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# Rational approximation to values of $G$-functions, and their expansions in integer bases

S. Fischler and T. Rivoal

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**Abstract**

Building upon previous works of André and Chudnovsky, we prove a general result concerning the approximations of values at rational points $a/b$ of any $G$-function $F$ with rational Taylor coefficients by fractions of the form $n/(B \cdot b^m)$, where the integer $B$ is fixed. As a corollary, we show that if $F$ is not in $\mathbb{Q}(z)$, then for any $\varepsilon > 0$, $|F(a/b) - n/b^m| \geq 1/b^{m(1+\varepsilon)}$ provided $b$ and $m$ are large enough with respect to $a$, $\varepsilon$ and $F$. This enables us to obtain a new result on the repetition of patterns in the $b$-ary expansion of $F(a/b)$ when $b \geq 2$. In particular, defining $\mathcal{N}(n)$ as the number of consecutive equal digits in the $b$-ary expansion of $F(a/b^s)$ starting from the $n$-th digit, we prove that $\limsup_n \mathcal{N}(n)/n \leq \varepsilon$ provided the integer $s \geq 1$ is such that $b^s$ is large enough with respect to $a$, $\varepsilon$ and $F$. This is a step towards the conjecture that this limit should be equal to 0 whenever $F(a/b)$ is an irrational number. All our results are effective.

## 1 Introduction

This paper deals with the approximations of values of $G$-functions at rational points by rational numbers with denominator a power of a fixed integer; an important motivation is that periods are conjecturally values of $G$-functions (see [18, Section 2.2]). Before stating our results, we recall some important results in the Diophantine theory of $G$-functions, as well as of $E$-functions, even though no new result will be given for the latter. Throughout the paper we fix an embedding of $\overline{\mathbb{Q}}$ into $\mathbb{C}$.

**Definition 1.** *A $G$-function $F$ is a power series $F(z) = \sum_{n=0}^{\infty} a_n z^n$ such that the coefficients $a_n$ are algebraic numbers and there exists $C > 0$ such that, for any $n \geq 1$:*

*$(i)$ the maximum of the moduli of the conjugates of $a_n$ is $\leq C^n$.*

*$(ii)$ there exists a sequence of rational integers $d_n$, with $|d_n| \leq C^n$, such that $d_n a_m$ is an algebraic integer for all $m \leq n$.*

*$(iii)$ $F(z)$ satisfies an homogeneous linear differential equation with coefficients in $\overline{\mathbb{Q}}(z)$.*

*An $E$-function is a power series $F(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n$ with algebraic coefficients and satisfying the same conditions $(i), (ii)$ and $(iii)$.*

Siegel's original definition [24] of $E$ and $G$-functions is slightly more general but it is believed to define the same functions as above. It is a fact that the Diophantine theory of $G$-functions is not as fully developped as that of $E$-functions. There is no general theorem about the transcendance of values of $G$-functions, but results like the following one.

**Theorem 1** (Chudnovsky [16]). *Let $Y(z) = {}^t(F_1(z), \dots, F_n(z))$ be a vector of $G$-functions solution of a differential system $Y'(z) = A(z)Y(z)$, where $A(z) \in M_n(\overline{\mathbb{Q}}(z))$. Assume that $F_1(z)$, ..., $F_n(z)$ are $\mathbb{C}(z)$-algebraically independent.*

*Then for any $d$, there exists $C = C(Y, d) > 0$ such that, for any algebraic number $\alpha \neq 0$ of degree $d$ with $|\alpha| < \exp(-C \log(H(\alpha))^{\frac{4n}{4n+1}})$, there does not exist a polynomial relation between the values $F_1(\alpha), \dots, F_n(\alpha)$ over $\mathbb{Q}(\alpha)$ of degree $d$.*

Here, $H(\alpha)$ is the naive height of $\alpha$, i.e. the maximum of the modulus of the coefficients of the (normalized) minimal polynomial of $\alpha$ over $\mathbb{Q}$. Chudnovky's theorem refines the works of Bombieri [12] and Galoshkin [19]. André generalized Chudnovsky's theorem to the case of an inhomogenous system $Y'(z) = A(z)Y(z) + B(z)$, $A(z), B(z) \in M_n(\overline{\mathbb{Q}}(z))$, with a similar condition on $\alpha$ and $H(\alpha)$; see [4, pp. 130-138] when the place $v$ is archimedean. André and Chudnovky's theorems are still essentially the best known today in this generality but they are far from being transcendance or algebraic independence statements. We recall that, in fact, it is not even known if there exist three algebraically independent $G$-values. ([1])

On the other hand, the situation is essentially best possible for $E$-functions.

**Theorem 2** (Siegel-Shidlovsky [24, 23]). *Let $Y(z) = {}^t(F_1(z), \dots, F_n(z))$ be a vector of $E$-functions solution of a differential system $Y'(z) = B(z)Y(z)$, where $B(z) \in M_n(\overline{\mathbb{Q}}(z))$. Denote the common denominator of the entries of $B(z)$ by $T(z)$. Then, for any $\xi \in \overline{\mathbb{Q}}$ such that $\xi T(\xi) \neq 0$, we have*

$$\deg \operatorname{tr}_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}}\big(F_1(\xi), \dots, F_n(\xi)\big) = \deg \operatorname{tr}_{\overline{\mathbb{Q}}(z)} \overline{\mathbb{Q}}(z)\big(F_1(z), \dots, F_n(z)\big).$$

Note that Beukers [11] was even able to describe very precisely the nature of the numerical algebraic relations when the transcendence degree is not maximal.

A lot of work has been devoted to improvements of Theorem 1, or alike, for classical $G$-functions like the polylogarithms $\sum_{n=1}^{\infty} z^n/n^s$, or to determine weaker conditions for the irrationality of the values of $G$-functions at rational points. From a qualitative point of view, the result is the following.

**Theorem 3.** *Let $F$ be a $G$-function with rational Taylor coefficients such that $F(z) \notin \mathbb{Q}(z)$. Then there exist some positive constants $C_1$ and $C_2$, depending only on $F$, with the following property. Let $a \neq 0$ and $b \geq 1$ be integers such that*

$$b > (C_1|a|)^{C_2}. \tag{1.1}$$

[1] There exist examples of two algebraically independent $G$-values, for instance $\pi$ and $\Gamma(1/3)^3$, or $\pi$ and $\Gamma(1/4)^4$. This was first proved by Chudnovsky with a method not related to $G$-functions, but André [5] obtained a proof with certain Gauss' hypergeometric functions, which are $G$-functions. André's method is very specific and has not been generalized.

*Then $F(a/b)$ is irrational.*

This result follows from Theorem I in [16, 17], together with an irrationality measure; see also [19]. This measure and the value of $C_2$ have been improved by Zudilin [26], under further assumptions on $F$. He obtains the following result (in a more precise form).

**Theorem 4** (Zudilin [26])**.** *Let $n \geq 2$ and $Y(z) = {}^t(F_1(z), \ldots, F_n(z))$ be a vector of $G$-functions solution of a differential system $Y'(z) = A(z)Y(z) + B(z)$, where $A(z), B(z) \in M_n(\mathbb{C}(z))$. Assume that either $n = 2$ and $1, F_1(z), F_2(z)$ are $\mathbb{C}(z)$-linearly independent, or that $n \geq 3$ and $F_1(z), \ldots, F_n(z)$ are $\mathbb{C}(z)$-algebraically independent. Let $\varepsilon > 0$, $a \in \mathbb{Z}$, $a \neq 0$. Let $b$ and $q$ be sufficiently large positive integers, in terms of the $F_j$'s, $a$ and $\varepsilon$; then $F_j(a/b)$ is an irrational number and for any integer $p$, we have*

$$\left| F_j\left(\frac{a}{b}\right) - \frac{p}{q} \right| \geq \frac{1}{q^{2+\varepsilon}}, \quad j = 1, \ldots, N. \tag{1.2}$$

Zudilin's proof follows Shidlovsky's ineffective approach to zero estimates (see [23, p. 93, Lemma 8]). It is likely that using an effective method instead (see [4, Appendix of Chapter III], [9] and [15]), one would make Theorem 4 effective. We mention that Zudilin [25] also obtained similar irrationality measures for the values of $E$-functions at any non-zero rational point.

We now come to our main result. Roughly speaking, it is an improvement of Zudilin's exponent $2+\varepsilon$ in (1.2) when $q$ is restricted to integers of the form $b^m$, or of a slightly more general type. In this case, the exponent drops from $2+\varepsilon$ to $1+\varepsilon$. We first state a more precise version, without $\varepsilon$.

**Theorem 5.** *Let $F$ be a $G$-function with rational Taylor coefficients and with $F(z) \notin \mathbb{Q}(z)$, and $t \geq 0$. Then there exist some positive effectively computable constants $c_1$, $c_2$, $c_3$, $c_4$, depending only on $F$ (and $t$ as well for $c_3$), such that the following property holds. Let $a \neq 0$ and $b, B \geq 1$ be integers such that*

$$b > (c_1|a|)^{c_2} \text{ and } B \leq b^t. \tag{1.3}$$

*Then for any $n \in \mathbb{Z}$ and any $m \geq c_3 \frac{\log(b)}{\log(|a|+1)}$ we have*

$$\left| F\left(\frac{a}{b}\right) - \frac{n}{B \cdot b^m} \right| \geq \frac{1}{B \cdot b^m \cdot (|a|+1)^{c_4 m}}. \tag{1.4}$$

In the case of the dilogarithm $\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}$, our proof provides $c_1 = 4e^{66}$, $c_2 = 12$ and $c_4 = 10^6$. We did not try to compute $c_3$ because it is useless for the application stated in Theorem 6 below, but this could be done in principle. Needless to say, these values are far from the best possible but this is not the point of this paper. For related results, but restricted only to the $G$-functions $(1-z)^\alpha$ and $\log(1-z)$, see [7, 8, 10] and [21] respectively. We point out that Theorem 5 is effective, because an effective zero estimate (due to André) is used. In contrast with Zudilin's theorem, we only need to assume that $F(z) \notin \mathbb{Q}(z)$. On

the other hand, we did not try to compare $(c_1, c_2)$ with the constants $(C_1, C_2)$ of Theorem 3, so that Theorem 5 might apply in some cases to rational values $F(a/b)$. If under (1.3), $F(a/b)$ is a rational number $u/v$ with coprime integers $u$ and $v$, then (1.4) proves that the denominator $v$ is not of the form $B \cdot b^m$ with $B \leq b^t$. Consequently, in the $b$-ary expansion of $F(a/b)$, the period $\ell$ is greater than $t$ because $v$ divides $b^m(b^\ell - 1)$ for some $m$.

Given $\varepsilon > 0$ and assuming that $m \geq 2t/\varepsilon$ and $b > (|a|+1)^{2c_4/\varepsilon}$, we derive the following corollary. It is also effective and it will be used to prove Theorem 6 below.

**Corollary 1.** *Let $F$ be a $G$-function with rational Taylor coefficients and with $F(z) \notin \mathbb{Q}(z)$, $\varepsilon > 0$, $t \geq 0$ and $a \in \mathbb{Z}$, $a \neq 0$. Let $b$ and $m$ be positive integers, sufficiently large in terms of $F$, $\varepsilon$, $a$ (and $t$ for $m$). Then for any integers $n$ and $B$ with $1 \leq B \leq b^t$, we have*

$$\left| F\left(\frac{a}{b}\right) - \frac{n}{B \cdot b^m} \right| \geq \frac{1}{b^{m(1+\varepsilon)}}.$$

We don't know if some analogues of Theorem 5 and Corollary 1 hold when $F$ is supposed to be an $E$-function. This is surprising because, as we indicated above, the Diophantine theory of $E$-values is much more advanced than that of $G$-values. Our method is inoperant for $E$-functions and we could not find any way to fix it. We explain the reason for this unusual advantage of $G$-functions in the final Section 5. We also explain there that an analogue of Theorem 5 holds for $1/F(z)$ instead of $F(z)$ under a less general assumption on the $G$-function $F(z)$.

The quality of restricted rational approximants as in Theorem 5 and Corollary 1 can be measured (when $t = 0$) by a Diophantine exponent $v_b$ studied in [3]. Given $\xi \in \mathbb{R} \setminus \mathbb{Q}$, $v_b(\xi)$ is the infimum of the real numbers $\mu$ such that $|\xi - \frac{n}{b^m}| \geq b^{-m(1+\mu)}$ for any $n \in \mathbb{Z}$ and any sufficiently large $m$. With this notation, the special case $t = 0$ of Corollary 1 reads $v_b(F(a/b)) \leq \varepsilon$. The metric properties of this Diophantine exponent are studied in [3, Section 7]: with respect to Lebesgue measure, almost all real numbers $\xi$ satisfy $v_b(\xi) = 0$ for any $b \geq 2$, and given $b \geq 2$ the set of $\xi$ such that $v_b(\xi) \geq \varepsilon$ has Hausdorff dimension $\frac{1}{1+\varepsilon}$. Therefore Theorem 5 and Corollary 1 are a step towards the conjecture that values of $G$-functions behave like generic real numbers with respect to rational approximation.

Our results have interesting consequences on the nature of the $b$-ary expansions of values of $G$-functions; this is a class of numbers for which very few such results are known (see [13, 14]). Let $b$, $t$ be integers with $b \geq 2$ and $t \geq 1$, and let $\xi \in \mathbb{R} \setminus \mathbb{Q}$. We denote by $0.a_1a_2a_3\ldots$ the expansion in base $b$ of the fractional part of $\xi$. For any $n \geq 1$, let $\mathcal{N}_b(\xi, t, n)$ denote the largest integer $\ell$ such that $(a_na_{n+1}\ldots a_{n+t-1})^\ell$ is a prefix of the infinite word $a_na_{n+1}a_{n+2}\ldots$. In other words, it is the number of times the pattern $a_na_{n+1}\ldots a_{n+t-1}$ is repeated starting from $a_n$. Obviously $\mathcal{N}_b(\xi, t, n) \geq 1$, and $\mathcal{N}_b(\xi, t, n)$ is finite since $\xi$ is irrational. If $t = 1$, $\mathcal{N}_b(\xi, t, n)$ is simply the number of consecutive equal digits in the expansion of $\xi$, starting from $a_n$. For almost all real numbers $\xi$ with respect to Lebesgue measure, $\lim_{n\to\infty} \frac{1}{n}\mathcal{N}_b(\xi, t, n) = 0$.

If $\xi$ is a rational number with a period of length greater than $t$ in its $b$-ary expansion, then $\mathcal{N}_b(\xi, t, n)$ is also finite. We mentioned above that under the hypothesis of Theorem 5,

if $F(a/b)$ is a rational number then the period of its $b$-ary expansion is greater than $t$. The theorem below, which is effective, applies also to this (presumably exceptional) case.

**Theorem 6.** *Let $F$ be a $G$-function with rational Taylor coefficients and with $F(z) \notin \mathbb{Q}(z)$, $\varepsilon > 0$, and $a \in \mathbb{Z}$, $a \neq 0$. Let $b \geq 2$. Then for any $s \geq 1$ such that $b^s$ is sufficiently large in terms of $F$, $\varepsilon$, and $a$, we have for any $t \geq 1$:*

$$\limsup_{n\to\infty} \frac{1}{n} \mathcal{N}_b\big(F(a/b^s), t, n\big) \leq \varepsilon/t.$$

In the case of the dilogarithm, this result applies to $\mathrm{Li}_2(1/b^s)$ for $a = 1$, any fixed $\varepsilon \in (0,1)$, any $t \geq 1$ and any $b \geq 2$ provided $s \geq 10^7/\varepsilon$. A similar bound for this upper limit, but with $1+\varepsilon$ instead of $\varepsilon$, follows from (and under the assumptions of) Theorem 4. Conjecturally, we have $\lim_n \frac{1}{n}\mathcal{N}_b(\xi, t, n) = 0$ whenever $\xi$ is a transcendental value of a $G$-function, but it seems that the only such $\xi$ for which the upper bound $\limsup_n \frac{1}{n}\mathcal{N}_b(\xi, t, n) < 1$ was known are values of the logarithm (using a slight generalization of the results of [21] to allow a parameter $t$ as in Theorem 5 and Corollary 1).

When $\xi$ is an irrational algebraic number, Ridout's theorem [20] yields $v_b(\xi) = 0$ and $\lim_n \frac{1}{n}\mathcal{N}_b(\xi, t, n) = 0$ for any $b$ and any $t$. It is not effective: for a general real algebraic number $\xi$, given $b$, $t$ and $\varepsilon > 0$, no explicit value of $N(\xi, b, t, \varepsilon)$ is known such that $\mathcal{N}_b(\xi, t, n) \leq \varepsilon n$ for any $n \geq N(\xi, b, t, \varepsilon)$. On the contrary, if $\xi = F(a/b^s)$ then Theorem 6 provides such an explicit value provided $b^s$ is large enough – recall that algebraic functions which are holomorphic at 0 are $G$-functions. However if $\xi$ is fixed then Theorem 6 applies only if $\varepsilon$ is not too small: we do not really get an effective version of Ridout's theorem for $\xi$. For other results concerning the $b$-ary expansions of algebraic numbers, we refer the reader to [2, 6, 22].

We proved in [18] that any real algebraic number is equal to $F(1)$ for some algebraic $G$-function $F$ with rational coefficients and radius of convergence arbitrarily large. Unfortunately, we do not have a control on the growth of the sequence of denominators of the coefficients of $F$, which is important in the computation of the constants in Theorem 5. Therefore, we cannot prove that any real algebraic number can be realized as a $G$-value $F(a/b)$ to which Theorem 5 applies.

Finally, let us explain the basic reason behind our improvement on Zudilin's exponent. To estimate the difference $|F(\frac{a}{b}) - \frac{p}{q}|$ using the methods of this article, we need at some point to find a lower bound for a certain difference $D = |\frac{p}{q} - \frac{u}{b^k v}|$ between two distinct rationals ($k \in \mathbb{N}$, $p, q, u, v \in \mathbb{Z}$). When $q$ could be anything, the best we can say is that, trivially, $D \geq (b^k q v)^{-1}$; however, if we know in advance that $q = b^K$ then we can improve the trivial bound to $D \geq (b^{\max(K,k)} v)^{-1}$ and we save a factor of $b^{\min(K,k)}$ in the process. The fraction $\frac{u}{b^k v}$ is obtained by constructing (inexplicit) Padé approximants of type II to $F(z)$ and the other $G$-functions appearing in a differential system of order 1 satisfied by $F$. Inexplicit Padé approximation is a classical tool in the Diophantine theory of $G$-functions.

Our main new ingredient is the use of non-diagonal Padé type approximants, i.e. the polynomials are made to have different degrees, which creates the factor $b^k$ we need. This

idea seems to have been introduced in [10] in a particular case; we use it in its full generality. To illustrate its importance for Theorem 6, we remark that if one tries to compute an irrationality measure for $F(a/b)$ under the assumptions of Theorem 3 with the method of the present paper, one gets an irrationality exponent not smaller than $N+1+\varepsilon$, where $N$ is the least integer such that $1, F(z), \ldots, F^{(N)}(z)$ are linearly dependent over $\mathbb{Q}(z)$.

The structure of this paper is as follows. Section 2 is devoted to general results on Padé-type approximation, and Section 3 to the proof of Theorem 5. At last, we deduce Theorem 6 in Section 4 and conclude with some remarks in Section 5.

# 2 Non diagonal Padé-type approximation

We gather in this section known results and preparatory computations that will be used in Section 3 to prove Theorem 5.

## 2.1 Setting and zero estimate

Let $F_1(z), \ldots, F_N(z)$ be $G$-functions with rational coefficients. We let $F_0(z) = 1$ and assume that $F_0, F_1, \ldots, F_N$ are linearly independent over $\mathbb{Q}(z)$. We assume also that $Y(z) = {}^t\big(1, F_1(z), \ldots, F_N(z)\big)$ is a solution of a differential system of order 1

$$Y'(z) = A(z)Y(z) \tag{2.1}$$

where $A(z) \in M_{N+1,N+1}(\mathbb{Q}(z))$ is a matrix of which the rows and columns are numbered from 0 to $N$.

Let $\mathcal{D}(z)$ be a non-zero polynomial in $\mathbb{Z}[z]$ such that $\mathcal{D}(z)A(z) \in M_{N+1,N+1}(\mathbb{Z}[z])$. Let $d \in \mathbb{N}$ be such that

$$\deg \mathcal{D}(z) \leq d \quad \text{and} \quad \deg \mathcal{D}(z)A_{i,j}(z) \leq d-1$$

for any coefficient $A_{i,j}(z)$ of $A(z)$. We observe that $\mathcal{D}(z)$ is not a constant polynomial because if $A(z)$ has polynomial entries, the system (2.1) cannot have a non-zero vector of solutions formed of $G$-functions; therefore $d \geq 1$.

For any integers $p, q, h$ such that $p \geq q \geq Nh \geq 0$, we can find $N$ polynomials $P_1(z), \ldots, P_N(z) \in \mathbb{Q}[z]$ of degree $\leq p$ and $Q(z) \in \mathbb{Q}[z]$ of degree $\leq q$, such that the order at $z = 0$ of

$$R_j(z) := Q(z)F_j(z) - P_j(z)$$

is at least $p+h+1$ for all $j = 1, \ldots, N$. In particular, $Q(z)$ is not identically zero. We say that $(Q; P_1, \ldots, P_N)$ is a Padé-type approximant of type II $[q; p, \ldots, p; p+h+1]$ of $(F_1, \ldots, F_N)$. It is not unique in general.

In what follows it is convenient to let $P_0(z) = Q(z)$ and $R_0(z) = 0$, even though they do not play exactly the same role as the other $P_j$'s and $R_j$'s.

Set $\mathbf{P}(z) = {}^t\big(P_0(z), \dots, P_N(z)\big)$ and $\mathbf{R}(z) = {}^t\big(R_0(z), \dots, R_N(z)\big)$. Following Chudnovsky [16, 17], for $k \geq 0$ we define $\mathbf{P}_k(z) := {}^t\big(P_{0,k}(z), \dots, P_{N,k}(z)\big) \in \mathbb{Q}[z]^{N+1}$ and $\mathbf{R}_k(z) := {}^t\big(R_{0,k}(z), \dots, R_{N,k}(z)\big) \in \mathbb{Q}[[z]]^{N+1}$ by

$$\mathbf{P}_k(z) := \frac{1}{k!}\mathcal{D}(z)^k\Big(\frac{d}{dz} - A(z)\Big)^k\mathbf{P}(z), \tag{2.2}$$

$$\mathbf{R}_k(z) := \frac{1}{k!}\mathcal{D}(z)^k\Big(\frac{d}{dz} - A(z)\Big)^k\mathbf{R}(z).$$

Now recall that $F_0 = 1, F_1, \dots, F_N$ are linearly independent, so that the matrix $A(z)$ is uniquely determined by these functions and the zero-th row of $A$ is identically zero. Therefore we obtain the formula

$$Q_k(z) = \frac{1}{k!}\mathcal{D}(z)^k Q^{(k)}(z) \tag{2.3}$$

where $Q_k(z) := P_{0,k}(z)$. An important property is that if $Q(z) \in \mathbb{Z}[z]$, then $Q_k(z) \in \mathbb{Z}[z]$ for any $k$ because $\frac{1}{k!}(z^j)^{(k)} = \binom{j}{k}z^{j-k}$. Moreover we have for any $k \in \mathbb{N}$ and any $j$:

$$\deg Q_k \leq q + (d-1)k \text{ and } \deg P_{j,k} \leq p + (d-1)k.$$

We shall make use of the following results. Part $(i)$ follows easily from the bounds on the degrees of $Q_k$ and $P_{j,k}$ and the relation $R_{j,k} = Q_kF_j - P_{j,k}$: see [17, §2]. Part $(ii)$ is the difficult one: it is a refinement and correction by André [4, p. 115] of Chudnovsky's zero estimate [16, 17]. The fact that $F_1, \dots, F_N$ are $G$-functions is used only to make the constant in $(ii)$ effective.

**Theorem 7** (Chudnovsky, André)**.** *Let $(Q; P_1, \dots, P_N)$ be a Padé-type approximant of type II $[q; p, \dots, p; p+h+1]$ of $(F_1, \dots, F_N)$; recall that $F_0(z) = 1$, $F_1(z)$, $\dots$, $F_N(z)$ are $\mathbb{Q}(z)$-linearly independent $G$-functions with rational coefficients. Then:*

*$(i)$ For any $k \geq 0$ such that $h \geq kd$, $(Q_k; P_{1,k}, \dots, P_{N,k})$ is a Padé-type approximant*

$$[q + k(d-1); p + k(d-1), \dots, p + k(d-1); p + h + 1 - k]$$

*of $(F_1, \dots, F_N)$.*

*$(ii)$ The determinant*

$$\Delta_N(z) := \begin{vmatrix} Q_0(z) & \cdots & Q_N(z) \\ P_{1,0}(z) & \cdots & P_{1,N}(z) \\ \vdots & \vdots & \vdots \\ P_{N,0}(z) & \cdots & P_{N,N}(z) \end{vmatrix}$$

*is not identically zero provided $h \geq h_0$, where $h_0$ is a positive constant, which depends only on $F_1$, $\dots$, $F_N$ and can be computed explicitly.*

Let us deduce precisely this result from André's theorem. Given distinct integers $i, j \in \{1, \ldots, N\}$ we have

$$\begin{aligned}\mathrm{ord}_0(P_iF_j - P_jF_i) &= -\mathrm{ord}_0(Q) + \mathrm{ord}_0(P_i(QF_j - P_j) - P_j(QF_i - P_i)) \\ &\geq -\mathrm{ord}_0(Q) + \min(\mathrm{ord}_0(P_i), \mathrm{ord}_0(P_j)) + p + h + 1 \\ &\geq p + h + 1\end{aligned}$$

because $\mathrm{ord}_0(P_i) \geq \mathrm{ord}_0(Q)$ for any $i$. Since we also have $\mathrm{ord}_0(QF_j - P_j) \geq p + h + 1$, André's zero estimate [4, p. 115] applies as soon as $h$ is greater than the constant he denotes by $c_0(\Lambda)$, that we call $h_0$ here. Moreover $h_0$ is effective: see [4, Exercise 2, p. 126]. For future reference, we notice that $(i)$ and $(ii)$ can be combined as soon as $h \geq \max(h_0, Nd)$; this will be the case below.

## 2.2 Construction of the Padé approximants

Let us explain precisely now the construction of the $P_j$'s and of $Q$. First of all, we set

$$F_j(z) = \sum_{n=0}^{\infty} f_{j,n} z^n.$$

Since the $F_j$'s are $G$-functions, there exist a sequence of integers $d_n > 0$ and a constant $D > 0$ such that $d_n f_{j,n} \in \mathbb{Z}$ and $d_n \leq D^{n+1}$, and also a constant $C > 0$ such that $|f_{j,n}| \leq C^{n+1}$ for all $n \geq 0$ and all $j$. Let us write $P_j(z) = \sum_{n=0}^{p} u_{j,n} z^n$ for $1 \leq j \leq N$ and $Q(z) = \sum_{n=0}^{q} v_n z^n$. By definition of the $P_j$'s and of $Q$, we have the equations

$$\sum_{k=0}^{q} f_{j,n-k} v_k = 0, \qquad n = p+1, \ldots, p+h, \ j = 1, \ldots, N \tag{2.4}$$

and

$$\sum_{k=0}^{\min(n,q)} f_{j,n-k} v_k = u_{j,n}, \qquad n = 0, \ldots, p, \ j = 1, \ldots, N. \tag{2.5}$$

Multiplying Eq. (2.4) by $d_{p+h}$, we obtain a system of $Nh$ equations in the $q+1$ unknowns $v_0, \ldots, v_q$, with integer coefficients $d_{p+h} f_{j,0}, \ldots, d_{p+h} f_{j,p+h}$ bounded in absolute value by $(CD)^{p+h+1}$. Since $q + 1 > Nh$, Siegel's lemma (see for instance [23, Lemma 11, Chapter 3, p. 102]) implies the existence of a non-zero vector of solutions $(v_0, \ldots, v_q) \in \mathbb{Z}^{q+1}$ such that

$$|v_k| \leq 1 + (q(CD)^{p+h+1})^{\frac{Nh}{q+1-Nh}}, \quad k = 0, \ldots, q. \tag{2.6}$$

From Eq. (2.5), we see that $d_p P_j(z) \in \mathbb{Z}[z]$ for $j = 1, \ldots, N$. Let $H(A)$ denote the maximum of the moduli of the coefficients of a polynomial $A(z)$ with real coefficients. Since $CD \geq 1$, Eq. (2.6) implies that

$$H(Q) \leq 2(q(CD)^{p+h+1})^{\frac{Nh}{q+1-Nh}}. \tag{2.7}$$

### 2.3 Properties of $Q_k$, $P_{j,k}$ and $R_{j,k}$

In this section, we collect some informations we shall freely use in the proof of Theorem 5.

From the estimate (2.7), we can bound the coefficients of $Q_k(z) \in \mathbb{Z}[z]$ for any $k \geq 0$. We recall that

$$H(A_1 \cdots A_n) \leq \binom{d_1 + \cdots + d_n}{n-1} H(A_1) \cdots H(A_n)$$

as soon as $\deg A_j \leq d_j$. We thus have

$$\begin{aligned} H(Q_k) &\leq \frac{1}{k!}\binom{q+(d-1)k}{k} H(\mathcal{D})^k H(Q^{(k)}) \\ &\leq 2^{2q+(d-1)k+1} H(\mathcal{D})^k (q(CD)^{p+h+1})^{\frac{Nh}{q+1-Nh}}. \end{aligned} \tag{2.8}$$

Let us now bound $R_{j,k}(z)$ for $1 \leq j \leq N$. Letting $Q_k(z) = \sum_{n=0}^{q+(d-1)k} v_n^{(k)} z^n$ and recalling that $R_{j,k} = Q_k F_j - P_{j,k}$, Lemma 7 $(i)$ yields

$$R_{j,k}(z) = \sum_{n=p+h+1-k}^{\infty} \Big( \sum_{\ell=0}^{\min(n,q+k(d-1))} f_{j,n-\ell} v_\ell^{(k)} \Big) z^n$$

from which we deduce that, for $|z| < 1/C$:

$$\begin{aligned} |R_{j,k}(z)| &\leq H(Q_k)(q+k(d-1)+1) \max(1,C)^{q+k(d-1)} \sum_{n=p+h+1-k}^{\infty} C^n |z|^n \\ &\leq \frac{H(Q_k)(q+k(d-1)+1)}{1-C|z|} \max(1,C)^{q+k(d-1)} (C|z|)^{p+h+1-k}. \end{aligned} \tag{2.9}$$

Finally, for $j = 1, \ldots, N$, letting $P_{j,k}(z) = \sum_{n=0}^{p+(d-1)k} u_{j,n}^{(k)} z^n$, we have

$$\sum_{m=0}^{\min(n,q+(d-1)k)} f_{j,n-m} v_m^{(k)} = u_{j,n}^{(k)}, \qquad n = 0, \ldots, p+(d-1)k.$$

It follows that $d_{p+(d-1)k} P_{j,k}(z) \in \mathbb{Z}[z]$ for all $k \geq 0$ and $1 \leq j \leq N$.

# 3 Proof of Theorem 5

We split the proof into two parts: in Section 3.1 we prove a general (and technical) result, namely Eq. (3.5), from which Theorem 5 will be deduced in Section 3.2.

## 3.1 Main part of the proof

We keep the notation and assumptions of Section 2 concerning $F_1, \ldots, F_N$, $A$, $\mathcal{D}$, $d$, $C$, $D$. Without loss of generality, we may assume that $C \geq 1$.

We fix $t, x, y \in \mathbb{R}$ and $a, b, B, m, n, h \in \mathbb{Z}$ such that $b, m \geq 1$ and

$$1 \leq \left|\frac{a}{b}\right| < \min\left(\frac{1}{2C}, \frac{1}{H(\mathcal{D})}\right), \quad 1 \leq B \leq b^t, \quad 0 < y < \frac{1}{d}, \quad x > N + y.$$

We also assume that $h$ is sufficiently large; in precise terms, we assume that

$$h \geq \max\left(h_0, 8N^2d^3, \frac{m}{x - N - y}\right)$$

where $h_0$ is the constant in Lemma 7, and we shall also assume below (just before Eq. (3.4)) that $h$ is greater than some other positive constant that could be effectively computed in terms of $F_1, \ldots, F_N$.

We let $\beta = b^{t/h}$ and make one more assumption on these parameters, namely Eq. (3.4) below. At last, we fix an integer $j \in \{1, \ldots, N\}$. Then we shall deduce a lower bound on $\left|F_j(\frac{a}{b}) - \frac{n}{B \cdot b^m}\right|$, namely Eq. (3.5).

Changing $z$ to $-z$ in all functions $F_1, \ldots, F_N$, we may assume that $a > 0$.

Let $z_0 \neq 0$ be a rational root of $\mathcal{D}$; let us write $z_0 = r_0/r_1$ with coprime integers $r_0$, $r_1$. Then $r_1$ divides the leading coefficient of $\mathcal{D}$, so that $|r_1| \leq H(\mathcal{D})$ and $|z_0| \geq \frac{1}{|r_1|} \geq \frac{1}{H(\mathcal{D})}$. Therefore $a/b$ is not a root of $\mathcal{D}$.

To apply the constructions of Section 2 we let

$$p = \lfloor xh \rfloor \quad \text{and} \quad q = \lfloor (N + y)h \rfloor,$$

so that

$$p \geq q + m.$$

Let us choose $k$ now. The determinant $\Delta_N(z)$ of Lemma 7 $(ii)$ has degree at most $q + Np + (d-1)N(N+1)/2$. We use the vanishing properties of Lemma 7 $(i)$ (since $h \geq Nd$) by susbtracting $F_i(z)$ times the zero-th row from the $i$-th row, for any $1 \leq i \leq N$. We obtain that $\Delta_N(z)$ vanishes at 0 with multiplicity at least $N(p+h+1) - N(N+1)/2$. Therefore we have

$$\Delta_N(z) = z^{N(p+h+1) - \frac{N(N+1)}{2}} \widetilde{\Delta}_N(z)$$

where $\widetilde{\Delta}_N(z)$ has degree $\leq \ell_0$, with $\ell_0 = q - N(h+1) + dN(N+1)/2$, and is not identically zero. Since $a/b$ is different from 0, its multiplicity as a root of $\widetilde{\Delta}_N(z)$ is at most $\ell_0$. Following the proof of [17, Theorem 4.1], we deduce that the matrix

$$\begin{pmatrix} Q_0(a/b) & \cdots & Q_{N+\ell_0}(a/b) \\ P_{1,0}(a/b) & \cdots & P_{1,N+\ell_0}(a/b) \\ \vdots & \vdots & \vdots \\ P_{N,0}(a/b) & \cdots & P_{N,N+\ell_0}(a/b) \end{pmatrix}$$

has rank $N+1$. Therefore we have $nQ_k(a/b) - Bb^m P_{j,k}(a/b) \neq 0$ for some integer $k$, with

$$k \leq \ell_0 + N = q - Nh + dN(N+1)/2;$$

recall that $j$ is fixed in this proof.

By construction of the polynomials $P_{j,k}(z)$ and $Q_k(z)$, there exist two integers $U_{j,k}, V_k$ such that $P_{j,k}(a/b) = U_{j,k}/(d_{p+(d-1)k} b^{p+(d-1)k})$ and $Q_k(a/b) = V_k/b^{q+(d-1)k}$. We deduce that

$$\xi = d_{p+(d-1)k} b^{p+(d-1)k} \big(nQ_k(a/b) - Bb^m P_{j,k}(a/b)\big)$$

is a non-zero integer (since $p \geq q$). Moreover we have assumed that $p \geq q+m$ so that $\xi$ is a multiple of $b^m$, and thus

$$|\xi| \geq b^m.$$

On the other hand we have

$$\xi = d_{p+(d-1)k} b^{p+(d-1)k} \Big(Q_k(a/b)\big(n - Bb^m F_j(a/b)\big) - Bb^m \big(P_{j,k}(a/b) - Q_k(a/b) F_j(a/b)\big)\Big)$$

so that

$$|\xi| \leq d_{p+(d-1)k} b^{p+(d-1)k} \Big(\big|Q_k(a/b)\big| \cdot \big|n - Bb^m F_j(a/b)\big| + Bb^m \big|R_{j,k}(a/b)\big|\Big).$$

Comparing this upper bound and the lower bound $|\xi| \geq b^m$ we obtain

$$\big|Q_k(a/b)\big| \cdot \big|n - Bb^m F_j(a/b)\big| \geq d_{p+(d-1)k}^{-1} b^{-p-(d-1)k+m} - Bb^m \big|R_{j,k}(a/b)\big|. \tag{3.1}$$

We shall prove below that under a suitable assumption (namely Eq. (3.4)) we have

$$\big|R_{j,k}(a/b)\big| < \frac{1}{2} d_{p+(d-1)k}^{-1} b^{-p-(d-1)k} B^{-1} \tag{3.2}$$

so that the right hand-side of Eq. (3.1) is positive, and $Q_k(a/b) \neq 0$. Moreover Eq. (3.1) yields

$$\Big|F_j\Big(\frac{a}{b}\Big) - \frac{n}{B \cdot b^m}\Big| \geq \frac{d_{p+(d-1)k}^{-1} b^{-p-(d-1)k}}{2B|Q_k(a/b)|}. \tag{3.3}$$

Now, recall that

$$p = \lfloor xh \rfloor, \ q = \lfloor (N+y)h \rfloor, \text{ and } k \leq yh + \frac{dN(N+1)}{2}.$$

Let us denote by $\mathcal{O}(1)$ any positive quantity that can be bounded (explicitly) in terms of $F_1, \ldots, F_N$; such a bound may involve, among others, $d$, $N$, $\mathcal{D}$, $C$ or $D$. We recall that $C, D \geq 1$ and notice that $\frac{Nh}{q+1-Nh} \leq \frac{N}{y}$. Then Eq. (2.8) yields

$$H(Q_k) \leq \big(2^{2(N+y)+(d-1)y} H(\mathcal{D})^y (CD)^{(x+1)N/y}\big)^h \cdot (CD(N+y)h)^{N/y} \cdot \mathcal{O}(1)$$

so that Eq. (2.9) provides, since $Ca/b < 1/2$:

$$\begin{aligned}&\big|R_{j,k}(a/b)\big| \leq \\ &\big(2^{2(N+y)+(d-1)y}H(\mathcal{D})^y(CD)^{(x+1)N/y}C^{N+dy}(Ca/b)^{x+1-y}\big)^h \cdot (CD(N+dy)h)^{1+N/y} \cdot \mathcal{O}(1).\end{aligned}$$

Now let us assume, for simplicity, that $y \geq \frac{1}{8d}$. Then we have $(d-1)k \leq dyh$ since $hy \geq N^2d^2$, so that

$$d_{p+(d-1)k}b^{p+(d-1)k}B \leq \big(\beta(bD)^{x+dy}\big)^h.$$

Therefore (3.2) holds if $h$ is larger than some effectively computable constant (depending only on $F_1, \ldots, F_N$) and if

$$y \geq \frac{1}{8d} \text{ and } 2^{2(N+y)+(d-1)y}H(\mathcal{D})^y(CD)^{(x+1)N/y}C^{N+dy}(Ca/b)^{x+1-y}(bD)^{x+dy}\beta < \frac{1}{2}. \quad (3.4)$$

Moreover, since $a/b < 1$, we have

$$\begin{aligned}\big|Q_k(a/b)\big| &\leq (q+(d-1)k+1)H(Q_k) \\ &\leq \Big(2^{2(N+y)+(d-1)y}H(\mathcal{D})^y(CD)^{(x+1)N/y}\Big)^h \cdot (CD(N+dy)h)^{1+N/y} \cdot \mathcal{O}(1)\end{aligned}$$

so that Eq. (3.3) yields finally (since $1/y \leq 8d$):

$$\begin{aligned}&\Big|F_j\Big(\frac{a}{b}\Big) - \frac{n}{B \cdot b^m}\Big| \geq \\ &\qquad \Big(\beta^{-1}(bD)^{-x-dy}2^{-2(N+y)-(d-1)y}H(\mathcal{D})^{-y}(CD)^{-(x+1)N/y}\Big)^h \cdot h^{-9Nd} \cdot \mathcal{O}(1). \quad (3.5)\end{aligned}$$

This is a very general lower bound and in the next section we will proceed to a suitable choice of the parameters.

## 3.2 Choice of the parameters and conclusion

In this section we prove Theorem 5 by applying the proof of Section 3.1 to suitable parameters.

Let $F$ be a $G$-function with $F(z) \notin \mathbb{Q}(z)$. Let $F_0(z) = 1$, and denote by $N$ the least positive integer for which there exist $a_0(z), \ldots, a_N(z) \in \mathbb{Q}(z)$ such that

$$F^{(N)}(z) = a_N(z)F^{(N-1)}(z) + \ldots + a_2(z)F'(z) + a_1(z)F(z) + a_0(z).$$

We have $N \geq 1$, and $N = 1$ may hold (it does for instance with $F(z) = \log(1-z)$). By construction and since $F(z) \notin \mathbb{Q}(z)$, the $G$-functions $F_0 = 1$, $F_1 = F$, $F_2 = F'$, $\ldots$, $F_N = F^{(N-1)}$ are linearly independent over $\mathbb{Q}(z)$. We are in position to apply the results of Sections 2 and 3.1; as in Section 3.1 we may assume $a > 0$ and $C \geq 1$.

We let

$$c_1 = 4H(\mathcal{D})(CD)^{8Nd+1}C \text{ and } c_2 = 3(N+2).$$

We take $c_5 = \max(h_0, h_1, h_2, 8N^2d^3, 4t)$ where $h_0$ is the constant implied in Lemma 7 $(ii)$, $h_1$ is the effectively computable constant defined just before Eq. (3.4), and $h_2$ is another effectively computable constant to be defined below; these three constants depend only on $F$ and $t$. Then we assume

$$m \geq \frac{c_5}{3}\frac{\log(b)}{\log(c_1 a)}; \tag{3.6}$$

this is a consequence of our assumption $m \geq c_3 \frac{\log(b)}{\log(a+1)}$ provided $c_3$ is large enough.

We choose

$$y = \frac{1}{4(d+1)}, \qquad x = \frac{1}{3}\frac{\log(b)}{\log(c_1 a)} \qquad \text{and} \qquad h = \left\lfloor \frac{m}{x-N-1} \right\rfloor.$$

Then (1.3) implies $x \geq N+2$, and (3.6) yields $h \geq c_5$.

Let us check that (3.4) holds. We notice that $\beta \leq b^{1/4}$ since $h \geq c_5 \geq 4t$. Since $y = \frac{1}{4(d+1)}$ and $x \geq N+1$, the left hand side of (3.4) is less than

$$2^{2N+1} H(\mathcal{D})(CD)^{8Nd(x+1)} C^{x+N+2} D^{x+1} a^{x+1} b^{-1/2} \leq (c_1 a)^{x+1} b^{-1/2} \leq 1/2;$$

indeed we have used the definition of $x$ and the lower bound $\frac{\log(b)}{\log(c_1 a)} \geq 9$ which follows from $x \geq N+2 \geq 3$. Therefore all the assumptions made in Section 3.1 hold.

We set

$$c_6 = D^{1+\frac{d}{(N+2)(d+1)}} 2^{\frac{8N+1}{4N+8}} H(\mathcal{D})^{\frac{1}{4(N+2)(d+1)}} (CD)^{\frac{4N(N+3)(d+1)}{N+2}}.$$

Using (3.5) and the various conditions on $x$ and $y$, we readily obtain

$$\left| F\left(\frac{a}{b}\right) - \frac{n}{B \cdot b^m} \right| \geq \beta^{-h} \left( b^{x+dy} c_6^x \right)^{-h} \cdot h^{-9Nd} \cdot \mathcal{O}(1) \geq B^{-1} (bc_6)^{-\frac{x+1}{x-N-1} \cdot m} \tag{3.7}$$

provided $h \geq h_2$, where $h_2$ is effective. Now we have

$$\frac{x+1}{x-N-1} = 1 + \frac{N+2}{x} \cdot \frac{1}{1 - \frac{N+1}{x}} \leq 1 + 3(N+2)^2 \frac{\log(c_1 a)}{\log(b)}$$

because $x = \frac{1}{3}\frac{\log(b)}{\log(c_1 a)} \geq N+2$. It is trivial matter to check that for any $u \geq 1$ and any $v \geq e$, we have $\log(uv) \leq 2\log(u+1)\log(v)$. Since $c_1 \geq 4 > e$ (because we always have $H(\mathcal{D}) \geq 1$, $C \geq 1$ and $D \geq 1$), we can apply this with $u = a$, $v = c_1$ and we get

$$\frac{x+1}{x-N-1} \leq 1 + c_7 \frac{\log(a+1)}{\log b}$$

with $c_7 = 6(N+2)^2 \log(c_1)$. Hence, we deduce from (3.7) that

$$\left| F\left(\frac{a}{b}\right) - \frac{n}{B \cdot b^m} \right| \geq \frac{1}{B \cdot b^m (a+1)^{c_8 m} c_6^m} \geq \frac{1}{B \cdot b^m (a+1)^{c_4 m}}$$

where

$$c_8 = \frac{\log(2c_6)}{\log(2)} c_7 \quad \text{and} \quad c_4 = c_8 + \frac{\log(c_6)}{\log(2)}.$$

This completes the proof of Theorem 5.

*Remark.* Let us compute the constants $c_1, c_2$ and $c_4$ in the case of the $G$-function $\mathrm{Li}_2$. The vector $Y(z) = {}^t\big(1, \mathrm{Li}_1(z), \mathrm{Li}_2(z)\big)$ is solution of the differential system

$$Y'(z) = \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{1-z} & 0 & 0 \\ 0 & \frac{1}{z} & 0 \end{pmatrix} Y(z).$$

Hence $\mathcal{D}(z) = z(1-z)$, $H(\mathcal{D}) = 1$, $d = 2$, $N = 2$, $C = 1$ and $D = e^2$. With the constants as defined in the proof just above, we obtain $c_1 = 4e^{66}$, $c_2 = 12$ and

$$c_4 = \frac{1201779}{48} + \frac{1185019}{3\log(2)} + 396\log(2) < 10^{5.78}.$$

The constant $c_3$ could be computed as well, but we did not try to do so because it is not important for the application to Theorem 6.

It follows that Theorem 5 can be applied with $F(z) = \mathrm{Li}_2(z)$ when $b \geq e^{809}|a|^{12} > 0$. Furthermore, Theorem 6 can be applied for any integer $s$ such that $b^s \geq e^{809}|a|^{12} > 0$ and $b^s \geq (|a|+1)^{2c_4/\varepsilon}$. In particular, if $a = 1$, $b \geq 2$ and $0 < \varepsilon < 1$, Theorem 6 applies to $\mathrm{Li}_2(1/b^s)$ for any integer $s \geq 10^{6.08}/\varepsilon$. We have not tried to optimize our general constants which in this case could be decreased.

# 4 Proof of Theorem 6

In this section we deduce Theorem 6 from Corollary 1 stated in the introduction.

Let $\xi = F(a/b^s)$, $q_n = b^{n-1}(b^t - 1)$, and

$$p_n = (b^t - 1)\lfloor b^{n-1}\xi \rfloor + a_n b^{t-1} + a_{n-1}b^{t-2} + \ldots + a_{n+t-1}.$$

Then the $b$-ary expansion of

$$\frac{p_n}{q_n} = \frac{\lfloor b^{n-1}\xi \rfloor}{b^{n-1}} + \frac{a_n b^{t-1} + a_{n-1}b^{t-2} + \ldots + a_{n+t-1}}{b^{n-1}(b^t - 1)}$$

has the same $n + t\mathcal{N}_b(\xi, t, n) - 1$ first digits as the $b$-ary expansion of $\xi$. Therefore we have

$$\left| \xi - \frac{p_n}{q_n} \right| \leq \frac{b-1}{b^{n+t\mathcal{N}_b(\xi,t,n)}}.$$

Now Corollary 1 with $b^s$ for $b$, $B = b^t - 1$ and $m = \lfloor \frac{n-1}{s} \rfloor$ yields

$$\left| \xi - \frac{p_n}{q_n} \right| \geq \frac{1}{b^{\lfloor \frac{n-1}{s} \rfloor s(1+\varepsilon)}}.$$

The comparison of both inequalities enables us to conclude the proof.

# 5 Concluding remarks

In Section 2, we assumed that the degrees of the polynomials satisfy $p \geq q$ and in fact $p \geq q + m$, which was crucial to prove Theorem 5. The case $q \geq p$ also provides some informations, but not in the exact situation of Theorem 7. Indeed, with the notation of Section 2.1 the polynomials $P_{j,k}(z)$ with $1 \leq j \leq N$ depend on $P_1(z), \ldots, P_N(z)$ and also on $P_0(z) = Q(z)$ (see Eq. (2.2)). In order to be able to bound the degree of $P_{j,k}(z)$ in terms of $p$ only (independently of $q$), we need to deduce from (2.2) a relation analogous to (2.3), namely an expression for the polynomials $P_{j,k}(z)$ in terms of $P_1(z), \ldots, P_N(z)$ only. This follows easily under the additional assumption that the zero-th row of $A(z)$ is identically zero, i.e. that ${}^t(F_1, \ldots, F_N)$ is a solution of an homogeneous linear differential system. Following the same method as in the case $p \geq q$, this enables us to prove the following result.

**Theorem 8.** *Let $F$ be a $G$-function with rational Taylor coefficients and $t \geq 0$. Let us assume that $F(z)$ is solution of a linear differential equation of order $N$ with coefficients in $\mathbb{Q}(z)$ and that $1, F(z), F'(z), \ldots, F^{(N-1)}(z)$ are linearly independent over $\mathbb{Q}(z)$. Then there exist some positive effectively computable constants $\widetilde{c}_1$, $\widetilde{c}_2$, $\widetilde{c}_3$, $\widetilde{c}_4$, depending only on $F$ (and $t$ as well for $\widetilde{c}_3$), such that the following property holds. Let $a \neq 0$ and $b, B \geq 1$ be integers such that*

$$b > (\widetilde{c}_1 |a|)^{\widetilde{c}_2} \text{ and } B \leq b^t. \tag{5.1}$$

*Then $F(a/b) \neq 0$ and for any $n \in \mathbb{Z}$ and any $m \geq \widetilde{c}_3 \frac{\log(b)}{\log(|a|+1)}$, we have*

$$\left| \frac{1}{F(\frac{a}{b})} - \frac{n}{B \cdot b^m} \right| \geq \frac{1}{B \cdot b^m \cdot (|a|+1)^{\widetilde{c}_4 m}}. \tag{5.2}$$

Analogues of Corollary 1 and Theorem 6 for $1/F(a/b)$ hold as well. These results can be applied to the functions $\log(1-z) + \sqrt{1-z}$ and $\sqrt{1-z}\log(1-z)$ for instance, but not to $\log(1-z)$. Actually the proof of Theorem 8 (and of all other results in this paper) can be generalized to number fields, at least to multiply $B$ with a fixed non-zero algebraic number (and all implied constants would depend on this number), by replacing the algebraic number $\xi$ defined in Section 3.1 with its norm over the rationals. Applying Theorem 8 to $\sqrt{1-z}\log(1-z)$ with $B$ multiplied by $\sqrt{1-a/b}$ and canceling out this factor shows that Theorem 5, Corollary 1 and Theorem 6 hold with $1/\log(1-a/b)$ instead of $F(a/b)$.

A natural problem is to obtain an analogue of Theorem 5 when the $F_j$'s are $E$-functions and not $G$-functions. With the same notations as in Section 2, the polynomials $Q_k$ would still have integer coefficients, but the denominators of the coefficients of the polynomials $P_{j,k}$ would no longer be bounded by $d_{p+(d-1)k}$ but by $(p+(d-1)k)!d_{p+(d-1)k}$. As the reader may check, this cancels the benefits of having non-diagonal Padé type approximants if we follow the same method of proof as in Section 3. We don't know if this problem can be fixed to prove analogues of Theorems 5 and 6 for $E$-functions. Very few results are known on $b$-ary expansions of values of $E$-functions (see [1], [13], [14]). From a conjectural point

of view, the situation is not clear either: values of $E$-functions *do not* behave like generic numbers with respect to rational approximation, as the continued fraction expansion of $e$ shows.

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