Numerical approximation of some time optimal control problems

Marius Tucsnak, Julie Valein, Chi-Ting Wu

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Abstract

In this work we study the numerical approximation of the solutions of a class of abstract parabolic time optimal control problems. Our main results assert that, provided that the target is a closed ball centered at the origin and of positive radius, the optimal time and the optimal controls of the approximate time optimal problems converge to the optimal time and to the optimal controls of the original problem. In order to prove our main theorem, we provide a nonsmooth data error estimate for abstract parabolic systems.

Keywords. distributed parameter systems, optimal control, numerical approximation

1. Introduction

Time optimal control of infinite dimensional systems is a subject of growing interest, motivated by numerous applications in domains such as guidance of complex systems or temperature regulation in large buildings. In recent year, using new tools from infinite dimensional systems theory, the literature devoted to this topic grew in a considerable manner (see [1]-[11] and references therein). The specific case of time optimal control for systems governed by parabolic PDE's has numerous applications, from which we quote optimization of building thermal storage (see, for instance [7] and references therein).

The aim of this paper is to study the approximation of the solutions of time optimal control problems for a class of infinite dimensional linear systems by projecting the original problem on an appropriate family of finite dimensional spaces. This is a delicate question since, as shown in the above mentioned references, time optimal controls are usually highly oscillating functions (due to the bang-bang property). As far as we know, the only papers having already investigated this issue are [15] and [16], which investigated finite elements approximation for systems governed by the heat equation.

To be more precise, let X and U be real Hilbert spaces, and let $A_0 : \mathscr{D}(A_0) \to X$ be a strictly positive operator with compact resolvents. It is known that $-A_0$ generates an exponentially stable analytic semigroup, denoted by $\mathbb{T}$. For $\gamma > 0$ we denote by X_γ the space $\mathscr{D}(A_0^\gamma)$, endowed with the graph norm. For $\gamma < 0$, X_γ stands for the dual of $X_{-\gamma}$ with respect to the pivot space X. We also introduce an operator $B \in \mathscr{L}(U, X_{-\alpha})$ with $0 \leqslant \alpha \leqslant \frac{1}{2}$, called *control operator*. In this paper we consider time optimal control problems for the following system,

$$\begin{aligned} \dot{z}(t) + A_0 z(t) &= Bu(t) \qquad (t \geq 0), & (1)\\ z(0) &= z_0 \qquad (z_0 \in X), & (2)\end{aligned}$$

where $u \in L^\infty([0,\infty[;U)$. Using the notation in [12], the solution of (1)-(2) writes :

$$z(t) = \mathbb{T}_t z_0 + \Phi_t u,$$

where

$$\Phi_t u = \int_0^t \mathbb{T}_{t-\sigma} B u(\sigma) d\sigma.$$

Given $\varepsilon > 0$, denote by $\bar{B}(0,\varepsilon)$ the closed ball centered in zero and of radius ε in X. We consider the time optimal control problem which consists in determining the smallest $\tau_0^* > 0$ such that there exists u with $\|u\|_{L^\infty([0,\tau];U)} \leqslant 1$ and where the solution z of (1)-(2) satisfies $z(\tau) \in \bar{B}(0,\varepsilon)$. The corresponding optimal control is denoted by u_0^*.

Denote

$$\mathscr{U}_{\rm ad} = \{u \in L^\infty([0,\infty[;U) \ \mid\ \|u\|_{L^\infty([0,\infty[;U)} \leqslant 1\}.$$

We call $u \in \mathscr{U}_{\rm ad}$ an admissible control if there exists $\tau > 0$ such that $\mathbb{T}_\tau z_0 + \Phi_\tau u \in \bar{B}(0,\varepsilon)$. It is well-known that the above optimal time τ_0^* and optimal control u_0^*

*[1]Institut Élie Cartan de Lorraine, INRIA, Nancy-Université de Lorraine, CNRS, B.P. 70239, F-54506 Vandoeuvre-ls-Nancy Cedex, France (marius.tucsnak@univ-lorraine.fr, julie.valein@univ-lorraine.fr, chi-ting.wu@univ-lorraine.fr)

exist and that, under additional assumptions, they are unique.

Let $(V_h)_{h>0}$ be a family of finite dimensional subspaces of $X_{\frac{1}{2}}$ and let $U_h = B^*V_h$. These spaces are normed spaces endowed with the restriction of the norm of $X_{\frac{1}{2}}$ (resp. U). We denote P_h (resp. Q_h) the orthogonal projector from X onto V_h (resp. U onto U_h). For each $h > 0$, we consider the following system:

$$\begin{aligned} \dot{z}_h(t) + A_h z_h(t) &= B_h u_h(t) \qquad (t \geq 0), && (3)\\ z_h(0) &= P_h z_0, && (4) \end{aligned}$$

where $(A_h)_{h>0}$ is defined by

$$\langle A_h \varphi, \psi \rangle = < A_0^{\frac{1}{2}} \varphi, A_0^{\frac{1}{2}} \psi >, \qquad (5)$$

for every $\varphi, \psi \in V_h$. Moreover, $B_h \in \mathscr{L}(U, V_h)$ is defined by:

$$\langle B_h u, \varphi \rangle = \langle u, B^* \varphi \rangle_U, \qquad (6)$$

for every $\varphi \in V_h,\ u \in U$. The above system is the Galerkin approximation of (1)-(2).

Denote by $\bar{B}_h(0, \varepsilon)$ the closed ball centered in zero in V_h with radius ε. For each $h > 0$, we consider the time optimal control problem for the above system (3)-(4) which is to determine the smallest $\tau_h^* > 0$ such that there exists u_h with $\|u_h\|_{L^\infty([0,\tau];U_h)} \leqslant 1$ and $z_h(\tau) \in \bar{B}_h(0, \varepsilon)$. Moreover, we aim to determine the corresponding optimal controls u_h^*.

The goal of this work is to study the convergence of τ_h^* to τ_0^* and of u_h^* to u_0^* when $h \to 0$. To this aim, we need appropriate assumptions on the approximation properties of the spaces $(V_h)_{h>0}$ and $(U_h)_{h>0}$. More precisely, we assume that there exist $\theta > 0,\ h_1 > 0,\ C > 0,\ 0 \leqslant \beta \leqslant \alpha$, such that for every $h \in (0, h_1)$ and $0 \leqslant \gamma \leqslant 1$, we have:

$(C1)$ $\quad \|x - P_h x\|_X \leqslant C h^{\theta\gamma} \|x\|_\gamma$ for every $x \in X_\gamma$.

$(C2)$ $\quad \|(I - P_h)B\|_{\mathscr{L}(U,X)} \leqslant C h^{\theta(1-\beta)}$.

$(C3)$ $\quad \|P_h B\|_{\mathscr{L}(U,V_h)} \leqslant C h^{-\theta\beta}$.

$(C4)$ $\quad \lim_{h\to 0} \|Q_h u - u\|_U = 0$ for every $u \in U$.

Note that assumptions $(C1) - (C4)$ are very natural when applying our results to systems governed by parabolic partial differential equations. They are satisfied, in particular, by all the usual approximations schemes based on finite elements or finite differences. To be more precise, with assumptions $(C1) - (C4)$, we can prove the following approximation results with rough initial data: $\forall z_0 \in X, \exists\ \hat{h} > 0, C > 0, s.t.\ \forall h \in (0, \hat{h}), \forall t \geqslant 0,$

$$\begin{aligned} \|z(t, z_0, u) - z_h(t, P_h z_0, u)\| \leqslant C h^\theta t^{-1} \|z_0\|_X \\ + C h^{\theta(1-\beta)} |\ln h|\ \|u\|_{L^\infty([0,T];U)}. \end{aligned} \qquad (7)$$

We refer to [9] and [2] for more details about the above approximation.

We are now in position to state the main results of this paper:

Theorem 1. *With the above notation and assumptions, assume that $z_0 \in X$, $\|z_0\| > \varepsilon$ and that $(C1) - (C4)$ hold. Then $\lim_{h\to 0} \tau_h^* = \tau_0^*$.*

Theorem 2. *With the above notation and assumptions, assume that if $z \in X$ is such that the measure of the set of those $t \geqslant 0$ such that $B^* \mathbb{T}_t z = 0$ is strictly positive then $z = 0$. Then we have*

$$u_h^* \to u_0^* \text{ strongly in } L^2([0, \tau^*]; U).$$

2. Sketch of the proof.

2.1. Proof of Theorem 1

The proof of our main results requires several steps, which will be briefly described in this section.

Lemma 1. *With the above notation and assumptions, let λ_1 (resp. $\lambda_{1,h}$) be the smallest eigenvalue of A_0 (resp. of A_h). Denote $z_{0,h} = P_h z_0$. Then*

(i) $\ \tau_0^*(z_0) \leqslant \frac{ln(\|z_0\|/\varepsilon)}{\lambda_1}, \qquad (z_0 \in X).$

(ii) $\ \tau_h^*(z_{0,h}) \leqslant \frac{ln(\|z_{0,h}\|/\varepsilon)}{\lambda_{1,h}}, \qquad (z_{0,h} \in V_h).$

(iii) $\ \lambda_1 \leqslant \lambda_{1,h}.$

Proof. We prove at first (i). It suffices to notice that:

$$\|z(t, z_0, 0)\| \leqslant e^{-\lambda_1 t} \|z_0\|.$$

Then, by taking $t = \frac{ln(\|z_0\|/\varepsilon)}{\lambda_1}$, we have: $\|z(t, z_0, 0)\| \leqslant \varepsilon$. This proves (i).

A similar argument shows that (ii) also holds.

We end by proving (iii). In fact, this inequality is easily deduced by the min-max formula:

$$\lambda_1 = \min_{z \in X_{\frac{1}{2}}} \frac{\|A_0^{\frac{1}{2}} z\|^2}{\|z\|^2}$$

and

$$\lambda_{1,h} = \min_{z \in V_h} \frac{\|A_0^{\frac{1}{2}} z\|^2}{\|z\|^2}.$$

We also need the following result

Lemma 2. *With the notation and assumptions in Lemma 1, for every $z_0 \in X$, $\|z_0\| > \varepsilon$, there exist $c, C > 0, \tilde{h} > 0$ such that for any $h \in (0, \tilde{h})$, we have*

$$c \leqslant \tau_h^*(P_h z_0) \leqslant C,$$

where $C = \frac{2ln(\|z_0\|/\varepsilon)}{\lambda_1}$.

Proof. We begin by proving that $\tau_h^*(P_h z_0)$ is bounded from below. Suppose by contradiction that $\lim_{h\to 0} \tau_h^*(P_h z_0) = 0$. By the continuity of $t \mapsto z_h(t)$, we have:

$$\lim_{h\to 0} \|z_h(\tau_h^*, P_h z_0, u_h^*) - z_h(0, P_h z_0, u_h^*)\|$$

$$= \lim_{h\to 0} \|z_h(\tau_h^*, P_h z_0, u_h^*) - P_h z_0\| = 0.$$

Using the fact that $\|z_h(\tau_h^*, P_h z_0, u_h^*)\| \leqslant \varepsilon$, it is clear that $\lim_{h\to 0} \|P_h z_0\| \leqslant \varepsilon$. However, with $(C1)$ it is clear that:

$$\lim_{h\to 0} \|P_h z_0 - z_0\| = 0,$$

which leads to the contradiction with the fact that $\|z_0\| > \varepsilon$.

We prove now that $\tau_h^*(P_h z_0)$ is bounded from above. This is obvious by using Lemma 1, since

$$\tau_h^*(P_h z_0) \leqslant \frac{ln(\|P_h z_0\|/\varepsilon)}{\lambda_{1,h}} \leqslant \frac{2ln(\|z_0\|/\varepsilon)}{\lambda_1} < +\infty.$$

Proof of Theorem 1. It suffices to prove the following two inequalities:

$$\liminf_{h\to 0} \tau_h^* \geqslant \tau_0^*, \tag{8}$$

$$\limsup_{h\to 0} \tau_h^* \leqslant \tau_0^*. \tag{9}$$

We begin by proving (8). We first notice that, for every $T > 0$ and $u \in \mathscr{U}_{ad}$:

$$\tau_0^*(z_0) \leqslant T + \tau_0^*(z(T, z_0, u)). \tag{10}$$

By (7), we have

$$\|z(\tau_h^*, z_0, u_h^*) - z_h(\tau_h^*, P_h z_0, u_h^*)\|$$

$$\leqslant C h^\theta \tau_h^{*-1} \|z_0\|_X + C h^{\theta(1-\beta)} |\ln h| \; \|u\|_{L^\infty([0,T];U)}.$$

This leads to:

$$\|z(\tau_h^*, z_0, u_h^*)\| \leqslant \varepsilon + C h^\theta \tau_h^{*-1} \|z_0\|_X$$

$$+ C h^{\theta(1-\beta)} |\ln h| \; \|u\|_{L^\infty([0,T];U)}$$

$$\leqslant \varepsilon + C h^\theta \tau_h^{*-1} + C h^{\theta(1-\beta)} |\ln h|.$$

Denote $\bar{z_0} = z(\tau_h^*, z_0, u_h^*)$. According to (10) with $T = \tau_h^*$, we have:

$$\tau_0^*(z_0) \leqslant \tau_h^* + \tau_0^*(\bar{z_0}).$$

In fact, $u_h^* \in L^\infty([0, +\infty[; U_h) \subset L^\infty([0, +\infty[; U)$ and $\|u_h^*(t)\| \leqslant 1$ which means that u_h^* is an admissible control for the original system.

Then, according to Lemma 1, we have:

$$\tau_0^* \leqslant \tau_h^* + \frac{ln((\varepsilon + C h^\theta \tau_h^{*-1} + C h^{\theta(1-\beta)} |\ln h|)/\varepsilon)}{\lambda_1}$$

$$\leqslant \tau_h^* + \frac{C h^\theta}{\lambda_1 \varepsilon} \tau_h^{*-1} + C \frac{h^{\theta(1-\beta)} |\ln h|}{\lambda_1 \varepsilon}. \tag{11}$$

Thus, (8) can be deduced by taking h to zero and by the fact that $\lim_{h\to 0} \tau_h^* > c > 0$ (Lemma 2).

We now prove the second inequality (9). We have:

$$\|z_h(\tau_0^*, P_h z_0, Q_h u^*) - z(\tau_0^*, z_0, u^*)\|$$

$$\leqslant \|z_h(\tau_0^*, P_h z_0, Q_h u^*) - z_h(\tau_0^*, P_h z_0, u^*)\|$$

$$+ \|z_h(\tau_0^*, P_h z_0, u^*) - z(\tau_0^*, z_0, u^*)\|$$

$$\leqslant \|z_h(\tau_0^*, P_h z_0, Q_h u^*) - z_h(\tau_0^*, P_h z_0, u^*)\|$$

$$+ C h^\theta \tau_0^{*-1} + C h^{\theta(1-\beta)} |\ln h|.$$

Set $f(h) = \|z_h(\tau_0^*, P_h z_0, Q_h u^*) - z_h(\tau_0^*, P_h z_0, u^*)\|$.
We notice that $\lim_{h\to 0} f(h) = 0$. Indeed,

$$\lim_{h\to 0} \|z_h(\tau_0^*, P_h z_0, Q_h u^*) - z_h(\tau_0^*, P_h z_0, u^*)\|$$

$$= \lim_{h\to 0} \|\Phi_{\tau_0^*, h}(u^* - Q_h u^*)\|.$$

Since $\Phi_{\tau_0^*, h} \in \mathscr{L}(L^2(0, \tau_0^*; U), X)$ (by the admissibility assumption upon B), this leads to:

$$\lim_{h\to 0} f(h) \leqslant K \lim_{h\to 0} \|Q_h u^* - u^*\|_{L^2(0,\tau_0^*;U)} \to 0,$$

using $(C4)$. Thus, we have:

$$\|z_h(\tau_0^*, P_h z_0, Q_h u_h^*)\| \leqslant \varepsilon + f(h)$$

$$+ C h^\theta \tau_0^{*-1} + C h^{\theta(1-\beta)} |\ln h|.$$

By the similar argument as in (11), we have:

$$\tau_h^* \leqslant \tau_0^*$$

$$+ \frac{\ln((\varepsilon + f(h) + C h^\theta \tau_0^{*-1} + C h^{\theta(1-\beta)} |\ln h|)/\varepsilon)}{\lambda_{1,h}}$$

$$\leqslant \tau_0^* + \frac{f(h) + C h^\theta \tau_0^{*-1} + C h^{\theta(1-\beta)} |\ln h|}{\lambda_1 \varepsilon}.$$

This leads to inequality (9) by letting h tend to zero.

2.2. Proof of Theorem 2

Before giving the proof, we recall a standard energy estimate.

Lemma 3. *Assume that $z_0 \in X_{\frac{1}{2}-\alpha}$. Then, there exists $c>0$ such that*

$$\|z(\tau)\|^2_{\frac{1}{2}-\alpha} + \int_0^\tau \left(\|\dot z(s)\|^2_{-\alpha} + \|z(s)\|^2_{1-\alpha}\right) \mathrm{d}s$$

$$\leqslant C\left(\int_0^\tau \|Bu(s)\|^2_{-\alpha} ds + \|z_0\|^2_{\frac{1}{2}-\alpha}\right).$$

Proof of Theorem 2.

Denote $T = 2\frac{ln(\|z_0\|/\varepsilon)}{\lambda_1}$. It is clear that $\tau_h^* \leqslant T$ for all $h>0$ and $\tau_0^* \leqslant T$. We extend $(u_h^*)_h$ and (u_0^*) to time T by zero.

Since $\|u_h^*\|_{L^\infty(0,T;U)} \leqslant 1$, there exist a control $\bar u \in L^\infty(0,T;U)$ and a subsequence $(h_n)_n \to 0$, such that :

$$u^*_{h_n} \to \bar u \quad \text{weakly}* \quad in \quad L^\infty(0,T;U).$$

Now we prove that $\bar u = u_0^*$.

The main step here is to prove the following convergence property :

$$\|z_{h_n}(\tau^*_{h_n}, u^*_{h_n}, P_h z_0) - z(\tau_0^*, \bar u, z_0)\| \to 0. \tag{12}$$

Indeed, since $\bar B(0,\varepsilon)$ is complete (notice that $\bar B_h(0,\varepsilon) \subset \bar B(0,\varepsilon)$), (12) leads to $z(\tau_0^*, \bar u, z_0) \in \bar B(0,\varepsilon)$. Then, by the uniqueness of the time optimal control, we deduce that $\bar u = u_0^*$.

Now we prove (12). We have :

$$\|z_{h_n}(\tau^*_{h_n}, u^*_{h_n}, P_h z_0) - z(\tau_0^*, \bar u, z_0)\|$$

$$\leqslant \|z_{h_n}(\tau^*_{h_n}, u^*_{h_n}, P_h z_0) - z(\tau^*_{h_n}, u^*_{h_n}, z_0)\| \tag{13}$$

$$+\|z(\tau^*_{h_n}, u^*_{h_n}, z_0) - z(\tau^*_0, u^*_{h_n}, z_0)\| \tag{14}$$

$$+\|z(\tau^*_0, u^*_{h_n}, z_0) - z(\tau^*_0, \bar u, z_0)\|. \tag{15}$$

Now we prove that these three parts converge to zero in order to deduce (12).

It is clear that (13) converges to zero using the error estimate (7).

Moreover, since $t \mapsto z(t,u,z_0)$ is continuous and $\tau^*_{h_n} \to \tau^*_0$, (14) converges to zero.

It remains to prove that (15) converges to zero. For that, denote $\psi(t) = z(t,\bar u, z_0)$ and $\psi_n(t) = z(t, u^*_{h_n}, z_0)$.

Then by Lemma 3, we know that $(\psi_n)_n$ is a bounded sequence in :

$$W = C(0,T;X_{\frac{1}{2}-\alpha}) \cap L^2(0,T;X_{1-\alpha})$$

$$\cap W^{1,2}(0,T;X_{-\alpha}).$$

Using a generalized Aubin-Lions Theorem (see [13, Cor. 4, p.85])) we deduce that :

$$\exists\, \widetilde\psi \in C([0,T];X)\ s.t.,$$

$$\psi_n \to \tilde\psi \text{ strongly in } C(0,T;X)$$

and

$$\psi_n \to \tilde\psi \text{ weakly in } W.$$

Now we prove that $\widetilde\psi = \psi$. We know that $(\psi_n)_n$ satisfies :

$$\begin{aligned} \dot\psi_n &= A\psi_n + Bu^*_{h_n}, \\ \psi_n(0) &= z_0. \end{aligned}$$

We prove then that $\dot\psi_n \to \dot{\tilde\psi}$ weakly in $L^2(0,T;X_{-\alpha})$, $A\psi_n \to A\widetilde\psi$ weakly in $L^2(0,T;X_{-\alpha})$, $Bu^*_{h_n} \to B\bar u$ weakly in $L^2([0,T];X)$ and $\tilde\psi(0) = z_0$.

The first two convergences are clear since $\psi_n \to \tilde\psi$ weakly in W.

Moreover, $u^*_{h_n} \overset{*}{\rightharpoonup} \bar u$ weakly $*$ in $L^\infty([0,T];U)$ implies that $u^*_{h_n} \rightharpoonup \bar u$ weakly in $L^2([0,T];U)$. Thus, $Bu^*_{h_n} \rightharpoonup B\bar u$ in $L^2([0,T];X)$.

It remains to prove that $\tilde\psi(0) = z_0$. Indeed, we know $z_0 = \psi_n(0) \to \tilde\psi(0)$, since $\psi_n \to \tilde\psi$ strongly in $C(0,T;X)$.

Consequently, $\tilde\psi$ satisfies :

$$\begin{aligned} \dot{\tilde\psi} &= A\tilde\psi + B\bar u, \\ \tilde\psi(0) &= z_0, \end{aligned}$$

which implies that $\tilde\psi = \psi$.

This leads to the fact that (15) converges to zero.

Thus, we have :

$$u_h^* \to u_0^* \text{ weakly } * \text{ in } L^\infty(0,T;U). \tag{16}$$

We deduce immediately that :

$$u_h^* \to u_0^* \text{ weakly in } L^2(0,T;U).$$

At last, since both u_h^* and u_0^* are bang-bang controls, we have $\lim_{h\to 0}\|u_h^*\|_{L^2(0,T;U)} = \|u_0^*\|_{L^2(0,T;U)}$. This leads to the strong convergence in $L^2(0,T;U)$ and ends the proof.

3. Example.

We consider here 1-D heat equation over $[0,1]$ with internal control over $[\frac{1}{3},\frac{2}{3}]$, more precisely, for every $t \geq 0$,

$$\begin{aligned} \dot z(t,x) &= \partial^2_{xx} z(t,x) \\ &\quad + \chi_{[\frac{1}{3},\frac{2}{3}]}(x) u(t,x) \quad (x\in[0,1], t\geq 0), \end{aligned} \tag{17}$$

$$z(t,0) = z(t,1) = 0, \quad (t \geq 0), \tag{18}$$

$$z(0,x) = 2sin(\pi x), (x \in [0,1]), \tag{19}$$

where $\chi_{[\frac{1}{3},\frac{2}{3}]}$ is the characteristic function of the interval $[\frac{1}{3},\frac{2}{3}]$. Obviously, (17)-(19) has the form (1)-(2) by taking $A_0 = -\partial^2_{xx}$ with Dirichlet boundary conditions of domain $D(A_0) = H^1_0(0,1) \cap H^2(0,1)$ on $X = L^2(0,1)$.

The control operator $B \in \mathscr{L}(U,X)$ (here $\alpha = 0$) is defined by:

$$B\varphi = \chi_{[\frac{1}{3},\frac{2}{3}]}\tilde{\varphi},$$

where $U = L^2(\frac{1}{3},\frac{2}{3})$ and where $\tilde{\varphi}$ is the extension of φ outside $[\frac{1}{3},\frac{2}{3}]$.

Now we consider the space semi-discrete approximation of (17)-(19) derived by the finite difference method. More precisely, for $N \in \mathbb{N}^*$ given and $h = \frac{1}{N+1}$, let $z_i(t)$ an approximation of $z(t,ih)$. We consider the following scheme:

$$\dot{z}_i(t) = \frac{z^n_{i+1} - 2z^n_i + z^n_{i-1}}{h^2} + B_h u_i(t),$$

$$z_0(t) = z_{N+1}(t) = 0,$$

$$z_i(0) = 2sin(\pi ih),$$

where $B_h u_i = u_i$ if $ih \in [\frac{1}{3},\frac{2}{3}]$ and 0 otherwise. If we denote the unknown $z_h(t) = (z_i(t))^T_{1\leqslant i\leqslant N}$, the above scheme can be rewritten in the vector form as (3)-(4).

It is well known that that $(C1)-(C4)$ are satisfied with $\beta = 0$ and $\theta = 1$ (see for example in [14])

According to Theorem 1, for every $z_0 \in X$, $\|z_0\|_X \geqslant \varepsilon$, we have $\lim_{h\to 0} \tau^*_h = \tau^*_0$. We test this scheme in Matlab and have the following result:

N	5	10	20	30	40	50
τ^*_h	0.0344	0.0336	0.0331	0.0332	0.0331	0.0330

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