



**HAL**  
open science

# FUNCTION VARIATIONAL PRINCIPLES AND NORMED MINIMIZERS

Mihai Turinici

► **To cite this version:**

Mihai Turinici. FUNCTION VARIATIONAL PRINCIPLES AND NORMED MINIMIZERS. 2015.  
hal-01244862

**HAL Id: hal-01244862**

**<https://hal.science/hal-01244862>**

Preprint submitted on 16 Dec 2015

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# FUNCTION VARIATIONAL PRINCIPLES AND NORMED MINIMIZERS

MIHAI TURINICI

ABSTRACT. The function variational principle due to El Amrouss [Rev. Col. Mat., 40 (2006), 1-14] may be obtained in a simplified manner. Further applications to existence of minimizers for Gâteaux differentiable bounded from below lsc functions over Hilbert spaces are then provided.

## 1. INTRODUCTION

Let  $(M, d)$  be a complete metric space; and  $\varphi : M \rightarrow R$  be  $(M, d)$ -regular; i.e.,

(a01)  $\varphi$  is bounded below ( $\inf \varphi(M) > -\infty$ )

(a02)  $\varphi$  is  $d$ -lsc on  $M$  ( $\liminf_n \varphi(x_n) \geq \varphi(x)$ , whenever  $x_n \xrightarrow{d} x$ ).

Define for each  $\theta \geq 0$

$$\text{ulev}(\varphi; M; \theta) = \{x \in M; \varphi(x) \leq \inf \varphi(M) + \theta\};$$

this will be referred to as the  $\theta$ -upper level set of  $\varphi$  with respect to  $M$ ; in particular,  $\text{ulev}(\varphi; M; 0)$  is nothing else than the (global) minimizers set of  $\varphi$  over  $M$ . The following 1974 statement in Ekeland [9] (referred to as Ekeland's variational principle; in short: EVP) is well known.

**Theorem 1.1.** *Let  $\varepsilon > 0$  be given; as well as some  $u \in \text{ulev}(\varphi; M; \varepsilon)$ . Then, for each  $\delta > 0$ , there exists  $v = v(\varepsilon, u; \delta) \in M$  with*

$$(11-a) \quad (\varepsilon/\delta)d(u, v) \leq \varphi(u) - \varphi(v); \text{ hence } [\varphi(u) \geq \varphi(v), d(u, v) \leq \delta]$$

$$(11-b) \quad (\varepsilon/\delta)d(v, x) > \varphi(v) - \varphi(x), \text{ for all } x \in M \setminus \{v\}.$$

This principle found some basic applications to control and optimization, generalized differential calculus, critical point theory and global analysis; we refer to the 1979 paper by Ekeland [10] for a survey of these. So, it cannot be surprising that, soon after its formulation, many extensions of (EVP) were proposed. For example, the *abstract (order)* one starts from the fact that, with respect to the Brøndsted order [6] (on  $M$ )

$$(x, y \in M): x \leq y \text{ iff } (\varepsilon/\delta)d(x, y) \leq \varphi(x) - \varphi(y)$$

the point  $v \in M$  appearing in the second conclusion above is *maximal*; so that, (EVP) is nothing but a variant of the Zorn-Bourbaki maximal principle [31, 4]; see Hyers et al [14, Ch 5] for a number of technical aspects. The *dimensional* way of extension refers to the ambient space  $(R)$  of  $\varphi(M)$  being substituted by a (topological or not) vector space; an account of the results in this area is to be found in the 2003 monograph by Goepfert et al [13, Ch 3]. Further, the *(pseudo) metrical*

---

2010 *Mathematics Subject Classification.* 49J53 (Primary), 54H25 (Secondary).

*Key words and phrases.* Metric space, bounded from below lsc function, variational principle, normal and comparison function, maximal element, Zorn order, inductive and regular order, Hilbert space, Gâteaux derivative, minimizer, Palais-Smale condition.

one consists in conditions imposed to the ambient metric over  $M$  being relaxed. The basic result in this direction was obtained in 1992 by Tataru [22], via Ekeland type techniques; subsequent extensions of it may be found in the 1996 paper by Kada et al [15]. Finally, we must add to this list the *functional* extension of (EVP) obtained in 1997 by Zhong [30] (and referred to as Zhong's Variational Principle; in short: ZVP). Let again  $\varphi : M \rightarrow R$  be a  $(M, d)$ -regular function. Further, take a *normal* function  $t \mapsto b(t)$  from  $R_+ := [0, \infty[$  to  $R_+^0 := ]0, \infty[$ ; i.e.,

- (a03)  $b(\cdot)$  is increasing and continuous
- (a04)  $B(\infty) = \infty$ , where  $B(t) = \int_0^t (1/b(\tau))d\tau$ ,  $t \geq 0$ .

**Theorem 1.2.** *Let  $\varepsilon > 0$  be given, as well as some  $u \in \text{ulev}(\varphi; M; \varepsilon)$ . Further, let  $x_0 \in M$  and  $\delta, \rho > 0$  be taken according to*

$$\delta \leq B(r + \rho) - B(r), \text{ where } r := d(x_0, u).$$

*There exists then  $v = v(\varepsilon, u; x_0, \delta, \rho)$  in  $M$ , with*

- (12-a)  $\varphi(u) \geq \varphi(v)$ ,  $d(x_0, v) \leq r + \rho$
- (12-b)  $(\varepsilon/\delta)d(v, x)/b(d(x_0, v)) > \varphi(v) - \varphi(x)$ , for each  $x \in M \setminus \{v\}$ .

Now, evidently, (ZVP) includes (for  $b = 1$ ,  $x_0 = u$ , and  $\delta = \rho$ ) the local version of (EVP) based upon (11-a) (the second half). The relative form of the same, based upon (11-a) (the first half) also holds (but indirectly); see Bao and Khanh [2] for details. Summing up, (ZVP) includes (EVP); but the provided argument is rather involved. A simplification of the proposed reasoning was given in Turinici [26], by a technique developed in Park and Bae [20]; note that, as a consequence of this, (ZVP) is nothing but a logical equivalent of (EVP).

Recently, a local functional version of (EVP) was established by El Amrouss [11]. Let  $\varphi : M \rightarrow R$  be a  $(M, d)$ -regular function. Further, let the function  $a : R_+ \rightarrow R_+^0$  be *admissible*, in the sense

- (a05)  $a(\cdot)$  is increasing and continuous
- (a06)  $a(\cdot)$  is a comparison function, of order  $k > 0$
- ( $\forall q \geq k, \exists \lambda, \mu \geq 0 : a((t+1)s) \leq a(t)[\lambda s^q + \mu], \forall t, s \geq 0$ ).

Denote also, for  $u \in M$ ,  $\rho > 0$ ,

$$M[u, \rho] = \{x \in M; d(u, x) \leq \rho\}, M(u, \rho) = \{x \in M; d(u, x) < \rho\}$$

(referred to as: the *closed/open* sphere with center  $x$  and radius  $\rho$ ). Finally, take some  $x_0 \in M$  and some function  $\gamma : M \rightarrow R_+^0$ , with

- (a07)  $u \mapsto \gamma(u)/(1 + d(x_0, u))$  is bounded on  $M$ .

The following result (referred to as El Amrouss Ordering Principle; in short: EAOP) is now available.

**Theorem 1.3.** *Let  $\varepsilon, \delta > 0$  be given, as well as some  $u \in \text{ulev}(\varphi; M; \varepsilon)$ . There exists then a sequence  $(z_n; n \geq 0)$  in  $M[u, \gamma(u)]$  and a point  $v \in M[u, \gamma(u)]$ , with the properties*

- (13-a)  $z_0 = u$ ,  $\lim_n z_n = v$  and  $(d(x_0, z_n); n \geq 0)$  is ascending
- (13-b)  $\sum_{n=0}^j d(z_n, z_{n+1})/a(d(x_0, z_{n+1})) < 2\delta$ , for all  $j \geq 0$
- (13-c)  $\varphi(u) \geq \varphi(v)$  and  $d(u, v) \leq \min\{\gamma(u), \delta a(d(x_0, v))\}$
- (13-d)  $(\varepsilon/\delta)d(v, w)/a(d(x_0, w)) \geq \varphi(v) - \varphi(w)$ ,  
for all  $w \in M[u, \gamma(u)] \setminus M(x_0, d(x_0, v))$ .

In particular, the constant function  $a = 1$  is admissible. Note that the variational conclusion (13-d) above (retainable for certain points of a closed sphere in  $M$ ) is effectively stronger than the variational conclusion (12-b) (valid for all elements of  $M$ ). Despite this local character, (EAOP) found some nice applications in Variational Analysis; see the quoted paper for details. So, a technical analysis of its basic lines may be not without profit. It is our aim in this exposition to show (in Section 3) that a simplification of this reasoning is possible, by reducing (EAOP) to the metrical version of (EVP) in Turinici [23] (cf. Section 2). This, among others, allows a re-consideration of the admissibility concept; in fact, we simply show that the comparison type condition (a06) may be dropped. Further, Section 4 is devoted to a differential version of our main result, in Hilbert spaces. Finally, in Section 5, an application of the differential result above to existence of minimizers for Gâteaux differentiable regular functions acting over Hilbert spaces is considered. Some other aspects of this problem will be discussed elsewhere.

## 2. METRICAL ORDERING PRINCIPLES

Let  $(M, d)$  be a metric space; and  $(\preceq)$  be an *order* (reflexive, transitive, anti-symmetric relation) over it; the resulting triple  $(M, d; \preceq)$  will be termed an *ordered metric space*. A point  $z \in M$  is called  $(\preceq)$ -*maximal*, when

$$w \in M, z \preceq w \implies z = w;$$

and the order  $(\preceq)$  is termed a *Zorn* one, when:

$$\text{for each } x \in M \text{ there exists a } (\preceq)\text{-maximal } z \in M \text{ with } x \preceq z.$$

For a number of both practical and theoretical reasons, it would be useful to establish under which conditions is such a property retainable. The standard way of solving this question is based upon the *chains* (i.e.: *totally ordered* subsets) of the structure  $(M, \preceq)$ ; cf. Bourbaki [4]. However, under the precise metrical setting, a denumerable version of such principles is more appropriate for our purposes. This will necessitate a few conventions and auxiliary facts. Let  $(x_n)$  be a sequence in  $M$ ; we shall term it *ascending* (resp.: *descending*), if

$$x_i \preceq x_j \text{ (resp.: } x_i \succeq x_j) \text{ when } i \leq j.$$

Further, a point  $u \in M$  is called an *upper bound* of our sequence  $(x_n)$ , provided

$$x_n \preceq u, \text{ for all } n \text{ (written as: } (x_n) \preceq u);$$

when such elements  $u$  exist, we say that  $(x_n)$  is *bounded above* (modulo  $(\preceq)$ ).

The following 1984 answer to the above problem (referred to as: *metrical Zorn-Bourbaki principle*; in short: (ZB-m)) is provided in Turinici [24]:

**Proposition 2.1.** *Suppose that*

- (b01)  $(\preceq)$  is inductive:  
each ascending sequence is bounded above (modulo  $(\preceq)$ )
- (b02)  $(\preceq)$  is regular: each ascending sequence is a  $d$ -Cauchy one.

*Then,  $(\preceq)$  is a Zorn order.*

Note that the semimetric variant of (ZB-m) includes the 1976 ordering principle due to Brezis and Browder [5] (in short: BB); which in turn, includes Ekeland's Variational principle [10] (i.e.: EVP). It is worth remarking that, by developments like in Cârjă et al [8, Ch 2, Sect 2.1], (ZB-m) is deductible from the Principle of Dependent Choices (in short: DC) due to Bernays [3] and Tarski [21]; see also Wolk

[29]. Finally, (EVP) includes (DC), as results from the 1987 paper by Brunner [7], and its refinement in Turinici [28]. Summing up, we have

$$(DC) \implies (ZB-m) \implies (BB) \implies (EVP) \implies (DC);$$

whence, all members of this inclusion chain are mutually equivalent. This assertion comprises as well the variational statements in Altman [1], Kang and Park [16], or Turinici [25]; we do not give details.

A basic particular case of these developments is the following. Let again  $(M, d; \preceq)$  be an ordered metric space. Call the subset  $Z \subseteq M$ ,  $(\preceq)$ -closed if:

the limit of each ascending (modulo  $(\preceq)$ ) sequence in  $Z$  belongs to  $Z$ .

For example, this holds whenever  $Z$  is closed; but the reciprocal is not in general true. Further, let us say that  $(\preceq)$  is *self-closed*, when;

$M(x, \preceq) := \{y \in M; x \preceq y\}$  is  $(\preceq)$ -closed, for each  $x \in M$ ;  
or, equivalently:

the limit of each ascending sequence is an upper bound of it (modulo  $(\preceq)$ ).

In particular, this holds whenever (cf. Nachbin [18, Appendix]):

$(\preceq)$  is *closed from the right*:  $M(x, \preceq)$  is closed, for each  $x \in M$ ;

but, the converse is not in general valid. Finally, call  $d$ ,  $(\preceq)$ -complete if

each ascending (modulo  $(\preceq)$ )  $d$ -Cauchy sequence in  $M$  is  $d$ -convergent.

For example, this always happens when  $d$  is complete (in the usual sense). The reciprocal is not in general true; just take  $M = ]0, 1]$ , endowed with the standard metric and order.

The following maximal type statement (referred to as: *strong metrical Zorn-Bourbaki principle*; in short: (ZB-m-s)) established in 1981 by Turinici [23], will be useful for us.

**Proposition 2.2.** *Suppose that*

- (b03)  $(\preceq)$  is regular and self-closed
- (b04)  $d$  is  $(\preceq)$ -complete (over  $M$ ).

*Then, for each  $u \in M$  there exists  $v \in M$  with*

- (22-1)  $u \preceq v$  (i.e.:  $v \in M(u, \preceq)$ )
- (22-2)  $w \in M$  and  $v \preceq w$  imply  $v = w$ .

The proof consists in verifying that, under (b03) (the second half) and (b04), the ambient order  $(\preceq)$  is inductive; we do not give details.

### 3. EXTENSION OF (EAOP)

With this information at hand, we may now return to the posed questions of introductory part.

Let  $(X, d)$  be a complete metric space; and  $\varphi : X \rightarrow \mathbb{R}$  be a function with the  $(X, d)$ -regular properties; i.e.:

- (c01)  $\varphi$  is bounded below ( $\inf \varphi(X) > -\infty$ )
- (c02)  $\varphi$  is  $d$ -lsc on  $X$  ( $\liminf_n \varphi(x_n) \geq \varphi(x)$ , whenever  $x_n \xrightarrow{d} x$ ).

Further, let the function  $a : \mathbb{R}_+ \rightarrow \mathbb{R}_+^0$  be taken as

- (c03)  $a(\cdot)$  is increasing ( $t_1 \leq t_2$  implies  $a(t_1) \leq a(t_2)$ ).

Finally, take some point  $x_0 \in X$  (fixed in the sequel).

(A) Let  $(\leq)$  stand for the relation (in  $X$ )

$$(z, w \in X): z \leq w \text{ iff } d(x_0, z) \leq d(x_0, w).$$

Clearly, it is reflexive and transitive; i.e., it is a *quasi-order* on  $X$ . Further, let us fix  $\varepsilon, \delta > 0$  and consider the relation

$$(z, w \in X): z \perp w \text{ iff } (\varepsilon/\delta)d(z, w)/a(d(x_0, w)) \leq \varphi(z) - \varphi(w).$$

For the moment,  $(\perp)$  is reflexive and antisymmetric. Further properties of this (and the preceding) object are contained in

**Lemma 3.1.** *The couple  $(\leq, \perp)$  is transitively compatible, in the sense*

$$u \leq v \leq w, u \perp v, v \perp w \implies u \perp w.$$

*Proof.* Let  $u, v, w \in X$  be as in the premise of this implication; that is

$$\begin{aligned} \text{(i)} \quad & d(x_0, u) \leq d(x_0, v) \leq d(x_0, w) \\ \text{(ii)} \quad & (\varepsilon/\delta)d(u, v)/a(d(x_0, v)) \leq \varphi(u) - \varphi(v), \\ & (\varepsilon/\delta)d(v, w)/a(d(x_0, w)) \leq \varphi(v) - \varphi(w). \end{aligned}$$

By the increasing property of  $a(\cdot)$ , we have

$$a(d(x_0, v)) \leq a(d(x_0, w)) \text{ [whence, } 1/a(d(x_0, w)) \leq 1/a(d(x_0, v))\text{]};$$

so that, the second half of working condition gives

$$\begin{aligned} (\varepsilon/\delta)d(u, v)/a(d(x_0, w)) &\leq \varphi(u) - \varphi(v), \\ (\varepsilon/\delta)d(v, w)/a(d(x_0, w)) &\leq \varphi(v) - \varphi(w); \end{aligned}$$

wherefrom (by the triangular property of  $d$ )

$$\begin{aligned} (\varepsilon/\delta)d(u, w)/a(d(x_0, w)) &\leq \\ (\varepsilon/\delta)(d(u, v) + d(v, w))/a(d(x_0, w)) &\leq \varphi(u) - \varphi(w); \end{aligned}$$

i.e.,  $u \perp w$ ; hence the conclusion.  $\square$

As a consequence of this, the "product" relation  $(\preceq)$  over  $X$

$$(z, w \in X): z \preceq w \text{ iff } z \leq w, z \perp w$$

is reflexive, transitive and antisymmetric; hence, an ordering on  $X$ . Our next objective is to establish that the strong metrical Zorn-Bourbaki principle (ZB-m-s) is applicable on each structure  $(M, \preceq; d)$ ; where  $M$  is a (nonempty) closed bounded part of  $X$ . This is clearly the case with the  $(\preceq)$ -completeness property; because (as  $M$  is closed) we necessarily have

$$d \text{ is complete on } M; \text{ hence, } d \text{ is } (\preceq)\text{-complete on } M.$$

So, it remains to verify that

$$(\preceq) \text{ is regular and self-closed (over this class of subsets).}$$

**(B)** We start with the verification of regularity condition. A positive answer to this is given in

**Lemma 3.2.** *The introduced order  $(\preceq)$  is boundedly regular; i.e.,*

$$\text{each ascending (modulo } (\preceq)\text{) bounded sequence in } X \text{ is } d\text{-Cauchy.}$$

*And, as such,  $(\preceq)$  is regular over each (nonempty) closed bounded part  $M$  of  $X$ .*

*Proof.* Let  $(y_n)$  be some bounded sequence in  $X$  with

$$y_n \preceq y_{n+1}, \forall n; \text{ or, equivalently: } y_n \preceq y_m, \text{ whenever } n \leq m.$$

By the very definition of our order  $(\preceq)$ , this amounts to the couple of conditions below being fulfilled

(asc-1)  $d(x_0, y_n) \leq d(x_0, y_m)$ , whenever  $n \leq m$

(asc-2)  $(\varepsilon/\delta)d(y_n, y_{n+1})/a(d(x_0, y_{n+1})) \leq \varphi(y_n) - \varphi(y_{n+1}), \forall n$ .

The boundedness condition upon  $(y_n)$  means

$$\mu := \sup\{d(y_i, y_j); i \leq j\} < \infty;$$

hence

$$d(x_0, y_n) \leq d(x_0, y_0) + d(y_0, y_n) \leq \nu := d(x_0, y_0) + \mu, \forall n.$$

Combining with (asc-2) yields (by the increasing property of  $a(\cdot)$ )

$$(\varepsilon/\delta)d(y_n, y_{n+1})/a(\nu) \leq \varphi(y_n) - \varphi(y_{n+1}), \forall n.$$

The (real) sequence  $(\varphi(y_n))$  is descending and bounded from below; hence

the series  $\sum_n (\varphi(y_n) - \varphi(y_{n+1}))$  converges (in  $R_+$ ).

This, added to the previous relation, assures us that

the series  $\sum_n d(y_n, y_{n+1})$  converges (in  $R_+$ );

whence,  $(y_n)$  is  $d$ -Cauchy; as claimed.  $\square$

(C) Finally, we are passing to the verification of self-closeness condition. An appropriate answer is contained in

**Lemma 3.3.** *Under the imposed conditions,*

( $\preceq$ ) *is self-closed on  $X$ ;*

hence, a fortiori,

( $\preceq$ ) *is self-closed over each (nonempty) closed bounded part  $M$  of  $X$ .*

*Proof.* Let  $(y_n)$  be some ascending (modulo ( $\preceq$ )) sequence in  $X$ ; i.e.,

conditions (asc-1) and (asc-2) (see above) are fulfilled.

In addition, assume that (for some  $y \in X$ )

$$y_n \xrightarrow{d} y \text{ (i.e.: } d(y_n, y) \rightarrow 0) \text{ as } n \rightarrow \infty.$$

By (asc-1) and continuity of  $u \mapsto d(x_0, u)$ , one gets (passing to limit as  $m \rightarrow \infty$ )

$$d(x_0, y_n) \leq d(x_0, y) \text{ (i.e.: } y_n \preceq y), \text{ for all } n.$$

So, combining with (asc-2) (and  $a(\cdot)$  being increasing)

$$(\varepsilon/\delta)d(y_n, y_{n+1})/a(d(x_0, y)) \leq \varphi(y_n) - \varphi(y_{n+1}), \text{ for all } n.$$

This, by the triangular property of  $d$ , gives

$$(\varepsilon/\delta)d(y_n, y_m)/a(d(x_0, y)) \leq \varphi(y_n) - \varphi(y_m), \text{ for } n \leq m.$$

Passing to  $\liminf$  as  $m \rightarrow \infty$ , one derives (by the  $d$ -lsc property of  $\varphi$ )

$$(\varepsilon/\delta)d(y_n, y)/a(d(x_0, y)) \leq \varphi(y_n) - \varphi(y) \text{ (i.e.: } y_n \preceq y), \text{ for all } n.$$

Hence,  $(y_n) \preceq y$ ; and this ends the argument.  $\square$

We are now in position to formulate an appropriate variational result involving these data. Let the general conditions above be accepted. Precisely, let  $(X, d)$  be a complete metric space; and  $\varphi : X \rightarrow R$  be a function with

(regu)  $\varphi$  is  $(X, d)$ -regular (see above).

Remember that, for each  $\theta \geq 0$  we denoted

$$\text{ulev}(\varphi; X; \theta) = \{x \in X; \varphi(x) \leq \inf \varphi(X) + \theta\};$$

this will be referred to as the  $\theta$ -upper level set of  $\varphi$  with respect to  $X$ ; in particular,  $\text{ulev}(\varphi; X; 0)$  is nothing else than the (global) minimizers set of  $\varphi$  over  $X$ . Further, let the function  $a : R_+ \rightarrow R_+^0$  be chosen according to

(incr)  $a(\cdot)$  is increasing ( $t_1 \leq t_2$  implies  $a(t_1) \leq a(t_2)$ ).

Finally, pick some point  $x_0 \in X$ ; and let  $u \mapsto \gamma(u)$  be a function from  $X$  to  $R_+^0$ . The following extended El Amrouss Ordering Principle (in short: (EAOP-ext)) is now available:

**Theorem 3.1.** *Let  $\varepsilon, \delta > 0$  be given, as well as some point  $u \in \text{ulev}(\varphi; X; \varepsilon)$ . There exists then another point  $v = v(\varepsilon, \delta; u) \in X[u, \gamma(u)]$ , with*

(31-a)  $d(x_0, u) \leq d(x_0, v)$ ,  $(\varepsilon/\delta)d(u, v)/a(d(x_0, v)) \leq \varphi(u) - \varphi(v)$ ;  
hence  $d(u, v) \leq \min\{\gamma(u), \delta a(d(x_0, v))\}$

(31-b)  $w \in X[u, \gamma(u)] \setminus \{v\}$ ,  $d(x_0, v) \leq d(x_0, w)$   
imply  $(\varepsilon/\delta)d(v, w)/a(d(x_0, w)) > \varphi(v) - \varphi(w)$ ;  
hence,  $(\varepsilon/\delta)d(v, w)/a(d(x_0, v)) > \varphi(v) - \varphi(w)$ .

*Proof.* Denote for simplicity  $M = X[u, \gamma(u)]$ , where  $u$  is as before. Clearly,  $M$  is closed bounded; as well as nonempty (since  $u \in M$ ). Let also  $(\preceq)$  stands for the (product) order

$(z, w \in X)$ :  $z \preceq w$  iff  $z \leq w$  (i.e.:  $d(x_0, z) \leq d(x_0, w)$ ), and  
 $z \perp w$  (i.e.:  $(\varepsilon/\delta)d(z, w)/a(d(x_0, w)) \leq \varphi(z) - \varphi(w)$ ).

By the preliminary facts above, the strong metrical Zorn-Bourbaki principle (ZB-m-s) applies to  $(M, d; \preceq)$ ; so that (the restriction to  $M$  of)  $(\preceq)$  is a Zorn order (over  $M$ ). This means that, for the starting point  $u \in M$ , there exists another point  $v = v(u) \in M$ , with

(31-c)  $u \preceq v$  (i.e.:  $u \leq v$  and  $u \perp v$ )

(31-d)  $v \preceq w \in M \implies v = w$  (i.e.:  $w \in M, v \leq w, v \perp w \implies v = w$ ).

The former of these means, by definition

(31-e)  $d(x_0, u) \leq d(x_0, v)$ ,  $(\varepsilon/\delta)d(u, v)/a(d(x_0, v)) \leq \varphi(u) - \varphi(v)$ .

Note that, as a consequence of its second half,

$$\varphi_* \leq \varphi(v) \leq \varphi(u), \text{ where } \varphi_* := \inf \varphi(X);$$

so that (by the choice of  $u$ )

$$\varphi(u) - \varphi(v) \leq \varphi(u) - \varphi_* \leq \varepsilon.$$

This, again combined with (31-e) (the second half), gives

$$d(u, v) \leq \delta a(d(x_0, v)); \text{ hence } d(u, v) \leq \min\{\gamma(u), \delta a(d(x_0, v))\};$$

which, along with (31-e), is just our first conclusion in the statement. On the other hand, (31-d) may be written as (by definition)

(31-f) whenever  $w \in M$  fulfills  $d(x_0, v) \leq d(x_0, w)$  and  
 $(\varepsilon/\delta)d(v, w)/a(d(x_0, w)) \leq \varphi(v) - \varphi(w)$ , then  $v = w$ ;

or, equivalently,

for each  $w \in M \setminus \{v\}$  with  $d(x_0, v) \leq d(x_0, w)$ ,  
we must have  $(\varepsilon/\delta)d(v, w)/a(d(x_0, w)) > \varphi(v) - \varphi(w)$ ;

which is just the first half of second conclusion in the statement. The second half of the same is immediate, in view of



$$\begin{aligned} d(x_0, v) \leq d(x_0, w) &\implies a(d(x_0, v)) \leq a(d(x_0, w)) \\ &\implies 1/a(d(x_0, v)) \geq 1/a(d(x_0, w)). \end{aligned}$$

The proof is thereby complete.  $\square$

Technically speaking, this result may be viewed as a refinement of El Amrouss Ordering Principle (EAOP). Remember that, in the quoted statement, the extra regularity conditions below were considered (in our notations)

- (c04)  $a(\cdot)$  is continuous on  $R_+$
- (c05)  $a(\cdot)$  is a comparison function, of order  $k > 0$
- ( $\forall q \geq k, \exists \lambda, \mu \geq 0 : a((t+1)s) \leq a(t)[\lambda s^q + \mu], \forall t, s \geq 0$ )
- (c06)  $u \mapsto \gamma(u)/(1+d(x_0, u))$  is bounded on  $X$ .

The proposed argument tells us that, in the refined principle (EAOP-ext), all these regularity conditions may be dropped. This may have some theoretical impact upon the quoted result; but, in general, not a practical one.

Finally, the obtained "relaxed" principle (EAOP-ext) allows us (in a limited sense) some comparison type operations with Zhong's Variational Principle (ZVP). For, as precise, a direct consequence of conclusion (31-b) is (by the increasing property of  $a(\cdot)$ )

$$\begin{aligned} (31-g) \quad w \in X[u, \gamma(u)] \setminus \{v\}, \quad d(x_0, v) \leq d(x_0, w) \\ \text{imply } (\varepsilon/\delta)d(v, w)/a(d(x_0, v)) > \varphi(v) - \varphi(w). \end{aligned}$$

This, added to the fact that

$$(12-b) \implies (31-g) \text{ under } b = a,$$

tells us that the variant of (EAOP-ext) – with (31-g) in place of (31-b) – is obtainable from (ZVP), whenever

$$(c06) \text{ the function } a(\cdot) \text{ is normal (see above).}$$

However, a general inclusion of this type is not accessible for the moment. Further aspects will be discussed elsewhere.

#### 4. DIFFERENTIAL VERSIONS

The usefulness of our main result above is to be judged in a differential context only. For, as we shall see further, this setting has an essential role in solving the minimizers problem for such (nonlinear) functions.

Having these precise, it is our aim in the following to establish a differential version of extended El Amrouss Ordering Principle (EAOP-ext), within the class of Gâteaux differentiable regular maps acting on a Hilbert space. Note that, such a context is not the most general one; but, for the applications to be considered, this will suffice.

Let  $H$  be a (real) Hilbert space with respect to the scalar product  $(x, y) \mapsto \langle x, y \rangle$ . As usually, we denote by  $\|\cdot\|$  the norm induced by this scalar product; and by  $d$ , its associated metric:

$$(\|x\| = \langle x, x \rangle^{1/2}; x \in H), \quad (d(x, y) = \|x - y\|; x, y \in H);$$

remember that, by the very definition of our structure,

$$(H, \|\cdot\|) \text{ is a Banach space; hence, } (H, d) \text{ is a complete metric space.}$$

Put also, for simplicity,

$$\begin{aligned} H_1 &= \{h \in H; \|h\| \leq 1\} \text{ (the unitary closed sphere in } H) \\ \partial H_1 &:= \{h \in H_1; \|h\| = 1\} \text{ (the boundary of } H_1). \end{aligned}$$

Let the function  $\varphi : H \rightarrow R$  be  $(H, d)$ -regular; i.e. (see above)

(d01)  $\varphi$  is bounded below ( $\inf \varphi(H) > -\infty$ )

(d02)  $\varphi$  is  $d$ -lsc on  $H$  ( $\liminf_n \varphi(x_n) \geq \varphi(x)$ , whenever  $x_n \xrightarrow{d} x$ ).

Define for each  $\theta \geq 0$

$$\text{ulev}(\varphi, H; \theta) = \{x \in H; \varphi(x) \leq \inf \varphi(H) + \theta\};$$

this will be referred to as the  $\theta$ -upper level set of  $\varphi$  with respect to  $H$ ; in particular,  $\text{ulev}(\varphi; H; 0)$  is nothing else than the (global) minimizers set of  $\varphi$  over  $H$ . The differential setting we just evoked is to be introduced as follows. Let  $z \in H$  be arbitrary fixed. We say that  $\varphi$  is *Gâteaux differentiable* at  $z$  when there exists an element  $\varphi'(z) \in H$  (the *Gâteaux differential* of  $\varphi$  at  $z$ ), with

$$\lim_{t \rightarrow 0^+} (1/t)(\varphi(z + th) - \varphi(z)) = \langle \varphi'(z), h \rangle, \text{ for all } h \in H.$$

Suppose that (in addition to (d01)+(d02))

(d03)  $\varphi$  is Gâteaux differentiable over  $H$ :

$\varphi'(z)$  exists (cf. this definition),  $\forall z \in H$ .

Denote, for each  $z \in H$ ,

$$C(z) = \{h \in H; \langle z, h \rangle \leq 0\}; C_1(z) = C(z) \cap \partial H_1.$$

Clearly,  $C(z)$  is a *convex cone* in  $H$ ; i.e.:

$$C(z) + C(z) \subseteq C(z), R_+ C(z) \subseteq C(z).$$

On the other hand,  $C_1(z)$  is always non-degenerate; precisely,

$$C_1(0) = \partial H_1; C_1(z) \ni -z/\|z\| \neq 0, z \in H \setminus \{0\}.$$

Denote also (for each  $z$  in  $H$ )

$$\nu(\varphi'(z)) = \sup\{\langle \varphi'(z), h \rangle; h \in C_1(z)\}, |\nu|(\varphi'(z)) = \max\{\nu(\varphi'(z)), 0\}.$$

Note that, in view of

$$|\langle \varphi'(z), h \rangle| \leq \|\varphi'(z)\| \cdot \|h\| \leq \|\varphi'(z)\|, \forall h \in C_1(z), \forall z \in H,$$

we must have (for all such  $z$ )

$$\nu(\varphi'(z)) \in R; \text{ hence, } |\nu|(\varphi'(z)) \in R_+.$$

Finally, take some function  $a : R_+ \rightarrow R_+^0$ , according to

(d04)  $a(\cdot)$  is increasing ( $t_1 \leq t_2$  implies  $a(t_1) \leq a(t_2)$ );

and introduce the mappings (from  $H$  to  $R_+^0$ )

$$\beta(u) = \|u\| + 1, \gamma(u) = 2\beta(u) = 2\|u\| + 2, u \in H.$$

As a direct application of extended El Amrouss Ordering Principle (EAOP-ext), we have the following statement, referred to as: differential El Amrouss Ordering Principle (in short: (EAOP-dif)):

**Theorem 4.1.** *Let  $\varepsilon > 0$  be given, as well as some  $u \in \text{ulev}(\varphi; H; \varepsilon)$ . Further, let  $\delta > 0$  be taken according to*

$$(d05) \quad \delta < \beta(u)/a(3\beta(u)) \text{ (hence, } \delta a(3\beta(u)) < \beta(u)\text{)}.$$

*There exists then  $v = v(\varepsilon, u, \delta) \in H$ , with the properties*

$$(41-a) \quad \|u\| \leq \|v\|, \varphi(u) \geq \varphi(v), \text{ and } \|u - v\| \leq \min\{\gamma(u), \delta a(\|v\|)\}$$

$$(41-b) \quad \|u - v\| < \beta(u) < \gamma(u); \text{ hence, } v \in H(u, \beta(u)) \subseteq H(u, \gamma(u))$$

$$(41-c) \quad (\varepsilon/\delta)t/a(\|v\|) > \varphi(v) - \varphi(v - th), \text{ for } 0 < t < \beta(u),$$

*whenever  $h \in C_1(v)$  (i.e.:  $h \in \partial H_1$  and  $\langle v, h \rangle \leq 0$ )*

$$(41-d) \quad \varepsilon/\delta a(\|v\|) \geq |\nu|(\varphi'(v)) \geq \nu(\varphi'(v)).$$

*Proof.* Choose in the following

$$x_0 = 0 \text{ (the null element of } H\text{).}$$

By the imposed conditions, (EAOP-ext) is applicable to the data  $(H, d; \varphi; a; \gamma)$ . So, for  $(\varepsilon, u)$  and  $\delta$  as before, there exists  $v = v(\varepsilon, u, \delta) \in H[u, \gamma(u)]$ , with the properties given by its conclusion; i.e.,

$$\begin{aligned} (41\text{-e}) \quad & \|u\| \leq \|v\|, (\varepsilon/\delta)\|u - v\|/a(\|v\|) \leq \varphi(u) - \varphi(v); \\ & \text{hence } \|u - v\| \leq \min\{\gamma(u), \delta a(\|v\|)\} \\ (41\text{-f}) \quad & w \in H[u, \gamma(u)] \setminus \{v\}, \|v\| \leq \|w\| \\ & \text{imply } (\varepsilon/\delta)\|v - w\|/a(\|w\|) > \varphi(v) - \varphi(w); \\ & \text{hence, } (\varepsilon/\delta)\|v - w\|/a(\|v\|) > \varphi(v) - \varphi(w). \end{aligned}$$

By the former of these [i.e.: (41-e)], one gets (41-a). As a direct consequence,

$$\|v\| \leq \|v - u\| + \|u\| \leq \gamma(u) + \|u\| \leq 3\|u\| + 2 < 3\beta(u);$$

and this, in combination with (d05), gives (via (41-a) above)

$$\|u - v\| \leq \delta a(3\beta(u)) < \beta(u) < \gamma(u);$$

i.e., (41-b) is verified. This, in particular, tells us that  $v$  is an interior point of  $H[u, \gamma(u)]$ ; moreover (from the above), we must have

$$v + th \in H[u, \gamma(u)], \quad \forall t \in [-\beta(u), \beta(u)], \forall h \in \partial H_1.$$

In fact, let the couple  $(t, h)$  be as before; then

$$\|v + th - u\| \leq \|v - u\| + |t| < \beta(u) + \beta(u) = \gamma(u);$$

and the claim follows. On the other hand,

$$\begin{aligned} & \|v - th\| > \|v\|, \text{ for } 0 < t < \beta(u), \\ & \text{whenever } h \in C_1(v) \text{ (i.e.: } h \in \partial H_1 \text{ and } \langle v, h \rangle \leq 0); \end{aligned}$$

because (from the properties of scalar product)

$$\|v - th\|^2 = \|v\|^2 + t^2 - 2t\langle v, h \rangle > \|v\|^2, \text{ for all such } (t, h).$$

Putting these together gives (44-c), if one takes (41-f) into account. Finally, let  $h \in C_1(v)$  be arbitrary fixed; hence,  $h \in \partial H_1$ ,  $\langle v, h \rangle \leq 0$ . From (41-c), one gets an evaluation like

$$\varepsilon/\delta a(\|v\|) > (1/t)(\varphi(v) - \varphi(v - th)), \text{ when } 0 < t < \beta(u).$$

So, passing to limit as  $t \rightarrow 0+$  and taking the Gâteaux differentiable property of  $\varphi$  into account, one derives

$$\varepsilon/\delta a(\|v\|) \geq \langle \varphi'(v), h \rangle.$$

This, by the arbitrariness of  $h$  in  $C_1(v)$ , yields

$$\varepsilon/\delta a(\|v\|) \geq \nu(\varphi'(v)); \text{ hence, } \varepsilon/\delta a(\|v\|) \geq |\nu|(\varphi'(v));$$

and establishes the final conclusion (41-d) in the statement.  $\square$

The differential principle (EAOP-dif) is comparable with a related one in El Amrouss and Tsouli [12]. However, some basic differences between these results must be noted.

**I)** As precise, the extra regularity conditions imposed by the quoted authors

$$\begin{aligned} (d06) \quad & a(\cdot) \text{ is continuous (in the usual sense)} \\ (d07) \quad & a(\cdot) \text{ is a comparison function, of order } k > 0 \\ & (\forall q \geq k, \exists \lambda, \mu \geq 0 : a((t+1)s) \leq a(t)[\lambda s^q + \mu], \forall t, s \geq 0) \end{aligned}$$

are not needed here.

(II) The final differential relation above is written by the quoted authors as

$$(41\text{-d-var}) \quad \varepsilon/\delta a(\|v\|) \geq \|\varphi'(v)\|.$$

Formally, this is better than (41-d) above, in view of

$$\|\varphi'(z)\| \geq |\nu|(\varphi'(z)), \quad \forall z \in H.$$

Unfortunately, (41-d-var) is not true under authors' directional context; and this is retainable as well for our statement we just exposed.

Note finally that some extensions of these results are possible, in the class of quasi-ordered normed spaces; see Turinici [27] for details.

## 5. EXISTENCE OF MINIMIZERS

In the following, an application of the result above to existence of minimizers for Gâteaux differentiable regular functionals is considered. The basic instrument for our investigations is the well known Palais-Smale condition [19].

Let  $H$  be a (real) Hilbert space with respect to the scalar product  $(x, y) \mapsto \langle x, y \rangle$ . Further, let  $\varphi : H \rightarrow R$  be a  $(H, d)$ -regular function; i.e.,

$$(e01) \quad \varphi \text{ is bounded below } (\inf \varphi(X) > -\infty)$$

$$(e02) \quad \varphi \text{ is } d\text{-lsc on } X \text{ (} \liminf_n \varphi(x_n) \geq \varphi(x), \text{ whenever } x_n \xrightarrow{d} x \text{)}.$$

Remember that, for each  $\theta \geq 0$  we denoted

$$\text{ulev}(\varphi; H; \theta) = \{x \in H; \varphi(x) \leq \inf \varphi(H) + \theta\};$$

this will be referred to as the  $\theta$ -upper level set of  $\varphi$  with respect to  $H$ ; in particular,  $\text{ulev}(\varphi; H; 0)$  is nothing else than the (global) minimizers set of  $\varphi$  over  $H$ . In addition, suppose that

$$(e03) \quad \varphi \text{ is Gâteaux differentiable over } H.$$

Further, take some function  $a : R_+ \rightarrow R_+^0$  according to

$$(e04) \quad a(\cdot) \text{ is increasing } (t_1 \leq t_2 \text{ implies } a(t_1) \leq a(t_2)).$$

Roughly speaking, the differential El Amrouss Ordering Principle (EAOP-dif) we just established is a local one; because, given the starting point  $u \in H$ , the associated variational point  $v = v(u)$  is to be found in a (closed) sphere  $H[u, \gamma(u)]$  around  $u$ . But, for an appropriate solving of our problem, a global version of this ordering principle, relative to the class of (closed) bounded parts of  $H$  is needed. Call the subset  $K$  of  $H$ , *admissible* (modulo  $\varphi$ ), provided

$$\text{ulev}(\varphi; H; \varepsilon) \cap K \text{ is nonempty, for each } \varepsilon > 0;$$

hence, necessarily,  $K$  is nonempty. Assume in the following that

$$(e05) \quad (H, \varphi) \text{ is admissible:}$$

there exist bounded admissible (modulo  $\varphi$ ) parts of  $H$ .

Fix in the following such an object,  $K$ ; as well as some number  $\delta > 0$  with

$$(e06) \quad \delta < 1/a(3(\rho + 1)); \text{ where } \rho = \sup\{\|u\|; u \in K\}.$$

A direct consequence of this is the following. Let us introduce the mappings

$$\beta(u) = \|u\| + 1, \quad \gamma(u) = 2\beta(u) = 2\|u\| + 2, \quad u \in H.$$

By the imposed condition, we have

$$\beta(u) \leq \rho + 1, \quad \gamma(u) \leq 2\rho + 2, \quad \forall u \in K;$$

and this yields (via  $a(\cdot)$ =increasing)

$\delta < 1/a(3\beta(u))$  (hence,  $\delta a(3\beta(u)) < 1$ ), for each  $u \in K$ .

The following global differential type variational statement (referred to as: global differential El Amrouss Ordering Principle; in short: (EAOP-dif-g)) is our main step towards the desired answer.

**Theorem 5.1.** *Let  $\varepsilon > 0$  be given, as well as some  $u \in \text{ulev}(\varphi; H; \varepsilon) \cap K$ . Further, let  $\delta > 0$  be taken as before. There exists then some  $v \in H$ , such that*

- (51-a)  $\varphi_* \leq \varphi(v) \leq \varphi(u) \leq \varphi_* + \varepsilon$ , where  $\varphi_* := \inf \varphi(H)$
- (51-b)  $\|u - v\| \leq \beta(u)$ ; hence  $\|v\| \leq \|u\| + \beta(u) \leq 2\rho + 1$
- (51-c)  $|\nu|(\varphi'(v))a(\|v\|) \leq \varepsilon/\delta$ .

*Proof.* Let  $\varepsilon > 0$ ,  $u \in \text{ulev}(\varphi; H; \varepsilon) \cap K$  and  $\delta > 0$  be taken as before. From the above relations involving the constant  $\delta > 0$  and the couple of functions  $(a(\cdot), \beta(\cdot))$ , it follows that the (local) differential El Amrouss Ordering Principle (EAOP-dif) is applicable to the data  $(H; \varepsilon, u; \delta; \beta(\cdot), \gamma(\cdot))$ ; and gives us all conclusions in the statement. The proof is thereby complete.  $\square$

In particular, when  $a : R_+ \rightarrow R_+^0$  fulfills the extra conditions

- (ec-1)  $a(\cdot)$  is continuous (in the usual sense)
- (ec-2)  $a(\cdot)$  is a comparison function, of order  $k > 0$
- ( $\forall q \geq k, \exists \lambda, \mu \geq 0 : a((t+1)s) \leq a(t)[\lambda s^q + \mu], \forall t, s \geq 0$ )

the global differential El Amrouss Ordering Principle (EAOP-dif-g) is nothing but the statement in El-Amrouss [11] proved under different methods. However, we must say that its differential conclusion

$$(51-d) \quad \|\varphi'(v)\|a(\|v\|) \leq \varepsilon/\delta$$

is not true under author's directional context; we do not give further details.

An application of (EAOP-dif-g) to existence of (global) minimizers for the function  $\varphi$  may now be given along the lines below. Let us say that  $(H, \varphi)$  satisfies the *Palais-Smale condition* (modulo  $a$ ), when

- (PS-a) each bounded sequence  $(x_n) \subseteq H$  such that  $\varphi(x_n) \rightarrow \varphi_*$  and  $|\nu|(\varphi'(x_n))a(\|x_n\|) \rightarrow 0$  has a convergent subsequence.

In particular, when  $a = 1$ , this is referred to as  $(H, \varphi)$  fulfilling the *standard Palais-Smale condition*:

- (PS) each bounded sequence  $(x_n) \subseteq H$  such that  $\varphi(x_n) \rightarrow \varphi_*$  and  $|\nu|(\varphi'(x_n)) \rightarrow 0$  has a convergent subsequence.

Concerning the relationship between these, note that (under the choice of our function  $a(\cdot)$ ), the following double inequality holds

$$|\nu|(\varphi'(x_n))a(0) \leq |\nu|(\varphi'(x_n))a(\|x_n\|) \leq |\nu|(\varphi'(\|x_n\|))a(\sigma), \quad \forall n;$$

where  $\sigma := \sup\{\|x_n\|; n \geq 0\}$ . This tells us that

$$(|\nu|(\varphi'(x_n))a(\|x_n\|) \rightarrow 0) \iff (|\nu|(\varphi'(x_n)) \rightarrow 0);$$

so that

- (PS-a) is equivalent with (PS).

We are now in position to formulate the announced answer. Let the general conditions of this section be admitted. Precisely, take a function  $\varphi : H \rightarrow R$ , endowed with the properties:

- $\varphi$  is  $(H, d)$ -regular and Gâteaux differentiable (see above).

Moreover, let the function  $a : R_+ \rightarrow R_+^0$  be increasing.

**Theorem 5.2.** *Suppose, in addition, that  $(H, \varphi)$  is admissible and fulfills the Palais-Smale condition (PS-a) (or, equivalently, the standard Palais-Smale condition (PS)). Then,  $\varphi$  admits at least one minimizer on  $H$ .*

*Proof.* As  $(H, \varphi)$  is admissible, it admits at least a bounded admissible subset  $K$ , in the sense:

- (adm-1)  $K$  is bounded; hence,  $\rho := \sup\{\|u\|; u \in K\} < \infty$
- (adm-2)  $K$  is admissible:  $\text{ulev}(\varphi; H; \varepsilon) \cap K \neq \emptyset, \forall \varepsilon > 0$ .

Further, pick the number  $\delta > 0$  in accordance with

$$\delta < 1/a(3(\rho + 1)); \text{ hence, } \delta a(3(\rho + 1)) < 1.$$

For the arbitrary fixed  $\varepsilon > 0$ , take some (starting) point  $u_\varepsilon \in \text{ulev}(\varphi; H; \varepsilon) \cap K$ . From the global differential El Amrouss Ordering Principle (EAOP-dif-g), there must be some associated point  $v_\varepsilon \in H$  fulfilling its conclusions (51-a)-(51-c); i.e. (for any such  $\varepsilon > 0$ )

- (52-a)  $\varphi_* \leq \varphi(v_\varepsilon) \leq \varphi(u_\varepsilon) \leq \varphi_* + \varepsilon$ , where  $\varphi_* := \inf \varphi(H)$
- (52-b)  $\|u_\varepsilon - v_\varepsilon\| \leq \beta(u_\varepsilon)$ ; hence  $\|v_\varepsilon\| \leq \|u_\varepsilon\| + \beta(u_\varepsilon) \leq 2\rho + 1$
- (52-c)  $|\nu|(\varphi'(v_\varepsilon))a(\|v_\varepsilon\|) \leq \varepsilon/\delta$ .

In particular, taking the sequence  $(\varepsilon_n = 2^{-n}; n \geq 0)$ , it results that, for each  $n \geq 0$  and each starting  $u_n := u_{\varepsilon_n} \in \text{ulev}(\varphi; H; \varepsilon_n) \cap K$ , there exists  $v_n := v_{\varepsilon_n}$  in  $H$  with

- (52-d)  $\varphi_* \leq \varphi(v_n) \leq \varphi(u_n) \leq \varphi_* + 2^{-n}$
- (52-e)  $\|u_n - v_n\| \leq \beta(u_n)$ ; hence  $\|v_n\| \leq \|u_n\| + \beta(u_n) \leq 2\rho + 1$
- (52-f)  $|\nu|(\varphi'(v_n))a(\|v_n\|) \leq 2^{-n}/\delta$ .

Now, (52-d)+(52-e) give us that  $(v_n)$  is a bounded sequence in  $H$  with  $\varphi(v_n) \rightarrow \varphi_*$ ; and, from (52-f), one derives that  $|\nu|(\varphi'(v_n))a(\|v_n\|) \rightarrow 0$ . This, along with the Palais-Smale condition (PS-a), yields a subsequence  $(y_n := v_{i(n)})$  of  $(v_n)$ , with:

$$y_n \xrightarrow{d} y, \text{ for some } y \in H.$$

Combining with the  $d$ -lsc condition imposed upon  $\varphi$ , we derive

$$\varphi_* \leq \varphi(y) \leq \lim_n \varphi(y_n) = \varphi_*; \text{ hence } \varphi(y) = \varphi_*;$$

or, in other words:  $y \in H$  is a minimizer for  $\varphi$ . The proof is thereby complete.  $\square$

In particular, under the extra regularity conditions

- (er-1)  $a(\cdot)$  is continuous (in the usual sense)
- (er-2)  $a(\cdot)$  is a comparison function of order  $k > 0$  (see above)

the obtained existence result yields a related statement in El-Amrouss [11]. Further aspects may be found in Motreanu et al [17, Ch 5].

## REFERENCES

- [1] M. Altman, *A generalization of the Brezis-Browder principle on ordered sets*, Nonlin. Anal., 6 (1982), 157-165.
- [2] T. Q. Bao and P. Q. Khanh, *Are several recent generalizations of Ekeland's variational principle more general than the original principle?*, Acta Math. Vietnamica, 28 (2003), 345-350.
- [3] P. Bernays, *A system of axiomatic set theory: Part III. Infinity and enumerability analysis*, J. Symbolic Logic, 7 (1942), 65-89.
- [4] N. Bourbaki, *Sur le théorème de Zorn*, Archiv Math., 2 (1949/1950), 434-437.

- [5] H. Brezis and F. E. Browder, *A general principle on ordered sets in nonlinear functional analysis*, Advances Math., 21 (1976), 355-364.
- [6] A. Brøndsted, *Fixed points and partial orders*, Proc. Amer. Math. Soc., 60 (1976), 365-366.
- [7] N. Brunner, *Topologische Maximalprinzipien*, Zeitschr. Math. Logik Grundl. Math., 33 (1987), 135-139.
- [8] O. Cârjă, M. Necula and I. I. Vrabie, *Viability, Invariance and Applications*, North Holland Math. Studies vol. 207, Elsevier, Amsterdam, 2007.
- [9] I. Ekeland, *On the variational principle*, J. Math. Anal. Appl., 47 (1974), 324-353.
- [10] I. Ekeland, *Nonconvex minimization problems*, Bull. Amer. Math. Soc. (New Series), 1 (1979), 443-474.
- [11] A. R. El Amrouss, *Variantes du principe variationnel d'Ekeland et applications*, Revista Colomb. Mat., 40 (2006), 1-14.
- [12] A. R. El Amrouss and N. Tsouli, *A generalization of Ekeland's variational principle with applications*, Electronic J. Diff. Eqs., Conference 14, (2006), 173-180.
- [13] A. Goepfert, H. Riahi, C. Tammer and C. Zălinescu, *Variational Methods in Partially Ordered Spaces*, Canad. Math. Soc. Books Math. vol. 17, Springer, New York, 2003.
- [14] D. H. Hyers, G. Isac and T. M. Rassias, *Topics in Nonlinear Analysis and Applications*, World Sci. Publ., Singapore, 1997.
- [15] O. Kada, T. Suzuki and W. Takahashi, *Nonconvex minimization theorems and fixed point theorems in complete metric spaces*, Math. Japonica, 44 (1996), 381-391.
- [16] B. G. Kang and S. Park, *On generalized ordering principles in nonlinear analysis*, Nonlin. Anal., 14 (1990), 159-165.
- [17] D. Motreanu, V. V. Motreanu, and N. Papageorgiou, *Topological and Variational Methods with Applications to Nonlinear Boundary Value Problems*, Springer, New York, 2014.
- [18] L. Nachbin, *Topology and Order*, van Nostrand, Princeton (N.J.), 1965.
- [19] R. S. Palais and S. Smale, *A generalized Morse theory*, Bull. Amer. Math. Soc., 70 (1964), 165-171.
- [20] S. Park and J. S. Bae, *On the Ray-Walker extension of the Caristi-Kirk fixed point theorem*, Nonlin. Anal., 9 (1985), 1135-1136.
- [21] A. Tarski, *Axiomatic and algebraic aspects of two theorems on sums of cardinals*, Fund. Math., 35 (1948), 79-104.
- [22] D. Tataru, *Viscosity solutions of Hamilton-Jacobi equations with unbounded nonlinear terms*, J. Math. Anal. Appl., 163 (1992), 345-392.
- [23] M. Turinici, *Maximality principles and mean value theorems*, An. Acad. Brasil. Cienc., 53 (1981), 653-655.
- [24] M. Turinici, *A generalization of Altman's ordering principle*, Proc. Amer. Math. Soc., 90 (1984), 128-132.
- [25] M. Turinici, *Metric variants of the Brezis-Browder ordering principle*, Demonstr. Math., 22 (1989), 213-228.
- [26] M. Turinici, *Function variational principles and coercivity*, J. Math. Anal. Appl., 304 (2005), 236-248.
- [27] M. Turinici, *Normed coercivity for monotone functionals*, Romai, 7 (No. 2) (2011), 169-179.
- [28] M. Turinici, *Sequential maximality principles*, in "Mathematics Without Boundaries" (T. M. Rassias and P. M. Pardalos, Eds.), pp. 515-548, Springer, New York, 2014.
- [29] E. S. Wolk, *On the principle of dependent choices and some forms of Zorn's lemma*, Canad. Math. Bull., 26 (1983), 365-367.
- [30] C. K. Zhong, *A generalization of Ekeland's variational principle and application to the study of the relation between the weak P.S. condition and coercivity*, Nonlin. Anal., 29 (1997), 1421-1431.
- [31] M. Zorn, *A remark on method in transfinite algebra*, Bull. Amer. Math. Soc., 41 (1935), 667-670.

"A. MYLLER" MATHEMATICAL SEMINAR; "A. I. CUZA" UNIVERSITY; 700506 IAȘI, ROMANIA  
 E-mail address: mturi@uaic.ro