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SECOND-ORDER ASYMPTOTIC EXPANSION FOR THE DISTRIBUTION OF PARTICLES IN A BRANCHING RANDOM WALK WITH A RANDOM ENVIRONMENT IN TIME

ZHIQIANG GAO AND QUANSHENG LIU

ABSTRACT. We consider a branching random walk in which offspring distribution and moving laws both depend on an independent and identically distributed environment indexed by the time. For $A \subset \mathbb{R}$, let $Z_n(A)$ be the number of particles of generation n located in A . We give the second-order asymptotic expansion for the counting measure $Z_n(\cdot)$ with appropriate normalization.

1. INTRODUCTION

The branching random walks (BRW) has been widely studied by many people. It consists of two main ingredients: branching processes and random walks. The model is of great importance and is closely related to many fields such as multiplicative cascades, infinite particle systems, Quicksort algorithms, random fractals, and Gaussian free fields (see e.g.[28, 29, 40]).

In this article, we aim to develop the asymptotic expansions in the central limit theorem for a branching random walk with a time-dependent random environment. The goal is twofold. On the one hand, although central limit theorems for branching random walks have been well studied and the asymptotic expansions for lattice BRW were considered recently by [18], the asymptotic expansions (even second order) in central limit theorems for nonlattice BRW are unknown. On the other hand, we will perform our research in a more general framework, i.e. for a branching random walk with a random environment in time, which is a natural generalization of classical BRW formulated in Harris[20].

The central limit theorem for branching random walks has been studied since 1963 by Harris([20, Chapter III. §16]), who initiated the question and conjectured the theorem; then this conjecture was proved in various forms and for various models in [2, 7, 17, 24–26, 31, 37, 38]. Later for branching Wiener processes (also for branching random motion with simple symmetric walk), Révész (1994, [33]) investigated the speed of above convergence and conjectured the exact convergence rate, which was confirmed by Chen(2001, [10])(see also [39] for another proof) and was extended to a general strong non-lattice case including non-Gaussian displacements by Gao and Liu(2014,[16]). Révész, Rosen and

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Shi (2005, [34]) gave large time asymptotic expansion in local limit theorem of branching Winer processes. The exact convergence rate obtained in [10, 16] can be viewed as the first order asymptotic expansion in the central limit theorem for the models therein. Inspired by those work, a natural question is what about the second asymptotic expansion(or bigger order).

In this article, we obtain the second asymptotic expansion for *a branching random walk with a random environment in time*. This model first appeared in Biggins (2004, [8]) as a particular case of a general framework, and some related limit theorems were surveyed in Liu (2007, [30]). The reader may refer to [6, 9, 11–13, 19, 21, 31, 38] for other models of *branching random walks in random environments*. For the model presented here, Gao, Liu and Wang (2014,[17]) showed central limit theorems on this model and further Gao and Liu (2014,[16]) figured out the first order asymptotic expansion. The study is a continuation of that in [16, 17].

The article is organized as follows. In Section 2, we recall the model *branching random walk with a random environment in time* and introduce the basic assumptions and notation, then we give the second-order asymptotic expansion for the distribution of particles for the model in Theorem 2.1. We give the proof of Theorem 2.1 in Section 3

2. SECOND-ORDER ASYMPTOTIC FOR BRWRE

2.1. Description of the model. The model so called *a branching random walk with a random environment in time* can be formulated as follows [16, 17]. Let (Θ, \mathbb{p}) be a probability space, and $(\Theta^{\mathbb{N}}, \mathbb{p}^{\otimes \mathbb{N}}) = (\Omega, \tau)$ be the corresponding product space. Let θ be the usual shift transformation on Ω . For a sequence $\xi \in \Omega$, we denote $\xi = (\xi_1, \xi_2, \dots)$, where ξ_k are the k -th coordinate function on Ω . Then $\xi = (\xi_n)$ will serve as an independent and identically distributed environment. Let θ be the usual shift transformation on $\Theta^{\mathbb{N}}$: $\theta(\xi_0, \xi_1, \dots) = (\xi_1, \xi_2, \dots)$. Each realization of ξ_n corresponds to two probability distributions: the offspring distribution $p(\xi_n) = (p_0(\xi_n), p_1(\xi_n), \dots)$ on $\mathbb{N} = \{0, 1, \dots\}$, and the moving distribution $G(\xi_n)$ on \mathbb{R} .

Given the environment $\xi = (\xi_n)$, the process is a branching random walk in varying environment, which evolves according to the following rules:

- At time 0, an initial particle \emptyset of generation 0 is located at the origin $S_{\emptyset} = 0$;
- At time 1, \emptyset is replace by $N = N_{\emptyset}$ new particles of generation 1, and for $1 \leq i \leq N$, each particle $\emptyset i$ moves to $S_{\emptyset i} = S_{\emptyset} + L_i$, where N, L_1, L_2, \dots are mutually independent, N has the law $p(\xi_0)$, and each L_i has the law $G(\xi_0)$.
- At time $n+1$, each particle $u = u_1 u_2 \dots u_n$ of generation n is replaced by N_u new particles of generation $n+1$, with displacements $L_{u1}, L_{u2}, \dots, L_{uN_u}$. That means for $1 \leq i \leq N_u$, each particle $u i$ moves to $S_{u i} = S_u + L_{u i}$, where $N_u, L_{u1}, L_{u2}, \dots$ are mutually independent, N_u has the law $p(\xi_n)$, and each $L_{u i}$ has the same law $G(\xi_n)$.

By definition, given the environment ξ , the random variables N_u and L_u , indexed by all the finite sequences u of positive integers, are independent of each other. For each realization $\xi \in \Theta^{\mathbb{N}}$ of the environment sequence, let $(\Gamma, \mathcal{G}, \mathbb{P}_{\xi})$ be the probability space under which the process is defined (when the environment ξ is fixed to the given realization).

The probability \mathbb{P}_ξ is usually called *quenched law*. The total probability space can be formulated as the product space $(\Theta^{\mathbb{N}} \times \Gamma, \mathcal{E}^{\mathbb{N}} \otimes \mathcal{G}, \mathbb{P})$, where $\mathbb{P} = \mathbb{E}(\delta_\xi \otimes \mathbb{P}_\xi)$ with δ_ξ the Dirac measure at ξ and \mathbb{E} the expectation with respect to the random variable ξ , so that for all measurable and positive g defined on $\Theta^{\mathbb{N}} \times \Gamma$, we have

$$\int_{\Theta^{\mathbb{N}} \times \Gamma} g(x, y) d\mathbb{P}(x, y) = \mathbb{E} \int_{\Gamma} g(\xi, y) d\mathbb{P}_\xi(y).$$

The total probability \mathbb{P} is usually called *annealed law*. The quenched law \mathbb{P}_ξ may be considered to be the conditional probability of \mathbb{P} given ξ . The expectation with respect to \mathbb{P} will still be denoted by \mathbb{E} ; there will be no confusion for reason of consistence. The expectation with respect to \mathbb{P}_ξ will be denoted by \mathbb{E}_ξ .

Let \mathbb{T} be the genealogical tree with $\{N_u\}$ as defining elements. By definition, we have: (a) $\emptyset \in \mathbb{T}$; (b) $ui \in \mathbb{T}$ implies $u \in \mathbb{T}$; (c) if $u \in \mathbb{T}$, then $ui \in \mathbb{T}$ if and only if $1 \leq i \leq N_u$. Let $\mathbb{T}_n = \{u \in \mathbb{T} : |u| = n\}$ be the set of particles of generation n , where $|u|$ denotes the length of the sequence u and represents the number of generation to which u belongs.

2.2. The main results. Let $Z_n(\cdot)$ be the counting measure of particles of generation n : for $B \subset \mathbb{R}$,

$$Z_n(B) = \sum_{u \in \mathbb{T}_n} \mathbf{1}_B(S_u).$$

Then $\{Z_n(\mathbb{R})\}$ constitutes a branching process in a random environment (see e.g. [3, 4, 35]). For $n \geq 0$, let \widehat{N}_n (resp. \widehat{L}_n) be a random variable with distribution $p(\xi_n)$ (resp. $G(\xi_n)$) under the law \mathbb{P}_ξ , and define

$$m_n = m(\xi_n) = \mathbb{E}_\xi \widehat{N}_n, \quad \Pi_n = m_0 \cdots m_{n-1}, \quad \Pi_0 = 1.$$

Throughout the paper, we shall always assume the following conditions:

$$\mathbb{E} \ln m_0 > 0 \quad \text{and} \quad \mathbb{E} \left(\frac{1}{m_0} \widehat{N}_0 \left(\ln^+ \widehat{N}_0 \right)^{1+\lambda} \right) < \infty, \quad (2.1)$$

where the value of $\lambda > 0$ will be specified in the hypothesis of the theorem. Under these conditions, the underlying branching process is *supercritical* and the number of the particles tends to infinity with positive probability ([36]). Moreover, it is well known that in supercritical case, the normalized sequence

$$W_n = \Pi_n^{-1} Z_n(\mathbb{R}), \quad n \geq 1$$

constitutes a martingale with respect to the filtration \mathcal{F}_n defined by:

$$\mathcal{F}_0 = \{\emptyset, \Omega\}, \quad \mathcal{F}_n = \sigma(\xi, N_u : |u| < n), \quad \text{for } n \geq 1.$$

Under (2.1), the limit

$$W = \lim_n W_n$$

exists a.s. with $\mathbb{E}W = 1$ (see for example [4]); $W > 0$ almost surely (a.s.) on the explosion event $\{Z_n(\mathbb{R}) \rightarrow \infty\}$ ([36]).

For $n \geq 0$, define

$$l_n = \mathbb{E}_\xi \widehat{L}_n, \quad \sigma_n^{(\nu)} = \mathbb{E}_\xi \left(\widehat{L}_n - l_n \right)^\nu, \quad \text{for } \nu \geq 2;$$

$$\ell_n = \sum_{k=0}^{n-1} l_k, \quad s_n^{(\nu)} = \sum_{k=0}^{n-1} \sigma_k^{(\nu)}, \quad \text{for } \nu \geq 2, \quad s_n = (s_n^{(2)})^{1/2}.$$

We will need the following conditions on the motion of particles:

$$\mathbb{P}\left(\limsup_{|t| \rightarrow \infty} |\mathbb{E}_\xi e^{it\widehat{L}_0}| < 1\right) > 0 \quad \text{and} \quad \mathbb{E}(|\widehat{L}_0|^\eta) < \infty, \quad (2.2)$$

where the value of $\eta > 1$ is to be specified in the hypothesis of the theorems. The first hypothesis means that Cramér's condition about the characteristic function of \widehat{L}_0 holds with positive probability.

Let $\{N_{1,n}\}$ and $\{N_{2,n}\}$ be two sequences of random variables, defined respectively by

$$N_{1,n} = \frac{1}{\prod_n} \sum_{u \in \mathbb{T}_n} (S_u - \ell_n) \quad \text{and} \quad N_{2,n} = \frac{1}{\prod_n} \sum_{u \in \mathbb{T}_n} ((S_u - \ell_n)^2 - s_n^2).$$

Due to [16, Proposition 2.1 and 2.2], they are martingales with respect to the filtration (\mathcal{D}_n) defined by

$$\mathcal{D}_0 = \{\emptyset, \Omega\}, \quad \mathcal{D}_n = \sigma(\xi, N_u, L_{ui} : i \geq 1, |u| < n), \quad \text{for } n \geq 1;$$

and if the conditions (2.1) and (2.2) hold for $\lambda > 2, \eta > 4$, then these two martingales converges a.s. :

$$V_1 := \lim_{n \rightarrow \infty} N_{1,n} \quad \text{and} \quad V_2 := \lim_{n \rightarrow \infty} N_{2,n} \quad \text{exist a.s. in } \mathbb{R}. \quad (2.3)$$

Set

$$Z_n(t) = Z_n((-\infty, t]), \quad \phi(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}, \quad \Phi(t) = \int_{-\infty}^t \phi(x) dx, \quad t \in \mathbb{R}.$$

as usual, we denote by $H_m(\cdot)$ the Chebyshev-Hermite polynomial of degree m , that is

$$H_m(x) = m! \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \frac{(-1)^k x^{m-2k}}{k!(m-2k)!2^k}.$$

More precisely, we need the following polynomials:

$$\begin{aligned} H_0(x) &= 1, \\ H_1(x) &= x, \\ H_2(x) &= x^2 - 1, \\ H_3(x) &= x^3 - 3x, \\ H_4(x) &= x^4 - 6x^2 + 3, \\ H_5(x) &= x^5 - 10x^3 + 15x. \end{aligned}$$

Then we can state our main result as follows:

Theorem 2.1. *Assume (2.1) for $\lambda > 18$, (2.2) for $\eta > 24$ and $\mathbb{E}m_0^{-\delta} < \infty$ for some $\delta > 0$. Then for $t \in \mathbb{R}$, as $n \rightarrow \infty$,*

$$\frac{1}{\prod_n} Z_n(\ell_n + s_n t) = \Phi(t)W - \frac{s_n^{(3)}}{6s_n^3} H_2(t)\phi(t)W - \frac{1}{s_n} \phi(t)V_1 + \frac{1}{n} \mathcal{R}_n(t) + o\left(\frac{1}{n}\right) \text{ a.s.}, \quad (2.4)$$

where

$$\mathcal{R}_n(t) = n \left[-\frac{1}{2s_n^2} H_1(t) \phi(t) V_2 - \frac{s_n^{(3)}}{6s_n^4} H_3(t) \phi(t) V_1 - \left(\frac{(s_n^{(3)})^2}{72s_n^6} H_5(t) + \frac{\sum_{j=1}^n (\sigma_j^{(4)} - 3(\sigma_j^{(2)})^2)}{24s_n^4} H_3(t) \right) \phi(t) W \right]. \quad (2.5)$$

When we assume Θ is singleton (i.e. the constant environment) and we take the distribution of L as $\mathcal{N}(0, 1)$, the model becomes the branching Wiener process studied in [10] and we get a generalization of Chen's work. Note in this case, the condition (2.2) is automatically always valid for any $\eta > 0$.

Corollary 2.2. *For branching Wiener process, assume (2.1) for $\lambda > 18$. Then for $t \in \mathbb{R}$, as $n \rightarrow \infty$,*

$$\frac{1}{m^n} Z_n(\sqrt{nt}) = \Phi(t)W - \frac{1}{\sqrt{n}} \phi(t) V_1 - \frac{1}{2n} H_1(t) \phi(t) V_2 + o\left(\frac{1}{n}\right) \text{ a.s.}$$

Remark 2.3. This corollary gives the second order asymptotic expansion of the central limit theorem for a supercritical branching Wiener process, and it generalizes Chen's result [10] which may be viewed as the first order expansion.

For simplicity and without loss of generality, hereafter we will always assume that $l_n = 0$ (otherwise, we only need to replace L_{ui} by $L_{ui} - l_n$) and hence $\ell_n = 0$. In the following, we will use K_ξ as a constant depending on the environment, which may change from line to line.

3. PROOF OF THE MAIN THEOREM

3.1. Notation and A key decomposition. We first introduce some notation which will be used in the sequel.

In addition to the σ -fields \mathcal{F}_n and \mathcal{D}_n , the following σ -fields will also be used:

$$\mathcal{J}_0 = \{\emptyset, \Omega\}, \quad \mathcal{J}_n = \sigma(\xi_k, N_u, L_{ui} : k < n, i \geq 1, |u| < n) \text{ for } n \geq 1.$$

For conditional probabilities and expectations, we write:

$$\begin{aligned} \mathbb{P}_{\xi, n}(\cdot) &= \mathbb{P}_\xi(\cdot | \mathcal{D}_n), & \mathbb{E}_{\xi, n}(\cdot) &= \mathbb{E}_\xi(\cdot | \mathcal{D}_n); & \mathbb{P}_n(\cdot) &= \mathbb{P}(\cdot | \mathcal{J}_n), & \mathbb{E}_n(\cdot) &= \mathbb{E}(\cdot | \mathcal{J}_n); \\ \mathbb{P}_{\xi, \mathcal{F}_n}(\cdot) &= \mathbb{P}_\xi(\cdot | \mathcal{F}_n), & \mathbb{E}_{\xi, \mathcal{F}_n}(\cdot) &= \mathbb{E}_\xi(\cdot | \mathcal{F}_n). \end{aligned}$$

As usual, we write $\mathbb{N}^* = \{1, 2, 3, \dots\}$ and denote by

$$U = \bigcup_{n=0}^{\infty} (\mathbb{N}^*)^n$$

the set of all finite sequences, where $(\mathbb{N}^*)^0 = \{\emptyset\}$ contains the null sequence \emptyset .

For all $u \in U$, let $\mathbb{T}(u)$ be the shifted tree of \mathbb{T} at u with defining elements $\{N_{uv}\}$: we have 1) $\emptyset \in \mathbb{T}(u)$, 2) $vi \in \mathbb{T}(u) \Rightarrow v \in \mathbb{T}(u)$ and 3) if $v \in \mathbb{T}(u)$, then $vi \in \mathbb{T}(u)$ if and only if $1 \leq i \leq N_{uv}$. Define $\mathbb{T}_n(u) = \{v \in \mathbb{T}(u) : |v| = n\}$. Then $\mathbb{T} = \mathbb{T}(\emptyset)$ and $\mathbb{T}_n = \mathbb{T}_n(\emptyset)$.

For $u \in (\mathbb{N}^*)^k (k \geq 0)$ and $n \geq 1$, let S_u be the position of u and write

$$Z_n(u, B) = \sum_{v \in \mathbb{T}_n(u)} \mathbf{1}_B(S_{uv} - S_u), \quad Z_n(u, t) = Z_n(u, (-\infty, t]).$$

Then the law of $Z_n(u, B)$ under \mathbb{P}_ξ is the same as that of $Z_n(B)$ under $P_{\theta^k \xi}$. Define

$$\begin{aligned} W_n(u, B) &= Z_n(u, B) / \Pi_n(\theta^k \xi), & W_n(u, t) &= W_n(u, (-\infty, t]), \\ W_n(B) &= Z_n(B) / \Pi_n, & W_n(t) &= W_n((-\infty, t]). \end{aligned}$$

By definition, we have $\Pi_n(\theta^k \xi) = m_k \cdots m_{k+n-1}$, $Z_n(B) = Z_n(\emptyset, B)$, $W_n(B) = W_n(\emptyset, B)$, $W_n = W_n(\mathbb{R})$.

For each n , we choose an integer $k_n < n$ as follows. Let β be a real number such that $\max\{\frac{3}{\lambda}, \frac{4}{\eta}\} < \beta < \frac{1}{6}$ and set $k_n = \lfloor n^\beta \rfloor$.

Write

$$\mathbb{X}_n(t) = \frac{1}{\Pi_n} Z_n(s_n t) - \Phi(t)W + \frac{s_n^{(3)}}{6s_n^3} H_2(t)\phi(t)W + \frac{1}{s_n} \phi(t)V_1. \quad (3.1)$$

By virtue of

$$Z_n(s_n t) = \sum_{u \in \mathbb{T}_{k_n}} Z_{n-k_n}(u, s_n t - S_u),$$

we have the following important decomposition:

$$\mathbb{X}_n(t) = \mathbb{A}_n + \mathbb{B}_n + \mathbb{C}_n, \quad (3.2)$$

with

$$\begin{aligned} \mathbb{A}_n &= \frac{1}{\Pi_{k_n}} \sum_{u \in \mathbb{T}_{k_n}} [W_{n-k_n}(u, s_n t - S_u) - \mathbb{E}_{\xi, k_n} W_{n-k_n}(u, s_n t - S_u)], \\ \mathbb{B}_n &= \frac{1}{\Pi_{k_n}} \sum_{u \in \mathbb{T}_{k_n}} \left[\mathbb{E}_{\xi, k_n} W_{n-k_n}(u, s_n t - S_u) - \Phi(t) + \frac{s_n^{(3)}}{6s_n^3} H_2(t)\phi(t) + \frac{1}{s_n} \phi(t)S_u \right], \\ \mathbb{C}_n &= \Phi(t)(W_{k_n} - W) - \frac{s_n^{(3)}}{6s_n^3} H_2(t)\phi(t)(W_{k_n} - W) + \frac{1}{s_n} \phi(t)(V_1 - N_{1, k_n}). \end{aligned}$$

3.2. The Edgeworth expansion for sums of independent random variables. We next present the Edgeworth expansion for sums of independent random variables, which is needed to prove the main theorem. Let us recall the theorem used in this paper obtained by Bai and Zhao(1986, [5]), that generalizing the case for i.i.d random variables (cf. [32, P.159, Theorem 1]).

Let $\{X_j\}$ be independent random variables, satisfying for each $j \geq 1$

$$\mathbb{E}X_j = 0, \mathbb{E}|X_j|^k < \infty \text{ with some integer } k \geq 3. \quad (3.3)$$

We write $B_n^2 = \sum_{j=1}^n \mathbb{E}X_j^2$ and only consider the nontrivial case $B_n > 0$. Let $\gamma_{\nu j}$ be the ν -order cumulant of X_j for each $j \geq 1$. Write

$$\lambda_{\nu, n} = n^{(\nu-2)/2} B_n^{-\nu} \sum_{j=1}^n \gamma_{\nu j}, \quad \nu = 3, 4, \dots, k;$$

$$\begin{aligned} Q_{\nu,n}(x) &= \sum' (-1)^{\nu+2s} \Phi^{(\nu+2s)}(x) \prod_{m=1}^{\nu} \frac{1}{k_m!} \left(\frac{\lambda_{m+2,n}}{(m+2)!} \right)^{k_m} \\ &= -\phi(x) \sum' H_{\nu+2s-1}(x) \prod_{m=1}^{\nu} \frac{1}{k_m!} \left(\frac{\lambda_{m+2,n}}{(m+2)!} \right)^{k_m}, \end{aligned}$$

where the summation \sum' is carried out over all nonnegative integer solutions (k_1, \dots, k_ν) of the equations:

$$k_1 + \dots + k_\nu = s \quad \text{and} \quad k_1 + 2k_2 + \dots + \nu k_\nu = \nu.$$

For $1 \leq j \leq n$ and $x \in \mathbb{R}$, define

$$F_n(x) = \mathbb{P}\left(B_n^{-1} \sum_{j=1}^n X_j \leq x\right), \quad v_j(t) = \mathbb{E}e^{itX_j};$$

$$Y_{nj} = X_j \mathbf{1}_{\{|X_j| \leq B_n\}}, \quad Z_{nj}^{(x)} = X_j \mathbf{1}_{\{|X_j| \leq B_n(1+|x|)\}}, \quad W_{nj}^{(x)} = X_j \mathbf{1}_{\{|X_j| > B_n(1+|x|)\}}.$$

The Edgeworth expansion theorem can be stated as follows.

Lemma 3.1 ([5]). *Let $n \geq 1$ and X_1, \dots, X_n be a sequence of independent random variables satisfying (3.3) and $B_n > 0$. Then for the integer $k \geq 3$,*

$$\begin{aligned} |F_n(x) - \Phi(x) - \sum_{\nu=1}^{k-2} Q_{\nu n}(x) n^{-1/2}| &\leq C(k) \left\{ (1+|x|)^{-k} B_n^{-k} \sum_{j=1}^n \mathbb{E}|W_{nj}^{(x)}|^k + \right. \\ &\left. (1+|x|)^{-k-1} B_n^{-k-1} \sum_{j=1}^n \mathbb{E}|Z_{nj}^{(x)}|^{k+1} + (1+|x|)^{-k-1} n^{k(k+1)/2} \left(\sup_{|t| \geq \delta_n} \frac{1}{n} \sum_{j=1}^n |v_j(t)| + \frac{1}{2n} \right)^n \right\}, \end{aligned}$$

where $\delta_n = \frac{1}{12} B_n^2 \left(\sum_{j=1}^n \mathbb{E}|Y_{nj}|^3 \right)^{-1}$, $C(k) > 0$ is a constant depending only on k .

3.3. Proof of Main Theorem. To prove the main theorem, by use of the Birkhoff ergodic Theorem, we will prove the following equivalent form:

$$n\mathbb{X}_n(t) \xrightarrow{n \rightarrow \infty} \mathcal{R}(t) \quad \text{a.s.}, \quad (3.4)$$

where $\mathcal{R}(t) = \lim_{n \rightarrow \infty} \mathcal{R}_n(t)$ with

$$\begin{aligned} \mathcal{R}(t) &= -\frac{H_1(t)\phi(t)V_2}{2\mathbb{E}\sigma_0^{(2)}} - \frac{\mathbb{E}\sigma_0^{(3)}H_3(t)\phi(t)V_1}{6(\mathbb{E}\sigma_0^{(2)})^2} \\ &\quad \left(\frac{(\mathbb{E}\sigma_0^{(3)})^2 H_5(t)}{72(\mathbb{E}\sigma_0^{(2)})^3} + \frac{\mathbb{E}(\sigma_0^{(4)} - 3(\sigma_0^{(2)})^2)H_3(t)}{24(\mathbb{E}\sigma_0^{(2)})^2} \right) \phi(t)W. \end{aligned}$$

On the basis of the decomposition (3.2), we divide the proof of (3.4) into three lemmas.

Lemma 3.2. *Under the hypothesis of Theorem 2.1,*

$$n\mathbb{A}_n \xrightarrow{n \rightarrow \infty} 0 \quad \text{a.s.} \quad (3.5)$$

Lemma 3.3. *Under the hypothesis of Theorem 2.1,*

$$n\mathbb{B}_n \xrightarrow{n \rightarrow \infty} \mathcal{R}(t) \quad \text{a.s.} \quad (3.6)$$

Lemma 3.4. *Under the hypothesis of Theorem 2.1,*

$$n\mathbb{C}_n \xrightarrow{n \rightarrow \infty} 0 \quad a.s. \quad (3.7)$$

Proof of Lemma 3.2. For ease of notation, we define for $|u| = k_n$,

$$X_{n,u} = W_{n-k_n}(u, s_n t - S_u) - \mathbb{E}_{\xi, k_n} W_{n-k_n}(u, s_n t - S_u), \quad \bar{X}_{n,u} = X_{n,u} \mathbf{1}_{\{|X_{n,u}| < \Pi_{k_n}\}},$$

$$\bar{\mathbb{A}}_n = \frac{1}{\Pi_{k_n}} \sum_{u \in \mathbb{T}_{k_n}} \bar{X}_{n,u}.$$

Then we see that $|X_{n,u}| \leq W_{n-k_n}(u) + 1$.

To prove Lemma 3.2, we will use the extended Borel-Cantelli Lemma. We can obtain the required result once we prove that $\forall \varepsilon > 0$,

$$\sum_{n=1}^{\infty} \mathbb{P}_{k_n}(|n\mathbb{A}_n| > 2\varepsilon) < \infty. \quad (3.8)$$

Notice that

$$\begin{aligned} & \mathbb{P}_{k_n}(|\mathbb{A}_n| > \frac{2\varepsilon}{n}) \\ & \leq \mathbb{P}_{k_n}(\mathbb{A}_n \neq \bar{\mathbb{A}}_n) + \mathbb{P}_{k_n}(|\bar{\mathbb{A}}_n - \mathbb{E}_{\xi, k_n} \bar{\mathbb{A}}_n| > \frac{\varepsilon}{n}) + \mathbb{P}_{k_n}(|\mathbb{E}_{\xi, k_n} \bar{\mathbb{A}}_n| > \frac{\varepsilon}{n}). \end{aligned}$$

We will proceed the proof in 3 steps.

Step 1 We first prove that

$$\sum_{n=1}^{\infty} \mathbb{P}_{k_n}(\mathbb{A}_n \neq \bar{\mathbb{A}}_n) < \infty. \quad (3.9)$$

To this end, define

$$W^* = \sup_n W_n,$$

and we need the following result :

Lemma 3.5. (*[27, Th. 1.2]*) *Assume (2.1) for some $\lambda > 0$ and $\mathbb{E}m_0^{-\delta} < \infty$ for some $\delta > 0$. Then*

$$\mathbb{E}(W^* + 1)(\ln(W^* + 1))^\lambda < \infty. \quad (3.10)$$

We observe that

$$\begin{aligned} \mathbb{P}_{k_n}(\mathbb{A}_n \neq \bar{\mathbb{A}}_n) & \leq \sum_{u \in \mathbb{T}_{k_n}} \mathbb{P}_{k_n}(X_{n,u} \neq \bar{X}_{n,u}) = \sum_{u \in \mathbb{T}_{k_n}} \mathbb{P}_{k_n}(|X_{n,u}| \geq \Pi_{k_n}) \\ & \leq \sum_{u \in \mathbb{T}_{k_n}} \mathbb{P}_{k_n}(W_{n-k_n}(u) + 1 \geq \Pi_{k_n}) \\ & = W_{k_n} \left[r_n \mathbb{P}(W_{n-k_n} + 1 \geq r_n) \right]_{r_n = \Pi_{k_n}} \\ & \leq W_{k_n} \left[\mathbb{E}((W_{n-k_n} + 1) \mathbf{1}_{\{W_{n-k_n} + 1 \geq r_n\}}) \right]_{r_n = \Pi_{k_n}} \\ & \leq W_{k_n} \left[\mathbb{E}((W^* + 1) \mathbf{1}_{\{W^* + 1 \geq r_n\}}) \right]_{r_n = \Pi_{k_n}} \\ & \leq W^* (\ln \Pi_{k_n})^{-\lambda} \mathbb{E}(W^* + 1)(\ln(W^* + 1))^\lambda \end{aligned}$$

$$\leq K_\xi W^* n^{-\lambda\beta} \mathbb{E}(W^* + 1)(\ln(W^* + 1))^\lambda,$$

where the last inequality holds since

$$\frac{1}{n} \ln \Pi_n \rightarrow \mathbb{E} \ln m_0 > 0 \text{ a.s.}, \quad (3.11)$$

and $k_n \sim n^\beta$. By the choice of β and Lemma 3.5, we obtain (3.9).

Step 2. We next prove that $\forall \varepsilon > 0$,

$$\sum_{n=1}^{\infty} \mathbb{P}_{k_n}(|\bar{\mathbb{A}}_n - \mathbb{E}_{\xi, k_n} \bar{\mathbb{A}}_n| > \frac{\varepsilon}{n}) < \infty. \quad (3.12)$$

Take a constant $b \in (1, e^{\mathbb{E} \ln m_0})$. Observe that $\forall u \in \mathbb{T}_{k_n}, n \geq 1$,

$$\begin{aligned} \mathbb{E}_{k_n} \bar{X}_{n,u}^2 &= \int_0^\infty 2x \mathbb{P}_{k_n}(|\bar{X}_{n,u}| > x) dx = 2 \int_0^\infty x \mathbb{P}_{k_n}(|X_{n,u}| \mathbf{1}_{\{|X_{n,u}| < \Pi_{k_n}\}} > x) dx \\ &\leq 2 \int_0^{\Pi_{k_n}} x \mathbb{P}_{k_n}(|W_{n-k_n}(u) + 1| > x) dx = 2 \int_0^{\Pi_{k_n}} x \mathbb{P}(|W_{n-k_n} + 1| > x) dx \\ &\leq 2 \int_0^{\Pi_{k_n}} x \mathbb{P}(W^* + 1 > x) dx \\ &\leq 2 \int_e^{\Pi_{k_n}} (\ln x)^{-\lambda} \mathbb{E}(W^* + 1)(\ln(W^* + 1))^\lambda dx + 9 \\ &\leq 2 \mathbb{E}(W^* + 1)(\ln(W^* + 1))^\lambda \left(\int_e^{b^{k_n}} (\ln x)^{-\lambda} dx + \int_{b^{k_n}}^{\Pi_{k_n}} (\ln x)^{-\lambda} dx \right) + 9 \\ &\leq 2 \mathbb{E}(W^* + 1)(\ln(W^* + 1))^\lambda (b^{k_n} + (\Pi_{k_n} - b^{k_n})(k_n \ln b)^{-\lambda}) + 9. \end{aligned}$$

Then we have that

$$\begin{aligned} &\sum_{n=1}^{\infty} \mathbb{P}_{k_n}(|\bar{\mathbb{A}}_n - \mathbb{E}_{\xi, k_n} \bar{\mathbb{A}}_n| > \frac{\varepsilon}{n}) \\ &= \sum_{n=1}^{\infty} \mathbb{E}_{k_n} \mathbb{P}_{\xi, k_n}(|\bar{\mathbb{A}}_n - \mathbb{E}_{\xi, k_n} \bar{\mathbb{A}}_n| > \frac{\varepsilon}{n}) \\ &\leq \varepsilon^{-2} \sum_{n=1}^{\infty} n^2 \mathbb{E}_{k_n} \left(\Pi_{k_n}^{-2} \sum_{u \in \mathbb{T}_{k_n}} \mathbb{E}_{\xi, k_n} \bar{X}_{n,u}^2 \right) = \varepsilon^{-2} \sum_{n=1}^{\infty} n^2 \left(\Pi_{k_n}^{-2} \sum_{u \in \mathbb{T}_{k_n}} \mathbb{E}_{k_n} \bar{X}_{n,u}^2 \right) \\ &\leq \varepsilon^{-2} \sum_{n=1}^{\infty} \frac{n^2 W_{k_n}}{\Pi_{k_n}} [2 \mathbb{E}(W^* + 1)(\ln(W^* + 1))^\lambda (b^{k_n} + (\Pi_{k_n} - b^{k_n})(k_n \ln b)^{-\lambda}) + 9] \\ &\leq 2 \varepsilon^{-2} W^* \mathbb{E}(W^* + 1)(\ln(W^* + 1))^\lambda \left(\sum_{n=1}^{\infty} \frac{n^2}{\Pi_{k_n}} b^{k_n} + \sum_{n=1}^{\infty} n^2 (k_n \ln b)^{-\lambda} \right) + 9 \varepsilon^{-2} W^* \sum_{n=1}^{\infty} \frac{n^2}{\Pi_{k_n}}. \end{aligned}$$

By (3.11) and $\lambda\beta > 3$, the three series in the last expression above converge under our hypothesis and hence (3.12) is proved.

Step 3. Observe

$$\begin{aligned}
& \mathbb{P}_{k_n} \left(|\mathbb{E}_{\xi, k_n} \bar{A}_n| > \frac{\varepsilon}{n} \right) \\
& \leq \frac{n}{\varepsilon} \mathbb{E}_{k_n} |\mathbb{E}_{\xi, k_n} \bar{A}_n| = \frac{n}{\varepsilon} \mathbb{E}_{k_n} \left| \frac{1}{\Pi_{k_n}} \sum_{u \in \mathbb{T}_{k_n}} \mathbb{E}_{\xi, k_n} \bar{X}_{n,u} \right| \\
& = \frac{n}{\varepsilon} \mathbb{E}_{k_n} \left| \frac{1}{\Pi_{k_n}} \sum_{u \in \mathbb{T}_{k_n}} (-\mathbb{E}_{\xi, k_n} X_{n,u} \mathbf{1}_{\{|X_{n,u}| \geq \Pi_{k_n}\}}) \right| \\
& \leq \frac{n}{\varepsilon} \frac{1}{\Pi_{k_n}} \sum_{u \in \mathbb{T}_{k_n}} \mathbb{E}_{k_n} (W_{n-k_n}(u) + 1) \mathbf{1}_{\{W_{n-k_n}(u) + 1 \geq \Pi_{k_n}\}} \\
& = \frac{n W_{k_n}}{\varepsilon} \left[\mathbb{E}(W_{n-k_n} + 1) \mathbf{1}_{\{W_{n-k_n} + 1 \geq r_n\}} \right]_{r_n = \Pi_{k_n}} \\
& \leq \frac{W^*}{\varepsilon} n \left[\mathbb{E}(W^* + 1) \mathbf{1}_{\{W^* + 1 \geq r_n\}} \right]_{r_n = \Pi_{k_n}} \\
& \leq \frac{W^*}{\varepsilon} \frac{n}{(\ln \Pi_{k_n})^\lambda} \mathbb{E}(W^* + 1) \ln^\lambda(W^* + 1) \\
& \leq \frac{W^*}{\varepsilon} K_\xi n^{1-\lambda\beta} \mathbb{E}(W^* + 1) \ln^\lambda(W^* + 1).
\end{aligned}$$

Then by (3.11) and $\lambda\beta > 2$, it follows that

$$\sum_{n=1}^{\infty} \mathbb{P}_{k_n} \left(|\mathbb{E}_{\xi, k_n} \bar{A}_n| > \frac{\varepsilon}{n} \right) < \infty.$$

Combining Steps 1-3, we obtain (3.8). Hence the lemma is proved. \square

Proof of Lemma 3.3. For ease of reference, we introduce some notation:

$$\begin{aligned}
\kappa_{1,n} &= \frac{1}{6} (s_n^2 - s_{k_n}^2)^{-3/2} (s_n^{(3)} - s_{k_n}^{(3)}), & D_1(x) &= -H_2(x)\phi(x), \\
\kappa_{2,n} &= \frac{1}{72} (s_n^2 - s_{k_n}^2)^{-3} (s_n^{(3)} - s_{k_n}^{(3)})^2, & D_2(x) &= -H_5(x)\phi(x), \\
\kappa_{3,n} &= \frac{1}{24} (s_n^2 - s_{k_n}^2)^{-2} \sum_{j=k_n}^{n-1} (\sigma_j^{(4)} - 3(\sigma_j^{(2)})^2), & D_3(x) &= -H_3(x)\phi(x).
\end{aligned}$$

By the properties of the Chebyshev-Hermite polynomials, we know that

$$D'_1(x) = H_3(x)\phi(x).$$

Observe that

$$\mathbb{B}_n = \mathbb{B}_{n1} + \mathbb{B}_{n2} + \mathbb{B}_{n3} + \mathbb{B}_{n4} + \mathbb{B}_{n5} + \mathbb{B}_{n6}, \tag{3.13}$$

where

$$\mathbb{B}_{n1} = \frac{1}{\Pi_{k_n}} \sum_{u \in \mathbb{T}_{k_n}} \mathbf{1}_{\{|S_u| > k_n\}} \left[\mathbb{E}_{\xi, k_n} W_{n-k_n}(u, s_n t - S_u) - \Phi(t) - \right.$$

$$\begin{aligned} & \left. \frac{s_n^{(3)}}{6s_n^3}(1-t^2)\phi(t) + \frac{1}{s_n}\phi(t)S_u \right], \\ \mathbb{B}_{n2} &= \frac{1}{\prod_{k_n}} \sum_{u \in \mathbb{T}_{k_n}} \mathbf{1}_{\{|S_u| \leq k_n\}} \left[\mathbb{E}_{\xi, k_n} W_{n-k_n}(u, s_n t - S_u) - \Phi\left(\frac{s_n t - S_u}{(s_n^2 - s_{k_n}^2)^{1/2}}\right) - \right. \\ & \qquad \qquad \qquad \left. \sum_{\nu=1}^3 \kappa_{\nu, n} D_\nu\left(\frac{s_n t - S_u}{(s_n^2 - s_{k_n}^2)^{1/2}}\right) \right], \\ \mathbb{B}_{n3} &= \frac{1}{\prod_{k_n}} \sum_{u \in \mathbb{T}_{k_n}} \mathbf{1}_{\{|S_u| \leq k_n\}} \left[\Phi\left(\frac{s_n t - S_u}{(s_n^2 - s_{k_n}^2)^{1/2}}\right) - \Phi(t) + \frac{1}{s_n}\phi(t)S_u \right], \\ \mathbb{B}_{n4} &= \frac{1}{\prod_{k_n}} \sum_{u \in \mathbb{T}_{k_n}} \mathbf{1}_{\{|S_u| \leq k_n\}} \left[\kappa_{1, n} D_1\left(\frac{s_n t - S_u}{(s_n^2 - s_{k_n}^2)^{1/2}}\right) - \frac{s_n^{(3)}}{6s_n^3} D_1(t) \right], \\ \mathbb{B}_{n5} &= \kappa_{2, n} \frac{1}{\prod_{k_n}} \sum_{u \in \mathbb{T}_{k_n}} \mathbf{1}_{\{|S_u| \leq k_n\}} D_2\left(\frac{s_n t - S_u}{(s_n^2 - s_{k_n}^2)^{1/2}}\right), \\ \mathbb{B}_{n6} &= \kappa_{3, n} \frac{1}{\prod_{k_n}} \sum_{u \in \mathbb{T}_{k_n}} \mathbf{1}_{\{|S_u| \leq k_n\}} D_3\left(\frac{s_n t - S_u}{(s_n^2 - s_{k_n}^2)^{1/2}}\right). \end{aligned}$$

The lemma will be proved once we show that a.s.

$$n\mathbb{B}_{n1} \xrightarrow{n \rightarrow \infty} 0, \quad (3.14)$$

$$n\mathbb{B}_{n2} \xrightarrow{n \rightarrow \infty} 0, \quad (3.15)$$

$$n\mathbb{B}_{n3} \xrightarrow{n \rightarrow \infty} -\frac{1}{2}(\mathbb{E}\sigma_0^{(2)})^{-1}t\phi(t)V_2, \quad (3.16)$$

$$n\mathbb{B}_{n4} \xrightarrow{n \rightarrow \infty} -\frac{1}{6}(\mathbb{E}\sigma_0^{(2)})^{-2}\mathbb{E}\sigma_0^{(3)}D_1'(t)V_1, \quad (3.17)$$

$$n\mathbb{B}_{n5} \xrightarrow{n \rightarrow \infty} \frac{1}{72}(\mathbb{E}\sigma_0^{(2)})^{-3}(\mathbb{E}\sigma_0^{(3)})^2D_2(t)W, \quad (3.18)$$

$$n\mathbb{B}_{n6} \xrightarrow{n \rightarrow \infty} \frac{1}{24}(\mathbb{E}\sigma_0^{(2)})^{-2}\mathbb{E}(\sigma_0^{(4)} - 3(\sigma_0^{(2)})^2)D_3(t)W. \quad (3.19)$$

We will prove these results subsequently.

We first prove (3.14). Since

$$\begin{aligned} |\mathbb{B}_{n1}| &\leq \frac{1}{\prod_{k_n}} \sum_{u \in \mathbb{T}_{k_n}} \mathbf{1}_{\{|S_u| > k_n\}} \left(1 + \Phi(t) + \frac{s_n^{(3)}}{6s_n^3}(1-t^2)\phi(t) \right) \\ & \qquad \qquad \qquad + \frac{1}{s_n}\phi(t) \frac{1}{\prod_{k_n}} \sum_{u \in \mathbb{T}_{k_n}} |S_u| \mathbf{1}_{\{|S_u| > k_n\}}, \end{aligned}$$

(3.14) will follow from that

$$\sqrt{n} \frac{1}{\prod_{k_n}} \sum_{u \in \mathbb{T}_{k_n}} |S_u| \mathbf{1}_{\{|S_u| > k_n\}} \xrightarrow{n \rightarrow \infty} 0 \text{ a.s.}; \quad n \frac{1}{\prod_{k_n}} \sum_{u \in \mathbb{T}_{k_n}} \mathbf{1}_{\{|S_u| > k_n\}} \xrightarrow{n \rightarrow \infty} 0 \text{ a.s.} \quad (3.20)$$

In order to prove (3.20), we first observe that

$$\begin{aligned}
& \mathbb{E} \left(\sum_{n=1}^{\infty} \sqrt{n} \frac{1}{\prod_{k_n}} \sum_{u \in \mathbb{T}_{k_n}} |S_u| \mathbf{1}_{\{|S_u| > k_n\}} \right) \\
&= \sum_{n=1}^{\infty} \sqrt{n} \mathbb{E} |\widehat{S}_{k_n}| \mathbf{1}_{\{|\widehat{S}_{k_n}| > k_n\}} \leq \sum_{n=1}^{\infty} \sqrt{n} k_n^{1-\eta} \mathbb{E} |\widehat{S}_{k_n}|^\eta \\
&\leq \sum_{n=1}^{\infty} \sqrt{n} k_n^{-\frac{\eta}{2}} \sum_{j=0}^{k_n-1} \mathbb{E} |\widehat{L}_j|^\eta = \sum_{n=1}^{\infty} \sqrt{n} k_n^{1-\frac{\eta}{2}} \mathbb{E} |\widehat{L}_0|^\eta \\
&< \sum_{n=1}^{\infty} n k_n^{-\frac{\eta}{2}} \mathbb{E} |\widehat{L}_0|^\eta, \\
& \mathbb{E} \left(\sum_{n=1}^{\infty} n \frac{1}{\prod_{k_n}} \sum_{u \in \mathbb{T}_{k_n}} \mathbf{1}_{\{|S_u| > k_n\}} \right) \\
&= \sum_{n=1}^{\infty} n \mathbb{E} \mathbf{1}_{\{|\widehat{S}_{k_n}| > k_n\}} \leq \sum_{n=1}^{\infty} n k_n^{-\eta} \mathbb{E} |\widehat{S}_{k_n}|^\eta \\
&\leq \sum_{n=1}^{\infty} n k_n^{-\frac{\eta}{2}-1} \sum_{j=0}^{k_n-1} \mathbb{E} |\widehat{L}_j|^\eta = \sum_{n=1}^{\infty} n k_n^{-\frac{\eta}{2}} \mathbb{E} |\widehat{L}_0|^\eta.
\end{aligned}$$

By the choice of β and k_n , $n k_n^{-\frac{\eta}{2}} < n^{-1}$ and hence the series in the right hand side of the above two expressions converge. So

$$\sum_{n=1}^{\infty} \sqrt{n} \frac{1}{\prod_{k_n}} \sum_{u \in \mathbb{T}_{k_n}} |S_u| \mathbf{1}_{\{|S_u| > k_n\}} < \infty, \quad \sum_{n=1}^{\infty} n \frac{1}{\prod_{k_n}} \sum_{u \in \mathbb{T}_{k_n}} \mathbf{1}_{\{|S_u| > k_n\}} < \infty \text{ a.s.},$$

which implies (3.20), and consequently (3.14) follows.

The proof of (3.15) will mainly be based on the following result about the asymptotic expansion of the distribution of the sum of random variables.

Proposition 3.6. *Under the hypothesis of Theorem 2.1, for a.e. ξ ,*

$$\varepsilon_n = n \sup_{x \in \mathbb{R}} \left| \mathbb{P}_\xi \left(\frac{\sum_{k=k_n}^{n-1} \widehat{L}_k}{(s_n^2 - s_{k_n}^2)^{1/2}} \leq x \right) - \Phi(x) - \sum_{\nu=1}^3 \kappa_{\nu,n} D_\nu(x) \right| \xrightarrow{n \rightarrow \infty} 0.$$

Proof. Let $X_k = 0$ for $0 \leq k \leq k_n - 1$ and $X_k = \widehat{L}_k$ for $k_n \leq k \leq n - 1$. Then the random variables $\{X_k\}$ are independent under \mathbb{P}_ξ . Denote by $v_k(\cdot)$ the characteristic function of X_k : $v_k(t) := \mathbb{E}_\xi e^{itX_k}$. Combining the Markov inequality with Lemma 3.1, we obtain the following result:

$$\begin{aligned}
& \sup_{x \in \mathbb{R}} \left| \mathbb{P}_\xi \left(\frac{\sum_{k=k_n}^{n-1} \widehat{L}_k}{(s_n^2 - s_{k_n}^2)^{1/2}} \leq x \right) - \Phi(x) - \sum_{\nu=1}^3 \kappa_{\nu,n} D_\nu(x) \right| \\
&\leq K_\xi \left\{ (s_n^2 - s_{k_n}^2)^{-\frac{5}{2}} \sum_{j=k_n}^{n-1} \mathbb{E}_\xi |\widehat{L}_j|^5 + n^5 \left(\sup_{|t| > T} \frac{1}{n} \left(k_n + \sum_{j=k_n}^{n-1} |v_j(t)| \right) + \frac{1}{2n} \right)^n \right\}.
\end{aligned}$$

By our conditions on the environment, we know that

$$\lim_{n \rightarrow \infty} n^{\frac{3}{2}} (s_n^2 - s_{k_n}^2)^{-\frac{5}{2}} \sum_{j=k_n}^{n-1} \mathbb{E}_\xi |\widehat{L}_k|^5 = \mathbb{E} |\widehat{L}_0|^5 / (\mathbb{E} \sigma_0^{(2)})^{\frac{5}{2}}. \quad (3.21)$$

By (2.2), \widehat{L}_n satisfies

$$\mathbb{P} \left(\limsup_{|t| \rightarrow \infty} |v_n(t)| < 1 \right) > 0.$$

So there exists a constant $c_n \leq 1$ depending on ξ_n such that

$$\sup_{|t| > T} |v_n(t)| \leq c_n \quad \text{and} \quad \mathbb{P}(c_n < 1) > 0.$$

Then $\mathbb{E} c_0 < 1$. By the Birkhoff ergodic theorem, we have

$$\sup_{|t| > T} \left(\frac{1}{n} \sum_{j=k_n}^{n-1} |v_j(t)| \right) \leq \frac{1}{n} \sum_{j=1}^{n-1} c_j \rightarrow \mathbb{E} c_0 < 1.$$

Then for n large enough,

$$\left(\sup_{|t| > T} \frac{1}{n} \left(k_n + \sum_{j=k_n}^{n-1} |v_j(t)| \right) + \frac{1}{2n} \right)^n = o(n^{-m}), \quad \forall m > 0. \quad (3.22)$$

The proposition comes from (3.21) and (3.22). \square

From this proposition, it follows that

$$n |\mathbb{B}_{n2}| \leq W_{k_n} \varepsilon_n \xrightarrow{n \rightarrow \infty} 0. \quad (3.23)$$

Hence (3.15) is proved.

Now we turn to the proof of (3.16). Observe

$$\mathbb{B}_{n3} = \mathbb{B}_{n31} + \mathbb{B}_{n32} + \mathbb{B}_{n33} + \mathbb{B}_{n34} + \mathbb{B}_{n35}, \quad (3.24)$$

with

$$\mathbb{B}_{n31} = \frac{1}{\prod_{k_n}} \sum_{u \in \mathbb{T}_{k_n}} \mathbf{1}_{\{|S_u| \leq k_n\}} \left[\Phi \left(\frac{s_n t - S_u}{(s_n^2 - s_{k_n}^2)^{1/2}} \right) - \Phi(t) - \phi(t) \left(\frac{s_n t - S_u}{(s_n^2 - s_{k_n}^2)^{1/2}} - t \right) + \frac{1}{2} t \phi(t) \left(\frac{s_n t - S_u}{(s_n^2 - s_{k_n}^2)^{1/2}} - t \right)^2 \right],$$

$$\mathbb{B}_{n32} = \left(\frac{1}{s_n} - \frac{1}{(s_n^2 - s_{k_n}^2)^{1/2}} \right) \phi(t) \frac{1}{\prod_{k_n}} \sum_{u \in \mathbb{T}_{k_n}} S_u \mathbf{1}_{\{|S_u| \leq k_n\}},$$

$$\mathbb{B}_{n33} = t \phi(t) \frac{1}{\prod_{k_n}} \sum_{u \in \mathbb{T}_{k_n}} \mathbf{1}_{\{|S_u| \leq k_n\}} \left[\left(\frac{s_n}{(s_n^2 - s_{k_n}^2)^{1/2}} - 1 \right) - \frac{1}{2} \left(\frac{s_n t - S_u}{(s_n^2 - s_{k_n}^2)^{1/2}} - t \right)^2 + \frac{1}{2(s_n^2 - s_{k_n}^2)} (S_u^2 - s_{k_n}^2) \right],$$

$$\mathbb{B}_{n34} = \frac{1}{2(s_n^2 - s_{k_n}^2)} t \phi(t) \frac{1}{\prod_{k_n}} \sum_{u \in \mathbb{T}_{k_n}} (S_u^2 - s_{k_n}^2) \mathbf{1}_{\{|S_u| > k_n\}},$$

$$\mathbb{B}_{n35} = -\frac{1}{2(s_n^2 - s_{k_n}^2)} t \phi(t) N_{2,k_n}.$$

We know that by the Birkhoff theorem, $s_n \sim \sqrt{n\mathbb{E}\sigma_0^{(2)}}$. By Taylor' expansion and $k_n < n^{\frac{1}{6}}$,

$$n|\mathbb{B}_{n31}| \leq 8W_{k_n} n \left[|t|^3 \left| \frac{s_n}{(s_n^2 - s_{k_n}^2)^{1/2}} - 1 \right|^3 + \frac{k_n^3}{(s_n^2 - s_{k_n}^2)^{3/2}} \right] \xrightarrow{n \rightarrow \infty} 0 \quad \text{a.s.} \quad (3.25)$$

Also by elementary calculus and the fact $s_n \sim \sqrt{n\mathbb{E}\sigma_0^{(2)}}$, we obtain that

$$n|\mathbb{B}_{n32}| \leq \phi(t) \frac{n s_{k_n}^2 k_n W_{k_n}}{s_n (s_n^2 - s_{k_n}^2)^{1/2} (s_n + (s_n^2 - s_{k_n}^2)^{1/2})} \xrightarrow{n \rightarrow \infty} 0 \quad \text{a.s.}, \quad (3.26)$$

$$\begin{aligned} n|\mathbb{B}_{n33}| \leq & |t| \phi(t) W_{k_n} \frac{n}{2(s_n^2 - s_{k_n}^2)} \left[\frac{(1+t^2)s_{k_n}^4}{(s_n + (s_n^2 - s_{k_n}^2)^{1/2})^2} \right. \\ & \left. + \frac{2k_n s_{k_n}^2 t}{s_n + (s_n^2 - s_{k_n}^2)^{1/2}} \right] \xrightarrow{n \rightarrow \infty} 0 \quad \text{a.s.}, \end{aligned} \quad (3.27)$$

$$n|\mathbb{B}_{n34}| \xrightarrow{n \rightarrow \infty} 0 \quad \text{a.s.}, \quad (\text{by similar arguments as in the proof of (3.20)}) \quad (3.28)$$

$$n\mathbb{B}_{n35} \xrightarrow{n \rightarrow \infty} -\frac{1}{2\mathbb{E}\sigma_0^{(2)}} t \phi(t) V_2 \quad \text{a.s.} \quad (3.29)$$

Combining (3.25)-(3.29), we get (3.16).

We prove (3.17) by using the following decomposition:

$$\mathbb{B}_{n4} = \mathbb{B}_{n41} + \mathbb{B}_{n42} + \mathbb{B}_{n43} + \mathbb{B}_{n44} + \mathbb{B}_{n45}, \quad (3.30)$$

with

$$\begin{aligned} \mathbb{B}_{n41} = & \kappa_{1,n} \frac{1}{\prod_{k_n}} \sum_{u \in \mathbb{T}_{k_n}} \mathbf{1}_{\{|S_u| \leq k_n\}} \left[D_1 \left(\frac{s_n t - S_u}{(s_n^2 - s_{k_n}^2)^{1/2}} \right) - D_1(t) \right. \\ & \left. - D_1'(t) \left(\frac{s_n t - S_u}{(s_n^2 - s_{k_n}^2)^{1/2}} - t \right) \right], \end{aligned}$$

$$\mathbb{B}_{n42} = t D_1'(t) \kappa_{1,n} \left[\frac{s_n}{(s_n^2 - s_{k_n}^2)^{1/2}} - 1 \right] \frac{1}{\prod_{k_n}} \sum_{u \in \mathbb{T}_{k_n}} \mathbf{1}_{\{|S_u| \leq k_n\}},$$

$$\mathbb{B}_{n43} = \left(\kappa_{1,n} - \frac{s_n^3}{6s_n^3} \right) D_1(t) \frac{1}{\prod_{k_n}} \sum_{u \in \mathbb{T}_{k_n}} \mathbf{1}_{\{|S_u| \leq k_n\}},$$

$$\mathbb{B}_{n44} = D_1'(t) \frac{\kappa_{1,n}}{(s_n^2 - s_{k_n}^2)^{1/2}} \frac{1}{\prod_{k_n}} \sum_{u \in \mathbb{T}_{k_n}} S_u \mathbf{1}_{\{|S_u| > k_n\}},$$

$$\mathbb{B}_{n45} = -D_1'(t) \frac{\kappa_{1,n}}{(s_n^2 - s_{k_n}^2)^{1/2}} N_{1,n}.$$

Using the Birkhoff ergodic theorem, we see that

$$\lim_{n \rightarrow \infty} n^{\frac{1}{2}} \kappa_{1,n} = \frac{1}{6} (\mathbb{E}\sigma_0^{(2)})^{-3/2} \mathbb{E}\sigma_0^{(3)} \quad \text{a.s.} \quad (3.31)$$

By Taylor' expansion, $k_n < n^{\frac{1}{6}}$ and (3.31),

$$n|\mathbb{B}_{n41}| \leq \frac{2n\kappa_{1,n}}{s_n^2 - s_{k_n}^2} \left[\frac{s_{k_n}^4 t^2}{(s_n + (s_n^2 - s_{k_n}^2)^{1/2})^2} + k_n^2 \right] W_{k_n} \xrightarrow{n \rightarrow \infty} 0 \quad \text{a.s.} \quad (3.32)$$

From elementary calculus and (3.31), it follows that

$$\begin{aligned} n|\mathbb{B}_{n42}| &\leq \frac{tD_1'(t)n\kappa_{1,n}s_{k_n}^2}{(s_n + (s_n^2 - s_{k_n}^2)^{1/2})(s_n^2 - s_{k_n}^2)^{1/2}} W_{k_n} \xrightarrow{n \rightarrow \infty} 0 \quad \text{a.s.} \\ n|\mathbb{B}_{n43}| &\xrightarrow{n \rightarrow \infty} 0 \quad \text{a.s.} , \\ n|\mathbb{B}_{n44}| &\xrightarrow{n \rightarrow \infty} 0 \quad \text{a.s.} , \quad (\text{by (3.20)}) \\ n\mathbb{B}_{n45} &\xrightarrow{n \rightarrow \infty} -\frac{1}{6} (\mathbb{E}\sigma_0^{(2)})^{-2} \mathbb{E}\sigma_0^{(3)} D_1'(t) V_1 \quad \text{a.s.} \end{aligned}$$

Hence (3.17) is proved.

To prove (3.18), we observe that

$$\mathbb{B}_{n5} = \mathbb{B}_{n51} + \mathbb{B}_{n52} + \mathbb{B}_{n53}, \quad (3.33)$$

with

$$\begin{aligned} \mathbb{B}_{n51} &= \kappa_{2,n} \frac{1}{\Pi_{k_n}} \sum_{u \in \mathbb{T}_{k_n}} \mathbf{1}_{\{|S_u| \leq k_n\}} \left[D_2 \left(\frac{s_n t - S_u}{(s_n^2 - s_{k_n}^2)^{1/2}} \right) - D_2(t) \right], \\ \mathbb{B}_{n52} &= -\kappa_{2,n} D_2(t) \frac{1}{\Pi_{k_n}} \sum_{u \in \mathbb{T}_{k_n}} \mathbf{1}_{\{|S_u| > k_n\}}, \\ \mathbb{B}_{n53} &= \kappa_{2,n} D_2(t) W_{k_n}. \end{aligned}$$

The Birkhoff ergodic theorem gives that

$$\lim_{n \rightarrow \infty} n\kappa_{2,n} = \frac{1}{72} (\mathbb{E}\sigma_0^{(2)})^{-3} (\mathbb{E}\sigma_0^{(3)})^2. \quad (3.34)$$

Then

$$\begin{aligned} n|\mathbb{B}_{n51}| &\leq n\kappa_{2,n} W_{k_n} \sup_{|y| \leq k_n} \left| D_2 \left(\frac{s_n t - y}{(s_n^2 - s_{k_n}^2)^{1/2}} \right) - D_2(t) \right| \xrightarrow{n \rightarrow \infty} 0 \quad \text{a.s.} , \\ n\mathbb{B}_{n52} &\xrightarrow{n \rightarrow \infty} 0 \quad \text{a.s.} , \quad (\text{by (3.20)}) \\ n\mathbb{B}_{n53} &\xrightarrow{n \rightarrow \infty} \frac{1}{72} (\mathbb{E}\sigma_0^{(2)})^{-3} (\mathbb{E}\sigma_0^{(3)})^2 D_2(t) W, \end{aligned}$$

and hence (3.18) follows.

The proof of (3.19) is similar to that of (3.18) and we omit the details.

Now Lemma 3.3 has been proved. \square

The proof of Lemma 3.4 will be based on the following results.

Proposition 3.7 ([22]). *Assume the condition (2.1). Then*

$$W - W_n = o(n^{-\lambda}) \quad a.s.$$

Proposition 3.8. *Assume (2.1) and $\mathbb{E}(\ln^- m_0)^{1+\lambda} < \infty$ for some $\lambda > 4$, and $\mathbb{E}(|\widehat{L}_0|^\eta) < \infty$ for some $\eta > 2$. Then for each $\lambda' < \lambda - 1$,*

$$N_{1,n} - V_1 = o(n^{-\lambda'}). \quad (3.35)$$

Proof of Proposition 3.8. The argument is inspired by Asmussen(1976, [1]). The key idea is to find a proper truncation to show the convergence of the series $\sum_n a_n(N_{1,n+1} - N_{1,n})$ with suitable a_n , which gives the information on the convergence rate of $V_1 - N_{1,n}$. The proof relies on the following lemma.

Lemma 3.9. ([1], Lemma 2). *Let $\{\alpha_n, \beta_n, n \geq 1\}$ be sequences of real numbers. If $0 < \alpha_n \nearrow \infty$, and the series $\sum_{n=1}^{\infty} \alpha_n \beta_n$ converges, then*

$$\sum_{n=N}^{\infty} \beta_n = o\left(\frac{1}{\alpha_N}\right).$$

Using this lemma, we will obtain (3.35) once we prove that the series

$$\sum_{n=1}^{\infty} n^{\lambda'}(N_{1,n+1} - N_{1,n}) \quad \text{converges a.s.} \quad (3.36)$$

To this end, we shall use a truncating argument. We start by introducing some notation:

$$I_{1,n} := \frac{1}{\prod_n} \sum_{u \in \mathbb{T}_n} S_u(N_u/m_n - 1), \quad I_{1,n} := \frac{1}{\prod_n} \sum_{u \in \mathbb{T}_n} S_u(N_u/m_n - 1) \mathbf{1}_{\{N_u/m_n \leq n^{-\lambda'} \prod_n\}},$$

$$I_{2,n} := \frac{1}{\prod_n} \sum_{u \in \mathbb{T}_n} Y_u, \quad I'_{2,n} = \frac{1}{\prod_n} \sum_{u \in \mathbb{T}_n} Y_u \mathbf{1}_{\{|Y_u| \leq n^{-\lambda'} \prod_n\}} \quad \text{with} \quad Y_u = \frac{1}{m_{|u|}} \sum_{i=1}^{N_u} L_{ui}.$$

For $u \in \mathbb{T}_n$, let \widehat{N}_n be the generic random variable of N_u , i.e. \widehat{N}_n has the same distribution with N_u .

We shall prove the convergence a.s. of (3.36) by showing that both of the series

$$\sum_{n=1}^{\infty} n^{\lambda'} I_{1,n} \quad \text{and} \quad \sum_{n=1}^{\infty} n^{\lambda'} I_{2,n} \quad \text{converge a.s.} \quad (3.37)$$

We only need to show that the following series converges a.s.: for $q = 1, 2$,

$$\sum_{n=1}^{\infty} n^{\lambda'} (I_{q,n} - I'_{q,n}), \quad \sum_{n=1}^{\infty} n^{\lambda'} (I'_{q,n} - \mathbb{E}_{\xi, \mathcal{D}_n} I'_{q,n}), \quad \text{and} \quad \sum_{n=1}^{\infty} n^{\lambda'} \mathbb{E}_{\xi, \mathcal{D}_n} I'_{q,n}.$$

For the first series ($q = 1$), we observe that

$$\begin{aligned} \mathbb{E}_{\xi} n^{\lambda'} |I_{1,n} - I'_{1,n}| &= \mathbb{E}_{\xi} n^{\lambda'} \left| \frac{1}{\prod_n} \sum_{u \in \mathbb{T}_n} S_u \left(\frac{N_u}{m_n} - 1 \right) \mathbf{1}_{\{N_u/m_n > n^{-\lambda'} \prod_n\}} \right| \\ &\leq n^{\lambda'} \mathbb{E}_{\xi} \frac{1}{\prod_n} \sum_{u \in \mathbb{T}_n} \mathbb{E}_{\xi} |S_u| \mathbb{E}_{\xi} \left(\frac{N_u}{m_n} + 1 \right) \mathbf{1}_{\{N_u/m_n > n^{-\lambda'} \prod_n\}} \end{aligned}$$

$$\begin{aligned}
&\leq K_\xi n^{1+\lambda'} \mathbb{E}_\xi \left(\frac{\widehat{N}_n}{m_n} + 1 \right) \mathbf{1}_{\{\widehat{N}_n/m_n > n^{-\lambda'} \Pi_n\}} \\
&\leq K_\xi \frac{n^{1+\lambda'}}{\ln^{1+\lambda}(n^{-\lambda'}(\Pi_n + 1))} \mathbb{E}_\xi \left(\frac{\widehat{N}_n}{m_n} + 1 \right) \left(\ln \left(1 + \frac{\widehat{N}_n}{m_n} \right) \right)^{1+\lambda} \\
&\leq K_\xi n^{\lambda'-\lambda} \mathbb{E}_\xi \left(\frac{\widehat{N}_n}{m_n} + 1 \right) \left(\ln \left(1 + \frac{\widehat{N}_n}{m_n} \right) \right)^{1+\lambda}.
\end{aligned}$$

We see that for $\lambda > 1$,

$$\begin{aligned}
&\mathbb{E} \sum_{n=1}^{\infty} n^{\lambda'-\lambda} \left[\mathbb{E}_\xi \frac{\widehat{N}_n}{m_n} (\ln^+ \widehat{N}_n)^{1+\lambda} + (\ln^- m_n)^{1+\lambda} \right] \\
&= \sum_{n=1}^{\infty} n^{\lambda'-\lambda} \left[\mathbb{E} \frac{\widehat{N}_0}{m_0} (\ln^+ \widehat{N}_0)^{1+\lambda} + \mathbb{E} (\ln^- m_0)^{1+\lambda} \right] < \infty,
\end{aligned}$$

which implies that

$$\sum_{n=1}^{\infty} n^{\lambda'-\lambda} \left[\mathbb{E}_\xi \frac{\widehat{N}_n}{m_n} (\ln^+ \widehat{N}_n)^{1+\lambda} + (\ln^- m_n)^{1+\lambda} \right] < \infty \text{ a.s.} \quad (3.38)$$

Therefore

$$\begin{aligned}
\mathbb{E}_\xi \left| \sum_{n=1}^{\infty} n^{\lambda'} (I_{1,n} - I'_{1,n}) \right| &\leq \sum_{n=1}^{\infty} n^{\lambda'} \mathbb{E}_\xi |I_{1,n} - I'_{1,n}| < \infty, \\
\mathbb{E}_\xi \left| \sum_{n=1}^{\infty} n^{\lambda'} \mathbb{E}_{\xi, \mathcal{D}_n} I'_{1,n} \right| &= \mathbb{E}_\xi \left| \sum_{n=1}^{\infty} n^{\lambda'} \mathbb{E}_{\xi, \mathcal{D}_n} (I_{1,n} - I'_{1,n}) \right| \leq \sum_{n=1}^{\infty} n^{\lambda'} \mathbb{E}_\xi |I_{1,n} - I'_{1,n}| < \infty.
\end{aligned}$$

It follows that the series $\sum_{n=1}^{\infty} n^{\lambda'} (I_{1,n} - I'_{1,n})$ and $\sum_{n=1}^{\infty} n^{\lambda'} \mathbb{E}_{\xi, \mathcal{D}_n} I'_{1,n}$ converge a.s.

Observe that $\sum_{k=1}^n n^{\lambda'} (I'_{1,k} - \mathbb{E}_{\xi, \mathcal{D}_k} I'_{1,k})$ is a martingale w.r.t. $\{\mathcal{D}_{n+1}\}$. By the a.s. convergence of an L^2 bounded martingale (see e.g. [14, P. 251, Ex. 4.9]), we prove the convergence a.s. of the series $\sum_{n=1}^{\infty} n^{\lambda'} (I'_{1,n} - \mathbb{E}_{\xi, \mathcal{D}_n} I'_{1,n})$ by showing that of the series

$$\sum_{n=1}^{\infty} n^{2\lambda'} \mathbb{E}_\xi (I'_{1,n} - \mathbb{E}_{\xi, \mathcal{D}_n} I'_{1,n})^2.$$

This immediately results from (3.38) and the following estimate:

$$\begin{aligned}
&\mathbb{E}_\xi n^{2\lambda'} (I'_{1,n} - \mathbb{E}_{\xi, \mathcal{D}_n} I'_{1,n})^2 \\
&= n^{2\lambda'} \mathbb{E}_\xi \left[\frac{1}{\Pi_n} \sum_{u \in \mathbb{T}_n} S_u \left(\left(\frac{N_u}{m_n} - 1 \right) \mathbf{1}_{\{\frac{N_u}{m_n} \leq \frac{\Pi_n}{n^{\lambda'}}\}} - \mathbb{E}_{\xi, \mathcal{D}_n} \left(\frac{N_u}{m_n} - 1 \right) \mathbf{1}_{\{\frac{N_u}{m_n} \leq \frac{\Pi_n}{n^{\lambda'}}\}} \right) \right]^2 \\
&= n^{2\lambda'} \mathbb{E}_\xi \left[\frac{1}{\Pi_n^2} \sum_{u \in \mathbb{T}_n} S_u^2 \mathbb{E}_{\xi, \mathcal{D}_n} \left(\left(\frac{N_u}{m_n} - 1 \right) \mathbf{1}_{\{\frac{N_u}{m_n} \leq \frac{\Pi_n}{n^{\lambda'}}\}} - \mathbb{E}_{\xi, \mathcal{D}_n} \left(\frac{N_u}{m_n} - 1 \right) \mathbf{1}_{\{\frac{N_u}{m_n} \leq \frac{\Pi_n}{n^{\lambda'}}\}} \right) \right]^2 \\
&\leq n^{2\lambda'} \mathbb{E}_\xi \left[\frac{1}{\Pi_n^2} \sum_{u \in \mathbb{T}_n} S_u^2 \mathbb{E}_\xi \left(\frac{N_u}{m_n} - 1 \right)^2 \mathbf{1}_{\{\frac{N_u}{m_n} \leq \frac{\Pi_n}{n^{\lambda'}}\}} \right]
\end{aligned}$$

$$\begin{aligned}
&\leq n^{2\lambda'} \mathbb{E}_\xi \left[\frac{1}{\prod_n^2} \sum_{u \in \mathbb{T}_n} \mathbb{E}_\xi S_u^2 \left[\mathbb{E}_\xi \left(\frac{N_u}{m_n} \right)^2 \mathbf{1}_{\left\{ \frac{N_u}{m_n} \leq \frac{\Pi_n}{n^{\lambda'}} \right\}} + 3 \right] \right] \\
&= n^{2\lambda'} \frac{s_n^2}{\prod_n} \mathbb{E}_\xi \left(\frac{\widehat{N}_n}{m_n} \right)^2 \mathbf{1}_{\left\{ \frac{\widehat{N}_n}{m_n} \leq \frac{\Pi_n}{n^{\lambda'}} \right\}} + \frac{3s_n^2}{\prod_n} \\
&= n^{2\lambda'} \frac{s_n^2}{\prod_n} \mathbb{E}_\xi \left[\left(\frac{\widehat{N}_n}{m_n} \right)^2 \left(\mathbf{1}_{\left\{ \frac{\widehat{N}_n}{m_n} \leq \min(e^{2\lambda}, \frac{\Pi_n}{n^{\lambda'}}) \right\}} + \mathbf{1}_{\left\{ e^{2\lambda} < \frac{\widehat{N}_n}{m_n} \leq \frac{\Pi_n}{n^{\lambda'}} \right\}} \right) \right] + n^{2\lambda'} \frac{3s_n^2}{\prod_n} \\
&\leq n^{2\lambda'} (3 + e^{4\lambda}) \frac{s_n^2}{\prod_n} + \\
&\quad n^{2\lambda'} \frac{s_n^2}{\prod_n} \left(\frac{\Pi_n}{n^{\lambda'}} + 1 \right) \ln^{-1-\lambda} \left(1 + \frac{\Pi_n}{n^{\lambda'}} \right) \mathbb{E}_\xi \left(\frac{\widehat{N}_n}{m_n} \right)^2 \left(\left(1 + \frac{\widehat{N}_n}{m_n} \right) \ln^{-(1+\lambda)} \left(1 + \frac{\widehat{N}_n}{m_n} \right) \right)^{-1} \\
&\quad \text{(because } x(\ln x)^{-1-\lambda} \text{ is increasing for } x > e^{2\lambda} \text{)} \\
&\leq (3 + e^{4\lambda}) \frac{n^{2\lambda'} s_n^2}{\prod_n} + K_\xi \frac{n^{\lambda'} s_n^2}{\ln^{1+\lambda} \left(1 + \frac{\Pi_n}{n^{\lambda'}} \right)} \mathbb{E}_\xi \frac{\widehat{N}_n}{m_n} \ln^{1+\lambda} \left(1 + \frac{\widehat{N}_n}{m_n} \right) \\
&\leq \frac{K_\xi n^{2\lambda'+1}}{\prod_n} + K_\xi \frac{1}{n^{\lambda-\lambda'}} \mathbb{E}_\xi \frac{\widehat{N}_n}{m_n} \ln^{1+\lambda} \left(1 + \frac{\widehat{N}_n}{m_n} \right).
\end{aligned}$$

Combining the above results, we see that the series $\sum I_{1,n}$ converges a.s.

Next we turn to the proof of the convergence a.s. of the series $\sum n^{\lambda'} I_{2,n}$.

To begin with, we prove that

$$\mathbb{E}_\xi |Y_u| (\ln^+ |Y_u|)^{1+\lambda} \leq K_\xi n + K_\xi n (\ln^- m_n)^{1+\lambda}. \quad (3.39)$$

This follows from the fact:

$$\begin{aligned}
\mathbb{E}_\xi |Y_u| (\ln^+ |Y_u|)^{1+\lambda} &\leq \mathbb{E}_\xi \left[\frac{1}{m_n} \left| \sum_{i=1}^{N_u} L_{ui} \right| \left(\ln^+ \left| \sum_{i=1}^{N_u} L_{ui} \right| + \ln^- m_n \right)^{1+\lambda} \right] \\
&\leq \mathbb{E}_\xi \left[\frac{1}{m_n} \left| \sum_{i=1}^{N_u} L_{ui} \right| 2^\lambda \left(\left(\ln^+ \left| \sum_{i=1}^{N_u} L_{ui} \right| \right)^{1+\lambda} + (\ln^- m_n)^{1+\lambda} \right) \right] \\
&\stackrel{(\text{??})}{\leq} K_\lambda \frac{1}{m_n} \mathbb{E}_\xi \left| \sum_{i=1}^{N_u} L_{ui} \right|^2 + 2^\lambda (\ln^- m_n)^{1+\lambda} \frac{1}{m_n} \mathbb{E}_\xi \left| \sum_{i=1}^{N_u} L_{ui} \right| \\
&\leq K_\lambda \frac{1}{m_n} \mathbb{E}_\xi \sum_{i=1}^{N_u} \mathbb{E}_\xi |L_{ui}|^2 + 2^\lambda (\ln^- m_n)^{1+\lambda} \frac{1}{m_n} \mathbb{E}_\xi \sum_{i=1}^{N_u} \mathbb{E}_\xi |L_{ui}| \\
&\leq K_\xi n + K_\xi n (\ln^- m_n)^{1+\lambda}.
\end{aligned}$$

Observe that

$$\begin{aligned}
\mathbb{E}_\xi n^{\lambda'} |I_{2,n} - I'_{2,n}| &= n^{\lambda'} \mathbb{E}_\xi \left| \frac{1}{\prod_n} \sum_{u \in \mathbb{T}_n} Y_u \mathbf{1}_{\left\{ |Y_u| > \frac{\Pi_n}{n^{\lambda'}} \right\}} \right| \\
&\leq n^{\lambda'} \mathbb{E}_\xi \frac{1}{\prod_n} \sum_{u \in \mathbb{T}_n} |Y_u| \mathbf{1}_{\left\{ |Y_u| > \frac{\Pi_n}{n^{\lambda'}} \right\}}
\end{aligned}$$

$$\begin{aligned}
&\leq n^{\lambda'} \mathbb{E}_\xi \frac{1}{\Pi_n (\ln \frac{\Pi_n}{n^{\lambda'}})^{1+\lambda}} \sum_{u \in \mathbb{T}_n} \mathbb{E}_\xi |Y_u| (\ln^+ |Y_u|)^{1+\lambda} \\
&\stackrel{(3.39)}{\leq} \frac{K_\xi n + K_\xi n (\ln^- m_n)^{1+\lambda}}{(\ln \frac{\Pi_n}{n^{\lambda'}})^{1+\lambda}} \\
&\leq K_\xi n^{\lambda'-\lambda} (1 + (\ln^- m_n)^{1+\lambda})
\end{aligned}$$

By (3.38), the series $\sum n^{\lambda'-\lambda} (1 + (\ln^- m_n)^{1+\lambda})$ converges a.s. Thus

$$\begin{aligned}
\mathbb{E}_\xi \left| \sum_{n=1}^{\infty} n^{\lambda'} (I_{2,n} - I'_{2,n}) \right| &\leq \sum_{n=1}^{\infty} n^{\lambda'} \mathbb{E}_\xi |I_{2,n} - I'_{2,n}| < \infty, \\
\mathbb{E}_\xi \left| \sum_{n=1}^{\infty} n^{\lambda'} \mathbb{E}_{\xi, \mathcal{D}_n} I'_{2,n} \right| &= \mathbb{E}_\xi \left| \sum_{n=1}^{\infty} n^{\lambda'} \mathbb{E}_{\xi, \mathcal{D}_n} (I_{2,n} - I'_{2,n}) \right| \leq \sum_{n=1}^{\infty} n^{\lambda'} \mathbb{E}_\xi |I_{2,n} - I'_{2,n}| < \infty.
\end{aligned}$$

This implies the convergence a.s. of the series $\sum_{n=1}^{\infty} n^{\lambda'} (I_{2,n} - I'_{2,n})$ and $\sum_{n=1}^{\infty} n^{\lambda'} \mathbb{E}_{\xi, \mathcal{D}_n} I'_{2,n}$.

To prove the convergence a.s. of the series $\sum_{n=1}^{\infty} n^{\lambda'} (I'_{2,n} - \mathbb{E}_{\xi, \mathcal{D}_n} I'_{2,n})$, we only need to show the convergence of the series: $\sum_{n=1}^{\infty} \mathbb{E}_\xi n^{2\lambda'} (I'_{2,n} - \mathbb{E}_{\xi, \mathcal{D}_n} I'_{2,n})^2$. This is implied by (3.38) the following observation:

$$\begin{aligned}
\mathbb{E}_\xi n^{2\lambda'} (I'_{2,n} - \mathbb{E}_{\xi, \mathcal{D}_n} I'_{2,n})^2 &= \mathbb{E}_\xi \frac{n^{2\lambda'}}{\Pi_n^2} \sum_{u \in \mathbb{T}_n} \mathbb{E}_\xi \left(|Y_u| \mathbf{1}_{\{|Y_u| \leq \frac{\Pi_n}{n^{\lambda'}}\}} - (\mathbb{E}_\xi |Y_u| \mathbf{1}_{\{|Y_u| \leq \frac{\Pi_n}{n^{\lambda'}}\}}) \right)^2 \\
&\leq \mathbb{E}_\xi \frac{n^{2\lambda'}}{\Pi_n^2} \sum_{u \in \mathbb{T}_n} \mathbb{E}_\xi \left(|Y_u|^2 \mathbf{1}_{\{|Y_u| \leq \frac{\Pi_n}{n^{\lambda'}}\}} \right) \\
&\leq \mathbb{E}_\xi \frac{n^{2\lambda'}}{\Pi_n^2} \sum_{u \in \mathbb{T}_n} \mathbb{E}_\xi \left(|Y_u|^2 \mathbf{1}_{\{|Y_u| \leq \min(e^{2\lambda}, \frac{\Pi_n}{n^{\lambda'}})\}} + |Y_u|^2 \mathbf{1}_{\{e^{2\lambda} < |Y_u| \leq \frac{\Pi_n}{n^{\lambda'}}\}} \right) \\
&\leq \frac{e^{4\lambda} n^{2\lambda'}}{\Pi_n} + \frac{n^{2\lambda'}}{\Pi_n^2} \frac{\Pi_n}{n^{\lambda'}} (\ln \frac{\Pi_n}{n^{\lambda'}})^{-1-\lambda} \mathbb{E}_\xi \sum_{u \in \mathbb{T}_n} \mathbb{E}_\xi |Y_u|^2 (|Y_u| (\ln^+ |Y_u|)^{-1-\lambda})^{-1} \\
&\quad (\text{because } x(\ln x)^{-1-\lambda} \text{ is increasing for } x > e^{2\lambda}) \\
&= \frac{e^{4\lambda} n^{2\lambda'}}{\Pi_n} + \frac{n^{\lambda'}}{\Pi_n (\ln \frac{\Pi_n}{n^{\lambda'}})^{1+\lambda}} \mathbb{E}_\xi \sum_{u \in \mathbb{T}_n} \mathbb{E}_\xi |Y_u| (\ln^+ |Y_u|)^{1+\lambda} \\
&\stackrel{(3.39)}{\leq} \frac{e^{4\lambda} n^{2\lambda'}}{\Pi_n} + \frac{K_\xi n^{1+\lambda'} (1 + (\ln^- m_n)^{1+\lambda})}{(\ln \frac{\Pi_n}{n^{\lambda'}})^{1+\lambda}} \\
&\leq \frac{e^{4\lambda} n^{2\lambda'}}{\Pi_n} + K_\xi n^{\lambda'-\lambda} (1 + (\ln^- m_n)^{1+\lambda}).
\end{aligned}$$

Combining the above results, we see that the series $\sum I_{2,n}$ converges a.s.

Therefore we have proved (3.37) and hence the proof of the Proposition is completed. \square

Proof of Lemma 3.4. Observe that

$$n\mathbb{C}_n = n(W_{k_n} - W) \left(\Phi(t) + \frac{s_n^{(3)}}{6s_n^3} (1 - t^2) \phi(t) \right) + \frac{\sqrt{n}}{s_n} \sqrt{n} (V_1 - N_{1,k_n}) \phi(t).$$

Since $\lambda\beta > 1$, $2\beta(\lambda - 1) > 1$, we can take λ' such that $\frac{1}{2\beta} < \lambda' < \lambda - 1$. Hence Lemma 3.4 follows from Propositions 3.7 and 3.8. \square

Now the main theorem follows from (3.2) and Lemmas 3.2 – 3.4.

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