NO RIEMANN-HURWITZ FORMULA FOR THE p-RANKS OF RELATIVE CLASS GROUPS

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ABSTRACT. We disprove, by means of numerical examples, the existence of a Riemann-Hurwitz formula for the p-ranks of relative class groups in a p-ramified p-extension K/k of number fields of CM-type containing μ_p. In the cyclic case of degree p, under some assumptions on the p-class group of k, we prove some properties of the Galois structure of the p-class group of K; but we have found, through numerical experimentation, that some theoretical group structures do not exist in this particular situation, and we justify this fact. Then we show, in this context, that Kida's formula on lambda invariants is valid for the p-ranks if and only if the p-class group of K is reduced to the group of ambiguous classes, which is of course not always the case.

December 14, 2015

1. Generalities

In 1980, Y. Kida [Ki] proved an analogue of the Riemann-Hurwitz formula for the minus part of the Iwasawa λ-invariant of the cyclotomic $\mathbb{Z}_p$-extension k_∞ of an algebraic number field k of CM-type with maximal totally real subfield k^+:

$$\lambda^-(K) - 1 = [K_\infty : k_\infty]\ (\lambda^-(k) - 1) + \sum_{v \nmid p} (e_v(K_\infty^+/k_\infty^+) - 1),$$

where K/k is a finite p-extension of CM-fields containing the group μ_p of pth roots of unity and where e_v denotes the ramification index of the place v. When K/k is p-ramified (i.e., unramified outside p) and such that $K \cap k_\infty = k$ the formula reduces to:

$$\lambda^-(K) - 1 = [K : k]\,(\lambda^-(k) - 1).$$

Many generalizations where given as in [JaMa], [JaMi], [Sch], among many others.

An interesting question is to ask if such a formula can be valid for the p-ranks of the relative ideal class groups in a p-extension K/k (e.g. K/k cyclic of degree p).

In a work, using Iwasawa theory and published in 1996 by K. Wingberg [W], such a formula is proposed (Theorem 2.1, Corollary 2.2) in a very general framwork and applied to the case of a p-ramified p-extension K/k of CM-fields containing μ_p.

As many people remarked, we can be astonished by a result which is not really "arithmetical" since many of our class groups investigations (as in [Gr2], [Gr3]) show that such a "regularity" only happens at infinity (Iwasawa theory). The proof of Kida's formula given by W. Sinnott [Sin], using p-adic L-functions, is probably the most appropriate to see the transition from one aspect to the other.

Indeed, to find a relation between the p-ranks (for instance in a cyclic extension K/k of degree p) depends for $G := \mathrm{Gal}(K/k) =: \langle\sigma\rangle$ on non obvious structures of finite $\mathbb{Z}_p[G]$-modules M provided with an arithmetical norm $\mathrm{N}_{K/k}$ and a transfert map $j_{K/k}$ (with $j_{K/k} \circ \mathrm{N}_{K/k} = 1 + \sigma + \ldots + \sigma^{p-1}$), where the filtration of the $M_i := \{h \in M,\ h^{(\sigma-1)^i} = 1\}$ plays an important non-algebraic role because all the orders $\#(M_{i+1}/M_i)$ depend on *arithmetical local normic computations* by means of formulas, given in [Gr2], similar to that of the case $i = 0$ of ambiguous ideal classes (see Section 3).

1991 *Mathematics Subject Classification.* Primary 11R ; Secondary ,11R29 11R37, 11S15, 11S20 .

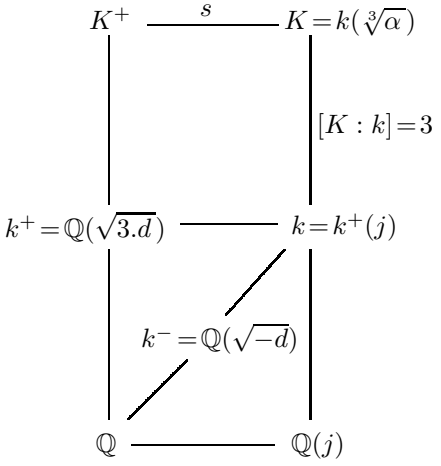
Key words and phrases. ideal class groups; class field theory; ambiguous classes; p-ramification; Riemann-Hurwitz formula; Kida's formula.

To be more convincing, we have given numerical computations and we shall see that it is not difficult to conjecture that there are infinitely many counterexamples to a formula, for the relative p-ranks, such as $r_K^- - 1 = p\,(r_k^- - 1)$ wich may be true in some cases.

2. A numerical counterexample

Using PARI (from [P]), we give a program which can be used by the reader to compute easily for the case $p = 3$ in a biquadratique field containing a primitive 3th root of unity j, and such that its 3-class group is of order 3.

2.1. **Definition and assumptions.** Consider the following diagram:



We recall, in this particular context, the hypothesis of the statement of Corollary 2.2 (ii) of [W] and we shall suppose these conditions satisfied in all the sequel. For any field F, let $\mathcal{C}\ell_F$ be its 3-class group and let $\mathcal{C}\ell_F^{\pm}$ be its two usual components when F is a CM-field. We denote by H_F the 3-Hilbert class field of F.

Hypothesis 2.1. (i) $d > 0$, d squarefree, $d \not\equiv 0 \pmod 3$,
(ii) $p = 3$ does not split in $k/\mathbb{Q}$ (hence $d \equiv 1 \pmod 3$),
(iii) K^+/k^+ is a 3-ramified cubic cyclic extension and $K = K^+(j)$,
(iv) K^+ is not contained in the cyclotomic $\mathbb{Z}_3$-extension of k^+,
(v) $\mathcal{C}\ell_{k^+} = 1$ & $\mathcal{C}\ell_{k^-} \simeq \mathbb{Z}/3\mathbb{Z}$.

From (v), the ambiguous class number formula implies $\mathcal{C}\ell_{K^+} = 1$ (Lemma 2.2).

Starting from a p-ramified cubic cyclic extension K^+/k^+, the associated Kummer extension K/k is defined by $\sqrt[3]{\alpha}$ such that $\alpha \in k^\times \setminus k^{\times 3}$, $(\alpha) = \mathfrak{a}^3$ for an ideal $\mathfrak{a}$ of k, and $\alpha^{s+1} \in k^{\times 3}$ where $s \in \mathrm{Gal}(K/K^+)$ is the complex conjugation (usual decomposition criterion of a Kummer extension over a subfield).

If the 3-rank of the 3-class group $\mathcal{C}\ell_k^- \simeq \mathcal{C}\ell_{k^-}$ is r^-, the 3-rank of the Galois group of the maximal Abelian 3-ramified 3-extension of k^+ is $r^- + 1$ when 3 does not split in k (see [Gr1], Proposition III.4.2.2 for the general statement). From (iv), necessarily $r^- \geq 1$. Here we suppose $\#\mathcal{C}\ell_{k^-} = 3$, hence $r^- + 1 = 2$ (this is also equivalent to the non-nullity of the 3-torsion subgroup $\mathcal{T}_3$, of the above Galois group, whose order is in our context $\#\mathcal{T}_3 \sim \log_3(\varepsilon)/\sqrt{3.d}$ where ε is the fundamental unit of k^+, see [Gr1], Remark III.2.6.5 (i)). We shall precise the choice of the extension K^+/k^+ (i.e., α) as follows:

Let α in k^- be such that $(\alpha) = \mathfrak{a}^3$ ($\mathfrak{a}$ must be a non-principal ideal since E_{k^-} is trivial); this defines a canonical cubic cyclic extension K^+ of k^+ and the numbers $\alpha\, j$, $\alpha\, j^2$ define the two other cubic cyclic extensions K_1^+, K_2^+ of k^+, distinct from the first step of the cyclotomic $\mathbb{Z}_3$-extension K_0^+ of k^+ defined by j.

Lemma 2.2. *(i) We have* $\#\mathcal{C}\ell_{K^+} = 1$ & $\#\mathcal{C}\ell_K^G = \#\mathcal{C}\ell_k = 3$, *where* $G = \mathrm{Gal}(K/k)$.
(ii) We have $\#\mathcal{C}\ell_{H_k} = 1$.

Proof. (i) Using the Chevalley's formula in K^+/k^+ (see e.g. [Gr1], Lemma II.6.1.2) with a trivial 3-class group for k^+, the formula reduces to

$$\#\mathcal{C}\ell_{K^+}^G = \frac{3}{3\,.\,(E_{k^+} : E_{k^+} \cap \mathrm{N}_{K^+/k^+} K^{+\times})} = 1,$$

since the product of ramification indices is equal to 3 ($\mathcal{C}\ell_{H_k} = 1$ implies that K^+/k^+ is necessarily totally ramified at the single prime above 3 of k^+).

The same formula in K/k is $\#\mathcal{C}\ell_K^G = \dfrac{\#\mathcal{C}\ell_k \times 3}{3\,.\,(E_k : E_k \cap \mathrm{N}_{K/k} K^\times)}$ with $E_k = \langle \varepsilon, j\rangle$, where ε is the fundamental unit of k^+; but, as for K^+/k^+, there is by assumption a single place of k ramified in K/k, thus, using the product formula, the Hasse norm theorem shows that all these units are local norms everywhere hence global norms. So $\#\mathcal{C}\ell_K^G = 3$ since $\#\mathcal{C}\ell_k = \#\mathcal{C}\ell_{k^-} = 3$.

(ii) We have $\#\mathcal{C}\ell_{H_k}^{G'} = \dfrac{\#\mathcal{C}\ell_k}{3\,.\,(E_k : E_k \cap \mathrm{N}_{H_k/k} H_k^\times)} = 1$ where $G' := \mathrm{Gal}(H_k/k)$. □

Corollary 2.3. *We have $\mathcal{C}\ell_K = \mathcal{C}\ell_K^-$ since $\mathcal{C}\ell_K^+ \simeq \mathcal{C}\ell_{K^+} = 1$.*

2.2. **Numerical data.** We have a first example with $d = 211 \equiv 1 \pmod 3$ and $\alpha = \frac{17+\sqrt{-211}}{2}$ where $(\alpha) = \mathfrak{p}_5^3$ for a non-principal prime ideal dividing 5 in k^-.

The class number of k^+ is 1 and that of k^- is 3, which is coherent with the fact that the fundamental unit $\varepsilon = 440772247 + 17519124\,\sqrt{3.211}$ of k^+ is 3-primary (indeed, $\varepsilon \equiv 1 + 3.\sqrt{3.211} \pmod 9$), which implies that H_{k^-} is given via $k(\sqrt[3]{\varepsilon})/k$ which is decomposed by means of an unramified cubic cyclic extension of k^-.

So all the five conditions (i) to (v) are fulfilled.

The PARI program (see Section 5) gives in "*component*$(H, 5)$" the class number and the structure of the whole class group of K; the program needs an irreducible polynomial defining K; it is given by "$P = polcompositum(x^2 + x + 1, Q)$" where $Q = x^6 - 17\,x^3 + 5^3$ is the irreducible polynomial of $\sqrt[3]{\alpha}$ over $\mathbb{Q}$ (the general formula is $Q = x^6 - \mathrm{Tr}_{k^-/\mathbb{Q}}(\alpha)\,x^3 + \mathrm{N}_{k^-/\mathbb{Q}}(\alpha)$); one obtains:

$P = x^{12} - 6\,x^{11} + 21\,x^{10} - 84\,x^9 + 243\,x^8 - 432\,x^7 + 1037\,x^6 - 1896\,x^5 - 204\,x^4 - 966\,x^3 + 5949\,x^2 + 4905\,x + 11881.$

2.3. **Conclusion.** The program gives $\mathcal{C}\ell_K \simeq \mathbb{Z}/9\,\mathbb{Z} \times \mathbb{Z}/3\,\mathbb{Z}$ for a class number equal to 27. This yields the 3-rank $R^- = 2$ of $\mathcal{C}\ell_K^-$ when the 3-rank r^- of $\mathcal{C}\ell_k^-$ is equal to 1, which is incompatible with the formula

$$R^- - 1 = 3 \times (r^- - 1).$$

But, this "Riemann-Hurwitz formula" is valid if and only if $\mathcal{C}\ell_K = \mathcal{C}\ell_K^G \simeq \mathbb{Z}/3\,\mathbb{Z}$ (no exceptional 3-classes). Such a case is also very frequent (see § 5.1).

3. Some structural results

Denote by M a finite $\mathbb{Z}_p[\Gamma]$-module, where we assume that Γ is an Abelian Galois group of the form $G \times g$, where $G = \mathrm{Gal}(K/k) =: \langle\sigma\rangle$ is cyclic of order p and $g \simeq \mathrm{Gal}(k/k_0)$ (of order prime to p), where k_0 is a suitable subfield of k (so that $K = k\,K_0$ with $K_0 := K^g$). The existence of g allows us to take isotypic components of M (as the $\pm$-components when the fields are of CM-type). In our example, $g = \langle s\rangle$, $k_0 = k^+$ and $K_0 = K^+$.

We have $M/M^p = M/M^{1+\sigma+\ldots+\sigma^{p-1}-\Omega}$ where $\Omega = u\,(\sigma-1)^{p-1}$ for an inversible element u of the group algebra $\mathbb{Z}_p[G]$ ([Gr3], Proposition 4.1); in our case $p = 3$, $\Omega = \sigma^2(\sigma-1)^2$. We shall use:

$$\omega := \sigma\,(\sigma - 1), \quad \text{such that } \omega^2 \equiv 3 \pmod \nu, \quad \text{where } \nu := 1 + \sigma + \sigma^2.$$

For our purpose we shall have $M = \mathcal{C}\ell_K^-$ (we refere to [Gr3], Chap. IV, A, § 2).

By class field theory, when K/k is totally ramified at the unique $\mathfrak{p} \mid 3$, the arithmetical norm $\mathrm{N}_{K/k} : \mathcal{C}\ell_K^- \longrightarrow \mathcal{C}\ell_k^-$ is surjective.

Another important fact for the structure of $\mathcal{C}\ell_K$, in our particular context, is that the class of order 3 of k capitulates in K because the equality $(\alpha)=\mathfrak{a}^3$ becomes $(\sqrt[3]{\alpha})=(\mathfrak{a})_K$ in K (the transfert map $j_{K/k}:\mathcal{C}\ell_k^-\longrightarrow\mathcal{C}\ell_K^-$ is not injective). This has the following tricky consequence:

$$\mathrm{N}_{K/k}(\mathcal{C}\ell_K^-)=\mathcal{C}\ell_k^- \quad\&\quad (\mathcal{C}\ell_K^-)^\nu=1.$$

Return to the general case $M=\mathcal{C}\ell_K$ for any prime p and suppose $\#M^G=p$. Put $M_i:=\{h\in M,\ h^{(\sigma-1)^i}=1\}$, $i\geq 0$, and let n be the least integer i such that $M_i=M$.

From the exact sequence $1\to M_1=M^G\longrightarrow M_{i+1}\xrightarrow{\omega}M_{i+1}^\omega\subseteq M_i\to 1$, with $\#M^G=p$, we obtain that $M_{i+1}^\omega=M_i$ and $\#(M_{i+1}/M_i)=p$ for $i=0,\ldots,n-1$. From [Gr3], Proposition 4.1 and Corollaire 4.3, assuming $M^\nu=1$ and $\#(M_{i+1}/M_i)=p$ for all $i<n$, we obtain the following structure of $\mathbb{Z}$-module:

$$M\simeq(\mathbb{Z}/p^{a+1}\mathbb{Z})^b\times(\mathbb{Z}/p^a\mathbb{Z})^{p-1-b},$$

where $n=a\,(p-1)+b$, $0\leq b\leq p-2$. This implies (assuming $M^+=1$) that the p-rank R^- of M^- is $R^-=p-1$ if $a\geq 1$ and $R^-=b$ if $a=0$ (i.e., $b=n\leq p-2$).

So, in the case $r^-=1$, we have the Riemann-Hurwitz formula $R^--1=p\,(r^--1)$ if and only if $b=n=1$ which is equivalent to $M^-=M^{-G}$. Otherwise, R^- can take any value in $[1,p-1]$, even if $r^-=1$.

This general isomorphism comes from the formula ([Gr2], Corollaire 2.8):

$$\#\big(M_{i+1}/M_i\big)^G=\frac{\#\mathcal{C}\ell_k\times\prod e_v}{[K:k]\,.\,\#\mathrm{N}_{K/k}(M_i)\,.\,(\Lambda_i:\Lambda_i\cap\mathrm{N}_{K/k}(K^\times))},$$

where $\mathrm{N}_{K/k}(M_i):=\mathcal{C}\ell_k(\mathrm{N}_{K/k}(I_i))$ for a suitable ideal group I_i such that $\mathcal{C}\ell_K(I_i)=M_i$, and where $\Lambda_i=\{x\in k^\times,\ (x)\in\mathrm{N}_{K/k}(I_i)\}$.

In our case there is a single ramified place $\mathfrak{p}\,|\,3$ and the elements of Λ_i, being norms of ideals, are everywhere local norms except perhaps at $\mathfrak{p}$; so $(\Lambda_i:\Lambda_i\cap\mathrm{N}_{K/k}(K^\times))=1$ under the product formula of class field theory and the Hasse norm theorem, and $\#\big(M_{i+1}/M_i\big)^G=\frac{3}{\#\mathrm{N}_{K/k}(M_i)}$.

In our numerical example with $d=211$, we get necessarily $a=b=1$, $n=3$, giving the structure $\mathbb{Z}/9\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}$.

For more structural results when $M^\nu\neq 1$, see [Gr3], Chap. IV, § 2, Proposition 4.3 valid for any $p\geq 2$. In our biquadratic case and $p=3$, we obtain interesting structures for which a theoretical study should be improved. From the general PARI program we have obtained the following numerical examples:

(i) For $d=1759$, for which $\#\mathcal{C}\ell_{k^+}=1$, $\mathcal{C}\ell_{k^-}\simeq\mathbb{Z}/27\mathbb{Z}$, and $\alpha=37+20\sqrt{-d}$ of norm 89^3, the structure is $\mathcal{C}\ell_K^-\simeq\mathbb{Z}/27\mathbb{Z}$ (i.e., $\mathcal{C}\ell_K^-=(\mathcal{C}\ell_K^-)^G$).

(ii) For $d=2047$, for which $\#\mathcal{C}\ell_{k^+}=1$, $\mathcal{C}\ell_{k^-}\simeq\mathbb{Z}/9\mathbb{Z}$, and $\alpha=332+11\sqrt{-d}$ of norm 71^3, the structure is $\mathcal{C}\ell_K^-\simeq\mathbb{Z}/9\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}$.

(iii) For $d=1579$, for which $\#\mathcal{C}\ell_{k^+}=1$, $\mathcal{C}\ell_{k^-}\simeq\mathbb{Z}/9\mathbb{Z}$, and $\alpha=\frac{1}{2}(115+3\sqrt{-d})$ of norm 19^3, the structure is $\mathcal{C}\ell_K^-\simeq\mathbb{Z}/27\mathbb{Z}\times\mathbb{Z}/9\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}$.

4. The structure $M\simeq\mathbb{Z}/3^a\mathbb{Z}\times\mathbb{Z}/3^a\mathbb{Z}$ does not exist

Of course, we keep the same numerical assumptions about the 3-class groups of k^+ and k^- (especially $\mathcal{C}\ell_{k^-}\simeq\mathbb{Z}/3\mathbb{Z}$), the non-splitting of 3 in $k/\mathbb{Q}$, and the Kummer construction of the 3-ramified cubic cyclic extension K/k.

Theorem 4.1. *Under all the Hypothesis 2.1, we get*

$$\mathcal{C}\ell_K=\mathcal{C}\ell_K^-\simeq\mathbb{Z}/3^{a+1}\mathbb{Z}\times\mathbb{Z}/3^a\mathbb{Z},$$

for some $a\geq 0$. The case $a=0$ is equivalent to $\mathcal{C}\ell_K=\mathcal{C}\ell_K^G$.

Proof. Let $n \geq 1$ be the least integer such that $M_n = M := \mathcal{C}\ell_K$. From the relations $M_{i+1}/M_i \simeq \mathbb{Z}/3\mathbb{Z}$, for $0 \leq i \leq n-1$, we get $\#M = 3^n$.

Let $h_n \in M$ be such that $\mathrm{N}_{K/k}(h_n)$ generates $\mathcal{C}\ell_k$ (equivalent to $h_n \in M_n \setminus M_{n-1}$). Since $M_{i+1}^{\omega} = M_i$ for $0 \leq i \leq n-1$, M_n is the $\mathbb{Z}_p[G]$-module generated by h_n and for all i, $0 \leq i \leq n-1$, $h_{n-i} := h_n^{\omega^i}$ generates M_{n-i}. We have $h^{\omega^2} = h^3$ for all $h \in M$.

Let m, $1 \leq m \leq n$; the structure of M_m is given (as for M, see Section 3) by:

$$M_m \simeq (\mathbb{Z}/3^{a_m+1}\mathbb{Z})^{b_m} \times (\mathbb{Z}/3^{a_m}\mathbb{Z})^{2-b_m}, \;\; m = 2\,a_m + b_m,\, b_m \in \{0,1\}.$$

(i) Case $m = 2\,e$. Then $M_m \simeq \mathbb{Z}/3^e\mathbb{Z} \times \mathbb{Z}/3^e\mathbb{Z}$.

(ii) Case $m = 2\,e+1$. Then $M_m \simeq \mathbb{Z}/3^{e+1}\mathbb{Z} \times \mathbb{Z}/3^e\mathbb{Z}$.

With the previous notations we have, for the two cases:

$$M_m = \langle h_m \rangle \oplus \langle h_{m-1} \rangle, \quad \text{as } \mathbb{Z}\text{-module.}$$

Indeed, suppose that $h = h_m^A = h_{m-1}^B$, $A, B \in \mathbb{Z}$. Then $h_m^{A - B\,\omega} = 1$. Since h_m is anihilated by ω^m and not by ω^{m-1}, we get $A - B\,\omega \in (\omega^m)$.

In the case $m = 2e$, $\omega^m \equiv 3^e \pmod{\nu}$ giving in the algebra $\mathbb{Z}_3[G] = \mathbb{Z}_3[\omega]$, the relations $A \equiv B \equiv 0 \pmod{3^e}$, hence $h = 1$. Since $\#M_{2e} = 3^{2e}$, the elements h_m and h_{m-1} are independent of order 3^e.

In the case $m = 2e+1$, $\omega^m \equiv 3^e\omega \pmod{\nu}$ and $A - B\,\omega \equiv 0 \pmod{3^e\omega}$, giving $A = 3^e\,A'$ and $B = 3^e\,B'$, then $A' - B'\omega \equiv 0 \pmod{\omega\,\mathbb{Z}_3[\omega]}$, which implies $A' \equiv 0 \pmod 3$ hence the result in this case with h_m of order 3^{e+1} and h_{m-1} of order 3^e.

Suppose that the structure of M is $M_{2e} \simeq \mathbb{Z}/3^e\mathbb{Z} \times \mathbb{Z}/3^e\mathbb{Z}$, $e \geq 1$, and let F be the subfield of H_K fixed by $M_{2(e-1)} = M^3$; so F is a cubic cyclic extension of KH_k and $\mathrm{Gal}(F/K) \simeq \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.

The 3-extension F/k is Galois: indeed, if σ is a generator of $\mathrm{Gal}(K/k)$, the action of σ on $\mathrm{Gal}(F/K) = \mathrm{Gal}(H_K/K)/M^3$ is given, via the correspondence of class field theory, by the action of σ on $\overline{h}_{2e} := h_{2e}M^3$ and on $\overline{h}_{2e-1} := h_{2e-1}M^3$. From $\omega = \sigma\,(\sigma - 1) = \sigma^{-2}\,(\sigma - 1)$, we have $\overline{h}_{2e}^{\sigma} = \overline{h}_{2e} \times \overline{h}_{2e-1}^{\sigma^2}$, and σ acts on $\overline{h}_{2e-1}$ by $\overline{h}_{2e-1}^{\sigma} = h_{2e-1}^{\sigma}M^3$; but $h_{2e-1}^{\sigma(\sigma-1)} = h_{2(e-1)} \in M^3$, so $h_{2e-1}^{\sigma-1} \in M^3$ and $\overline{h}_{2e-1}^{\sigma} = \overline{h}_{2e-1}$, hence the result and the fact that F/H_k is Galois. But a group of order p^2 (p prime) is Abelian and F/H_k is the direct compositum of KH_k and L such that L is the decomposition field at 3 giving a cyclic extension L/H_k, unramified of degree 3. But we know (Lemma 2.2) that the 3-class group of H_k is trivial (contradiction). □

Remarks 4.2. (i) Since 3 is non-split in $k/\mathbb{Q}$, the unique prime ideal $\mathfrak{P} \,|\, 3$ in K is principal: indeed, $\mathfrak{P}^{1+s} = \mathfrak{P}^2$ gives the square of the extension of $\mathfrak{P}^+ \,|\, 3$ in K^+ which is 3-principal (Lemma 2.2); so $\mathcal{C}\ell_K(\mathfrak{P}) = 1$. By class field theory, $\mathfrak{P}$ splits completely in H_K/K.

(ii) In the case n even for the above reasoning, an analog of the field F does not exist as Galois field over H_k.

(iii) Note that the parameter a can probably take any value; we have for instance obtained the following example:

For $d = 12058$, for which $\#\mathcal{C}\ell_{k^+} = 2$, $\mathcal{C}\ell_{k^-} \simeq \mathbb{Z}/3\mathbb{Z}$, and $\alpha = 989 + 26\sqrt{-d}$ of norm 209^3, the structure is $\mathcal{C}\ell_K^- \simeq \mathbb{Z}/3^4\mathbb{Z} \times \mathbb{Z}/3^3\mathbb{Z}$. A polynomial defining K is:
$P = x^{12} - 6\,x^{11} + 21\,x^{10} - 4006\,x^9 + 17892\,x^8 - 35730\,x^7 + 22212821\,x^6 - 66531354\,x^5 - 113482743\,x^4 - 35777798264\,x^3 + 54059937672\,x^2 + 54106942656\,x + 83308554531904$.

For $d = 86942$, for which $\#\mathcal{C}\ell_{k^+} = 4$, $\mathcal{C}\ell_{k^-} \simeq \mathbb{Z}/3\mathbb{Z}$, and $\alpha = 557 + 3\sqrt{-d}$ of norm 103^3, the structure is $\mathcal{C}\ell_K^- \simeq \mathbb{Z}/3^5\mathbb{Z} \times \mathbb{Z}/3^4\mathbb{Z}$. A polynomial defining K is:
$P = x^{12} - 6\,x^{11} + 21\,x^{10} - 2278\,x^9 + 10116\,x^8 - 20178\,x^7 + 3449985\,x^6 - 10289502\,x^5 - 8954865\,x^4 - 2399550304\,x^3 + 3642928674\,x^2 + 3641624304\,x + 1191621124996$.

In conclusion, the structure of $\mathcal{C}\ell_K^-$ strongly depends on the hypothesis on the order of $\mathcal{C}\ell_k^-$ and not only of its 3-rank.

5. Numerical results

We give explicit numerical computations of $\mathcal{C}\ell_K$, for various biquadratics fields k satisfying the conditions (i) to (v) assumed in the Hypothesis 2.1.

The following PARI program gives in "$component(H,5)$" the class number and the structure of the whole class group $\mathcal{C}\ell_K$ of K in the form

$$[classnumber, [c_1, \ldots, c_\lambda]]$$

such that the class group is isomorphic to $\bigoplus_{i=1}^{\lambda} \mathbb{Z}/c_i\mathbb{Z}$.

For simplicity we compute an α being an integer without non-trivial rational divisor. So (α) is the cube of an ideal if and only if $\mathrm{N}_{k^-/\mathbb{Q}}(\alpha) \in \mathbb{Q}^{\times 3}$. Then the irreducible polynomial defining K is given by $P = polcompositum(x^2+x+1, Q)$ where $Q = x^6 - 2\,a\,x^3 + (a^2 + d\,b^2)$ or $Q = x^6 - a\,x^3 + (a^2 + d\,b^2)/4$, where $\alpha = a + b\sqrt{-d}$ or $\dfrac{a+b\sqrt{-d}}{2}$, respectively.

In all the examples, we recall that the 3-class group of k^+ is trivial, and that of k^- is of order 3 exactely. Thus, the 3-class group of K^+ is trivial and $\mathcal{C}\ell_K^- = \mathcal{C}\ell_K$; moreover, $\mathcal{C}\ell_K^G$ is of order 3. So the case $n=1$ yields to $a=0, b=1$ ($\mathcal{C}\ell_K = \mathcal{C}\ell_K^G$), the case $n=3$ yields to $a=1, b=1$, and so on.

```
allocatemem(1000000000)
{d = 1; while(d < 5 * 10^3, d = d + 3; if(core(d) == d,
D = 3 * d; if(Mod(D, 4)! = 1, D = 4 * D); h = qfbclassno(D); if(Mod(h, 3)! = 0,
Dm = -d; if(Mod(Dm, 4)! = 1, Dm = 4 * Dm); hm = qfbclassno(Dm);
if(Mod(hm, 3) == 0&Mod(hm, 9)! = 0,
for(b = 1, 10^2, for(a = 1, 10^3, if(gcd(a, b) == 1, T = 2 * a; N = a^2 + d * b^2;
if(Mod(-d, 4) == 1&Mod(a * b, 2)! = 0, T = T/2; N = N/4);
if(floor(N^(1/3))^3 - N == 0, Q = x^6 - T * x^3 + N;
P = polcompositum(x^2 + x + 1, Q); R = component(P, 1);
H = bnrinit(bnfinit(R, 1), 1); F = component(H, 5); G = component(F, 1);
if(Mod(G, 3) == 0&Mod(G, 9)! = 0,
print(""); print("d = ", d); print("a = ", a); print("b = ", b);
print("hm = ", hm, "h = ", h); print(P);
print("classgroup : ", F); a = 10^3; b = 10^2))))))))))}
```

The instruction "if(Mod(G, 3)==0 & Mod(G, 9)!=0" must be adapted to the relevant needed structure $(3^{a+1}, 3^a)$.

We give below an extract of the numerical examples we have obtained; for a complete table, please see the Section 5 of:

`https://www.researchgate.net/publication/286452614`

5.1. Case $\mathcal{C}\ell_K \simeq \mathbb{Z}/3\mathbb{Z}$ $(a=0)$.

This implies $\mathcal{C}\ell_K = \mathcal{C}\ell_K^G \simeq \mathbb{Z}/3\mathbb{Z}$:

$d = 31$
$a = 1,\ b = 1$
$\#\mathcal{C}\ell_{k^-} = 3,\ \#\mathcal{C}\ell_{k^+} = 1$
$P = x^{12} - 6\,x^{11} + 21\,x^{10} - 52\,x^9 + 99\,x^8 - 144\,x^7 + 179\,x^6 - 186\,x^5 - 33\,x^4 + 268\,x^3 - 87\,x^2 - 24\,x + 64$
$classgroup : [3, [3]]$

$d = 61$
$a = 8,\ \ b = 1$
$\#\mathcal{C}\ell_{k^-} = 6,\ \#\mathcal{C}\ell_{k^+} = 2$
$P = x^{12} - 6\,x^{11} + 21\,x^{10} - 82\,x^9 + 234\,x^8 - 414\,x^7 + 983\,x^6 - 1788\,x^5 - 393\,x^4 - 506\,x^3 + 5394\,x^2 + 4620\,x + 12100$
$classgroup : [12, [6, 2]]$

$\vdots$

$d = 913$
$a = 321,\ \ b = 4$
$\#\mathcal{C}\ell_{k^-} = 12,\ \#\mathcal{C}\ell_{k^+} = 8$
$P = x^{12} - 6\,x^{11} + 21\,x^{10} - 1334\,x^9 + 5868\,x^8 - 11682\,x^7 + 661085\,x^6 - 1948290\,x^5 + 702561\,x^4 - 149227072\,x^3 + 227288688\,x^2 + 224655360\,x + 13690872064$
$classgroup : [768, [24, 4, 4, 2]]$

$d = 970$
$a = 563,\ \ b = 20$
$\#\mathcal{A}_{k^-} = 12,\ \#\mathcal{A}_{k^+} = 4$
$P = x^{12} - 6\,x^{11} + 21\,x^{10} - 2302\,x^9 + 10224\,x^8 - 20394\,x^7 + 2701601\,x^6 - 8043702\,x^5 - 2977323\,x^4 - 1568242964\,x^3 + 2378397756\,x^2 + 2373361968\,x + 495396376336$
$classgroup$: $[600, [30, 10, 2]]$

5.2. **Case** $\mathcal{C}\ell_K \simeq \mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ $(a = 1)$.

$d = 211$
$a = 17,\ \ b = 1$
$\#\mathcal{A}_{k^-} = 3,\ \#\mathcal{A}_{k^+} = 1$
$P = x^{12} - 6\,x^{11} + 21\,x^{10} - 84\,x^9 + 243\,x^8 - 432\,x^7 + 1037\,x^6 - 1896\,x^5 - 204\,x^4 - 966\,x^3 + 5949\,x^2 + 4905\,x + 11881$
$classgroup$: $[27, [9, 3]]$

$d = 214$
$a = 89,\ \ b = 6$
$\#\mathcal{A}_{k^-} = 6,\ \#\mathcal{A}_{k^+} = 2$
$P = x^{12} - 6\,x^{11} + 21\,x^{10} - 406\,x^9 + 1692\,x^8 - 3330\,x^7 + 66813\,x^6 - 190530\,x^5 - 45783\,x^4 - 5155600\,x^3 + 8296296\,x^2 + 8156544\,x + 238640704$
$classgroup$: $[54, [18, 3]]$

$\vdots$

$d = 4531$
$a = 403,\ \ b = 5$
$\#\mathcal{A}_{k^-} = 12,\ \#\mathcal{A}_{k^+} = 2$
$P = x^{12} - 6\,x^{11} + 21\,x^{10} - 856\,x^9 + 3717\,x^8 - 7380\,x^7 + 308855\,x^6 - 904506\,x^5 - 62898\,x^4 - 53921936\,x^3 + 83258895\,x^2 + 82428357\,x + 4694853361$
$classgroup$: $[1728, [36, 12, 4]]$

$d = 4639$
$a = 361,\ \ b = 2$
$\#\mathcal{A}_{k^-} = 51,\ \#\mathcal{A}_{k^+} = 1$
$P = x^{12} - 6\,x^{11} + 21\,x^{10} - 1494\,x^9 + 6588\,x^8 - 13122\,x^7 + 834341\,x^6 - 2463738\,x^5 + 888141\,x^4 - 212657184\,x^3 + 323349156\,x^2 + 320016960\,x + 21950200336$
$classgroup$: $[7344, [612, 12]]$

5.3. **Case** $\mathcal{C}\ell_K \simeq \mathbb{Z}/27\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z}$ $(a = 2)$.

$d = 1141$
$a = 449,\ \ b = 8$
$\#\mathcal{A}_{k^-} = 24,\ \#\mathcal{A}_{k^+} = 4$
$P = x^{12} - 6\,x^{11} + 21\,x^{10} - 1846\,x^9 + 8172\,x^8 - 16290\,x^7 + 1374653\,x^6 - 4075170\,x^5 + 711057\,x^4 - 487867520\,x^3 + 740542656\,x^2 + 735780864\,x + 74927017984$
$classgroup$: $[7776, [216, 18, 2]]$

$d = 1174$
$a = 21,\ \ b = 5$
$\#\mathcal{A}_{k^-} = 30,\ \#\mathcal{A}_{k^+} = 2$
$P = x^{12} - 6\,x^{11} + 21\,x^{10} - 134\,x^9 + 468\,x^8 - 882\,x^7 + 62369\,x^6 - 184542\,x^5 - 436569\,x^4 - 1322320\,x^3 + 3316650\,x^2 + 3570000\,x + 885062500$
$classgroup$: $[2430, [270, 9]]$

$\vdots$

$d = 4087$
$a = 357,\ \ b = 8$
$\#\mathcal{A}_{k^-} = 30,\ \#\mathcal{A}_{k^+} = 2$
$P = x^{12} - 6\,x^{11} + 21\,x^{10} - 1478\,x^9 + 6516\,x^8 - 12978\,x^7 + 1302965\,x^6 - 3870042\,x^5 - 2782815\,x^4 - 543509224\,x^3 + 830485104\,x^2 + 829417344\,x + 150779996416$
$classgroup$: $[19440, [270, 18, 2, 2]]$

$d = 4567$
$a = 195,\ \ b = 1$
$\#\mathcal{A}_{k^-} = 33,\ \#\mathcal{A}_{k^+} = 7$
$P = x^{12} - 6\,x^{11} + 21\,x^{10} - 440\,x^9 + 1845\,x^8 - 3636\,x^7 + 63557\,x^6 - 179844\,x^5 + 66765\,x^4 - 3988930\,x^3 + 6294021\,x^2 + 6052866\,x + 109286116$
$classgroup$: $[18711, [2079, 9]]$

5.4. **Case $\mathcal{C}\ell_K \simeq \mathbb{Z}/81\mathbb{Z} \times \mathbb{Z}/27\mathbb{Z}$ $(a = 3)$.**

$d = 12058$
$a = 989,\ \ b = 26$
$\#\mathcal{C}\ell_{k^-} = 42,\ \#\mathcal{C}\ell_{k^+} = 2$
$P = x^{12} - 6\,x^{11} + 21\,x^{10} - 4006\,x^9 + 17892\,x^8 - 35730\,x^7 + 22212821\,x^6 - 66531354\,x^5 - 113482743\,x^4 - 35777798264\,x^3 + 54059937672\,x^2 + 54106942656\,x + 83308554531904$
classgroup : $[30618, [1134, 27]]$

$d = 15607$
$a = 534,\ \ b = 1$
$\#\mathcal{C}\ell_{k^-} = 39,\ \#\mathcal{C}\ell_{k^+} = 1$
$P = x^{12} - 6\,x^{11} + 21\,x^{10} - 2186\,x^9 + 9702\,x^8 - 19350\,x^7 + 1764719\,x^6 - 5236188\,x^5 + 2322777\,x^4 - 638361238\,x^3 + 965957748\,x^2 + 958427808\,x + 89817692416$
classgroup : $[28431, [1053, 27]]$

$\vdots$

$d = 45517$
$a = 845,\ \ b = 6$
$\#\mathcal{C}\ell_{k^-} = 120\#\mathcal{C}\ell_{k^+} = 4$
$P = x^{12} - 6\,x^{11} + 21\,x^{10} - 3430\,x^9 + 15300\,x^8 - 30546\,x^7 + 7597005\,x^6 - 22699458\,x^5 - 18168075\,x^4 - 7877764840\,x^3 + 11909686236\,x^2 + 11905200672\,x + 5526956498704$
classgroup : $[174960, [3240, 54]]$

$d = 47194$
$a = 293,\ \ b = 2$
$\#\mathcal{C}\ell_{k^-} = 120\#\mathcal{C}\ell_{k^+} = 4$
$P = x^{12} - 6\,x^{11} + 21\,x^{10} - 1222\,x^9 + 5364\,x^8 - 10674\,x^7 + 905093\,x^6 - 2683338\,x^5 - 2064183\,x^4 - 313267016\,x^3 + 480721224\,x^2 + 480118080\,x + 75097921600$
classgroup : $[699840, [3240, 54, 2, 2]]$

Acknowledgments. I thank Christian Maire for telling me about difficulties with the techniques developed in [W], Thong Nguyen Quang Do and Jean-François Jaulent for similar conversations and advices about it.

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