General time elapsed neuron network model: well-posedness and strong connectivity regime

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GENERAL TIME ELAPSED NEURON NETWORK MODEL: WELL-POSEDNESS AND STRONG CONNECTIVITY REGIME

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Abstract. For large fully connected neuron networks, we study the dynamics of homogenous assemblies of interacting neurons described by time elapsed models, indicating how the time elapsed since the last discharge construct the probability density of neurons. Through the spectral analysis theory for semigroups in Banach spaces developed recently in [6, 9], on the one hand, we prove the existence and the uniqueness of the weak solution in the whole connectivity regime as well as the parallel results on the long time behavior of solutions obtained in [10] under general assumptions on the firing rate and the delay distribution. On the other hand, we extend those similar results obtained in [11, 12] in the case without delay to the case taking delay into account and both in the weak and the strong connectivity regime with a particular step function firing rate. Our approach uses the spectral analysis theory for semigroups in Banach spaces developed recently by the first author and collaborators.

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Keywords. Neuron networks, time elapsed dynamics, semigroup spectral analysis, strong connectivity, exponential asymptotic stability.

sec:Intro

1. Introduction

The information transmission and processing mechanism in the nervous systems relies on the quantity of electrical pulses as the reflect to incoming stimulations, during which the neurons experience a period of recalcitrance called discharge time before reactive. In this work, we shall focus on the model describing the neuronal dynamics in accordance with this kind of discharge time which has been introduced and studied in [3, 11, 12]. In order to show the response to the recovery of the neuronal membranes after each discharge, the model consider an instantaneous firing rate depending on the time elapsed since last discharge as well as the inputs of neurons. This sort of models are also regarded as a mean field limit of finite number of neuron network models referred to [1, 2, 14, 13].

For a local time (or internal clock) $x \geq 0$ corresponding to the elapsed time since the last discharge, we consider the dynamic of the neuronal network with the density number of neurons $f = f(t,x) \geq 0$ in state $x \geq 0$ at time $t \geq 0$, given by the following nonlinear time elapsed (or of age structured type) evolution equation

eq:ASM

$$\partial_t f = -\partial_x f - a(x, \varepsilon\, m(t)) f =: \mathcal{L}_{\varepsilon m(t)} f, \tag{1.1a}$$

$$f(t,0) = p(t), \quad f(0,x) = f_0(x), \tag{1.1b}$$

where $a(x, \varepsilon\, \mu) \geq 0$ denotes the firing rate of a neuron in state x and in an environment $\mu \geq 0$ formed by the global neuronal activity with a network connectivity parameter $\varepsilon \geq 0$ corresponding to the strength of the interactions. The total density of neurons $p(t)$ undergoing a discharge at time t is defined through

$$p(t) := \mathcal{P}[f(t); m(t)],$$

where

$$\mathcal{P}[g, \mu] = \mathcal{P}_\varepsilon[g, \mu] := \int_0^\infty a(x, \varepsilon\mu) g(x) \mathrm{d}x,$$

while the global neuronal activity $m(t)$ at time $t \geq 0$ taking into account the interactions among the neurons resulting from earlier discharges is given by

eq:delay

$$m(t) := \int_0^\infty p(t-y) b(\mathrm{d}y), \tag{1.2}$$

where the delay distribution b is a probability measure considering the persistence of the electric activity to those discharges in the network. In the sequel, we will consider the two following situations respectively:

- The *case without delay*, when $b = \delta_0$ then $m(t) = p(t)$.
- The *case with delay*, when b is smooth.

Observe that the solution f of the time elapsed equation (1.1) satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^\infty f(t,x) \mathrm{d}x = f(t,0) - \int_0^\infty a(x, \varepsilon m(t)) f(t,x) \mathrm{d}x = 0,$$

in both the cases, which implies the conservation of the total density number of neurons (also called *mass* in the sequel) permitting us to normalize it to be 1. Then we assume in the sequel

$$\langle f(t,\cdot) \rangle = \langle f_0 \rangle = 1, \quad \forall t \geq 0, \quad \langle g \rangle := \int_0^\infty g(x) \mathrm{d}x.$$

eq:StSt

We define a couple $(F_\varepsilon, M_\varepsilon)$ as a corresponding steady state, which satisfy

eq:StSt1

$$0 = -\partial_x F_\varepsilon - a(x, \varepsilon\, M_\varepsilon) F_\varepsilon = \mathcal{L}_{\varepsilon M_\varepsilon} F_\varepsilon, \tag{1.3a}$$

$$F_\varepsilon(0) = M_\varepsilon, \quad \langle F_\varepsilon \rangle = 1. \tag{1.3b}$$

Noticing that the associated network activity and the discharge activity are equal constants for a steady state as $\langle b \rangle = 1$.

Our main purpose in this paper is to prove the existence and uniqueness of the solution to the time elapsed evolution equation (1.1) no matter which $\varepsilon > 0$, which possesses the exponential asymptotic stability in strong connectivity regime, which is a range of connectivity parameter $\varepsilon \in [\varepsilon_1, \infty)$, ε_1 large enough, such that the equations (1.1) and (1.3) do not possess intense nonlinearity. In order to conclude those results, it is necessary to give the following mathematical assumptions on the firing rate a and on the delay distribution b.

We make the physically reasonable assumptions

hyp:a1 (1.4) $$\partial_x a \geq 0, \quad a' = \partial_\mu a \geq 0,$$

hyp:a2 (1.5) $$0 < a_0 := \lim_{x\to\infty} a(x,0) \leq \lim_{x,\,\mu\to\infty} a(x,\mu) =: a_1 < \infty,$$

and the smoothness assumption

hyp:a3 (1.6) $$a \in W^{1,\infty}(\mathbb{R}^2_+).$$

as well as the decay assumption, for a.e. $x \geq 0$,

hyp:a4 (1.7) $$\varepsilon \sup_{x\geq 0} \partial_\mu a(x, \varepsilon\mu) \to 0, \quad \text{as } \varepsilon \to \infty.$$

In the case with delay, we assume that the delay distribution $b(\mathrm{d}y) = b(y)\mathrm{d}y$ has the exponential bound and satisfies the smoothness condition

hyp:del (1.8) $$\exists \delta > 0, \quad \int_0^\infty e^{\delta y}\,(b(y) + |b'(y)|)\,\mathrm{d}y < \infty.$$

The above assumptions permit the existence and uniqueness of the solution to the nonlinear problem (eq:ASM) thanks to the Banach fixed-point theorem, as well as the existence of corresponding steady state, which is unique in the strong connectivity regime.

th:EaU **Theorem 1.1.** *Assume (hyp:a1)-(hyp:a2)-(hyp:a3)-(hyp:a4), then for any $\varepsilon > 0$, there exists a unique nonnegative and mass conserving weak solution $f \in C(\mathbb{R}_+; L^1(\mathbb{R}_+))$ to the evolution equation (eq:ASM) for some functions m, $p \in C([0,\infty))$.*

th:SS **Theorem 1.2.** *Assume (hyp:a1)-(hyp:a2)-(hyp:a3)-(hyp:a4). For any $\varepsilon \geq 0$, there exists at least one solution $(F_\varepsilon(x), M_\varepsilon) \in BV(\mathbb{R}_+) \times \mathbb{R}_+$ to the stationary problem (eq:StSt), where $BV(\mathbb{R}_+)$ represents the space of bounded variation measures. Moreover, there exists $\varepsilon_1 > 0$, large enough, such that the steady state is unique for any $\varepsilon \in (\varepsilon_1, +\infty]$.*

We are able to conclude our main results, the exponential long time stability again as those in [10] in the weak connectivity regime, which extends the result in [11, 12] in the case without delay and for a *step function* firing rate a given by

StepFctStructure (1.9) $$a(x,\mu) = \mathbf{1}_{x>\sigma(\mu)}, \quad \sigma, \sigma^{-1} \in W^{1,\infty}(\mathbb{R}_+), \quad \sigma' \leq 0,$$

to the case with delay rather than the indescribable stability and tackling in the two cases at the same time with firing rates a satisfying (hyp:a1)-(hyp:a4) in the strong connectivity regime.

th:MR **Theorem 1.3.** *We assume that the firing rate a satisfies (hyp:a1), (hyp:a2), (hyp:a3) and (hyp:a4). We also assume that the delay distribution b satisfies $b = \delta_0$ or (hyp:del). There exists $\varepsilon_1 > 0$, large enough, such that for any $\varepsilon \in (\varepsilon_1, +\infty)$ the steady state $(F_\varepsilon, M_\varepsilon)$ is unique. There also exist some constants $\alpha < 0$, $C \geq 1$, $\eta > 0$ and $\zeta_\varepsilon \to 0$ as $\varepsilon \to \infty$ such that for any connectivity parameter $\varepsilon \in (\varepsilon_1, +\infty)$ and any initial datum $0 \leq f_0 \in L^1$ with mass 1 and such that $\|f_0 - F_\varepsilon\|_{L^1} \leq \eta/\zeta_\varepsilon$, the (unique, positive and mass conserving) solution f to the evolution equation (eq:ASM) satisfies*

$$\|f(t,.) - F_\varepsilon\|_{L^1} \leq Ce^{\alpha t}, \qquad \forall\, t \geq 0.$$

In order to study the asymptotic convergence to an equilibrium for the homogeneous inelastic Boltzman equation, the strategy of "perturbation of semigroup spectral gap" is first introduced in [7]. Inspired by its rencent application to a neuron network equation

in [8], we linearize the equation around a stationary state $(F_\varepsilon, M_\varepsilon, M_\varepsilon)$ on the variation $(g, n, q) = (f, m, p) - (F_\varepsilon, M_\varepsilon, M_\varepsilon)$, such that

$$\partial_t g = -\partial_x g - a(x, \varepsilon\, M_\varepsilon) g - n(t)\, \varepsilon (\partial_\mu a)(x, \varepsilon M_\varepsilon) F_\varepsilon, \tag{1.10a}$$

$$g(t,0) = q(t), \quad g(0,x) = g_0(x), \tag{1.10b}$$

with

$$q(t) = \int_0^\infty a(x, \varepsilon\, M_\varepsilon) g \,\mathrm{d}x + n(t)\, \varepsilon \int_0^\infty (\partial_\mu a)(x, \varepsilon M_\varepsilon) F_\varepsilon \,\mathrm{d}x \tag{1.11}$$

and

$$n(t) := \int_0^\infty q(t-y) b(\mathrm{d}y), \tag{1.12}$$

which induces the linearized generator Λ_ε and the associated semigroup S_{Λ_ε} by regarding the boundary condition as a source term. As in [16, 4, 9, 6], we split the operator Λ_ε by

$$\Lambda_\varepsilon = \mathcal{A}_\varepsilon + \mathcal{B}_\varepsilon,$$

where $\mathcal{B}_\varepsilon$ is α-hypodissipative, $\alpha < 0$ while $\mathcal{A}_\varepsilon$ is bounded and much more regular. Then a particular version of the Spectral Mapping Theorem of [9, 6] and the Weyl's Theorem of [16, 4, 9, 6] imply the semigroup S_{Λ_ε} as a finite dimensional dominant part. Besides, the Krein-Rutman Theorem established in [9, 6] tells that the stationary state $(F_\infty, M_\infty, M_\infty)$ is unique and the semigroup S_∞ is exponentially stable because of its positivity. Then a perturbative argument developed in [7, 15, 6] extends the exponential stability to the semigroup S_ε in the strong connectivity regime, which implies Theorem 1.3 by a somewhat classical nonlinear exponential stability argument. In the delay case, we replace the delay equation (1.2) by a simple age equation to form an autonomous system with the linearized equation to generate a semigroup and follow the same strategy. However, the firing rate(1.9) in [11, 12] does not fall in the class of those considered above because condition (1.6) is not met. Therefore, in order to accurate the stability in the case with delay obtained in [11, 12], we consider a more concise linear equation around the stationary state instead of linearizing the equation with the similar approach.

This article is organized by the following plan. In Section 2, we demonstrate the existence and uniqueness of the solution and the stationary state result. In Section 3, we introduce the strategy and establish Theorem 1.3 in the case without delay, meanwhile the case with delay in section 4. In section 5, we establish Theorem 1.3 again both in the weak regime and the strong regime for the particular firing rate (1.9).

2. Existence and the steady state

2.1. Existence of the solution.

To conclude the existence of a solution to (1.1), we are going to apply a fixed point argument with the benefit of the following lemma.

Lemma 2.1. *For any $m \in L^\infty([0,T])$ with a corresponding f satisfying the equation (1.1), consider the application $\mathcal{J}$: $L^\infty([0,T]) \to L^\infty([0,T])$, $\mathcal{J}(m)(t) := \int_0^t p(t-y) b(\mathrm{d}y)$, where $p(t) = \int_0^\infty a(x, \varepsilon\, m(t)) f(x,t) \mathrm{d}x$. Then there exist $T > 0$ and $0 < C < 1$ such that the estimate*

$$\|\mathcal{J}(m_1) - \mathcal{J}(m_2)\|_{L^\infty([0,T])} \le C \|m_1 - m_2\|_{L^\infty([0,T])} \tag{2.1}$$

holds for all $(m_1, m_2) \in L^\infty([0,T])$ and for any $\varepsilon > 0$.

Proof of Lemma 2.1. Thanks to the explicit formula with respect to x or t, $f(x,t)$ satisfying the equation (1.1) can be expressed respectively by

$$f(x,t) = f_0(x-t)e^{-\int_0^t a(s+x-t,\varepsilon m(s))\mathrm{d}s}, \qquad \forall\, x \geq t \tag{2.2}$$

and

$$f(x,t) = p(t-x)e^{-\int_0^x a(s,\varepsilon m(s+t-x))\mathrm{d}s}, \qquad \forall\, x \leq t. \tag{2.3}$$

We denote f_i, $i=1,\,2$ as two solutions of the equation (1.1)

$$\begin{cases} \partial_t f_i(x,t) + \partial_x f_i(x,t) + a(x,\varepsilon m_i(t))f_i(x,t) = 0, \\ f_i(0,t) = p_i(t) = \int_0^\infty a(x,\varepsilon m_i(t))f_i(x,t)\mathrm{d}x, \end{cases} \tag{2.4}$$

with the same initial data f_0. According to (2.2) and (2.3), we rewrite $p_i(t)$ as

$$p_i(t) = \int_0^t a(x,\varepsilon m_i(t))p_i(t-x)e^{-\int_0^x a(s,\varepsilon m(s+t-x))\mathrm{d}s}\mathrm{d}x \tag{2.5}$$

$$+ \int_t^\infty a(x,\varepsilon m_i(t))f_0(x-t)e^{-\int_0^t a(s+x-t,\varepsilon m(s))\mathrm{d}s}\mathrm{d}x. \tag{2.6}$$

We split $p1(t) - p_2(t)$ into two items $I_1(t)$ and $I_2(t)$ with

$$\begin{aligned} I_1(t) = \int_0^t \Big(& p_1(t-x)a(x,\varepsilon m_1(t))e^{-\int_0^x a(s,\varepsilon m_1(s+t-x))\mathrm{d}s} \\ & - p_2(t-x)a(x,m_2(t))e^{-\int_0^x a(s,\varepsilon m_2(s+t-x))\mathrm{d}s}\Big)\mathrm{d}x \end{aligned}$$

and $I_2(t)$ as the remainder. In order to control the first item, we divide it into three parts as $I_1(t) = I_{1,1}(t) + I_{1,2}(t) + I_{1,3}(t)$, where

$$I_{1,1}(t) = \int_0^t (p_1-p_2)(t-x)a(x,\varepsilon m_1(t))e^{-\int_0^x a(s,\varepsilon m_1(s+t-x))\mathrm{d}s}\mathrm{d}x,$$

$$I_{1,2}(t) = \int_0^t p_2(t-x)\big(a(x,\varepsilon m_1(t)) - a(x,\varepsilon m_2(t))\big)e^{-\int_0^x a(s,\varepsilon m_1(s+t-x))\mathrm{d}s}\mathrm{d}x,$$

$$I_{1,3}(t) = \int_0^t p_2(t-x)a(x,m_2(t))\big(e^{-\int_0^x a(s,\varepsilon m_1(s+t-x))\mathrm{d}s} - e^{-\int_0^x a(s,\varepsilon m_2(s+t-x))\mathrm{d}s}\big)\mathrm{d}x.$$

Clearly, we have the estimates

$$\|I_{1,1}\|_{L^\infty([0,T])} \leq a_1 T\|p_1-p_2\|_{L^\infty([0,T])}$$

and

$$\|I_{1,2}\|_{L^\infty([0,T])} \leq \varepsilon a_1\|a'\|_{L^\infty_x}T\|m_1-m_2\|_{L^\infty([0,T])}.$$

Since there exists a constant C such that

$$\begin{aligned} & \left|e^{-\int_0^x a(s,\varepsilon m_1(s+t-x))\mathrm{d}s} - e^{-\int_0^x a(s,\varepsilon m_2(s+t-x))\mathrm{d}s}\right| \\ \leq\ & C\int_0^x |a(s,\varepsilon m_1(s+t-x)) - a(s,\varepsilon m_2(s+t-x))|\mathrm{d}s, \end{aligned}$$

which leads to the estimate

$$\|I_{1,3}\|_{L^\infty([0,T])} \leq \varepsilon a_1^2\|a'\|_{L^\infty_x}\frac{T^2}{2}\|m_1-m_2\|_{L^\infty[0,T]}.$$

From the assumption (1.7), there exists ε_1 large enough such that $\varepsilon\|a'\|_{L^\infty_x} \leq 1$, for any $\varepsilon \in [\varepsilon_1, +\infty)$. Denoting $\eta := \max\{\varepsilon_1,\,1\}$, we deduce that

$$\|I_1\|_{L^\infty([0,T])} \leq a_1 T\|p_1-p_2\|_{L^\infty([0,T])} + \eta(C_1T^2 + C_2T)\|m_1-m_2\|_{L^\infty([0,T])}, \tag{2.7}$$

for any $\varepsilon>0$. On the other hand, we have the item $I_2(t)$ as

$$\begin{aligned} I_2(t) &= \int_t^\infty f_0(x-t)\Big(a(x,\varepsilon\, m_1(t))e^{-\int_0^t a(s+x-t,\varepsilon\, m_1(s))\mathrm{d}s} \\ &\quad -a(x,\varepsilon\, m_2(t))e^{-\int_0^t a(s+x-t,\varepsilon\, m_2(s))\mathrm{d}s}\Big)\mathrm{d}x \end{aligned}$$

Similarly to estimating I_1, we deduce that

ineq:EI2 (2.8) $$\|I_2\|_{L^\infty([0,T])} \le \eta(C_3T+C_4)\|m_1-m_2\|_{L^\infty([0,T])}.$$

From the above estimates (2.7) and (2.8), it turns out that

$$\|p_1-p_2\|_{L^\infty([0,T])} \le a_1T\|p_1-p_2\|_{L^\infty([0,T])} + \eta(C_1T^2+C_2'T+C_3')\|m_1-m_2\|_{L^\infty([0,T])},$$

which implies

ineq:ep (2.9) $$\|p_1-p_2\|_{L^\infty([0,T])} \le \eta C\|m_1-m_2\|_{L^\infty([0,T])}$$

when a_1T less than 1. Hence, from (2.9) and the fact that

$$\mathcal{J}(m_1)(t)-\mathcal{J}(m_2)(t) = \int_0^t (p_1-p_2)(t-y)b(\mathrm{d}y),$$

we obviously deduce

$$\|\mathcal{J}(m_1)-\mathcal{J}(m_2)\|_{L^\infty([0,T])} \le \eta CT\|m_1-m_2\|_{L^\infty([0,T])}.$$

By taking T small enough such that $\eta CT<1$, we attain our conclusion. □

Proof of Theorem 1.1. From Lemma 2.1, for any $\varepsilon>0$, there is a $T>0$ which does not depend upon the initial data such that the application $\mathcal{J}$ admits a unique fixed point $m(t)$ on $[0,T]$ then the corresponding $f(t,x)$ on $[0,T]\times\mathbb{R}^d$, which is the unique solution to the equation (1.1), according to the Banach-Picard fixed point theorem. Iterating on T, we deduce the global existence and uniqueness of the solution (f,m) to equation (1.1). □

2.2. **The stationary problem.** Now we present the proof of the steady state in the strong connectivity regime.

Proof. Step 1. Existence. From the assumption (1.5), we deduce that for any $x\ge 0$, $\mu\ge 0$, there exists $x_0\in[0,\infty)$ such that $a(x,\mu)\ge\frac{a_0}{2}$. Denoting

$$A(x,\mu) := \int_0^x a(y,\mu)\mathrm{d}y, \quad \forall, x, m\ge 0,$$

we naturally estimate that

ineq:EoA (2.10) $$\frac{a_0}{2}(x-x_0)_+ \le A(x,\mu) \le a_1x, \quad \forall\, x\ge 0,\ \mu\ge 0.$$

For any $m\ge 0$, the equation (1.3a) can be solved by

$$F_{\varepsilon,m}(x) := T_m e^{-A(x,\varepsilon m)},$$

whose mass conservation gives

$$T_m^{-1} = \int_0^\infty e^{-A(x,\varepsilon m)}\mathrm{d}x.$$

Then the existence of the solution is equivalent to find $m=M_\varepsilon$ satisfying $m=F_{\varepsilon,m}(0)=T_m$. Considering

$$\Psi(\varepsilon,m) = mT_m^{-1} := m\int_0^\infty e^{-A(x,\varepsilon m)}\mathrm{d}x,$$

it is merely necessary to find $M_\varepsilon \geq 0$ such that

eq:mnc
$$\Psi(\varepsilon, M_\varepsilon) = 1. \tag{2.11}$$

From the Lebesgue dominated convergence theorem, the function $\Psi(\varepsilon, .)$ is continuous. In addition to the fact that $\Psi(0) = 0$ and $\Psi(\infty) = \infty$, the intermediate value theorem implies the existence immediately.

Step 2. Uniqueness in the strong connectivity regime. Obviously, $M_\infty := (\int_0^\infty e^{-A(x,\infty)}\mathrm{d}x)^{-1} \in (0, \infty)$ is the unique solution to $\Psi(\infty, M_\infty) = 1$. It is clear that

$$\frac{\partial}{\partial m}\Psi(\varepsilon, m) = \int_0^\infty e^{-A(x,\varepsilon m)}\big(1 - m\int_0^x \varepsilon\partial_\mu a(y, \varepsilon m)\mathrm{d}y\big)\mathrm{d}x,$$

is continuous with respect to the two variables because of (hyp:a3 1.6), which implies that $\Psi \in C^1$. Coupled with that

$$\frac{\partial}{\partial m}\Psi(\varepsilon, m)|_{\varepsilon=\infty} = \int_0^\infty e^{-A(x,\infty)}\mathrm{d}x > 0,$$

we conclude from the implicit function theorem that there exists $\varepsilon_1 > 0$, large enough, such that the equation (eq:mnc 2.11) has a unique solution for any $\varepsilon \in (\varepsilon_1, +\infty]$. □

3. Case without delay

sec:WithoutDelay

In this section, we conclude our main result Theorem th:MR 1.3 gradually in the case without delay.

3.1. **Linearized equation and structure of the spectrum.**

We introduce the linearized equation on the variation (g, n) around the steady state $(F_\varepsilon, M_\varepsilon)$ given by

$$\begin{aligned}
&\partial_t g + \partial_x g + a_\varepsilon g + a'_\varepsilon F_\varepsilon n(t) = 0,\\
&g(t, 0) = n(t) = \int_0^\infty (a_\varepsilon g + a'_\varepsilon F_\varepsilon n(t))\,\mathrm{d}x, \quad g(0, x) = g_0(x),
\end{aligned}$$

with notes $a_\varepsilon := a(x, \varepsilon M_\varepsilon)$ and $a'_\varepsilon := \varepsilon\,(\partial_\mu a)(x, \varepsilon M_\varepsilon)$ for simplification. According to the assumption (hyp:a4 1.7), there exists $\varepsilon_1 > 0$, large enough, such that

$$\forall\, \varepsilon \in (\varepsilon_1, \infty) \qquad \kappa := \int_0^\infty a'_\varepsilon F_\varepsilon \mathrm{d}x < 1,$$

permitting to define

def:Nepsg
$$\mathcal{M}_\varepsilon[g] := (1 - \kappa)^{-1}\int_0^\infty a_\varepsilon g\,\mathrm{d}x. \tag{3.1}$$

by considering the boundary condition as a source term, we rewrite the linearized equation as

eq:ASMDL
$$\partial_t g = \Lambda_\varepsilon g := -\partial_x g - a_\varepsilon g - a'_\varepsilon F_\varepsilon \mathcal{M}_\varepsilon[g] + \delta_{x=0}\mathcal{M}_\varepsilon[g], \tag{3.2}$$

acting on the space of bounded Radon measures

$$X := M^1(\mathbb{R}_+) = \{g \in (C_0(\mathbb{R}))';\ \mathrm{supp}\, g \subset \mathbb{R}_+\},$$

endowed with the weak $*$ topology $\sigma(M^1, C_0)$, where C_0 represents the space of continuous functions converging to 0 at infinity. The spectral analysis theory referred to [5, 6] (KatoBook,Mbook*) indicates the structure of the spectrum and the associated semigroup S_{Λ_ε}.

th:MRe **Theorem 3.1.** *Assume (H.4)-(H.5)-(H.6)-(H.7) and define $\alpha := -a_0/2 < 0$. The operator Λ_ε is the generator of a weakly $*$ continuous semigroup S_{Λ_ε} acting on X. Moreover, there exists a finite rank projector $\Pi_{\Lambda_\varepsilon,\alpha}$ which commutes with S_{Λ_ε}, an integer $j \geq 0$ and some complex numbers*

$$\xi_1, ..., \xi_j \in \Delta_\alpha := \{z \in \mathbb{C},\ \Re e\, z > \alpha\},$$

such that on $E_1 := \Pi_{\Lambda_\varepsilon,\alpha} X$ the restricted operator satisfies

$$\Sigma(\Lambda_{\varepsilon|E_1}) \cap \Delta_\alpha = \{\xi_1, ..., \xi_j\}$$

(with the convention $\Sigma(\Lambda_{\varepsilon|E_1}) \cap \Delta_\alpha = \emptyset$ when $j = 0$) and for any $a > \alpha$ there exists a constant C_a such that the remainder semigroup satisfies

$$\|S_{\Lambda_\varepsilon}(I - \Pi_{\Lambda_\varepsilon,\alpha})\|_{\mathscr{B}(X)} \leq C_a e^{at}, \quad \forall\, t \geq 0.$$

In order to apply the Spectral Mapping Theorem of [9, 6] and the Weyl's Theorem of [16, 4, 9, 6], we split the operator Λ_ε as $\Lambda_\varepsilon = \mathcal{A}_\varepsilon + \mathcal{B}_\varepsilon$ defined on X by

def:Aeps (3.3) $$\mathcal{A}_\varepsilon g := \mu_\varepsilon \mathcal{M}_\varepsilon[g], \quad \mu_\varepsilon := \delta_0 - a'_\varepsilon F_\varepsilon,$$

def:Beps (3.4) $$\mathcal{B}_\varepsilon g := -\partial_x g - a_\varepsilon g.$$

As in the weak connectivity regime in [10], the properties of the two auxiliary operators still hold in the strong one, which implies Theorem 3.1 directly.

lem:Lm1 **Lemma 3.2.** *Assume that a satisfies (H.4)-(H.5)-(H.7), then the operators $\mathcal{A}_\varepsilon$ and $\mathcal{B}_\varepsilon$ satisfy the following properties.*

(i) *$\mathcal{A}_\varepsilon \in \mathscr{B}(X, Y)$, where $Y = \mathbb{C}\mu_\varepsilon \subset X$ with compact embedding.*
(ii) *$S_{\mathcal{B}_\varepsilon}$ is α-hypodissipative in X.*
(iii) *The family of operators $S_{\mathcal{B}_\varepsilon} * \mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}$ satisfies*

$$\|(S_{\mathcal{B}_\varepsilon} * \mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})(t)\|_{X \to BV} \leq C e^{\alpha t}, \quad \forall t \geq 0.$$

3.2. **Strong connectivity regime - exponential stability of the linearized equation.** When the network connectivity parameter ε goes to infinity, the linearized time elapsed operator simplifies

eq:TLC (3.5) $$\Lambda_\infty g = -\partial_x g - a(x, \infty) g + \delta_{x=0} \mathcal{M}_\infty[g],$$

where $\mathcal{M}_\infty[g] = \int_0^\infty a(x, \infty) g(x) \mathrm{d}x$. Similarly to the limited case $\varepsilon = 0$ in [10], the semigroup S_{Λ_∞} is also positive and the operator $-\Lambda_\infty$ is also strong maximum while the Kato's inequality still holds in the limited connectivity. Then we conclude the following evolution estimate benefiting from the Theorem 3.1.

th:lc **Theorem 3.3.** *There exist some constants $\alpha < 0$ and $C > 0$ such that $\Sigma(\Lambda_\infty) \cap \Delta_\alpha = \{0\}$ and for any $g_0 \in X$, $\langle g_0 \rangle = 0$, there holds*

eq:lc (3.6) $$\|S_{\Lambda_\infty}(t) g_0\|_X \leq C e^{\alpha t} \|g_0\|_X, \quad \forall\, t \geq 0.$$

We extend the exponential stability property in the limited case to the strong connectivity regime.

th:CWR **Theorem 3.4.** *There exist some constants $\varepsilon_1 > 0$, $\alpha < 0$ and $C > 0$ such that for any $\varepsilon \in [\varepsilon_1, \infty]$ there hold $\Sigma(\Lambda_\varepsilon) \cap \Delta_\alpha = \{0\}$ and*

eq:CWR (3.7) $$\|S_{\Lambda_\varepsilon}(t) g_0\|_X \leq C e^{\alpha t} \|g_0\|_X, \quad \forall\, t \geq 0,$$

for any $g_0 \in X$, $\langle g_0 \rangle = 0$.

The proof uses the stability theory for semigroups developed in Kato's book [5] and revisited in [7, 15, 6]. Now, we present several results needed in the proof of Theorem 3.4.

Proof of Theorem 3.4. With the definitions (3.1), (3.3) and (3.4) of $\mathcal{M}_\varepsilon$, $\mathcal{A}_\varepsilon$ and $\mathcal{B}_\varepsilon$, we have

$$(\mathcal{B}_\varepsilon - \mathcal{B}_\infty)g = (a(x,\infty) - a(x,\varepsilon M_\varepsilon))g$$

and

$$(\mathcal{A}_\varepsilon - \mathcal{A}_\infty)g = (\mathcal{M}_\varepsilon[g] - \mathcal{M}_\infty[g])\,\delta_0 - \varepsilon(\partial_\mu a)(x,\varepsilon M_\varepsilon)\,F_\varepsilon\,\mathcal{M}_\varepsilon[g].$$

From the decay assumption (1.7), there exists positive $\zeta_\varepsilon \to 0$, as $\varepsilon \to +\infty$, such that $|\varepsilon\partial_\mu a(x,\varepsilon M_\varepsilon)| < \zeta_\varepsilon$, for ε large enough. Together with the smoothness assumption (1.6), we deduce that

eq:Lambdaeps-0

$$\|\mathcal{B}_\varepsilon - \mathcal{B}_\infty\|_{\mathscr{B}(X)} + \|\mathcal{A}_\varepsilon - \mathcal{A}_\infty\|_{\mathscr{B}(X)} \le C\,\zeta_\varepsilon. \tag{3.8}$$

In the strong connectivity regime, the eigenprojector Π_ε satisfies similarly that

$$\|\Pi_\varepsilon - \Pi_\infty\|_{\mathscr{B}(X)} < 1.$$

From the classical result [5, Section I.4.6] (or more explicitly [15, Lemma 2.18]), we deduce that there exists $\xi_\varepsilon \in \Delta_\alpha$ such that

$$\Sigma(\Lambda_\varepsilon) \cap \Delta_\alpha = \{\xi_\varepsilon\}, \quad \xi_\varepsilon \text{ is a simple eigenvalue,}$$

for any $\varepsilon \in [\varepsilon_1, +\infty]$ (up to take a larger real number $\varepsilon_1 > 0$). We conclude by observing that $\xi_\varepsilon = 0$ because $1 \in X'$ and $\Lambda_\varepsilon^* 1 = 0$ (which is nothing but the mass conservation). □

3.3. **Strong connectivity regime - nonlinear exponential stability.** Now, we focus on the nonlinear exponential stability of the solution to the evolution equation (1.1) in the case without delay. We start with an auxiliary result. We define the function $\Phi : L^1(\mathbb{R}_+) \times \mathbb{R} \to \mathbb{R}$ by

eq:Phi

$$\Phi[g,\mu] := \int_0^\infty a(x,\varepsilon\mu)g(x)\,\mathrm{d}x - \mu. \tag{3.9}$$

We denote by W_1 the optimal transportation Monge-Kantorovich-Wasserstein distance on the probability measures set $\mathbf{P}(\mathbb{R}_+)$ associated to the distance $d(x,y) = |x-y| \wedge 1$, or equivalently defined by

$$\forall\, f,g \in \mathbf{P}(\mathbb{R}_+), \quad W_1(f,g) := \sup_{\varphi, \|\varphi\|_{W^{1,\infty}} \le 1} \int_0^\infty (f-g)\,\varphi.$$

lem:varphig

Lemma 3.5. *Assume (1.6). There exists $\varepsilon_1 > 0$ and for any $\varepsilon \in (\varepsilon_1, +\infty)$ there exists a function $\varphi_\varepsilon : \mathbf{P}(\mathbb{R}) \to \mathbb{R}$ which is Lipschitz continuous for the weak topology of probability measures and such that $\mu = \varphi_\varepsilon[g]$ is the unique solution to the equation*

$$\mu \in \mathbb{R}_+, \quad \Phi(g,\mu) = 0.$$

Proof of Lemma 3.5. Step 1. Existence. For any $g \in \mathbf{P}(\mathbb{R})$ we have $\Phi(g,0) > 0$ and for any $g \in \mathbf{P}(\mathbb{R})$ and $\mu \ge 0$, we have

$$\Phi(g,\mu) \le a_1 - \mu,$$

so that $\Phi(g,\mu) < 0$ for $\mu > a_1$. By the intermediate value theorem and the continuity property of Φ, for any fixed $g \in \mathbf{P}(\mathbb{R}_+)$ and $\varepsilon \ge 0$, there exists at least one solution $\mu \in (0, a_1]$ to the equation $\Phi(g,\mu) = 0$.

Step 2. Uniqueness and Lipschitz continuity. Fix $f,g \in \mathbf{P}(\mathbb{R}_+)$ and consider $\mu,\nu \in \mathbb{R}_+$ such that

$$\Phi(f,\mu) = \Phi(g,\nu) = 0.$$

We have

$$\nu-\mu=\int_0^\infty a(x,\varepsilon\nu)(g-f)+\int_0^\infty (a(x,\varepsilon\nu)-a(x,\varepsilon\mu))f,$$

with

$$\Big|\int_0^\infty a(x,\varepsilon\nu)(g-f)\Big|\le \|a(\cdot,\varepsilon\nu)\|_{W^{1,\infty}}\, W_1(g,f),$$

and

$$\Big|\int_0^\infty \big(a(x,\varepsilon\nu)-a(x,\varepsilon\mu)\big)f\Big|\le \|a(\cdot,\varepsilon\nu)-a(\cdot,\varepsilon\mu)\|_{L^\infty}\le \zeta_\varepsilon|\mu-\nu|.$$

We then obtain

eq:mu-nu

$$|\mu-\nu|\,(1-\zeta_\varepsilon)\le \|a(\cdot,\varepsilon\nu)\|_{W^{1,\infty}}\, W_1(g,f), \tag{3.10}$$

and we may fix $\varepsilon_1>0$ such that $1-\zeta_{\varepsilon_1}\|\partial_\mu a\|_{L^\infty}\in(0,1)$, $\varepsilon\in[\varepsilon_1,+\infty]$. On the one hand, for $f=g$, we deduce that $\mu=\nu$ and that uniquely defines the mapping $\varphi_\varepsilon[g]:=\mu$. On the other hand, the function is Lipschitz continuous because of (3.10). □

We come back to the proof of our main result Theorem 1.3 in the case without delay.

Proof of Theorem 1.3 in the case without delay. We split the proof into two steps.

Step 1. New formulation. We start giving a new formulation of the solutions to the evolution and stationary equations in the strong connectivity regime $\varepsilon\in[\varepsilon_1,\infty)$, where ε_1 is defined in Lemma 3.5. For a given initial datum $0\le f_0\in L^1_1(\mathbb{R}_+)$ with unit mass the solution $f\in C([0,\infty);L^1_1(\mathbb{R}_+))$ to the evolution equation (1.1) and the solution F_ε to the stationary equation (1.3) clearly satisfy

$$\begin{aligned}\partial_t f+\partial_x f+a(\varepsilon\varphi[f])f=0,\qquad & f(t,0)=\varphi[f(t,\cdot)],\\ \partial_x F+a(\varepsilon M)F=0,\qquad & F(0)=M=\varphi[F],\end{aligned}$$

where here and below the ε and x dependency is often removed without risk of misleading.

We introduce the variation function $g:=f-F$ which satisfies the PDE

$$\partial_t g=-\partial_x g-a(\varepsilon M)g-\varepsilon a'(\varepsilon M)F\,\mathcal{M}[g]-Q[g] \tag{3.11}$$

with

$$Q[g]:=a(\varepsilon\varphi[f])f-a(\varepsilon\varphi[F])F-a(\varepsilon\varphi[F])g-\varepsilon a'(\varepsilon\varphi[F])F\mathcal{M}[g]$$

where $\mathcal{M}=\mathcal{M}_\varepsilon$ is defined in (3.1). The above PDE is complemented with the boundary condition

$$g(t,0)\;=\;\varphi[f(t,\cdot)]-\varphi[F],$$

and we may write again

$$\varphi[f]-\varphi[F]=\mathcal{M}[g]+\mathcal{Q}[g],\quad \mathcal{Q}[g]:=\langle Q[g]\rangle.$$

As a consequence, we have proved that the variation function g satisfies the equation

eq:dtg=Lambda+Z

$$\partial_t g=\Lambda_\varepsilon g+Z[g],\quad Z[g]:=-Q[g]+\delta_0\mathcal{Q}[g]. \tag{3.12}$$

Step 2. The nonlinear term. With the fact that f is mass conserved, $\|F\|_X=1$ and the assumption (1.7), we estimate that

$$\begin{aligned}\|Q[g]\|_X\;&=\;\|a(\varepsilon\varphi[f])f-a(\varepsilon\varphi[F])f-\varepsilon a'(\varepsilon\varphi[F])F\,\mathcal{M}[g]\|_X\\ &\le\;\varepsilon\|a'\|_{L^\infty_x}\|f\|_X\big|\varphi[f]-\varphi[F]\big|+\varepsilon\|a'\|_{L^\infty_x}\|F\|_X\mathcal{M}[g]\\ &\lesssim\;\zeta_\varepsilon\big(\mathcal{M}[g]+\|Q[g]\|_X\big)+\zeta_\varepsilon\mathcal{M}[g],\end{aligned}$$

where $\zeta_\varepsilon\to0$ as $\varepsilon\to+\infty$. Considering that

$$\mathcal{M}[g]\le a_1(1-\kappa)^{-1}\|g\|X\lesssim\|g\|_X,$$

from the above inequality, we deduce that

$$\|Q[g]\| \lesssim \zeta_\varepsilon \mathcal{M}[g] \lesssim \zeta_\varepsilon \|g\|_X,$$

with ε large enough. We then obtain

$$\|Z[g]\|_X \leq 2\|Q[g]\|_X \lesssim \zeta_\varepsilon \|g\|_X$$

Step 3. Decay estimate. Thanks to the Duhamel formula, the solution g to the evolution equation (3.12) satisfies

$$g(t) = S_{\Lambda_\varepsilon}(t)(g_0) + \int_0^t S_{\Lambda_\varepsilon}(t-s) Z[g(s)]\,\mathrm{d}s.$$

Benefiting from Theorem 3.4 and the second step, we deduce

$$\begin{aligned} \|g(t)\|_X &\leq C\, e^{\alpha t}\, \|g_0\|_X + \int_0^t C\, e^{\alpha(t-s)}\, \|Z[g(s)]\|_X\,\mathrm{d}s \\ &\lesssim e^{\alpha t}\, \|g_0\|_X + \zeta_\varepsilon \int_0^t e^{\alpha(t-s)}\, \|g(s)\|_X\,\mathrm{d}s, \end{aligned}$$

for any $t \geq 0$ and for some constant $\alpha < 0$, independent of $\varepsilon \in [\varepsilon_1, +\infty)$. Thanks to the Gronwall's lemma (for linear integral inequality), we have

$$\begin{aligned} \|g(t)\|_X &\lesssim e^{\alpha t}\|g_0\|_X + \zeta_\varepsilon \|g_0\|_X \int_0^t e^{\alpha t} \exp\{\int_s^t e^{\alpha(t-r)}\mathrm{d}r\}\mathrm{d}s \\ &\lesssim e^{\alpha t}\|g_0\|_X + \zeta_\varepsilon t e^{\alpha t}\|g_0\|_X \\ &\lesssim e^{\alpha' t}\|g_0\|_X, \end{aligned}$$

for some constant $\alpha < \alpha' < 0$. □

4. Case with delay

sec:WithDelay

This section is devoted to the proof of our main result, Theorem 1.3, in the case with delay by following the same strategy as in the case without delay but adaptation the functional framework. We have already proved in Theorem 1.2 the existence of a unique stationary solution $(F_\varepsilon, M_\varepsilon)$ in the strong connectivity regime and we may then focus on the evolution equation.

4.1. Linearized equation and structure of the spectrum.

In order to write as a time autonomous equation the linearized equation (1.10)-(1.11)-(1.12), we introduce the following intermediate evolution equation on a function $v = v(t, y)$

eq:Eov

$$\partial_t v + \partial_y v = 0, \quad v(t,0) = q(t), \quad v(0,y) = 0, \tag{4.1}$$

where $y \geq 0$ represent the local time for the network activity. That last equation can be solved with the characteristics method

$$v(t,y) = q(t-y)\mathbf{1}_{0\leq y\leq t}.$$

Therefore, equation (1.12) on the variation $n(t)$ of network activity writes

$$n(t) = \mathcal{D}[v(t)], \quad \mathcal{D}[v] := \int_0^\infty v(y) b(\mathrm{d}y),$$

and then equation (1.11) on the variation $q(t)$ of discharging neurons writes

$$q(t) = \mathcal{O}_\varepsilon[g(t), v(t)],$$

with

$$\mathcal{O}_\varepsilon[g,v] := \mathcal{N}_\varepsilon[g] + \kappa_\varepsilon\,\mathcal{D}[v],$$
$$\mathcal{N}_\varepsilon[g] := \int_0^\infty a_\varepsilon(M_\varepsilon) g\,\mathrm{d}x, \quad \kappa_\varepsilon := \int_0^\infty a'_\varepsilon(M_\varepsilon) F_\varepsilon\,\mathrm{d}x.$$

As a consequence, we may rewrite the linear system (1.10)-(1.11)-(1.12), as the autonomous system

eq:AS
$$\partial_t (g,v) = \mathscr{L}_\varepsilon(g,v), \tag{4.2}$$

where the operator $\mathscr{L}_\varepsilon = (\mathscr{L}^1_\varepsilon, \mathscr{L}^2_\varepsilon)$ is defined by

$$\begin{aligned}\mathscr{L}^1_\varepsilon(g,v) &:= -\partial_x g - a_\varepsilon g - a'_\varepsilon F_\varepsilon \mathcal{D}[v] + \delta_{x=0}\mathcal{O}_\varepsilon[g,v],\\ \mathscr{L}^2_\varepsilon(g,v) &:= -\partial_y v + \delta_{y=0}\mathcal{O}_\varepsilon[g,v],\end{aligned}$$

in the space

$$X = X_1 \times X_2 := M^1(\mathbb{R}_+) \times M^1(\mathbb{R}_+,\mu)$$

with $\mu(x) = e^{-\delta x}$ and $\delta > 0$ is the same as in the condition (1.8).

th:MRe1
Theorem 4.1. *Assume (1.4)-(1.5)-(1.6)-(1.7) and (1.8). The conclusions of Theorem 3.1 holds true with* $\alpha := \max\{-a_0/2, -\delta\} < 0$.

The result follows from the Spectral Mapping theorem and the Weyl's Theorem established in [9, 6] by introducing a convenient splitting of the operator $\mathscr{L}_\varepsilon$. More precisely, we write $\mathscr{L}_\varepsilon = \mathcal{A}_\varepsilon + \mathcal{B}_\varepsilon$ with

$$\mathcal{B}_\varepsilon(g,v) = \begin{pmatrix}\mathcal{B}^1_\varepsilon(g,v)\\ \mathcal{B}^2_\varepsilon(g,v)\end{pmatrix} = \begin{pmatrix}-\partial_x g - a_\varepsilon g\\ -\partial_y v\end{pmatrix}$$

and

$$\mathcal{A}_\varepsilon(g,v) = \begin{pmatrix}\mathcal{A}^1_\varepsilon(g,v)\\ \mathcal{A}^2_\varepsilon(g,v)\end{pmatrix} = \begin{pmatrix}-a'_\varepsilon F_\varepsilon \mathcal{D}[v] + \delta_{x=0}\mathcal{O}_\varepsilon[g,v]\\ \delta_{y=0}\mathcal{O}_\varepsilon[g,v]\end{pmatrix}.$$

Taking $Y = (\mathbb{C}\delta_0 + BV(\mathbb{R}_+)) \times \mathbb{C}\delta_0 \subset X$, the operators $\mathcal{A}_\varepsilon$ and $\mathcal{B}_\varepsilon$ possess the same properties as in Lemma 3.2.

4.2. **Strong connectivity regime - exponential stability of the linearized equation.** When the network connectivity parameter goes to the limit, $\varepsilon = +\infty$, the linearized operator simplifies into

eq:lc1
$$\mathscr{L}_\infty \begin{pmatrix}g\\ v\end{pmatrix} = \begin{pmatrix}-\partial_x g - a(x,\infty) g + \delta_{x=0}\mathcal{O}_\infty[g,v]\\ -\partial_y v + \delta_{y=0}\mathcal{O}_\infty[g,v]\end{pmatrix}, \tag{4.3}$$

where $\mathcal{O}_\infty[g,v] = \mathcal{N}_\infty[g] = \int_0^\infty a(x,\infty)g(x)\mathrm{d}x$. From Theorem 3.3 and the Duhamel formula

$$v(t) = S_{\mathcal{B}^2_\infty}(t)v_0 + \int +0^t S_{\mathcal{B}^2_\infty}(t-s)\mathcal{A}^2_\infty\Big(g(s),v(s)\Big)\mathrm{d}s,$$

the exponential asymptotic estimate holds for the second component of the operator $\mathscr{L}_\infty$.

th:lc1
Theorem 4.2. *There exist some constants* $\alpha < 0$ *and* $C > 0$ *such that* $\Sigma(\mathscr{L}_\infty) \cap \Delta_\alpha = \{0\}$ *and for any* $(g_0,\, v_0) \in X$, $\langle g_0\rangle = 0$, *there holds*

eq:lc1
$$\|S_{\mathscr{L}_\infty}(t)(g_0,\, v_0)\|_X \le C e^{\alpha t}\,\|(g_0,\, v_0)\|_X, \quad \forall\, t \ge 0. \tag{4.4}$$

Then we extend the geometry structure of the spectrum of the linearized time elapsed equation to the strong connectivity regime taking delay into account.

th:CWR1 **Theorem 4.3.** *There exists some constants $\varepsilon_1 > 0$, $C \geq 1$ and $\alpha < 0$ such that for any $\varepsilon \in [\varepsilon_1, +\infty]$ there holds $\Sigma(\mathscr{L}_\varepsilon) \cap \Delta_\alpha = \{0\}$ and*

eq:Cwd1
$$\|S_{\mathscr{L}_\varepsilon}(t)(g_0, v_0)\|_X \leq Ce^{\alpha t}\|(g_0, v_0)\|_X, \tag{4.5}$$

for any $(g_0, v_0) \in X$ such that $\langle g_0 \rangle = 0$.

Proof. Similarly, we proceed exactly as in the proof of Theorem th:CWR 3.4 and deduce the continuity of the operator $\mathscr{L}_\varepsilon$ to $\mathscr{L}_*$ in the strong connectivity regime, with whose help, we conclude that (see also again [15])

$$\Sigma(\mathscr{L}_\varepsilon) \cap \Delta_\alpha = \{\xi_\varepsilon\},$$

with $|\xi_\varepsilon| \leq O(\zeta_\varepsilon)$ and ξ_ε is algebraically simple. We observe that

$$\mathscr{L}_\varepsilon^* \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} \partial_x \varphi - a_\varepsilon \varphi + a_\varepsilon(\varphi(0) + \psi(0)) \\ \partial_y \psi + \kappa_\varepsilon b\, \psi(0) + \kappa_\varepsilon b\, \varphi(0) - b \int a_\varepsilon' F_\varepsilon\, \varphi \, \mathrm{d}x \end{pmatrix},$$

from which we deduce that $\mathscr{L}^*(1,0) = 0$. Then $0 \in \Sigma(\mathscr{L}_\varepsilon^*)$ and $\xi_\varepsilon = 0$. Moreover, the orthogonality condition $\langle g_0 \rangle = \langle (g_0, v_0), (1,0) \rangle_{X,X'} = 0$ implies that the exponential estimate (eq:Cwd1 4.5) holds. □

4.3. **Strong connectivity regime - nonlinear exponential stability.** We finally come back on the nonlinear problem and we present the proof of the second part of our main result for the case with delay.

Proof of Theorem th:MR 1.3 in case with delay. We write the system as

$$\begin{aligned} \partial_t f &= -\partial_x f - a_\varepsilon(\mathcal{D}[u]) f + \delta_0 \mathbf{P}[f, \mathcal{D}[u]] \\ \partial_t u &= -\partial_y u + \delta_0 \mathbf{P}[f, \mathcal{D}[u]], \end{aligned}$$

with

$$\mathbf{P}[f, m] = \int a(m) f, \quad \mathcal{D}[u] = \int bu.$$

We recall that the steady state (F, U), $U := M\mathbf{1}_{y \geq 0}$, satisfies

$$\begin{aligned} 0 &= -\partial_x F - a_\varepsilon(M) F + \delta_0 M \\ 0 &= -\partial_y U + \delta_0 M, \quad M = \mathcal{D}[U] = \mathbf{P}[F, \mathcal{D}[U]]. \end{aligned}$$

We introduce the variation $g := f - F$ and $v = u - U$. The equation on g is

$$\begin{aligned} \partial_t g &= -\partial_x g - a_\varepsilon(\mathcal{D}[u]) f + a_\varepsilon(M) F + \delta_0(\mathbf{P}[f, \mathcal{D}[u]] - \mathbf{P}[F, \mathcal{D}[U]]) \\ &= -\partial_x g - a_\varepsilon(M) g - a_\varepsilon' F \mathcal{D}[v] - Q[g, v] + \delta_0 \mathcal{O}[g, v] + \delta_0 \mathcal{Q}[g, v] \\ &= \mathscr{L}_\varepsilon^1(g, v) + \mathcal{Z}^1[g, v], \end{aligned}$$

with

$$Q[g, v] := a_\varepsilon(\mathcal{D}[u]) f - a_\varepsilon(M) F - a_\varepsilon(M) g - a_\varepsilon' F \mathcal{D}[v]$$

and $\mathcal{Q}[g, v] = \langle Q[g, v] \rangle$, $\mathcal{Z}^1[g, v] := -Q[g, v] + \delta_0 \mathcal{Q}[g, v]$. The equation on v is

$$\begin{aligned} \partial_t v &= -\partial_y v + \delta_0(\mathbf{P}[f, \mathcal{D}[u]] - \mathbf{P}[F, \mathcal{D}[U]]) \\ &= -\partial_y v + \delta_0 \mathcal{O}[g, v] + \delta_0 \mathcal{Q}[g, v] \\ &= \mathscr{L}_\varepsilon^2(g, v) + \mathcal{Z}^2[g, v], \quad \mathcal{Z}^2[g, v] := \delta_0 \mathcal{Q}[g, v]. \end{aligned}$$

Since $\|f\|_{X_1} = 1$ and the assumption (hyp:a4 1.7), we estimate that

$$\begin{aligned} \|Q[g, v]\|_{X_1} &= a_\varepsilon(\mathcal{D}[u]) f - a_\varepsilon(M) f - a_\varepsilon' F \mathcal{D}[v] \\ &\leq \varepsilon \|a'\|_{L_x^\infty} \|f\|_{X_1} \big| \mathcal{D}[u] - \mathcal{D}[U] \big| - \varepsilon \|a'\|_{L_x^\infty} \mathcal{D}[v] \\ &\lesssim \zeta_\varepsilon \|v\|_{X_2}, \end{aligned}$$

which implies $\|Z[g,v]\|_X \lesssim \zeta_\varepsilon \|(g,v)\|_X$. The associated Duhamel formula writes

$$(g(t),v(t)) = S_{\mathscr{L}_\varepsilon}(t)(g_0,v_0) + \int_0^t S_{\mathscr{L}_\varepsilon}(t-s)\mathcal{Z}[g(s),v(s)]\,\mathrm{d}s.$$

Using the above estimate for the nonlinear term and the Gronwall's lemma, we conclude as in the proof of Theorem th:MR 1.3. □

5. Step function firing rate

sec:StepFctFR

In this section, we focus on the nonlinear time elapsed model in PK1,PK2 [11, 12] for a particular step function firing rate given by

eq:FR

eq:FR1 (5.1a) $$a(x,\mu) = \mathbf{1}_{x>\sigma(\varepsilon\mu)}, \quad \sigma,\, \sigma^{-1} \in W^{1,\infty}(\mathbb{R}_+), \quad \sigma' \le 0,$$

eq:FR2 (5.1b) $$\sigma(0) = \sigma_+, \quad \sigma(\infty) = \sigma_- < \sigma_+ < 1.$$

In the strong connectivity regime, we need the additional decay rate assumptions

hyp:s1 (5.2) $$\limsup_{\varepsilon\to\infty} \sup_{\mu\in I_1(\sigma)} \frac{-\varepsilon\sigma'(\varepsilon\mu)}{\big(1+\sigma(\varepsilon\mu)\big)^2} < 1,$$

hyp:s2 (5.3) $$\limsup_{\varepsilon\to\infty} \sup_{\mu\in I_2(\sigma)} \varepsilon\sigma'(\varepsilon\mu) = 0.$$

where $I_1(\sigma) = [(1+\sigma_+)^{-1}, (1+\sigma_-)^{-1}]$, and $I_2(\sigma) = [1-\sigma_+, 1]$. We consider the dynamic of the neuron network (eq:ASM 1.1) completed with an initial probability density f_0 satisfying

eq:ID (5.4) $$0 \le f_0 \le 1, \quad \int_0^\infty f_0(x)\mathrm{d}x = 1.$$

Obviously, the solution f of the time elapsed equation (eq:ASM 1.1) corresponding to the firing rate (eq:FR1 5.1a) is still mass conserved, and we naturally renormalize that mass. From PK1 [11], the model (eq:ASM 1.1) with the step function firing rate (eq:FR 5.1) admits a steady state as well as a unique solution.

th:SSFR
eq:SPFR

Theorem 5.1. *For any $\varepsilon \ge 0$, there exists at least one unit mass steady solution $(F_\varepsilon, M_\varepsilon)$ to the stationary problem*

(5.5a) $$\partial_x F_\varepsilon + \mathbf{1}_{x>\sigma(\varepsilon M_\varepsilon)} F_\varepsilon = 0,$$

(5.5b) $$F_\varepsilon(0) = M_\varepsilon, \quad 0 \le F_\varepsilon \le 1, \quad \langle F_\varepsilon \rangle = 1.$$

Moreover, there exists positive ε_0 small enough, such that the above solution is unique for any $\varepsilon \in [0,\varepsilon_0)$. Furthermore, assume (hyp:s1 5.2), then there exists ε_1 large enough, such that the uniqueness holds for any $\varepsilon \in (\varepsilon_1, +\infty]$

th:EU

Theorem 5.2. *Assume (eq:ID 5.4), then for any $\varepsilon > 0$, there exists a unique solution (f,m) to the equations (eq:ASM 1.1) with the step function firing rate satisfying (eq:FR 5.1), which satisfies that*

$$0 \le f(t,x) \le 1, \quad \forall t,x \ge 0,$$

$$1-\sigma_+ \le m(t) \le 1, \quad \forall t \ge 0.$$

By applying the same method, we conclude again Theorem th:MR 1.3 as well as the asymptotic exponential stability in the weak regime obtained in MSWD [10] for the model with the particular firing rate (eq:FR 5.1), which accurate the results in PK1 [11].

th:MRFR **Theorem 5.3.** *We assume that the firing rate a satisfies (5.1) and the delay distribution b satisfies $b = \delta_0$ or (1.8). There exists $\varepsilon_0 > 0$ ($\varepsilon_1 > 0$), small (large) enough, such that for any $\varepsilon \in (0, \varepsilon_0)$ ($\varepsilon \in (\varepsilon_1, +\infty)$, with the assumption (5.2)), the steady state $(F_\varepsilon, M_\varepsilon)$ is unique. There also exists some constants $\alpha < 0$, $C \geq 1$ and $\eta > 0$ (besides $\zeta_\varepsilon \to 0$, as $\varepsilon \to \infty$ under the assumption (5.3)) such that for any connectivity parameter $\varepsilon \in (0, \varepsilon_0)$ ($\varepsilon \in (\varepsilon_1, +\infty)$) and any initial datum $f_0 \in L^1$ with mass 1 satisfying $0 \leq f_0 \leq 1$ and such that $\|f_0 - F_\varepsilon\|_{L^1} \leq \eta/\varepsilon$ ($\leq \eta/\zeta_\varepsilon$), the solution f to the evolution equation (1.1) satisfies*

$$\|f(t,.) - F_\varepsilon\|_{L^1} \leq Ce^{\alpha t}, \qquad \forall\, t \geq 0.$$

Failing to construct the linearized equations (1.10) and (1.11), we replace them with another more concise linear equation for the variation functions $(g, n, q) = (f, m, p) - (F_\varepsilon, M_\varepsilon, M_\varepsilon)$ around a stationary state $(F_\varepsilon, M_\varepsilon)$, which writes

eq:ASMFRlin1

$$\partial_t g = -\partial_x g - g\mathbf{1}_{x>\sigma_\varepsilon}, \tag{5.6a}$$

$$g(t,0) = q(t), \quad g(0,x) = g_0(x), \tag{5.6b}$$

where here and below we note $\sigma_\varepsilon := \sigma(\varepsilon M_\varepsilon)$ for simplicity, with

eq:ASMFRlin2

$$q(t) = \mathcal{P}[g, M_\varepsilon] = \int_0^\infty g\mathbf{1}_{x>\sigma_\varepsilon}\mathrm{d}x. \tag{5.7}$$

c:FRWithoutDelay

5.1. **Case without delay.** When $b = \delta_0$, the system (5.6) goes to

$$\begin{aligned} &\partial_t g + \partial_x g + g\mathbf{1}_{x>\sigma_\varepsilon} = 0,\\ &g(t,0) = \mathcal{N}_\varepsilon[g(t,\cdot)] := \mathcal{P}[g, M_\varepsilon],\\ &g(0,x) = g_0(x). \end{aligned}$$

We rewriting similarly the equation as

$$\partial_t g = \Lambda_\varepsilon g := -\partial_x g - g\mathbf{1}_{x>\sigma_\varepsilon} + \delta_{x=0}\mathcal{N}_\varepsilon[g], \tag{5.8}$$

acting on the space of X defined by

$$X := L^p_0(\mathbb{R}_+) = \{h \in L^p(\mathbb{R}_+);\ \langle h\rangle = 0\}, \quad 1 \leq p \leq \infty,$$

and possessing the asymptotic convergence to the equilibrium.

th:MReFR **Theorem 5.4.** *Assume (5.1) (with (5.3)) and define the operator Λ_ε the generator of a weakly $*$ continuous semigroup S_{Λ_ε} acting on X endowed with the weak $*$ topology $\sigma(L^p_0, L^{p'})$. Moreover, there exist some constants $\varepsilon_0 > 0$ small enough or $\varepsilon_1 > 0$ large enough and $C > 0$, such that for any $\varepsilon \in [0, \varepsilon_0]$ or $\varepsilon \in [\varepsilon_1, \infty]$, the structure of the spectrum satisfies $\Sigma(\Lambda_\varepsilon) \cap \Delta_{-1} = \{0\}$, where $\Delta_{-1} := \{z \in \mathbb{C},\ \Re\mathrm{e}\, z > -1\}$ and there holds*

eq:lcFR

$$\|S_{\Lambda_\varepsilon}(t)g_0\|_X \leq Ce^{-t}\|g_0\|_X, \quad \forall\, t \geq 0, \tag{5.9}$$

for any $g_0 \in X$, $\langle g_0\rangle = 0$.

To conclude our main result again for the particular firing rate, we split the operator Λ_ε as $\Lambda_\varepsilon = \mathcal{A}_\varepsilon + \mathcal{B}_\varepsilon$ with $\mathcal{A}_\varepsilon$ and $\mathcal{B}_\varepsilon$ defined on X by

eq:FRAB

$$\mathcal{A}_\varepsilon g := \delta_0\mathcal{N}_\varepsilon[g], \quad \mathcal{B}_\varepsilon g := -\partial_x g - g\mathbf{1}_{x>\sigma_\varepsilon}. \tag{5.10}$$

Then, Theorem 5.4 is naturally a consequence under the deduction of the proof of Theorem 3.1 with the following properties of the two auxiliary operators.

lem:LmFR1 **Lemma 5.5.** *Assume (5.1) (with (5.3)). The operators $\mathcal{A}_\varepsilon$ and $\mathcal{B}_\varepsilon$ satisfy the following properties.*

(i) *$\mathcal{A}_\varepsilon \in \mathscr{B}(X, Y)$, where $Y = \mathbb{C}\delta_0 \subset X$ with compact embedding;*

(ii) $S_{\mathcal{B}_\varepsilon}(t)$ *is* -1*-hypodissipative in* X*, namely there exists* C *such that*

$$\|S_{\mathcal{B}_\varepsilon}(t)\|_{X\to X} \le C\, e^{-t}, \quad \forall\, t \ge 0.$$

(iii) *the family of operators* $S_{\mathcal{B}_\varepsilon} * \mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}$ *satisfies*

$$\|(S_{\mathcal{B}_\varepsilon} * \mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})(t)\|_{X\to L^{p'}} \le Ce^{-t}, \quad \forall t \ge 0.$$

Proof of Lemma lem:LmFR1 *5.5.* In order to shorten notation, we note

$$A(x) := \int_0^x \mathbf{1}_{y>\sigma_\varepsilon}\mathrm{d}y = (x-\sigma_\varepsilon)_+.$$

(i) We obtain $\mathcal{A}_\varepsilon \in \mathscr{B}(X,Y)$ from the fact that $\mathcal{N}_\varepsilon[\cdot] \in \mathscr{B}(X,\mathbb{R})$ because of (hyp:a1 II.4) ((hyp:s2 5.3)).

(ii) We write $S_{\mathcal{B}_\varepsilon}$ with the explicit formula

eq:defSBeps1 (5.11) $$S_{\mathcal{B}_\varepsilon}(t)g(x) = e^{-A(x)}g(x-t)\mathbf{1}_{x-t\ge 0} =: S(t);$$

We have

$$\begin{aligned}\|S_{\mathcal{B}_\varepsilon}(t)g\|_X &= \|e^{-A(x+t)_+}g(x)\|_X\\ &\le \|e^{-(t-\sigma_\varepsilon)_+}g(x)\|_X\\ &\le Ce^{-t}\|g(x)\|_X,\end{aligned}$$

with $C = e^{\sigma_\varepsilon} > 0$.

(iii) We have

$$\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}(t)g = \delta_0\, N(t),$$

with $N(t) := \mathcal{N}_\varepsilon[S(t)]$. Moreover, we have

$$|N(t)| \le \int_0^\infty e^{-x}\,|g(x-t)|\mathbf{1}_{x-t\ge 0}\,\mathrm{d}x \le C\,e^{-t}\|g\|_X$$

for any $t \ge 0$. We deduce

$$\begin{aligned}(S_{\mathcal{B}_\varepsilon} * \mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})(t)g(x) &= \int_0^t (S_{\mathcal{B}_\varepsilon}(s)\delta_0)(x)N(t-s)\,\mathrm{d}s\\ &= \int_0^t e^{-A(x)}\,\delta_{x-s=0}N(t-s)\mathbf{1}_{x-s\ge 0}\,\mathrm{d}s\\ &= e^{-A(x)}\,N(t-x)\mathbf{1}_{t-x\ge 0}.\end{aligned}$$

With direct computation, we have

$$\|(S_{\mathcal{B}_\varepsilon} * \mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})(t)g\|_{L^{p'}} \le C\,e^{-t}\|g\|_X.$$

□

To go further on the nonlinear equation (eq:ASM I.1) with the firing rate satisfying (eq:FR 5.1), we reconsider the function (eq:Phi 3.9) and the optimal transportation Monge-Kantorovich-Wasserstein distance to deduce that

lem:varphigFR **Lemma 5.6.** *Assume* (eq:FR1 5.1a) *(with* (hyp:s2 5.3)*). There exists* $\varepsilon_0 > 0$ ($\varepsilon_1 > 0$) *small (large) enough, such that for any* $\varepsilon \in (0,\varepsilon_0)$ ($\varepsilon \in (\varepsilon_1,\infty)$)*, there exists a function* $\varphi_\varepsilon : P(\mathbb{R}) \to \mathbb{R}$ *which is Lipschitz continuous for the weak topology of probability measures and such that* $\mu = \varphi_\varepsilon[g]$ *is the unique solution to the equation*

$$\mu \in \mathbb{R}_+, \quad \Phi(g,\mu) = 0.$$

Proof of Lemma 5.6. For any fixed $g \in P(\mathbb{R}_+)$ and $\varepsilon \geq 0$, there exists clearly at least one solution $\mu \in (0,1]$ to the equation $\Phi(g,\mu) = 0$. Fix $f, g \in P(\mathbb{R}_+)$ and consider $\mu, \nu \in \mathbb{R}_+$ such that

$$\Phi(f,\mu) = \Phi(g,\nu) = 0.$$

considering

$$\nu - \mu = \int_0^\infty (g-f)\mathbf{1}_{x>\sigma(\varepsilon\nu)} + \int_0^\infty f(\mathbf{1}_{x>\sigma(\varepsilon\nu)} - \mathbf{1}_{x>\sigma(\varepsilon\mu)}),$$

with the fact that

$$\Big|\int_0^\infty (g-f)\mathbf{1}_{x>\sigma(\varepsilon\nu)}\Big| \leq W_1(g,f),$$

and

$$\Big|\int_0^\infty f(\mathbf{1}_{x>\sigma(\varepsilon\nu)} - \mathbf{1}_{x>\sigma(\varepsilon\mu)})\mathrm{d}x\Big| \leq |\sigma(\varepsilon\nu) - \sigma(\varepsilon\mu)| \leq \varepsilon\|\sigma'\|_\infty|\mu-\nu|,$$

we deduce obviously that

eq:mu-nuFR

$$|\mu-\nu|\,(1-\varepsilon\|\sigma'\|_\infty) \leq\ W_1(g,f), \tag{5.12}$$

and we may fix $\varepsilon_0 > 0$ ($\varepsilon_1 > 0$) such that $1-\varepsilon\|\sigma'\|_{L^\infty} \in (0,1)$, $\varepsilon \in [0,\varepsilon_0]$ ($\varepsilon \in [\varepsilon_1,\infty]$), which permits the uniqueness and Lipschitz continuity then the well-defined mapping $\varphi_\varepsilon[g] := \mu$.

The above mapping admits our proof of Theorem 5.3 in the case without delay.

Proof of Theorem 5.3. With the above mapping, we present the new formulation of the solutions to the evolution and stationary equations in the weak (strong) connectivity regime $\varepsilon \in (0,\varepsilon_0]$ ($\varepsilon \in [\varepsilon_1,\infty)$), where ε_0 (ε_1) is defined in Lemma 5.6. For any given initial datum $f_0 \in L^p(\mathbb{R}_+)$, $1 \leq p \leq \infty$ satisfying (5.4) the solution $f \in C([0,\infty);L^p(\mathbb{R}_+))$, $1 \leq p \leq \infty$, $0 \leq f \leq 1$ to the evolution equation (1.1) with the particular firing rate (5.1) and the steady state F_ε clearly satisfy

$$\begin{aligned} \partial_t f + \partial_x f + f\mathbf{1}_{x>\sigma(\varepsilon\varphi[f])} = 0, \qquad & f(t,0) = \varphi[f(t,\cdot)], \\ \partial_x F_\varepsilon + F_\varepsilon\mathbf{1}_{x>\sigma_\varepsilon} = 0, \qquad & F_\varepsilon(0) = M_\varepsilon = \varphi[F_\varepsilon]. \end{aligned}$$

Then the variation function $g := f - F_\varepsilon$ satisfies the equation

$$\partial_t g = \Lambda_\varepsilon g + Z_\varepsilon[g], \quad Z_\varepsilon[g] := -Q_\varepsilon[g] + \delta_0\langle Q_\varepsilon[g]\rangle,$$

with the item

$$Q_\varepsilon[g] = \mathrm{sgn}(\mathcal{N}_\varepsilon[g])(g+F_\varepsilon)\mathbf{1}_{\mathscr{I}[\mathcal{N}_\varepsilon[g]]},$$

where the interval

$$\mathscr{I}(n) := (\sigma(\varepsilon M_\varepsilon + \varepsilon n_+), \sigma(\varepsilon M_\varepsilon + \varepsilon n_-)].$$

We observe that

$$\begin{aligned} \|Q_\varepsilon[g]\|_X &= \|\,\mathrm{sgn}(\mathcal{N}_\varepsilon)(g+F_\varepsilon)\mathbf{1}_{\mathscr{I}[\mathcal{N}_\varepsilon]}\|_X \leq \big|\mathscr{I}[\mathcal{N}_\varepsilon]\big| \\ &= \big(\sigma(\varepsilon M_\varepsilon - \varepsilon\mathcal{N}_{\varepsilon-}) - \sigma(\varepsilon M_\varepsilon + \varepsilon\mathcal{N}_{\varepsilon+})\big) \\ &\leq C\,\varepsilon\|\sigma'\|_\infty\big|\mathcal{N}_\varepsilon[g]\big| \\ &\leq C\,\varepsilon(\zeta_\varepsilon)\|g\|_X, \end{aligned}$$

which implies immediately that $\|Z_\varepsilon[g]\|_X \le \varepsilon\,(\zeta_\varepsilon)\,C\|g\|_X$. The Duhamel formula tells

$$\begin{aligned}\|g(t)\|_X &\le Ce^{-t}\|g_0\|_X + \int_0^t Ce^{s-t}\|Z_\varepsilon[g(s)]\|_X\mathrm{d}s\\ &\le Ce^{-t}\|g_0\|_X + \varepsilon\,(\zeta_\varepsilon)\,C\int_0^t e^{s-t}\|g(s)\|_X\mathrm{d}s.\end{aligned}$$

We conclude the asymptotic exponential stability from the Gronwall's lemma. □

sec:WithDelayFR

5.2. **Case with delay.** We introduce the intermediate evolution equation (eq:Eov 4.1) again to write the linear equation (eq:ASM eq:ASM1 eq:ASM2 eq:ASM1lin3 5.6)-(5.7)- (1.12) as a time autonomous system

eq:ASFR

$$\partial_t(g,v) = \mathscr{L}_\varepsilon(g,v), \tag{5.13}$$

where the operator $\mathscr{L}_\varepsilon = (\mathscr{L}^1_\varepsilon, \mathscr{L}^2_\varepsilon)$ is defined by

$$\begin{aligned}\mathscr{L}^1_\varepsilon(g,v) &:= -\partial_x g - g\mathbf{1}_{x>\sigma_\varepsilon} + \delta_{x=0}\mathcal{O}_\varepsilon[g,v],\\ \mathscr{L}^2_\varepsilon(g,v) &:= -\partial_y v + \delta_{y=0}\mathcal{O}_\varepsilon[g,v],\end{aligned}$$

with the boundary term

$$\mathcal{O}_\varepsilon[g(t),v(t)] := \mathcal{N}_\varepsilon[g] = \int_0^\infty g\mathbf{1}_{x>\sigma_\varepsilon}\,\mathrm{d}x,$$

in the space

$$X = X_1 \times X_2 := L^p_0(\mathbb{R}_+) \times L^p(\mathbb{R}_+,\mu)$$

with $1 \le p \le \infty$ and $\mu(x) = e^{-\delta x}$, $\delta > 0$ is the same as in the condition (hyp:del 1.8). We extend the exponential stability from the single equation of g to the above autonomous system.

th:MReFR1

Theorem 5.7. *Assume* (eq:FR1 5.1a)-(eq:FR2 5.1b)(hyp:del 1.8) *(with* (hyp:s2 5.3)*). There exist some constants $\varepsilon_0 > 0$ ($\varepsilon_1 > 0$), $C \ge 1$ and $\alpha < 0$ such that for any $\varepsilon \in [0,\varepsilon_0]$ $\varepsilon \in [\varepsilon_1,\infty]$ there holds $\Sigma(\mathscr{L}_\varepsilon) \cap \Delta_\alpha = \{0\}$ and*

CwdFR

$$\|S_{\mathscr{L}_\varepsilon}(t)(g_0,v_0)\|_X \le Ce^{\alpha t}\|(g_0,v_0)\|_X, \tag{5.14}$$

for any $(g_0,v_0) \in X$, s.t. $\langle (g_0,v_0),(1,0)\rangle_{X,X'} = 0$.

The extension follows from the Spectral Mapping theorem and the Weyl's Theorem by introducing a convenient splitting of the operator $\mathscr{L}_\varepsilon$ as $\mathscr{L}_\varepsilon = \mathcal{A}_\varepsilon + \mathcal{B}_\varepsilon$ with

$$\mathcal{B}_\varepsilon(g,v) = \begin{pmatrix}\mathcal{B}^1_\varepsilon(g,v)\\ \mathcal{B}^2_\varepsilon(g,v)\end{pmatrix} = \begin{pmatrix}-\partial_x g - g\mathbf{1}_{x>\sigma_\varepsilon}\\ -\partial_y v\end{pmatrix}$$

and

$$\mathcal{A}_\varepsilon(g,v) = \begin{pmatrix}\mathcal{A}^1_\varepsilon(g,v)\\ \mathcal{A}^2_\varepsilon(g,v)\end{pmatrix} = \begin{pmatrix}\delta_{x=0}\mathcal{O}_\varepsilon[g,v]\\ \delta_{y=0}\mathcal{O}_\varepsilon[g,v]\end{pmatrix},$$

and there is nothing left except whether the following properties are met by the operators $\mathcal{A}_\varepsilon$ and $\mathcal{B}_\varepsilon$.

lem:LmFR2

Lemma 5.8. *Assume* (eq:FR1 5.1a)-(eq:FR2 5.1b)(hyp:del 1.8) *(with* (hyp:s2 5.3)*). Then the two operators satisfy*

(i) *$\mathcal{A}_\varepsilon \in \mathscr{B}(X,Y)$, where $Y = \mathbb{C}\delta_0 \times \mathbb{C}\delta_0 \subset X$ with compact embedding;*
(ii) *$S_{\mathcal{B}_\varepsilon}(t)$ is α-hypodissipative in X;*
(iii) *the family of operators $S_{\mathcal{B}_\varepsilon} * \mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}$ satisfies*

$$\|(S_{\mathcal{B}_\varepsilon} * \mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})(t)\|_{X\to L^{p'}\times L^{p'}(\mu)} \le Ce^{\alpha t}, \quad \forall t \ge 0.$$

Proof. (i) It is an immediate consequence of the fact that $\mathcal{D} \in \mathscr{B}(X_2;\mathbb{R})$ (because of (1.8)) and $\mathcal{N}_\varepsilon \in \mathscr{B}(X_1;\mathbb{R})$.

(ii) Since $S_{\mathcal{B}^1_\varepsilon}$ is nothing but the semigroup $S_{\mathcal{B}_\varepsilon}$ defined in (5.11) which is -1-dissipative thanks to Lemma 5.5-(ii), we just have to prove the dissipativity of the translation semigroup $S_{\mathcal{B}^2_\varepsilon}$ which is given by the explicit formula $[S_{\mathcal{B}^2_\varepsilon}(t)v](y) = v(y-t)\mathbf{1}_{y-t\geq 0}$. That follows from

$$\|S_{\mathcal{B}^2_\varepsilon}(t)v\|_{X_2} = \Big(\int_0^\infty \big(|v(y-t)|\mathbf{1}_{y-t\geq 0}\, e^{-\delta y}\big)^p\, dy\Big)^{\frac{1}{p}} = e^{-\delta t}\|v\|_{X_2},$$

for any $v \in X_2$ and any $t \geq 0$. Then, we choose $\alpha := \max\{-1, -\delta\}$.

(iii) Clearly, we have

$$\mathcal{A}^1_\varepsilon S_{\mathcal{B}_\varepsilon}(t)(g,v)(x) = \delta_{x=0}N(t), \tag{5.15}$$

$$\mathcal{A}^2_\varepsilon S_{\mathcal{B}_\varepsilon}(t)(g,v)(y) = \delta_{y=0}N(t), \tag{5.16}$$

with

$$N(t) := \mathcal{N}_\varepsilon[S^1_{\mathcal{B}_\varepsilon}(t)g] = \int_0^\infty e^{-A(x)}g(x-t)\mathbf{1}_{x-t\geq 0}\mathbf{1}_{x>\sigma_\varepsilon}\mathrm{d}x,$$

where here and below the ε dependency is often removed without risk of misleading. With the similar argument in the proof of Lemma 5.5, we deduce

$$\begin{aligned}\|S_{\mathcal{B}^1_\varepsilon} * \mathcal{A}^1_\varepsilon S_{\mathcal{B}_\varepsilon}(t)(g,v)(x)\|_{L^{p'}} &\leq Ce^{\alpha t}\|(g,v)\|_X\\ \|S_{\mathcal{B}^2_\varepsilon} * \mathcal{A}^2_\varepsilon S_{\mathcal{B}_\varepsilon}(t)(g,v)(y)\|_{L^{p'}(\mu)} &\leq Ce^{\alpha t}\|(g,v)\|_X.\end{aligned}$$

Thus, the announced estimate holds for the family of operators $S_{\mathcal{B}_\varepsilon} * \mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}$. □

Proof of Theorem 5.7. Since $\mathscr{L}^1_\varepsilon = \Lambda_\varepsilon$, we have already proved that $g(t) := S_{\mathscr{L}^1_\varepsilon}(t)g_0$ satisfies $\|g(t)\| \leq Ce^{-t}\|g_0\|_{X_1}$ for any $t \geq 0$ from Theorem 3.3. We then just focus on $\mathscr{L}^2_\varepsilon$. The Duhamel formula associated to the equation $\partial_t v = \mathscr{L}^2_\varepsilon(g,v)$ writes

$$v(t) = S_{\mathcal{B}^2_\varepsilon}(t)v_0 + \int_0^t S_{\mathcal{B}^2_\varepsilon}(t-s)\mathcal{A}^2_\varepsilon\big(g(s),v(s)\big)\,\mathrm{d}s.$$

Using the already known estimate on $g(t)$, we deduce

$$\begin{aligned}\|S_{\mathscr{L}^2_\varepsilon}v_0(t)\|_{X_2} &= \|v(t)\|_{X_2} \leq \|S_{\mathcal{B}^2_\varepsilon}(t)v_0\|_{X_2} + \int_0^t \|S_{\mathcal{B}^2_\varepsilon}(t-s)\delta_0\mathcal{N}_\varepsilon[g(s)]\|_{X_2}\,\mathrm{d}s\\ &\leq e^{-\delta t}\|v_0\|_{X_2} + \int_0^t e^{-\delta(t-s)}C\,e^{-s}\|g_0\|_{X_1}\,\mathrm{d}s\\ &\leq C\,e^{\alpha t}\|(g_0,\,v_0)\|_X\end{aligned}$$

for some $0 > \alpha > \max\{-1, -\delta\}$, which yields our conclusion. □

Now, we complete the rest part of the proof to describe the stability in the case with delay more precisely compared to that in [11].

Proof of Theorem 5.3 in case with delay. We write the system as

$$\begin{aligned}\partial_t f &= -\partial_x f - f\mathbf{1}_{x>\sigma(\varepsilon\mathcal{D}[u])} + \delta_0\mathbf{P}[f,\mathcal{D}[u]]\\ \partial_t u &= -\partial_y u + \delta_0\mathbf{P}[f,\mathcal{D}[u]]\end{aligned}$$

with

$$\mathbf{P}[f,m] = \int a(m)f, \quad \mathcal{D}[u] = \int bu.$$

We recall that the steady state (F,U), $U := M\mathbf{1}_{y\geq 0}$, satisfies

$$\begin{aligned} 0 &= -\partial_x F - F\mathbf{1}_{x>\sigma(\varepsilon M)} + \delta_0 M \\ 0 &= -\partial_y U + \delta_0 M, \quad M = \mathcal{D}[U] = \mathbf{P}[F, \mathcal{D}[U]]. \end{aligned}$$

We introduce the variation $g := f - F$ and $v = u - U$. The equation on g is

$$\begin{aligned} \partial_t g &= -\partial_x g - f\mathbf{1}_{x>\sigma(\varepsilon\mathcal{D}[u])} + F\mathbf{1}_{x>\sigma(\varepsilon M)} + \delta_0(\mathbf{P}[f,\mathcal{D}[u]] - \mathbf{P}[F,\mathcal{D}[U]]) \\ &= \mathscr{L}^1_\varepsilon(g,v) - \mathcal{Q}[g,v] + \delta_0\langle\mathcal{Q}[g,v]\rangle \\ &= \mathscr{L}^1_\varepsilon(g,v) + \mathcal{Z}^1[g,v], \end{aligned}$$

with

$$\mathcal{Q}[g,v] := \operatorname{sgn}\big(\mathcal{D}[v]\big)(g + F_\varepsilon)\mathbf{1}_{\mathscr{I}[\mathcal{D}[v]]}.$$

The equation on v is

$$\begin{aligned} \partial_t v &= -\partial_y v + \delta_0(\mathbf{P}[f,\mathcal{D}[u]] - \mathbf{P}[F,\mathcal{D}[U]]) \\ &= -\partial_y v + \delta_0\mathcal{O}[g,v] + \delta_0\langle\mathcal{Q}[g,v]\rangle \\ &= \mathscr{L}^2_\varepsilon(g,v) + \mathcal{Z}^2[g,v], \quad \mathcal{Z}^2[g,v] := \delta_0\langle\mathcal{Q}[g,v]\rangle. \end{aligned}$$

We observe that

$$\begin{aligned} \|\mathcal{Q}[g,v]\|_{L^p} &= \|\operatorname{sgn}(\mathcal{D}[v])g\mathbf{1}_{\mathscr{I}[\mathcal{D}[v]]}\|_{L^p} \leq \big|\mathscr{I}[\mathcal{D}[v]]\big| \\ &= \big(\sigma(\varepsilon M - \varepsilon\mathcal{D}[v]_-) - \sigma(\varepsilon M + \varepsilon\mathcal{D}[v]_+)\big) \\ &\leq C\varepsilon\,\|\sigma'\|_\infty\big|\mathcal{D}[v]\big| \\ &\leq \varepsilon\,(\zeta_\varepsilon)\|v\|_{X_2}, \end{aligned}$$

which implies immediately that $\|\mathcal{Z}^1[g,v]\|_{X_1} \leq \varepsilon\,(\zeta_\varepsilon)\,C\|(g,v)\|_X$ and $\|\mathcal{Z}^2[g,v]\|_{X_2} \leq \varepsilon\,(\zeta_\varepsilon)\,C\|(g,v)\|_X$. We write the Duhamel formula

$$(g(t),v(t)) = S_{\mathscr{L}_\varepsilon}(t)(g_0,v_0) + \int_0^t S_{\mathscr{L}_\varepsilon}(t-s)\mathcal{Z}[g(s),v(s)]\,\mathrm{d}s.$$

and thanks to the Gronwall's Lemma, we conclude the exponential asymptotic stability of Theorem th:MRFR 5.3 in the case with delay. □

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