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# General time elapsed neuron network model: well-posedness and strong connectivity regime

S. MISCHLER & Q. WENG

## Abstract

The time elapsed models, mathematically describing the probability density of neurons structured by the distributions of times elapsed since the last charge, are designed to analyse the firing activity of a homogenous assembly of neuron network. Under general assumption on the firing rate and the delay distribution, we prove the existence and the uniqueness of the weak solution in the whole connectivity regime as well as the parallel results obtained in [10], including the uniqueness of the steady state and its nonlinear exponential stability in the strong connectivity regime. The result generalizes more explicit asymptotic behaviours than those obtained in [11, 12] in the strong connectivity regime. Our approach uses the spectral analysis theory for semigroups in Banach spaces developed recently by the first author and collaborators.

**Keywords.** Neuron networks, time elapsed dynamics, semigroup spectral analysis, strong connectivity, long time asymptotic.

## 1 Introduction

The information transmission and processing mechanism in the nervous systems relies on the quantity of electrical pulses as the response to incoming stimulations, during which the neuron experience a period of recalcitrance called discharge time before reactive. In this work, we shall focus on the model describing the neuronal dynamics in accordance with this kind of discharge time which has been introduced and studied in [3, 11, 12]. In order to reflect the recovery of the neuronal membranes after each discharge, the

model consider an instantaneous firing rate depending on the time elapsed since last discharge as well as the inputs of neurons. This sort of models are also regarded as a mean field limit of finite number of neuron network models referred to [1, 2, 14, 13].

We consider the neuronal network described by the density number of neurons  $f = f(t, x) \geq 0$  in state  $x \geq 0$  at time  $t \geq 0$ , where  $x$  is a local time (or internal clock) corresponding to the elapsed time since the last discharge. The dynamic of the neuron network is given by the following nonlinear time elapsed (or of age structured type) evolution equation

$$\partial_t f = -\partial_x f - a(x, \varepsilon m(t))f =: \mathcal{L}_{\varepsilon m(t)} f, \quad (1.1a)$$

$$f(t, 0) = p(t), \quad f(0, x) = f_0(x). \quad (1.1b)$$

Here  $a(x, \varepsilon \mu) \geq 0$  represents the firing rate of a neuron in the state  $x$  and in an environment  $\mu \geq 0$  formed by the global network activity and a network connectivity parameter  $\varepsilon \geq 0$ . The function  $p(t)$  represents the total density of neurons undergoing a discharge at time  $t$  and is defined through

$$p(t) := \mathcal{P}[f(t); m(t)],$$

where

$$\mathcal{P}[g, \mu] = \mathcal{P}_\varepsilon[g, \mu] := \int_0^\infty a(x, \varepsilon \mu) g(x) dx.$$

The function  $m(t)$  represents the global neuronal activity at time  $t \geq 0$  resulting from earlier discharges with the interactions among the neurons taken into account and is given by

$$m(t) := \int_0^\infty p(t-y)b(dy),$$

where the delay distribution  $b$  is a probability measure which takes into account the persistence of the electric activity in the network resulting from discharges. In the sequel, we will consider the two following situations :

- The *case without delay*, when  $b = \delta_0$  and then  $m(t) = p(t)$ .
- The *case with delay*, when  $b$  is a smooth function.

Notice that in both cases, the solution  $f$  of the time elapsed equation (1.1) satisfies

$$\frac{d}{dt} \int_0^\infty f(t, x) dx = f(t, 0) - \int_0^\infty a(x, \varepsilon m(t)) f(t, x) dx = 0.$$

As a consequence, the total density number of neurons (also called *mass* in the sequel) is conserved and we can normalize that mass to be 1. In other words, we may always assume

$$\langle f(t, \cdot) \rangle = \langle f_0 \rangle = 1, \quad \forall t \geq 0, \quad \langle g \rangle := \int_0^\infty g(x) dx.$$

A (normalized) steady state for the time elapsed evolution equation (1.1) is a couple  $(F_\varepsilon, M_\varepsilon)$  of a density number of neurons  $F_\varepsilon = F_\varepsilon(x) \geq 0$  and a network activity  $M_\varepsilon \geq 0$  such that

$$0 = -\partial_x F_\varepsilon - a(x, \varepsilon M_\varepsilon) F_\varepsilon = \mathcal{L}_{\varepsilon M_\varepsilon} F_\varepsilon, \quad (1.2a)$$

$$F_\varepsilon(0) = M_\varepsilon, \quad \langle F_\varepsilon \rangle = 1. \quad (1.2b)$$

It is worth emphasizing that for a steady state the associated network activity and discharge activity are two equal constants because of the normalization of the delay distribution, i.e.  $\langle b \rangle = 1$ .

Our main purpose in this paper is to prove the existence and uniqueness of the solution to the time elapsed evolution equation (1.1) no matter which  $\varepsilon > 0$ , which possesses the exponential asymptotic stability in strong connectivity regime. Before stating that result, let us present the precise mathematical assumptions we will need on the firing rate  $a$  and on the delay distribution  $b$ .

We make the physically reasonable assumption

$$\partial_x a \geq 0, \quad a' = \partial_\mu a \geq 0, \quad (1.3)$$

$$0 < a_0 := \lim_{x \rightarrow \infty} a(x, 0) \leq \lim_{x, \mu \rightarrow \infty} a(x, \mu) =: a_1 < \infty, \quad (1.4)$$

and the smoothness assumption

$$a \in W^{2, \infty}(\mathbb{R}_+^2). \quad (1.5)$$

as well as the decay assumption, for a.e.  $x \geq 0$ ,

$$\varepsilon \partial_\mu a(x, \varepsilon \mu) \rightarrow 0, \quad \varepsilon^2 \partial_{\mu\mu}^2 a(x, \varepsilon \mu) \rightarrow 0, \quad \text{as } \varepsilon \rightarrow \infty. \quad (1.6)$$

In the delay case, we assume that  $b(dy) = b(y) dy$  satisfies the smoothness and lost of memory conditions

$$\exists \delta > 0, \quad \int_0^\infty e^{\delta y} (b(y) + |b'(y)|) dy < \infty. \quad (1.7)$$

The above assumptions permit the existence and uniqueness of the solution to the nonlinear problem (1.1) thanks to the Banach fixed-point theorem, as well as the existence of corresponding steady state, which is unique in the strong connectivity regime. The proofs are presented in the following part.

**Theorem 1.1.** *Assume (1.3)-(1.4)-(1.5)-(1.6), then for any  $\varepsilon > 0$ , there exists a unique nonnegative and mass conserving weak solution  $f \in C(\mathbb{R}_+; L^1(\mathbb{R}_+))$  to the evolution equation (1.1) for some functions  $m, p \in C([0, \infty))$ .*

**Theorem 1.2.** *Assume (1.3)-(1.4)-(1.5)-(1.6). For any  $\varepsilon \geq 0$ , there exists at least one solution  $(F_\varepsilon(x), M_\varepsilon) \in BV(\mathbb{R}_+) \times \mathbb{R}_+$  to the stationary problem (1.2), where  $BV(\mathbb{R}_+)$  represents the space of bounded variation measures. Moreover, there exists  $\varepsilon_1 > 0$ , large enough, such that the above solution is unique for any  $\varepsilon \in (\varepsilon_1, +\infty]$ .*

The main result we establish in the paper is the following long-time asymptotic result on the solutions in the strong connectivity regime parallel to those in the weak one referred to [10].

**Theorem 1.3.** *We assume that the firing rate  $a$  satisfies (1.3), (1.4), (1.5) and (1.6). We also assume that the delay distribution  $b$  satisfies  $b = \delta_0$  or (1.7). There exists  $\varepsilon_1 > 0$ , large enough, such that for any  $\varepsilon \in (\varepsilon_1, +\infty)$  the steady state  $(F_\varepsilon, M_\varepsilon)$  is unique. There also exist some constants  $\alpha < 0$ ,  $C \geq 1$ ,  $\eta > 0$  and  $\zeta_\varepsilon \rightarrow 0$  as  $\varepsilon \rightarrow \infty$  such that for any connectivity parameter  $\varepsilon \in (\varepsilon_1, +\infty)$  and any initial datum  $0 \leq f_0 \in L^1$  with mass 1 and such that  $\|f_0 - F_\varepsilon\|_{L^1} \leq \eta/\zeta_\varepsilon$ , the (unique, positive and mass conserving) solution  $f$  to the evolution equation (1.1) satisfies*

$$\|f(t, \cdot) - F_\varepsilon\|_{L^1} \leq Ce^{\alpha t}, \quad \forall t \geq 0.$$

Our proof follows a strategy of “perturbation of semigroup spectral gap” initiated in [7] for studying long time convergence to the equilibrium for the homogeneous inelastic Boltzmann equation and used recently in [8] for a neuron network equation. More precisely, we introduce the linearized equation for the variation functions  $(g, n, q) = (f, m, p) - (F_\varepsilon, M_\varepsilon, M_\varepsilon)$  around a stationary state  $(F_\varepsilon, M_\varepsilon, M_\varepsilon)$ , which writes

$$\partial_t g = -\partial_x g - a(x, \varepsilon M_\varepsilon)g - n(t) \varepsilon (\partial_\mu a)(x, \varepsilon M_\varepsilon) F_\varepsilon, \quad (1.8a)$$

$$g(t, 0) = q(t), \quad g(0, x) = g_0(x), \quad (1.8b)$$

with

$$q(t) = \int_0^\infty a(x, \varepsilon M_\varepsilon) g \, dx + n(t) \varepsilon \int_0^\infty (\partial_\mu a)(x, \varepsilon M_\varepsilon) F_\varepsilon \, dx \quad (1.9)$$

and

$$n(t) := \int_0^\infty q(t-y)b(dy). \quad (1.10)$$

We associate to that linear evolution equation a generator  $\Lambda_\varepsilon$  (which acts on an appropriate space to be specified in the two cases without and with delay) and its semigroup  $S_{\Lambda_\varepsilon}$ . It turns out that we may split the operator  $\Lambda_\varepsilon$  as

$$\Lambda_\varepsilon = \mathcal{A}_\varepsilon + \mathcal{B}_\varepsilon,$$

for some  $\alpha$ -hypodissipative operator  $\mathcal{B}_\varepsilon$ ,  $\alpha < 0$ , and some bounded and  $\mathcal{B}_\varepsilon$ -power regular operator  $\mathcal{A}_\varepsilon$  as defined in [16, 4, 9, 6]. In particular, the version of the Spectral Mapping Theorem of [9, 6] and the version of the Weyl's Theorem of [16, 4, 9, 6] imply that the semigroup  $S_{\Lambda_\varepsilon}$  as a finite dimensional dominant part. Moreover, the semigroup  $S_\infty$  being positive, we may use the Krein-Rutman Theorem established in [9, 6] in order to get that the stationary state  $(F_\infty, M_\infty, M_\infty)$  is unique and exponentially stable. Using next a perturbative argument developed in [7, 15, 6], we get that the unique stationary state  $(F_\varepsilon, M_\varepsilon, M_\varepsilon)$  is also exponentially stable in the strong connectivity regime. We conclude the proof of Theorem 1.3 by a somewhat classical nonlinear exponential stability argument.

This article is organized by the following plan. In Section 2, we prove the existence and uniqueness of the solution and the stationary state result. In Section 3, we introduce the strategy and establish Theorem 1.3 in the case without delay. In section 4, we establish Theorem 1.3 in the case with delay.

## 2 Existence and the steady state

### 2.1 Existence of the solution

To conclude the existence of a solution to (1.1), we are going to apply a fixed point argument with the benefit of the following lemma.

**Lemma 2.1.** *For any  $m \in L^\infty([0, T])$  with a corresponding  $f$  satisfying the equation (1.1), consider the application  $\mathcal{J}: L^\infty([0, T]) \rightarrow L^\infty([0, T])$ ,*

$\mathcal{J}(m)(t) := \int_0^t p(t-y)b(dy)$ , where  $p(t) = \int_0^\infty a(x, \varepsilon m(t))f(x, t)dx$ . Then there exist  $T > 0$  and  $0 < C < 1$  such that the estimate

$$\|\mathcal{J}(m_1) - \mathcal{J}(m_2)\|_{L^\infty([0, T])} \leq C\|m_1 - m_2\|_{L^\infty([0, T])} \quad (2.1)$$

holds for all  $(m_1, m_2) \in L^\infty([0, T])$  and for any  $\varepsilon > 0$ .

*Proof of Lemma 2.1.* We use the characteristics. For all  $x \geq 0$ , the function  $\tilde{f}_x(t) := f(x+t, t)$  satisfies the equation

$$\frac{d}{dt}\tilde{f}_x(t) = -a(x+t, \varepsilon m(t))\tilde{f}_x(t),$$

which tells that  $\tilde{f}_x(t) = f_0(x)e^{-\int_0^t a(s+x, \varepsilon m(s))ds}$  and thus

$$f(x, t) = f_0(x-t)e^{-\int_0^t a(s+x-t, \varepsilon m(s))ds}, \quad \forall x \geq t. \quad (2.2)$$

On the other hand, for all  $t \geq 0$ , we note  $\tilde{f}_t(x) := f(x, x+t)$  satisfying

$$\frac{d}{dx}\tilde{f}_t(x) = -a(x, \varepsilon m(x+t))\tilde{f}_t(x),$$

and we deduce similarly as before that

$$f(x, t) = p(t-x)e^{-\int_0^x a(s, \varepsilon m(s+t-x))ds}, \quad \forall x \leq t. \quad (2.3)$$

We denote  $f_i$ ,  $i = 1, 2$  as the two solutions of the equation

$$\begin{cases} \partial_t f_i(x, t) + \partial_x f_i(x, t) + a(x, \varepsilon m_i(t))f_i(x, t) = 0, \\ f_i(0, t) = p_i(t) = \int_0^\infty a(x, \varepsilon m_i(t))f_i(x, t)dx, \end{cases} \quad (2.4)$$

with the same initial data  $f_0$ . From (2.2) and (2.3), we deduce that

$$p_i(t) = \int_0^t a(x, \varepsilon m_i(t))p_i(t-x)e^{-\int_0^x a(s, \varepsilon m(s+t-x))ds}dx \quad (2.5)$$

$$+ \int_t^\infty a(x, \varepsilon m_i(t))f_0(x-t)e^{-\int_0^t a(s+x-t, \varepsilon m(s))ds}dx. \quad (2.6)$$

Now, we deduce that

$$\mathcal{J}(m_1)(t) - \mathcal{J}(m_2)(t) = \int_0^t (p_1 - p_2)(t-y)b(dy) = \int_0^t (H + E)(t-y)b(dy),$$

In order to control  $H$ , we split it into three parts as  $H = H_1 + H_2 + H_3$ , where

$$H_1 = \int_0^t (p_1 - p_2)(t-x) a(x, \varepsilon m_1(t)) e^{-\int_0^x a(s, \varepsilon m_1(s+t-x)) ds} dx,$$

$$H_2 = \int_0^t p_2(t-x) (a(x, \varepsilon m_1(t)) - a(x, \varepsilon m_2(t))) e^{-\int_0^x a(s, \varepsilon m_1(s+t-x)) ds} dx,$$

$$H_3 = \int_0^t p_2(t-x) a(x, m_2(t)) (e^{-\int_0^x a(s, \varepsilon m_1(s+t-x)) ds} - e^{-\int_0^x a(s, \varepsilon m_2(s+t-x)) ds}) dx.$$

It is clear to attain the estimates

$$\|H_1\|_{L^\infty([0, T])} \leq a_1 T \|p_1 - p_2\|_{L^\infty([0, T])}$$

and

$$\|H_2\|_{L^\infty([0, T])} \leq \varepsilon a_1 \|a'\|_\infty T \|m_1 - m_2\|_{L^\infty([0, T])}.$$

To deal with the rest term  $H_3$ , there exists a constant  $C$  such that

$$\begin{aligned} & \left| e^{-\int_0^x a(s, \varepsilon m_1(s+t-x)) ds} - e^{-\int_0^x a(s, \varepsilon m_2(s+t-x)) ds} \right| \\ & \leq C \int_0^x |a(s, \varepsilon m_1(s+t-x)) - a(s, \varepsilon m_2(s+t-x))| ds, \end{aligned}$$

which leads to that

$$\|H_3\|_{L^\infty([0, T])} \leq \varepsilon a_1^2 \|a'\|_\infty \frac{T^2}{2} \|m_1 - m_2\|_{L^\infty[0, T]}.$$

From the assumption (1.6), there exists  $\varepsilon_1$  large enough such that  $\varepsilon \|a'\|_\infty \leq 1$ , for any  $\varepsilon \in [\varepsilon_1, +\infty)$ . Denoting  $\eta := \max\{\varepsilon_1, 1\}$ , we deduce that

$$\|H\|_{L^\infty([0, T])} \leq a_1 T \|p_1 - p_2\|_{L^\infty([0, T])} + \eta (C_1 T^2 + C_2 T) \|m_1 - m_2\|_{L^\infty([0, T])}, \quad (2.7)$$

for any  $\varepsilon > 0$ . On the other hand, we have the term  $E$  as

$$\begin{aligned} E(t) &= \int_t^\infty f_0(x-t) \left( a(x, \varepsilon m_1(t)) e^{-\int_0^t a(s+x-t, \varepsilon m_1(s)) ds} \right. \\ & \quad \left. - a(x, \varepsilon m_2(t)) e^{-\int_0^t a(s+x-t, \varepsilon m_2(s)) ds} \right) dx \end{aligned}$$



Similarly to the estimates of  $H$ , we deduce that

$$\|E\|_{L^\infty([0,T])} \leq \eta(C_3T + C_4)\|m_1 - m_2\|_{L^\infty([0,T])}. \quad (2.8)$$

From the above estimates (2.7) and (2.8), it turns out that

$$\|p_1 - p_2\|_{L^\infty([0,T])} \leq a_1T\|p_1 - p_2\|_{L^\infty([0,T])} + \eta(C_1T^2 + C_2'T + C_3')\|m_1 - m_2\|_{L^\infty([0,T])},$$

which implies

$$\|p_1 - p_2\|_{L^\infty([0,T])} \leq \eta C\|m_1 - m_2\|_{L^\infty([0,T])} \quad (2.9)$$

when  $a_1T$  less than 1. Hence, it is clear to conclude from (2.9) that

$$\|\mathcal{J}(m_1) - \mathcal{J}(m_2)\|_{L^\infty([0,T])} \leq \eta CT\|m_1 - m_2\|_{L^\infty([0,T])}.$$

By taking  $T$  small enough such that  $\eta CT < 1$ , we attain our conclusion.  $\square$

*Proof of Theorem 1.1.* From Lemma 2.1, for any  $\varepsilon > 0$ , there is a  $T > 0$  which does not depend upon the initial data such that the application  $\mathcal{J}$  admits a unique fixed point  $m(t)$  on  $[0, T]$  then the corresponding  $f(t, x)$  on  $[0, T] \times \mathbb{R}^d$ , which is the unique solution to the equation (1.1), according to the Banach-Picard fixed point theorem. Iterating on  $T$ , we deduce the global existence and uniqueness of the solution  $(f, m)$  to equation (1.1).  $\square$

## 2.2 The stationary problem

Now we present the proof of the steady state in the strong connectivity regime.

*Proof. Step 1.* We prove the existence of a solution. We set

$$A(x, m) := \int_0^x a(y, m) dy, \quad \forall x, m \geq 0.$$

For any  $m \geq 0$ , we can solve the equation (1.2a), by writing

$$F_{\varepsilon, m}(x) := T_m e^{-A(x, \varepsilon m)},$$

where,  $T_m \geq 0$  is chosen in order that  $F_{\varepsilon, m}$  satisfies the mass normalized condition, namely

$$T_m^{-1} = \int_0^\infty e^{-A(x, \varepsilon m)} dx.$$

In order to conclude the existence of a solution, we just have to find a real number  $m = M_\varepsilon$  such that  $m = F_{\varepsilon,m}(0) = T_m$ . Equivalently, we need to find  $M_\varepsilon \geq 0$  such that

$$\Phi(\varepsilon, M_\varepsilon) = 1, \quad (2.10)$$

where

$$\Phi(\varepsilon, m) = mT_m^{-1} := m \int_0^\infty e^{-A(x,\varepsilon m)} dx.$$

From the assumption (1.4) of  $a$ , there exists  $x_0 \in [0, \infty)$  such that  $a(x, \mu) \geq \frac{a_0}{2}$ , for any  $x \geq 0$ ,  $\mu \geq 0$ , and therefore

$$\frac{a_0}{2}(x - x_0)_+ \leq A(x, \mu) \leq a_1 x, \quad \forall x \geq 0, \forall \mu \geq 0. \quad (2.11)$$

We deduce that  $\Phi(\varepsilon, \cdot)$  is a continuous function (from the Lebesgue dominated convergence theorem) and that  $\Phi(0) = 0$ ,  $\Phi(\infty) = \infty$ . From the intermediate value theorem, we immediately conclude.

*Step 2. We prove the uniqueness of the solution in the strong connectivity regime.* Obviously, there exists a unique  $M_\infty := (\int_0^\infty e^{-A(x,\infty)} dx)^{-1} \in (0, \infty)$  such that  $\Phi(\infty, M_\infty) = 1$ . Moreover, we compute

$$\frac{\partial}{\partial m} \Phi(\varepsilon, m) = \int_0^\infty e^{-A(x,\varepsilon m)} \left(1 - m \int_0^x \varepsilon \partial_\mu a(y, \varepsilon m) dy\right) dx,$$

which is continuous as a function of the two variables because of (1.5). We then easily obtain that  $\Phi \in C^1$ . Since moreover

$$\frac{\partial}{\partial m} \Phi(\varepsilon, m)|_{\varepsilon=\infty} = \int_0^\infty e^{-A(x,\infty)} dx > 0,$$

the implicit function theorem implies that there exists  $\varepsilon_1 > 0$ , large enough, such that the equation (2.10) has a unique solution for any  $\varepsilon \in (\varepsilon_1, +\infty]$ .  $\square$

### 3 Case without delay

In this section, we conclude our main result Theorem 1.3 gradually in the case without delay.

### 3.1 Linearized equation and structure of the spectrum

To go one step further, we introduce the linearized equation around the stationary solution  $(F_\varepsilon, M_\varepsilon)$ . On the variation  $(g, n)$ , the linearized equation writes

$$\begin{aligned} \partial_t g + \partial_x g + a_\varepsilon g + a'_\varepsilon F_\varepsilon n(t) &= 0, \\ g(t, 0) = n(t) &= \int_0^\infty (a_\varepsilon g + a'_\varepsilon F_\varepsilon n(t)) dx, \quad g(0, x) = g_0(x), \end{aligned}$$

with  $a_\varepsilon := a(x, \varepsilon M_\varepsilon)$ ,  $a'_\varepsilon := \varepsilon (\partial_\mu a)(x, \varepsilon M_\varepsilon)$ . According to the assumption (1.6), there exists  $\varepsilon_1 > 0$ , large enough, such that

$$\forall \varepsilon \in (\varepsilon_1, \infty) \quad \kappa := \int_0^\infty a'_\varepsilon F_\varepsilon dx < 1,$$

we may define

$$\mathcal{M}_\varepsilon[g] := (1 - \kappa)^{-1} \int_0^\infty a_\varepsilon g dx, \quad (3.1)$$

and the linearized equation is then equivalent to

$$\partial_t g + \partial_x g + a_\varepsilon g + a'_\varepsilon F_\varepsilon \mathcal{M}_\varepsilon[g(t, \cdot)] = 0, \quad (3.2)$$

$$g(t, 0) = \mathcal{M}_\varepsilon[g(t, \cdot)], \quad g(0, x) = g_0(x). \quad (3.3)$$

By regarding the boundary term as a source term, here we reconsider the equation as

$$\partial_t g = \Lambda_\varepsilon g := -\partial_x g - a_\varepsilon g - a'_\varepsilon F_\varepsilon \mathcal{M}_\varepsilon[g] + \delta_{x=0} \mathcal{M}_\varepsilon[g], \quad (3.4)$$

acting on the space of bounded Radon measures

$$X := M^1(\mathbb{R}_+) = \{g \in (C_0(\mathbb{R}))'; \text{supp } g \subset \mathbb{R}_+\},$$

endowed with the weak \* topology  $\sigma(M^1, C_0)$ .

**Theorem 3.1.** *Assume (1.3)-(1.4)-(1.5)-(1.6) and define  $\alpha := -a_0/2 < 0$ . The operator  $\Lambda_\varepsilon$  is the generator of a weakly \* continuous semigroup  $S_{\Lambda_\varepsilon}$  acting on  $X$  endowed with the weak \* topology  $\sigma(M^1, C_0)$ . Moreover, there exists a finite rank projector  $\Pi_{\Lambda_\varepsilon, \alpha}$  which commutes with  $S_{\Lambda_\varepsilon}$ , an integer  $j \geq 0$  and some complex numbers*

$$\xi_1, \dots, \xi_j \in \Delta_\alpha := \{z \in \mathbb{C}, \Re z > \alpha\},$$

such that on  $E_1 := \Pi_{\Lambda_\varepsilon, \alpha} X$  the restricted operator satisfies

$$\Sigma(\Lambda_{\varepsilon|E_1}) \cap \Delta_\alpha = \{\xi_1, \dots, \xi_j\}$$

(with the convention  $\Sigma(\Lambda_{\varepsilon|E_1}) \cap \Delta_\alpha = \emptyset$  when  $j = 0$ ) and for any  $a > \alpha$  there exists a constant  $C_a$  such that the remainder semigroup satisfies

$$\|S_{\Lambda_\varepsilon}(I - \Pi_{\Lambda_\varepsilon, \alpha})\|_{\mathcal{B}(X)} \leq C_a e^{at}, \quad \forall t \geq 0.$$

In order to conclude the spectral gap estimate, we split the operator  $\Lambda_\varepsilon$  as  $\Lambda_\varepsilon = \mathcal{A}_\varepsilon + \mathcal{B}_\varepsilon$  defined on  $X$  by

$$\mathcal{A}_\varepsilon g := \mu_\varepsilon \mathcal{M}_\varepsilon[g], \quad \mu_\varepsilon := \delta_0 - a'_\varepsilon F_\varepsilon, \quad (3.5)$$

$$\mathcal{B}_\varepsilon g := -\partial_x g - a_\varepsilon g, \quad (3.6)$$

for which can apply the Spectral Mapping Theorem of [9, 6] and the Weyl's Theorem of [16, 4, 9, 6]. As in the weak connectivity regime in [10], the properties of the two auxiliary operators still hold in the strong one.

**Lemma 3.2.** *Assume that  $a$  satisfies conditions (1.3)-(1.4)-(1.6). The operators  $\mathcal{A}_\varepsilon$  and  $\mathcal{B}_\varepsilon$  satisfy the following properties.*

- (i)  $\mathcal{A}_\varepsilon \in \mathcal{B}(X, Y)$ , where  $Y = \mathbb{C}\mu_\varepsilon \subset X$  with compact embedding.
- (ii)  $S_{\mathcal{B}_\varepsilon}$  is  $\alpha$ -hypodissipative in  $X$ .
- (iii) The family of operators  $S_{\mathcal{B}_\varepsilon} * \mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}$  satisfies

$$\|(S_{\mathcal{B}_\varepsilon} * \mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})(t)\|_{X \rightarrow BV} \leq C e^{\alpha t}, \quad \forall t \geq 0.$$

Then the proof of Theorem 3.1 is a direct consequence of those properties.

## 3.2 Strong connectivity regime - exponential stability of the linearized equation

When the network connectivity parameter  $\varepsilon$  goes to infinity, the linearized time elapsed operator simplifies

$$\Lambda_\infty g = -\partial_x g - a(x, \infty)g + \delta_{x=0} \mathcal{M}_\infty[g], \quad (3.7)$$

where  $\mathcal{M}_\infty[g] = \int_0^\infty a(x, \infty)g(x)dx$ . Similarly to the limited case  $\varepsilon = 0$  in [10], the semigroup  $S_{\Lambda_\infty}$  is also positive and the operator  $-\Lambda_\infty$  is also strong maximum while the Kato's inequality still holds in the limited connectivity. Then we conclude the following evolution estimate benefiting from the Theorem 3.1.

**Theorem 3.3.** *There exist some constants  $\alpha < 0$  and  $C > 0$  such that  $\Sigma(\Lambda_\infty) \cap \Delta_\alpha = \{0\}$  and for any  $g_0 \in X$ ,  $\langle g_0 \rangle = 0$ , there holds*

$$\|S_{\Lambda_\infty}(t)g_0\|_X \leq Ce^{\alpha t} \|g_0\|_X, \quad \forall t \geq 0. \quad (3.8)$$

We extend the exponential stability property in the limited case to the strong connectivity regime.

**Theorem 3.4.** *There exist some constants  $\varepsilon_1 > 0$ ,  $\alpha < 0$  and  $C > 0$  such that for any  $\varepsilon \in [\varepsilon_1, \infty]$  there hold  $\Sigma(\Lambda_\varepsilon) \cap \Delta_\alpha = \{0\}$  and*

$$\|S_{\Lambda_\varepsilon}(t)g_0\|_X \leq Ce^{\alpha t} \|g_0\|_X, \quad \forall t \geq 0, \quad (3.9)$$

for any  $g_0 \in X$ ,  $\langle g_0 \rangle = 0$ .

The proof uses the stability theory for semigroups developed in Kato's book [5] and revisited in [7, 15, 6]. Now, we present several results needed in the proof of Theorem 3.4.

*Proof of Theorem 3.4.* With the definitions (3.1), (3.5) and (3.6) of  $\mathcal{M}_\varepsilon$ ,  $\mathcal{A}_\varepsilon$  and  $\mathcal{B}_\varepsilon$ , we have

$$(\mathcal{B}_\varepsilon - \mathcal{B}_\infty)g = (a(x, \infty) - a(x, \varepsilon M_\varepsilon))g$$

and

$$(\mathcal{A}_\varepsilon - \mathcal{A}_\infty)g = (\mathcal{M}_\varepsilon[g] - \mathcal{M}_\infty[g])\delta_0 - \varepsilon(\partial_\mu a)(x, \varepsilon M_\varepsilon)F_\varepsilon \mathcal{M}_\varepsilon[g].$$

From the decay assumption (1.6), there exists positive  $\zeta_\varepsilon \rightarrow 0$ , as  $\varepsilon \rightarrow +\infty$ , such that  $|\varepsilon \partial_\mu a(x, \mu M_\varepsilon)| < \zeta_\varepsilon$ , for  $\varepsilon$  large enough. Together with the smoothness assumption (1.5), we deduce that

$$\|\mathcal{B}_\varepsilon - \mathcal{B}_\infty\|_{\mathcal{B}(X)} + \|\mathcal{A}_\varepsilon - \mathcal{A}_\infty\|_{\mathcal{B}(X)} \leq C \zeta_\varepsilon. \quad (3.10)$$

In the strong connectivity regime, the eigenprojector  $\Pi_\varepsilon$  satisfies similarly that

$$\|\Pi_\varepsilon - \Pi_\infty\|_{\mathcal{B}(X)} < 1.$$

From the classical result [5, Section I.4.6] (or more explicitly [15, Lemma 2.18]), we deduce that there exists  $\xi_\varepsilon \in \Delta_\alpha$  such that

$$\Sigma(\Lambda_\varepsilon) \cap \Delta_\alpha = \{\xi_\varepsilon\}, \quad \xi_\varepsilon \text{ is a simple eigenvalue,}$$

for any  $\varepsilon \in [\varepsilon_1, +\infty]$  (up to take a larger real number  $\varepsilon_1 > 0$ ). We conclude by observing that  $\xi_\varepsilon = 0$  because  $1 \in X'$  and  $\Lambda_\varepsilon^* 1 = 0$  (which is nothing but the mass conservation).  $\square$

### 3.3 Strong connectivity regime - nonlinear exponential stability

Now, we focus on the nonlinear exponential stability of the solution to the evolution equation (1.1) in the case without delay. We start with an auxiliary result. We define the function  $\Phi : L^1(\mathbb{R}_+) \times \mathbb{R} \rightarrow \mathbb{R}$  by

$$\Phi[g, \mu] := \int_0^\infty a(x, \varepsilon\mu)g(x) dx - \mu.$$

We denote by  $W_1$  the optimal transportation Monge-Kantorovich-Wasserstein distance on the probability measures set  $\mathbf{P}(\mathbb{R}_+)$  associated to the distance  $d(x, y) = |x - y| \wedge 1$ , or equivalently defined by

$$\forall f, g \in \mathbf{P}(\mathbb{R}_+), \quad W_1(f, g) := \sup_{\varphi, \|\varphi\|_{W^{1,\infty}} \leq 1} \int_0^\infty (f - g) \varphi.$$

**Lemma 3.5.** *Assume (1.5). There exists  $\varepsilon_1 > 0$  and for any  $\varepsilon \in (\varepsilon_1, +\infty)$  there exists a function  $\varphi_\varepsilon : \mathbf{P}(\mathbb{R}) \rightarrow \mathbb{R}$  which is Lipschitz continuous for the weak topology of probability measures and such that  $\mu = \varphi_\varepsilon[g]$  is the unique solution to the equation*

$$\mu \in \mathbb{R}_+, \quad \Phi(g, \mu) = 0.$$

*Proof of Lemma 3.5. Step 1. Existence.* For any  $g \in \mathbf{P}(\mathbb{R})$  we have  $\Phi(g, 0) > 0$  and for any  $g \in \mathbf{P}(\mathbb{R})$  and  $\mu \geq 0$ , we have

$$\Phi(g, \mu) \leq \|a\|_{L^\infty} - \mu,$$

so that  $\Phi(g, \mu) < 0$  for  $\mu > \|a\|_{L^\infty}$ . By the intermediate value theorem and the continuity property of  $\Phi$ , for any fixed  $g \in \mathbf{P}(\mathbb{R}_+)$  and  $\varepsilon \geq 0$ , there exists at least one solution  $\mu \in (0, \|a\|_{L^\infty}]$  to the equation  $\Phi(g, \mu) = 0$ .

*Step 2. Uniqueness and Lipschitz continuity.* Fix  $f, g \in \mathbf{P}(\mathbb{R}_+)$  and consider  $\mu, \nu \in \mathbb{R}_+$  such that

$$\Phi(f, \mu) = \Phi(g, \nu) = 0.$$

We have

$$\nu - \mu = \int_0^\infty a(x, \varepsilon\nu)(g - f) + \int_0^\infty (a(x, \varepsilon\nu) - a(x, \varepsilon\mu))f,$$

with

$$\left| \int_0^\infty a(x, \varepsilon\nu)(g - f) \right| \leq \|a(\cdot, \varepsilon\nu)\|_{W^{1,\infty}} W_1(g, f),$$

and

$$\left| \int_0^\infty (a(x, \varepsilon\nu) - a(x, \varepsilon\mu))f \right| \leq \|a(\cdot, \varepsilon\nu) - a(\cdot, \varepsilon\mu)\|_{L^\infty} \leq \zeta_\varepsilon |\mu - \nu|.$$

We then obtain

$$|\mu - \nu| (1 - \zeta_\varepsilon) \leq \|a(\cdot, \varepsilon\nu)\|_{W^{1,\infty}} W_1(g, f), \quad (3.11)$$

and we may fix  $\varepsilon_1 > 0$  such that  $1 - \zeta_{\varepsilon_1} \|\partial_\mu a\|_{L^\infty} \in (0, 1)$ ,  $\varepsilon \in [\varepsilon_1, +\infty)$ . On the one hand, for  $f = g$ , we deduce that  $\mu = \nu$  and that uniquely defines the mapping  $\varphi_\varepsilon[g] := \mu$ . On the other hand, the function is Lipschitz continuous because of (3.11).  $\square$

We also recall the following classical Gronwall's type lemma, with the proof referred to [10].

**Lemma 3.6.** *Assume that  $u \in C([0, \infty); \mathbb{R}_+)$  satisfies the integral inequality*

$$u(t) \leq C_1 e^{at} u_0 + C_2 \int_0^t e^{a(t-s)} u(s)^2 ds, \quad \forall t > 0,$$

for some constants  $C_1 \geq 1$ ,  $C_2, u_0 \geq 0$  and  $a < 0$ . Under the smallness assumption

$$a + 2C_2 u_0 < 0,$$

there holds

$$u(t) \leq \left( 1 + \frac{C_1 u_0 C_2}{|a + 2C_2 u_0|} \right) C_1 e^{at} u_0, \quad \forall t \geq 0.$$

We come back to the proof of our main result Theorem 1.3 in the case without delay.

*Proof of Theorem 1.3 in the case without delay.* We split the proof into two steps.

*Step 1. New formulation.* We start giving a new formulation of the solutions to the evolution and stationary equations in the strong connectivity regime  $\varepsilon \in [\varepsilon_1, \infty)$ , where  $\varepsilon_1$  is defined in Lemma 3.5. For a given initial datum  $0 \leq f_0 \in L^1(\mathbb{R}_+)$  with unit mass the solution  $f \in C([0, \infty); L^1(\mathbb{R}_+))$  to the

evolution equation (1.1) and the solution  $F_\varepsilon$  to the stationary equation (1.2) clearly satisfy

$$\begin{aligned}\partial_t f + \partial_x f + a(\varepsilon\varphi[f])f &= 0, & f(t, 0) &= \varphi[f(t, \cdot)], \\ \partial_x F + a(\varepsilon M)F &= 0, & F(0) &= M = \varphi[F],\end{aligned}$$

where here and below the  $\varepsilon$  and  $x$  dependency is often removed without risk of misleading.

We introduce the variation function  $g := f - F$  which satisfies the PDE

$$\partial_t g = -\partial_x g - a(\varepsilon M)g - \varepsilon a'(\varepsilon M)F \mathcal{M}[g] - Q[g] \quad (3.12)$$

with

$$Q[g] := a(\varepsilon\varphi[f])f - a(\varepsilon\varphi[F])F - a(\varepsilon\varphi[F])g - \varepsilon a'(\varepsilon\varphi[F])F \mathcal{M}[g],$$

where  $\mathcal{M} = \mathcal{M}_\varepsilon$  is defined in (3.1). The above PDE is complemented with the boundary condition

$$g(t, 0) = \varphi[f(t, \cdot)] - \varphi[F],$$

and we may write again

$$\varphi[f] - \varphi[F] = \mathcal{M}[g] + \mathcal{Q}[g], \quad \mathcal{Q}[g] := \langle Q[g] \rangle.$$

As a consequence, we have proved that the variation function  $g$  satisfies the equation

$$\partial_t g = \Lambda_\varepsilon g + Z[g], \quad Z[g] := -Q[g] + \delta_0 \mathcal{Q}[g]. \quad (3.13)$$

*Step 2. The nonlinear term.* On the one hand, we obviously have

$$\langle Z[g] \rangle = 0, \quad \forall g \in M^1(\mathbb{R}_+). \quad (3.14)$$

On the other hand, in order to get an estimate on the nonlinear term  $Z[g]$ , we introduce the notation

$$\psi(u) = a(x, \varepsilon m_u) f_u,$$

where, for some fixed  $g \in \mathbf{P}(\mathbb{R}_+)$ ,  $\langle g \rangle = 0$ , we have set

$$f := F + g, \quad f_u := u f + (1 - u)F, \quad m_u := \varphi[f_u].$$



We first notice that  $\psi(0) = a(\varepsilon\varphi[F])F$  and  $\psi(1) = a(\varepsilon\varphi[f])f$ . Second, we have

$$\psi'(u) = a'_\varepsilon(m_u)f_u m'_u + a_\varepsilon(m_u)g. \quad (3.15)$$

In order to compute  $m'_u$ , we differentiate with respect to  $u$  the identity

$$m_u = \int_0^\infty a_\varepsilon(m_u)f_u dx,$$

and we have

$$m'_u = \int_0^\infty a'_\varepsilon(m_u)f_u dx m'_u + \int_0^\infty a_\varepsilon(m_u) g dx,$$

which implies

$$m'_u = \left(1 - \int_0^\infty a'_\varepsilon(m_u)f_u dx\right)^{-1} \int_0^\infty a_\varepsilon(m_u) g dx. \quad (3.16)$$

We may thus observe that  $m'_0 = \mathcal{M}[g]$ , so that  $\psi'(0) = a'_\varepsilon(M)F\mathcal{M}_\varepsilon[g] + a_\varepsilon(M)g$ , and therefore

$$Q[g] = \psi(1) - \psi(0) - \psi'(0).$$

Third, from (3.15), we have

$$\psi''(u) = a''_\varepsilon(m_u)f_u (m'_u)^2 + 2a'_\varepsilon(m_u)gm'_u + a'_\varepsilon(m_u)f_u m''_u,$$

and from (3.16), we have

$$\begin{aligned} m''(u) &= 2\left(1 - \int_0^\infty a'_\varepsilon f_u\right)^{-2} \int_0^\infty a_\varepsilon g \int_0^\infty a'_\varepsilon g \\ &\quad + 2\left(1 - \int_0^\infty a'_\varepsilon f_u\right)^{-3} \int_0^\infty a''_\varepsilon f \left(\int_0^\infty a_\varepsilon g\right)^2. \end{aligned}$$

From (1.6), it is clearly that there exists  $\eta_\varepsilon \rightarrow 0$ , as  $\varepsilon \rightarrow +\infty$ , such that  $\|a''_\varepsilon\| < \eta_\varepsilon$  for  $\varepsilon$  large enough. Then in the strong connectivity regime  $\varepsilon \in [\varepsilon_1, +\infty)$ ,  $\zeta_{\varepsilon_1} < 1$ , we get the bound

$$\begin{aligned} \|\psi''(u)\|_X &\leq \|a''_\varepsilon\|_\infty |m'_u|^2 + 2\|a'_\varepsilon\|_\infty \|g\|_X |m'_u| + \|a'_\varepsilon\|_\infty |m''_u| \\ &\leq \frac{\eta_\varepsilon \|a\|_\infty^2}{(1 - \zeta_\varepsilon)^2} \|g\|_X^2 + 2\frac{\zeta_\varepsilon \|a\|_\infty}{1 - \zeta_\varepsilon} \|g\|_X^2 \\ &\quad + 2\frac{\zeta_\varepsilon^2 \|a\|_\infty}{(1 - \zeta_\varepsilon)^2} \|g\|_X^2 + 2\frac{\eta_\varepsilon \zeta_\varepsilon \|a\|_\infty}{(1 - \zeta_\varepsilon)^3} \|g\|_X^2 \\ &\leq (\eta_\varepsilon + \zeta_\varepsilon) K \|g\|_X^2, \end{aligned}$$

for some constant  $K \in (0, \infty)$ . Using the Taylor expansion

$$Q[g] = \psi(1) - \psi(0) - \psi'(0) = \int_0^1 (1-u)\psi''(u)du,$$

we then obtain

$$\|Z[g]\|_X \leq 2\|Q[g]\|_X \leq \int_0^1 (1-u)\|\psi''(u)\|_X du \leq C\|g\|^2.$$

*Step 3. Decay estimate.* Thanks to the Duhamel formula, the solution  $g$  to the evolution equation (3.13) satisfies

$$g(t) = S_{\Lambda_\varepsilon}(t)(f_0 - F) + \int_0^t S_{\Lambda_\varepsilon}(t-s)Z[g(s)] ds.$$

Using Theorem 3.4 and the second step, we deduce

$$\begin{aligned} \|g(t)\|_X &\leq C e^{\alpha t} \|g_0\|_X + \int_0^t C e^{\alpha(t-s)} \|Z[g(s)]\|_X ds \\ &\leq C e^{\alpha t} \|g_0\|_X + C(\eta_\varepsilon + \zeta_\varepsilon) K \int_0^t e^{\alpha(t-s)} \|g(s)\|_X^2 ds, \end{aligned}$$

for any  $t \geq 0$  and for some constant  $C \geq 1$ ,  $\alpha < 0$ , independent of  $\varepsilon \in [\varepsilon_1, +\infty)$ . Observing that  $\|g(t)\|_X = \|g(t)\|_{L^1} \in C([0, \infty))$ , we conclude thanks to Lemma 3.6.  $\square$

## 4 Case with delay

This section is devoted to the proof of our main result, Theorem 1.3, in the case with delay by following the same strategy as in the case without delay but adaptation the functional framework. We have already proved in Theorem 1.2 the existence of a unique stationary solution  $(F_\varepsilon, M_\varepsilon)$  in the strong connectivity regime and we may then focus on the evolution equation.

### 4.1 Linearized equation and structure of the spectrum

In order to write as a time autonomous equation the linearized equation (1.8)-(1.9)-(1.10), we introduce the following intermediate evolution equation on a function  $v = v(t, y)$

$$\partial_t v + \partial_y v = 0, \quad v(t, 0) = q(t), \quad v(0, y) = 0, \quad (4.1)$$

where  $y \geq 0$  represent the local time for the network activity. That last equation can be solved with the characteristics method

$$v(t, y) = q(t - y) \mathbf{1}_{0 \leq y \leq t}.$$

Therefore, equation (1.10) on the variation  $n(t)$  of network activity writes

$$n(t) = \mathcal{D}[v(t)], \quad \mathcal{D}[v] := \int_0^\infty v(y) b(dy),$$

and then equation (1.9) on the variation  $q(t)$  of discharging neurons writes

$$q(t) = \mathcal{O}_\varepsilon[g(t), v(t)],$$

with

$$\begin{aligned} \mathcal{O}_\varepsilon[g, v] &:= \mathcal{N}_\varepsilon[g] + \kappa_\varepsilon \mathcal{D}[v], \\ \mathcal{N}_\varepsilon[g] &:= \int_0^\infty a_\varepsilon(M_\varepsilon) g \, dx, \quad \kappa_\varepsilon := \int_0^\infty a'_\varepsilon(M_\varepsilon) F_\varepsilon \, dx. \end{aligned}$$

As a consequence, we may rewrite the linear system (1.8)-(1.9)-(1.10), as the autonomous system

$$\partial_t(g, v) = \mathcal{L}_\varepsilon(g, v), \tag{4.2}$$

where the operator  $\mathcal{L}_\varepsilon = (\mathcal{L}_\varepsilon^1, \mathcal{L}_\varepsilon^2)$  is defined by

$$\begin{aligned} \mathcal{L}_\varepsilon^1(g, v) &:= -\partial_x g - a_\varepsilon g - a'_\varepsilon F_\varepsilon \mathcal{D}[v] + \delta_{x=0} \mathcal{O}_\varepsilon[g, v], \\ \mathcal{L}_\varepsilon^2(g, v) &:= -\partial_y v + \delta_{y=0} \mathcal{O}_\varepsilon[g, v], \end{aligned}$$

in the space

$$X = X_1 \times X_2 := M^1(\mathbb{R}_+) \times M^1(\mathbb{R}_+, \mu)$$

with  $\mu(x) = e^{-\delta x}$  and  $\delta > 0$  is the same as in the condition (1.7).

**Theorem 4.1.** *Assume (1.3)-(1.4)-(1.5)-(1.6) and (1.7). The conclusions of Theorem 3.1 holds true with  $\alpha := \max\{-a_0/2, -\delta\} < 0$ .*

The result follows from the Spectral Mapping theorem and the Weyl's Theorem established in [9, 6] by introducing a convenient splitting of the operator  $\mathcal{L}_\varepsilon$ . More precisely, we write  $\mathcal{L}_\varepsilon = \mathcal{A}_\varepsilon + \mathcal{B}_\varepsilon$  with

$$\mathcal{B}_\varepsilon(g, v) = \begin{pmatrix} \mathcal{B}_\varepsilon^1(g, v) \\ \mathcal{B}_\varepsilon^2(g, v) \end{pmatrix} = \begin{pmatrix} -\partial_x g - a_\varepsilon g \\ -\partial_y v \end{pmatrix}$$

and

$$\mathcal{A}_\varepsilon(g, v) = \begin{pmatrix} \mathcal{A}_\varepsilon^1(g, v) \\ \mathcal{A}_\varepsilon^2(g, v) \end{pmatrix} = \begin{pmatrix} -a'_\varepsilon F_\varepsilon \mathcal{D}[v] + \delta_{x=0} \mathcal{O}_\varepsilon[g, v] \\ \delta_{y=0} \mathcal{O}_\varepsilon[g, v] \end{pmatrix}.$$

Taking  $Y = (\mathbb{C}\delta_0 + BV(\mathbb{R}_+)) \times \mathbb{C}\delta_0 \subset X$ , the operators  $\mathcal{A}_\varepsilon$  and  $\mathcal{B}_\varepsilon$  possess the same properties as in Lemma 3.2.

## 4.2 Strong connectivity regime - exponential stability of the linearized equation

When the network connectivity parameter goes to the limit,  $\varepsilon = +\infty$ , the linearized operator simplifies into

$$\mathcal{L}_\infty \begin{pmatrix} g \\ v \end{pmatrix} = \begin{pmatrix} -\partial_x g - a(x, \infty)g + \delta_{x=0} \mathcal{O}_\infty[g, v] \\ -\partial_y v + \delta_{y=0} \mathcal{O}_\infty[g, v] \end{pmatrix}, \quad (4.3)$$

where  $\mathcal{O}_\infty[g, v] = \mathcal{N}_\infty[g] = \int_0^\infty a(x, \infty)g(x)dx$ . From the Theorem 3.3 and the Duhamel formula

$$v(t) = S_{\mathcal{B}_\infty^2}(t)v_0 + \int_0^t S_{\mathcal{B}_\infty^2}(t-s)\mathcal{A}_\infty^2(g(s), v(s)) ds,$$

the evolution estimate holds for the second component of the operator  $\mathcal{L}_\varepsilon$ .

**Theorem 4.2.** *There exist some constants  $\alpha < 0$  and  $C > 0$  such that  $\Sigma(\mathcal{L}_\infty) \cap \Delta_\alpha = \{0\}$  and for any  $(g_0, v_0) \in X$ ,  $\langle g_0 \rangle = 0$ , there holds*

$$\|S_{\mathcal{L}_\infty}(t)(g_0, v_0)\|_X \leq Ce^{\alpha t} \|(g_0, v_0)\|_X, \quad \forall t \geq 0. \quad (4.4)$$

Then we extend the geometry structure of the spectrum of the linearized time elapsed equation to the strong connectivity regime taking delay into account.

**Theorem 4.3.** *There exists some constants  $\varepsilon_1 > 0$ ,  $C \geq 1$  and  $\alpha < 0$  such that for any  $\varepsilon \in [\varepsilon_1, +\infty]$  there holds  $\Sigma(\mathcal{L}_\varepsilon) \cap \Delta_\alpha = \{0\}$  and*

$$\|S_{\mathcal{L}_\varepsilon}(t)(g_0, v_0)\|_X \leq Ce^{\alpha t} \|(g_0, v_0)\|_X, \quad (4.5)$$

for any  $(g_0, v_0) \in X$  such that  $\langle g_0 \rangle = 0$ .

*Proof.* Similarly, we proceed exactly as in the proof of Theorem 3.4 and deduce the continuity of the operator  $\mathcal{L}_\varepsilon$  to  $\varepsilon$  in the strong connectivity regime, with whose help, we conclude that (see also again [15])

$$\Sigma(\mathcal{L}_\varepsilon) \cap \Delta_\alpha = \{\xi_\varepsilon\},$$

with  $|\xi_\varepsilon| \leq O(\zeta_\varepsilon)$  and  $\xi_\varepsilon$  is algebraically simple. We observe that

$$\mathcal{L}_\varepsilon^* \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} \partial_x \varphi - a_\varepsilon \varphi + a_\varepsilon(\varphi(0) + \psi(0)) \\ \partial_y \psi + \kappa_\varepsilon b \psi(0) + \kappa_\varepsilon b \varphi(0) - b \int a'_\varepsilon F_\varepsilon \varphi dx \end{pmatrix},$$

from which we deduce that  $\mathcal{L}^*(1, 0) = 0$ . Then  $0 \in \Sigma(\mathcal{L}_\varepsilon^*)$  and  $\xi_\varepsilon = 0$ . Moreover, the orthogonality condition  $\langle g_0 \rangle = \langle (g_0, v_0), (1, 0) \rangle_{X, X'} = 0$  implies that the exponential estimate (4.5) holds.  $\square$

### 4.3 Strong connectivity regime - nonlinear exponential stability

We finally come back on the nonlinear problem and we present the proof of the second part of our main result for the case with delay.

*Proof of Theorem 1.3 in case with delay.* We write the system as

$$\begin{aligned} \partial_t f &= -\partial_x f - a_\varepsilon(\mathcal{D}[u])f + \delta_0 \mathcal{P}[f, \mathcal{D}[u]] \\ \partial_t u &= -\partial_y u + \delta_0 \mathcal{P}[f, \mathcal{D}[u]], \end{aligned}$$

with

$$\mathcal{P}[f, m] = \int a(m)f, \quad \mathcal{D}[u] = \int bu.$$

We recall that the steady state  $(F, U)$ ,  $U := M\mathbf{1}_{y \geq 0}$ , satisfies

$$\begin{aligned} 0 &= -\partial_x F - a_\varepsilon(M)F + \delta_0 M \\ 0 &= -\partial_y U + \delta_0 M, \quad M = \mathcal{D}[U] = \mathcal{P}[F, \mathcal{D}[U]]. \end{aligned}$$

We introduce the variation  $g := f - F$  and  $v = u - U$ . The equation on  $g$  is

$$\begin{aligned} \partial_t g &= -\partial_x g - a_\varepsilon(\mathcal{D}[u])f + a_\varepsilon(M)F + \delta_0(\mathcal{P}[f, \mathcal{D}[u]] - \mathcal{P}[F, \mathcal{D}[U]]) \\ &= -\partial_x g - a_\varepsilon(M)g - a'_\varepsilon F \mathcal{D}[v] - Q[g, v] + \delta_0 \mathcal{O}[g, v] + \delta_0 \mathcal{Q}[g, v] \\ &= \mathcal{L}_\varepsilon^1(g, v) + \mathcal{Z}^1[g, v], \end{aligned}$$

with

$$\begin{aligned} Q[g, v] &:= a_\varepsilon(M)F - a_\varepsilon(\mathcal{D}[u])f + a_\varepsilon(M)g + a'_\varepsilon F\mathcal{D}[v] \\ &= \Phi(0) - \Phi(1) + \Phi'(0), \end{aligned}$$

where  $\Phi(k) = a_\varepsilon(\mathcal{D}[k u + (1 - k)U])(k f + (1 - k)F)$  and  $\mathcal{Q}[g, v] = \langle Q[g, v] \rangle$ ,  $\mathcal{Z}^1[g, v] := -Q[g, v] + \delta_0 \mathcal{Q}[g, v]$ . The equation on  $v$  is

$$\begin{aligned} \partial_t v &= -\partial_y v + \delta_0(\mathcal{P}[f, \mathcal{D}[u]] - \mathcal{P}[F, \mathcal{D}[U]]) \\ &= -\partial_y v + \delta_0 \mathcal{O}[g, v] + \delta_0 \mathcal{Q}[g, v] \\ &= \mathcal{L}_\varepsilon^2(g, v) + \mathcal{Z}^2[g, v], \quad \mathcal{Z}^2[g, v] := \delta_0 \mathcal{Q}[g, v]. \end{aligned}$$

We then write the associated Duhamel formula

$$(g(t), v(t)) = S_{\mathcal{L}_\varepsilon}(t)(g_0, v_0) + \int_0^t S_{\mathcal{L}_\varepsilon}(t - s) \mathcal{Z}[g(s), v(s)] ds.$$

Because  $\|\mathcal{Z}[g, v]\|_X \leq C \|(g, v)\|_X^2$  we may conclude as in the proof of Theorem 1.3.  $\square$

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