HAL
open science

General time elapsed neuron network model: well-posedness and strong connectivity regime

Q Weng

To cite this version:

Q Weng. General time elapsed neuron network model: well-posedness and strong connectivity regime. 2016. hal-01243163v3

HAL Id: hal-01243163

https://hal.science/hal-01243163v3

Preprint submitted on 17 Nov 2016

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

GENERAL TIME ELAPSED NEURON NETWORK MODEL: WELL-POSEDNESS AND STRONG CONNECTIVITY REGIME

Q. WENG

Abstract. For large fully connected neuron networks, we study the dynamics of homogenous assemblies of interacting neurons described by time elapsed models, indicating how the time elapsed since the last discharge construct the probability density of neurons. Through the spectral analysis theory for semigroups in Banach spaces developed recently in [6, 9], on the one hand, we prove the existence and the uniqueness of the weak solution in the whole connectivity regime as well as the parallel results on the long time behavior of solutions obtained in [10] under general assumptions on the firing rate and the delay distribution. On the other hand, we extend those similar results obtained in [11, 12] in the case without delay to the case taking delay into account and both in the weak and the strong connectivity regime with a particular step function firing rate.

Version of November 17, 2016

Contents

1. Introduction 1
2. Existence and the steady state 5
3. Case without delay 8
4. Case with delay 16
5. Step function firing rate 20
References 24

Keywords. Neuron networks, time elapsed dynamics, semigroup spectral analysis, strong connectivity, exponential asymptotic stability.

1. Introduction

The information transmission and processing mechanism in the nervous systems relies on the quantity of electrical pulses as the reflect to incoming stimulations, during which the neurons experience a period of recalcitrance called discharge time before reactive. In this work, we shall focus on the model describing the neuronal dynamics in accordance with this kind of discharge time which has been introduced and studied in [3, 11, 12]. In order to show the response to the recovery of the neuronal membranes after each discharge, the model consider an instantaneous firing rate depending on the time elapsed since last discharge as well as the inputs of neurons. This sort of models are also regarded as a mean field limit of finite number of neuron network models referred to [1, 2, 14, 13].

For a local time (or internal clock) $x \geq 0$ corresponding to the elapsed time since the last discharge, we consider the dynamic of the neuronal network with the density number

of neurons $f = f(t,x) \geq 0$ in state $x \geq 0$ at time $t \geq 0$, given by the following nonlinear time elapsed (or of age structured type) evolution equation

$$\partial_t f = -\partial_x f - a(x, \varepsilon\, m(t)) f =: \mathcal{L}_{\varepsilon m(t)} f, \tag{1.1a}$$

$$f(t,0) = p(t), \quad f(0,x) = f_0(x), \tag{1.1b}$$

where $a(x, \varepsilon\, \mu) \geq 0$ denotes the firing rate of a neuron in state x and in an environment $\mu \geq 0$ formed by the global neuronal activity with a network connectivity parameter $\varepsilon \geq 0$ corresponding to the strength of the interactions. The total density of neurons $p(t)$ undergoing a discharge at time t is defined through

$$p(t) := \mathcal{P}[f(t); m(t)],$$

where

$$\mathcal{P}[g, \mu] = \mathcal{P}_\varepsilon[g, \mu] := \int_0^\infty a(x, \varepsilon\mu) g(x) \mathrm{d}x,$$

while the global neuronal activity $m(t)$ at time $t \geq 0$ taking into account the interactions among the neurons resulting from earlier discharges is given by

$$m(t) := \int_0^\infty p(t-y) b(\mathrm{d}y), \tag{1.2}$$

where the delay distribution b is a probability measure considering the persistence of the electric activity to those discharges in the network. In the sequel, we will consider the two following situations respectively:

- The *case without delay*, when $b = \delta_0$ then $m(t) = p(t)$.
- The *case with delay*, when b is smooth.

Observe that the solution f of the time elapsed equation (1.1) satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^\infty f(t,x) \mathrm{d}x = f(t,0) - \int_0^\infty a(x, \varepsilon m(t)) f(t,x) \mathrm{d}x = 0,$$

in both the cases, which implies the conservation of the total density number of neurons (also called *mass* in the sequel) permitting us to normalize it to be 1. Then we assume in the sequel

$$\langle f(t,\cdot) \rangle = \langle f_0 \rangle = 1, \quad \forall t \geq 0, \quad \langle g \rangle := \int_0^\infty g(x) \mathrm{d}x.$$

We define a couple $(F_\varepsilon, M_\varepsilon)$ as a corresponding steady state, which satisfy

$$0 = -\partial_x F_\varepsilon - a(x, \varepsilon\, M_\varepsilon) F_\varepsilon = \mathcal{L}_{\varepsilon M_\varepsilon} F_\varepsilon, \tag{1.3a}$$

$$F_\varepsilon(0) = M_\varepsilon, \quad \langle F_\varepsilon \rangle = 1. \tag{1.3b}$$

Noticing that the associated network activity and the discharge activity are equal constants for a steady state as $\langle b \rangle = 1$.

Our main purpose in this paper is to prove the existence and uniqueness of the solution to the time elapsed evolution equation (1.1) no matter which $\varepsilon > 0$. Furthermore, we obtain the exponential asymptotic stability in strong connectivity regime, which is a range of connectivity parameter $\varepsilon \in [\varepsilon_1, \infty)$, with ε_1 large enough, such that the equations (1.1) and (1.3) do not possess intense nonlinearity. We are also able to extend the result in [11, 12] in the case without delay for a *step function* firing rate a to the case with delay rather than the indescribable stability. In order to conclude those results, it is necessary to give the following mathematical assumptions on the firing rate a and on the delay distribution b.

We make the physically reasonable assumptions

$$\partial_x a \geq 0, \quad a' = \partial_\mu a \geq 0, \tag{1.4}$$

$$0 < a_0 := \lim_{x\to\infty} a(x,0) \leq \lim_{x,\,\mu\to\infty} a(x,\mu) =: a_1 < \infty, \tag{1.5}$$

one particular example of the firing rate is the "step function" one

$$a(x,\mu) = \mathbf{1}_{x>\sigma(\mu)}, \quad \sigma' \leq 0, \tag{1.6}$$

$$\sigma(0) = \sigma_+, \quad \sigma(\infty) = \sigma_- < \sigma_+ < 1, \tag{1.7}$$

associated with some continuity assumption

$$a \in W^{1,\infty}(\mathbb{R}_+^2), \tag{1.8}$$

or particularly

$$\sigma, \sigma^{-1} \in W^{1,\infty}(\mathbb{R}_+), \tag{1.9}$$

In the strong connectivity regime, we consider the decay assumption on the two cases, for a.e. $x \geq 0$,

$$\varepsilon \sup_{x\geq 0} \partial_\mu a(x,\varepsilon\mu) \to 0, \quad \text{as } \varepsilon \to \infty, \tag{1.10}$$

$$\limsup_{\varepsilon\to\infty} \sup_{\mu\in[1-\sigma_+,1]} \varepsilon|\sigma'(\varepsilon\mu)| = 0. \tag{1.11}$$

In the case with delay, we assume that the delay distribution $b(\mathrm{d}y) = b(y)\mathrm{d}y$ has the exponential bound and satisfies the smoothness condition

$$\exists \delta > 0, \quad \int_0^\infty e^{\delta y}\left(b(y) + |b'(y)|\right)\mathrm{d}y < \infty. \tag{1.12}$$

The above assumptions permit the existence and uniqueness of the solution to the nonlinear problem (1.1) thanks to the Banach fixed-point theorem.

Theorem 1.1. *On the one hand, for a smooth firing rate, assuming* (1.4)-(1.5)-(1.8)-(1.10)*, then for any* $f_0 \in L^1(\mathbb{R}_+)$ *and any* $\varepsilon > 0$*, there exists a unique nonnegative and mass conserving weak solution* $f \in C(\mathbb{R}_+; L^1(\mathbb{R}_+))$ *to the evolution equation* (1.1) *for some functions* $m, p \in C([0,\infty))$*. On the other hand, for a step function firing rate, assuming* (1.6)-(1.7)-(1.9)-(1.11)*, then for any* $f_0 \in L^1 \cap L^\infty(\mathbb{R}_+)$ *and* $0 \leq f_0 \leq 1$*, the parallel conclusion holds with the unique solution* (f, m) *satisfying that*

$$0 \leq f(t,x) \leq 1, \quad \forall t, x \geq 0,$$
$$1 - \sigma_+ \leq m(t) \leq 1, \quad \forall t \geq 0.$$

For any $\varepsilon > 0$, there also exits a corresponding steady state, which is unique additionally in the weak or strong connectivity regime.

Theorem 1.2. *On the one hand, for a smooth firing rate, assuming* (1.4)-(1.5)-(1.8)-(1.10)*, then for any* $\varepsilon \geq 0$*, there exists at least one pair of solutions* $(F_\varepsilon(x), M_\varepsilon) \in W^{1,\infty}(\mathbb{R}_+) \times \mathbb{R}_+$ *to the stationary problem* (1.3) *such that*

$$0 \leq F_\varepsilon(x) \lesssim e^{\frac{-a_0}{2}x}, \quad |F_\varepsilon'(x)| \lesssim e^{\frac{-a_0}{2}x}, \quad x \geq 0. \tag{1.13}$$

On the other hand, for a step function firing rate, the existence of the steady state holds under the assumptions (1.6)-(1.7)-(1.9)-(1.11)*, which satisfy*

$$0 \leq F_\varepsilon \leq 1, \quad 1 - \sigma_+ \leq M_\varepsilon \leq 1.$$

Moreover, there exists $\varepsilon_0 > 0$ small enough or $\varepsilon_1 > 0$ large enough, such that the above steady state is unique for any $\varepsilon \in [0, \varepsilon_0) \cup (\varepsilon_1, +\infty]$.

We conclude the exponential long time stability in the strong connectivity regime combined with those in the weak one in [10] as our main result.

Theorem 1.3. *We assume that the firing rate a satisfies* (1.4)-(1.5)-(1.8)-(1.10) *or* (1.6)-(1.7)-(1.9)-(1.11)*. We also assume that the delay distribution b satisfies $b = \delta_0$ or* (1.12)*. There exists $\varepsilon_0 > 0$ ($\varepsilon_1 > 0$), small (large) enough, such that for any $\varepsilon \in (0, \varepsilon_0)$ ($\varepsilon \in (\varepsilon_1, +\infty)$) the steady state $(F_\varepsilon, M_\varepsilon)$ is unique. There also exist some constants $\alpha < 0$, $C \geq 1$ and $\eta > 0$ (besides $\zeta_\varepsilon \to 0$ as $\varepsilon \to \infty$) such that for any connectivity parameter $\varepsilon \in (0, \varepsilon_0)$ ($\varepsilon \in (\varepsilon_1, +\infty)$) and for any unit mass initial datum $0 \leq f_0 \in L^1$ or in the case of step function firing rate for any unit mass initial datum $f_0 \in L^p$, $1 \leq p \leq \infty$, satisfying $0 \leq f_0 \leq 1$ additionally, such that $\|f_0 - F_\varepsilon\|_{L^1} \leq \eta/\varepsilon$ ($\leq \eta/\zeta_\varepsilon$), then the (unique, positive and mass conserving) solution f to the evolution equation* (1.1) *satisfies*

$$\|f(t,.) - F_\varepsilon\|_{L^1} \leq Ce^{\alpha t}, \qquad \forall\, t \geq 0.$$

In order to study the asymptotic convergence to an equilibrium for the homogeneous inelastic Boltzman equation, the strategy of "perturbation of semigroup spectral gap" is first introduced in [7]. Inspired by its recent application to a neuron network equation in [8], we linearize the equation around a stationary state $(F_\varepsilon, M_\varepsilon, M_\varepsilon)$ on the variation $(g, n, q) = (f, m, p) - (F_\varepsilon, M_\varepsilon, M_\varepsilon)$, such that

$$\partial_t g = -\partial_x g - a(x, \varepsilon\, M_\varepsilon) g - n(t)\, \varepsilon (\partial_\mu a)(x, \varepsilon M_\varepsilon) F_\varepsilon, \tag{1.14a}$$

$$g(t, 0) = q(t), \quad g(0, x) = g_0(x), \tag{1.14b}$$

with

$$q(t) = \int_0^\infty a(x, \varepsilon\, M_\varepsilon) g \,\mathrm{d}x + n(t)\, \varepsilon \int_0^\infty (\partial_\mu a)(x, \varepsilon M_\varepsilon) F_\varepsilon \,\mathrm{d}x \tag{1.15}$$

and

$$n(t) := \int_0^\infty q(t - y) b(\mathrm{d}y), \tag{1.16}$$

while for the step function firing rate (1.6) in [11, 12], it is impossible to linearize the model because of the failure in meeting the condition (1.8). Insteadly, we introduce another more concise linear equation around the steady state on the variation (g, n, p), which writes

$$\partial_t g = -\partial_x g - g \mathbf{1}_{x > \sigma_\varepsilon}, \tag{1.17a}$$

$$g(t, 0) = q(t), \quad g(0, x) = g_0(x), \tag{1.17b}$$

where here and below we note $\sigma_\varepsilon := \sigma(\varepsilon M_\varepsilon)$ for simplicity, with

$$q(t) = \mathcal{P}[g, M_\varepsilon] = \int_0^\infty g \mathbf{1}_{x > \sigma_\varepsilon} \mathrm{d}x. \tag{1.18}$$

By regarding the boundary condition as a source term, we construct the linear generator Λ_ε and the associated semigroup S_{Λ_ε} respectively from the above linear equations to apply the spectral analysis. As in [16, 4, 9, 6], we split the operator Λ_ε into two parts, one of which is α-hypodissipative, $\alpha < 0$, denoted by $\mathcal{B}_\varepsilon$ while the other one is bounded and $\mathcal{B}_\varepsilon$- power regular, denoted as $\mathcal{A}_\varepsilon$. Benefiting from this split, the semigroup S_{Λ_ε} admits a finite dimensional dominant part, thanks to a particular version of the Spectral Mapping Theorem in [9, 6] and the Weyl's Theorem in [16, 4, 9, 6]. As $\varepsilon \to \infty$, the limited semigroup S_∞ becomes positive because of the vanishment of the items with $n(t)$ in (1.14)

and (1.15), which permits the Krein-Rutman Theorem established in [9, 6] to imply that the steady state (F_∞, M_∞) possesses the exponential stability. And so does the stationary state $(F_\varepsilon, M_\varepsilon)$ in the strong connectivity regime after a perturbative argument developed in [7, 15, 6]. Then we extend the exponential stability to our main result Theorem 1.3 in the case without delay by the analysis on the rest term of the linear equation (1.14) or (1.14) compared to the original nonlinear equation. As for the delay case, we replace the delay equation (1.2) by a simple age equation to form an autonomous system with the linear equation (1.14) or (1.14) to generate a semigroup and follow the same strategy.

Actually, the previous works [11, 12] shows the asymptotic stability with simpler explicitly expression and appropriate norm benefiting from the choice of the step function firing rate (1.6). Different from those approach, we take consideration of more realistic and flexible firing rate with more abstract method, which allows us to obtain the dissipativity of the corresponding linear operators without the explicitly exhibited norm. In particular, we are able to establish the existence and uniqueness of weak solutions first as the complement of the results in [10] in the weak connectivity regime and adapt our approach to the strong connectivity regime as well as to the step function firing rate to generalize the stability results obtained in [11, 12] in the case without delay to the case considering of the delay term.

This article is organized by the following plan. In Section 2, we demonstrate the existence and uniqueness of the solution and the stationary state result. In the strong connectivity regime, we introduce the strategy and establish Theorem 1.3 in the case without delay in Section 3, meanwhile the case with delay in section 4. In section 5, we establish Theorem 1.3 again both in the weak regime and the strong regime for the model with the step function firing rate (1.6).

2. Existence and the steady state

2.1. **Existence of the solution.** To establish the existence of a solution to (1.1), we are going to apply a fixed point argument with the benefit of the following lemma.

Lemma 2.1. *Assuming the firing rate a satisfies* (1.4)-(1.5)-(1.8)-(1.10) *for any $m \in L^\infty([0,\,T])$ and $f \in C(\mathbb{R}_+; L^1(\mathbb{R}_+))$ satisfying the equation* (1.1)*, consider the application $\mathcal{J}\colon L^\infty([0,\,T]) \to L^\infty([0,\,T])$,*

$$\mathcal{J}(m)(t) := \int_0^t p(t-y)b(\mathrm{d}y), \quad \textit{with} \quad p(t) = \int_0^\infty a(x, \varepsilon\, m(t)) f(x,t)\mathrm{d}x,$$

then there exist $T > 0$ and $0 < C < 1$ such that the estimate

$$\|\mathcal{J}(m_1) - \mathcal{J}(m_2)\|_{L^\infty([0,\,T])} \le C\|m_1 - m_2\|_{L^\infty([0,\,T])} \tag{2.1}$$

holds for all $(m_1,\, m_2) \in L^\infty([0,\,T])$ and for any $\varepsilon > 0$.

Proof of Lemma 2.1. Thanks to the method of characteristics with respect to x or t, a solution $f(t,x)$ to the equation (1.1) can be expressed as

$$f(x,t) = f_0(x-t)e^{-\int_0^t a(s+x-t, \varepsilon\, m(s))\mathrm{d}s}, \qquad \forall\, x \ge t \tag{2.2}$$

or

$$f(x,t) = p(t-x)e^{-\int_0^x a(s, \varepsilon\, m(s+t-x))\mathrm{d}s}, \qquad \forall\, x \le t. \tag{2.3}$$

We denote f_i, $i=1,\,2$, two solutions to the equation

$$\begin{cases}\partial_t f_i(x,t)+\partial_x f_i(x,t)+a(x,\varepsilon\, m_i(t))f_i(x,t)=0,\\ f_i(0,t)=p_i(t)=\displaystyle\int_0^\infty a(x,\varepsilon\, m_i(t))f_i(x,t)\mathrm{d}x,\end{cases}\tag{2.4}$$

with the same initial data f_0. We rewrite $p_i(t)$ corresponding to two expressions (2.2) and (2.3) as two global activity functions

$$\begin{aligned}p_i(t) &= \int_0^t a(x,\varepsilon\, m_i(t))p_i(t-x)e^{-\int_0^x a(s,\varepsilon\, m_i(s+t-x))\mathrm{d}s}\mathrm{d}x\\ &+ \int_t^\infty a(x,\varepsilon\, m_i(t))f_0(x-t)e^{-\int_0^t a(s+x-t,\varepsilon\, m_i(s))\mathrm{d}s}\mathrm{d}x.\end{aligned}$$

We split $p_1(t)-p_2(t)$ into two items $I_1(t)$ and $I_2(t)$ with

$$\begin{aligned}I_1(t) &= \int_0^t \Big(p_1(t-x)a(x,\varepsilon\, m_1(t))e^{-\int_0^x a(s,\varepsilon\, m_1(s+t-x))\mathrm{d}s}\\ &\quad -p_2(t-x)a(x,m_2(t))e^{-\int_0^x a(s,\varepsilon\, m_2(s+t-x))\mathrm{d}s}\Big)\mathrm{d}x\end{aligned}$$

and $I_2(t)$ as the remainder. In order to control the first item, we divide it into three parts as $I_1(t)=I_{1,1}(t)+I_{1,2}(t)+I_{1,3}(t)$, where

$$\begin{aligned}I_{1,1}(t)&=\int_0^t (p_1-p_2)(t-x)a(x,\varepsilon\, m_1(t))e^{-\int_0^x a(s,\varepsilon\, m_1(s+t-x))\mathrm{d}s}\mathrm{d}x,\\ I_{1,2}(t)&=\int_0^t p_2(t-x)\big(a(x,\varepsilon\, m_1(t))-a(x,\varepsilon\, m_2(t))\big)e^{-\int_0^x a(s,\varepsilon\, m_1(s+t-x))\mathrm{d}s}\mathrm{d}x,\\ I_{1,3}(t)&=\int_0^t p_2(t-x)a(x,m_2(t))\big(e^{-\int_0^x a(s,\varepsilon\, m_1(s+t-x))\mathrm{d}s}-e^{-\int_0^x a(s,\varepsilon\, m_2(s+t-x))\mathrm{d}s}\big)\mathrm{d}x.\end{aligned}$$

Clearly, we have the estimates

$$\|I_{1,1}\|_{L^\infty([0,T])}\le a_1T\|p_1-p_2\|_{L^\infty([0,T])}$$

and

$$\|I_{1,2}\|_{L^\infty([0,T])}\le \varepsilon a_1\|\partial_\mu a\|_{L^\infty_x}T\|m_1-m_2\|_{L^\infty([0,T])}.$$

Since there exists a constant C such that

$$\begin{aligned}&\Big|e^{-\int_0^x a(s,\varepsilon\, m_1(s+t-x))\mathrm{d}s}-e^{-\int_0^x a(s,\varepsilon\, m_2(s+t-x))\mathrm{d}s}\Big|\\ \le\ & C\int_0^x |a(s,\varepsilon\, m_1(s+t-x))-a(s,\varepsilon\, m_2(s+t-x))|\mathrm{d}s,\end{aligned}$$

which leads to the estimate

$$\|I_{1,3}\|_{L^\infty([0,T])}\le \varepsilon a_1^2\|\partial_\mu a\|_{L^\infty_x}\frac{T^2}{2}\|m_1-m_2\|_{L^\infty[0,T]}.$$

From the assumption (1.10), there exists ε_1 large enough such that $\varepsilon\|a'\|_{L^\infty_x}\le 1$, for any $\varepsilon\in[\varepsilon_1,+\infty)$. Denoting $\eta:=\max\{\varepsilon_1,\,1\}$, we deduce that

$$\|I_1\|_{L^\infty([0,T])}\le a_1T\|p_1-p_2\|_{L^\infty([0,T])}+\eta(C_1T^2+C_2T)\|m_1-m_2\|_{L^\infty([0,T])},\tag{2.5}$$

for any $\varepsilon > 0$. On the other hand, we have the item $I_2(t)$ as

$$\begin{aligned} I_2(t) &= \int_t^\infty f_0(x-t)\Big(a(x,\varepsilon\, m_1(t))e^{-\int_0^t a(s+x-t,\varepsilon\, m_1(s))\mathrm{d}s} \\ &\quad -a(x,\varepsilon\, m_2(t))e^{-\int_0^t a(s+x-t,\varepsilon\, m_2(s))\mathrm{d}s}\Big)\mathrm{d}x \\ &= \int_0^\infty f_0(x)\Big(a(x+t,\varepsilon\, m_1(t))e^{-\int_0^t a(s+x,\varepsilon\, m_1(s))\mathrm{d}s} \\ &\quad -a(x+t,\varepsilon\, m_2(t))e^{-\int_0^t a(s+x,\varepsilon\, m_2(s))\mathrm{d}s}\Big)\mathrm{d}x \end{aligned}$$

Clearly, we have

$$a(x+t,\varepsilon\, m(t)) = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t a(s+x,\varepsilon\, m(s))\mathrm{d}s,$$

which implies

$$\begin{aligned} &\Big|a(x+t,\varepsilon\, m_1(t))e^{-\int_0^t a(s+x,\varepsilon\, m_1(s))\mathrm{d}s} - a(x+t,\varepsilon\, m_2(t))e^{-\int_0^t a(s+x,\varepsilon\, m_2(s))\mathrm{d}s}\Big| \\ \leq\ & C\int_0^t |a(s+x,\varepsilon\, m_1(s)) - a(s+x,\varepsilon\, m_2(s))|\mathrm{d}s, \end{aligned}$$

for some constant $C > 0$. Similarly to estimating I_1, we deduce that

$$\|I_2\|_{L^\infty([0,T])} \leq C_3\eta T\|m_1 - m_2\|_{L^\infty([0,T])}. \tag{2.6}$$

From the above estimates (2.5) and (2.6), it turns out that

$$\|p_1 - p_2\|_{L^\infty([0,T])} \leq a_1 T\|p_1 - p_2\|_{L^\infty([0,T])} + \eta T(C_1 T + C_2')\|m_1 - m_2\|_{L^\infty([0,T])},$$

which implies

$$\|p_1 - p_2\|_{L^\infty([0,T])} \leq \eta CT\|m_1 - m_2\|_{L^\infty([0,T])} \tag{2.7}$$

when $a_1 T$ less than 1. Hence, in the case without delay, we have

$$\|\mathcal{J}(m_1) - \mathcal{J}(m_2)\|_{L^\infty([0,T])} = \|p_1 - p_2\|_{L^\infty([0,T])} \leq \eta CT\|m_1 - m_2\|_{L^\infty([0,T])},$$

while taking the delay into account with the fact that

$$\mathcal{J}(m_1)(t) - \mathcal{J}(m_2)(t) = \int_0^t (p_1 - p_2)(t-y)b(\mathrm{d}y),$$

we obviously deduce

$$\|\mathcal{J}(m_1) - \mathcal{J}(m_2)\|_{L^\infty([0,T])} \leq \eta CT^2\|m_1 - m_2\|_{L^\infty([0,T])}$$

from (2.7). By taking T small enough such that $\eta CT^2 < \eta CT < 1$, we finally attain our estimate (2.1). □

Proof of Theorem 1.1. From Lemma 2.1, for any $\varepsilon > 0$, there is a $T > 0$ which does not depend upon the initial data such that the application $\mathcal{J}$ admits a unique fixed point $m(t)$ on $[0,T]$ then the corresponding $f(t,x)$ on $[0,T]\times\mathbb{R}^d$, which is the unique solution to the equation (1.1), according to the Banach-Picard fixed point theorem. Iterating on T, we deduce the global existence and uniqueness of the solution (f,m) to equation (1.1). □

2.2. **The stationary problem.** Now we present the proof of the steady state in the strong connectivity regime.

Proof of Theorem 1.2. Step 1. Existence. From the assumption (1.5), we deduce that for any $x \geq 0$, $\mu \geq 0$, there exists $x_0 \in [0, \infty)$ such that $a(x, \mu) \geq \frac{a_0}{2}$. Denoting

$$A(x, \mu) := \int_0^x a(y, \mu) \mathrm{d}y, \quad \forall, x, m \geq 0,$$

we naturally estimate that

$$\frac{a_0}{2}(x - x_0)_+ \leq A(x, \mu) \leq a_1 x, \quad \forall\, x \geq 0,\ \mu \geq 0. \tag{2.8}$$

For any $m \geq 0$, the equation (1.3a) can be solved by

$$F_{\varepsilon,m}(x) := T_m e^{-A(x,\varepsilon m)},$$

whose mass conservation gives

$$T_m^{-1} = \int_0^\infty e^{-A(x,\varepsilon m)} \mathrm{d}x.$$

Then the existence of the solution is equivalent to find $m = M_\varepsilon$ satisfying $m = F_{\varepsilon,m}(0) = T_m$. Considering

$$\Psi(\varepsilon, m) = m T_m^{-1} := m \int_0^\infty e^{-A(x,\varepsilon m)} \mathrm{d}x,$$

it is merely necessary to find $M_\varepsilon \geq 0$ such that

$$\Psi(\varepsilon, M_\varepsilon) = 1. \tag{2.9}$$

From the Lebesgue dominated convergence theorem, the function $\Psi(\varepsilon, .)$ is continuous. In addition to the fact that $\Psi(0) = 0$ and $\Psi(\infty) = \infty$, the intermediate value theorem implies the existence immediately. The inequality (2.8) shows the estimates (1.13) clearly.

Step 2. Uniqueness in the strong connectivity regime. Obviously,

$$M_\infty := (\int_0^\infty e^{-A(x,\infty)} \mathrm{d}x)^{-1} \in (0, \infty)$$

is the unique solution to $\Psi(\infty, M_\infty) = 1$. It is clear that

$$\frac{\partial}{\partial m}\Psi(\varepsilon, m) = \int_0^\infty e^{-A(x,\varepsilon m)} \big(1 - m \int_0^x \varepsilon \partial_\mu a(y, \varepsilon m) \mathrm{d}y\big) \mathrm{d}x,$$

is continuous with respect to the two variables because of (1.8), which implies that $\Psi \in C^1$. Coupled with that

$$\frac{\partial}{\partial m}\Psi(\varepsilon, m)|_{\varepsilon=\infty} = \int_0^\infty e^{-A(x,\infty)} \mathrm{d}x > 0,$$

we conclude from the implicit function theorem that there exists $\varepsilon_1 > 0$, large enough, such that the equation (2.9) has a unique solution for any $\varepsilon \in (\varepsilon_1, +\infty]$. □

3. Case without delay

In this section, we conclude our main result Theorem 1.3 gradually in the case without delay.

3.1. **Linearized equation and structure of the spectrum.** We introduce the linearized equation on the variation (g, n) around the steady state $(F_\varepsilon, M_\varepsilon)$ given by

$$\partial_t g + \partial_x g + a_\varepsilon g + a'_\varepsilon F_\varepsilon n(t) = 0,$$
$$g(t,0) = n(t) = \int_0^\infty (a_\varepsilon g + a'_\varepsilon F_\varepsilon n(t))\,\mathrm{d}x, \quad g(0,x) = g_0(x),$$

with notes $a_\varepsilon := a(x, \varepsilon M_\varepsilon)$ and $a'_\varepsilon := \varepsilon\,(\partial_\mu a)(x, \varepsilon M_\varepsilon)$ for simplification. According to the assumption (1.10), there exists $\varepsilon_1 > 0$, large enough, such that

$$\forall\, \varepsilon \in (\varepsilon_1, \infty) \qquad \kappa := \int_0^\infty a'_\varepsilon F_\varepsilon \mathrm{d}x < 1,$$

permitting to define

$$n(t) = \mathcal{M}_\varepsilon[g] := (1-\kappa)^{-1} \int_0^\infty a_\varepsilon g\,\mathrm{d}x. \tag{3.1}$$

We consider the operator L_ε to the above linearized euqation given by

$$L_\varepsilon g := \partial_x g - a_\varepsilon g - a'_\varepsilon F_\varepsilon \mathcal{M}_\varepsilon[g]$$

in domain

$$D(L_\varepsilon) := \{g \in W^{1,1}(\mathbb{R}_+), g(0) = \mathcal{M}_\varepsilon[g]\}$$

generating the semigroup S_{L_ε} on space $X := L^1(\mathbb{R}_+)$. Then for any initial datum $g_0 \in X$, the weak solution of the linearized equation is given by $g(t) = S_{L_\varepsilon}(t) g_0$. By regarding the boundary condition as a source term, we rewrite the linearized equation as

$$\partial_t g = \Lambda_\varepsilon g := -\partial_x g - a_\varepsilon g - a'_\varepsilon F_\varepsilon \mathcal{M}_\varepsilon[g] + \delta_{x=0}\mathcal{M}_\varepsilon[g], \tag{3.2}$$

with the associated semigroup S_{Λ_ε}, acting on the space of bounded Radon measures

$$\mathcal{X} := M^1(\mathbb{R}_+) = \{g \in (C_0(\mathbb{R}))';\ \operatorname{supp} g \subset \mathbb{R}_+\},$$

endowed with the weak $*$ topology $\sigma(M^1, C_0)$, where C_0 represents the space of continuous functions converging to 0 at infinity. From the duality of $S_{\Lambda^*_\varepsilon}$, we deduce $S_{\Lambda_\varepsilon}|_X = S_{L_\varepsilon}$. The spectral analysis theory referred to [5, 6] indicates the structure of the spectrum denoting by $\Sigma(\Lambda_\varepsilon)$ and the associated semigroup S_{Λ_ε}.

Theorem 3.1. *Assume* (1.4)-(1.5)-(1.8)-(1.10) *and define* $\alpha := -a_0/2 < 0$. *The operator* Λ_ε *is the generator of a weakly* $*$ *continuous semigroup* S_{Λ_ε} *acting on* $\mathcal{X}$. *Moreover, there exists a finite rank projector* $\Pi_{\Lambda_\varepsilon,\alpha}$ *which commutes with* S_{Λ_ε}, *an integer* $j \geq 0$ *and some complex numbers*

$$\xi_1, ..., \xi_j \in \Delta_\alpha := \{z \in \mathbb{C},\ \Re e\, z > \alpha\},$$

such that on $E_1 := \Pi_{\Lambda_\varepsilon,\alpha}\mathcal{X}$ *the restricted operator satisfies*

$$\Sigma(\Lambda_{\varepsilon|E_1}) \cap \Delta_\alpha = \{\xi_1, ..., \xi_j\}$$

(with the convention $\Sigma(\Lambda_{\varepsilon|E_1}) \cap \Delta_\alpha = \emptyset$ *when* $j = 0$*) and for any* $a > \alpha$ *there exists a constant* C_a *such that the remainder semigroup satisfies*

$$\|S_{\Lambda_\varepsilon}(I - \Pi_{\Lambda_\varepsilon,\alpha})\|_{\mathscr{B}(\mathcal{X})} \leq C_a e^{at}, \quad \forall\, t \geq 0.$$

In order to apply the Spectral Mapping Theorem of [9, 6] and the Weyl's Theorem of [16, 4, 9, 6], we split the operator Λ_ε as $\Lambda_\varepsilon = \mathcal{A}_\varepsilon + \mathcal{B}_\varepsilon$ defined on $\mathcal{X}$ by

$$\mathcal{A}_\varepsilon g := \mu_\varepsilon \mathcal{M}_\varepsilon[g], \quad \mu_\varepsilon := \delta_0 - a'_\varepsilon F_\varepsilon, \tag{3.3}$$

$$\mathcal{B}_\varepsilon g := -\partial_x g - a_\varepsilon g. \tag{3.4}$$

As in the weak connectivity regime in [10], the properties of the two auxiliary operators still hold in the strong one, which implies Theorem 3.1.

Lemma 3.2. *Assume that a satisfies* (1.4)-(1.5)-(1.10)*, then the operators $\mathcal{A}_\varepsilon$ and $\mathcal{B}_\varepsilon$ satisfy the following properties.*

(i) $\mathcal{A}_\varepsilon \in \mathscr{B}(W^{-1,1}(\mathbb{R}_+), \mathcal{X})$.
(ii) $S_{\mathcal{B}_\varepsilon}$ *is α-hypodissipative both in X and $\mathcal{X}$.*
(iii) *The family of operators $S_{\mathcal{B}_\varepsilon} * \mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}$ satisfies*

$$\|(S_{\mathcal{B}_\varepsilon} * \mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})(t)\|_{\mathcal{X}\to Y} \leq C_a e^{at}, \quad \forall a > \alpha.$$

for some positive constant C_a, where $Y := BV(\mathbb{R}_+) \cap L^1_1(\mathbb{R}_+)$ with $BV(\mathbb{R}_+)$ representing the space of bounded variation measures while $L^1_1(\mathbb{R}_+)$ denoting the Lebesgue space weighted by $\langle x\rangle := (1+|x|^2)^{1/2}$.

Proof. (i) Under the assumption (1.8) and (1.10), there exists some constant $K > 0$ such that $|\varepsilon\partial_\mu a| < K$, for any $\varepsilon \geq 0$, and we naturally have $\mathcal{M}_\varepsilon[\cdot] \in \mathscr{B}(W^{-1,1}(\mathbb{R}_+), \mathbb{R})$, which implies that $\mu_\varepsilon \in \mathscr{B}(\mathbb{R}, \mathcal{X})$. Then, we conclude as $\mathcal{A}_\varepsilon = \mu_\varepsilon \mathcal{M}_\varepsilon[\cdot]$.

(ii) Here and below, we mark $A(x, \varepsilon M_\varepsilon)$ as $A(x)$ without obscurity. The explicit formula gives

$$S_{\mathcal{B}_\varepsilon}(t)g(x) = e^{A(x-t)-A(x)} g(x-t)\mathbf{1}_{x-t\geq 0}.$$

From the assumption (1.5), there exists $x_1 \in [0,\infty)$ such that $A(x) \geq \frac{3}{4}a_0(x-x_1)_+$, which implies

$$A(x-t) - A(x) \leq C\, e^{3\beta t}, \quad 0 \leq t \leq x, \tag{3.5}$$

where $C = e^{3a_0x_1/4}$ and $\beta = -a_0/4 < 0$. Next, we conclude

$$\|S_{\mathcal{B}_\varepsilon}(t)g\|_X \leq C\, e^{3\beta t}\|g\|_X, \quad t \geq 0,$$

with From (weakly *) density argument, we also have the same estimate in $\mathcal{X}$. Then, $S_{\mathcal{B}_\varepsilon}$ is $\alpha-$ hypodissipative both in X and $\mathcal{X}$, as $\alpha > 3\beta$.

(iii) We denote

$$N(t) := \mathcal{M}_\varepsilon[S_{\mathcal{B}_\varepsilon}(t)g] = (1-\kappa)^{-1}\int_0^\infty a_\varepsilon e^{A(x-t)-A(x)} g(x-t)\mathbf{1}_{x-t\geq 0}\mathrm{d}x.$$

From the assumption (1.8), $N \in C^1_b(\mathbb{R}_+)$, we compute

$$\begin{aligned} N'(t) &= (1-\kappa)^{-1}\int_0^\infty \partial_x\Big(a_\varepsilon e^{-A(x)}\Big)e^{A(x-t)} g(x-t)\mathbf{1}_{x-t\geq 0}\mathrm{d}x \\ &= (1-\kappa)^{-1}\int_0^\infty (a_\varepsilon' - a_\varepsilon^2)e^{A(x-t)-A(x)} g(x-t)\mathbf{1}_{x-t\geq 0}\mathrm{d}x. \end{aligned}$$

The inequality (3.5) together with the assumption (1.8) and (1.10) permit us to have the following estimates

$$|N(t)| \leq (1-\kappa)^{-1}Ca_1\int_0^\infty e^{3\beta t}g(x)\mathrm{d}x \lesssim e^{3\beta t}\|g\|_{\mathcal{X}}, \tag{3.6}$$

$$|N'(t)| \leq (1-\kappa)^{-1}C(K+a_1^2)\int_0^\infty e^{3\beta t}g(x)\mathrm{d}x \lesssim e^{3\beta t}\|g\|_{\mathcal{X}}. \tag{3.7}$$

We continue to analyse the operator,

$$\begin{aligned}(S_{\mathcal{B}_\varepsilon} * \mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})(t)g(x) &= \int_0^t (S_{\mathcal{B}_\varepsilon}(s)\mu_\varepsilon)(x)N(t-s)\mathrm{d}s \\ &= \int_0^t e^{A(x-s)-A(x)}\mu_\varepsilon(x-s)N(t-s)\mathbf{1}_{t-s\geq 0} \\ &= e^{-A(x)}(\nu_\varepsilon * \check{N}_t)(x),\end{aligned}$$

where $\nu_\varepsilon = e^A\mu_\varepsilon$ and $\check{N}_t(\cdot) = N(t-\cdot)$. And its partial derivative with respect to x is

$$\begin{aligned}\partial_x(S_{\mathcal{B}_\varepsilon} * \mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})(t)g &= -a_\varepsilon e^{-A(x)}(\nu_\varepsilon * \check{N}_t)(x) - e^{-A(x)}(\nu_\varepsilon * \check{N}_t')(x) \\ &\quad -e^{-A(x)}\nu_\varepsilon(x-t)N(0)\mathbf{1}_{x-t\geq 0} + e^{-A(x)}\nu_\varepsilon(x)N(t).\end{aligned}$$

Using the inequality (3.5),

$$\begin{aligned}\|e^{-A(x)}(\nu_\varepsilon * \check{N}_t)(x)\|_{L^1_1} &\leq \|e^{\beta x}\langle x\rangle\|_{L^1}\|e^{-A(x)}(\nu_\varepsilon * \check{N}_t)e^{-\beta x}\|_{L^\infty} \\ &\lesssim \Big\|\int_0^t e^{3\beta s}|\mu_\varepsilon(x-s)N(t-s)|e^{-\beta x}\mathrm{d}s\Big\|_{L^\infty} \\ &\lesssim e^{2\beta t}\|(|\mu_\varepsilon|e^{-\beta}) * (|N|\check{e^{-2\beta}})_t\|_{L^\infty} \\ &\lesssim e^{\alpha t}\|\mu_\varepsilon e^{-\beta}\|_{\mathcal{X}}\|Ne^{-2\beta}\|_{L^\infty},\end{aligned}$$

since $\|e^{\beta x}\langle x\rangle\|_{L^1} < \infty$. Then from the assumption (1.10) as well as the estimates (1.13) and (3.6), we get

$$\|S_{\mathcal{B}_\varepsilon} * \mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})(t)g\|_{L^1_1} \lesssim e^{\alpha t}(1 + K\|F_\varepsilon e^{-\beta}\|_{\mathcal{X}})\|e^{\beta}\|_{L^\infty}\|g\|_{\mathcal{X}} \lesssim e^{\alpha t}\|g\|_{\mathcal{X}}. \tag{3.8}$$

Similarly, from (3.7), we have

$$\begin{aligned}\|e^{-A(x)}(\nu_\varepsilon * \check{N}_t')(x)\|_{\mathcal{X}} &\lesssim \int_0^t\int_0^\infty e^{3\beta s}|\mu_\varepsilon(x-s)||N'(t-s)|\mathrm{d}s \\ &\lesssim e^{\alpha t}\|\mu_\varepsilon\|_{\mathcal{X}}\int_0^t e^{\beta s}\|g\|_{\mathcal{X}}\mathrm{d}s \\ &\lesssim e^{\alpha t}\|g\|_{\mathcal{X}},\end{aligned}$$

which implies

$$\begin{aligned}\|\partial_x(S_{\mathcal{B}_\varepsilon} * \mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})(t)g\|_{\mathcal{X}} &\lesssim a_1\|e^{-A}(\nu_\varepsilon * \check{N}_t)\|_{\mathcal{X}} + \|e^{-A}(\nu_\varepsilon * \check{N}_t')\|_{\mathcal{X}} \\ &\quad +e^{3\beta t}\|e^{3\beta}\mu_\varepsilon\|_{\mathcal{X}}\|g\|_{\mathcal{X}} + e^{3\beta t}\|e^{-A}\nu_\varepsilon\|_{\mathcal{X}}\|g\|_{\mathcal{X}} \\ &\lesssim e^{\alpha t}\|g\|_{\mathcal{X}}.\end{aligned}$$

We finally conclude (iii) from the above estimate together with (3.8). □

3.2. **Strong connectivity regime - exponential stability of the linearized equation.** Under the assumption (1.10), when the network connectivity parameter ε goes to infinity, the linearized time elapsed operator is simplified as

$$\Lambda_\infty g = -\partial_x g - a(x,\infty)g + \delta_{x=0}\mathcal{M}_\infty[g], \tag{3.9}$$

where $\mathcal{M}_\infty[g] = \int_0^\infty a(x,\infty)g(x)\mathrm{d}x$. Similarly to the vanishing limited case $\varepsilon = 0$ in [10], we also obtain the following properties.

Lemma 3.3. *In the limited case $\varepsilon = \infty$, the operator Λ_∞ and the associated semigroup S_{Λ_∞} satisfy*

(i) S_{Λ_∞} *is positive, i.e.*

$$S_{\Lambda_\infty}(t)g \in \mathcal{X}_+, \quad \forall g \in \mathcal{X}_+, \quad \forall t \geq 0.$$

(ii) $-\Lambda_\infty$ *possesses the strong maximum principle, i.e. for any given* $\mu \in \mathbb{R}$ *and any nontrivial* $g \in D(\Lambda_\infty) \cap \mathcal{X}_+$, *there holds*

$$(-\Lambda_\infty + \mu)g \geq 0 \text{ implies } g > 0.$$

(iii) Λ_∞ *satisfies the complex Kato's inequality, i.e.*

$$\Re e(\operatorname{sgn} g)\Lambda_\infty g \leq \Lambda_\infty |g|, \quad \forall g \in D(\Lambda_\infty^2).$$

Then we conclude the following exponential asymptotic stability.

Theorem 3.4. *There exist some constants* $\alpha < 0$ *and* $C > 0$ *such that* $\Sigma(\Lambda_\infty) \cap \Delta_\alpha = \{0\}$ *and for any* $g_0 \in X$, $\langle g_0 \rangle = 0$, *there holds*

$$\|S_{\Lambda_\infty}(t)g_0\|_X \leq Ce^{\alpha t}\|g_0\|_X, \quad \forall\, t \geq 0. \tag{3.10}$$

Proof. Theorem 1.2 shows that there exists at least one nontrivial $F_\infty \geq 0$ as the eigenvector to 0 and the associated dual eigenvector is $\psi = 1$. From Lemma 3.3-(ii)&(iii), we deduce that the eigenvalue 0 is simple and the associated eigenspace is $\mathrm{Vect}(F_\infty)$. While Lemma 3.3-(i)&(ii) imply that 0 is the only eigenvalue with nonnegative real part. We then conclude from Theorem 3.1. □

We extend the exponential stability property in the limited case to the strong connectivity regime through a perturbation argument.

Theorem 3.5. *There exist some constants* $\varepsilon_1 > 0$, $\alpha < 0$ *and* $C > 0$ *such that for any* $\varepsilon \in [\varepsilon_1, \infty]$ *there hold* $\Sigma(\Lambda_\varepsilon) \cap \Delta_\alpha = \{0\}$ *and*

$$\|S_{\Lambda_\varepsilon}(t)g_0\|_X \leq Ce^{\alpha t}\|g_0\|_X, \quad \forall\, t \geq 0, \tag{3.11}$$

for any $g_0 \in X$, $\langle g_0 \rangle = 0$.

The proof uses the stability theory for semigroups developed in Kato's book [5] and revisited in [7, 15, 6]. Now, we present several results needed in the proof of Theorem 3.5.

Proof. Step 1. Continuity of the operator. Directly from the definitions (3.1), (3.3) and (3.4) of $\mathcal{M}_\varepsilon$, $\mathcal{A}_\varepsilon$ and $\mathcal{B}_\varepsilon$, we have

$$(\mathcal{B}_\varepsilon - \mathcal{B}_\infty)g = (a(x,\infty) - a(x,\varepsilon M_\varepsilon))g$$

and

$$(\mathcal{A}_\varepsilon - \mathcal{A}_\infty)g = (\mathcal{M}_\varepsilon[g] - \mathcal{M}_\infty[g])\,\delta_0 - \varepsilon(\partial_\mu a)(x,\varepsilon M_\varepsilon)\,F_\varepsilon\,\mathcal{M}_\varepsilon[g].$$

From the decay assumption (1.10), there exists positive $\zeta_\varepsilon \to 0$, as $\varepsilon \to +\infty$, such that $|\varepsilon\partial_\mu a(x,\varepsilon M_\varepsilon)| < \zeta_\varepsilon$, for ε large enough. Together with the smoothness assumption (1.8), we deduce that

$$\|\mathcal{B}_\varepsilon - \mathcal{B}_\infty\|_{\mathscr{B}(X)} + \|\mathcal{A}_\varepsilon - \mathcal{A}_\infty\|_{\mathscr{B}(X)} \leq C\,\zeta_\varepsilon, \tag{3.12}$$

in the strong connectivity regime.

Step 2. Perturbation argument. Similarly to [10], we present the sketch as the argument in the proof of [15, Theorem 2.15] (see also [5, 7, 6]). Define

$$K_\varepsilon(z) := (R_{\mathcal{B}_\varepsilon}(z)\mathcal{A}_\varepsilon)^2 R_{\Lambda_\infty}(z)(\Lambda_\varepsilon - \Lambda_\infty) \in \mathscr{B}(\mathcal{X}, X).$$

From Lemma 3.2-(i)&(ii) as well as (3.12), there exist $\varepsilon_1 > 0$ large enough and $C > 0$, such that for any $z \in \Delta_\alpha \backslash B(0,\eta)$, for some $0 < \eta < |\alpha|$ and any $\varepsilon \in [\varepsilon_1, \infty]$, the operator $K_\varepsilon(z)$ satisfies

$$\|K_\varepsilon(z)\|_{\mathscr{B}(X)} \leq C\zeta_\varepsilon \leq C\zeta_{\varepsilon_1} < 1, \tag{3.13}$$

which permits us to well define $(1 - K_\varepsilon(z))^{-1}$ in $\mathscr{B}(X)$. From the Duhamel formula and the inverse Laplace transform, we have

$$R_{\Lambda_\varepsilon} = R_{\mathcal{B}_\varepsilon} - R_{\mathcal{B}_\varepsilon}\mathcal{A}_\varepsilon R_{\mathcal{B}_\varepsilon} + (R_{\mathcal{B}_\varepsilon}\mathcal{A}_\varepsilon)^2 R_{\Lambda_\varepsilon}.$$

From the definition of K_ε, we directly get

$$(I - K_\varepsilon)R_{\Lambda_\varepsilon} = R_{\mathcal{B}_\varepsilon} - R_{\mathcal{B}_\varepsilon}\mathcal{A}_\varepsilon R_{\mathcal{B}_\varepsilon} + (R_{\mathcal{B}_\varepsilon}\mathcal{A}_\varepsilon)^2 R_{\Lambda_\infty}.$$

From (3.13) and since for any $\varepsilon \in [\varepsilon_1, \infty]$, all the terms in the RHS of the above expression are clearly uniformly bounded in $\mathscr{B}(\mathcal{X}, X)$ on $\Delta_\alpha \backslash B(0,\eta)$, we deduce that

$$\Sigma(\Lambda_\varepsilon) \cap \Delta_\alpha \subset B(0,\eta).$$

Thanks to the unique continuity principle for holomorphic functions, we deduce that $R_{\Lambda_\varepsilon}(b)|_X = R_{L_\varepsilon}(b)$ for $b \in \mathbb{R}$ large enough, which implies that $R_{\Lambda_\varepsilon}(z)|_X = R_{L_\varepsilon}(z)$ for any $\Delta_\alpha \backslash B(0,\eta)$. By mean of Dunford integral (see [5, Section III.6.4] or [4, 6]), we express the eigenprojector Π_ε as

$$\begin{aligned}
\Pi_\varepsilon &= \frac{i}{2\pi}\int_{|z|=\eta} R_{\Lambda_\varepsilon}(z)\,\mathrm{d}z \\
&= \frac{i}{2\pi}\int_{|z|=\eta} (I - K_\varepsilon)R_{\Lambda_\varepsilon}\,\mathrm{d}z + \frac{i}{2\pi}\int_{|z|=\eta} K_\varepsilon R_{\Lambda_\varepsilon}\,\mathrm{d}z \\
&= \frac{i}{2\pi}\int_{|z|=\eta} (R_{\mathcal{B}_\varepsilon}\mathcal{A}_\varepsilon)^2 R_{\Lambda_\infty}\,dz + \frac{i}{2\pi}\int_{|z|=\eta} K_\varepsilon R_{\Lambda_\varepsilon}\,\mathrm{d}z,
\end{aligned}$$

where the contribution of holomorphic functions vanish. In a similar way, we have

$$\Pi_\infty = \frac{i}{2\pi}\int_{|z|=\eta} R_{\Lambda_\infty}(z)\,\mathrm{d}z = \frac{i}{2\pi}\int_{|z|=\eta} (R_{\mathcal{B}_\infty}\mathcal{A}_\infty)^2 R_{\Lambda_0}\,\mathrm{d}z.$$

Next, we compute that

$$\begin{aligned}
(R_{\mathcal{B}_\varepsilon}\mathcal{A}_\varepsilon)^2 - (R_{\mathcal{B}_\infty}\mathcal{A}_\infty)^2 &= R_{\mathcal{B}_\varepsilon}\mathcal{A}_\varepsilon R_{\mathcal{B}_\varepsilon}\{(\mathcal{A}_\varepsilon - \mathcal{A}_\infty) + (\mathcal{B}_\infty - \mathcal{B}_\varepsilon)R_{\mathcal{B}_\infty}\mathcal{A}_\infty\} \\
&\quad + R_{\mathcal{B}_\varepsilon}\{(\mathcal{A}_\varepsilon - \mathcal{A}_\infty) + \mathcal{A}_\infty(\mathcal{B}_\infty - \mathcal{B}_\varepsilon)R_{\mathcal{B}_\infty}\}R_{\mathcal{B}_\infty}\mathcal{A}_\infty.
\end{aligned}$$

From the above identity together with the estimates (3.12) and (3.13), we deduce that

$$\begin{aligned}
\|(\Pi_\varepsilon - \Pi_\infty)g\|_X &= \|(\Pi_\varepsilon - \Pi_\infty)g\|_{\mathcal{X}} \\
&\leq \frac{1}{2\pi}\int_{|z|=\eta} \|((R_{\mathcal{B}_\varepsilon}\mathcal{A}_\varepsilon)^2 - (R_{\mathcal{B}_\infty}\mathcal{A}_\infty)^2)R_{\Lambda_\infty}g\|_{\mathcal{X}}\,\mathrm{d}z \\
&\quad + \frac{1}{2\pi}\int_{|z|=\eta} \|K_\varepsilon R_{\Lambda_\varepsilon}g\|_{\mathcal{X}}\,\mathrm{d}z \\
&\leq C\,\zeta_\varepsilon\,\|g\|_{\mathcal{X}},
\end{aligned}$$

for any $g \in X$. Therefore, the eigenprojector Π_ε satisfies that

$$\|\Pi_\varepsilon - \Pi_\infty\|_{\mathscr{B}(X)} < 1, \quad \forall \varepsilon \in [\varepsilon_1, \infty].$$

Step 3. Spectral gap. From the classical result [5, Section I.4.6] (or more explicitly [15, Lemma 2.18]), we deduce that there exists a unique simple eigenvalue $\xi_\varepsilon \in \Delta_\alpha$ such that

$$\Sigma(\Lambda_\varepsilon) \cap \Delta_\alpha = \{\xi_\varepsilon\},$$

for any $\varepsilon \in [\varepsilon_1, +\infty]$. From the fact of mass conservation, we have $1 \in \mathcal{X}'$ and $\Lambda_\varepsilon^* 1 = 0$, which implies the conclusion since $\xi_\varepsilon = 0$. □

3.3. **Strong connectivity regime - nonlinear exponential stability.** Now, we come back to the nonlinear problem (1.1) in the case without delay as

$$m(t) = p(t) = \int_0^\infty a(x, \varepsilon m(t)) f(x)\, \mathrm{d}x.$$

In order to show that $m(t)$ is well defined, we recall the optimal transportation Monge-Kantorovich-Wasserstein distance on the probability measures set $\mathbf{P}(\mathbb{R}_+)$ associated to the distance $d(x, y) = |x - y| \wedge 1$, denoting as W_1 and given by

$$\forall\, f, g \in \mathbf{P}(\mathbb{R}_+), \quad W_1(f, g) := \sup_{\varphi, \|\varphi\|_{W^{1,\infty}} \le 1} \int_0^\infty (f - g)\, \varphi.$$

In addition, we define $\Phi : L^1(\mathbb{R}_+) \times \mathbb{R} \to \mathbb{R}$ as

$$\Phi[g, \mu] := \int_0^\infty a(x, \varepsilon\mu) g(x)\, \mathrm{d}x - \mu.$$

Lemma 3.6. *Under the assumption* (1.5)-(1.8)-(1.10)*, there exists $\varepsilon_1 > 0$ large enough such that for any $\varepsilon \in (\varepsilon_1, +\infty)$, the equation $\Phi(g, \mu) = 0$ towards μ has a unique nonnegative solution $\mu = \varphi_\varepsilon[g]$, where $\varphi_\varepsilon : \mathbf{P}(\mathbb{R}) \to \mathbb{R}$ is Lipschitz continuous for the weak topology of probability measures.*

Proof. Step 1. Existence. For any $g \in \mathbf{P}(\mathbb{R})$, we obviously have $\Phi(g, 0) > 0$ while for any $g \in \mathbf{P}(\mathbb{R})$ and $\mu > a_1$, we have

$$\Phi(g, \mu) \le a_1 - \mu < 0.$$

Thanks to the intermediate value theorem, for any fixed $g \in \mathbf{P}(\mathbb{R}_+)$ and any $\varepsilon \ge 0$, there exists at least one solution $\mu \in (0, a_1]$ to the equation $\Phi(g, \mu) = 0$ from the continuity property of Φ.

Step 2. Uniqueness and Lipschitz continuity. For any $f, g \in \mathbf{P}(\mathbb{R}_+)$, from Step 1, we are able to consider $\mu, \nu \in \mathbb{R}_+$ such that

$$\Phi(f, \mu) = \Phi(g, \nu) = 0,$$

which implies that

$$\nu - \mu = \int_0^\infty a(x, \varepsilon\nu)(g - f) + \int_0^\infty (a(x, \varepsilon\nu) - a(x, \varepsilon\mu)) f.$$

From the definition of W_1 and the assumption (1.10), we have

$$\begin{aligned} \Big| \int_0^\infty a(x, \varepsilon\nu)(g - f) \Big| &\le \|a(\cdot, \varepsilon\nu)\|_{W^{1,\infty}}\, W_1(g, f), \\ \Big| \int_0^\infty \big(a(x, \varepsilon\nu) - a(x, \varepsilon\mu)\big) f \Big| &\le \|a(\cdot, \varepsilon\nu) - a(\cdot, \varepsilon\mu)\|_{L^\infty} \le \zeta_\varepsilon |\mu - \nu|, \end{aligned}$$

where $\zeta_\varepsilon \to 0$ as $\varepsilon \to \infty$. We then take ε_1 large enough, such that

$$1 - \zeta_\varepsilon \in (0, 1), \quad \forall \varepsilon \in [\varepsilon_1, \infty],$$

which permitting us to obtain

$$|\mu-\nu|\,(1-\zeta_\varepsilon)\le \|a(\cdot,\varepsilon\nu)\|_{W^{1,\infty}}\,W_1(g,f). \tag{3.14}$$

On the one hand, when $f=g$, we immediately deduce that $\mu=\nu$, the uniqueness of the definition of the mapping $\varphi_\varepsilon[g]:=\mu$. On the other hand, we get the Lipschitz continuity of the function φ_ε directly from the inequation (3.14). □

We present the proof of our main result Theorem 1.3 in the case without delay.

Proof of Theorem 1.3 in the case without delay. We split the proof into three steps.
Step 1. New formulation. Benefiting from Lemma 3.6, in the strong connectivity regime $\varepsilon\in[\varepsilon_1,\infty)$, where ε_1 is the same as that in Lemma 3.6, we introduce a new formulation of the solution $f\in C([0,\infty);X)$ to the evolution equation (1.1) and the solution F_ε to the stationary problem (1.3) satisfying

$$\begin{aligned}\partial_t f+\partial_x f+a(\varepsilon\varphi[f])f=0, &\qquad f(t,0)=\varphi[f(t,\cdot)],\\ \partial_x F+a(\varepsilon M)F=0, &\qquad F(0)=M=\varphi[F],\end{aligned}$$

for a given unit mass initial datum $0\le f_0\in X$, where here and below the ε and x dependency is often removed without any confusion.

Next, we consider the variation function $g:=f-F$ satisfying

$$\partial_t g=-\partial_x g-a(\varepsilon M)g-\varepsilon a'(\varepsilon M)F\,\mathcal{M}[g]-Q[g], \tag{3.15}$$

where

$$Q[g]:=a(\varepsilon\varphi[f])f-a(\varepsilon\varphi[F])F-a(\varepsilon\varphi[F])g-\varepsilon a'(\varepsilon\varphi[F])F\mathcal{M}[g],$$

with $\mathcal{M}=\mathcal{M}_\varepsilon$ defined in (3.1), complemented with the boundary condition given by

$$\begin{aligned}g(t,0) &= \varphi[f(t,\cdot)]-\varphi[F]\\ &= \int_0^\infty a(\varepsilon\varphi[f])f-\int_0^\infty a(\varepsilon\varphi[F])F\\ &= \mathcal{M}[g]+\mathcal{Q}[g],\end{aligned}$$

where $\mathcal{Q}[g]:=\langle Q[g]\rangle$. Regarding the boundary condition as a source term again, we deduce that the variation function g satisfies the equation

$$\partial_t g=\Lambda_\varepsilon g+Z[g], \tag{3.16}$$

with the nonlinear term $Z[g]:=-Q[g]+\delta_0\mathcal{Q}[g]$.
Step 2. The nonlinear term. With the fact that f is mass conserved, $\|F\|_X=1$ and the assumption (1.10), we estimate that

$$\begin{aligned}\|Q[g]\|_X &= \|a(\varepsilon\varphi[f])f-a(\varepsilon\varphi[F])f-\varepsilon a'(\varepsilon\varphi[F])F\,\mathcal{M}[g]\|_X\\ &\le \varepsilon\|a'\|_{L^\infty_x}\|f\|_X\big|\varphi[f]-\varphi[F]\big|+\varepsilon\|a'\|_{L^\infty_x}\|F\|_X\mathcal{M}[g]\\ &\lesssim \zeta_\varepsilon\big(\mathcal{M}[g]+\|Q[g]\|_X\big)+\zeta_\varepsilon\mathcal{M}[g],\end{aligned}$$

where $\zeta_\varepsilon\to 0$ as $\varepsilon\to+\infty$. Considering that

$$\mathcal{M}[g]\le a_1(1-\kappa)^{-1}\|g\|X\lesssim\|g\|_X,$$

from the above inequality, we deduce that

$$\|Q[g]\|\lesssim\zeta_\varepsilon\mathcal{M}[g]\lesssim\zeta_\varepsilon\|g\|_X,$$

with ε large enough. We then obtain

$$\|Z[g]\|_X\le 2\|Q[g]\|_X\lesssim\zeta_\varepsilon\|g\|_X$$

Step 3. Decay estimate. Thanks to the Duhamel formula, the solution g to the evolution equation (3.16) satisfies

$$g(t) = S_{\Lambda_\varepsilon}(t)(g_0) + \int_0^t S_{\Lambda_\varepsilon}(t-s) Z[g(s)]\,\mathrm{d}s.$$

Benefiting from Theorem 3.5 and the second step, we deduce

$$\begin{aligned} \|g(t)\|_X &\leq C\, e^{\alpha t}\, \|g_0\|_X + \int_0^t C\, e^{\alpha(t-s)}\, \|Z[g(s)]\|_X\,\mathrm{d}s \\ &\lesssim e^{\alpha t}\, \|g_0\|_X + \zeta_\varepsilon \int_0^t e^{\alpha(t-s)}\, \|g(s)\|_X\,\mathrm{d}s, \end{aligned}$$

for any $t \geq 0$ and for some constant $\alpha < 0$, independent of $\varepsilon \in [\varepsilon_1, +\infty)$. Thanks to the Gronwall's lemma (for linear integral inequality), we have

$$\begin{aligned} \|g(t)\|_X &\lesssim e^{\alpha t}\|g_0\|_X + \zeta_\varepsilon \|g_0\|_X \int_0^t e^{\alpha t} \exp\{\int_s^t e^{\alpha(t-r)}\mathrm{d}r\}\mathrm{d}s \\ &\lesssim e^{\alpha t}\|g_0\|_X + \zeta_\varepsilon t e^{\alpha t}\|g_0\|_X \\ &\lesssim e^{\alpha' t}\|g_0\|_X, \end{aligned}$$

for some constant $\alpha < \alpha' < 0$. □

4. Case with delay

In this section, we conclude our main result Theorem 1.3, in the case with delay by following the same strategy as section 3 but with appropriate adaptation towards the boundary term. Recalling from Theorem 1.2, we already know that their exists a uinque stationary state $(F_\varepsilon, M_\varepsilon)$ in the strong connectivity regime, therefore, we start from the linearization of the evolution equation (1.1).

4.1. **Linearized equation and structure of the spectrum.** We still consider the variation functions $(g, n, q) = (f, m, p) - (F_\varepsilon, M_\varepsilon, M_\varepsilon)$ around the steady state. However, because of the failure to express $n(t)$ explicitly, we introduce the following intermediate evolution equation on a function $v = v(t, y)$

$$\partial_t v + \partial_y v = 0, \quad v(t, 0) = q(t), \quad v(0, y) = 0, \tag{4.1}$$

where $y \geq 0$ represent the local time for the network activity. Clearly, the above equation can be solved by the characteristics method as

$$v(t, y) = q(t - y)\mathbf{1}_{0 \leq y \leq t},$$

which simplifies the expression of the variation $n(t)$ of network activity in the equation (1.16), given by

$$n(t) = \mathcal{D}[v(t)], \quad \mathcal{D}[v] := \int_0^\infty v(y) b(\mathrm{d}y),$$

while the variation $q(t)$ of discharging neurons in the equation (1.15) is simplified as

$$q(t) = \mathcal{O}_\varepsilon[g(t), v(t)] := \mathcal{N}_\varepsilon[g(t)] + \kappa_\varepsilon\, \mathcal{D}[v(t)],$$

where

$$\mathcal{N}_\varepsilon[g] := \int_0^\infty a_\varepsilon(M_\varepsilon) g\,\mathrm{d}x, \quad \kappa_\varepsilon := \int_0^\infty a'_\varepsilon(M_\varepsilon) F_\varepsilon\,\mathrm{d}x.$$

Therefore, we may rewrite the linearized system (1.14)-(1.15)-(1.16) together with the evolution equation (4.1) as an autonomous system

$$\partial_t \begin{pmatrix} g \\ v \end{pmatrix} = \mathscr{L}_\varepsilon \begin{pmatrix} g \\ v \end{pmatrix} := \begin{pmatrix} -\partial_x g - a_\varepsilon g - a'_\varepsilon F_\varepsilon \mathcal{D}[v] \\ -\partial_y v \end{pmatrix}, \tag{4.2}$$

with the associated semigroup $S_{\mathscr{L}_\varepsilon}(t)$ acting on

$$X := X_1 \times X_2 := L^1(\mathbb{R}_+) \times L^1(\mathbb{R}_+, \mu),$$

where the measure $\mu(x) = e^{-\delta x}$ with the same $\delta > 0$ in the condition (1.12). The domain of the operator $\mathscr{L}_\varepsilon$ is given by

$$D(\mathscr{L}_\varepsilon) := \{(g, v) \in W^{1,1}(\mathbb{R}_+) \times W^{1,1}(\mathbb{R}_+, \mu);\ g(0) = v(0) = \mathcal{O}_\varepsilon[g, v]\}.$$

We also consider the boundary condition as a source term and rewrite the autonomous systemwe as

$$\partial_t (g, v) = \Lambda_\varepsilon (g, v),$$

where the generator $\Lambda_\varepsilon = (\Lambda^1_\varepsilon, \Lambda^2_\varepsilon)$ is given by

$$\begin{aligned} \Lambda^1_\varepsilon(g, v) &:= -\partial_x g - a_\varepsilon g - a'_\varepsilon F_\varepsilon \mathcal{D}[v] + \delta_{x=0} \mathcal{O}_\varepsilon[g, v], \\ \Lambda^2_\varepsilon(g, v) &:= -\partial_y v + \delta_{y=0} \mathcal{O}_\varepsilon[g, v]. \end{aligned}$$

and the associated semigroup $S_{\Lambda_\varepsilon}(t)$ acting on

$$\mathcal{X} := \mathcal{X}_1 \times \mathcal{X}_2 := M^1(\mathbb{R}_+) \times M^1(\mathbb{R}_+, \mu).$$

Similarly, we have $S_{\Lambda_\varepsilon}|_X = S_{\mathscr{L}_\varepsilon}$. Next, we are going to show that S_{Λ_ε} also possesses a suitable decomposition of a finite dimensional principal part as well as an exponential decaying remainder.

Theorem 4.1. *Under the assumptions* (1.4)-(1.5)-(1.8)-(1.10) *as well as the condition* (1.12) *and taking delay into account, the conclusions of Theorem 3.1 still holds true with* $\alpha := \max\{-a_0/2, -\delta\} < 0$.

The result also comes from the spectral analysis approach. In order to apply the Spectral Mapping theorem and the Weyl's Theorem established in [9, 6], we split the operator appropriately as $\Lambda_\varepsilon = \mathcal{A}_\varepsilon + \mathcal{B}_\varepsilon$ with

$$\mathcal{B}_\varepsilon(g, v) = \begin{pmatrix} \mathcal{B}^1_\varepsilon(g, v) \\ \mathcal{B}^2_\varepsilon(g, v) \end{pmatrix} = \begin{pmatrix} -\partial_x g - a_\varepsilon g \\ -\partial_y v \end{pmatrix}$$

and

$$\mathcal{A}_\varepsilon(g, v) = \begin{pmatrix} \mathcal{A}^1_\varepsilon(g, v) \\ \mathcal{A}^2_\varepsilon(g, v) \end{pmatrix} = \begin{pmatrix} -a'_\varepsilon F_\varepsilon \mathcal{D}[v] + \delta_{x=0} \mathcal{O}_\varepsilon[g, v] \\ \delta_{y=0} \mathcal{O}_\varepsilon[g, v] \end{pmatrix},$$

which hold the following parallel properties as those in Lemma 3.2.

Lemma 4.2. (i) $\mathcal{A}_\varepsilon \in \mathscr{B}(W^{-1,1}(\mathbb{R}_+) \times W^{-1,1}(\mathbb{R}_+, \mu), \mathcal{X})$.

(ii) $S_{\mathcal{B}_\varepsilon}(t)$ *is* α*-hypodissipative in both* X *and* $\mathcal{X}$*;*

(iii) *the family of operators* $S_{\mathcal{B}_\varepsilon} * \mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}$ *satisfies*

$$\|(S_{\mathcal{B}_\varepsilon} * \mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})(t)\|_{\mathscr{B}(\mathcal{X}, Y)} \le C_a e^{\alpha t}, \quad \forall a > \alpha, \quad \forall t \ge 0,$$

for some constant $C_a > 0$ *and with* $Y := Y_1 \times Y_2$*, where* $Y_1 = BV(\mathbb{R}_+) \cap L^1_1(R_+)$ *and* $Y_2 = BV(\mathbb{R}_+, \mu) \cap L^1_1(R_+, \mu)$.

We skip the proof and refer to the proof of Lemma 3.2 in [10] for more details.

4.2. **Strong connectivity regime - exponential stability of the linearized equation.** In the limited case, i.e. as the network connectivity parameter ε passes to the infinity, κ_ε vanishes from the assumption (1.10), which simplifies the linearized operator as

$$\Lambda_\infty \begin{pmatrix} g \\ v \end{pmatrix} = \begin{pmatrix} -\partial_x g - a(x,\infty)g + \delta_{x=0}\mathcal{O}_\infty[g,v] \\ -\partial_y v + \delta_{y=0}\mathcal{O}_\infty[g,v] \end{pmatrix}, \tag{4.3}$$

where $\mathcal{O}_\infty[g,v] = \mathcal{N}_\infty[g] = \int_0^\infty a(x,\infty)g(x)\mathrm{d}x$. We have already proven the exponential stability of the first component Λ^1_∞, then the Duhamel formula

$$v(t) = S_{\mathcal{B}^2_\infty}(t)v_0 + \int_0^t S_{\mathcal{B}^2_\infty}(t-s)\mathcal{A}^2_\infty\Big(g(s), v(s)\Big)\mathrm{d}s$$

implies the similar exponential asymptotic estimate for the second component Λ^2_∞. Together with Theorem 3.4, we have

Theorem 4.3. *There exist some constants $\alpha < 0$ and $C > 0$ such that $\Sigma(\Lambda_\infty) \cap \Delta_\alpha = \{0\}$ and for any $(g_0,\, v_0) \in X$, $\langle g_0 \rangle = 0$, there holds*

$$\|S_{\Lambda_\infty}(t)(g_0,\, v_0)\|_{\mathcal{X}} \le Ce^{\alpha t}\,\|(g_0,\, v_0)\|_{\mathcal{X}}, \quad \forall\, t \ge 0. \tag{4.4}$$

Then we extend the geometry structure of the spectrum of the linearized time elapsed equation in the limited case to the strong connectivity regime taking delay into account.

Theorem 4.4. *There exist some constants $\varepsilon_1 > 0$, $C \ge 1$ and $\alpha < 0$ such that for any $\varepsilon \in [\varepsilon_1, +\infty]$ there holds $\Sigma(\Lambda_\varepsilon) \cap \Delta_\alpha = \{0\}$ and*

$$\|S_{\Lambda_\varepsilon}(t)(g_0, v_0)\|_{\mathcal{X}} \le Ce^{\alpha t}\|(g_0, v_0)\|_{\mathcal{X}}, \tag{4.5}$$

for any $(g_0, v_0) \in \mathcal{X}$ such that $\langle g_0 \rangle = 0$.

Proof. We proceed in two steps.
Step 1. Continuity of the operator Λ_ε. For all $(g,v) \in \mathcal{X}$, we have

$$\Lambda_\varepsilon \begin{pmatrix} g \\ v \end{pmatrix} = \begin{pmatrix} -\partial_x g - a_\varepsilon g - a'_\varepsilon F_\varepsilon \mathcal{D}_\varepsilon[v] + \delta_{x=0}\mathcal{O}_\varepsilon[g,v] \\ -\partial_y v + \delta_{y=0}\mathcal{O}_\varepsilon[g,v] \end{pmatrix}, \tag{4.6a}$$

$$\Lambda_\infty \begin{pmatrix} g \\ v \end{pmatrix} = \begin{pmatrix} -\partial_x g - a(x,\infty)g + \delta_{x=0}\mathcal{O}_\infty[g,v] \\ -\partial_y v + \delta_{y=0}\mathcal{O}_\infty[g,v] \end{pmatrix}. \tag{4.6b}$$

Compute the difference between (4.6a) and (4.6b), we have

$$(\Lambda_\varepsilon - \Lambda_\infty)\begin{pmatrix} g \\ v \end{pmatrix} = \begin{pmatrix} (a(x,\infty) - a_\varepsilon)g - a'_\varepsilon F_\varepsilon \mathcal{D}_\varepsilon[v] + \delta_{x=0}\big(\mathcal{O}_\varepsilon[g,v] - \mathcal{O}_\infty[g,v]\big) \\ \delta_{y=0}\big(\mathcal{O}_\varepsilon[g,v] - \mathcal{O}_\infty[g,v]\big) \end{pmatrix}.$$

From the assumption (1.5) and (1.10), we deduce that

$$\begin{aligned} \|(\Lambda_\varepsilon - \Lambda_0)(g,v)\|_{\mathcal{X}} &= \|(a(\cdot,\infty) - a_\varepsilon)g\|_{\mathcal{X}_1} + \|a'_\varepsilon F_\varepsilon \mathcal{D}_\varepsilon[v]\|_{\mathcal{X}_1} + 2|\mathcal{O}_\varepsilon[g,v] - \mathcal{O}_\infty[g,v]| \\ &\le 3\|(a_\infty - a_\varepsilon)g\|_{\mathcal{X}_1} + 2\|a'_\varepsilon F_\varepsilon \mathcal{D}_\varepsilon[v]\|_{\mathcal{X}_2} \\ &\le 3\zeta_\varepsilon\|g\|_{\mathcal{X}_1} + 2a_1\zeta_\varepsilon(1-\zeta_\varepsilon)\|F_\varepsilon\|_{\mathcal{X}_1}\|v\|_{\mathcal{X}_2} \\ &\lesssim \zeta_\varepsilon\|(g,v)\|_{\mathcal{X}}, \end{aligned}$$

where $\zeta_\varepsilon \to 0$ as $\varepsilon \to \infty$, which implies the continuity of the operatorΛ_ε to ε in the strong connectivity regime.

Step 2. Extension to the strong connectivity regime. Similar to the proof of Theorem 3.5, we deduce that their exists $\varepsilon_1 > 0$ large enough, such that for any $\varepsilon \in [\varepsilon_1, \infty]$, the eigenprojector Π_ε satisfies

$$\|\Pi_\varepsilon - \Pi_\infty\|_{\mathscr{B}(X)} < 1,$$

which permits us to conclude that there exists ξ_ε satisfying $|\xi_\varepsilon| \leq O(\zeta_\varepsilon)$, such that

$$\Sigma(\Lambda_\varepsilon) \cap \Delta_\alpha = \{\xi_\varepsilon\}$$

and ξ_ε is algebraically simple (see in [5, Section I.4.6] and [15]). Since the dual operator Λ_ε^* is given by

$$\Lambda_\varepsilon^* \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} \partial_x \varphi - a_\varepsilon \varphi + a_\varepsilon(\varphi(0) + \psi(0)) \\ \partial_y \psi + \kappa_\varepsilon b\, \psi(0) + \kappa_\varepsilon b\, \varphi(0) - b \displaystyle\int a_\varepsilon' F_\varepsilon\, \varphi \,\mathrm{d}x \end{pmatrix},$$

for any $(\varphi, \psi) \in \mathcal{X}'$, observe that $\Lambda_\varepsilon^*(1,0) = 0$. Thus, $0 \in \Sigma(\Lambda_\varepsilon^*)$, which implies $\xi_\varepsilon = 0$. Together with the fact that $\langle g_0 \rangle = \langle (g_0, v_0), (1,0) \rangle_{\mathcal{X},\mathcal{X}'} = 0$, the exponential asymptotic estimate (4.5) holds. □

4.3. **Strong connectivity regime - nonlinear exponential stability.** Finally, we focus on the nonlinear problem taking delay into account and present the proof of the first part of our main result in the case with delay, neglecting the connectivity parameter ε most of time without any misleading.

Proof of Theorem 1.3 in case with delay. Inspired by the intermediate evolution equation (4.1), we rewrite the nonlinear problem as a system of (f, u), given by

$$\begin{aligned} \partial_t f &= -\partial_x f - a_\varepsilon(\mathcal{D}[u]) f + \delta_0 \mathbf{P}[f, \mathcal{D}[u]] \\ \partial_t u &= -\partial_y u + \delta_0 \mathbf{P}[f, \mathcal{D}[u]], \end{aligned}$$

where

$$\mathbf{P}[f, m] = \int a(m) f, \quad \mathcal{D}[u] = \int bu.$$

Denoting $U := M \mathbf{1}_{y \geq 0}$, the steady state (F, U) satisfies

$$\begin{aligned} 0 &= -\partial_x F - a_\varepsilon(M) F + \delta_0 M \\ 0 &= -\partial_y U + \delta_0 M, \quad M = \mathcal{D}[U] = \mathbf{P}[F, \mathcal{D}[U]]. \end{aligned}$$

Introducing the variation functions $g := f - F$ and $v = u - U$ again, we obtain the system of (g, v) as

$$\begin{aligned} \partial_t g &= -\partial_x g - a_\varepsilon(\mathcal{D}[u]) f + a_\varepsilon(M) F + \delta_0 (\mathbf{P}[f, \mathcal{D}[u]] - \mathbf{P}[F, \mathcal{D}[U]]) \\ &= -\partial_x g - a_\varepsilon(M) g - a_\varepsilon' F \mathcal{D}[v] - Q[g, v] + \delta_0 \mathcal{O}[g, v] + \delta_0 \mathcal{Q}[g, v] \\ &= \Lambda_\varepsilon^1(g, v) + \mathcal{Z}^1[g, v], \\ \partial_t v &= -\partial_y v + \delta_0 (\mathbf{P}[f, \mathcal{D}[u]] - \mathbf{P}[F, \mathcal{D}[U]]) \\ &= -\partial_y v + \delta_0 \mathcal{O}[g, v] + \delta_0 \mathcal{Q}[g, v] \\ &= \Lambda_\varepsilon^2(g, v) + \mathcal{Z}^2[g, v], \end{aligned}$$

where

$$\begin{aligned} Q[g, v] &:= a_\varepsilon(\mathcal{D}[u]) f - a_\varepsilon(M) F - a_\varepsilon(M) g - a_\varepsilon' F \mathcal{D}[v] \\ \mathcal{Q}[g, v] &:= \langle Q[g, v] \rangle, \end{aligned}$$

and the remainders are given by

$$\begin{aligned}\mathcal{Z}^1[g,v] &:= -Q[g,v]+\delta_0\mathcal{Q}[g,v],\\ \mathcal{Z}^2[g,v] &:= \delta_0\mathcal{Q}[g,v].\end{aligned}$$

From the mass conservation of f and the assumption (1.10), we deduce that

$$\begin{aligned}\|Q[g,v]\|_{X_1} &= a_\varepsilon(\mathcal{D}[u])f - a_\varepsilon(M)f - a'_\varepsilon F\mathcal{D}[v]\\ &\leq \zeta_\varepsilon\|f\|_{X_1}\Big|\mathcal{D}[u]-\mathcal{D}[U]\Big| - \zeta_\varepsilon\|\mathcal{D}[v]\\ &\lesssim \zeta_\varepsilon\|v\|_{X_2},\end{aligned}$$

where ζ_ε is the same as in Theorem 4.4. Denoting $\mathcal{Z}[g,v] := (\mathcal{Z}^1[g,v],\mathcal{Z}^2[g,v])$, we clearly have the estimate

$$\|\mathcal{Z}[g,v]\|_X \lesssim \zeta_\varepsilon\|(g,v)\|_X$$

The associated Duhamel formula writes

$$(g(t),v(t)) = S_{\Lambda_\varepsilon}(t)(g_0,v_0) + \int_0^t S_{\Lambda_\varepsilon}(t-s)\mathcal{Z}[g(s),v(s)]\,\mathrm{d}s.$$

Using the above estimate for the nonlinear term and the Gronwall's lemma, we conclude as in the proof of Theorem 1.3. □

5. Step function firing rate

In this section, we focus on the nonlinear time elapsed model in [11, 12] with a particular step function firing rate given by

$$a(x,\mu) = \mathbf{1}_{x>\sigma(\varepsilon\mu)}$$

satisfying (1.6)-(1.7)-(1.9)-(1.11). We consider the dynamic of the neuron network (1.1) completed with an initial probability density f_0 satisfying

$$0 \leq f_0 \leq 1, \quad \int_0^\infty f_0(x)\mathrm{d}x = 1. \tag{5.1}$$

Obviously, the solution f of the time elapsed equation (1.1) corresponding to the firing rate (1.6) is still mass conserved, and we naturally renormalize that mass. The previous work [11] shows that the model (1.1) with the step function firing rate (1.6) admits a steady state as well as a unique solution, that is to say the second part of Theorem 1.1 and Theorem 1.2. By applying the adapted above spectral analysis method, we conclude the results in Theorem 1.3 for a particular step function firing rate, which accurate the stability results in [11] in the case with delay. Failing to construct the linearized equations (1.14) and (1.15), we replace them with another more concise linear equation for the variation functions $(g,n,q) = (f,m,p) - (F_\varepsilon, M_\varepsilon, M_\varepsilon)$, which writes

$$\begin{aligned}&\partial_t g + \partial_x g + g\mathbf{1}_{x>\sigma_\varepsilon} = 0,\\ &g(t,0) = q(t) := \int_0^\infty g\mathbf{1}_{x>\sigma_\varepsilon}\mathrm{d}x, \quad g(0,x) = g_0(x),\end{aligned}$$

where here and below we note $\sigma_\varepsilon := \sigma(\varepsilon M_\varepsilon)$ for simplicity. We introduce the intermediate evolution equation (4.1) again to write the linear equation (1.17)-(1.18)- (1.16) as a time autonomous system

$$\partial_t(g,v) = \mathscr{L}_\varepsilon(g,v), \tag{5.2}$$

where the operator $\mathscr{L}_\varepsilon = (\mathscr{L}_\varepsilon^1, \mathscr{L}_\varepsilon^2)$ is defined by

$$\begin{aligned} \mathscr{L}_\varepsilon^1(g,v) &:= -\partial_x g - g\mathbf{1}_{x>\sigma_\varepsilon} + \delta_{x=0}\mathcal{R}_\varepsilon[g,v], \\ \mathscr{L}_\varepsilon^2(g,v) &:= -\partial_y v + \delta_{y=0}\mathcal{R}_\varepsilon[g,v], \end{aligned}$$

with the boundary term

$$\mathcal{R}_\varepsilon[g(t),v(t)] := \mathcal{P}[g,M_\varepsilon] = \int_0^\infty g\mathbf{1}_{x>\sigma_\varepsilon}\,\mathrm{d}x,$$

in the space

$$X = X_1 \times X_2 := L_0^p(\mathbb{R}_+) \times L^p(\mathbb{R}_+,\mu)$$

with $L_0^p(\mathbb{R}_+) = \{h \in L^p(\mathbb{R}_+);\ \langle h\rangle = 0\}$, $1 \le p \le \infty$, and $\mu(x) = e^{-\delta x}$, $\delta > 0$ is the same as in the condition (1.12). We extend the exponential stability from the single equation of g to the above autonomous system.

Theorem 5.1. *Assume* (1.6)-(1.7)-(1.9) *and* (1.12) *(with* (1.11)*). There exist some constants $\varepsilon_0 > 0$ ($\varepsilon_1 > 0$), $C \ge 1$ and $\alpha < 0$ such that for any $\varepsilon \in [0,\varepsilon_0]$ $\varepsilon \in [\varepsilon_1,\infty]$ there holds $\Sigma(\mathscr{L}_\varepsilon) \cap \Delta_\alpha = \{0\}$ and*

$$\|S_{\mathscr{L}_\varepsilon}(t)(g_0,v_0)\|_X \le Ce^{\alpha t}\|(g_0,v_0)\|_X, \tag{5.3}$$

for any $(g_0,v_0) \in X$, s.t. $\langle (g_0,v_0),(1,0)\rangle_{X,X'} = 0$.

The extension follows from the Spectral Mapping theorem and the Weyl's Theorem by introducing a convenient splitting of the operator $\mathscr{L}_\varepsilon$ as $\mathscr{L}_\varepsilon = \mathcal{A}_\varepsilon + \mathcal{B}_\varepsilon$ with

$$\mathcal{B}_\varepsilon(g,v) = \begin{pmatrix} \mathcal{B}_\varepsilon^1(g,v) \\ \mathcal{B}_\varepsilon^2(g,v) \end{pmatrix} = \begin{pmatrix} -\partial_x g - g\mathbf{1}_{x>\sigma_\varepsilon} \\ -\partial_y v \end{pmatrix}$$

and

$$\mathcal{A}_\varepsilon(g,v) = \begin{pmatrix} \mathcal{A}_\varepsilon^1(g,v) \\ \mathcal{A}_\varepsilon^2(g,v) \end{pmatrix} = \begin{pmatrix} \delta_{x=0}\mathcal{R}_\varepsilon[g,v] \\ \delta_{y=0}\mathcal{R}_\varepsilon[g,v] \end{pmatrix}.$$

Since the step function firing rate is no longer continuous, we have to consider the resolvent of the operator $\mathcal{B}_\varepsilon$.

Lemma 5.2. *Assume* (1.6)-(1.7)-(1.9) *and* (1.12) *(with* (1.11)*). Then the two operators satisfy*

(i) *$\mathcal{A}_\varepsilon \in \mathscr{B}(X,Y)$, where $Y = \mathbb{C}\delta_0 \times \mathbb{C}\delta_0 \subset X$ with compact embedding;*
(ii) *$S_{\mathcal{B}_\varepsilon}(t)$ is α-hypodissipative in X;*
(iii) *$(R_{\mathcal{B}_\varepsilon}(z)\mathcal{A}_\varepsilon)^2(z) \in \mathscr{B}(X)$, with bound in $\mathcal{O}(\langle z\rangle^{-1})$, $\forall z \in \Delta_{-1}$.*

Proof. In order to simplify the notation, we note

$$\rho(x) := \int_0^x \mathbf{1}_{y>\sigma_\varepsilon}\mathrm{d}y = (x-\sigma_\varepsilon)_+.$$

(i) It is an immediate consequence of the fact that $\mathcal{D} \in \mathscr{B}(X_2;\mathbb{R})$ (because of (1.12)) and $\mathcal{N}_\varepsilon \in \mathscr{B}(X_1;\mathbb{R})$.
(ii) We write $S_{\mathcal{B}_\varepsilon^1}$ and $S_{\mathcal{B}_\varepsilon^2}$ respectively with the explicit formula

$$\begin{aligned} S_{\mathcal{B}_\varepsilon^1}(t)g(x) &= e^{\rho(x-t)-\rho(x)}g(x-t)\mathbf{1}_{x-t\ge 0}, \\ S_{\mathcal{B}_\varepsilon^2}(t)v(y) &= v(y-t)\mathbf{1}_{y-t\ge 0}. \end{aligned}$$

We estimate that

$$\begin{aligned}\|S_{\mathcal{B}^1_\varepsilon}(t)g\|_{X_1} &= \|e^{\rho(x)-\rho(x+t)}g(x)\|_{X_1} = \|e^{(x-\sigma_\varepsilon)_+-(x+t-\sigma_\varepsilon)_+}g(x)\|_{X_1}\\ &\le Ce^{-t}\|g(x)\|_{X_1},\\ \|S_{\mathcal{B}^2_\varepsilon}(t)v\|_{X_2} &= \|e^{-\delta(y+t)}v(y)\|_{L^p} = e^{-\delta t}\|v\|_{X_2},\end{aligned}$$

for any $g \in X_1$ and $v \in X_2$ and any $t \ge 0$, which implies

$$\|S_{\mathcal{B}_\varepsilon}(t)\|_{X\to X} \le Ce^{\alpha t}, \quad t \ge 0,$$

by choosing $C := \max\{2e^{\sigma_\varepsilon}, 1\}$ and $\alpha := \max\{-1, -\delta\}$.
(iii) We have

$$S_{\mathcal{B}^1_\varepsilon}(t)\mathcal{A}_\varepsilon[g,v](x) = \delta_{x=t}\mathcal{R}_\varepsilon[g,v]e^{\rho(x-t)-\rho(x)},$$

and we denote

$$\begin{aligned}k_t(x) &:= \mathcal{A}_\varepsilon S_{\mathcal{B}^1_\varepsilon}(t)\mathcal{A}_\varepsilon[g,v](x) = \delta_{x=0}\int_0^\infty \delta_{x=t}e^{\rho(x-t)-\rho(x)}\mathbf{1}_{x>\sigma_\varepsilon}\mathrm{d}x\\ &= \delta_{x=0}e^{-\rho(t)}\mathbf{1}_{t>\sigma_\varepsilon}\mathcal{R}_\varepsilon[g,v].\end{aligned}$$

Finally, we obtain

$$\begin{aligned}(S_{\mathcal{B}^1_\varepsilon}\mathcal{A}_\varepsilon)^{(*2)}(t)[g,v](x) &= \int_0^t k_{t-s}(x-s)e^{\rho(x-s)-\rho(x)}\mathbf{1}_{x-s\ge 0}\mathrm{d}s\\ &= e^{-\rho(t-x)+\rho(0)-\rho(x)}\mathbf{1}_{t\ge x}\mathbf{1}_{t-x>\sigma_\varepsilon}\mathcal{R}_\varepsilon[g,v]\\ &= e^{-\rho(t-x)-\rho(x)}\mathbf{1}_{t-x>\sigma_\varepsilon}\mathcal{R}_\varepsilon[g,v].\end{aligned}$$

Denoting $\psi_t(x) := e^{-\rho(t-x)-\rho(x)}\mathbf{1}_{t-x>\sigma_\varepsilon}$, we compute its Laplace transform

$$\begin{aligned}\hat{\psi}(z) &= \int_0^\infty e^{-\rho(t-x)-\rho(x)}\mathbf{1}_{t-x>\sigma_\varepsilon}e^{-zt}\mathrm{d}t\\ &= e^{-\rho(x)-zx}\int_{\sigma_\varepsilon}^\infty e^{(t-\sigma_\varepsilon)_+-zt}\mathrm{d}t\\ &= e^{-\rho(x)-z(x+\sigma_\varepsilon)}\int_0^\infty e^{-(1+z)t}\mathrm{d}t\\ &= \frac{1}{1+z}e^{-\rho(x)-z(x+\sigma_\varepsilon)},\end{aligned}$$

with the estimate

$$\begin{aligned}\|\hat{\psi}(z)\|_{X_1} &\le \frac{1}{|1+z|}\int_0^\infty e^{-(x-\sigma_\varepsilon)_+-\Re e\, z(x+\sigma_\varepsilon)}\mathrm{d}x\\ &\le \frac{e^{\sigma_\varepsilon(1-\Re e\, z)}}{|1+z|(1+\Re e\, z)}.\end{aligned}$$

All together, we get

$$\|(R_{\mathcal{B}^1_\varepsilon}(z)\mathcal{A}_\varepsilon)^2(z)(g,v)\|_{X_1} \le \|\hat{\psi}(x)\|_{X_1}|\mathcal{R}_\varepsilon[g,v]| \le \frac{C}{\langle z\rangle}\|(g,v)\|_X,$$

for any $z \in \Delta_{-1}$ and some constant $C > 0$ and similar estimate for the second part

$$\|(R_{\mathcal{B}^2_\varepsilon}(z)\mathcal{A}_\varepsilon)^2(z)(g,v)\|_{X_2} \le \frac{C}{\langle z\rangle}\|(g,v)\|_X.$$

□

Proof of Theorem 5.1. From the factorization formula with the properties of the two auxiliarty operators in the above Lemma 5.2 and benefiting from the perturbation argument in [7, 15, 6], we deduce that for any $\langle g_0\rangle = 0$, there holds

$$\|g(t)\|_{X_1} = \|S_{\mathscr{L}^1_\varepsilon}(t)g_0\|_{X_1} \le Ce^{-t}\,\|g_0\|_{X_1}.$$

The Duhamel formula associated to the equation $\partial_t v = \mathscr{L}^2_\varepsilon(g,v)$ writes

$$v(t) = S_{\mathcal{B}^2_\varepsilon}(t)v_0 + \int_0^t S_{\mathcal{B}^2_\varepsilon}(t-s)\mathcal{A}^2_\varepsilon\big(g(s),v(s)\big)\,\mathrm{d}s.$$

Using the already known estimate on $g(t)$, we deduce

$$\begin{aligned}
\|S_{\mathscr{L}^2_\varepsilon}(t)v_0\|_{X_2} &= \|v(t)\|_{X_2} \le \|S_{\mathcal{B}^2_\varepsilon}(t)v_0\|_{X_2} + \int_0^t \|S_{\mathcal{B}^2_\varepsilon}(t-s)\delta_0\mathcal{N}_\varepsilon[g(s)]\|_{X_2}\,\mathrm{d}s\\
&\le e^{-\delta t}\|v_0\|_{X_2} + \int_0^t e^{-\delta(t-s)}C\,e^{-s}\|g_0\|_{X_1}\,\mathrm{d}s\\
&\le C\,e^{\alpha t}\|(g_0,\,v_0)\|_X
\end{aligned}$$

for some $0 > \alpha > \max\{-1,-\delta\}$, which yields our conclusion. □

Now, we complete the rest part of the proof of Theorem 1.3 to describe the stability in the case with delay more precisely compared to that in [11].

Proof of Theorem 1.3 for the step function firing rate in the case with delay. We write the system as

$$\begin{aligned}
\partial_t f &= -\partial_x f - f\mathbf{1}_{x>\sigma(\varepsilon\mathcal{D}[u])} + \delta_0\mathbf{P}[f,\varepsilon\mathcal{D}[u]]\\
\partial_t u &= -\partial_y u + \delta_0\mathbf{P}[f,\varepsilon\mathcal{D}[u]]
\end{aligned}$$

with

$$\mathbf{P}[f,m] = \int a(m)f, \quad \mathcal{D}[u] = \int bu.$$

We recall that the steady state (F,U), where $U := M\mathbf{1}_{y\ge 0}$, satisfies

$$\begin{aligned}
0 &= -\partial_x F - F\mathbf{1}_{x>\sigma(\varepsilon M)} + \delta_0 M\\
0 &= -\partial_y U + \delta_0 M, \quad M = \mathcal{D}[U] = \mathbf{P}[F,\varepsilon\mathcal{D}[U]].
\end{aligned}$$

We introduce the variation $g := f - F$ and $v = u - U$. The equation on g is

$$\begin{aligned}
\partial_t g &= -\partial_x g - f\mathbf{1}_{x>\sigma(\varepsilon\mathcal{D}[u])} + F\mathbf{1}_{x>\sigma(\varepsilon M)} + \delta_0(\mathbf{P}[f,\varepsilon\mathcal{D}[u]] - \mathbf{P}[F,\varepsilon\mathcal{D}[U]])\\
&= \mathscr{L}^1_\varepsilon(g,v) - \mathcal{Q}[g,v] + \delta_0\langle\mathcal{Q}[g,v]\rangle\\
&= \mathscr{L}^1_\varepsilon(g,v) + \mathcal{Z}^1[g,v],
\end{aligned}$$

with

$$\begin{aligned}
\mathcal{Z}^1[g,v] &= -\mathcal{Q}[g,v] + \delta_0\langle\mathcal{Q}[g,v]\rangle,\\
\mathcal{Q}[g,v] &:= \operatorname{sgn}\big(\mathcal{D}[v]\big)(g+F_\varepsilon)\mathbf{1}_{\mathscr{I}[\mathcal{D}[v]]},
\end{aligned}$$

where the interval

$$\mathscr{I}(n) := (\sigma(\varepsilon M + \varepsilon n_+), \sigma(\varepsilon M + \varepsilon n_-)].$$

The equation on v is

$$\begin{aligned}
\partial_t v &= -\partial_y v + \delta_0(\mathbf{P}[f,\varepsilon\mathcal{D}[u]] - \mathbf{P}[F,\varepsilon\mathcal{D}[U]])\\
&= -\partial_y v + \delta_0\mathcal{O}[g,v] + \delta_0\langle\mathcal{Q}[g,v]\rangle\\
&= \mathscr{L}^2_\varepsilon(g,v) + \mathcal{Z}^2[g,v],
\end{aligned}$$

with

$$\mathcal{Z}^2[g,v] := \delta_0 \langle \mathcal{Q}[g,v] \rangle.$$

We observe that

$$\begin{aligned}
\|\mathcal{Q}[g,v]\|_{L^p} &= \|\operatorname{sgn}(\mathcal{D}[v]) g \mathbf{1}_{\mathscr{I}[\mathcal{D}[v]]}\|_{L^p} \leq \left|\mathscr{I}[\mathcal{D}[v]]\right|^{1/p} \\
&= \left(\sigma(\varepsilon M - \varepsilon \mathcal{D}[v]_-) - \sigma(\varepsilon M + \varepsilon \mathcal{D}[v]_+)\right)^{1/p} \\
&\leq C\left(\varepsilon \|\sigma'\|_\infty \left|\mathcal{D}[v]\right|\right)^{1/p} \\
&\leq C\, \varepsilon(\zeta_\varepsilon) \|v\|_{X_2},
\end{aligned}$$

which implies immediately that

$$\begin{aligned}
\|\mathcal{Z}^1[g,v]\|_{X_1} &\leq \varepsilon\,(\zeta_\varepsilon)\, C \|(g,v)\|_X, \\
\|\mathcal{Z}^2[g,v]\|_{X_2} &\leq \varepsilon\,(\zeta_\varepsilon)\, C \|(g,v)\|_X.
\end{aligned}$$

We write the Duhamel formula

$$\big(g(t), v(t)\big) = S_{\mathscr{L}_\varepsilon}(t)(g_0, v_0) + \int_0^t S_{\mathscr{L}_\varepsilon}(t-s) \mathcal{Z}[g(s), v(s)]\, \mathrm{d}s.$$

and thanks to the Gronwall's Lemma, we conclude the exponential asymptotic stability of Theorem 1.3 for the step function firing rate in the case with delay. □

References

[1] De Masi, A., Galves, A., Löcherbach, E., and Presutti, E. Hydrodynamic limit for interacting neurons. *J. Stat. Phys. 158*, 4 (2015), 866–902.

[2] Fournier, N., and Löcherbach, E. On a toy model of interacting neurons. (2014) arXiv:1410.3263.

[3] Gerstner, W., and Kistler, W. M. *Spiking neuron models.* Cambridge University Press, Cambridge, 2002. Single neurons, populations, plasticity.

[4] Gualdani, M. P., Mischler, S., and Mouthot, C. Factorization of non-symmetric operators and exponential H-Theorem. arXiv:1006.5523, 2010.

[5] Kato, T. *Perturbation theory for linear operators.* Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition.

[6] Mischler, S. *Semigroups in Banach spaces - splitting approach for spectral analysis and asymptotics estimates.* work in progress.

[7] Mischler, S., and Mouhot, C. Stability, convergence to self-similarity and elastic limit for the Boltzmann equation for inelastic hard spheres. *Comm. Math. Phys. 288*, 2 (2009), 431–502.

[8] Mischler, S., Quiñinao, C., and Touboul, J. On a kinetic FitzHugh-Nagumo model of neuronal network. (2015) hal-01108872.

[9] Mischler, S., and Scher, J. Spectral analysis of semigroups and growth-fragmentation equations. accepted to publication in Ann. Inst. H. Poincaré Anal. Non Linéaire, hal-00877870.

[10] Mischler, S., and Weng, Q. Relaxation in time elapsed neuron network models in the weak connectivity regime. (2015) hal-01148645

[11] Pakdaman, K., Perthame, B., and Salort, D. Dynamics of a structured neuron population. *Nonlinearity 23*, 1 (2010), 55–75.

[12] Pakdaman, K., Perthame, B., and Salort, D. Relaxation and self-sustained oscillations in the time elapsed neuron network model. *SIAM J. Appl. Math. 73*, 3 (2013), 1260–1279.

[13] Quiñinao, C. A microscopic spiking neuronal network for the age-structured model. (2015) hal-01121061.

[14] Robert, P., and Touboul, J. D. On the dynamics of random neuronal networks. (2014) arXiv:1410.4072.

[15] Tristani, I. Boltzmann equation for granular media with thermal force in a weakly inhomogeneous setting. hal-00906770.

[16] Voigt, J. A perturbation theorem for the essential spectral radius of strongly continuous semigroups. *Monatsh. Math. 90*, 2 (1980), 153–161.

Qilong Weng

Université Paris-Dauphine
CEREMADE, UMR CNRS 7534
Place du Maréchal de Lattre de Tassigny 75775 Paris Cedex 16
FRANCE

e-mail: weng@ceremade.dauphine.fr