|| < b\,. \tag{7}$$

b is called an ultimate bound of the solutions. If a can be arbitrarily large, i.e. if $c = +\infty$, the solutions are said to be globally uniformly ultimately bounded.

Proposition 2. *If $A_k + A_k^T$ is negative definite then, for any $\epsilon > 0$, the differential inequality:*

$$||\ddot{\tau}_k - \ddot{\tau}^d_k||^2 < \epsilon\,, \tag{8}$$

results in $\eta_k(t)$ globally uniformly ultimately bounded. Moreover, for any $t \mapsto \varepsilon(t) > 0$ such that $\limsup_{t\to+\infty} \varepsilon(t) = 0$, the differential inequality:

$$||\ddot{\tau}_k - \ddot{\tau}_k^d||^2 < \varepsilon(t)\,, \tag{9}$$

implies, for every initial condition $\eta_k(0)$,

$$\eta_k(t) \xrightarrow[t\to+\infty]{} 0\,. \tag{10}$$

Proof. The inequality (8) can be rewritten as

$$||\dot{\eta_k} - A_k\eta_k|| = \left|\left|\begin{pmatrix} 0 \\ \ddot{\tau}_k - \ddot{\tau}_k^d \end{pmatrix}\right|\right| = ||\ddot{\tau}_k - \ddot{\tau}_k^d|| < \sqrt{\epsilon}\,, \tag{11}$$

which is equivalent to

$$\dot{\eta}_k = A_k\eta_k + \zeta(t)\,, \tag{12}$$

with $||\zeta(t)|| < \sqrt{\epsilon}$. Denoting $\mu(A_k)$ the *logarithmic norm* of A_k associated with the vector norm $||.||$, it can be shown [25] that (12) implies

$$||\eta_k(t)|| \leq e^{t\mu(A_k)}||\eta_k(0)|| + \int_0^t e^{(t-\theta)\mu(A_k)}||\zeta(\theta)||d\theta\,, \tag{13}$$

$$\leq e^{t\mu(A_k)}||\eta_k(0)|| + \int_0^t e^{(t-\theta)\mu(A_k)}\sqrt{\epsilon}d\theta\,, \tag{14}$$

$$= \left(||\eta_k(0)|| + \frac{\sqrt{\epsilon}}{\mu(A_k)}\right)e^{t\mu(A_k)} - \frac{\sqrt{\epsilon}}{\mu(A_k)}\,. \tag{15}$$

Let $\delta > 0$. We show that $\eta_k(t)$ is globally UUB with ultimate bound $-\frac{\sqrt{\epsilon}}{\mu(A_k)} + \delta$. So let $a > 0$. From (15), $||\eta_k(0)|| < a$ implies that

$$||\eta_k(t)|| < \left(a + \frac{\sqrt{\epsilon}}{\mu(A_k)}\right)e^{t\mu(A_k)} - \frac{\sqrt{\epsilon}}{\mu(A_k)}\,. \tag{16}$$

We also have $\mu(A_k) = \lambda_{\max}\left[\frac{1}{2}(A_k + A_k^T)\right]$ the maximum eigenvalue of $\frac{1}{2}(A_k + A_k^T)$ [26]. Since $A_k + A_k^T$ is negative definite, $\mu(A_k) < 0$, and hence the right-hand side of (16) goes to $-\frac{\sqrt{\epsilon}}{\mu(A_k)}$ as t goes to $+\infty$. Therefore there exists $T(a,\delta)$ such that $\forall t \geq T(a,\delta) : ||\eta_k(t)|| < -\frac{\sqrt{\epsilon}}{\mu(A_k)} + \delta$, and we can conclude that $\eta_k(t)$ is globally UUB with ultimate bound $-\frac{\sqrt{\epsilon}}{\mu(A_k)} + \delta$.

In the case of (9), we have similarly, denoting $\sigma(t) = \sup_{0<\theta<t}\epsilon(\theta)$,

$$||\eta_k(t)|| \leq \left(||\eta_k(0)|| + \frac{\sqrt{\sigma(t)}}{\mu(A_k)}\right)e^{t\mu(A_k)} - \frac{\sqrt{\sigma(t)}}{\mu(A_k)}\,. \tag{17}$$

Since $\sigma(t) \xrightarrow[t\to+\infty]{} 0$ and $\mu(A_k) < 0$ the right-hand side of (17) goes to 0 as t goes to $+\infty$ and therefore $\lim_{t\to+\infty}\eta_k(t) = 0$. □

Following this train of thought, it appears now that the multi-task problem can indeed be written as a multiobjective optimization problem as introduced earlier in this section

$$\text{“}\min_{x\in\mathcal{X}}\text{”}\, f(x) = (||\ddot{\tau}_1 - \ddot{\tau}_1^d||^2, \ldots, ||\ddot{\tau}_p - \ddot{\tau}_p^d||^2). \tag{18}$$

We thus provide in the following a complete characterization of all the efficient solutions of this problem.

B. Characterization of the Efficient Solutions

It can be shown that, to a certain extent that is precisely defined hereafter, all the efficient solutions of the multiobjective optimisation problem (1) can be obtained by solving single objective problems of the form

$$\min_{x\in\mathcal{X}} \sum_{k=1}^{p} w_k f_k(x)\,. \tag{19}$$

The problem (19) is called a weighted sum *scalarization* of the problem (1). Different results on the completeness of the characterization of the solutions of (1) can be obtained depending on whether we consider the non-identically null scalar weights w_k of (19) as only nonnegative or as (strictly) positive (i.e. whether $0 \leq w$ or $0 < w$ using the componentwise order notations of Section II-A). Let us denote the set of optimal points in $\mathcal{Y}$ that are spanned by the problems (19) in these two cases respectively as

$$\mathcal{S}_0(\mathcal{Y}) = \left\{ y^* \in \mathcal{Y} \mid \sum_{k=1}^{p} w_k y_k^* = \min_{y\in\mathcal{Y}} \sum_{k=1}^{p} w_k y_k,\ 0 \leq w \right\}, \tag{20}$$

$$\mathcal{S}(\mathcal{Y}) = \left\{ y^* \in \mathcal{Y} \mid \sum_{k=1}^{p} w_k y_k^* = \min_{y\in\mathcal{Y}} \sum_{k=1}^{p} w_k y_k,\ 0 < w \right\}. \tag{21}$$

We need a few more definitions to complete those already introduced in Section II-A. A solution $x^* \in \mathcal{X}$ is said to be a *weakly efficient* solution of (1) if $f(x^*)$ is *weakly nondominated* in $\mathcal{Y}$, that is, if there is no $x \in \mathcal{X}$ such that $f(x) < f(x^*)$. The set of all weakly nondominated points in $\mathcal{Y}$ is then denoted $\mathcal{Y}_{wN}$.

Theorem 2. $\mathcal{S}_0(\mathcal{Y}) \subset \mathcal{Y}_{wN}$.

Proof. let $y^* \in \mathcal{S}_0(\mathcal{Y})$. Then there exits $0 \leq w$ such that y^* minimizes $\sum_{k=0}^{p} w_k y_k$. Suppose that $y^* \notin \mathcal{Y}_{wN}$, then there exists y^0 such that $y_k^0 < y_k^*$ for all k in $\{1,\ldots,p\}$. Hence $\sum_{k=0}^{p} w_k y^0 < \sum_{k=0}^{p} w_k y^*$ since at least one of the weights is positive, which contradicts the optimality of y^*. □

For the converse inclusion we need the following definition:

Definition 2. *A set $\mathcal{Y}$ is said to be $\mathbb{R}^p_{\leqq}$-convex if $\mathcal{Y} + \mathbb{R}^p_{\leqq}$ is convex. $\mathbb{R}^p_{\leqq} = \{y \in \mathbb{R}^p \mid 0 \leqq y\}$ is the nonnegative orthant.*

Theorem 3. *If $\mathcal{Y}$ is $\mathbb{R}^p_{\leqq}$-convex then $\mathcal{S}_0(\mathcal{Y}) = \mathcal{Y}_{wN}$.*

Proof. See e.g. [20, Theorem 3.5 p. 69]. □

Thus we can see that under the conditions of Theorem 3 all weakly nondominated solutions of a multiobjective optimization problem can be obtained by weighted sum scalarizations with nonnegative weights. In our coming formulation of multi-task control we need the weights to be positive for the sake of stability. Thus we need stronger results, characterizing $\mathcal{S}(\mathcal{Y})$ rather than $\mathcal{S}_0(\mathcal{Y})$.

Theorem 4. $\mathcal{S}(\mathcal{Y}) \subset \mathcal{Y}_N$.

Proof. Similarly to the proof of Theorem 2, let $y^* \in \mathcal{S}(\mathcal{Y})$. Then there exists $0 < w$ such that y^* minimizes $\sum_{k=0}^{p} w_k y_k$.

Suppose that $y^* \notin \mathcal{Y}_N$, then there exists y^0 such that $y_k^0 \leq y_k^*$ for all k in $\{1,\dots,p\}$, with a strict inequality for at least one k_0. All the weights being positive we have $w_{k_0} y_{k_0}^0 < w_{k_0} y_{k_0}^*$. Hence $\sum_{k=0}^p w_k y^0 < \sum_{k=0}^p w_k y^*$, which contradicts the optimality of y^*. □

Unfortunately, the inclusion in Theorem 4 is too large, and the converse inclusion does not hold in general. In fact, it can be shown that the positive weights will only yield a set of so-called *properly efficient* solutions.

Definition 3. *A solution $x^* \in \mathcal{X}$ is called properly efficient if it is efficient and $\exists M > 0$ s.t. $\forall x \in \mathcal{X}$, $\forall i \in \{1,\dots,p\}$: $f_i(x) < f_i(x^*) \Rightarrow \exists j \in \{1,\dots,p\} \setminus \{i\}$ s.t. $f_j(x^*) < f_j(x)$ and*

$$\frac{f_i(x^*) - f_i(x)}{f_j(x) - f_j(x^*)} \leq M\,.$$

In that case the point $f(x^*)$ is said to be properly nondominated in $\mathcal{Y}$ and the set of all properly nondominated points of $\mathcal{Y}$ is denoted $\mathcal{Y}_{pN}$.

Using Definition 3, a tighter inclusion than that of Theorem 4 can be obtained:

Theorem 5 (Geoffrion (1968))**.** $\mathcal{S}(\mathcal{Y}) \subset \mathcal{Y}_{pN}$.

Proof. See e.g. [20, Theorem 3.11 p. 72]. See also the original work of Geoffrion in [27, Theorem 1]. □

The converse inclusion of Theorem 9 holds:

Theorem 6. *If $\mathcal{Y}$ is $\mathbb{R}^p_{\leqq}$-convex then $\mathcal{S}(\mathcal{Y}) = \mathcal{Y}_{pN}$.*

Proof. See e.g. [20, Theorem 3.13 p. 74]. □

Theorem 6 shows that only the properly efficient solutions of (1) can be attained with positive weights, and that this is the best we can achieve *exactly*. However, the following theorem, due to Hartley (1978), allows us to *approximate* any efficient solution with positive weight scalarization which will prove useful in our application.

Definition 4. *A set $\mathcal{Y}$ is said to be $\mathbb{R}^p_{\leqq}$-closed if $\mathcal{Y} + \mathbb{R}^p_{\leqq}$ is closed.*

Theorem 7 (Hartley (1978))**.** *If $\mathcal{Y}$ is nonempty, $\mathbb{R}^p_{\leqq}$-convex and $\mathbb{R}^p_{\leqq}$-closed then $\mathcal{Y}_N \subset \mathrm{cl}\,(\mathcal{S}(\mathcal{Y}))$.*

Proof. See e.g. [20, Theorem 3.17 p. 77]. See also the original work of Hartley in [28, Theorem 5.5]. □

Theorem 7 is a powerful tool that allows us to perform our desired approximation. Before applying it we will need the following lemma:

Lemma 1. *There is always at least one efficient solution of problem (1) that exactly realizes a given component of the ideal point y^I (2), i.e. $\forall k \in \{1,\dots,p\}$ $\exists y \in \mathcal{Y}_N$ s.t. $y_k = y_k^I$.*

Proof. Let k be a given index in $\{1,\dots,p\}$. Let $\mathcal{X}'$ denote the set $\mathcal{X}' = \{x \in \mathcal{X} \mid f_k(x) = y_k^I\}$, let $f' : \mathcal{X}' \to \mathbb{R}^{p-1}$ such that $f'(x) = (f_1(x),\dots,f_{k-1}(x),f_{k+1}(x),\dots,f_p(x))$ and let y' be any nondominated point of $\mathcal{Y}' = f'(\mathcal{X}')$. Then it is clear that y such that $y_k = y_k^I$ and $y_i = y'_i$ for $i \neq k$ satisfies the desired result. □

Now, we state the following corollary, supposing in the remainder of this section that the conditions of Theorem 7 are satisfied:

Corollary 1. *For any $\epsilon > 0$ and any index k, there exists a set of positive weights $0 < w$ such that $f_k(x^*) - y_k^I < \epsilon$, where x^* denotes a solution of problem (19).*

Proof. From Lemma 1 there exists $y \in \mathcal{Y}_N$ such that $y_k = y_k^I$. From Theorem 7 we then have $y \in \mathrm{cl}(\mathcal{S}(\mathcal{Y}))$ Since $\mathcal{Y}$ is finite-dimensional all norms are topologically equivalent and thus we can consider the ℓ^∞-norm $||.||_\infty$ for the closure definition $\mathrm{cl}(.)$. Therefore, there exists a sequence of elements $(y^l)_{l\in\mathbb{N}} \in \mathcal{Y}_N^{\mathbb{N}}$ such that $||y^l - y||_\infty \xrightarrow[l\to+\infty]{} 0$, and as such there exists $l_0 \in \mathbb{N}$ such that $||y^{l_0} - y||_\infty < \epsilon$. Finally we have $y_k^{l_0} - y_k^I = y_k^{l_0} - y_k \leq ||y^{l_0} - y||_\infty < \epsilon$ which shows the desired result. □

Applying Corollary 1 to problem (18) gives us:

Corollary 2. *If a given task τ_k is realizable exactly, i.e. $\exists x \in \mathcal{X}$ s.t. $\ddot{\tau}_k = \ddot{\tau}_k^d$, then it can be reached with weighted-sum scalarization of (18) with positive weights at any given precision, i.e. for any $\epsilon > 0$ there exists $0 < w$ such that*

$$||\ddot{\tau}_k(x^*) - \ddot{\tau}_k^d||^2 < \epsilon\,, \tag{22}$$

where x^* is the solution of the w-weighted sum scalarization of (18):

$$\min_{x\in\mathcal{X}} \sum_{l=1}^{p} w_l\, ||\ddot{\tau}_l(x) - \ddot{\tau}_l^d||^2\,. \tag{23}$$

Proof. Immediate from Corollary 1. □

In redundant manipulator control, one popular optimality notion is what is usually referred to as the *strict priority* ordering of the tasks (or sometimes *strict hierarchy*), which is *de facto* imposed by the nature of the method itself, i.e. the recursive pseudo-inversion of the task "constraint" and the projection in the null space of higher priority constrains [29], [30]. In the context of multiobjective optimization a similar notion is labelled under the term *lexicographic optimisation*

$$\operatorname*{lexmin}_{x\in\mathcal{X}} (f_1(x),\dots,f_p(x))\,, \tag{24}$$

which consists in finding a point $y^L \in \mathcal{Y}$ called the *lexicographic optimum* such that $\forall y \in \mathcal{Y}$ $y^L \leq_{\text{lex}} y$ where $\leq_{\text{lex}}$ denotes the *lexicographic order* (a total order) in $\mathbb{R}^p$.

Lemma 2. *The lexicographic optimum is one particular efficient solution of (1), i.e. $y^L \in \mathcal{Y}_N$.*

Proof. Suppose that $y^L \notin \mathcal{Y}_N$, then there exists $y \in \mathcal{Y}$ such that $y \leq y^L$, and thus the set $\left\{k \in \{1,\dots,p\} \mid y_k < y_k^L\right\}$ is nonempty. Let then $k_0 = \min\left\{k \in \{1,\dots,p\} \mid y_k < y_k^L\right\}$, since $y \leq y^L$ we have $y_k = y_k^L$ for $k \in \{1,\dots,k_0-1\}$ and $y_{k_0} < y_{k_0}^L$ and therefore $y <_{\text{lex}} y^L$, which contradicts the lexicographic optimality of y^L. □

Applying again Theorem 7 we get:

Corollary 3. *The lexicographic (strict priority) optimum can be approached at any given precision by positive weighted sum*

scalarization, i.e., for any $\epsilon > 0$ there exists a set of positive weights $0 < w$ such that $||f(x^*) - y^L|| < \epsilon$, where x^* is the solution of (23).

Proof. Similar to the proof of Corollary 1 from Lemma 2. □

We have now characterized the efficient solutions of (18) and justified the use of (23) for solving it. Propositions 1 and 2 give us some stability results in the state-space of the tasks $(\tau_k, \dot{\tau}_k)$, we study in the following the behavior of the system in the state-space of the generalized coordinates of the robot $(q, \dot{q})$.

III. Stability in the State Space of the Generalized Coordinates

In this section we restrict ourselves to the case in which $x = \ddot{q}$ and $\mathcal{X} = \mathbb{R}^p$. This would provide us with some insight on the general case that is more complex to study analytically and is out of the scope of this paper. We also consider task function regulation problems in which $t \mapsto \tau_k^r(t)$ are constant in time, and for ease of notation we denote their constant regulation values τ_k^r.

Our aim here is to study the behavior of the system of ordinary differential equations (ODEs) defined by

$$\ddot{q} = \operatorname{argmin} \sum_{k=1}^{p} w_k \left|\left|\ddot{\tau}_k - \ddot{\tau}_k^d\right|\right|^2 , \tag{25}$$

in the state space of $(q, \dot{q})$, where the weights are positive $0 < w$ following our analysis in Section II-B. As for related work concerning this section, see for example [31], [32] that study the stability of the strict priority inverse kinematics control approach, [10], [30] for the stability of strict priority inverse dynamics, [33], [34] for the stability of the weighted approach of a multi-task controller based on control Lyapunov functions (CLF).

We will base our argumentation below on the Lyapunov linearization method, so we propose to first introduce some general matrix differentiation concepts that we extensively use in the course of its application. This also allows us to introduce along the way the concept of the second derivative of the forward kinematics mapping (the "Jacobian of the Jacobian").

A. Matrix Differentiation Tools for the Lyapunov Linearization Method

Let us consider the nonlinear system:

$$\dot{\chi} = \varphi(\chi) , \tag{26}$$

with $\varphi(\chi^0) = 0$ and φ continuously differentiable at χ^0. Let $\Phi = \left.\frac{\partial \varphi}{\partial \chi}\right|_{\chi^0}$ denote the Jacobian matrix of φ at χ^0. We have the following Lyapunov linearization theorem:

Theorem 8 (Lyapunov (1892))**.** *The nonlinear system (26) is asymptotically stable at χ^0 if the linear system*

$$\dot{z} = \Phi z , \tag{27}$$

is asymptotically stable at 0.

Proof. See e.g. [24, Theorem 4.7 p. 139], [35, Corollary 26 p. 213], [36, Theorem 3.1 p. 55]. See also the theorem in the original work of Lyapunov translated to English in [37, Theorem I p. 556]. □

Theorem 8 gives a sufficient condition for the asymptotic stability of the equilibrium of (26), but it can also be extended to give a necessary and sufficient condition for the exponential stability of this equilibrium as follows:

Theorem 9. *The nonlinear system (26) is exponentially stable at χ^0 if and only if the linear system (27) is exponentially stable at 0.*

Proof. See, e.g. [35, Theorem 1 p. 246], [24, Corollary 4.3 p. 166] □

Finally we recall the following characterization of the asymptotic and exponential stability of autonomous linear systems:

Theorem 10. *The following statements are equivalent:*
(i) the linear system (27) is asymptotically stable at 0,
(ii) the linear system (27) is exponentially stable at 0,
(iii) the matrix Φ is stable.

Proof. See, e.g. [24, Theorem 4.5 p. 134] for $(i) \Leftrightarrow (iii)$, [35, Theorem 29 p. 197] for $(ii) \Leftrightarrow (iii)$. □

Hence, by Theorem 10, applying Theorems 8 or 9 amounts to studying the stability of the Jacobian matrix of φ. In our coming application in Section III-B the mapping φ includes in its expression the Jacobian matrices of the tasks J_k. Thus we need a tool to efficiently differentiate $J_k(q)$ with respect to q, that can somewhat be termed the "Jacobian of the Jacobian" (which is not to be confused with the notion of a *Hessian matrix* that is only defined for scalar functions). Unfortunately the expression

$$\text{“}\ \frac{\partial J_k(q)}{\partial q}\ \text{”} , \tag{28}$$

does not make sense and is not properly defined, since it involves the differentiation of a matrix with respect to a vector. Magnus and Neudecker (1985) proposed to use the following quantity that is thoroughly consistent with all the properties of the classical differentiation frameworks (in particular with the chain rule, the notion of the Jacobian, and Cauchy's rule of invariance) [38]:

$$G_k = DJ_k(q) = \frac{\partial \operatorname{vec} J_k(q)}{\partial q} . \tag{29}$$

The vec operator denotes the vectorization operator that consists for a matrix in stacking its columns as a vector, i.e.

$$\operatorname{vec} \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix} = \begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \\ \vdots \\ \vdots \\ a_{1m} \\ \vdots \\ a_{nm} \end{pmatrix} . \tag{30}$$

G_k in (29) appears thus as a regular "Jacobian" matrix of size $n \cdot n_k \times n$. We will make use of following definitions and properties

Theorem 11. *There exists a so-called commutation matrix K_{nm}, that is the $nm \times nm$ permutation matrix which transforms $\operatorname{vec} A^T$ into $\operatorname{vec} A$ for any $n \times m$ matrix A, i.e. $\forall A \in \mathbb{R}^{n\times m}$ $\operatorname{vec} A^T = K_{mn} \operatorname{vec} A$.*

Proof. See e.g. [39, Proposition 7.1.13 p. 402], [40, p. 54]. □

Denoting $\otimes$ the *Kronecker product*:

Theorem 12. *For any vector X and matrices A, B and C such that ABC is defined we have*

$$\begin{aligned} X &= \operatorname{vec} X\,, && (31)\\ \operatorname{vec}(ABC) &= (C^T \otimes A) \operatorname{vec} B\,, && (32)\\ \operatorname{vec}(AB) &= (I \otimes A) \operatorname{vec} B\,, && (33)\\ \operatorname{vec}(AB) &= (B^T \otimes I) \operatorname{vec} A\,. && (34) \end{aligned}$$

Proof. See e.g. [39, Proposition 7.1.9 p. 401 and Fact 7.4.6 p. 405]. □

Definition 5. *A vector function $f : S \subset \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $c \in \operatorname{int}(S)$ if there exists a matrix $A(c)$ such that, for u in a neighborhood of 0 in $\mathbb{R}^n$, we have*

$$f(c+u) = f(c) + A(c)u + o(||u||)\,. \quad (35)$$

If $A(c)$ exists it is unique and in that case the vector

$$df(c;u) = A(c)u\,, \quad (36)$$

is called the differential of f at c with increment u.

Theorem 13. *If f is differentiable at c then $A(c)$ defined in Definition 5 is the Jacobian matrix of f with respect to x (the variable of f) at c:*

$$A(c) = Df(c) = \left.\frac{\partial f}{\partial x}\right|_c\,. \quad (37)$$

Proof. See e.g. [40, Theorem 6 p. 98]. □

Magnus and Neudecker extended this definition of differentials to matrix functions:

Definition 6. *A matrix function $F : S \subset \mathbb{R}^{n\times m} \to \mathbb{R}^{p\times q}$ is differentiable at $C \in \operatorname{int}(S)$ if there exists a matrix $A(C) \in \mathbb{R}^{mn\times pq}$ such that, for U in a neighborhood of 0 in $\mathbb{R}^{n\times m}$, we have*

$$\operatorname{vec} F(C+U) = \operatorname{vec} F(c) + A(C) \operatorname{vec} U + o(||U||)\,. \quad (38)$$

If $A(C)$ exists it is unique and the $p \times q$ matrix $dF(C;U)$ defined by

$$\operatorname{vec} dF(C;U) = A(C) \operatorname{vec} U\,, \quad (39)$$

is called the differential of F at C with increment U.

Theorem 14. *If F is differentiable at C then $A(C)$ defined in Definition 6 is the Jacobian of $\operatorname{vec} F$ with respect to $\operatorname{vec} X$ (X denoting the variable of F) that we will also call the Jacobian of F at X*

$$A(C) = DF(C) = \left.\frac{\partial \operatorname{vec} F}{\partial \operatorname{vec} X}\right|_C\,. \quad (40)$$

Proof. See e.g. [40, Theorem 11 p. 108]. □

Definition 6 is consistent with Definition 5 and reduces to it when performing the conventional identification of vector and matrix spaces $\mathbb{R}^n \equiv \mathbb{R}^{n\times 1}$. The following theorem is called *Cauchy's rule of invariance* and is valid for either Definitions 5 and 6 of the differentials, in particular we state it here for the matrix differentials:

Theorem 15 (Cauchy's rule of invariance)**.** *If F is differentiable at C and G is differentiable at $B = F(C)$ then $H = G \circ F$ is differentiable at C and*

$$dH(C;U) = dG(B; dF(C;U))\,. \quad (41)$$

Proof. See e.g. [40, Theorem 13 p. 108]. □

Theorem 15 allows us to use the symbol dy to denote the differential of a vector or matrix function $y = g(t)$ as $dy = dg(t;dt)$ where dt is an arbitrary vector, since, if we change the variable $t = f(x)$ and denoting $h(x) = g(f(x)) = y$, we get following Cauchy's rule of invariance $dy = dh(x;dx) = dg(f(x); df(x;dx)) = dg(t;dt)$. Hence we shall even also use the notation $dg = dg(t;dt)$ without ambiguity. The following example of application of this rule that we use later illustrates this point:

Example 1. *The differentials of the mappings $GL_n(\mathbb{R}) \to \mathbb{R}^n$ $X \mapsto X^{-1}$; $\mathbb{R}^{n\times m} \to \mathbb{R}^{m\times n}$, $X \mapsto X^T$; and $\mathbb{R}^{n\times m} \to \mathbb{R}^{n\times n}$, $X \mapsto X^T X$ can be derived respectively as:*

$$\begin{aligned} d(X^{-1}) &= -X^{-1} dX X^{-1}\,, && (42)\\ d(X^T) &= K_{nm} dX\,, && (43)\\ d(X^T X) &= (K_{mm} + I_{m^2}) \left(I_m \otimes X^T\right) dX\,. && (44) \end{aligned}$$

Hence by Cauchy's rule of invariance we can write for $J_k(q)$ seen as a function of q:

$$dJ_k(q)^{-1} = -J_k^{-1} dJ_k(q) J_k^{-1} \quad \textit{(J(q) nonsingular)}\,, \quad (45)$$

$$d(J_k(q)^T) = K_{n_k n} dJ_k(q)\,, \quad (46)$$

$$d(J_k(q)^T J_k(q)) = \left(K_{n_k n_k} + I_{n_k^2}\right) \left(I_{n_k} \otimes J_k^T\right) dJ_k(q)\,. \quad (47)$$

Proof. See e.g. [40, Theorem 3 p. 71 and Chapter 9 Section 13 pp. 205-208]. □

We shall make use of these three formulas shortly hereafter.

By now, we introduced all the tools that we need for the coming developments in Section III-B.

B. Stability Analysis

We start with a single task case to illustrate our method in a simple setting, we then generalize the approach to multiple tasks. Note that some of the notations that will be used throughout the rest of the paper are introduced inside the proofs of this section.

Proposition 3. *Suppose $n_k = n$. The system:*

$$\ddot{q} = \operatorname{argmin} ||\ddot{\tau}_k - \ddot{\tau}_k^d||^2\,, \quad (48)$$

has an equilibrium if and only if there exists q^0 such that $g_k(q^0) = \tau_k^r$ and, in that case, if $J_k(q^0)$ is nonsingular then the equilibrium is exponentially stable in the state-space of $(q, \dot{q})$. More generally, the system:

$$\ddot{q} = \operatorname{argmin} ||\dot{\eta}_k - A_k \eta_k||^2 \,, \tag{49}$$

where A_k is stable, has an equilibrium if and only if there exists q^0 such that $g_k(q^0) = \tau_k^r$ and, in that case, if $J_k(q^0)$ is nonsingular then the equilibrium is exponentially stable in the state space of $(q, \dot{q})$.

Proof. Let us denote $\xi = (q, \dot{q})$ the state of the system (49). The variable ξ is related to η_k through the nonlinear "forward kinematics" mapping

$$\gamma_k : \xi \mapsto \eta_k = \gamma_k(\xi) = \begin{pmatrix} g_k(q) - \tau_k^r \\ J_k(q)\dot{q} \end{pmatrix} . \tag{50}$$

Let $\mathcal{J}_k(\xi)$ denote the Jacobian matrix of that mapping at ξ. From (50) it appears that $\mathcal{J}_k(\xi)$ is related to $J_k(q)$ through the following relation:

$$\mathcal{J}_k(\xi) = \begin{pmatrix} J_k(q) & 0 \\ \frac{\partial [J(q)\dot{q}]}{\partial q} & J_k(q) \end{pmatrix} . \tag{51}$$

From Proposition 1, the system (49) is equivalent to

$$\dot{\eta}_k = A_k \eta_k \,, \tag{52}$$

which has an equilibrium if and only if there exists q^0 such that $\eta_k = 0$, i.e. such that $g_k(q^0) = \tau_k^r$. In terms of ξ, (52) translates into the nonlinear *descriptor* system

$$\mathcal{J}_k(\xi)\dot{\xi} = A_k \gamma_k(\xi) \,. \tag{53}$$

Let $\xi^0 = (q^0, 0)$. Since $n = n_k$ and $J_k(q^0)$ is nonsingular, we can see from (51) that $\mathcal{J}_k(\xi^0)$ is a square $2n \times 2n$ lower block triangular matrix with rank $\operatorname{rank} \mathcal{J}_k(\xi^0) = \operatorname{rank} J_k(q^0) + \operatorname{rank} J_k(q^0) = 2n$, therefore $\mathcal{J}_k(\xi^0)$ is also nonsingular. Supposing now that the forward kinematics mapping is continuously differentiable, then the mapping $\mathcal{J} : \xi \mapsto \mathcal{J}_k(\xi)$ is continuous, and as such the inverse image of any open set of $\mathbb{R}^{2n\times 2n}$ under $\mathcal{J}$ is open. Since the $GL_{2n}(\mathbb{R})$ group is an open subset of $\mathbb{R}^{2n\times 2n}$, $\mathcal{J}^{-1}\big(GL_{2n}(\mathbb{R})\big)$ is an open set containing ξ^0, therefore there exists a neighborhood V of ξ^0 included in $\mathcal{J}^{-1}\big(GL_{2n}(\mathbb{R})\big)$. Finally, for any $\xi \in V$, $\mathcal{J}_k(\xi) = \mathcal{J}(\xi) \in GL_{2n}(\mathbb{R})$, and hence, in that neighborhood V, the descriptor system (53) takes the form of the nonlinear dynamical system:

$$\dot{\xi} = \mathcal{J}_k(\xi)^{-1} A_k \gamma_k(\xi) \,, \tag{54}$$

or, denoting ϕ_k the mapping $\phi_k : \xi \mapsto \mathcal{J}_k(\xi)^{-1} A_k \gamma_k(\xi)$,

$$\dot{\xi} = \phi_k(\xi) \,. \tag{55}$$

Before calculating the Jacobian of ϕ_k at ξ^0 in order to apply Theorem 8, we introduce the following matrix:

$$\Gamma_k = D\mathcal{J}_k(\xi) = \frac{\partial \operatorname{vec} \mathcal{J}_k}{\partial \xi} \,. \tag{56}$$

We have (we drop the dependencies on ξ when there is no ambiguity):

$$\begin{aligned} d\phi_k &= d[\mathcal{J}_k(\xi)^{-1} A_k \gamma_k(\xi)] \,, & (57)\\ &= d\mathcal{J}_k(\xi)^{-1} A_k \gamma_k + \mathcal{J}_k^{-1} A_k d\gamma_k(\xi) \,. & (58) \end{aligned}$$

Then

$$\begin{aligned} d\mathcal{J}_k(\xi)^{-1} A_k \gamma_k &= \operatorname{vec}\left[d\mathcal{J}_k(\xi)^{-1} A_k \gamma_k\right] \,, & (59)\\ &= \left(\gamma_k^T A_k^T \otimes I_{2n_k}\right) \operatorname{vec} d\mathcal{J}_k(\xi)^{-1} \,, & (60) \end{aligned}$$

and by (42)

$$\begin{aligned} \operatorname{vec} d\mathcal{J}_k(\xi)^{-1} &= \operatorname{vec}\left[-\mathcal{J}_k^{-1} d\mathcal{J}_k(\xi) \mathcal{J}_k^{-1}\right] \,, & (61)\\ &= -\left(\mathcal{J}_k^{-T} \otimes \mathcal{J}_k^{-1}\right) \operatorname{vec} d\mathcal{J}_k(\xi) \,, & (62)\\ &= -\left(\mathcal{J}_k^{-T} \otimes \mathcal{J}_k^{-1}\right) \Gamma_k d\xi \,. & (63) \end{aligned}$$

We also have

$$d\gamma_k(\xi) = \mathcal{J}_k d\xi. \tag{64}$$

Plugging (60), (63) and (64) into (58) yields

$$\begin{aligned} d\phi_k = \big[&- \left(\gamma_k^T A_k^T \otimes I_{2n_k}\right) \left(\mathcal{J}_k^{-T} \otimes \mathcal{J}_k^{-1}\right) \Gamma_k \\ &+ \mathcal{J}_k^{-1} A_k \mathcal{J}_k \big] \, d\xi \,, \end{aligned} \tag{65}$$

and, therefore, we get the expression of the Jacobian of ϕ_k:

$$\frac{\partial \phi_k}{\partial \xi} = -\left(\gamma_k^T A_k^T \otimes I_{2n_k}\right) \left(\mathcal{J}_k^{-T} \otimes \mathcal{J}_k^{-1}\right) \Gamma_k + \mathcal{J}_k^{-1} A_k \mathcal{J}_k \,. \tag{66}$$

At ξ^0 we have $\gamma_k(\xi^0) = 0$, and (66) simplifies into

$$\left.\frac{\partial \phi_k}{\partial \xi}\right|_{\xi^0} = \mathcal{J}_k(\xi^0)^{-1} A_k \mathcal{J}_k(\xi^0) \,, \tag{67}$$

which has the same eigenvalues as A_k. From Theorems 9 and 10 we conclude that (55) is exponentially stable. □

In the multi-task case we also propose to analytically linearize the system in the $(q, \dot{q})$ state space. We will always suppose in the following that at least one of the tasks k_0 is a full-configuration task $\tau_{k_0}(q) = q$, no matter how infinitesimally small its weight w_{k_0} is, as long as it remains positive $w_{k_0} > 0$. This is a non-restrictive assumption following the analysis in Section II-B.

Lemma 3. *If one of the tasks is a full-configuration task then for all ξ the matrix*

$$B(\xi) = \sum_{k=1}^{p} w_k \mathcal{J}_k(\xi)^T \mathcal{J}_k(\xi) \,, \tag{68}$$

is nonsingular.

Proof. $B(\xi)$ is clearly a symmetric positive matrix. Since one of the tasks τ_{k_0} is a full-configuration task $\tau_{k_0}(q) = q$, we have $J_{k_0}(q) = I_n$ and from (51) $\mathcal{J}_{k_0}(\xi) = I_{2n}$, therefore

$$B(\xi) = w_{k_0} I_{2n} + \sum_{\substack{k=1 \\ k\neq k_0}}^{p} w_k \mathcal{J}_k(\xi)^T \mathcal{J}_k(\xi) \,. \tag{69}$$

Since $w_{k_0} > 0$, $B(\xi)$ is positive definite and thus nonsingular. □

Proposition 4. *The system*

$$\dot{\xi} = \operatorname{argmin} \sum_{k=1}^{p} w_k ||\dot{\eta}_k - A_k \eta_k||^2 \,, \tag{70}$$

has an equilibrium if and only if there exists ξ^0 such that

$$\sum_{k=1}^{p} w_k \mathcal{J}_k(\xi^0)^T A_k \gamma_k(\xi^0) = 0 \,. \tag{71}$$

In that case, the equilibrium is exponentially stable if and only if the matrix

$$B^{-1} \sum_{k=1}^{p} w_k \Big(\big(\gamma_k^T A_k^T \otimes I_{2n_k}\big) K_{2n_k 2n} \Gamma_k + \mathcal{J}_k^T A_k \mathcal{J}_k \Big) \,, \tag{72}$$

evaluated at ξ^0 is stable.

Proof. The first order optimality condition for (70) is:

$$\frac{\partial}{\partial \dot{\xi}} \left[\sum_{k=1}^{p} w_k ||\dot{\eta}_k - A_k \eta_k||^2 \right] = 0 \,, \tag{73}$$

$$\Leftrightarrow \sum_{k=1}^{p} 2\, w_k \mathcal{J}_k^T (\dot{\eta}_k - A_k \eta_k) = 0 \,, \tag{74}$$

$$\Leftrightarrow \left[\sum_{k=1}^{p} w_k \mathcal{J}_k^T \mathcal{J}_k \right] \dot{\xi} = \sum_{k=1}^{p} w_k \mathcal{J}_k^T A_k \eta_k \,. \tag{75}$$

$B(\xi)$ being nonsingular from Lemma 3, (75) takes the form of the nonlinear system:

$$\dot{\xi} = B(\xi)^{-1} \sum_{k=1}^{p} w_k \mathcal{J}_k(\xi)^T A_k \gamma_k(\xi) \,, \tag{76}$$

which admits an equilibrium if and only if there exists ξ^0 such that:

$$\sum_{k=1}^{p} w_k \mathcal{J}_k(\xi^0)^T A_k \gamma_k(\xi^0) = 0 \,. \tag{77}$$

Let us linearize (76) around such an equilibrium. To do this we calculate the Jacobian of the mapping $\psi : \xi \mapsto B(\xi)^{-1} \sum_{k=1}^{p} w_k \mathcal{J}_k(\xi)^T A_k \gamma_k(\xi)$ using the differential-based treatment introduced in Section III-A. We have (dropping again the dependencies on ξ when appropriate):

$$d\psi = dB(\xi)^{-1} \sum_{k=1}^{p} w_k \mathcal{J}_k^T A_k \gamma_k + B^{-1} \sum_{k=1}^{p} w_k \left[d\mathcal{J}_k(\xi)^T A_k \gamma_k + \mathcal{J}_k^T A_k d\gamma_k(\xi) \right] \,. \tag{78}$$

Let us calculate each term of the right-hand side of (78) separately. To shorten the expressions let C denote the vector $C(\xi) = \sum_{k=1}^{p} w_k \mathcal{J}_k(\xi)^T A_k \gamma_k(\xi)$. We have, by (42),

$$dB(\xi)^{-1} C = -B^{-1} dB(\xi) B^{-1} C \,, \tag{79}$$

$$= \operatorname{vec}\left[-B^{-1} dB(\xi) B^{-1} C \right] \,, \tag{80}$$

$$= -\left(C^T B^{-T} \otimes B^{-1} \right) \operatorname{vec} dB(\xi) \,, \tag{81}$$

where

$$\operatorname{vec} dB(\xi) = d \operatorname{vec} B(\xi) \,, \tag{82}$$

$$= \sum_{k=1}^{p} w_k d \operatorname{vec} \mathcal{J}_k(\xi)^T \mathcal{J}_k(\xi) \,, \tag{83}$$

and by (44)

$$d \operatorname{vec} \mathcal{J}_k^T \mathcal{J}_k = \left(I_{4n_k^2} + K_{2n_k 2n_k} \right) \left(\mathcal{J}_k \otimes I_{2n_k} \right) d \operatorname{vec} \mathcal{J}_k \,, \tag{84}$$

with

$$d \operatorname{vec} \mathcal{J}_k(\xi) = \Gamma_k d\xi \,. \tag{85}$$

This gives us the first term in (78) as

$$dB(\xi)^{-1} C = -\left(C^T B^{-T} \otimes B^{-1} \right) \sum_{k=1}^{p} w_k \Big(I_{4n_k^2} + K_{2n_k 2n_k} \Big) \left(\mathcal{J}_k \otimes I_{2n_k} \right) \Gamma_k d\xi \,. \tag{86}$$

As for the other two terms we write, applying (43) for (89):

$$d\mathcal{J}_k(\xi)^T A_k \gamma_k = \operatorname{vec}\left[d\mathcal{J}_k(\xi)^T A_k \gamma_k \right] \,, \tag{87}$$

$$= \left(\gamma_k^T A_k^T \otimes I_{2n_k} \right) \operatorname{vec} d\mathcal{J}_k(\xi)^T \,, \tag{88}$$

$$= \left(\gamma_k^T A_k^T \otimes I_{2n_k} \right) K_{2n_k 2n} \operatorname{vec} d\mathcal{J}_k(\xi) \,, \tag{89}$$

$$= \left(\gamma_k^T A_k^T \otimes I_{2n_k} \right) K_{2n_k 2n} \Gamma_k d\xi \,, \tag{90}$$

and finally the last term

$$\mathcal{J}_k^T A_k d\gamma_k(\xi) = \mathcal{J}_k^T A_k \mathcal{J}_k d\xi \,. \tag{91}$$

Plugging (86), (90) and (91) into (78) gives us

$$d\psi = \left[-\left(C^T B^{-T} \otimes B^{-1} \right) \sum_{k=1}^{p} w_k \Big(I_{4n_k^2} + K_{2n_k 2n_k} \Big) \left(\mathcal{J}_k \otimes I_{2n_k} \right) \Gamma_k + B^{-1} \sum_{k=1}^{p} w_k \Big(\big(\gamma_k^T A_k^T \otimes I_{2n_k}\big) K_{2n_k 2n} \Gamma_k + \mathcal{J}_k^T A_k \mathcal{J}_k \Big) \right] d\xi \,, \tag{92}$$

from which we get the desired analytic expression of the Jacobian of the mapping ψ:

$$\frac{\partial \psi}{\partial \xi} = -\left(C^T B^{-T} \otimes B^{-1} \right) \sum_{k=1}^{p} w_k \Big(I_{4n_k^2} + K_{2n_k 2n_k} \Big) \left(\mathcal{J}_k \otimes I_{2n_k} \right) \Gamma_k + B^{-1} \sum_{k=1}^{p} w_k \Big(\big(\gamma_k^T A_k^T \otimes I_{2n_k}\big) K_{2n_k 2n} \Gamma_k + \mathcal{J}_k^T A_k \mathcal{J}_k \Big) \,. \tag{93}$$

At the equilibrium ξ^0 we have from (77) $C(\xi^0) = 0$, hence (93) simplifies into

$$\left. \frac{\partial \psi}{\partial \xi} \right|_{\xi^0} = B^{-1} \sum_{k=1}^{p} w_k \Big(\big(\gamma_k^T A_k^T \otimes I_{2n_k}\big) K_{2n_k 2n} \Gamma_k + \mathcal{J}_k^T A_k \mathcal{J}_k \Big) \,. \tag{94}$$

Thus, the equilibrium ξ^0 is exponentially stable if and only if this latter matrix is stable. □

Corollary 4. *If the tasks τ_k are ultimately realizable simultaneously, i.e. if there exists ξ^0 such that $\forall k \in \{1,\dots,p\}$ $\gamma_k(\xi^0) = 0$, then ξ^0 is an equilibrium of (70). In that case, a sufficient condition for ξ^0 to be exponentially stable is that the matrices $A_k + A_k^T$ are negative definite.*

Proof. If $\forall k \in \{1,\dots,p\}$ $\gamma_k(\xi^0) = 0$ then (71) holds, and by Proposition 4, ξ^0 is an equilibrium point of (70). Moreover, in that case, (72) simplifies into

$$\left[\sum_{k=1}^{p} w_k \mathcal{J}_k^T \mathcal{J}_k\right]^{-1} \sum_{k=1}^{p} w_k \mathcal{J}_k^T A_k \mathcal{J}_k = B^{-1}\mathcal{A}\,, \tag{95}$$

where we denoted

$$\mathcal{A} = \sum_{k=1}^{p} w_k \mathcal{J}_k^T A_k \mathcal{J}_k\,. \tag{96}$$

If we additionally suppose that $A_k + A_k^T$ are negative definite, then $\mathcal{A} + \mathcal{A}^T$ is also negative definite since

$$\mathcal{A} + \mathcal{A}^T = w_{k_0}\left(A_{k_0} + A_{k_0}^T\right) + \sum_{\substack{k=1 \\ k\neq k_0}}^{p} w_k \mathcal{J}_k^T \left(A_k + A_k^T\right)\mathcal{J}_k\,, \tag{97}$$

with $w_{k_0}\left(A_{k_0} + A_{k_0}^T\right)$ negative definite (since $w_{k_0} > 0$) and $\forall k \neq k_0$ $w_k \mathcal{J}_k^T \left(A_k + A_k^T\right)\mathcal{J}_k$ negative.

Furthermore, B being positive definite, $\mathcal{B} = B^{-1}$ is also symmetric positive definite. Any matrix congruent to a negative definite matrix is also a negative definite matrix and hence $\mathcal{B}\left(\mathcal{A}^T + \mathcal{A}\right)\mathcal{B}^T$ is negative definite. And given that

$$\mathcal{B}\left(\mathcal{A}^T + \mathcal{A}\right)\mathcal{B}^T = \mathcal{B}\left(\mathcal{A}^T + \mathcal{A}\right)\mathcal{B}\,, \quad (\mathcal{B} \text{ symmetric})\,, \tag{98}$$

$$= \mathcal{B}\mathcal{A}^T\mathcal{B} + \mathcal{B}\mathcal{A}\mathcal{B}\,, \tag{99}$$

$$= \mathcal{B}\left(\mathcal{B}\mathcal{A}\right)^T + \left(\mathcal{B}\mathcal{A}\right)\mathcal{B}\,, \tag{100}$$

then the pair of positive definite matrices $\mathcal{Q} = -\mathcal{B}\left(\mathcal{A}^T + \mathcal{A}\right)\mathcal{B}^T$ and $\mathcal{P} = \mathcal{B}$ satisfy the Lyapunov equation $\mathcal{P}(\mathcal{B}\mathcal{A})^T + (\mathcal{B}\mathcal{A})\mathcal{P} = -\mathcal{Q}$. Therefore, $\mathcal{B}\mathcal{A} = B^{-1}\mathcal{A}$ is stable. By Proposition 4 we conclude that ξ^0 is exponentially stable. □

We conclude this section on analytical stability analysis of the multiobjective optimization-based multi-task controls scheme and continue with the humanoid control case-study.

IV. Application to Humanoid Multi-Task Control

In this section we determine the nature of the control decision variable x and characterize the constraint set $\mathcal{X}$ in the humanoid control application case. We also cast the problem (23) as a linearly constrained QP inspired by approaches in the literature [41]–[44] and show some of its stability properties.

A. Physical and Mathematical Constraints

Constraints of the humanoid robot motion include its equation of motion, the non-slipping contact constraints (e.g. at the feet surfaces), the corresponding Couloumb friction constraints, and various bounds on the applicable torques, admissible ranges of joint angles, joint velocities, and collision-avoidance.

The equation of motion of a humanoid robot in a given contact phase is usually written:

$$M(q)\ddot{q} + N(q,\dot{q}) = Su + J^c(q)^T\lambda\,, \tag{101}$$

$$J^c(q)\ddot{q} + \dot{J}^c(q)\dot{q} = 0\,. \tag{102}$$

In direct comparison with the standard industrial manipulator's EOM:

$$M(q)\ddot{q} + N(q,\dot{q}) = u\,, \tag{103}$$

the humanoid system is characterized as being

1) *underactuated*: with a non-actuated *free-floating base* expressed in the fact the matrix S mapping the actuation to the DoFs is not square and in particular noninvertible,
2) *constrained*: from the non-slipping contact constraint equation (102) and the corresponding Lagrange multipliers λ in (101),
3) *redundant*: the number of DoFs of the robot is in general strictly greater than the number of DoFs of an individual task, which allow for multi-task control.

One additional constraint however has to be appended to the system (101)-(102) and yet is often omitted in many existing treatments of the problem, that is the Coulomb friction cone constraint which gives rise to the following system:

$$M(q)\ddot{q} + N(q,\dot{q}) = Su + J^c(q)^T\lambda\,, \tag{104}$$

$$J^c(q)\ddot{q} + \dot{J}^c(q)\dot{q} = 0\,, \tag{105}$$

$$\lambda \in \mathcal{C}\,, \tag{106}$$

$\mathcal{C}$ denoting a "Coulomb friction cone". We have used the latter quotation marks, to draw one's attention regarding the choice of the particular formulation of the constraint (105) which cannot be derived from any arbitrary holonomic constraint $h(q) = 0$ that expresses the fixation of the contact (with $\frac{\partial h}{\partial q} = J^c$) . For example, for any such constraint $h(q) = 0$, the constraint $||h(q)||^2 = 0$ would mathematically express the exact same constraint but would result in a different Jacobian and thus in Lagrange multipliers that would not satisfy the same mathematical relations.

In order for the constraint (106) to physically make sense, λ has to be the actual physical contact forces, not arbitrary constraint forces. For a point contact at a point a belonging to a planar surface $\mathcal{S}$ of the robot with normal $\nu_\mathcal{S}$, the physical contact force λ is associated with the constraint $J^a\dot{q} = 0$ where J^a is the Jacobian such that $\dot{a} = J^a\dot{q}$. In that case the Coulomb friction cone takes the following form:

$$\mathcal{C}_\mathcal{S} = \left\{\lambda \in \mathbb{R}^3 \mid \langle\lambda,\nu_\mathcal{S}\rangle > 0\,,\ ||\lambda - \langle\lambda,\nu_\mathcal{S}\rangle\nu_\mathcal{S}|| \leq \mu\langle\lambda,\nu_\mathcal{S}\rangle\right\}\,. \tag{107}$$

For distributed surface contact on a surface $\mathcal{S}$ we would have a *continuum* of forces and likewise constraints in a system of the form:

$$M(q)\ddot{q} + N(q,\dot{q}) = Su + \iint_{a\in\mathcal{S}} J^a(q)^T\lambda(a)d\mathcal{S}(a)\,, \tag{108}$$

$$\forall a \in \mathcal{S} \quad J^a(q)\ddot{q} + \dot{J}^a(q)\dot{q} = 0\,, \tag{109}$$

$$\forall a \in \mathcal{S} \quad \lambda(a) \in \mathcal{C}_\mathcal{S}\,, \tag{110}$$

This system can however be simplified according to the following theorem

Theorem 16. *If $\mathcal{S}$ is a convex polygon*

$$\mathcal{S} = \left\{ \sum_{i=1}^{s} \alpha_i a_i \mid \sum_{i=1}^{s} \alpha_i = 1 \right\}, \tag{111}$$

then we have the following equivalence

$$\begin{aligned} &\forall F \in \mathbb{R}^n : \\ &\quad \exists \lambda : \mathcal{S} \to \mathcal{C}_\mathcal{S} \text{ s.t. } F = \iint_{a \in \mathcal{S}} J^a(q)^T \lambda(a) d\mathcal{S}(a) \\ &\qquad\qquad \Leftrightarrow \\ &\quad \exists (\lambda_1, \dots, \lambda_s) \in [\mathcal{C}_\mathcal{S}]^s \text{ s.t. } F = \sum_{i=1}^{s} J^{a_i}(q)^T \lambda_i \,. \end{aligned} \tag{112}$$

Proof. See e.g. [45, Proposition 1]. □

Additionally, if we stay under the conditions of Theorem 16, it is clear that

$$(109) \Leftrightarrow \forall i \in \{1, \dots, s\} \; J^{a_i}(q)\ddot{q} + \dot{J}^{a_i}(q)\dot{q} = 0 \,, \tag{113}$$

$$\Leftrightarrow \qquad J^\mathcal{S}(q)\ddot{q} + \dot{J}^\mathcal{S}\dot{q} = 0 \,, \tag{114}$$

where $J^\mathcal{S}$ denotes the rotational and translational Jacobian of any frame rigidly attached to $\mathcal{S}$. This latter remark together with Theorem 16 allows us to rewrite the continuum system of equations (108) to (110) in the following equivalent finite system form:

$$M(q)\ddot{q} + N(q, \dot{q}) = Su + \sum_{i=1}^{s} J^{a_i}(q)^T \lambda_i \,, \tag{115}$$

$$J^\mathcal{S}(q)\ddot{q} + \dot{J}^\mathcal{S}\dot{q} = 0 \tag{116}$$

$$\forall i \in \{1, \dots, s\} \quad \lambda_i \in \mathcal{C}_\mathcal{S} \,. \tag{117}$$

B. Structural Constraints

We write here the structural constraints using the weak componentwise order notation for vector inequalities as follows

$$u_{\min} \leqq u \leqq u_{\max} \,, \tag{118}$$

$$q_{\min} \leqq q \leqq q_{\max} \,, \tag{119}$$

$$\dot{q}_{\min} \leqq \dot{q} \leqq \dot{q}_{\max} \,, \tag{120}$$

and the collision avoidance between two bodies based on a velocity damper formulation

$$\dot{d} \geq -\kappa \frac{d - \delta_s}{\delta_i - \delta_s} \,, \tag{121}$$

where d denotes the distance between the two bodies and δ_i, δ_s, κ, respectively, an influence distance, a security distance, and a damping constant (see [46], [47] for details on this particular formulation).

C. Casting the problem as a QP

In order to cast the problem as a QP we conservatively approximate the friction cone $\mathcal{C}_\mathcal{S}$ with an inscribed polyhedral cone $\hat{\mathcal{C}}_\mathcal{S}$ [48]. Let $\mathcal{C}$ denote the matrix of the set of the polyhedral cone generators' coordinates in the world frame, and let c denote the number of generators, $\mathcal{C} \in \mathbb{R}^{3 \times c}$, then we have $\lambda \in \hat{\mathcal{C}}_\mathcal{S}$ if and only if $\exists \hat{\lambda} \in \mathbb{R}^c_\geq$ s.t. $\lambda = \mathcal{C}\hat{\lambda}$. The system (115) to (117) becomes:

$$M(q)\ddot{q} + N(q, \dot{q}) = Su + \sum_{i=1}^{s} J^{a_i}(q)^T \mathcal{C} \hat{\lambda}_i \,, \tag{122}$$

$$J^\mathcal{S}(q)\ddot{q} + \dot{J}^\mathcal{S}\dot{q} = 0 \,, \tag{123}$$

$$\forall i \in \{1, \dots, s\} \quad 0 \leqq \hat{\lambda}_i \,. \tag{124}$$

We also rewrite the constraints (118) to (121) respectively as follows:

$$u_{\min} \leqq u \leqq u_{\max} \,, \tag{125}$$

$$\frac{\dot{q}_{\min} - \dot{q}}{\Delta t} \leqq \ddot{q} \leqq \frac{\dot{q}_{\max} - \dot{q}}{\Delta t} \,, \tag{126}$$

$$\frac{q_{\min} - q - \dot{q}\Delta t}{\frac{1}{2}\Delta t^2} \leqq \ddot{q} \leqq \frac{q_{\max} - q - \dot{q}\Delta t}{\frac{1}{2}\Delta t^2} \,, \tag{127}$$

$$\ddot{d} \geq \frac{1}{\Delta t} \left(-\xi \frac{d - \delta_s}{\delta_i - \delta_s} - \dot{d} \right) . \tag{128}$$

where Δt is a fixed parameter (e.g. control time-step). Finally we enforce the compactness of the feasible set by setting an arbitrarily large bound on $\hat{\lambda}$

$$\hat{\lambda} \leqq \hat{\lambda}_{\max} \,. \tag{129}$$

It can now be seen that setting the control decision variable as $x = (\ddot{q}, u, \hat{\lambda}) \in \mathbb{R}^{2n-6+s \cdot c}$, the set of equations and inequalities (122) to (129) defining the feasible set $\mathcal{X} \subset \mathbb{R}^{2n-6+s \cdot c}$ are linear in x, i.e. $\mathcal{X}$ is a an intersection of closed halfspaces. Let $H_e x = b_e$ denote the set of equations (122) and (123) and $H_i x \leqq b_i$ denote the set of inequalities (124) to (129).

$$\mathcal{X} = \left\{ x = (\ddot{q}, u, \hat{\lambda}) \in \mathbb{R}^{2n-6+s \cdot c} \mid (122) \text{ to } (129) \right\}, \tag{130}$$

$$= \left\{ x \in \mathbb{R}^{2n-6+s \cdot c} \mid H_e x = b_e, \, H_i x \leqq b_i \right\}. \tag{131}$$

Denoting the matrix

$$K(q) = \begin{pmatrix} J^{a_1}(q)^T \mathcal{C} & \cdots & J^{a_s}(q)^T \mathcal{C} \end{pmatrix} \in \mathbb{R}^{n \times s \cdot c} \,, \tag{132}$$

we have, in particular,

$$H_e = \begin{pmatrix} M(q) & K(q) & S \\ J^\mathcal{S}(q) & 0 & 0 \end{pmatrix} . \tag{133}$$

To the set of tasks $\tau_1, \dots, \tau_p$, of which we recall that the task τ_{k_0} is a full-configuration task $\tau_{k_0} = g_{k_0}(q) = q$, we append two additional components in the vector optimization problem (18):

$$\text{“}\min_{x \in \mathcal{X}}\text{”} f(x) = (||\ddot{\tau}_1 - \ddot{\tau}_1^d||^2, \dots, ||\ddot{\tau}_p - \ddot{\tau}_p^d||^2, ||u||^2, ||\hat{\lambda}||^2) \,. \tag{134}$$

We show now that the conditions of Theorem 7 hold. We shall invoke the following two theorems, reusing the notations of Section II:

Theorem 17. *A sufficient condition for the $\mathbb{R}^p_{\leqq}$-convexity of $\mathcal{Y} = f(\mathcal{X})$ is that $\mathcal{X}$ is convex and the functions $f_1, \ldots, f_p$ are convex.*

Proof. See e.g. [21, Proposition 2.1.22 p. 15]. □

Theorem 18. *Let Y^+ denote the extended recession cone of a set Y, defined as*

$$Y^+ = \left\{ y' \mid \exists (\beta^k) \in \mathbb{R}^{\mathbb{N}},\ \exists (y^k) \in Y^{\mathbb{N}},\ \beta^k > 0, \right. \\ \left. \text{s.t. } \beta^k \xrightarrow[k\to+\infty]{} 0,\ \beta^k y^k \xrightarrow[k\to+\infty]{} y' \right\}. \quad (135)$$

Let Y_1 and Y_2 be two nonempty closed sets. If

$$Y_1^+ \cap (-Y_2^+) = \{0\}, \quad (136)$$

then $Y_1 + Y_2$ is closed.

Proof. See e.g. [21, Lemma 3.2.3 p. 52]. □

We can now prove the following:

Proposition 5. *if $\mathcal{X}$ is nonempty then the conditions of Theorem 7 hold for the problem (134).*

Proof. We recall that in finite dimension compactness is equivalent to simultaneous closedness and boundedness. Since $\mathcal{X}$ is closed as the intersection of a finite number of closed halfspaces, and $\mathcal{X}$ is bounded by the constraints (124), (125), (126), (129), $\mathcal{X}$ is compact. f in (134) being continuous, $\mathcal{Y} = f(\mathcal{X})$ is therefore compact, which implies that it is closed and bounded.

The extended recession cone of a bounded set is $\{0\}$ by [21, Lemma 3.2.1 p. 52], thus $\mathcal{Y}^+ = \{0\}$, and hence $\mathcal{Y}^+ \cap (-\mathbb{R}^{p+}_{\leqq}) = \{0\}$. Since $\mathcal{Y}$ and $\mathbb{R}^p_{\leqq}$ are closed, by Theorem 18 $\mathcal{Y} + \mathbb{R}^p_{\leqq}$ is closed, i.e. $\mathcal{Y}$ is $\mathbb{R}^p_{\leqq}$-closed.

Moreover, $\mathcal{X}$ is convex as the intersection of a finite number of closed halfspaces which are convex sets, and the functions $f_1, \ldots, f_{p+2}$ in (134) are convex, then by Theorem 17 $\mathcal{Y}$ is $\mathbb{R}^p_{\leqq}$-convex. □

With Proposition 5 we can now safely consider the weighted-sum scalarization of (134) with strictly positive weights $0 < w$ without sacrificing the completeness of all the achievable task behaviours:

$$\min_{x\in\mathcal{X}} \sum_{k=1}^{p} w_k ||\ddot{\tau}_1 - \ddot{\tau}_p^d||^2 + w_{p+1}||u||^2 + w_{p+2}||\hat{\lambda}||^2 . \quad (137)$$

Problem (137) is a quadratic program of the form:

$$\begin{aligned} &\min_x\ x^T Q x + l^T x, \\ &\text{subject to } H_e x = b_e,\ H_i x \leqq b_i , \end{aligned} \quad (138)$$

where, in particular:

$$Q = \begin{pmatrix} \sum_{k=1}^{p} w_k J_k^T J_k & 0 & 0 \\ 0 & w_{p+1} I_{n-6} & 0 \\ 0 & 0 & w_{p+2} I_{s\cdot c} \end{pmatrix}. \quad (139)$$

D. Stability of the QP

To conclude this section we study some stability properties of the QP (138). Related work for a different control approach can be found for example in [34]. We are interested in the questions of existence, uniqueness and continuity of the solution, as well as robustness to perturbations and modeling uncertainties. We will take as a first assumption the nonemptiness of $\mathcal{X}$ (i.e. the feasibility of the problem) at a given initial state ξ^0. Other assumptions we will make is the full row rank condition of the matrix H_e in (133), i.e. $\operatorname{rank} H_e = n + 6$, and the *regularity* of the system

$$H_e x = b_e,\ H_i x \leqq b_i . \quad (140)$$

Definition 7. *The system of equations and inequalities (140) is said to be regular if H_e has full row rank and there exists x such that $H_e x = b_e$ and $H_i x < b_i$.*

Lemma 4. *Q is symmetric positive definite. Moreover, for any perturbation resulting from the updating of the state $(q, \dot{q})$ or from uncertainty in the model, the perturbed matrix $Q + \delta Q$ remains positive definite.*

Proof. Isolating the configuration task τ_{k_0} in (139) we get:

$$Q = \begin{pmatrix} w_{k_0} I_n & 0 & 0 \\ 0 & w_{p+1} I_{n-6} & 0 \\ 0 & 0 & w_{p+2} I_{s\cdot c} \end{pmatrix} \\ + \sum_{\substack{k=1 \\ k\neq k_0}}^{p} \begin{pmatrix} w_k J_k^T J_k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (141)$$

Since $0 < w$, we have in particular $w_{k_0}, w_{p+1}, w_{p+2} > 0$ and therefore Q is symmetric positive definite. The perturbations of the state and the model would affect only J_k for $k \neq k_0$ in the right-hand side of (141), with $(J_k + \delta J_k)^T (J_k + \delta J_k)$ remaining positive, and therefore $Q + \delta Q$ remains positive definite. □

Proposition 6. *If $\mathcal{X}$ is nonempty then (138) reaches a minimum at a unique point, i.e. the solution exists and is unique.*

Proof. The set $\mathcal{X}$ being compact and the mapping $\mathfrak{F} : x \mapsto x^T Q x + l^T x$ being continuous, from the extreme value theorem (138) has a minimum. $\mathfrak{F}$ being strictly convex from Q positive definite by Lemma 4, the minimizer is unique. □

Proposition 7. *A sufficient condition for the full row rank condition of H_e is that $\operatorname{rank}\begin{pmatrix} K(q) & S \end{pmatrix} = n$ (i.e. the contact forces completely make up for the underactuation).*

Proof. Let $L(q) = \begin{pmatrix} K(q) & S \end{pmatrix}$. We have

$$n + 6 \geq \operatorname{rank} H_e = \operatorname{rank} \begin{pmatrix} M(q) & L(q) \\ J^{\mathrm{S}}(q) & 0 \end{pmatrix}, \quad (142)$$

$$\geq \operatorname{rank} L(q) + \operatorname{rank} J^{\mathrm{S}}(q), \quad (143)$$

$$= \operatorname{rank} L(q) + 6 . \quad (144)$$

Therefore $\operatorname{rank} H_e = n + 6$ if $\operatorname{rank} L(q) = n$. □

Proposition 8. *Let x^0 denote the solution of (138) at an initial point ξ^0. If the system (140) is regular, then there exists*

$\epsilon_1 > 0$ and $\mathcal{K}_1 > 0$ such that, for any update of the state ξ or modelling error (in particular, in $M(q)$, $N(q,\dot{q})$, and the various Jacobians of the robot) the perturbed system

$$(H_e + \delta H_e)x = b_e + \delta b_e,\ (H_i + \delta H_i)x \leqq b_i + \delta b_i\,, \quad (145)$$

remains solvable and regular for those perturbations $(\delta H_e, \delta H_i, \delta b_e, \delta b_i)$ such that

$$\left|\left|\begin{pmatrix}\delta H_e\\ \delta H_i\end{pmatrix}\right|\right| + \left|\left|\begin{pmatrix}\delta b_e\\ \delta b_i\end{pmatrix}\right|\right| \leq \epsilon_1\,, \quad (146)$$

and, denoting x any solution of (145) with $\delta x = x - x^0$, we have

$$||\delta x|| \leq \mathcal{K}_1 \left(\left|\left|\begin{pmatrix}\delta H_e\\ \delta H_i\end{pmatrix}\right|\right| + \left|\left|\begin{pmatrix}\delta b_e\\ \delta b_i\end{pmatrix}\right|\right|\right) \max\{1, ||x^0||\}(1 + ||x^0||)\,. \quad (147)$$

Proof. This is a direct application of [49, Corollary 7] since the conditions of the latter Corollary are all satisfied in the present case. [49, Corollary 7] is itself a direct consequence of the original work of Robinson [50, Theorem 1]. See also the discussion in [51] and in [52]. □

Proposition 9. *Let $p = (\delta Q, \delta l, \delta H_e, \delta H_i, \delta b_e, \delta b_i)$ denote a perturbation of the QP (138). We suppose that H_e and $H_e + \delta H_e$ are both full row rank and that the system (140) is regular at the initial state ξ^0. Then there exists $\epsilon_2 > 0$ and $\mathcal{K}_2 > 0$ such that the solution $x^* = x^0 + \delta x$ of the perturbed QP*

$$\min_x\ x^T(Q + \delta Q)x + (l + \delta l)^T x,$$
$$\text{subject to } (H_e + \delta H_e)x = b_e + \delta b_e,\ (H_i + \delta H_i)x \leqq b_i + \delta b_i\,, \quad (148)$$

exists and is unique and satisfies, whenever $||p||_\infty < \epsilon_2$

$$||x^* - x^0|| < \mathcal{K}_2 ||p||_\infty\,. \quad (149)$$

Proof. Our aim here is to apply [53, Theorem 4.4]. We thus shall show that the hypotheses [53, Equations (3.1) to (3.4)] hold. First, we know that the conditions of Proposition 8 hold, thus the first conclusion we can draw from that Proposition is that there exists $\epsilon_1 > 0$ such that the system (145) is regular and solvable whenever (146) hold. Hence both feasible sets of (138) and (148) are nonempty under (146), which constitutes the first of the needed hypotheses. The other hypotheses are already satisfied by our assumptions and therefore we can apply [53, Theorem 4.4], from which we deduce that, under (146), there exist $\epsilon_1' > 0$ and $\mathcal{K}_2$ such that if $||p||_\infty < \epsilon_1'$ and x' is any solution that minimizes (148) we have $||x^0 - x'|| < ||p||_\infty$. From Lemma 4 $Q + \delta Q$ is positive definite and thus x' is unique and we denote it x^*. Take now

$$\epsilon_2 = \min\left\{\frac{\epsilon_1}{4}, \epsilon_1'\right\}\,. \quad (150)$$

We have

$$||p||_\infty < \epsilon_2 \quad \Rightarrow \quad (146) \text{ and } ||p||_\infty < \epsilon_1'\,. \quad (151)$$

We finally conclude that if $||p||_\infty < \epsilon_2$ then $||x^* - x^0|| < \mathcal{K}_2||p||_\infty$. □

Corollary 5. *In the context and with the notations of Proposition 9 the mapping $p \mapsto x^*$ is well defined on a neighborhood of 0 and continuous at 0.*

Proof. Immediate from Proposition 9. □

V. Conclusion

We have demonstrated that the essence of the multi-task control problem can be effectively captured by the multi-objective optimization formal framework. We discussed the pertinence of scalarizing the vector optimization problem as a weighted sum with positive weights and proved that the positive-weight scalarization does indeed satisfy a completeness property with respect to all the efficient solutions, the popular lexicographic solution being one of them. We studied the Lyapunov stability of the feedback system resulting from such a weighted-sum scalarization scheme and proposed some necessary and/or sufficient conditions for the exponential stability of the equilibrium points of the systems. Finally we applied the study to the particular case of the humanoid robot. We demonstrated that in that case the positive weighted-sum scalarization leads to a linearly-constrained positive definite quadratic problem that is stable and well-behaved under the stated regularity conditions. Future work is dedicated to translating some of the non-constructive pure existence proofs of this paper, proposed essentially as theoretical foundation layers, into practical weight tuning algorithms, which still constitutes an open problem and an active research topic.

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