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► **To cite this version:**

Harmonie Leduc, Dimitri Peaucelle, Christelle Pittet-Mechin. Adaptive Control LMI-based design for descriptor systems rational in the uncertainties. IFAC International Workshop on Adaptation and Learning in Control and Signal Processing (ALCOSP), Jun 2016, Eindhoven, Netherlands. hal-01243247

**HAL Id: hal-01243247**

**<https://hal.science/hal-01243247v1>**

Submitted on 14 Dec 2015

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# Adaptive Control LMI-based design for descriptor systems rational in the uncertainties

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## Abstract

Uncertain systems are considered. They are represented in a descriptor form, where the matrices have an affine dependence on the uncertain parameter. S-variable approach for the design of a robust adaptive control feedback loop is presented. The only requirement to build such an adaptive law is robust stability of the closed-loop system by a static gain. No assumption about passivity of the system is made. Asymptotic stability of the given adaptive control is proved using Lyapunov arguments, and gain adaptation parameters are tunable by linear matrix inequality based convex optimization. An application to the attitude control of a microsatellite of the CNES Myriade series illustrates the results.

## 1 Introduction

The parameters of a system cannot be well known and might be subject to important variations. Adaptive control theory proposes to deal with this issue by making the gains of its law time-varying, depending on the real time measurements. But can robustness be proved? For example using robust control methods?

They are two types of adaptive control approaches: in an indirect adaptive scheme, the gains of the controller evolve with relation to an estimation of the parameters of the system. The idea of such an estimator has been introduced in [6]. But the indirect scheme does not fit well with uncertain systems and its implementation is complex, as highlighted in [14]. For these reasons, we choose to use the

direct adaptive scheme, where the gains are directly modified according to the measured outputs, making its implementation very simple. The counterpart is that it is based on strong hypothesis, as the passivity of the system to be controlled ([4]). Moreover, noise on the measurements tends to push the gains of the controller to infinity. To tackle this issue, [5] and [7] propose the so-called  $\sigma$ -modification to achieve changes in the dynamics based on the measured outputs.

In robust control community, the effectiveness of LMI-based methods has been widely proved ([2]), but only a few works use them in the context of adaptive control of uncertain systems. In [10], only certain systems are treated, whereas in [9] and [16], some assumptions are made about the uncertainties, but they are not always verifiable; [1] designs a simple adaptive controller, which does not require the knowledge of the system dynamics.

In this paper, we deal with direct adaptive control of uncertain systems, and the controllers are designed using LMI-based methods. The paper has three main contributions: First, the passivity of the system is not required. Second, we use the recent results of descriptor systems, that applies for systems rational in the uncertainties ([15] and [3]). The third major contribution of this paper is the establishment of new results proving that adaptive law has improved (at least no worse) robustness, compared to a given static feedback controller.

The paper is organized as follows: First, we justify our choice to use a descriptor representation. In section III, we design adaptive controllers, with no worse and improved robustness respectively. An

application is given in section IV. Finally, we give some conclusions and outlooks for future work.

**Notation.**  $\mathbf{I}$  stands for the identity matrix.  $\{1; V\}$  is the set of all the integers between 1 and  $V$ .  $A^T$  is the transpose of the matrix  $A$ .  $A^S$  stands for the symmetric matrix  $A + A^T$ .  $A(\preceq) \prec B$  is the matrix inequality stating that  $A - B$  is negative (semi-)definite. If  $A \in \mathbb{R}^{n \times m}$  and  $\text{rank}A = r$ ,  $A^\perp$  is a full rank matrix such that  $A^\perp \in \mathbb{R}^{(n-r) \times n}$  and  $A^\perp A = 0$ .  $A^\circ$  is a full rank matrix such that  $A^\circ \in \mathbb{R}^{m \times r}$  and  $AA^\circ$  is full rank.

## 2 Preliminaries about descriptor systems

A system can be represented with the following descriptor form

$$E_{xx}\dot{x}(t) + E_{x\pi}\pi(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) \quad (1)$$

where  $x \in \mathbb{R}^{n_x}$  is the state of the system,  $u \in \mathbb{R}^{n_u}$  is the control input,  $\pi \in \mathbb{R}^{n_\pi}$  is an auxiliary signal (see 4.).  $E_{xx} \in \mathbb{R}^{n_x \times n_x}$ ,  $E_{x\pi} \in \mathbb{R}^{n_x \times n_\pi}$ ,  $A \in \mathbb{R}^{n_x \times n_x}$  and  $B \in \mathbb{R}^{n_x \times n_u}$  define the system.

One of the main advantages of descriptor systems is that they derive directly from physical representations. Moreover they happen to be well suited for dealing with uncertainties. The following result is stated in [12] and generalized in [3]:

**Theorem 2.1** *Assume a parameter-dependent descriptor model*

$$\bar{E}_{xx}(\delta)\dot{x}(t) + \bar{E}_{x\pi}(\delta)\bar{\pi}(t) = \bar{A}(\delta)x(t) + \bar{B}(\delta)u(t) \\ y(t) = Cx(t), \quad \delta \in \Delta^V \quad (2)$$

where  $\Delta^V := \{\delta \in \mathbb{R}^V : \delta \geq 0, \mathbf{1}^T \delta = 1\}$  and the  $\delta$ -dependent matrices are rational with respect to the components of the uncertain vector  $\delta$ . Then, there always exists another parameter-dependent descriptor model

$$E_{xx}(\delta)\dot{x}(t) + E_{x\pi}(\delta)\pi(t) = A(\delta)x(t) + B(\delta)u(t) \\ y(t) = Cx(t), \quad \delta \in \Delta^V \quad (3)$$

in which the  $\delta$ -dependent matrices are affine functions of  $\delta$ , that is  $E_{xx}(\delta) = \sum_{v=1}^V \delta_v E_{xx}^{[v]}$ ,  $E_{x\pi}(\delta) = \sum_{v=1}^V \delta_v E_{x\pi}^{[v]}$ ,  $A(\delta) = \sum_{v=1}^V \delta_v A^{[v]}$  and  $B(\delta) = \sum_{v=1}^V \delta_v B^{[v]}$ ,  $E_{xx}^{[v]}$ ,  $E_{x\pi}^{[v]}$ ,  $A^{[v]}$  and  $B^{[v]}$  being the values of the matrices of the system on the  $V$  vertices of  $\delta$ . Descriptor representations allow to handle rational systems as if affine in the uncertainties, which is a key point.

In all the following, we consider that the matrices which describe the system are affine functions of the uncertain parameter  $\delta$ . In order to get a condition of stability for systems of the form of (3), we suppose the following assumption holds:

**Assumption 1:** It is assumed that

$$[E_{xx}(\delta) \quad E_{x\pi}(\delta)] = E_1(\delta) [E_{2xx} \quad E_{2x\pi}] \quad (4)$$

where  $E_1(\delta) = \sum_{v=1}^V \delta_v E_1^{[v]}$  is full column rank for all  $\delta \in \Delta^V$ .

Assumption 1 means that the potential impulsive and non dynamic modes of system (3) do not depend on the uncertainty  $\delta$ .

We can now recall the result of [3] for uncertain descriptor systems:

**Theorem 2.2** *Under assumption 1, let  $E_2 = E_{2x\pi}^\perp E_{2xx}$ . The system (3) is robustly stable if there exist matrices  $\hat{P}^{[v]} = \hat{P}^{[v]T}$ ,  $\hat{Y}^{[v]}$  and  $\hat{S}$  such that the following conditions hold for all  $v \in \{1; V\}$ :*

$$(E_2 E_2^\circ)^T \hat{P}^{[v]} (E_2 E_2^\circ) \succ 0 \quad (5)$$

$$\begin{bmatrix} 0 & \hat{P}_e^{[v]T} \\ \hat{P}_e^{[v]} & 0 \end{bmatrix} + \left\{ \hat{S} \begin{bmatrix} E_1^{[v]} & -A^{[v]} \end{bmatrix} \right\}^S \prec 0 \quad (6)$$

where  $\hat{P}_e^{[v]} = (E_2^T \hat{P}^{[v]} + \hat{Y}^{[v]T} E_2^\perp) E_{2x\pi}^\perp$ .

By stability, we mean boundedness and convergence of  $E_2 x$  and the absence of impulsive modes, see [3] for details.

**Remark:** Condition of Theorem 2.2 only requires that (6) is satisfied for all  $v \in \{1; V\}$ . By convexity, it implies that it holds for all  $\delta \in \Delta^V$  with parameter dependent matrices  $\hat{P}(\delta) = \sum_{v=1}^V \delta_v \hat{P}^{[v]}$  and  $\hat{Y}(\delta) = \sum_{v=1}^V \delta_v \hat{Y}^{[v]}$ .  $\hat{P}(\delta)$  defines a parameter-dependent quadratic Lyapunov function for the plant.

### 3 LMI-based robust adaptive control design

The main result of this paper aims at designing an adaptive law which stabilizes the system (3) for every value of the uncertain vector  $\delta$ , under the following assumption:

**Assumption 2:** Under assumption 1, let  $u(t) = K_0 y(t)$  be a static output feedback. It is assumed that there exist matrices  $\hat{P}^{[v]} = \hat{P}^{[v]T}$ ,  $\hat{Y}^{[v]}$  and  $\hat{S}$  such that for all  $v \in \{1; V\}$ , conditions of Theorem 2.2 hold for the closed-loop system.

The proposed adaptive law consists in replacing the static feedback by a structured time-varying control

$$u(t) = (K_0 + LK(t)R)y(t) \quad (7)$$

where  $L$  and  $R$  are partitioned with appropriate dimensions such that  $LKR = \sum_{k=1}^{\bar{k}} L_k K_k R_k$ .  $K(t) = \text{diag}(K_1(t), K_2(t), \dots)$ ,  $L = [L_1 \ L_2 \ \dots]$ ,  $R^T = [R_1^T \ R_2^T \ \dots]$  and the adaptation is driven by

$$\begin{aligned} \dot{K}_k(t) &= \text{Proj}_{D_k}(K_k(t), W_k(t)) \\ W_k(t) &= \gamma_k (-G_k y(t)(R_k y(t))^T - \sigma_k K_k(t)). \end{aligned} \quad (8)$$

where  $D_k$  defines an ellipsoidal set  $\mathcal{E}_k$ :

$$K_k \in \mathcal{E}_k \Leftrightarrow \text{Tr}(K_k^T D_k K_k) \leq 1 \quad (9)$$

and  $\text{Proj}_{D_k}$  is the operator defined as in [13]. When the gain  $K_k$  is inside the set, the operator outputs  $\dot{K}_k = W_k$ , and when  $K_k$  is at the border of the set,

the operator aims at pushing it inside the set, so that the gains cannot exit the set:

$$\text{Proj}_{D_k}(K_k, W_k) = W_k - H_k$$

where  $H_k$  is such that

$$\begin{aligned} H_k &= 0 \text{ if } K_k \in \mathcal{E}_k \\ \text{else s.t. } \begin{cases} \text{Tr}(\dot{K}_k^T D_k K_k) \leq 0 \\ \text{Tr}((K_k - F_k)^T H_k) \geq 0 \quad \forall F_k \in \mathcal{E}_k \end{cases} \end{aligned} \quad (10)$$

The definition of the operator guarantees that  $K_k$  remains bounded, with a bound inversely proportional to the square-root of  $\|D_k\|$ . Notice that if the gains are scalar, (8) can be implemented as a saturated integrator.

The adaptation is driven by the first term of (8)  $-G_k y(t)(R_k y(t))^T$ , whereas the second term  $-\sigma_k K_k(t)$  contains a forgetting factor which allows the gain  $K_k$  to tend to 0 when the output signal  $y$  is zero. The factor  $\gamma_k$  determines the speed of adaptation of the gain. The issue is to find some appropriate values for  $D = \text{diag}(D_1, \dots, D_{\bar{k}})$ ,  $G^T = [G_1^T \ G_2^T \ \dots]$ ,  $\Gamma = \text{diag}(\gamma_1, \dots, \gamma_{\bar{k}})$  and  $\sigma \text{diag}(\sigma_1, \dots, \sigma_{\bar{k}})$  such that the system with adaptive control is robustly stable.

#### 3.1 Adaptive add-on design with no worse robustness

**Theorem 3.1** *If assumption 2 is satisfied for system (3), then there exist matrices  $P^{[v]}$ ,  $Y^{[v]}$ ,  $S$ ,  $G^T = [G_1^T \ G_2^T \ \dots]$ ,  $D = \text{diag}(D_1, \dots, D_{\bar{k}})$  and  $\epsilon > 0$  such that the following equation holds  $\forall v \in \{1; V\}$ :*

$$\begin{aligned} M^{[v]} &= \begin{bmatrix} 0 & P_e^{[v]T} & 0 \\ P_e^{[v]} & \epsilon E_2^T E_2 + 2C^T R^T R C & -C^T G^T \\ 0 & -GC & -2D \end{bmatrix} \\ &+ \left\{ S \begin{bmatrix} E_1^{[v]} & -A_c^{[v]} & -B^{[v]} L \end{bmatrix} \right\}^S \prec 0 \end{aligned} \quad (11)$$

where  $A_c^{[v]} = A^{[v]} + B^{[v]} K_0 C$ , and  $P_e^{[v]} = (E_2^T P^{[v]} + Y^{[v]T} E_2^\perp) E_{2x\pi}^\perp$ .

Besides, the solution is such that the adaptive control (8) stabilizes the plant whatever positive values of  $\sigma_k$ ,  $\gamma_k$  and for all  $\delta \in \Delta^V$ .

**Proof 1:** If assumption 2 holds, we prove that (11) is satisfied for all  $v \in \{1; V\}$ . For a given  $v \in \{1; V\}$ , by a small perturbation argument on the condition of assumption 2, and whatever a priori choice of  $G$ , there exist  $\hat{\epsilon}^{[v]} > 0$  and  $\tilde{\epsilon}^{[v]} > 0$  such that

$$\begin{bmatrix} 0 & \hat{P}_e^{[v]T} \\ \hat{P}_e^{[v]} & 0 \end{bmatrix} + \left\{ \hat{S}_1 \begin{bmatrix} E_1^{[v]} & -A_c^{[v]} \end{bmatrix} \right\}^S \prec N^{[v]} \quad (12)$$

where

$$\begin{aligned} N^{[v]} = & \begin{bmatrix} 0 & 0 \\ 0 & -\hat{\epsilon}^{[v]} E_2^T E_2 - \tilde{\epsilon}^{[v]} C^T R^T R C \end{bmatrix} \\ & - \left( S_1 B^{[v]} L - \begin{bmatrix} 0 \\ C^T G^T \end{bmatrix} \right) \hat{\epsilon}^{[v]} I \\ & \left( S_1 B^{[v]} L - \begin{bmatrix} 0 \\ C^T G^T \end{bmatrix} \right)^T \end{aligned} \quad (13)$$

Take  $\hat{\epsilon} = \min_v \hat{\epsilon}^{[v]}$ ,  $\tilde{\epsilon} = \min_v \tilde{\epsilon}^{[v]}$ . Multiply the new inequality by  $2/\tilde{\epsilon}$  and take  $\epsilon = 2\hat{\epsilon}/\tilde{\epsilon}$ ,  $P_e^{[v]} = (2/\tilde{\epsilon})\hat{P}_e^{[v]}$ ,  $Y^{[v]} = (2/\tilde{\epsilon})\hat{Y}^{[v]}$  and  $D = (\tilde{\epsilon}/4\hat{\epsilon})I$ . The Schur complement of the result gives (11) with  $S = \begin{bmatrix} S_1 \\ 0 \end{bmatrix}$ .

It now remains to prove the second and last part of the theorem: stability with adaptive control. First, we use the fact that the  $\delta$ -dependent matrices are affine functions of  $\delta$ . By convexity, equation (11) is equivalent to

$$\sum_{v=1}^V \delta_v M^{[v]} \prec 0 \quad \forall \delta_v \geq 0$$

or, by linearity, to  $M(\delta) \prec 0$ , where  $E_1(\delta)$  replaces  $E_1^{[v]}$ ,  $P(\delta) + \sum_{v=1}^V \delta_v P^{[v]}$  replaces  $P^{[v]}$ ,  $Y(\delta) = \sum_{v=1}^V \delta_v Y^{[v]}$  replaces  $Y^{[v]}$  and  $A_c(\delta) = A(\delta) + B(\delta)K_0C$ .

Then, let us notice that since along the closed-loop system trajectories,  $E_{xx}(\delta)\dot{x} + E_{x\pi}(\delta)\pi = (A(\delta) + B(\delta)LKRC)x$ , pre and post multiplying  $\{S[E_1(\delta) - A_c(\delta) - B(\delta)L]\}^S$  by  $(\dot{x}^T E_{2xx}^T + \pi^T E_{2x\pi}^T \ x^T \ x^T C^T R^T K^T)$  and its transpose respectively gives zero. Therefore, if we pre and post multiply (11) by

$(\dot{x}^T E_{2xx}^T + \pi^T E_{2x\pi}^T \ x^T \ x^T C^T R^T K^T)$  and its transpose respectively, it remains:

$$\begin{aligned} 2x^T P_e (E_{2xx}\dot{x} + E_{2x\pi}\pi) - 2x^T C^T R^T R C x \\ - x^T C^T R^T K^T G C x + \epsilon x^T E_2^T E_2 x \\ - 2x^T C^T G^T K R C x \leq 0. \end{aligned}$$

Then, due to the expression of  $P_e$  and the fact that  $y = Cx$ ,

$$\begin{aligned} 2x^T E_2^T P E_2 \dot{x} - 2y^T G^T K R y \\ + 2y^T R^T (I - K^T D K) R y \leq -\epsilon x^T E_2^T E_2 x. \end{aligned}$$

Moreover, the fact that  $K$  and  $D$  are block-diagonal implies that  $(I - K^T D K)$  is also block-diagonal, with  $K^T D K \succ 0$  and  $\text{Tr}(K^T D K) \leq 1$  involving that  $(I - K^T D K) \succ 0$ .

Hence,

$$2x^T E_2^T P E_2 \dot{x} - 2y^T G^T K R y \leq -\epsilon x^T E_2^T E_2 x. \quad (14)$$

Keeping this in mind we consider the Lyapunov function defined by:

$$V(x, K) = x^T E_2^T P E_2 x + \text{Tr}(K^T \Gamma^{-1} K)$$

where  $\Gamma = \text{diag}(\gamma_1 I_{m_1}, \dots, \gamma_{\bar{k}} I_{m_{\bar{k}}})$  and  $m_k$  is the number of columns of  $L_k$ . Taking  $\sigma = \text{diag}(\sigma_1 I_{m_1}, \dots, \sigma_{\bar{k}} I_{m_{\bar{k}}})$  and  $H = \text{diag}(H_1, \dots, H_{\bar{k}})$ , its derivative reads as:

$$\begin{aligned} \dot{V}(x, K) &= 2x^T E_2^T P E_2 \dot{x} + 2\text{Tr}(K^T \Gamma^{-1} \dot{K}) \\ &= 2x^T E_2^T P E_2 \dot{x} - 2\text{Tr}(K^T (Gy)(Ry)^T) \\ &\quad - 2\text{Tr}(K^T (\sigma K + \Gamma^{-1} H)) \end{aligned}$$

Furthermore, the trace operator properties, the structure of  $\Gamma$  and the properties of  $H$  give:

$$\text{Tr}(K^T (Gy)(Ry)^T) = y^T G^T K R y$$

$$\text{Tr}(K^T \Gamma^{-1} H) \geq 0$$

and then, using (14)

$$\dot{V}(x, K) \leq -\epsilon x^T E_2^T E_2 x - \sum_{k=1}^{\bar{k}} \sigma_k \text{Tr}(K_k^T K_k).$$

Consequently, the adaptive closed-loop system is asymptotically stable;  $E_2 x$  converges to zero and  $K_k$  converges to 0 for all  $k$ . The property holds for any  $\delta \in \Delta^V$  and hence is robust.

### 3.2 Adaptive add-on analysis, improved robustness

With Theorem 3.1, we only know that the adaptive law is no worse than the static control. The following theorem aims at improving it.

**Theorem 3.2** *Consider the following matrix inequalities with  $\check{P}^{[v]} \succ 0$ ,  $\check{Y}^{[v]}$  and  $\check{\epsilon} > 0$  for all  $v \in \{1; V\}$ :*

$$\begin{bmatrix} 0 & \check{P}_e^{[v]T} & 0 \\ \check{P}_e^{[v]} & \check{N} & -(GC)^T \\ 0 & -GC & -2D \end{bmatrix} + \left\{ S \begin{bmatrix} E_1^{[v]} & -A_{c\Delta}^{[v]} & -B^{[v]}L \end{bmatrix} \right\}^S \prec 0 \quad (15)$$

and

$$\begin{bmatrix} T_k & F_k^T D_k \\ D_k F_k & D_k \end{bmatrix} \succeq 0, \text{Tr}(T_k) \leq 1 \quad \forall k \in \{1; \bar{k}\} \quad (16)$$

where  $\check{N} = \check{\epsilon} E_2^T E_2 + 2C^T R^T R C + \{C^T R^T FGC\}^S$ ,  $A_{c\Delta}^{[v]} = A^{[v]} + B^{[v]}(K_0 + LFR)C$  and  $\check{P}_e^{[v]} = (E_2^T \check{P}^{[v]} + \check{Y}^{[v]T} E_2^\perp) E_{2x\pi}^\perp$  for all  $v \in \{1; V\}$ .

These constraints are such that:

- (i) For fixed  $K_0$ ,  $G$ ,  $S$  and  $D = \text{diag}(D_1, \dots, D_{\bar{k}})$  the constraints are LMI in  $\check{P}^{[v]}$ ,  $\check{Y}^{[v]}$ ,  $\check{\epsilon}$  and  $F$ .
- (ii) For  $K_0$ ,  $G$ ,  $S$  and  $D$  solution to constraints in Theorem 3.1 the LMIs are feasible.
- (iii) If the constraints are feasible, then  $F$  is such that  $F_k^T D_k F_k \preceq I$  and  $u(t) = (K_0 + LFR)y(t)$  stabilizes the plant (3).
- (iv) If the constraints are feasible, then whatever  $\gamma_k$  the adaptive control (8) quadratically stabilizes the set of the states  $x$  such that  $E_2 x = 0$  when all  $\sigma_k = 0$  and quadratically stabilizes a neighborhood of this same set when at least one  $\sigma_k > 0$ .

**Proof 2:** The proof of Theorem 3.2 is quite the same as the one of Theorem 3.1. The most important thing to notice is that the baseline control is no more required to be a quadratically stabilizing gain

in items (i), (iii) and (iv). Here, only a static add-on stabilizes the plant, but it is not needed to be known.

(i) is trivial.

To prove (ii) one has to notice that (15) is nothing else but (11) with  $\check{P} = P$ ,  $\check{Y} = Y$ ,  $F = 0$  and  $\check{\epsilon} = \epsilon$ , which proves the feasibility of (15).

To prove (iii), one has to notice on the one hand that  $F$  is forced to be in the same set as the adaptive gain  $K$ . Indeed, applying Schur complement to (16) gives

$$\text{Tr}(F_k^T D_k F_k) \preceq 1 \text{ for all } k \in \{1; \bar{k}\}.$$

On the other hand, if we denote  $S$  by  $S = [S_{1a}^T \ S_{1b}^T \ S_3^T]^T$ , (15) implies that

$$\begin{aligned} & \left\{ \begin{bmatrix} S_{1a} \\ S_{1b} \end{bmatrix} \begin{bmatrix} E_1^{[v]} & -A_{c\Delta}^{[v]} \end{bmatrix} \right\}^S + \begin{bmatrix} 0 & \check{P}_e^{[v]T} \\ \check{P}_e^{[v]} & 0 \end{bmatrix} \\ & \preceq \left\{ \begin{bmatrix} S_{1a} \\ S_{1b} \end{bmatrix} \begin{bmatrix} E_1^{[v]} & -A_{c\Delta}^{[v]} \end{bmatrix} \right\}^S \\ & + \begin{bmatrix} 0 & \check{P}_e^{[v]T} \\ \check{P}_e^{[v]} & \check{\epsilon} E_2^T E_2 + 2C^T R^T R C + \{C^T R^T FGC\}^S \end{bmatrix} \\ & \preceq 0 \end{aligned} \quad (17)$$

Using Theorem 2.2, this proves the stability of the closed-loop with static gain  $K_0 + LFR$ .

Now let us prove (iv). The generalization of (15) for all  $\delta \in \Delta^V$  is achieved as in the proof of Theorem 3.1. Still following the same lines as the second part of proof 1, multiplying (15) by  $(\dot{x}^T E_{2xx}^T + \pi^T E_{2x\pi}^T \ x^T \ x^T C^T R^T K^T)$  and its transpose respectively gives

$$\begin{aligned} 2x^T E_2 \check{P} E_2 \dot{x} - 2y^T R^T (K - F)^T G y \\ - 2y^T R^T (K^T D K - I) R y \leq -\check{\epsilon} x^T E_2^T E_2 x. \end{aligned} \quad (19)$$

where  $(K^T D K - I) \preceq 0$ . That result proves the stability with the adaptive add-on.

Here, the Lyapunov function is given by:

$$V(x, K) = x^T E_2 \check{P} E_2 x + \text{Tr}((K - F)^T \Gamma^{-1} (K - F)^T)$$

and its derivative writes as follows:

$$\begin{aligned}
\dot{V}(x, K) &= 2x^T E_2 \check{P} E_2 \dot{x} + 2\text{Tr}(\dot{K} \Gamma^{-1} (K - F)^T) \\
&= 2x^T E_2 \check{P} E_2 \dot{x} - 2\text{Tr}((G y y^T R^T + \sigma K) \\
&\quad (K - F)^T) - 2\text{Tr}(H \Gamma^{-1} (K - F)^T) \\
&= 2x^T E_2 \check{P} E_2 \dot{x} - 2y^T R^T (K - F)^T G y \\
&\quad - 2\sigma \text{Tr}(K^T (K - F)) - 2\Gamma^{-1} \text{Tr}((K - F)^T H) \\
&\leq -\check{\epsilon} x^T E_2^T E_2 x - 2 \sum \sigma_k \text{Tr}(K_k^T (K_k - F_k))
\end{aligned} \tag{20}$$

For the second row, we use the definition of the projection operator (10). For the third row, we use the fact that the three following properties are true for all matrices M and N with appropriate sizes and for every  $\lambda \in \mathbb{R}$ :

$$\text{Tr}(M + \lambda N) = \text{Tr}(M) + \lambda \text{Tr}(N)$$

$$\text{Tr}(MN) = \text{Tr}(NM); \text{Tr}(M) = \text{Tr}(M^T)$$

For the last row we use (19) and again the property of the projection operator (10).

The last row indicates that when all  $\sigma_k$  are zero, the derivative of the Lyapunov function is negative along trajectories as long as the state of the plant has not converged to  $E_2 x = 0$ . The Lyapunov theory allows to claim that in that case, the state  $x$  converges to  $E_2 x = 0$ .

However, the only thing we can say about the adaptive add-on gain is that if the baseline controller does not stabilize the system, the adaptive add-on may not converge to zero.

Now, consider the case when at least one  $\sigma_k$  is strictly positive. On the one hand, the last right term of (20) being bounded, there exists positive functions  $f_k(\|D_k\|)$ , decreasing when  $\|D_k\|$  grows, such that  $\dot{V}(x, K) \leq -\check{\epsilon} x^T E_2^T E_2 x + 2 \sum \sigma_k f_k(\|D_k\|)$ .

Hence we get  $\dot{V}(x, K) < 0$  as soon as

$$x^T E_2^T E_2 x > \frac{2}{\check{\epsilon}} \sum \sigma_k f_k(\|D_k\|).$$

On the other hand, the last term of  $V(x, K)$  is also bounded, we can find positive functions  $g_k(\|D_k\|)$ , decreasing when  $\|D_k\|$  grows, such that:

$$V(x, K) \leq \lambda_{\max}(\check{P}) x^T E_2^T E_2 x - \sum \gamma_k^{-1} g_k(\|D_k\|). \tag{21}$$

Then, if

$$x^T E_2^T E_2 x \geq \max\left(\frac{2}{\check{\epsilon}} \sum \sigma_k f_k; \lambda_{\max}^{-1}(\check{P}) \sum \gamma_k^{-1} g_k\right)$$

then  $x$  is outside an equipotential of the Lyapunov function and is such that  $\dot{V} < 0$ . This equipotential defines an attractor, and one can easily see that the larger any  $\sigma_k$  or  $\gamma_k$  is, the larger is the size of the attractor.

## 4 Robust design of the Deme-ter adaptive attitude control

In this section, Theorems 3.1 and 3.2 are applied to the CNES microsatellite Taranis, whose dynamics can be modeled as follows:

$$\begin{aligned}
J\ddot{\theta} + \sqrt{J}l\dot{\eta} &= u \\
\sqrt{J}l\ddot{\theta} + \dot{\eta} + 2\zeta\omega_n\dot{\eta} + \omega_n^2\eta &= 0
\end{aligned} \tag{22}$$

where  $\theta$  is the attitude of the satellite,  $\eta$  represents the flexible modes,  $J$  the inertia,  $l$  stands for the coupling between  $\theta$  and  $\eta$ ,  $\omega_n$  is the natural frequency of the flexible modes and  $\zeta$  the damping.  $l = 0.7582$  is known but  $\omega_n \in [0.2 \ 0.6] \times 2\pi$  (50% uncertainty),  $\zeta \in [5e^{-4} \ 5e^{-3}]$  (80% uncertainty) and  $\sqrt{J} \in [5.205 \ 7.041]$  (30% uncertainty for  $J$ ) are uncertain. The first step is to rewrite system (22) into a descriptor form in which the matrices are affine function of the uncertain parameter. To do it we introduce two exogenous signals  $\pi_1 = \sqrt{J}\ddot{\theta}$  and  $\pi_2 = 2\zeta\dot{\eta} + \omega_n\eta$  and we consider that the uncertain parameter is  $\delta = [\omega_n \ \zeta \ \sqrt{J}]^T$ . Thus we get the following descriptor system:

$$\begin{aligned}
&\begin{bmatrix} 0 & 0 & 0 & l\sqrt{J} \\ 0 & \sqrt{J} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & l\sqrt{J} & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \dot{x} + \begin{bmatrix} \sqrt{J} & 0 \\ -1 & 0 \\ 0 & 0 \\ 0 & \omega_n \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \pi \\
&= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \omega_n & 2\zeta \\ 0 & 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u, \tag{23}
\end{aligned}$$

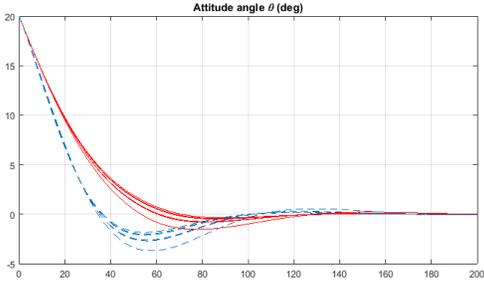


Figure 1: Attitude angle. Dashed: without adaptation. Solid: with adaptation

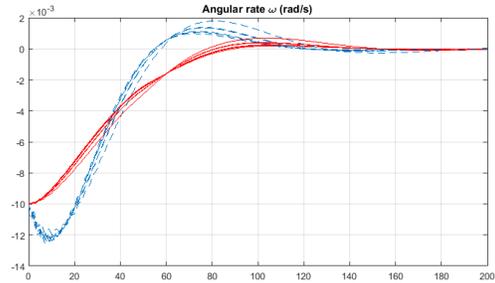


Figure 2: Angular rate. Dashed: without adaptation. Solid: with adaptation

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x$$

where matrices  $E_{xx}(\delta)$ ,  $E_{x\pi}(\delta)$ ,  $A(\delta)$ ,  $B(\delta)$  and  $C$  are easily identifiable. As  $[E_{xx}(\delta) \ E_{x\pi}(\delta)]$  is square and full rank, assumption 1 holds with  $E_1(\delta) = [E_{xx}(\delta) \ E_{x\pi}(\delta)]$  and  $[E_{2xx} \ E_{2x\pi}] = I_6$ . As  $\delta \in \mathbb{R}^3$ ,  $2^3 = 8$  models define the system. Assumption 2 is tested and satisfied.

#### 4.1 Adaptive control with no worse robustness

Theorem 3.1 is applied to get numerical values for adaptive law parameters. For more details about the parametrization of the law, see [11] and [8].

Simulation results with and without this adaptive control are plotted with solid and dashed lines respectively in Fig. 1 to Fig. 4, under the same initial conditions and with several random values of the uncertain parameter within the polytope of its extremal values.

Fig. 1 and Fig. 2 show that the adaptive control allows a faster convergence of the states. The overshoots in the attitude angle  $\theta$  and the angular rate  $\omega$  have disappeared with the adaptive law.

Time variations of the adaptive gains  $K_\theta$  and  $K_\omega$  are plotted in Fig. 3 and Fig. 4. The dashed curves are constant since they correspond to a static control law. The solid lines show non negligible variations of

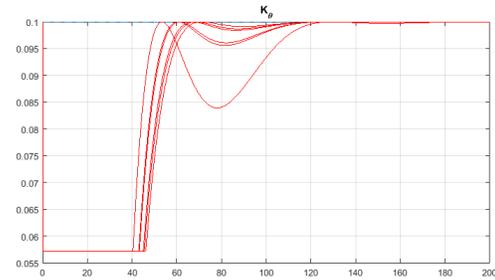


Figure 3: Adaptive gain for the attitude angle. Dashed: without adaptation. Solid: with adaptation

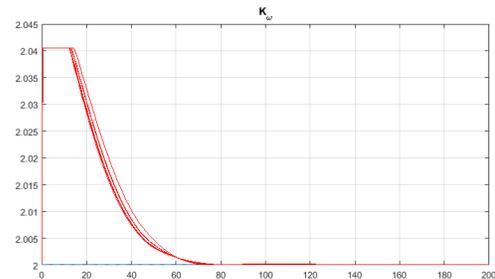


Figure 4: Adaptive gain for the angular rate. Dashed: without adaptation. Solid: with adaptation

the gains.

Moreover, setting the  $\sigma_k$  values allow to tune the convergence speed of the states, and the variation rate of the gains evolve with the values of  $\gamma_k$ .  $\gamma_k$  must be chosen large enough to allow the adaptation, but small enough to have reasonable and then implementable values for  $\dot{K}_k$ . For more details, one can refer to [10].

## 4.2 Adaptive control with improved robustness

Theorem 3.2 is an improvement in terms of admissible values of the uncertainties: due to the fact that the baseline control is no more required to stabilize all the realizations of the system, we can expect that (15) and (16) are feasible for a larger set of uncertainties than the one given in the previous part. This use of Theorem 3.2 is relevant in the context of TARANIS attitude control, since the inertia  $J$  of the satellite is ill known and can be subject to more important uncertainties than those given above.

Tests have been achieved with the same set of admissible dampings  $\zeta$  and natural frequencies of the flexible modes  $\omega_n$ . We aim at finding the biggest set of inertias for which (15) and (16) are feasible. Results yield the set [23.61 51,37]. Compared to the original set of uncertainties [26.27 48.71], which was the largest set for which LMIs of Theorem 3.1 are feasible, Theorem 3.2 allows an extension of the set of 23%.

Beyond that, the controller built solving (15) and (16) can be used to stabilize other sets of uncertainties. To be clear, let us apply this feature to system (23). Figure 5 can be read as follows: first, the static control with  $F = [0.1 \ 2]$  robustly stabilizes system (23) with a nominal inertia increased by 75% compared to the real nominal one, and with a  $\pm 30\%$  uncertainty. Tests show that beyond this new nominal inertia and/or beyond an uncertainty of 30%, assumption 2 is no more satisfied (blue interval in Figure 5). But with the adaptive control designed using Theorem 3.2, the uncertainty on the inertia can be pushed up to 90% (red interval in Figure 5).

These applications show that Theorem 3.2 provides



Figure 5: Admissible sets of uncertainty on the inertia using robust adaptive control

a non negligible improvement in terms of robustness compared to Theorem 3.1, all of these based on the fact that we gain asymptotic stability of a neighborhood of the equilibrium point at the expense of relaxing the goal of asymptotic stability of the equilibrium itself.

## 5 Conclusion

After recalling some important results about uncertain descriptor systems, we have established in this paper a SV-LMI based result to design a robust adaptive control law. A similar result but with improved robustness has also been proved, and two main uses have been highlighted. Both theorems have been applied to satellite attitude control. To go further, it should be studied if such improved robustness control law could have some drawbacks or not.

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