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Abstract

The quantitative verification of Probabilistic Automata (PA) is undecidable in general. Unary PA are a simpler model where the choice of action is fixed. Still, the quantitative verification problem is open and known to be as hard as Skolem's problem, a problem on linear recurrence sequences, whose decidability is open for at least 40 years. In this paper, we approach this problem by studying the languages generated by unary PAs (as defined below), whose regularity would entail the decidability of quantitative verification.

Given an initial distribution, we represent the trajectory of a unary PA over time as an infinite word over a finite alphabet, where the n^{th} letter represents a probability range after n steps. We extend this to a language of trajectories (a set of words), one trajectory for each initial distribution from a (possibly infinite) set. We show that if the eigenvalues of the transition matrix associated with the unary PA are all distinct positive real numbers, then the language is *effectively regular*. Further, we show that this result is at the boundary of regularity, as non-regular languages can be generated when the restrictions are even slightly relaxed. The regular representation of the language allows us to reason about more general properties, e.g., robustness of a regular property in a neighbourhood around a given distribution.

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1 Introduction

Markov decision processes (MDPs for short) are a standard model for describing probabilistic systems with nondeterminism. The system or controller has a strategy according to which it chooses an action at every step, which is then performed according to a probability distribution defined over the set of possible resultant states. The usual question is whether some property (e.g. reaching a set of *Goal* states) can be achieved with probability at least some threshold γ.

In many interesting settings, the controller cannot observe the state in which it operates or only has partial information regarding the state (Partially Observable MDPs, POMDPs). Probabilistic automata (PAs for short) [21, 20] form the subclass of POMDPs where the controller cannot observe anything. The problem of whether there is a strategy to reach *Goal* with probability at least a threshold γ (also called a cut-point) is already undecidable [5]. Even approximating this probability has been shown undecidable in PAs [13]. In fact, deciding whether there exists a sequence of strategies with probability arbitrarily close to

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$\gamma = 1$ is already undecidable [9], and only very restricted subclasses are known to ensure decidability [8, 7].

A line of work, which we follow, is to consider unary PAs [6, 22], where the alphabet has a single letter. That is, there is a unique strategy, and the model is essentially a Markov chain. Surprisingly, the 'simple' problem of whether there exists a finite number of steps after which the probability to be in *Goal* is higher than the threshold $\gamma \in (0, 1)$ is open and has recently been shown [3] to be as hard as the so-called *Skolem's problem*, which is a long-standing open problem on linear recurrence sequences [14, 12, 16]. One way to tackle the problem is to approximate it, asking whether for all ϵ there exists a number of steps n_ϵ after which the probability to be in *Goal* is at least $\gamma - \epsilon$. The decidability and precise complexity of this problem has been explored in [6]. A more general approximation scheme, valid for much more general questions which can be expressed in some LTL logic, has also been tackled by generating a regular language of *approximated* behaviors [1].

In this paper, we study classes for which the language of *exact* behaviors is (ω-)regular, allowing for the exact resolution of any regular question (e. g. checking any LTL$_{\mathcal{I}}$ formula [1, 2]). We define the trajectory from a given initial distribution as an (infinite) word over the alphabet $\{A, B\}$. The n^{th} letter of a trajectory being A (for Above, respectively, B for Below) represents that after n steps the probability to be in *Goal* is greater than or equal to (respectively lesser than) the threshold γ. Further, we consider the language of a unary PA as the set of trajectories (words) ranging over a (possibly infinite) set of initial distributions. Thus, we can answer questions such as: does there exist a trajectory from the set of initial distributions satisfying a regular property or do all trajectories satisfy it. We can also tackle more complicated questions such as robustness wrt. a given initial distribution δ_{init}: does a regular property hold for all initial distributions "around" δ_{init}.

As motivation, consider a population of yeast under osmotic stress [15]. The stress level of the population can be studied through a protein which can be marked (by a chemical reagent). For the sake of illustration, consider the following simplistic model of a Markov Chain M_{yeast} with the protein being in 3 different discrete states (namely the concentration of the protein being high (state 1), medium (state 2) and low (state 3)). The transition matrix, also denoted M_{yeast}, gives the proportion of yeast moving from one protein concentration level to another one, in one time step (say, 15 seconds). For instance, 20% of the yeast with low protein concentration will have high protein concentration at the next time step. The marker can be observed optically when the concentration of the protein is high. We know that the original proportion of yeast in state 1 is 1/3 (by counting the marked yeast population), but we are unsure of the mix between low and medium. The initial set of distributions is thus $Init_{yeast} = \{(1/3, x, 2/3 - x) \mid 0 \leq x \leq 2/3\}$. The language of M_{yeast} will tell us how the population evolves wrt the number of marked yeast being above or below the threshold $\gamma_{yeast} = 5/12$, depending on the initial distribution in $Init_{yeast}$. Now, suppose an experiment with yeasts reveals that there are at first less than 5/12 of *marked* yeast (i.e. with high concentration of proteins), then more than 5/12 of marked yeast, and eventually less than 5/12 of marked yeasts. That is, the trajectory is B for a while, then A for a while, then it stabilises at B. Let us call this property as (P_{yeast}) (note that this is a regular property). We are interested in checking whether our simplistic model exhibits at least one trajectory with the property (P_{yeast}), and if yes, the range of initial values generating trajectories with this property.

$$M_{yeast} = \begin{pmatrix} 0.8 & 0.1 & 0.2 \\ 0.1 & 0.8 & 0.1 \\ 0.1 & 0.1 & 0.7 \end{pmatrix}$$

Property of eigenvalues of Markov chain	Regular language	Ultimately periodic traj.
Distinct, positive real numbers	✓ (Thm.4)	✓ (from below)
Distinct, roots of real numbers	× (Thm.10)	✓ (Prop.1)
Distinct	× (from above)	× ([2], Thm.3)

Table 1 A summary of the results in this paper.

Our contributions as depicted in table 1 are the following: if the eigenvalues of the transition (row-stochastic) matrix associated with the unary PA are distinct roots of real numbers, then any trajectory from a given initial distribution is ultimately periodic. This is tight, in the sense that, there are examples of trajectories which are not ultimately periodic even for unary PAs with 3 states [2, 22] (with some eigenvalue not root of any real number). Our main result is that if, further, the eigenvalues are distinct positive real numbers, then the language generated by a unary PA starting from a convex polytope of initial distributions is *effectively regular*. Surprisingly, this result is also tight: there exist unary PA with eigenvalues being distinct roots of real numbers (starting from a convex initial set) which generate a non-regular language, as we show in Section 6. Due to space constraints, we only present the main ideas in this paper. Full proofs and details can be found in the technical report at [4].

The proof of our main regularity result is surprisingly hard to obtain. First, for each trajectory ρ, one obtains easily a number of steps n_ρ after which the trajectory is constant. However, there is in general no bound on n_ρ *uniform* over all ρ in the language. Thus, while every trajectory is simple to describe, the language turns out to be in general much more complex. We prove that the language does have a representation as a finite union of languages of the form $wA^iA^*B^*A^*B^*A^*\cdots B^*A^\omega$ with a bounded number of alternations. Our method computes effectively the language of M_{yeast}, as M_{yeast} has positive real eigenvalues, answering the question whether there exists an initial trajectory s.t. property (P_{yeast}) holds.

2 Preliminaries and definitions

▶ **Definition** 1. A Probabilistic Automaton (PA) $\mathcal{A}$ is a tuple $(Q, \Sigma, (M_\sigma)_{\sigma\in\Sigma}, Goal)$, where Q is a finite set of states, Σ is a finite alphabet, $Goal \subseteq Q$, and M_σ is the $|Q| \times |Q|$ transition stochastic matrix for each letter $\sigma \in \Sigma$. The PA is called *unary* PA (uPA for short) if $|\Sigma| = 1$.

For a unary PA $\mathcal{A}$ on alphabet $\{\sigma\}$, there is a unique transition matrix $M = M_\sigma$ of $Q \times Q$ with value in $[0, 1]$. For all $x \in Q$, we have $\sum_{y\in Q} M(x, y) = 1$. In other words, M is the Markov chain on set of states Q associated with $\mathcal{A}$.

A distribution δ over Q is a function $\delta : Q \to [0, 1]$ such that $\sum_{q\in Q} \delta(q) = 1$. Given M associated with a uPA, we denote by $M\delta$ the distribution given by $M\delta(q) = \sum_{q'\in Q} \delta(q')M(q', q)$ for all $q \in Q$. Notice that, considering δ and $M\delta$ as row-vectors, this corresponds to performing the matrix multiplication. That is, we consider M as a transformer of probabilities, as in [11, 1]: $(M\delta)(q)$ represents exactly the probability to be in q after applying M once, knowing that the initial distribution is δ. Inductively, $(M^n\delta)(q)$ represents the probability to be in q after applying n times M, knowing that the initial distribution is δ. We now review literature relating several problems on uPA with the Skolem's problem, named after the Skolem-Mahler-Lech Theorem [12],[14].

2.1 Relation with the Skolem problem

We start by defining three basic problems which have been studied extensively in different contexts. Given an initial distribution δ_0 and a uPA $\mathcal{A}$ with Matrix M, target states $Goal$

and threshold γ:

Existence problem: does there exist $n \in \mathbb{N}$ such that the probability to be in *Goal* after n iterations of M from δ_0 is γ (i.e., $\sum_{q \in Goal}(M^n \delta_0)(q) = \gamma$) ?

Positivity problem: for all $n \in \mathbb{N}$ is the probability to be in *Goal* after n iterations of M from δ_0 at least γ (i.e., $\sum_{q \in Goal}(M^n \delta_0)(q) \geq \gamma$)?

Ultimate Positivity problem: does there exist $n \in \mathbb{N}$ s.t., for all $m \geq n$, the probability to be in *Goal* after m iterations of M from δ_0 is at least γ (i.e., $\sum_{q \in Goal}(M^m \delta_0)(q) \geq \gamma$)?

Note that all these problems are defined from a fix initial distribution δ_0. These problems for PAs are specific instances of problems over general recurrence sequences, that have been extensively studied [16, 10]. It turns out that the existence for the special PA case is as hard as the existence (Skolem) problem over general recurrence sequences as shown in [3].

▶ **Theorem** 1. [3, 10] For general unary PAs, the existence and positivity are as hard as the Skolem's problem.

The positivity result comes from the interreducibility of Skolem's problem and the positivity problem for general recurrence sequences [10]. The decidability of Skolem has been open for 40 years, and it has been shown that solving positivity, ultimate positivity or existence for general uPAs even for a small number of states (<50, depending on the problem considered) would entail major breakthroughs in diophantine approximations [18].

2.2 Simple unary PAs

In order to obtain decidability, we will consider restrictions over the matrix M associated with the uPA. The first restriction, fairly standard, is that M has distinct eigenvalues, which makes M diagonalizable.

▶ **Definition** 2. A stochastic matrix is *simple* if all its eigenvalues are distinct. A uPA is *simple* if its associated transition matrix is.

Some decidability results [19, 17] have been proved in the case of distinct eigenvalues for variants of the Skolem, which implies the following for *simple* uPAs:

▶ **Theorem** 2. For simple unary PAs, ultimate positivity is decidable [19].
For simple unary PAs with at most 9 states, positivity is decidable [17].

We will consider the *simple* uPA restriction. Notice that the decidability restrictions in Theorem 2 for these two closely related problems have led to two different papers [17],[19] in the same conference, using different techniques. As we want to answer in a uniform way any regular question (subsuming among others the above three problems and regular properties such as (P_{yeast})) for uPAs of all sizes, we will later impose more restrictions. We start with the simple well-known observation that a simple unary PA has a unique stationary distribution.

▶ **Lemma 1.** *Let M be a simple stochastic matrix. Then there exists a unique distribution δ_{stat} such that $M\delta_{stat} = \delta_{stat}$.*

Proof. We give a sketch of proof here. We will later get an analytical explanation of this result. We have $M\delta = \delta$ iff $(M - Id)\delta = 0$. As M is diagonalizable and 1 is a eigenvalue of M of multiplicity 1, we have $Ker(M - Id)$ is of dimension 1. The intersection of distributions and of $Ker(M - Id)$ is of dimension 0, that is, it is a single point. ◀

As usual with PAs, we consider the probability to be in the set of states *Goal*, that is $\sum_{q \in Goal}(M^n \delta)(q)$. We consider only one threshold γ, for simplicity. In fact, the case of multiple thresholds reduces to this case, since the behavior is non-trivial for only one threshold, namely $\gamma_{stat} = \sum_{q \in Goal} \delta_{stat}(q)$ (see [4] for details).

2.3 Trajectories and ultimate periodicity

We want to know whether the n^{th} distribution $M^n\delta$ of the trajectory starting in distribution $\delta \in Init$ is above the hyperplane defined by $\sum_{q \in Goal} x_q = \gamma$, i.e., whether $\sum_{q \in Goal}[M^n\delta](q) \geq \gamma$. We will write $\rho_\delta(n) = A$ (Above) for $\sum_{q \in Goal}[M^n\delta](q) \geq \gamma$, and $\rho_\delta(n) = B$ (Below) else.

▶ **Definition 3.** The *trajectory* $\rho_\delta = \rho_0\rho_1 \cdots \in \{A, B\}^\omega$ from a distribution δ is the infinite word with $\rho_n = \rho_\delta(n)$ for all $n \in \mathbb{N}$.

We write the eigenvalues of M as $p_0, \ldots, p_k$ with $||p_i|| \geq ||p_j||$ for all $i < j$. Notice that $k + 1 = |Q|$ the number of states (as the uPA is simple). It is a standard result that all eigenvalues of Markov chains have modulus at most 1, and at least one eigenvalue is 1. We fix $p_0 = 1$. Now, as M is simple, it is also diagonalizable. Thus, there exists $a_i(\delta) \in \mathbb{C}$ (see [4] for further details) such that:

$$(1) \quad \rho_\delta(n) = A \text{ iff } \sum_{i=0}^{k} a_i(\delta) p_i^n \geq \gamma$$

In the following, we denote $u_\delta(n) = \sum_{i=0}^{k} a_i(\delta) p_i^n$ for all $n \in \mathbb{N}$. If ρ_δ is (effectively) ultimately periodic (i.e, of the form uv^ω), every (omega) regular property, such as existence, positivity and ultimate positivity is decidable (and are in fact easy to check). Unfortunately, this is not always the case, even for small simple unary PAs.

▶ **Theorem 3.** [2] There exists an initial distribution δ_0 and simple unary PA $\mathcal{A}$ with 3 states, and coefficients and threshold in $\mathbb{Q}$, such that ρ_{δ_0} is not ultimately periodic.

Proof Sketch. The unary PA is given by: $Goal = \{1\}$ is the first state, $\gamma = \frac{1}{3}$ and the associated matrix M_0 and initial distribution δ_0 are:

$$M_0 = \begin{pmatrix} 0.6 & 0.1 & 0.3 \\ 0.3 & 0.6 & 0.1 \\ 0.1 & 0.3 & 0.6 \end{pmatrix} \text{ and } \delta_0 = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{2} \end{pmatrix}$$

The reason the trajectory is not ultimately periodic follows from the fact that the eigenvalues of M_0 are 1, $r_0 e^{i\theta_0}$ and $r_0 e^{-i\theta_0}$ with $r_0 = \sqrt{19}/10$ and $\theta_0 = cos^{-1}(4/\sqrt{19})$. ◀

An easy way to obtain ultimately periodic trajectories is to restrict to eigenvalues v which are roots of real numbers, that is, there exists $n \in \mathbb{N} \setminus \{0\}$ with $v^n \in \mathbb{R}$.

▶ **Proposition 1.** Let $\mathcal{A}$ be a simple unary PA with eigenvalues $(p_i)_{i \leq m}$ all roots of real numbers. Then ρ_δ is ultimately periodic for all distributions δ. The (ultimate) period of ρ_δ can be chosen as any $m \in \mathbb{N} \setminus \{0\}$ such that p_i^m is a positive real number for all $i \leq m$.

Now, for a finite state (Büchi) automaton $\mathcal{B}$ over the alphabet $\{A, B\}$, the membership problem, of whether a given single trajectory $\rho_\delta \in \mathcal{L}(\mathcal{B})$, is decidable. As it is easy to obtain a (small) automaton $\mathcal{B}$ for each of the existence, positivity and ultimate positivity problem such that this problem is true iff $\rho_\delta \in \mathcal{L}(\mathcal{B})$, we obtain:

▶ **Proposition 2.** Let $\mathcal{A}$ be a simple unary PA with eigenvalues all roots of real numbers. Let δ_0 be a distribution. Then the existence, positivity and ultimate positivity problems from initial distribution δ_0 are decidable.

Note that Propositions 1 and 2 hold even when the matrix associated with the PA is diagonalizable, but not necessarily simple.

3 Language of a unary PA

Using automata-based methods allows us to consider more complex problems, where the initial distribution is not fixed. We define the set *Init* of initial distributions as a convex polytope, that is the convex hull of a finite number of distributions.

▶ **Definition 4.** The language of a unary PA $\mathcal{A}$ wrt. the set of initial distributions *Init* is $\mathcal{L}(Init, \mathcal{A}) = \{\rho_\delta \mid \delta \in Init\} \subseteq \{A, B\}^\omega$.

Note that A and B, and the language, depend on the threshold γ. As we assumed this threshold value to be fixed, the language only depends on $\mathcal{A}$ and *Init*. As $\mathcal{A}$ is often clear from the context, we will often write $\mathcal{L}(Init)$ instead of $\mathcal{L}(Init, \mathcal{A})$. For the yeast example $M = M_{yeast}$, we have eigenvalues $1; 0.7; 0.6$:

$$M \cdot \begin{pmatrix} 5/12 \\ 1/3 \\ 1/4 \end{pmatrix} = 1 \begin{pmatrix} 5/12 \\ 1/3 \\ 1/4 \end{pmatrix};\ M \cdot \begin{pmatrix} 5/12 \\ -5/12 \\ 0 \end{pmatrix} = 0.7 \begin{pmatrix} 5/12 \\ -5/12 \\ 0 \end{pmatrix};\ M \cdot \begin{pmatrix} 5/12 \\ 0 \\ -5/12 \end{pmatrix} = 0.6 \begin{pmatrix} 5/12 \\ 0 \\ -5/12 \end{pmatrix}$$

We can decompose two initial distributions $\delta_1, \delta_2 \in Init_{yeast}$ on the eigenvector basis:

$$\begin{pmatrix} 1/3 \\ 1/4 \\ 5/12 \end{pmatrix} = \begin{pmatrix} 5/12 \\ 1/3 \\ 1/4 \end{pmatrix} + \frac{1}{5} \begin{pmatrix} 5/12 \\ -5/12 \\ 0 \end{pmatrix} - \frac{2}{5} \begin{pmatrix} 5/12 \\ 0 \\ -5/12 \end{pmatrix};\quad \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix} = \begin{pmatrix} 5/12 \\ 1/3 \\ 1/4 \end{pmatrix} - \frac{1}{5} \begin{pmatrix} 5/12 \\ 0 \\ -5/12 \end{pmatrix}$$

Projecting on the first component, we have $\rho_{\delta_1}(n) = A$ iff $\frac{1}{12}0.7^n - \frac{1}{6}0.6^n \geq 0$, that is $\rho_{\delta_1} = B^4 A^\omega$. Also, $\rho_{\delta_2}(n) = A$ iff $-\frac{1}{12}0.6^n \geq 0$, that is $\rho_{\delta_2} = B^\omega$. With the techniques developed in the following, we can prove more generally that, for all $n \in \mathbb{N}$, we can find an ϵ s.t., $\delta = (1/3\ \ 1/3-\epsilon\ \ 1/3+\epsilon)^T$ has trajectory $\rho_\delta = B^n A^\omega$, and that $\mathcal{L}(Init_{yeast}) = B^* A^\omega \cup B^\omega$. Thus, property (P_{yeast}), from Introduction, does not hold for every initial distribution.

In general, if $\mathcal{L}(Init, \mathcal{A})$ is regular, then any regular question will be decidable. For instance, if $\mathcal{L}(Init, \mathcal{A})$ is regular, then it is decidable whether there exists $\delta_0 \in Init$ such that the existence problem is true for $\mathcal{A}, \delta_0$. One can also ask whether for a given convex polytope Q, some property (such as positivity) expressed e.g. with $LTL_{\mathcal{I}}$ [1] is true. Taking δ in the interior of Q, this corresponds to checking the robustness of the property around δ.

Clearly, simple PA $\mathcal{A}$ does not ensure the regularity of $\mathcal{L}(Init, \mathcal{A})$ because of Theorem 3 (by choosing $Init = \{\delta_0\}$ which is a convex polytope). Surprisingly, restricting eigenvalues to be *distinct* and *roots of real numbers* does not ensure regularity either (see Section 6). In the following, we thus take a stronger restriction: we assume that the eigenvalues of M are *distinct* and *positive real numbers*. That is, $p_0 = 1 > p_1 > \cdots > p_k \geq 0$ with $k + 1 = |Q|$ the number of states. From Proposition 1, we obtain as corollary that for all δ_0, we have either $\rho_{\delta_0} = wA^\omega$ or $\rho_{\delta_0} = wB^\omega$ for w a finite word of $\{A, B\}^*$:

▶ **Corollary 2.** *Let M be a simple (or just diagonalizable) stochastic matrix with positive real eigenvalues. Then every trajectory ρ_{δ_0} is ultimately constant.*

However, the language $\mathcal{L}(Init_{yeast}, M_{yeast})$ shows that $\mathcal{L}(Init, \mathcal{A})$ is not always of the simple form $\bigcup_{w \in W_A} wA^\omega \cup \bigcup_{w \in W_B} wB^\omega$, for W_A, W_B two finite sets of finite words over $\{A, B\}^*$. Nevertheless, in the next two sections, we succeed in proving the regularity of $\mathcal{L}(Init, \mathcal{A})$, which is our main result:

▶ **Theorem 4.** Let $\mathcal{A}$ be a unary PA with distinct positive real eigenvalues, and *Init* be a convex polytope of (initial) distributions. Then, $\mathcal{L}(Init, \mathcal{A})$ is effectively regular.

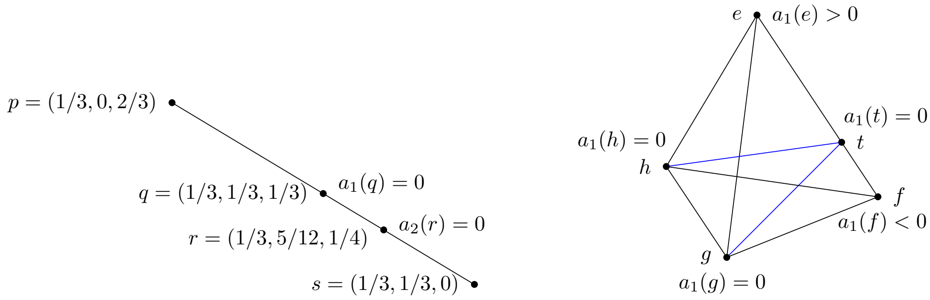

Figure 1 Breaking into convex polytopes with constant signs

Note that the hypotheses of Theorem 4 are decidable for $\mathcal{A}$ with rational coefficients. Indeed, it suffices to use linear algebra to compute the eigenvalues and vectors, and check whether their complex part is null. Further the proof carries through even when the matrix of $\mathcal{A}$ is diagonalizable (though we tackle just the simple case here). We also show that this result is tight, i.e., relaxing the hypothesis any further leads to non-regularity (see Section 6).

3.1 Partition of the set *Init* of initial distributions

Recall that we write $u_\delta(n) := \sum_{i=0}^{k} a_i(\delta)p_i^n$, where $a_i(\delta)$ are given by Equation (1) from the previous section. Because the eigenvalues are real numbers, $a_i(\delta)$ is a real number for every i and δ. Notice that a_i is a linear function in δ, that is, $a_i(\alpha\delta_1 + \beta\delta_2) = \alpha a_i(\delta_1) + \beta a_i(\delta_2)$. The trajectory ρ_δ depends crucially on the sign of $a_0(\delta)$, and if $a_0(\delta) = 0$, on the sign of $a_1(\delta)$, etc. First, let $L_i = \{\delta \mid a_0(\delta) = \cdots = a_i(\delta) = 0\}$. This is a vector space (i.e., it is in $\mathbb{R}^k$ and contains the space of distributions over Q), as for any $\nu_1, \nu_2 \in \mathbb{R}^k$, we have $\nu_1, \nu_2 \in L_i$ implies that any linear combination $\alpha\delta_1 + \beta\delta_2 \in L_i$ (since $a_i(\nu)$ is linear in ν, and the kernel of a linear function is a vector space).

We will divide the space of distributions into a finite set $\mathcal{H}$ of convex polytopes $H \in \mathcal{H}$ to keep the sign of each a_i constant on each polytope. Each $H \in \mathcal{H}$ satisfies that for all $e, f \in H$, for all $i \leq k$, we have $a_i(e), a_i(f)$ do not have different signs (either one is 0, or both are positive or both are negative). This can be done since $a_i(\nu)$ is continuous (as it is linear). This is pictorially represented in the left of Figure 1. For instance, we divide $Init_{yeast}$ into three polytopes: $\{(1/3, y, 2/3 - y) \mid y \leq 1/3\}$ and $\{(1/3, y, 2/3 - y) \mid 1/3 \leq y \leq 5/12\}$ and $\{(1/3, y, 2/3 - y) \mid y \geq 5/12\}$ as for $\delta = (1/3, 1/3, 1/3)$ we have $a_0(\delta) = 1$, $a_1(\delta) = 0$ (and $a_2(\delta) = -1/5$) and for $\delta = (1/3, 5/12, 1/4)$ we have $a_0(\delta) = 1$, $a_1(\delta) = -1/5$, $a_2(\delta) = 0$.

In general, we can assume that each of $H \in \mathcal{H}$ is the convex hull of $k + 2$ points (else we divide further: this can be done as the space has dimension $k + 1$). Consider the right part of Figure 1. Let *Init* be the convex hull of points e, f, g, h (in three dimensions) and $a_0(x) = 0$ and $a_2(x) > 0$ for all $x \in \{e, f, g, h, t\}$. Hence the sign of each trajectory ultimately depends upon $a_1(x)$. In the example, $a_1(g) = a_1(h) = 0$ while $a_1(e) > 0 > a_1(f)$. Then there is a point t between e and f for which $a_1(t) = 0$ (in fact, $t = |a_1(f)|/(|a_1(e)| + |a_1(f)|)e + |a_1(e)|/(|a_1(e)| + |a_1(f)|)f$). We have $L_1 \cap Init$ is the convex hull of h, g, t. We break *Init* into two convex polytopes, the convex hull of h, g, t, e and the convex hull of h, g, t, f.

Let $H \in \mathcal{H}$. We let P be the finite set of (at most $k + 2$) extremities of H. In particular, H is the convex hull of P. Now it suffices to show that the language $\mathcal{L}(H)$ (taking H as the initial set of distributions) of each of these convex polytopes H is regular to prove that the language $\mathcal{L}(Init) = \bigcup_{H \in \mathcal{H}} \mathcal{L}(H)$ is regular.

3.2 High level description of the proof

The proof of the regularity of the language $\mathcal{L}(H)$ starting from the convex polytope H is performed as follows. We first prove that there exists a N_{max} such that the ultimate language (after N_{max} steps) of H is effectively regular using analytical techniques.

▶ **Definition 5.** Given N_{max}, the *ultimate language* from a convex polytope H is defined as $\mathcal{L}_{ult}^{N_{max}}(H) = \{v \mid \exists w \in \{A,B\}^{N_{max}}, wv \in \mathcal{L}(H)\}$.

In the next section (Corollary 6), we show that this ultimate language $\mathcal{L}_{ult}^{N_{max}}(H)$ is regular, of the form $A^*B^* \cdots B^*A^\omega \cup A^*B^* \cdots A^*B^\omega$ with a bounded number of switches between A and B's. However, while for each prefix $w \in \{A,B\}^{N_{max}}$, the set H_w of initial distributions in H whose trajectory starts with w is a convex polytope; the language $\mathcal{L}(H_w)$ from H_w can be complex to represent. It is not in general $w\mathcal{L}_{ult}^{N_{max}}(H)$, but a strict subset.

In Section 5 (Lemma 9), we prove that the language $\mathcal{L}(H')$ associated with some carefully defined convex polytope $H' \subseteq H$ is a regular language, of the form $\bigcup_{w\in W} wA^iA^*B^* \cdots B^*A^\omega \cup wA^iA^*B^* \cdots A^*B^\omega$ for a finite set W. Further, removing H' from H gives rise to a finite number of convex polytopes with a smaller number of “sign-changes”, as formally defined in the next section. Hence we can apply the arguments inductively (requiring potentially to change the N_{max} considered). Finally, the union of these languages gives the desired regularity characterization for $\mathcal{L}(H)$.

4 Ultimate Language

4.1 Limited number of switches.

We first show that the ultimate language $\mathcal{L}_{ult}^{N_{max}}(H)$ is included into $A^*B^*A^* \cdots A^*B^\omega \cup A^*B^*A^* \cdots B^*A^\omega$ for some $N_{max} \in \mathbb{N}$, with a limited number of switches between A and B depending on properties of the set P of extremities of H.

We start by considering the generalisation of a sequence u_δ to a function over positive reals, and we will abuse the notation u_δ to denote both the sequence and the real function.

▶ **Definition 6.** A function of type $k \in \mathbb{N}$ is a function of the form $u : \mathbb{R}_{>0} \to \mathbb{R}$, with $u(x) = \sum_{j=0}^{k} \alpha_j p_j^x$, where $p_0 > \cdots > p_k > 0$.

Now, let $u : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a continuous function. We can associate with function u the (infinite) word $L(u) \in \{A,B\}^\omega, L(u) = (a_0a_1\ldots)$, where for all $n \in \mathbb{N}$, a_n is defined as $a_n = A$ if $u(n) \geq 0$ and $a_n = B$ otherwise. We have easily that $\rho_\delta = L(u_\delta)$. Knowing the zeros of u_δ and its sign before and after the zeros, defines uniquely the trajectory ρ_δ.

For example, let u be such that it has four zeros: $u(N-0.04) = u(N+10.3) = u(N+20) = u(N+35) = 0$ for some integer N. Assume that $u(0) < 0, u(N+1) > 0, u(N+11) < 0, u(N+30) < 0$ and $u(N+40) > 0$. Thus, by continuity of u, u is strictly negative on $[0, N-1]$, strictly positive on $[N, N+10]$, non-positive on $[N+11, N+34]$ and non-negative on $[N+35, \infty)$. Thus the associated trajectory $\rho_\delta = B^NA^{11}B^{24}A^\omega$.

Hence, it is important to analyze the zeros of functions u_δ. If the number of zeros is bounded, then the number of alternations between A's and B's in any trajectory ρ_δ from $\delta \in H$ will be bounded. In fact, it is a standard result (which we do not use hence do not reprove here) that every type k function u has at most k zeros. We now show a more precise bound on the number of zeros. Namely, for the convex hull H' of a finite set P' of distributions in H, the number of alternations between A's and B's in H' is limited by the number of alternations of the sign of the dominant coefficients of the distributions in P'.

Let $z \in \mathbb{N}$. For $i \in \{0, \ldots, z\}$, let $u^i(x) := a_0^i p_0^x + a_1^i p_1^x + \cdots + a_k^i p_k^x$, with $p_0 > p_1 > p_2 > \ldots > p_k > 0$, representing for instance the functions associated with the $z+1$ extremities of H'. We denote $dom(u^i)$ the dominant coefficient of u^i, that is the smallest integer j with $a_j^i \neq 0$. We reorder $(u^i)_{i \in \{0,\ldots,z\}}$ such that $dom(u^i) \leq dom(u^{i+1})$ for all $i < z$. We denote $sign\_dom(u^i) \in \{+1, -1\}$ as the sign of $dom(u^i)$. We will assume, as for H, that for all i, i', j, a_j^i and $a_j^{i'}$ have the same sign. We let $Z(u^0, \cdots, u^z) = |\{i \leq z-1 \mid sign\_dom(u^i) \neq sign\_dom(u^{i+1})\}|$. That is, $Z(u^0, \cdots, u^z)$ is the number of switches of sign between the dominant terms of u^i and u^{i+1}. We have $0 \leq Z(u^0, \cdots, u^z) \leq z$. Notice that as for $dom(u^i) = dom(u^j)$, we have $sign\_dom(u^i) = sign\_dom(u^j)$, $Z(u^0, \cdots, u^z)$ does not depend upon the choice in the ordering of $(u^i)_{i \in \{0,\ldots,z\}}$. We can now give a bound on the number of zeros of functions which are convex combinations of $u^0 \cdots u^z$.

▶ **Lemma 3.** *Let $u^0 \cdots u^z$ be $z+1$ type k functions. There exists a $N_{max} \in \mathbb{N}$ such that for all $\lambda_i \in [0,1]$ with $\sum_i \lambda_i = 1$, denoting $u(x) = \sum_{i=0}^{z} \lambda_i u^i(x)$, $u(x)$ has at most $Z(u^0, \cdots, u^z)$ zeros after N_{max}. Further, if $u(x)$ has exactly $Z(u^0, \cdots, u^z)$ zeros after N_{max}, then its sign changes exactly $Z(u^0, \cdots, u^z)$ times (that is, no zero is a local maximum/minimum).*

In other words, we show that $u(x)$ behaves like a polynomial of degree $Z(u^0, \cdots, u^z)$ (as it has $Z(u^0, \cdots, u^z)$ dominating terms), although it has degree $k > Z(u^0, \cdots, u^z)$. In fact, we prove that for $\ell = dom(u^i)$, the coefficients $a_j^i p_j^x$ for all $j > \ell$ play a negligible role wrt. $a_\ell^i p_\ell^x$.

Let $H \in \mathcal{H}$, and P its finite set of extremal points. We can apply Lemma 3 to $u^0, \ldots, u^z$, the functions associated with the points of P (in decreasing order of dominating coefficient), and obtain a N_{max}. Now, since P is finite, the trajectories from P are ultimately constant, hence there exists N_y such that for all $i \leq y$, the trajectory of u^i is wA^ω or wB^ω for some $w \in \{A,B\}^{N_y}$. We define N_H to be the maximum of N_y and N_{max}. With this bound on the number of zeros, we deduce the following inclusion for the ultimate language $L_{ult}^{N_H}(H)$:

▶ **Corollary 4.** *Let $y = Z(u^0, \ldots, u^z)$. The ultimate language $L_{ult}^{N_H}(H) \subseteq C_1^* \cdots C_{y-1}^* C_y^\omega \cup C_1^* \cdots C_{y-1}^* C_{y-1}^\omega$ for $\{C_i, C_{i+1}\} = \{A, B\}$ for all $i < y$; and $C_y = A$ iff $sign\_dom(u^0)$ is positive.*

We can have 4 different sequences for $C_1^* \cdots C_{y-1}^* C_y^\omega$ with $\{C_i, C_{i+1}\} = \{A, B\}$, depending on the first and last letters C_1, C_y (or equivalently, C_y and parity of y which determines C_1).

The proof of our main result on regularity of $\mathcal{L}(H)$ will proceed by induction over the *switching-dimension* $Z(H)$ of H which we define as $Z(H) = Z(u^0, \ldots, u^z)$. Notice that we could define the switching dimension for any convex set (not necessarily a polytope) whenever the sign of $a_i(\delta)$ does not change within the convex set. Finally, we also define $sign\_dom(H) = sign\_dom(u^0)$.

4.2 Characterization of the Ultimate Language.

We now show that the ultimate language of H is exactly $\mathcal{L}_{ult}^{N_H}(H) = A^* B^* A^* \cdots A^* B^\omega \cup A^* B^* A^* \cdots B^* A^\omega$, with at most $Z(H)$ switches of signs. We will state the associated technical Lemma 5 in the more general settings of "faces" as defined below, as it will be useful in the next section. Let P be the finite set of extremal points of a H. We call $(f^0, \ldots, f^y) \subseteq P$ a *face* of H if $Z(v^0, \ldots, v^y) = y = Z(H)$ for the functions $(v^0, \ldots, v^y)$ associated with the extremal points $(f^0, \ldots, f^y)$. Notice that denoting H' the convex hull of F, we can choose $N_{H'} = N_H$ (which is not the case for H' an arbitrary polytope included into H).

▶ **Lemma 5.** *Given a face $(f^0, \ldots, f^y) \subseteq P$ of H with associated functions v^i, we have, for all $n_1, n_2, \ldots, n_y \in \mathbb{N}$ there exist $\lambda_i \in [0,1]$ with $\sum_i \lambda_i = 1$, such that denoting $\widetilde{v}(x) = \sum_{i=1}^{y} \lambda_i v^i(x)$, $L(\widetilde{v}) = wA^{n_1} B^{n_2} \ldots B^{n_y} A^\omega$ (for y even) for some prefix $w \in \{A,B\}^{N_H}$.*

That is, for all $n_1, \cdots n_y$, one can find a prefix w of size N_H and a point δ in the convex hull of $e^1, \cdots e^y$, such that $\rho_\delta = wA^{n_1}B^{n_2}\cdots B^{n_y}A^\omega$ (assuming the correct parity of y). Let H' be the convex hull of $f^0, \ldots, f^y$. Hence $Z(H') = Z(H)$. Then, the ultimate language of H' (i.e., the language after prefixes of size N_H associated with y) contains $A^*B^* \ldots B^*A^\omega$ with y switches between A and B, which is the converse of Corollary 4. We can thus deduce the following about the ultimate language:

▶ **Corollary 6.** *$L_{ult}^{N_H}(H) = L_{ult}^{N_H}(H') = C_1^*C_2^* \ldots C_y^*A^\omega \cup C_1^*C_2^* \ldots C_{y-1}^*B^\omega$ with $\{C_i, C_{i+1}\} = \{A, B\}$.*

Proof. We first prove the result for $L_{ult}^{N_H}(H')$. We can apply lemma 5 to H' and lemma 3 to H'. We obtain the first part of the union. Now, let $H'' \subseteq H'$ be the convex hull of $e^1, \cdots, e^y$ (that is excluding e^0). Each point δ in $H' \setminus H''$ has a trajectory which ends with A^ω, as $dom(u_\delta) = dom(v^1)$, and thus $sign\_dom(u_\delta) = sign\_dom(v^1)$ by construction of H (and $H' \subseteq H$). Thus the points with trajectory ending with B^ω are in H'', and applying lemma 3, we know that their ultimate trajectory has at most $y-1$ switches. Applying lemma 5 to H'', we obtain the second hand of the union. Now, $L_{ult}^{N_H}(H') \subseteq L_{ult}^{N_H}(H)$, and $L_{ult}^{N_H}(H) \subseteq C_1^*C_2^* \ldots C_y^*A^\omega \cup C_1^*C_2^* \ldots C_{y-1}^*B^\omega$ by Corollary 4. ◀

However, we cannot immediately conclude that $\mathcal{L}(H)$ is regular. Though N_H is finite, computable and there are a finite number of prefixes w of size N_H, we need to show that the subset of $\mathcal{L}_{ult}^{N_H}(H)$ appearing after a given $w \in \{A, B\}^{N_H}$ is (effectively) regular. This is what we do formally in the following section.

5 Regularity of the Language

Let $\{e^0, \cdots, e^z\} = P$ the extremal points of H. Let u^p the function associated with each $e^p \in P$. We denote $y = Z(H) = Z((u^p)_{p \leq z})$. We will show the regularity of $\mathcal{L}(H)$ using an induction on $Z(H)$.

For $Z(H) = 0$, the regularity of $\mathcal{L}(H)$ is trivial as all the dominant coefficients have the same sign. Thus, by Corollary 4, the ultimate language is $\mathcal{L}_{ult}^{N_H}(H) = A^\omega$ and then the language is $\mathcal{L}(H) = \bigcup_{w \in W} wA^\omega$; or the ultimate language is $\mathcal{L}_{ult}^{N_H}(H) = B^\omega$ and the language is $\mathcal{L}(H) = \bigcup_{w \in W} wB^\omega$, for a finite set of $W \subseteq \{A, B\}^{N_H}$.

For $w \in \{A, B\}^{N_H}$, consider $H_w = \{\delta \in H \mid \rho_\delta = wv\}$, i.e., the language of words which begin with the prefix w. It is easy to see that $H_w \subseteq H$ is a polytope. Hence $Z(H_w) \leq Z(H)$. Observe that $\mathcal{L}(H) = \bigcup_{w \in \{A,B\}^{N_H}} \mathcal{L}(H_w)$. To show the regularity of $\mathcal{L}(H)$, we show the regularity of $\mathcal{L}(H_w)$ for each of the finitely many $w \in \{A, B\}^{N_H}$. For each $w \in \{A, B\}^{N_H}$, we have two cases: either $Z(H_w) < Z(H)$; then we apply the induction hypothesis and we are done. Or else, $Z(H_w) = Z(H) = y$. In this case, the sketch of proof is as follows:

- We show that there exists J such that for all $i \leq y$ and all $j \geq J$, we have a point h_j^i in H_w with trajectory $wC_1^jC_2C_3 \cdots C_{i-1}C_i^\omega$. This is shown by applying lemma 5 to each face $(f^0, \ldots, f^y)$ of H and then using convexity arguments and the fact that $Z(H_w) = Z(H)$.
- Subsequently, denoting H' the convex hull of $h_J^0 \cdots h_J^y$, we will deduce that $\mathcal{L}(H')$ is a regular language of the form $wC_1^JC_1^*C_2^*C_3^* \cdots C_{i-1}^*C_i^\omega$,
- Partitioning $H_w \setminus H'$ into a finite set of polytopes, we obtain polytopes of lower switching-dimensions, which have regular languages by induction.
- We conclude since the finite union of these regular languages is a regular language, namely $\mathcal{L}(H_w)$.

We now formalize the above proof sketch in a sequence of lemmas, whose details can be found in [4]. For all faces F of H, applying Lemma 5 gives for all $j \in \mathbb{N}$, a point $g_j(F)$ of the convex hull of F with trajectory $w_j C_1^j C_2 C_3 \cdots C_y^\omega$, for some $w_j \in \{A,B\}^{N_H}$. We now prove that (g_j) converges towards f^y, the point of F with lowest dominant term.

▶ **Lemma 7.** *For every face $F = (f^0, \ldots, f^y)$ of H, $(g_j(F))_{j \in \mathbb{N}}$ converges towards f^y as j tends to infinity.*

For all j, we consider $F(y,j)$ the convex hull of $\{g_j(F) \mid F \text{ is a face of } H\}$. Every point of $F(y,j)$ has trajectory $w' C_1^j C_2 C_3 \cdots C_y^\omega$ for some $w' \in \{A,B\}^{N_H}$. We then show by convexity that H_2 intersects $F(y,j)$, i.e., it has a point with trajectory $w' C_1^j C_2 C_3 \ldots C_y^\omega$.

▶ **Lemma 8.** *For $w \in \{A,B\}^{N_H}$ with $Z(H_w) = Z(H)$, there exists J s.t. for all $j > J$, $F(y,j) \cap H_w \neq \emptyset$.*

Similarly, for all $i \leq y$ we can define a polytope $F(i,j)$. All the points in $F(i,j)$ have trajectory $w' C_1^j C_2 C_3 \cdots C_i^\omega$ for some $w' \in \{A,B\}^{N_H}$. We can find a J_i and a point $h_j^i \in H_w$ with trajectory $w C_1^j C_2 C_3 \cdots C_i^\omega$ for all $i \leq y$ and all $j > J_i$. Now, as the number of $i \leq y$ is bounded, one can find such a J uniform over all $i \leq y$ (by taking maximum over all i).

Consider $F(J)$ the convex hull of $F(0,J), \ldots, F(y,J)$. By convexity, all the points in $F(J)$ have their n-th letters of trajectory as C_1 for all $n \in [N_H + 1 \cdots N_H + J]$, since this is true for all points of $F(i,J)$. Hence, the language of $H_w \cap F(J)$ is included into $w C_1^J C_1^* C_2^* \cdots C_y^\omega \cup w C_1^J C_1^* C_2^* \cdots C_{y-1}^\omega$, because of the bound on the number of alternations after N_H of trajectories from points of H (Lemma 3). We show now that we have equality.

▶ **Lemma 9.** *The language of the convex hull of $\{h_J^0, \ldots, h_J^y\}$ is exactly $w C_1^J C_1^* C_2^* C_3^* \cdots C_{y-1}^* C_y^\omega \cup w C_1^J C_1^* C_2^* C_3^* \cdots C_{y-2}^* C_{y-1}^\omega$.*

Hence the language of $H_w \cap F(J)$ is $w C_1^i C_1^* \cdots C_y^\omega \cup w C_1^i C_1^* \cdots C_{y-1}^\omega$.

Next, we note that the set $H_w \setminus F(J)$ may not be convex. However, one can partition $H_w \setminus F(J)$ into a finite number of convex polytopes. Now, let G be a convex polytope in $H_w \setminus F(J)$. We want to show that $Z(G) < Z(H_w) = Z(H) = y$. Indeed, else, one could apply Lemma 8 to $G_w = G$ and for some J' obtain $F(i,j) \cap G \neq \emptyset$ for any $j > J'$, which contradicts G being a convex set in $H_w \setminus F(J)$.

Hence one can compute the language of every G inductively, and each of them is regular. Finally, this leads to the regularity of $\mathcal{L}(H_w)$ by finite union, and to the regularity of $\mathcal{L}(H)$, and again by finite union to the regularity of $\mathcal{L}(Init)$. This concludes our proof of the main regularity result, i.e., Theorem 4.

6 Non-regularity of the symbolic dynamics

In this section, we will prove that symbolic dynamics of uPA can produce non-regular languages even when eigenvalues of the transition matrix are distinct roots of real numbers. We prove this by constructing such a uPA and choosing the set of initial distributions carefully. Consider a uPA $\mathcal{A}_1$ with 7 states $q_1, \ldots q_7$, $Goal = \{q_7\}$, and the following transition matrix:

$$M_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{1}{512} & \frac{8r+3}{512} & \frac{3+3r}{64} & \frac{13+16r}{128} & \frac{9+2r}{32} & \frac{1+4r}{16} & \frac{1-r}{2} \end{bmatrix}$$

where $r = cos(\pi/8) = \frac{\sqrt{\sqrt{2}+2}}{2}$. Eigenvalues of M_1 are 1, $\frac{1}{2}e^{\pm i\pi/2}$, $\frac{1}{2\sqrt{2}}e^{\pm i3\pi/4}$ and $\frac{1}{4}e^{\pm i7\pi/8}$, which are distinct roots of real numbers. We choose $\gamma = \sum_{q \in Goal} \delta_{stat}(q) = \delta_{stat}(q_7) = \frac{512}{65(17+8cos(\frac{\pi}{8}))}$ (for any other choice of γ, the language is regular).

Let δ be the initial distribution and $M_1^n\delta$ be the distribution after n steps of M_1. We consider a basis of eigenvectors such that the eigenvector corresponding to eigenvalue 1 is the stationary distribution and the remaining eigenvectors are normalized such that the 7^{th} component (corresponding to the *Goal* state) of each of them is 1. This is possible as the 7^{th} component of each eigenvector of M_1 is non-zero. Now, by eigenvalue decomposition:

$$M_1^n\delta(7) = \mu_0 + \frac{\mu_1}{2^n}(e^{ni\pi/2}+e^{-ni\pi/2}) + \frac{\mu_2}{(2\sqrt{2})^n}\left(e^{n3i\pi/4}+e^{-n3i\pi/4}\right) + \frac{\mu_3}{4^n}(e^{n7i\pi/8}+e^{-n7i\pi/8})$$

where $\mu_0 = \gamma$ and δ written in the eigenvector basis is $(1, \mu_1, \mu_1, \mu_2, \mu_2, \mu_3, \mu_3)$.

Consider the initial set of distributions *Init* to be the line segment $(P1, P2)$ where $P1 = (1, a, a, b, b, c, c)$ and $P2 = (1, 0, 0, b, b, c, c)$ in the eigenvector basis, where $a = \frac{cos(3\pi/8)}{2^{46}\sqrt{2}}$, $b = \frac{1}{2^{20}}$, $c = \frac{1}{2^{11}cos(3\pi/8)}$. These values are chosen so that μ_0 dominates over the other terms in the above equation, which ensures that $P1$ and $P2$ correspond to valid distributions in the standard basis. Note that *Init* is the set of convex combinations of distributions $P1$ and $P2$. Now, we can show our main theorem of this section.

▶ **Theorem 10.** *$\mathcal{L}(Init, \mathcal{A}_1)$ is not regular.*

Proof sketch. Let $L = \mathcal{L}(Init, \mathcal{A}_1)$. For $x, y, z, k \in \mathbb{N}$, we define $L^k_{x,y,z} = \{w \in \Sigma^\omega \mid \exists w' \in L,\ \forall i \in \mathbb{N}, w'_{k(i-1)+x} = w_{3(i-1)+1}, w'_{k(i-1)+y} = w_{3(i-1)+2}, w'_{k(i-1)+z} = w_{3(i-1)+3}\}$. That is, for every $a_1a_2a_3 \ldots \in L$, $a_xa_ya_za_{k+x}a_{k+y}a_{k+z} \ldots \in L^k_{x,y,z}$ where $x, y, z \leq k$. It is easy to see that if $L^k_{x,y,z}$ is non-regular, so is L. Now we can show that $L^{16}_{2,3,4} = \{(ABB)^2(AAB)^{y+g(\mu_1,\mu_2,\mu_3)}(BAB)^y(BAA)^w : y \geq 0\}$. As the range of y is $[1, \infty)$ and $g(\mu_1, \mu_2, \mu_3)$ is a bounded function, hence $L^{16}_{2,3,4}$ is not regular. Thus, L is not regular which completes the proof. ◀

7 Conclusion

Though unary Probabilistic Automata (or Markov Chains) are a simple formalism, there are still many basic problems, whose decidability is open and thought to be very hard. Indeed, it is surprising yet significant that even after assuming strong hypotheses, their behaviors cannot be described easily. In this paper, we proposed a class of unary probabilistic automata, for which all properties of some logic, e.g. $LTL_{\mathcal{I}}$ are decidable even considering an infinite set of initial distributions. This allows for instance to check for the robustness of the behavior wrt. a given property (e.g. positivity) for behaviors around a given initial distribution. Further, while we proved our results with respect to a single hyperplane (above is A, below is B), we can generalize these to more general settings as well. Finally, we showed that relaxing the assumptions immediately leads to non-regularity.

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