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Antoine Benoit

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# Geometric optics expansions for hyperbolic corner problems II : huge amplification phenomenon.

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## Abstract

In this article we are interested in the rigorous construction of geometric optics expansions for weakly well-posed hyperbolic corner problems. More precisely we focus on the case where selfinteracting phases occur and where one of them is exactly the phase where the uniform Kreiss-Lopatinskii condition fails. We show that the associated WKB expansion suffers arbitrarily many amplifications before a fixed finite time. As a consequence, we show that such a corner problem can not be weakly well-posed even at the price of a huge loss of derivatives. The new result, in that framework, is that the violent instability (or Hadamard instability) does not come from the degeneracy of the weak Kreiss-Lopatinskii condition, but of the accumulation of arbitrarily many weak instabilities.

AMS subject classification : 35L04, 78A05

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## 1 Introduction

The main study in this article is the rigorous construction of geometric optics expansions for hyperbolic corner problem that is to say, problems reading :

$$\begin{cases} L(\partial)u := \partial_t u + A_1 \partial_1 u + A_2 \partial_2 u = f, & \text{for } x_1, x_2 > 0 \\ B_1 u|_{x_1=0} = g_1, & \text{on } x_1 > 0, \\ B_2 u|_{x_2=0} = g_2, & \text{on } x_2 > 0, \\ u|_{t \leq 0} = 0, \end{cases} \quad (1)$$

where the matrices  $A_i \in \mathbf{M}_N(\mathbb{R})$  and where  $B_i \in \mathbf{M}_{p_i \times N}(\mathbb{R})$  (the values of  $p_1$  and  $p_2$  will be made precise in Assumption 2.2).

Such expansions, under the uniform Kreiss-Lopatinskii condition (that is to say the necessary and sufficient condition for strong well-posedness of the hyperbolic boundary value problem in the

half space (see [Kre70] for more details) have already been studied in [SS75] and more recently by the author in [Ben15] and [Ben]. More precisely in [SS75] the authors give precious intuitions and some elements of proof about this construction which are used in [Ben15]-[Ben] to construct rigorously the expansions.

In particular a new phenomenon, proper to the corner problem, has been investigated, the self-interaction phenomenon. In more details, this phenomenon induces that some phases in the WKB expansion can generate themselves after a suitable number of reflections on the sides of the quarter space (see [Ben15]-[Ben]). On one hand, in terms of the geometry of the characteristic variety, such systems contain a rectangle whose corners are elements of the characteristic variety (with suitable group velocities) and whose side are parallel to the axis of the frequency space. On the other hand, in terms of the resolution of the WKB cascade, a new amplitude equation, whose provenance is intrinsically linked with the uniform Kreiss-Lopatinskii condition has to be solved to initialize the resolution.

We are here interested in corner problems whose, one of the boundary condition, to fix the ideas, let us say  $B_1$ , does not satisfy the uniform Kreiss-Lopatinskii condition (the other boundary condition is assumed to be as "nice" as necessary, that is it satisfies the uniform Kreiss-Lopatinskii condition or even it is strictly dissipative see [BGS07]). The litterature about the hyperbolic boundary value problem in the half space tells us that there are four possible kinds of degeneracy of the uniform Kreiss-Lopatinskii condition relying on the structure of the resolvent matrix at the frequency where the uniform Kreiss-Lopatinskii condition breaks down (see Definition 2.1). Namely, the degeneracies of the uniform Kreiss-Lopatinskii condition can occur in the so-called hyperbolic, elliptic, mixed or glancing regions.

Concerning boundary value problems in the half space for which the uniform Kreiss-Lopatinskii condition is violated, the construction of the associated geometric optics expansion has already been made for three of the four possible cases. We refer to [CG10] for the construction when the degeneracy takes place in the hyperbolic region and to [Ben14] and [Mar10] for the construction associated to a degeneracy in the elliptic or in the mixed region.

We will here focus our attention on a degeneracy of the uniform Kreiss-Lopatinskii condition in the hyperbolic region, that is to say to problems in the so-called  $WR$  class in the sense of [BGRSZ02]. Some conjectures about the behaviours of the geometric optics expansions when the degeneracy of the uniform Kreiss-Lopatinskii condition occurs in the elliptic or in the mixed region can be found in paragraph 7.

In [CG10], the authors construct the geometric optics expansions for such  $WR$  problems in the half space and show that if the source terms are in order one compared to the small parameter  $\varepsilon$  which encodes the high oscillating behaviour, then the leading term in the geometric optics expansion is of order zero compared to  $\varepsilon$ . Then, they used this construction to show that the energy estimate with losses of derivatives established by [Cou05] :

$$\|u\|_{L^2(\Omega_T)}^2 + \|u|_{x_1=0}\|_{L^2(\Omega_T)}^2 \leq C_T \left( \|f\|_{L^2_{x_1}(H^1(\partial\Omega_{1,T}))}^2 + \|g_1|_{x_1=0}\|_{(H^1(\partial\Omega_{1,T}))}^2 \right),$$

is sharp in term of losses of derivatives.

Concerning corner problems, the litterature is much more poor. Indeed, the full characterization of strongly well-posed problems is, to our knowledge, not achieved yet. So, the full characterization

of weakly well-posed corner problems seems, in the author opinion, to be a long range problem. As a consequence the fact that we construct the geometric optics expansions for these (expected to be) weakly well-posed problems can be seen as a "Majda's project" (see [Maj83b] and [Maj83a]). That is to say, that the author believes that the use of geometric optics expansions can give some intuition about what can be the good number of losses of derivatives in the associated corner problems.

In this article, we will focus our attention on the particular case of corner problems in the  $WR$  class whose section of the characteristic variety contains a loop. We will also assume that one of the frequencies for which the uniform Kreiss-Lopatinskii condition breaks down is associated to such a selfinteracting phases. Our purpose is thus to construct the geometric optics expansion for such a corner problem. As we will see in section 5, for any arbitrarily big  $M \in \mathbb{N}$ , the fact that the phase for which the uniform Kreiss-Lopatinskii condition breaks down is generated an arbitrarily number of times will imply a leading term in the geometric optics expansion of order zero, compared with the small parameter  $\varepsilon$ , when one considers a source term on the boundary  $\{x_1 = 0\}$  of order  $M + 1$  in terms of  $\varepsilon$ .

Using this geometric optics expansion, we will show that such a corner problem can not be (even weakly) well-posed on a finite interval of time because it losses an arbitrarily big number of derivatives. In term of losses of derivatives, it is the worst case that we can imagine because we have a violent or Hadamard instability.

The paper is organized as follows : in Section 2, we give some classical definitions and introduce some notations. In Section 3 we give a formal study of our problem and we describe what are the expected phases and the associated amplitudes in the WKB expansion. Then in Section 4, we give a general framework in which the previous formal discussion becomes rigorous. This section uses the same tools as these introduced in [Ben] to describe, with precision, the set of expected phases in the WKB expansion. However, we believe that it is important to recall these tools for a sake of completeness.

Section 5 is devoted to the construction of the geometric optics expansion and is the main section of the paper. The most difficult part of the construction is to find a way to initialize the resolution of the WKB cascade of equations. Indeed in [Ben] the new amplitude equation whose resolution permits to initialize the WKB expansion was, as already mentionned, intrinsically linked with the uniform Kreiss-Lopatinskii condition. As a consequence to initialize the resolution of the WKB cascade, we need a new amplitude equation when the Kreiss-Lopatinskii uniform condition fails. This equation comes from an adaptation of the method described in [CG10]. The hardest point to handle with is that in [CG10], due to the "nice" geometry of the half space, it was possible to determine all the outgoing phases (which act as source terms in the equation determining the amplitude for which the uniform Kreiss-Lopatinskii breaks down) before all the others. However, this is not true anymore in the quarter space geometry. But we found a new equation for the initialization of the WKB, it is given in paragraph 5.1. The resolution of this equation is made in subparagraphs 5.1.4 and 5.1.6. The resolution is made after a "necessary" reformulation of this equation, in view to show that this equation can in fact be rewritten under the particular form  $(I - \mathbb{T})u = G$ , for some operator  $\mathbb{T}$ . This rewrite is made in paragraph 5.1.2.<sup>1</sup>

Section 6 contains the proof of our main result, that is that a corner problem in the  $WR$  class for one side which admits a loop in the section of its characteristic variety and for which one of

---

<sup>1</sup>Let us note that the particular structure  $(I - \mathbb{T})u = G$  was already the structure obtained in [Ben]. Moreover, this structure will be very important in the following proof because it permits to write the solution  $u$  as the sum of compositions of  $\mathbb{T})u$  and thus to express the solution  $u$  in term of the sum of wave packets.

the element of the loop violates the uniform Kreiss-Lopatinskii condition can not be (even) weakly well-posed. It is not really a merry result because it tell us that, in such a framework, there is no hope to solve the corner problem. However, the author believe that this result is interesting in itself because it gives the first examples, in our knowledge, of ill-posed hyperbolic boundary value problems for which the ill-posedness is due to the accumulation of weak instabilities and not to the failure of the weak Kreiss-Lopatinskii condition. Moreover the instability phenomenon in this framework is much more sneaky.

At last, Section 7 gives examples of such ill-posed corner problems and some (more optimistic) conjectures about what should be the leading orders sizes (and so the number of losses of derivatives) in the more favorable cases where the uniform Kreiss-Lopatinskii condition breaks down outside the loop.

## 2 Notations and assumptions

### 2.1 About the operator $L(\partial)$ .

Let

$$\Omega := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0, x_2 \geq 0\}, \quad \partial\Omega_1 := \Omega \cap \{x_1 = 0\}, \quad \text{and} \quad \partial\Omega_2 := \Omega \cap \{x_2 = 0\},$$

be the quarter space and both of its edges. For  $T > 0$ , we will denote :

$$\Omega_T := ]-\infty, T] \times \Omega, \quad \partial\Omega_{1,T} := ]-\infty, T] \times \partial\Omega_1, \quad \text{and} \quad \partial\Omega_{2,T} := ]-\infty, T] \times \partial\Omega_2.$$

Hereof  $\mathcal{L}$  will denote the symbol of the differential operator  $L(\partial)$ . It is defined for  $\tau \in \mathbb{R}$  and  $\xi \in \mathbb{R}^2$  by :

$$\mathcal{L}(\tau, \xi) := \tau I + \sum_{j=1}^2 \xi_j A_j.$$

The characteristic variety  $V$  of  $L(\partial)$  is thus given by :

$$V := \{(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^2 \mid \det \mathcal{L}(\tau, \xi) = 0\}.$$

In this article we choose to restrict our subject, in view to save some notations, to strictly hyperbolic operators. The following constructions of the geometric optics expansions should also operate in the framework of constantly hyperbolic operators. However this simplifying assumption will make the analysis of Section 5 slightly easier. We will give in the core of the proof some comments about the expected modifications about the proof for constantly hyperbolic operators. We thus assume the following property on  $L(\partial)$  :

**Assumption 2.1** *There exist  $N$  real valued functions, denoted by  $\lambda_j$ , analytic on  $\mathbb{R}^2 \setminus \{0\}$  such that :*

$$\forall \xi \in \mathbb{S}^1, \quad \det \mathcal{L}(\tau, \xi) = \prod_{j=1}^N (\tau + \lambda_j(\xi)),$$

where the eigenvalues  $\lambda_j(\xi)$  satisfy  $\lambda_1(\xi) < \dots < \lambda_N(\xi)$ .

We also assume that the boundary of  $\Omega$  is non-characteristic, and that the matrices  $B_1$  and  $B_2$  induce the good number of boundary conditions, that is to say :

**Assumption 2.2** *The matrices  $A_1, A_2$  are invertible. Moreover  $p_1$  (resp.  $p_2$ ), the number of lines of  $B_1$  (resp.  $B_2$ ), equals the number of strictly positive eigenvalues of  $A_1$  (resp.  $A_2$ ). At last  $B_1$  and  $B_2$  are assumed to be of maximal rank.*

Under Assumption 2.2, we can define the resolvent matrices :

$$\mathcal{A}_1(\zeta) := -A_1^{-1}(\sigma I + i\eta A_2) \text{ and } \mathcal{A}_2(\zeta) := -A_2^{-1}(\sigma I + i\eta A_1),$$

where  $\zeta$  denotes an element of the frequency space :

$$\Xi := \{\zeta := (\sigma = \gamma + i\tau, \eta) \in \mathbb{C} \times \mathbb{R}, \gamma \geq 0\} \setminus \{(0, 0)\}.$$

For convenience, we also introduce  $\Xi_0$  the boundary of  $\Xi$  :

$$\Xi_0 := \Xi \cap \{\gamma = 0\}.$$

and the notation :

$$\text{for } j = 1, 2, \zeta_j := (\sigma, \xi_{3-j}),$$

For  $j = 1, 2$ ,  $\zeta_j \in (\Xi \setminus \Xi_0)$ , we denote by  $E_j^s(\zeta_j)$  the stable subspace of  $\mathcal{A}_j(\zeta_j)$  and  $E_j^u(\zeta_j)$  its unstable subspace. These spaces are well-defined according to [Her63]. For all  $\zeta_j \in (\Xi \setminus \Xi_0)$ , the stable subspace  $E_j^s(\zeta_j)$  has dimension  $p_j$ , while  $E_j^u(\zeta_j)$  has dimension  $N - p_j$ . Let us recall the following theorem due to Kreiss [Kre70]<sup>2</sup> :

**Theorem 2.1** *Under Assumptions 2.1 and 2.2, for all  $\underline{\zeta} \in \Xi$ , there exists a neighborhood  $\mathcal{V}$  of  $\underline{\zeta}$  in  $\Xi$ , integers  $L_1, L_2 \geq 1$ , two partitions  $N = \nu_{1,1} + \dots + \nu_{1,L_1} = \nu_{2,1} + \dots + \nu_{2,L_2}$  with  $\nu_{1,l}, \nu_{2,l} \geq 1$ , and two invertible matrices  $T_1, T_2$ , regular on  $\mathcal{V}$  such that :*

$$\begin{aligned} \forall \zeta \in \mathcal{V}, \quad T_1(\zeta)^{-1} \mathcal{A}_1(\zeta) T_1(\zeta) &= \text{diag}(\mathcal{A}_{1,1}(\zeta), \dots, \mathcal{A}_{1,L_1}(\zeta)), \\ T_2(\zeta)^{-1} \mathcal{A}_2(\zeta) T_2(\zeta) &= \text{diag}(\mathcal{A}_{2,1}(\zeta), \dots, \mathcal{A}_{2,L_2}(\zeta)), \end{aligned}$$

where the blocks  $\mathcal{A}_{j,l}(\zeta)$  have size  $\nu_{j,l}$  and satisfy one of the following alternatives :

- i) All the elements in the spectrum of  $\mathcal{A}_{j,l}(\zeta)$  have positive real part.
- ii) All the elements in the spectrum of  $\mathcal{A}_{j,l}(\zeta)$  have negative real part.
- iii)  $\nu_{j,l} = 1$ ,  $\mathcal{A}_{j,l}(\zeta) \in i\mathbb{R}$ ,  $\partial_\gamma \mathcal{A}_{j,l}(\zeta) \in \mathbb{R} \setminus \{0\}$ , and  $\mathcal{A}_{j,l}(\zeta) \in i\mathbb{R}$  for all  $\zeta \in \mathcal{V} \cap \Xi_0$ .
- iv)  $\nu_{j,l} > 1$ ,  $\exists k_{j,l} \in i\mathbb{R}$  such that

$$\mathcal{A}_{j,l}(\zeta) = \begin{bmatrix} k_{j,l} & i & 0 \\ & \ddots & i \\ 0 & & k_{j,l} \end{bmatrix},$$

the coefficient in the lower left corner of  $\partial_\gamma \mathcal{A}_{j,l}(\zeta)$  is real and non-zero, and moreover  $\mathcal{A}_{j,l}(\zeta) \in i\mathbf{M}_{\nu_{j,l}}(\mathbb{R})$  for all  $\zeta \in \mathcal{V} \cap \Xi_0$ .

Thanks to this Theorem it is possible to describe the four kinds of frequencies, for each part of the boundary  $\partial\Omega$  :

**Definition 2.1** *For  $j = 1, 2$ , we denote by :*

- 1)  $\mathcal{E}_j$  the set of elliptic frequencies, that is to say the set of  $\underline{\zeta} \in \Xi_0$  such that Theorem 2.1 for the matrix  $\mathcal{A}_j(\underline{\zeta})$  is satisfied with one block of type i) and one block of type ii) only.

<sup>2</sup>For constantly (resp. geometrically regular) hyperbolic operators, this theorem remains true, see [Mét00] (resp. [MZ05]).

2)  $\mathcal{H}_j$  the set of hyperbolic frequencies, that is to say the set of  $\underline{\zeta} \in \Xi_0$  such that Theorem 2.1 for the matrix  $\mathcal{A}_j(\underline{\zeta})$  is satisfied with blocks of type iii) only.

3)  $\mathcal{E}\mathcal{H}_j$  the set of mixed frequencies, that is to say the set of  $\underline{\zeta} \in \Xi_0$  such that Theorem 2.1 for the matrix  $\mathcal{A}_j(\underline{\zeta})$  is satisfied with one block of type i), one of type ii) and at least one of type iii), but without block of type iv).

4)  $\mathcal{G}_j$  the set of glancing frequencies, that is to say the set of  $\underline{\zeta} \in \Xi_0$  such that Theorem 2.1 for the matrix  $\mathcal{A}_j(\underline{\zeta})$  is satisfied with at least one block of type iv).

Thus, by Definition,  $\Xi_0$  admits the following decomposition :

$$\Xi_0 = \mathcal{E}_j \cup \mathcal{E}\mathcal{H}_j \cup \mathcal{H}_j \cup \mathcal{G}_j.$$

The study made in [Kre70] shows that the subspaces  $E_1^s(\zeta_j)$  and  $E_2^s(\zeta_j)$  admit a continuous extension up to  $\Xi_0$ . Moreover, for  $j = 1, 2$ , for  $\underline{\zeta}_j \in \Xi_0 \setminus (\mathcal{G}_1 \cup \mathcal{G}_2)$  one can decompose :

$$\mathbb{C}^N = E_j^s(i\mathcal{T}, \underline{\xi}_{3-j}) \oplus E_j^u(i\mathcal{T}, \underline{\xi}_{3-j}), \quad (2)$$

where the spaces  $E_j^s$  and  $E_j^u$  can be decomposed as :

$$E_j^s(i\mathcal{T}, \underline{\xi}_{3-j}) = E_j^{s,e}(i\mathcal{T}, \underline{\xi}_{3-j}) \oplus E_j^{s,h}(i\mathcal{T}, \underline{\xi}_{3-j}), \quad E_j^u(i\mathcal{T}, \underline{\xi}_{3-j}) = E_j^{u,e}(i\mathcal{T}, \underline{\xi}_{3-j}) \oplus E_j^{u,h}(i\mathcal{T}, \underline{\xi}_{3-j}).$$

Here  $E_j^{s,e}(i\mathcal{T}, \underline{\xi}_{3-j})$  (resp.  $E_j^{u,e}(i\mathcal{T}, \underline{\xi}_{3-j})$ ) is the generalized eigenspace associated with eigenvalues of  $\mathcal{A}_j(i\mathcal{T}, \underline{\xi}_{3-j})$  with negative (resp. positive) real part, and where spaces  $E_j^{s,h}(i\mathcal{T}, \underline{\xi}_{3-j})$  and  $E_j^{u,h}(i\mathcal{T}, \underline{\xi}_{3-j})$  are sums of eigenspaces of  $\mathcal{A}_j(i\mathcal{T}, \underline{\xi}_{3-j})$  associated with some purely imaginary eigenvalues of  $\mathcal{A}_j(i\mathcal{T}, \underline{\xi}_{3-j})$ .

Moreover since the matrices  $A_1$  and  $A_2$  are invertible we can also write (2) in the following way : for  $j = 1, 2$

$$\mathbb{C}^N = A_j E_j^s(i\mathcal{T}, \underline{\xi}_{3-j}) \oplus A_j E_j^u(i\mathcal{T}, \underline{\xi}_{3-j}), \quad (3)$$

In fact, it is possible to give a more precise decomposition of the spaces  $E_j^{s,h}(\underline{\zeta}_j)$  and  $E_j^{u,h}(\underline{\zeta}_j)$ . Indeed, let  $\underline{\omega}_{m,j}$  be a purely imaginary eigenvalue of  $\mathcal{A}_j(\underline{\zeta}_j)$ , that is :

$$\det(\mathcal{T} + \eta A_1 + \underline{\omega}_{m,2} A_2) = \det(\mathcal{T} + \underline{\omega}_{m,1} A_1 + \eta A_2) = 0.$$

Then, using Assumption 2.1, there exists an index  $k_{m,j}$  such that :

$$\mathcal{T} + \lambda_{k_{m,2}}(\eta, \underline{\omega}_{m,2}) = \mathcal{T} + \lambda_{k_{m,1}}(\underline{\omega}_{m,1}, \eta) = 0,$$

where  $\lambda_{k_{m,j}}$  is smooth in both variables. Let us then introduce the following classification :

**Definition 2.2** *The set of incoming (resp. outgoing) phases for the side  $\partial\Omega_1$ , denoted by  $\mathfrak{I}_1$  (resp.  $\mathfrak{D}_1$ ), is the set of indices  $m$  such that the group velocity  $v_m := \nabla \lambda_{k_{m,1}}(\underline{\omega}_{m,1}, \eta)$  satisfies  $\partial_1 \lambda_{k_{m,1}}(\underline{\omega}_{m,1}, \eta) > 0$  (resp.  $\partial_1 \lambda_{k_{m,1}}(\underline{\omega}_{m,1}, \eta) < 0$ ). Similarly, the set of incoming (resp. outgoing) phases for the side  $\partial\Omega_2$ , denoted by  $\mathfrak{I}_2$  (resp.  $\mathfrak{D}_2$ ), is the set of indices  $m$  such that the group velocity  $v_m := \nabla \lambda_{k_{m,2}}(\eta, \underline{\omega}_{m,2})$  satisfies  $\partial_2 \lambda_{k_{m,2}}(\eta, \underline{\omega}_{m,2}) > 0$  (resp.  $\partial_2 \lambda_{k_{m,2}}(\eta, \underline{\omega}_{m,2}) < 0$ ).*

Thanks to this Definition, we can write the following decomposition of the stable and unstable components  $E_j^{s,h}(\underline{\zeta})$  and  $E_j^{u,h}(\underline{\zeta})$  :

**Lemma 2.1** *For all  $\underline{\zeta} \in \mathcal{H}_j \cup \mathcal{E}\mathcal{H}_j$ ,  $j = 1, 2$  there holds*

$$E_1^{s,h}(\underline{\zeta}) = \bigoplus_{m \in \mathfrak{I}_1} \ker \mathcal{L}(\mathcal{T}, \underline{\omega}_{m,1}, \eta), \quad E_1^{u,h}(\underline{\zeta}) = \bigoplus_{m \in \mathfrak{D}_1} \ker \mathcal{L}(\mathcal{T}, \underline{\omega}_{m,1}, \eta), \quad (4)$$

$$E_2^{s,h}(\underline{\zeta}) = \bigoplus_{m \in \mathfrak{I}_2} \ker \mathcal{L}(\mathcal{T}, \eta, \underline{\omega}_{m,2}), \quad E_2^{u,h}(\underline{\zeta}) = \bigoplus_{m \in \mathfrak{D}_2} \ker \mathcal{L}(\mathcal{T}, \eta, \underline{\omega}_{m,2}), \quad (5)$$

We refer, for example, to [CG10] or [Wil96] for a proof of this lemma.



## 2.2 About the boundary conditions.

Let us introduce the initial boundary value problem in the half space  $\{x_1 \geq 0, x_2 \in \mathbb{R}\}$  :

$$\begin{cases} L(\partial)u = f, \text{ on } \{x_1 \geq 0, x_2 \in \mathbb{R}\} \\ B_1 u|_{x_1=0} = g_1, \\ u|_{t \leq 0} = 0. \end{cases} \quad (6)$$

We recall the following result due to Kreiss [Kre70] which states that the boundary value problem (6) is strongly well-posed if and only if it satisfies the following condition :

**Definition 2.3** *We say that the system (6) satisfies the uniform Kreiss-Lopatinskii condition if it satisfies :*

$$\forall \zeta \in \Xi, \ker B_1 \cap E_1^s(\zeta) = \{0\},$$

where  $E_1^s(\zeta)$  denotes still the continuation of the stable subspace of the resolvent matrix  $\mathcal{A}_1(\zeta)$  up to  $\Xi_0$ .

**Definition 2.4** *We denote by  $\Upsilon$  the set of frequencies for which the ibvp (6) does not satisfy the uniform Kreiss-Lopatinskii condition i.e.*

$$\Upsilon := \{\zeta_1 \in \Xi : \ker B_1 \cap E_1^s(\zeta_1) \neq \{0\}\}.$$

Let us recall the following definition due to [BGRSZ02] :

**Definition 2.5** *The ibvp (6) is said to be in the WR class if it satisfies the following conditions :*

- i) *The ibvp (6) satisfies the weak Kreiss-Lopatinskii condition i.e.  $\Upsilon \cap (\Xi \setminus \Xi_0) = \emptyset$ .*
- ii)  *$\Upsilon \neq \emptyset$  and  $\Upsilon \subset \mathcal{H}_1$ .*
- iii) *For all  $\zeta \in \Upsilon$ , there is a neighborhood  $\mathcal{V}$  of  $\zeta$  in  $\Xi$ , a regular basis  $(E_{1,1}^s, \dots, E_{1,p_1}^s)(\zeta)$  of  $E_1^s(\zeta)$  on  $\mathcal{V}$ , an invertible  $p_1 \times p_1$  matrix  $P(\zeta)$ , regular on  $\mathcal{V}$  and a smooth real valued function  $\Theta$  such that*

$$\forall \zeta \in \mathcal{V}, B_1 [E_{1,1}^s, \dots, E_{1,p_1}^s](\zeta) = P(\zeta) \text{diag}(\gamma + i\Theta(\zeta), 1, \dots, 1).$$

*In particular, one can find a Lopatinskii's determinant under the form :*

$$\forall \zeta \in \mathcal{V}, \Delta(\zeta) = (\gamma + i\Theta(\zeta)) \det P(\zeta).$$

These definitions about hyperbolic boundary value problems in the half space motivate the following definition for the corner problem :

**Definition 2.6** *We say that the corner problem (1) is in the WR class for the side  $\partial\Omega_1$  if the boundary value problem (6) is in the WR class.*

In all this paper, the boundary condition for the side  $\partial\Omega_1$  will not satisfy the uniform Kreiss-Lopatinskii condition ( more precisely we will then work with corner problem in the WR class for the side  $\partial\Omega_1$ ) and then will lead to weak well-posedness. While, as mentioned in the introduction, the boundary condition for the side  $\partial\Omega_2$  will be chosen as "nice" as possible in terms of well-posedness. However, we will see in Theorem 3.1 that even if the boundary condition on  $\partial\Omega_2$  is in the most favorable class of strictly dissipative boundary conditions (see [BGS07]), it can not compensated the "bad" boundary condition on  $\partial\Omega_1$ .

To make things more precise, let us thus assume the following :

**Assumption 2.3** *The corner problem (1) is in the WR class for the side  $\partial\Omega_1$ .*

*The boundary condition on the side  $\partial\Omega_2$  is strictly dissipative or satisfies (at least) the uniform Kreiss-Lopatinskii condition. In the first case, that is to say that the following inequality holds :*

$$\forall v \in \ker B_2, \langle A_2 v, v \rangle < 0.$$

*While in the second case, it means that  $\forall \zeta \in \Xi$ , we have :*

$$\ker B_2 \cap E_2^s(\zeta) = \{0\}.$$

*In both cases (we refer to [BGS07] for a proof that strict dissipativity implies the uniform Kreiss-Lopatinskii condition) the restriction of  $B_2$  to  $E_2^s(\zeta)$  is invertible. We denote this inverse by  $\phi_2(\zeta)$ .*

When one studies geometric optics expansion for weakly well-posed boundary value problems, it is useful to define the following vectors (this definition comes from [CG10] and was also used later in [Ben14]) :

**Definition 2.7** *Let (1) be in the WR class for the side  $\partial\Omega_1$ , then there exists*

- $\diamond$  *a vector  $e \in \mathbb{C}^N \setminus \{0\}$  such that  $\ker B_1 \cap E_1^s(\underline{\zeta}) = \text{vect}(e)$ .*
- $\diamond$  *A vector  $b \in \mathbb{C}^{p_1} \setminus \{0\}$  such that  $b \cdot Bw = 0$ , for all  $w \in E_1^s(\underline{\zeta})$ .*

### 3 Formal phase generation process and main result.

#### 3.1 Phase generation process.

In this paragraph we give some elements about what are the expected phases in the WKB expansion and the associated amplitudes for corner problem in the WR class for the side  $\partial\Omega_1$ . We will not here give a precise description of the phase generation process in itself (we refer to [Ben] and [SS75] for a complete discussion) but we will focus on the expected sizes of the amplitudes according to the small parameter  $\varepsilon$ . More especially we will also discuss after how many time of travel the amplifications are expected.

Let us thus consider the hyperbolic corner problem :

$$\begin{cases} L(\partial)u^\varepsilon = 0, & \text{on } \Omega_T \\ B_1 u^\varepsilon|_{x_1=0} = g^\varepsilon, & \text{on } \partial\Omega_{1,T}, \\ B_2 u^\varepsilon|_{x_2=0} = 0, & \text{on } \partial\Omega_{2,T}, \\ u^\varepsilon|_{t \leq 0} = 0, \end{cases} \quad (7)$$

and we assume that (7) satisfies Assumptions (2.1)-(2.2) and (2.3). We also suppose that this corner problem admits four selfinteracting planar phases (see Assumption (4.1) or [Ben] for more details) namely

$$\varphi_{n_j}(t, x) := \underline{t}t + \xi_1^{n_j} x_1 + \xi_2^{n_j} x_2, \quad j = 1, \dots, 4,$$

and that the source term  $g^\varepsilon$  in (7) is chosen in such a way that it "turns on" the phase  $\varphi_{n_1}$ . Finally we assume that the phase  $\varphi_{n_1}$  is associated to the only frequency for which the uniform Kreiss-Lopatinskii condition is violated.

Let us choose  $g^\varepsilon$ , zero for negative times, with compact support away from the corner for all positive times (or at least a function which is zero on a neighborhood of the corner for all positive times). Then by finite speed of propagation arguments, the information carried by  $g^\varepsilon$  can not hit the side  $\partial\Omega_2$  immediately. So, at least during a short time, we can forget the boundary condition on

$\{x_2 = 0\}$  and see the corner problem (7) as a problem in the  $WR$  class for the half space  $\{x_1 > 0\}$ . It is thus natural to start by taking the ansatz for this problem lying in the half space, that is to say to consider the phases  $\varphi_{j,1}$  defined by

$$\varphi_{j,1}(t, x) := \tau t + \xi_1^{j,1} x_1 + \xi_2^{n_1} x_2, \quad j = 1, \dots, N,$$

where the  $(\xi_1^{j,1})$  are the roots in the  $\xi_1$  variable of the dispersion relation :

$$\det \mathcal{L}(d\varphi_j) = 0,$$

associated to incoming-outgoing or incoming-incoming group velocities (see [Ben] for more details). Let us stress that, by assumption, the phase  $\varphi_{n_1}$  is contained in the  $\varphi_{j,1}$ .

However the analysis of [CG10] tells us that when the uniform Kreiss-Lopatinskii condition breaks down, due to a transport phenomenon along the side  $\partial\Omega_1$ , the leading order is one order less than the order of the source term on the boundary  $\{x_1 = 0\}$ . More precisely, if the source term is of order one, then the leading order in the WKB expansion is of order zero. It can also be shown that this transport phenomenon along the side  $\partial\Omega_1$ , in the presence of a source term on the boundary, is immediately turned on. We thus expect that this property which comes from the study of the problem in the half space remains true for the corner problem (7), at least during a short time.

In all this paper, we will assume that the transport along the side  $\partial\Omega_1$  spreads the information away from the corner. If it is not the case then the information will hit the corner in finite time. Until now, we are not able to construct geometric optics expansions if such a situation occurs and we thus have a maximal time of existence for the geometric optics expansions after the time of impact. However let us remark that in the particular framework  $d = 2$ , then the velocity of the transport along the side  $\partial\Omega_1$  is explicitly computable. More precisely it is given by  $-\frac{\tau}{\xi_2^{n_1}}$  (see [CG10]). So in this particular setting asking that the transport phenomenon along the side  $\partial\Omega_1$  spreads the information away from the corner is equivalent to ask that  $\tau$  and  $\xi_2^{n_1}$  have opposite signs, which is easily verifiable in practice.

Then we study the reflections of the phase  $\varphi_{n_1}$  against the side  $\partial\Omega_2$ . A precise description of the reflections for the other amplitudes can be found in [Ben], but it is not as important as the generation of the phase  $\varphi_{n_2}$  by the phase  $\varphi_{n_1}$  for our current discussion.

The ray associated to the phase  $\varphi_{n_1}$  is incoming-outgoing so it hits the side  $\partial\Omega_2$  after a finite time of travel. However, when it happens, the striking ray still have its support away from the corner. Thus, once again, by finite speed of propagation arguments, at least during a small time, we can see the corner problem (7) as a problem in the half space  $\{x_2 > 0\}$  from which the information on the side  $\partial\Omega_2$  has been "turned on" by the incoming-outgoing phase  $\varphi_{n_1}$ . We thus add in the ansatz the amplitudes associated with the phases

$$\varphi_{j,2}(t, x) := \tau t + \xi_1^{n_1} x_1 + \xi_2^{j,2} x_2, \quad j = 1, \dots, N_2,$$

where the  $(\xi_2^{j,2})$  are the roots in the  $\xi_2$  variable of the dispersion relation :

$$\det \mathcal{L}(d\varphi_j) = 0.$$

Let us here insist on the fact that there can be less than  $N$  roots for this dispersion relation and that these roots can be complex valued. Indeed, the boundary value problem in the half space  $\{x_2 > 0\}$  is not assumed to be in the  $WR$  class. According to [Ben] we only consider the real roots

associated with outgoing-incoming or incoming-incoming group velocities and a class representative of the complex valued roots with positive imaginary part (see Section 4 or [Ben] for more details).

As the boundary condition on the side  $\partial\Omega_2$  is assumed to satisfy the uniform Kreiss-Lopatinskii condition, there is no transport phenomenon along the boundary and the leading order of the *WKB* expansion remains of order zero (for a source term on the boundary  $\partial\Omega_1$  of order one).

We can then repeat exactly the same arguments to show that the phases  $\varphi_{n_3}$  and  $\varphi_{n_4}$  are generated when we consider the reflections of the ray associated to  $\varphi_{n_2}$  on  $\partial\Omega_1$ <sup>3</sup> and the reflection of the ray associated to  $\varphi_{n_3}$  on  $\partial\Omega_2$ . It is then interesting to study the reflection of  $\varphi_{n_4}$  against the side  $\partial\Omega_1$ . This reflection starts after a strictly positive time  $t_0$  at a strictly positive distance of the corner  $y_0$  (the precise values of this parameters are given in Section 5 and will be fundamental in our proof). By selfinteraction phenomenon, we know that during this reflection the phase  $\varphi_{n_1}$  is generated again. However due to the degeneracy of uniform Kreiss-Lopatinskii condition for the phase  $\varphi_{n_1}$ , a new transport along the boundary is expected and shall induce a new amplification phenomenon. All the amplitudes in the ansatz then should lose one power of  $\varepsilon$ , that is to say that the amplitude of order zero becomes of order  $-1$ , the amplitude of order one becomes of order zero and so on.

As a consequence, if one starts with a source term of order two in  $\varepsilon$  then before the time  $t_0$ , the leading order of the geometric optics expansion is expected to be of order one, but after the time  $t_0$ , the leading order in this expansion is expected to be of order zero. Moreover the traces of the amplitudes associated to the phase  $\varphi_{n_1}$  are now expected to be zero for  $x_2$  less than  $y_0$ . We can thus repeat exactly the same arguments. That is to say that after another complete circuit around the loop, the amplitude associated to  $\varphi_{n_4}$  hits again the side  $\partial\Omega_1$  and regenerates the phase  $\varphi_{n_1}$ . A new amplification thus happens and all the amplitudes lose one order in terms of  $\varepsilon$ . The leading order in the geometric optics expansion is now expected to be of order zero for a source term on the boundary of order three.

So if one wants to construct a geometric optics expansion with  $M + 1$  amplifications before a fixed time  $T$ , it will be sufficient to choose the support of the source term  $g$  close enough of the corner to ensure that the rays have made  $M$  complete circuits around the loop before the time  $T$ .

Moreover, the fact that we choose to work with a fixed maximal time of resolution  $T$  and with a source term  $g$  having its support away from the corner permits us to assume that the number of generated phases in the phase generation process is finite and avoid the technical difficulties pointed in [Ben]. Indeed, in [Ben] it was shown that the phase generation process consists of considering sequences of phases with incoming-outgoing and then outgoing-incoming group velocities and to stop the sequence when we meet a phase with incoming-incoming group velocity. However, here each transport phenomenon from one side to the other takes some strictly positive time (which is explicitly computable) because the transported information has its support away from the corner. So when we apply the phase generation process we can stop the sequence as soon as the sum of the times needed to generate the considered phase is strictly more than the fixed maximal time  $T$ .

### 3.2 Main result.

In this paragraph we describe the main result of this paper. Roughly speaking, this result states that when one deals with a corner problem which do not satisfy the uniform Kreiss-Lopatinskii

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<sup>3</sup>Let us stress that during this reflection on the side  $\partial\Omega_1$  there is no amplification because the considered phases are  $\varphi_{n_2}$  and  $\varphi_{n_3}$  but not  $\varphi_{n_1}$ .

condition on each boundary then he should be careful. Indeed, Theorem 3.1 demonstrates that such a problem can, in some situations depending on the geometry of the characteristic variety, be ill-posed. The violent instability is, in this framework, caused by an accumulation of weak instabilities and differs from the failure of the weak Kreiss-Lopatinskii condition which is, to our knowledge, the only known example of violent instability for hyperbolic boundary value problem. Before to state the main theorem, let us define more precisely the terms of the previous discussion :

**Definition 3.1** *The corner problem (1) is said to be weakly well-posed (or to generate a weak instability) if there exists  $(K, K_1, K_2) \in \mathbb{R}^3 \setminus \{0\}$  such that for all source terms  $f \in H^K(\Omega_T)$ ,  $g_1 \in H^{K_1}(\partial\Omega_{1,T})$ ,  $g_2 \in H^{K_2}(\partial\Omega_{2,T})$ , the corner problem (1) admits a unique solution  $u \in L^2(\Omega_T)$  with traces  $u|_{x_1=0} \in L^2(\partial\Omega_{1,T})$ ,  $u|_{x_2=0} \in L^2(\partial\Omega_{2,T})$  satisfying the energy estimate :*

$$\|u\|_{L^2(\Omega_T)}^2 + \|u|_{x_1=0}\|_{L^2(\partial\Omega_{1,T})}^2 + \|u|_{x_2=0}\|_{L^2(\partial\Omega_{2,T})}^2 \leq C_T \left( \|f\|_{H^K(\Omega_T)}^2 + \|g_1\|_{H^{K_1}(\partial\Omega_{1,T})}^2 + \|g_2\|_{H^{K_2}(\partial\Omega_{2,T})}^2 \right),$$

for some positive constant  $C_T$ .

*The corner problem (1) is said to be ill-posed (or to generate a violent instability) if such integers do not exist.*

The main theorem of this paper is the following :

**Theorem 3.1** *Let (1) be a corner problem satisfying Assumptions 2.1-2.2-2.3-4.1 and 4.2 then (1) can not be weakly well-posed. In other words, for all  $K \in \mathbb{N}^*$ , one can find  $g_1 \in H^K(\partial\Omega_{1,T})$ , such that the energy estimate*

$$\|u\|_{L^2(\Omega_T)}^2 + \|u|_{x_1=0}\|_{L^2(\partial\Omega_{1,T})}^2 + \|u|_{x_2=0}\|_{L^2(\partial\Omega_{2,T})}^2 \leq C_T \left( \|f\|_{L^2(\Omega_T)}^2 + \|g_1\|_{H^K(\partial\Omega_{1,T})}^2 + \|g_2\|_{L^2(\partial\Omega_{2,T})}^2 \right),$$

*is violated.*

The proof of Theorem 3.1 is given in Section 6. This proof is based on the rigorous construction of a geometric optics expansion for the corner problem (1) with an arbitrary number of amplifications compared to the source term on the side  $\partial\Omega_1$ . This construction, which is the technical part of the proof, is made in Section 5.

## 4 General framework.

In this section we recall some notations and definitions used in [Ben] to describe rigorously the set of phases obtained by reflection against the side of the quarter space.

### 4.1 Definition of the frequency set and first properties.

Let us start with the definition of what we mean by a frequency set :

**Definition 4.1** *Let  $\mathcal{I}$  be a subset of  $\mathbb{N}$  and  $\underline{\tau} \in \mathbb{R}$ ,  $\underline{\tau} \neq 0$ . A set indexed by  $\mathcal{I}$ ,*

$$\mathcal{F} := \{f_i := (\underline{\tau}, \xi_1^i, \xi_2^i), i \in \mathcal{I}\},$$

*will be a set of frequencies for the corner problem (1) if for all  $i \in \mathcal{I}$ ,  $f_i$  satisfies*

$$\det \mathcal{L}(f_i) = 0,$$

and one of the following alternatives :

- i)  $\xi_1^i, \xi_2^i \in \mathbb{R}$ .
- ii)  $\xi_1^i \in (\mathbb{C} \setminus \mathbb{R}), \xi_2^i \in \mathbb{R}$  and  $\text{Im } \xi_1^i > 0$ .
- iii)  $\xi_2^i \in (\mathbb{C} \setminus \mathbb{R}), \xi_1^i \in \mathbb{R}$  and  $\text{Im } \xi_2^i > 0$ .

In all what follows, if  $\mathcal{F}$  is a frequency set for the corner problem (1), we will define :

$$\begin{aligned}\mathcal{F}_{os} &:= \{f_i \in \mathcal{F} \text{ satisfying } i)\}, \\ \mathcal{F}_{ev1} &:= \{f_i \in \mathcal{F} \text{ satisfying } ii)\}, \\ \mathcal{F}_{ev2} &:= \{f_i \in \mathcal{F} \text{ satisfying } iii)\}.\end{aligned}$$

It is clear that the sets  $\mathcal{F}_{os}, \mathcal{F}_{ev1}$  and  $\mathcal{F}_{ev2}$  give a partition of  $\mathcal{F}$ . Moreover to each  $f_i \in \mathcal{F}_{os}$ , we can associate a group velocity  $v_i := (v_{i,1}, v_{i,2})$ . Let us recall that the group velocity  $v_i$  is defined in Definition 2.2. The set  $\mathcal{F}_{os}$  can thus be decomposed as follows :

$$\begin{aligned}\mathcal{F}_{ii} &:= \{f_i \in \mathcal{F}_{os} \setminus v_{i,1}, v_{i,2} > 0\}, \quad \mathcal{F}_{io} := \{f_i \in \mathcal{F}_{os} \setminus v_{i,1} > 0, v_{i,2} < 0\}, \\ \mathcal{F}_{oi} &:= \{f_i \in \mathcal{F}_{os} \setminus v_{i,1} < 0, v_{i,2} > 0\}, \quad \mathcal{F}_{oo} := \{f_i \in \mathcal{F}_{os} \setminus v_{i,1} < 0, v_{i,2} < 0\}, \\ \mathcal{F}_g &:= \{f_i \in \mathcal{F}_{os} \setminus v_{i,1} = 0 \text{ or } v_{i,2} = 0\}.\end{aligned}$$

The partition of  $\mathcal{F}$  induces the following partition of  $\mathcal{I}$  :

$$\mathcal{I} = \mathcal{I}_g \cup \mathcal{I}_{oo} \cup \mathcal{I}_{io} \cup \mathcal{I}_{oi} \cup \mathcal{I}_{ii} \cup \mathcal{I}_{ev1} \cup \mathcal{I}_{ev2},$$

where we have denoted by  $\mathcal{I}_{io}$  (resp.  $g, oo, oi, ii, ev1, ev2$ ) the set of indices  $i \in \mathcal{I}$  such that the corresponding frequency  $f_i \in \mathcal{F}_{io}$  (resp.  $g, oo, oi, ii, ev1, ev2$ ).

From now on, the source term  $g^\varepsilon$  on the boundary in (1) reads :

$$g^\varepsilon(t, x_2) := e^{\frac{i}{\varepsilon}(\tau t + \xi_2 x_2)} \varepsilon^{M+1} g(t, x_2), \quad (8)$$

for some fixed integer  $M$  and where the amplitude  $g$  has its support in space away from the corner and is zero for negative times.

The following definition gives a precise framework for the phase generation process described in paragraph 3.1. More precisely, this definition qualifies the frequency set that contains all (and only) the frequencies linked with the expected non-zero amplitudes in the WKB expansion of the solution to the corner problem (1).

**Definition 4.2** *The corner problem (1) is said to be complete for reflections if there exists a set of frequencies  $\mathcal{F}$  satisfying the following properties :*

i)  $\mathcal{F}$  contains the real roots (in the variable  $\xi_1$ ) associated with incoming-outgoing or incoming-incoming group velocities and the complex roots with positive imaginary part, to the dispersion equation

$$\det \mathcal{L}(\tau, \xi_1, \xi_2) = 0.$$

ii)  $\mathcal{F}_g = \emptyset$ .

iii) If  $(\tau, \xi_1^i, \xi_2^i) \in \mathcal{F}_{io}$ , then  $\mathcal{F}$  contains all the roots (in the variable  $\xi_2$ ), denoted by  $\xi_2^p$ , to the dispersion relation  $\det \mathcal{L}(\tau, \xi_1^i, \xi_2) = 0$ , that satisfy one of the following two alternatives :

iii')  $\xi_2^p \in \mathbb{R}$  and the frequency  $(\tau, \xi_1^i, \xi_2^p)$  is associated with an outgoing-incoming group velocity or an incoming-incoming group velocity.

iii'')  $\text{Im } \xi_2^p > 0$ .

iv) If  $(\mathcal{I}, \xi_1^i, \xi_2^i) \in \mathcal{F}_{oi}$ , then  $\mathcal{F}$  contains all the roots (in the variable  $\xi_1$ ), denoted by  $\xi_1^p$ , to the dispersion relation  $\det \mathcal{L}(\mathcal{I}, \xi_1, \xi_2^i) = 0$ , that satisfy one of the following two alternatives :

iv')  $\xi_1^p \in \mathbb{R}$  and the frequency  $(\mathcal{I}, \xi_1^p, \xi_2^i)$  is associated with an incoming-outgoing or an incoming-incoming group velocity.

iv'')  $\text{Im } \xi_1^p > 0$ .

v)  $\mathcal{F}$  is minimal (for the inclusion) for the four preceding properties.

**Remark** Point i) imposes that the frequency set  $\mathcal{F}$  contains all the incoming phases for  $\partial\Omega_1$  that are induced by the source term  $g^\varepsilon$ .

Point iii) (resp. iv)) explains the generation by reflection on the side  $\partial\Omega_2$  (resp.  $\partial\Omega_1$ ) of a wave packet that emanates from the side  $\partial\Omega_1$  (resp.  $\partial\Omega_2$ ).

An immediate consequence of the minimality of  $\mathcal{F}$  is that  $\mathcal{F}_{oo}$  is empty. Moreover let us stress that according to the discussion made in paragraph 3.1, because we are dealing with finite time and a source term which has its support away from the corner, without loss of generality we can assume that  $\#\mathcal{F} < +\infty$ .

Let us recall that if the corner problem is complete for reflections, one can define two applications, defined on the index set  $\mathcal{I}$  and which give, in the output, the indices "in the direct vicinity" of the input index :

$$\Phi, \Psi : \mathcal{I} \longrightarrow \mathcal{P}_N(\mathcal{I}),$$

where  $\mathcal{P}_N(\mathcal{I})$  denotes the power set of  $\mathcal{I}$  with at most  $N$  elements. More precisely, the definitions of  $\Phi$  and  $\Psi$  are : for  $i \in \mathcal{I}$ ,  $f_i = (\mathcal{I}, \xi_1^i, \xi_2^i)$ ,

$$\Phi(i) := \left\{ j \in \mathcal{I} \mid \xi_2^j = \xi_2^i \right\} \text{ and } \Psi(i) := \left\{ j \in \mathcal{I} \mid \xi_1^j = \xi_1^i \right\}.$$

For convenience let us introduce the short notations :

$$\Phi^*(i) := \Phi(i) \setminus \{i\} \text{ and } \Psi^*(i) := \Psi(i) \setminus \{i\}$$

We refer to [Ben] for more details about the sets  $\Phi(i)$  (resp.  $\Psi(i)$ ) in terms of wave packet reflection and about the graph structure that they induce on the index set  $\mathcal{I}$ . The applications  $\Phi$  and  $\Psi$  have the following properties :

**Proposition 4.1** *If the corner problem (1) is complete for reflections, then  $\Phi$  and  $\Psi$  satisfy :*

i)  $\forall i \in \mathcal{I}, i \in \Psi(i), i \in \Phi(i)$ .

ii)  $\forall i \in \mathcal{I}, \forall j \in \Psi(i), \forall k \in \Phi(i)$  we have  $\Psi(i) = \Psi(j)$  and  $\Phi(i) = \Phi(k)$ .

iii)  $\forall i \in \mathcal{I}, \Phi(i) \cap \mathcal{I}_{ev2} = \emptyset$  and  $\Psi(i) \cap \mathcal{I}_{ev1} = \emptyset$ . And,  $\forall i \in \mathcal{I}_{ev1}, \forall j \in \mathcal{I}_{ev2}$ , we have  $\Psi(i) \subset \mathcal{I}_{ev1}, \Phi(i) \subset \mathcal{I}_{ev2}$ .

iv)  $\forall i \in \mathcal{I}_{os}, \#(\Phi(i) \cap \mathcal{I}_{ev1} \cap \mathcal{I}_{io} \cap \mathcal{I}_{ii}) \leq p_1$ , and  $\#(\Psi(i) \cap \mathcal{I}_{ev2} \cap \mathcal{I}_{oi} \cap \mathcal{I}_{ii}) \leq p_2$ .

v)  $\forall i \in \mathcal{I}$ , we have on one hand  $\forall i_1, i_2 \in \Phi(i), i_1 \neq i_2$  :

$$\Phi(i) \cap \Psi(i_1) = \{i_1\} \text{ and } \Psi(i_1) \cap \Psi(i_2) = \emptyset,$$

and on the other hand,  $\forall j_1, j_2 \in \Psi(i), j_1 \neq j_2$  :

$$\Psi(i) \cap \Phi(j_1) = \{j_1\} \text{ and } \Phi(j_1) \cap \Phi(j_2) = \emptyset.$$

We refer to [Ben] for a proof.

Thanks to applications  $\Phi$  and  $\Psi$  it is easy to define the notion of two linked indices in the graph structure of  $\mathcal{I}$  :

**Definition 4.3** *If  $\underline{i} \in \mathcal{I}_{io}$ , we say that the index  $j \in \mathcal{I}_{io} \cup \mathcal{I}_{ev1}$  (resp.  $j \in \mathcal{I}_{oi} \cup \mathcal{I}_{ev2}$ ) is linked with the index  $\underline{i}$ , if there exists  $p \in 2\mathbb{N}+1$  (resp.  $p \in 2\mathbb{N}$ ) and a sequence of indices  $l = (l_1, l_2, \dots, l_p) \in \mathcal{I}^p$  such that :*

$\alpha')$   $l_1 \in \Psi(\underline{i}) \cap \mathcal{I}_{oi}$ ,  $l_2 \in \Phi(l_1) \in \mathcal{I}_{io}$ , ...,  $j \in \Phi(l_p)$  (resp.  $j \in \Psi(l_p)$ ).

*We say that the index  $j \in \mathcal{I}_{ii}$  is linked with the index  $\underline{i}$ , if there is a sequence of indices  $l = (l_1, l_2, \dots, l_p) \in \mathcal{I}^p$  such that :*

$\beta')$   $l_1 \in \Psi(\underline{i}) \cap \mathcal{I}_{oi}$ ,  $l_2 \in \Phi(l_1) \cap \mathcal{I}_{io}$ , ...,  $\begin{cases} j \in \Phi(l_p), & p \text{ is odd,} \\ j \in \Psi(l_p), & p \text{ is even.} \end{cases}$

*If  $\underline{i} \in \mathcal{I}_{oi}$ , we say that the index  $j \in \mathcal{I}_{io} \cup \mathcal{I}_{ev1}$  (resp.  $j \in \mathcal{I}_{oi} \cup \mathcal{I}_{ev2}$ ) is linked with the index  $\underline{i}$ , if there exists  $p \in 2\mathbb{N}$  (resp.  $p \in 2\mathbb{N}+1$ ) and a sequence of indices  $l = (l_1, l_2, \dots, l_p) \in \mathcal{I}^p$  such that :*

$\alpha'')$   $l_1 \in \Phi(\underline{i}) \cap \mathcal{I}_{io}$ ,  $l_2 \in \Psi(l_1) \in \mathcal{I}_{oi}$ , ...,  $j \in \Phi(l_p)$  (resp.  $j \in \Psi(l_p)$ ).

*We say that the index  $j \in \mathcal{I}_{ii}$  is linked with the index  $\underline{i}$ , if there exists a sequence of indices  $l = (l_1, l_2, \dots, l_p) \in \mathcal{I}^p$  such that :*

$\beta'')$   $l_1 \in \Phi(\underline{i}) \cap \mathcal{I}_{io}$ ,  $l_2 \in \Psi(l_1) \cap \mathcal{I}_{oi}$ , ...,  $\begin{cases} j \in \Psi(l_p), & p \text{ is odd,} \\ j \in \Phi(l_p), & p \text{ is even.} \end{cases}$

Finally, if  $\underline{i} \in \mathcal{I}_{ii} \cup \mathcal{I}_{ev1} \cup \mathcal{I}_{ev2}$ , there is no element of  $\mathcal{I}$  linked with  $\underline{i}$ .

Moreover, we will say that an index  $j \in \mathcal{I}$  is linked with the index  $\underline{i}$  by a sequence of type  $H$  (for "horizontal") (resp.  $V$  (for "vertical")) and we will note  $i \xrightarrow{H} j$  (resp.  $i \xrightarrow{V} j$ ) if the sequence  $(\underline{i}, l_1, l_2, \dots, l_p, j)$  satisfies  $\alpha'')$  or  $\beta'')$  (resp.  $\alpha')$  or  $\beta')$ .

Applications  $\Phi$  and  $\Psi$  also enable us to define a set of class of representative for the complex valued frequencies (or evanescent frequencies) in view to treat these frequencies in a "monoblock" way. That is to say that for an index  $\underline{i} \in \mathcal{I}_{ev1}$  (resp.  $\underline{i} \in \mathcal{I}_{ev2}$ ), all the indices  $j \in \mathcal{I}_{ev1} \cap \Phi(\underline{i})$  (resp.  $j \in \mathcal{I}_{ev2} \cap \Psi(\underline{i})$ ) will contribute to a single vector valued amplitude. To write off the ansatz and to describe with enough precision the boundary conditions, it is useful to introduce the two equivalence relations  $\underset{\Phi}{\sim}$  and  $\underset{\Psi}{\sim}$  defined by :

$$i \underset{\Phi}{\sim} j \iff j \in \Phi(i), \text{ and } i \underset{\Psi}{\sim} j \iff j \in \Psi(i).$$

Let  $\mathfrak{C}_1$  (resp.  $\mathfrak{C}_2$ ) be the set of equivalence classes for the relation  $\underset{\Phi}{\sim}$  (resp.  $\underset{\Psi}{\sim}$ ), and  $\mathcal{R}_1$  (resp.  $\mathcal{R}_2$ ), be a set of class representative for  $\mathfrak{C}_1$  (resp.  $\mathfrak{C}_2$ ). So  $\mathcal{R}_1$  (resp.  $\mathcal{R}_2$ ) is a set of indices which include all the possible values for  $\xi_2$  (resp.  $\xi_1$ ) of the different frequencies. Let us define  $\mathfrak{R}_1$  and  $\mathfrak{R}_2$  by :

$$\mathfrak{R}_1 := \{i \in \mathcal{R}_1 \setminus \Phi(i) \cap \mathcal{I}_{ev1} \neq \emptyset\}, \quad (9)$$

$$\mathfrak{R}_2 := \{i \in \mathcal{R}_2 \setminus \Psi(i) \cap \mathcal{I}_{ev2} \neq \emptyset\}. \quad (10)$$

$\mathfrak{R}_1$  (resp.  $\mathfrak{R}_2$ ) is a set of class representative of the values in  $\xi_2$  (resp.  $\xi_1$ ) for which there is an evanescent mode for the side  $\partial\Omega_1$  (resp.  $\partial\Omega_2$ ). At last, without loss of generality, we can always assume that  $n_1 \in \mathcal{R}_2$ , in other words, we choose  $n_1$  as a class representative of its equivalence class.



To conclude, let us recall the following proposition which is an immediate consequence of Definitions 4.2 and 4.3.

**Proposition 4.2** *Let  $\mathcal{F}$  be a complete for reflections frequency set indexed by  $\mathcal{I}$ . Let  $\mathcal{I}_0$  be the set of indeces in  $\mathcal{I}$  generated by the source term  $g^\varepsilon$ , that is to say :*

$$\mathcal{I}_0 := \{i \in \mathcal{I}_{io} \cup \mathcal{I}_{ii} \cup \mathcal{I}_{ev1} \setminus \det \mathcal{L}(\underline{\tau}, \xi_1^i, \underline{\xi}_2) = 0\}.$$

Let  $\mathcal{I}_{\mathcal{R}}$  be the set of indeces in  $\mathcal{I}$  linked with one of the elements of  $\mathcal{I}_0$ . Then

$$\mathcal{I}_{\mathcal{R}} = \mathcal{I}.$$

## 4.2 Frequency sets with loop.

As in [Ben] we will assume that the considered corner problem admits a unique selfinteraction loop<sup>4</sup>. Let us recall the definition of such a loop :

**Definition 4.4** *Let  $i \in \mathcal{I}$ ,  $p \in 2\mathbb{N} + 1$  and  $\ell = (\ell_1, \dots, \ell_p) \in \mathcal{I}^p$ .*

- *We say that the index  $i \in \mathcal{I}$  admits a loop if there exists a sequence  $\ell$  satisfying :*

$$\ell_1 \in \Phi(i), \ell_2 \in \Psi(\ell_1), \dots, i \in \Psi(\ell_p).$$

- *A loop for an index  $i$  is said to be simple if the sequence  $\ell$  does not contain a periodically repeated subsequence.*
- *An index  $i \in \mathcal{I}_{io}$  (resp.  $i \in \mathcal{I}_{oi}$ ) admits a selfinteraction loop if  $i$  admits a simple loop and if the sequence  $(i, \ell, i)$  is of type  $V$  (resp.  $H$ ) according to Definition 4.3.*

Let us assume that :

**Assumption 4.1** *Let (1) be complete for reflections, we assume that the frequency set  $\mathcal{F}$  contains a unique loop, of size 3 and that this loop is a selfinteraction loop. More precisely, we ask that the following properties are satisfied :*

*vi)  $\exists (n_1, n_3) \in \mathcal{I}_{io}^2, (n_2, n_4) \in \mathcal{I}_{oi}^2$  such that*

$$n_2 \in \Psi(n_1), n_3 \in \Phi(n_2), n_4 \in \Psi(n_3), n_1 \in \Phi(n_4).$$

*vii) Let  $i \in \mathcal{I}$  an index with a loop  $\ell = (\ell_1, \dots, \ell_p)$ . Then  $p = 3$  and  $\{i, \ell_1, \ell_2, \ell_3\} = \{n_1, n_2, n_3, n_4\}$ .*

Let us define  $f^{n_j}$  the frequencies associated to the loop's elements. For  $j = 1, \dots, 4$  we write :

$$f^{n_1} := (\underline{\tau}, \underline{\xi}_1, \bar{\xi}_2), f^{n_2} := (\underline{\tau}, \underline{\xi}_1, \underline{\xi}_2), f^{n_3} := (\underline{\tau}, \bar{\xi}_1, \underline{\xi}_2) \text{ and } f^{n_4} := (\underline{\tau}, \bar{\xi}_1, \bar{\xi}_2). \quad (11)$$

We also assume that the only index of  $\mathcal{I}$  for which uniform Kreiss-Lopatinskii condition is violated is  $n_1$ . We summarize the previous requests on the set of indeces for the corner problem (1) in the following assumption (which specifies Assumption 2.3):

**Assumption 4.2** *The corner problem (1) is in the class  $WR$  for the side  $\partial\Omega_1$  and satisfies the uniform Kreiss-Lopatinskii condition on the side  $\partial\Omega_2$ . The set where the uniform Kreiss-Lopatinskii condition for the side  $\partial\Omega_1$  breaks down  $\Upsilon$  (see Definition 2.4) is given by  $\Upsilon = \left\{ \tilde{f}^{n_1} := (i\underline{\tau}, \underline{\xi}_1, \bar{\xi}_2) \right\}$ .*

*Moreover to make sure that the transport along the boundary spreads the information away from the corner we will ask that  $\underline{\tau}$  and  $\bar{\xi}_2$  has opposite signs.*

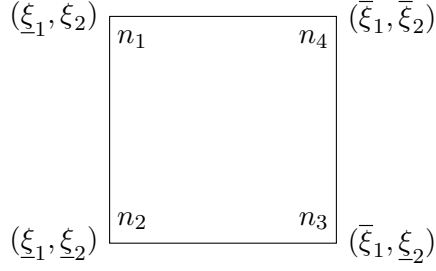


Figure 1: The coefficients of the loop's elements.

To conclude let us recall that when the frequency set  $\mathcal{F}$ , indexed by  $\mathcal{I}$ , is complete for the reflections and that when it admits a unique loop, then it follows from the definition of linked indices 4.3 that we have the following Propositions :

**Proposition 4.3** *Let  $i \in \mathcal{I}$  then there exists a unique type  $V$  sequence linking  $n_1$  to  $i$ .*

Moreover, one can write  $\mathcal{I} \setminus \{n_1, n_2, n_3, n_4\}$  as a partition :

$$(\mathcal{I} \setminus \{n_1, n_2, n_3, n_4\}) = (\cup_{l \in \mathbb{C}_1} A_{a_l}) \cup (\cup_{m \in \mathbb{C}_2} B_{b_m}) \cup (\cup_{q \in \mathbb{C}_3} C_{c_q}) \cup (\cup_{r \in \mathbb{C}_4} D_{d_r}), \quad (12)$$

where  $A_{a_l}$  denotes the set of indices  $i \in \mathcal{I}$  from which the type  $V$  sequence linking  $n_1$  to  $i$  starts by  $a_l$ ,  $B_{b_m}$  denotes the set of indices  $i \in \mathcal{I}$  from which the type  $V$  sequence linking  $n_1$  to  $i$  starts by  $(n_2, b_l)$ ,  $C_{c_q}$  denotes the set of indices  $i \in \mathcal{I}$  from which the type  $V$  sequence linking  $n_1$  to  $i$  starts by  $(n_2, n_3, c_q)$  and  $D_{d_r}$  is the set of indices  $i \in \mathcal{I}$  such that the type  $V$  sequence linking  $n_1$  to  $i$  starts by  $(n_2, n_3, n_4, d_r)$ .

**Proposition 4.4** *Let  $\mathcal{F}$  be complete for the reflections, under Assumption 4.1. Let  $\mathcal{I}$  be the index set ; then  $\Phi$  and  $\Psi$  satisfy, in addition to the properties of Proposition 4.1, the four extra properties :*

viii)

$$\begin{aligned} \Phi(n_1) \setminus \{n_2\} &\subset \mathcal{I}_{io} \cup \mathcal{I}_{ii} \cup \mathcal{I}_{ev1} \quad , \quad \Psi(n_1) \setminus \{n_1\} \subset \mathcal{I}_{oi} \cup \mathcal{I}_{ii} \cup \mathcal{I}_{ev2}, \\ \Phi(n_4) \setminus \{n_4\} &\subset \mathcal{I}_{io} \cup \mathcal{I}_{ii} \cup \mathcal{I}_{ev1} \quad , \quad \Psi(n_3) \setminus \{n_3\} \subset \mathcal{I}_{oi} \cup \mathcal{I}_{ii} \cup \mathcal{I}_{ev2}. \end{aligned}$$

ix) Let  $i \in \mathcal{I}_{ii} \cup \mathcal{I}_{ev1}$  and  $j \in \mathcal{I}_{ii} \cup \mathcal{I}_{ev2}$  then

$$\begin{aligned} i \in \Phi(n_1) &\implies \Psi(i) = \{i\} \quad , \quad j \in \Psi(n_1) \implies \Phi(j) = \{j\}, \\ i \in \Phi(n_4) &\implies \Psi(i) = \{i\} \quad , \quad j \in \Psi(n_3) \implies \Phi(j) = \{j\}. \end{aligned}$$

Let  $\underline{j} \in A_{\underline{a}} \setminus \{\underline{a}\}$ , we denote by  $\ell = (\ell_1, \dots, \ell_p)$  the sequence of type  $H$  linking  $\underline{j}$  to  $\underline{a}$ . Then, according to the parity of  $p$ , we have :

$x')$  If  $p \in 2\mathbb{N}$ , then  $\underline{j} \notin \mathcal{I}_{oi}$ . Moreover, if  $\underline{j} \in \mathcal{I}_{ev1} \cup \mathcal{I}_{ii}$  then  $\Psi(\underline{j}) = \{\underline{j}\}$ .

$x'')$  If  $p \in 2\mathbb{N} + 1$ , then  $\underline{j} \notin \mathcal{I}_{io}$ . Moreover, if  $\underline{j} \in \mathcal{I}_{ev2} \cup \mathcal{I}_{ii}$  then  $\Phi(\underline{j}) = \{\underline{j}\}$ .

Propositions 4.3 and 4.4 then permit to show that the set of indices  $\mathcal{I}$  has the "tree" structure depicted in the figure 2. To conclude this section we define the following matrices which will be useful during the construction of the geometric optics expansions.

<sup>4</sup>Compared to [Ben], for convenience, we have exchanged the indices  $n_2$  and  $n_4$ . It is due to the fact that here we are more interested in a precise description in terms of wave packets reflection than in [Ben].

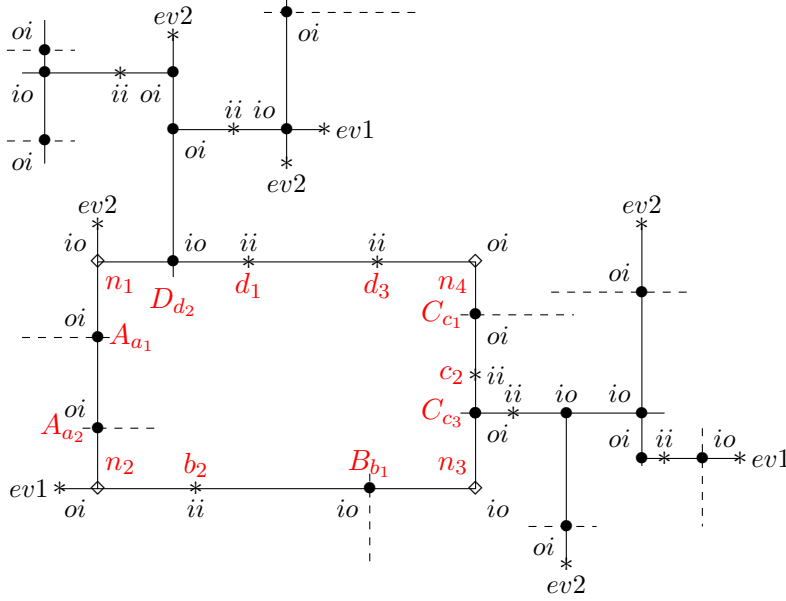


Figure 2: "Tree structure" of the frequency set  $\mathcal{F}$ .

**Definition 4.5** For  $j = 1, 2$  and  $k \in \mathfrak{R}_j$ . Let  $f^k = (\underline{\tau}, \xi_1^k, \xi_2^k)$  be the associated frequency with the index  $k$ . We define  $P_{ev,j}^k$  (resp.  $Q_{ev,j}^k$ ) the projector on  $E_j^{s,e}(i\underline{\tau}, \xi_{3-j}^k)$  (resp.  $A_j E_j^{s,e}(i\underline{\tau}, \xi_{3-j}^k)$ ) with respect to the decomposition (2) (resp. (3)).

For  $j = 1, 2$  and  $k \in \mathcal{I}_{os}$ . Let  $f^k = (\underline{\tau}, \xi_1^k, \xi_2^k)$  be the associated frequency with the index  $k$ . We define  $P_j^k$  (resp.  $Q_j^k$ ) the projector on  $\ker \mathcal{L}(f^k)$  (resp.  $A_j \ker \mathcal{L}(f^k)$ ) with respect to the decomposition (2) (resp. (3)). We also define  $R_j^k$  the partial-inverse of  $\mathcal{L}(f^k)$  uniquely determined by :

$$R_j^k \mathcal{L}(f^k) = I - P_j^k, \quad P_j^k R_j^k = R_j^k Q_j^k = 0. \quad (13)$$

## 5 Geometric optics expansions for selfinteracting problems in the $WR$ class.

The corner problem that we are now interested in reads :

$$\begin{cases} L(\partial)u^\varepsilon = 0, & \text{on } \Omega_T \\ B_1 u^\varepsilon|_{x_1=0} = g_M^\varepsilon, & \text{on } \partial\Omega_{1,T}, \\ B_2 u^\varepsilon|_{x_2=0} = 0, & \text{on } \partial\Omega_{2,T}, \\ u^\varepsilon|_{t \leq 0} = 0, \end{cases} \quad (14)$$

where the source term  $g_M^\varepsilon$  is given by :

$$g_M^\varepsilon(t, x_2) := \varepsilon^{M+1} e^{\frac{i}{\varepsilon}(\underline{\tau}t + \bar{\xi}_2 x_2)} g(t, x_2), \quad (15)$$

with  $\underline{\tau} > 0$  fixed,  $\bar{\xi}_2 < 0$  defined in (11) and where  $M \in \mathbb{N} \setminus \{0\}$  is fixed<sup>5</sup>. The function  $g$  is zero for negative time and has its support in space away from the corner for all positive times. More

<sup>5</sup>The case  $M = 0$  corresponds to the boundary value problem in the half space and will not be discussed here.

precisely, there exists  $\underline{y}_0 > 0$  such that :

$$\forall t \in \mathbb{R}_+, \forall x < \underline{y}_0, g(t, x) = 0.$$

The aim of this section is to construct the geometric optics expansion associated to the corner problem (14) up to a fixed time  $T > 0$ . More precisely, we want to show that, up to choose the support of the function  $g$  close enough to the corner, the geometric optics expansion suffers a number of amplifications that can be made arbitrarily large. Indeed, due to the presence of the loop, the amplification arising for the amplitude associated to  $n_1$  will be repeated at each cycle around the loop. More precisely, if one fix  $M$  in (15) and wants to show that the leading order in the WKB expansion is of order zero then it will be sufficient to choose  $g$  such that its support is close enough to the corner to ensure that we can make  $M$  complete circuits before the time  $T$ .

Let us give more details about these times of travel. We fix a point  $(0, \underline{y}^0) \in \partial\Omega_1$  satisfying  $g(0, \underline{y}^0) \neq 0$  and we draw the characteristic with group velocity  $v_{n_1}$  passing through  $(0, \underline{y}^0)$ . This characteristic intersects  $\partial\Omega_2$  in a point  $(\underline{x}^0, 0)$  after a certain time of travel  $t_1(\underline{y}^0)$ . Then we draw the characteristic with group velocity  $v_{n_2}$  passing through  $(\underline{x}^0, 0)$ . It intersects  $\partial\Omega_1$  in a point  $(0, \underline{y}^1)$  after a certain time  $t_2(\underline{y}^0)$ . We repeat the same process for the characteristics with group velocities  $v_{n_3}$  and  $v_{n_4}$ . Let  $(\underline{x}^1, 0)$ ,  $(0, \underline{y}^2)$  be the corresponding points of intersection and  $t_3(\underline{y}^0)$ ,  $t_4(\underline{y}^0)$  be the corresponding times of travel. An easy computation shows that :

$$\underline{y}^2 := \beta \underline{y}^0, \text{ and, } T_1(\underline{y}^0) := \sum_{j=1}^4 t_j(\underline{y}^0) = \alpha \underline{y}^0,$$

where  $\alpha$  and  $\beta$  are given by :

$$\alpha := -\frac{1}{v_{n_1,2}} \left( 1 - \frac{v_{n_1,1}}{v_{n_2,1}} + \frac{v_{n_1,1}v_{n_2,2}}{v_{n_2,1}v_{n_3,2}} - \frac{v_{n_1,1}v_{n_2,2}v_{n_3,1}}{v_{n_2,1}v_{n_3,2}v_{n_4,1}} \right), \quad (16)$$

$$\beta := \prod_{j=1}^4 \beta_j, \text{ with, } \beta_j := \begin{cases} \frac{v_{n_j,1}}{v_{n_j,2}}, & \text{if } j \text{ is odd,} \\ \frac{v_{n_j,2}}{v_{n_j,1}}, & \text{if } j \text{ is even,} \end{cases} \quad (17)$$

where the  $v_{n_j} := [v_{n_j,1} \ v_{n_j,2}]$  are the group velocities for the indices of the loop.

After  $m \in \mathbb{N}^*$  circuits around the loop the initial point  $(0, \underline{y}^0)$  comes back on  $\partial\Omega_1$  in  $(0, \underline{y}^{2m})$  after a total time of travel  $\underline{T}_m$ . It is easy to show that :

$$\underline{y}^{2m} := \beta^m \underline{y}^0, \text{ and, } \underline{T}_m = \underline{T}_m(\underline{y}^0) := \alpha \underline{y}^0 \sum_{k=0}^{m-1} \beta^k.$$

We point the fact that if  $\beta < 1$ , that is to say when the rays get closer and closer of the corner, the limit of the sum defining  $\underline{T}_m$  when  $m$  goes to infinity is finite. In other words, the rays reach the corner in a finite time  $T_{max} := \alpha \underline{y}^0 \sum_{k=0}^{\infty} \beta^k$ . In this particular situation we will thus assume that  $T < T_{max}$ .

It is clear that one can always choose  $\underline{y}^0$  in such a way that :

$$\underline{T}_M \leq T < \underline{T}_{M+1}.$$

As already mentionned in paragraph 3.1, if  $t \in [\underline{T}_k, \min(\underline{T}_{k+1}, T)[$  one expects that all the amplitudes of order less than  $M - k + 1$  are zero. Moreover one also expects that the traces on  $\partial\Omega_1$  of the amplitudes associated to the index  $n_1$  which are not identically zero vanish for  $x < \underline{y}^{2k}$ . Indeed, for time  $t \in [\underline{T}_k, \min(\underline{T}_{k+1}, T)[$ , the information has only made  $k$  complete circuits around the loop, so it can not have been amplified more than  $k$  times and can not have been transported under the threshold  $\underline{y}^{2k}$ . This observation motivates the following definition :

**Definition 5.1** Let us write, for  $n \in \mathbb{N}$ ,

$$u_{n,n_1}(t, x) := \nu_{n,n_1}(t, x)e + \check{v}_n(t, x),$$

where the vector  $e$  is defined in Definition 2.7 (this decomposition will be explained in paragraph 5.1) and where  $\check{v}_n \in \bigoplus_{j \in \Phi^*(n_1)} \ker \mathcal{L}(f^j)$ . Let  $M \in \mathbb{N}$  we distinguish the two following subcases :

if  $\beta \leq 1$ , we say that the sequence of amplitudes  $(u_{n,n_1})_{n \in \mathbb{N}}$  is in  $\mathcal{P}_{\leq}$  if for all  $n \in \mathbb{N}$ ,  $\nu_{n,n_1}|_{x_1=0}$  is in  $\mathcal{P}_{b,\leq}^{M-n}$ . Where the space  $\mathcal{P}_{b,\leq}^{M-n}$  is the set of functions  $\mu \in \mathcal{C}^\infty(]-\infty, T[, \mathcal{D}(\mathbb{R}_+))$  satisfying :

i) If  $M - n > 0$ , then for  $-\infty < t < \underline{T}_{M-n}$ ,  $\mu(t, x) = 0 \forall x \in \mathbb{R}_+$ , and for all  $k$  such that  $M - n \leq k \leq M$ , if  $\underline{T}_k \leq t < \min(T, \underline{T}_{k+1})$ ,  $\mu(t, x) = 0 \forall x < \underline{y}^{2k}$ .

ii) If  $M - n \leq 0$ , then  $\mu$  is zero for negative times, and for all  $k$  such that  $0 \leq k \leq M$ , if  $\underline{T}_k \leq t < \min(T, \underline{T}_{k+1})$ ,  $\mu(t, x) = 0 \forall x < \underline{y}^{2k}$ .

iii) If  $n \leq -1$  then  $\mathcal{P}_b^{M-n} = \{0\}$ .

If  $\beta > 1$ , we say that the sequence of amplitudes  $(u_{n,n_1})_{n \in \mathbb{N}}$  is in  $\mathcal{P}_{>}$  if for all  $n \in \mathbb{N}$ ,  $\nu_{n,n_1}|_{x_1=0}$  is in  $\mathcal{P}_{b,>}^{M-n}$ . Where the space  $\mathcal{P}_{b,>}^{M-n}$  is the set of functions  $\mu \in \mathcal{C}^\infty(]-\infty, T[, \mathcal{D}(\mathbb{R}_+))$  satisfying :

i) If  $M - n > 0$ , then for  $-\infty < t < \underline{T}_{M-n}$ ,  $\mu(t, x) = 0 \forall x \in \mathbb{R}_+$ , and for all  $k$  such that  $M - n \leq k \leq M$ , if  $\underline{T}_k \leq t < \min(T, \underline{T}_{k+1})$ ,  $\mu(t, x) = 0 \forall x < \underline{y}_0$ .

ii) If  $n \leq -1$  then  $\mathcal{P}_b^{M-n} = \{0\}$ .

The fact that we distinguish the profile space for  $\beta \leq 1$  and for  $\beta > 1$  is due to technical reasons which will be explained in paragraph 5.1.4. Notice that by definition we have  $\mathcal{P}_{b,\leq}^{M-n} = \mathcal{P}_{b,\leq}^0$  for all  $n > M$ .

We are now able to describe the expected space of profiles :

**Definition 5.2** When  $\beta \leq 1$ , a sequence  $(u_{n,k})_{n \in \mathbb{N}, k \in \mathcal{I}_{os}}$  is said to be in the space of profiles  $\mathbb{P}_{os,\leq}$  if  $(u_{n,n_1}) \in \mathcal{P}_{\leq}$  and if for all  $k \in \mathcal{I}_{os} \setminus \{n_1\}$ , for all  $n \in \mathbb{N}$ ,  $u_{n,k}$  lies in  $H^\infty(\Omega_T)$ .

When  $\beta > 1$ , a sequence  $(u_{n,k})_{n \in \mathbb{N}, k \in \mathcal{I}_{os}}$  is said to be in the space of profiles  $\mathbb{P}_{os,<}$  if  $(u_{n,n_1}) \in \mathcal{P}_{<}$  and if for all  $k \in \mathcal{I}_{os} \setminus \{n_1\}$ , for all  $n \in \mathbb{N}$ ,  $u_{n,k}$  lies in  $H^\infty(\Omega_T)$ .

For  $i = 1, 2$ , the set  $\mathbb{P}_{ev,i}$  of evanescent profiles for the side  $\partial\Omega_i$  is the set of  $U(t, x, X_i) \in H^\infty(\Omega_T \times \mathbb{R}_+)$  for which there exists a positive  $\delta$  such that  $e^{\delta X_i} U(t, x, X_i) \in H^\infty(\Omega_T \times \mathbb{R}_+)$ .

As already mentioned, we will have to consider three kinds of phases, the oscillating ones, the evanescent ones for the side  $\partial\Omega_1$  and the evanescent ones for the side  $\partial\Omega_2$ . These will be denoted by :

$$\begin{aligned} \varphi_k(t, x) &:= \langle (t, x), f_k \rangle, \quad f_k \in \mathcal{F}_{os}, \\ \psi_{k,1}(t, x_2) &:= \langle (t, 0, x_2), f_k \rangle, \quad f_k \in \mathcal{F}_{ev1} \cup \mathcal{F}_{os}, \\ \forall k \in \mathcal{I}_{os}, \psi_{k,2}(t, x_1) &:= \langle (t, x_1, 0), f_k \rangle, \quad f_k \in \mathcal{F}_{ev2} \cup \mathcal{F}_{os}. \end{aligned}$$

Once the expected phases are defined, we postulate the ansatz :

$$\begin{aligned} u^\varepsilon(t, x) &\sim \sum_{n \geq 0} \sum_{k \in \mathcal{I}_{os}} \varepsilon^n e^{\frac{i}{\varepsilon} \varphi_k(t, x)} u_{n,k}(t, x) \\ &+ \sum_{n \geq 0} \sum_{k \in \mathfrak{R}_1} \varepsilon^n e^{\frac{i}{\varepsilon} \psi_{k,1}(t, x_2)} U_{n,k,1} \left( t, x, \frac{x_1}{\varepsilon} \right) + \sum_{n \geq 0} \sum_{k \in \mathfrak{R}_2} \varepsilon^n e^{\frac{i}{\varepsilon} \psi_{k,2}(t, x_1)} U_{n,k,2} \left( t, x, \frac{x_2}{\varepsilon} \right), \end{aligned} \quad (18)$$

where  $(u_{n,k})_{n \in \mathbb{N}, k \in \mathcal{I}_{os}} \in \mathbb{P}_{os, \leq}$  (resp.  $\mathbb{P}_{os, <}$ ) for  $\beta \leq 1$  (resp.  $\beta > 1$ ), and for all  $n \in \mathbb{N}, \forall k \in \mathfrak{R}_1$  (resp.  $\mathfrak{R}_2$ ),  $U_{n,k,1}$  (resp.  $U_{n,k,2}$ ) is in  $\mathbb{P}_{ev,1}$  (resp.  $\mathbb{P}_{ev,2}$ ).

Plugging the ansatz (18) into the evolution equation of (14) lead us to the cascade of equations

:

$$\begin{cases} \mathcal{L}(d\varphi_k)u_{0,k} = 0, & \forall k \in \mathcal{I}_{os}, \\ i\mathcal{L}(d\varphi_k)u_{n+1,k} + L(\partial)u_{n,k} = 0, & \forall n \in \mathbb{N}, \forall k \in \mathcal{I}_{os}, \\ L_k(\partial_{X_1})U_{0,k,1} = 0, & \forall k \in \mathfrak{R}_1, \\ L_k(\partial_{X_1})U_{n+1,k,1} + L(\partial)U_{n,k,1} = 0, & \forall n \in \mathbb{N}, \forall k \in \mathfrak{R}_1, \\ L_k(\partial_{X_2})U_{0,k,2} = 0, & \forall k \in \mathfrak{R}_2, \\ L_k(\partial_{X_2})U_{n+1,k,2} + L(\partial)U_{n,k,2} = 0, & n \in \mathbb{N}, \forall k \in \mathfrak{R}_2, \end{cases} \quad (19)$$

where the operators of derivation in the fast variables  $L_k(\partial_{X_1})$  and  $L_k(\partial_{X_2})$  are defined by :

$$L_k(\partial_{X_1}) := A_1(\partial_{X_1} - \mathcal{A}_1(\underline{\tau}, \xi_2^k)) \text{ for } k \in \mathfrak{R}_1, \quad (20)$$

$$L_k(\partial_{X_2}) := A_2(\partial_{X_2} - \mathcal{A}_2(\underline{\tau}, \xi_1^k)) \text{ for } k \in \mathfrak{R}_2. \quad (21)$$

Then, plugging the ansatz (18) in the boundary conditions on  $\partial\Omega_1$  and on  $\partial\Omega_2$ , give :

$$\begin{cases} B_1 \left[ \sum_{j \in \Phi(n_1)} u_{n,j} \right]_{|x_1=0} = \delta_{n, M+1} g, & \forall n \in \mathbb{N} \\ B_1 \left[ \sum_{j \in \Phi(k) \cap \mathcal{I}_{os}} u_{n,j} + U_{n,k,1}|_{x_1=0} \right]_{|x_1=0} = 0, & n \in \mathbb{N}, \forall k \in (\mathcal{B}_1 \setminus \{n_1\}) \cap \mathfrak{R}_1, \\ B_1 \left[ \sum_{j \in \Phi(k)} u_{n,j} \right]_{|x_1=0} = 0, & n \in \mathbb{N}, \forall k \notin (\mathcal{B}_1 \setminus \{n_1\}) \cap \mathfrak{R}_1, \\ B_2 \left[ \sum_{j \in \Psi(k) \cap \mathcal{I}_{os}} u_{n,j} + U_{n,k,2}|_{x_2=0} \right]_{|x_2=0} = 0, & n \in \mathbb{N}, \forall k \in \mathfrak{R}_2, \\ B_2 \left[ \sum_{j \in \Psi(k)} u_{n,j} \right]_{|x_2=0} = 0, & n \in \mathbb{N}, \forall k \in \mathcal{B}_2 \setminus \mathfrak{R}_2, \end{cases} \quad (22)$$

where  $\delta_{n,p}$  denotes the Kronecker symbol.

Finally, plugging the ansatz (18) into the initial condition of (14), give :

$$\forall n \in \mathbb{N}, \begin{cases} u_{n,k}|_{t=0} = 0, & \forall k \in \mathcal{I}_{os}, \\ U_{n,k,1}|_{t=0} = 0, & \forall k \in \mathfrak{R}_1, \\ U_{n,k,2}|_{t=0} = 0, & \forall k \in \mathfrak{R}_2. \end{cases} \quad (23)$$

Thanks to the cascade of equations (19)-(22) and (23) we are now able to state our made result about the construction of the geometric optics expansions :

**Theorem 5.1** *Let  $T > 0$  and assume that the corner problem (14) satisfies Assumptions 2.1-2.2-4.1 and 4.2. Then :*

- *If  $\beta > 1$ , then for all  $M \in \mathbb{N}$  the corner problem (14) admits a WKB expansion under the form (18). More precisely there exists sequences of functions  $(u_{n,k})_{n \in \mathbb{N}, k \in \mathcal{I}_{os}} \in \mathbb{P}_{os, >}$ ,  $(U_{n,k,1}) \in \mathbb{P}_{ev,1}$ , and  $(U_{n,k,2}) \in \mathbb{P}_{ev,2}$  satisfying the cascades of equations (19)-(22) and (23). Moreover one can always choose  $g$  in such a way that the leading order in (18) is not identically zero.*

- *If  $\beta \leq 1$ , assume that  $T < T_{max}$  if  $\beta < 1$ , then for all  $M \in \mathbb{N}$  the corner problem (14) admits a WKB expansion under the form (18). More precisely there exists sequences of functions  $(u_{n,k})_{n \in \mathbb{N}, k \in \mathcal{I}_{os}} \in \mathbb{P}_{os, \leq}$ ,  $(U_{n,k,1}) \in \mathbb{P}_{ev,1}$ , and  $(U_{n,k,2}) \in \mathbb{P}_{ev,2}$  satisfying the cascades of equations*

(19)-(22) and (23). Moreover one can always choose  $g$  in such a way that the leading order in (18) is not identically zero.

The question is now to solve the cascades of equations (19)-(22) and (23). More precisely, to solve the cascades, we are looking for an order of resolution of the different equations and an equation which can be solved before all the other in view to initialize the resolution.

In [Ben] it is shown that to construct any amplitude in one of the "trees" (that is the sets composing the partition (12) of  $\mathcal{S}$  depicted in Figure 2) it is in fact sufficient, thanks to the uniqueness of the type  $V$  sequence linking any index of the "tree" to its root (see Proposition 4.3) and the uniform Kreiss-Lopatinskii condition, to know the amplitude associated to the root. Then it is shown that to know the amplitudes associated to the roots it is sufficient to know the amplitudes associated to the loop indices. Thus the determination of the loop's amplitudes was in [Ben] the construction needed to initialize the resolution of the cascade of equations.

The first equation of the cascade (19) implies that we have the well-known polarization condition for the oscillating amplitudes  $u_{0,k}$ , in particular for  $k = n_1$ , we have  $u_{0,n_1} \in \ker \mathcal{L}(d\varphi_{n_1})$ , in other words we can write :

$$P_1^{n_1} u_{0,n_1} = u_{0,n_1}. \quad (24)$$

But, thanks to Assumption 2.1,  $\ker \mathcal{L}(d\varphi_{n_1})$  is one dimensional. Assumption 4.2 then permits to write :

$$u_{0,n_1}(t, x) = \nu_{0,n_1}(t, x)e, \quad (25)$$

for some unknown scalar function  $\nu_{0,n_1}$  and where  $e$  is the vector defined in 2.7.<sup>6</sup> For other indices  $k \in \mathcal{I}_{os}$  we can always write :

$$u_{0,k} = \nu_{0,k}e_k, \quad (26)$$

where  $\nu_{0,k}$  is a scalar function and where  $e_k$  is a generator of  $\ker \mathcal{L}(d\varphi_k)$ .

More generally for any  $n$  let us introduce the following decomposition :

$$P_1^{n_1} u_{n,n_1}(t, x) = \nu_n(t, x)e, \quad (27)$$

where  $e$  is defined in Definition 2.7.

Next let us study the second equation of the cascade (19) written for  $n = 0$  and  $k = n_1$ . If we compose this equation by  $Q_1^{n_1}$  and use the fact that  $\ker Q_1^{n_1} = \text{Ran} \mathcal{L}(d\varphi_{n_1})$  thanks to the polarization condition (24) we obtain that  $u_{0,n_1}$  satisfies the equation :

$$Q_1^{n_1} L(\partial) P_1^{n_1} u_{0,n_1} = 0.$$

From Lax's lemma [Lax57], we have :

$$Q_1^{n_1} L(\partial) P_1^{n_1} u_{0,n_1} = (\partial_t + v_{n_1} \cdot \nabla_x) \nu_{0,n_1} = 0, \quad (28)$$

---

<sup>6</sup>Let us remark that (25) is not true anymore if the operator  $L(\partial)$  is hyperbolic with constant multiplicity. In that framework, (25) has to be replaced by a decomposition reading :

$$u_{0,n_1}(t, x) = \nu_{0,n_1}(t, x)e + \tilde{v}(t, x),$$

where  $\tilde{v}$  lies in some subspace of  $E_1^s(i_{\mathcal{T}}, \bar{\xi}_2)$  where we can apply Kreiss-Lopatinskii condition. However, it seems that in that case, one can always determine  $\tilde{v}$  by reiterating the arguments of [Ben] which lead to ask for the invertibility of an operator reading under the form  $(I - \mathbb{T})$ .

where we used the fact that  $Q_1^{n_1}$  induces an isomorphism from  $\text{Ran}P_1^{n_1}$  to  $\text{Ran}Q_1^{n_1}$ . Using the fact that the group velocity  $v_{n_1}$  is incoming-outgoing, to solve equation (28) we only need to determine the trace of  $\nu_{0,n_1}$  on  $\{x_1 = 0\}$ .

However, the degeneracy of the uniform Kreiss-Lopatinskii condition prevents to determine this trace by the easy classical way. Indeed, the only equation where the trace of  $\nu_{0,n_1}$  on  $\{x_1 = 0\}$  seems to appear is the boundary condition (22) written for  $n = 0$  that is :

$$B_1 \left[ \sum_{j \in \Phi(n_1)} u_{0,j} \right]_{|x_1=0} = 0.$$

From (25) and the fact that, by definition  $e \in \ker B_1$ , the boundary condition reads :

$$B_1 \left[ \sum_{j \in \Phi^*(n_1)} u_{0,j} \right]_{|x_1=0} = 0,$$

where we recall that  $\Phi^*(n_1)$  stands for  $\Phi(n_1) \setminus \{n_1\}$ . As a consequence this equation does not give any information about  $\nu_{0,n_1}|_{x_1=0}$ . So it seems that we shall find another way to determine  $\nu_{0,n_1}|_{x_1=0}$ . To do that, we will start by using the method of [CG10] which has been introduced to construct the geometric optics expansion for the initial boundary value problem in the half space and we will see how to adapt this method.

Before to recall the main ideas of the method of determination of  $\nu_{0,n_1}|_{x_1=0}$  by [CG10], let us remark the following fact. If one assumes that  $\nu_{0,n_1}|_{x_1=0}$  is known and write, once again in view to simplify the notations :

$$\nu_{0,n_1}|_{x_1=0}(t, x_2) := \mu_0(t, x_2), \quad (29)$$

then is it easy to determine the other amplitudes  $u_{0,k}$ , for  $k = n_2, \dots, n_4$ . Indeed, the transport equation determining  $\nu_{0,n_1}$  reads :

$$\begin{cases} (\partial_t + v_{n_1} \cdot \nabla_x) \nu_{0,n_1} = 0, \text{ on } \Omega_T, \\ \nu_{0,n_1}|_{x_1=0} = \mu_0, \text{ on } \partial\Omega_{1,T}, \\ \nu_{0,n_1}|_{t \leq 0} = 0. \end{cases}$$

Integrating along the characteristics we obtain that :

$$\nu_{0,n_1}(t, x_1, x_2) = \mu_0 \left( t - \frac{1}{v_{n_1,1}} x_1, x_2 - \beta_1^{-1} x_1 \right). \quad (30)$$

In particular, the trace of  $u_{0,n_1}$  on  $\{x_2 = 0\}$  is given by

$$u_{0,n_1}|_{x_2=0} = \nu_{0,n_1}(t, x_1, 0)e,$$

so, using the uniform Kreiss-Lopatinskii condition in the boundary condition (22) written for  $n = 0$  and  $k = n_2$  permit us to determine the trace on  $\{x_2 = 0\}$  of  $\nu_{0,n_2}$ . Integrating along the characteristics we obtain :

$$\nu_{0,n_2}(t, x_1, x_2) = -\mu_0(\mathbf{t}_2(t, x_1, x_2), \mathbf{x}_2(x_1, x_2)) \mathcal{L}_{n_2} e, \quad (31)$$

where

$$\mathbf{t}_2(t, x_1, x_2) := t - \frac{1}{v_{n_1,1}} x_1 - \left[ \frac{1}{v_{n_2,2}} - \frac{\beta_2^{-1}}{v_{n_1,1}} \right] x_2, \quad \mathbf{x}_2(x_1, x_2) := -\beta_1^{-1}(x_1 - \beta_2^{-1} x_2), \quad (32)$$



and

$$\mathcal{S}_{n_2} := P_2^{n_2} \phi^{n_2} B_2. \quad (33)$$

We can thus repeat exactly the same reasoning, applied to the amplitude  $u_{0,n_3}$  and then for the amplitude  $u_{0,n_4}$ , to obtain their values in terms of the unknown trace  $\mu_0$ . More precisely it follows that :

$$\nu_{0,n_3}(t, x_1, x_2) = \mu_0(\mathbf{t}_3(t, x_1, x_2), \mathbf{x}_3(x_1, x_2)) \mathcal{S}_{n_3} e, \quad (34)$$

$$\nu_{0,n_4}(t, x_1, x_2) = -\mu_0(\mathbf{t}_4(t, x_1, x_2), \mathbf{x}_4(x_1, x_2)) \mathcal{S}_{n_4} e, \quad (35)$$

where we set for  $\nu_{0,n_3}$  :

$$\mathcal{S}_{n_3} := P_1^{n_3} \phi^{n_3} B_1 \mathcal{S}_{n_2}, \quad (36)$$

$$\mathbf{t}_3(t, x_1, x_2) := t - Ax_1 - \left[ \frac{1}{v_{n_2,2}} - \frac{\beta_2^{-1}}{v_{n_1,1}} \right] x_2, \text{ and, } \mathbf{x}_3(x_1, x_2) := \prod_{j=1}^2 \beta_j^{-1} (x_2 - \beta_3^{-1} x_1), \quad (37)$$

with  $A$  a non meaningfull parameter introduced to simplify the notations. It is precisely given by :

$$A := \frac{1}{v_{n_3,1}} - \beta_3^{-1} \left( \frac{1}{v_{n_2,2}} - \frac{\beta_2^{-1}}{v_{n_1,1}} \right). \quad (38)$$

And for  $\nu_{0,n_4}$  :

$$\mathcal{S}_{n_4} := P_2^{n_4} \phi^{n_4} B_2 \mathcal{S}_{n_3}, \quad (39)$$

$$\mathbf{t}_4(t, x_1, x_2) := t - Ax_1 - \left[ \frac{1}{v_{n_4,2}} - \beta_4^{-1} A \right] x_2, \text{ and, } \mathbf{x}_4(x_1, x_2) := - \prod_{j=1}^3 \beta_j^{-1} (x_1 - \beta_4^{-1} x_2). \quad (40)$$

An important observation for what follows is to remark that the trace  $\nu_{0,n_4|_{x_1=0}}$  depends of the particular values,  $\mathbf{t}_4(t, 0, x_2)$  and  $\mathbf{x}_4(0, x_2)$ . An easy computation shows that the constant in front of  $x_2$  in  $\mathbf{t}_4(t, 0, x_2)$  can in fact be expressed in terms of the parameters  $\alpha$  and  $\beta$ , introduced in (16) and (17) and which encode the time needed to make one complete circuit around the loop. More precisely, we have :

$$\underline{\mathbf{t}}(t, x_2) := \mathbf{t}_4(t, 0, x_2) = t - \alpha\beta^{-1}x_2, \text{ and, } \underline{\mathbf{x}}(x_2) := \mathbf{x}_4(0, x_2) = \beta^{-1}x_2. \quad (41)$$

Let us also notice, because it will be important in paragraph 5.1.5, that the knowledge of  $u_{0,n_4}$  allow us to express all the amplitudes in  $\Phi^*(n_1)$  in terms of  $\mu_0$ . Indeed, Proposition 4.1 implies that these amplitudes (except  $n_4$ ) are in  $\mathcal{S}_{ii} \cup \mathcal{S}_{io}$ . So, let  $\underline{i} \in \Phi^*(n_1) \setminus \{n_4\}$  to determine  $u_{0,\underline{i}}$  for  $\underline{i} \in \mathcal{S}_{io}$  (resp.  $\underline{i} \in \mathcal{S}_{ii}$ ) we have to solve the transport equation :

$$\begin{cases} (\partial_t + v_{\underline{i}} \cdot \nabla_x) u_{0,\underline{i}} = 0, \\ B_1 \left[ \sum_{j \in \Phi(n_1) \setminus \{n_4\}} u_{0,j} \right]_{|x_1=0} = -B_1 u_{0,n_4|_{x_1=0}}, \\ u_{0,\underline{i}|_{t \leq 0}} = 0, \end{cases} \quad (42)$$

$$\left( \text{resp. } \begin{cases} (\partial_t + v_{\underline{i}} \cdot \nabla_x) u_{0,\underline{i}} = 0, \\ B_1 \left[ \sum_{j \in \Phi(n_1) \setminus \{n_4\}} u_{0,j} \right]_{|x_1=0} = -B_1 u_{0,n_4|_{x_1=0}}, \\ B_2 \left[ \sum_{j \in \Psi(n_2)} u_{0,j} \right]_{|x_2=0} = 0, \\ u_{0,\underline{i}|_{t \leq 0}} = 0, \end{cases} \right) \quad (43)$$

However, by assumption, the index  $\underline{i}$  is associated with a frequency for which the uniform Kreiss-Lopatinskii condition holds, so the boundary condition(s) of (42) (resp.(42)-(43)) can be written under the form :

$$u_{0,\underline{i}|_{x_1=0}} = -P_1^{\underline{i}} \phi_1^{\underline{i}} B_1 u_{0,n_4|_{x_1=0}}, \left( \text{resp. } \begin{cases} u_{0,\underline{i}|_{x_1=0}} = -P_1^{\underline{i}} \phi_1^{\underline{i}} B_1 u_{0,n_4|_{x_1=0}}, \\ u_{0,\underline{i}|_{x_2=0}} = 0. \end{cases} \right)$$

Using the expression of  $u_{0,n_4|_{x_1=0}}$  and integrating along the characteristics we obtain that :

$$u_{0,\underline{i}}(t, x) = \mu_0 \left( \underline{t} - \frac{1}{v_{i,1}} x_1, \underline{x} - \frac{v_{i,2}}{v_{i,1}} x_1 \right) (\mathcal{S}^{n_4} e) P_1^i \phi_1^i B_1 e_4. \quad (44)$$

### 5.1 Initialization of the resolution, determination of the loop's indices.

As already mentioned in the previous paragraph to determine  $\mu_0$  we will adapt the method of [CG10].

Let us study the boundary condition for the amplitude  $u_{1,n_1}$  that is (22) written for  $n = 1$  :

$$B_1 \left[ \sum_{j \in \Phi(n_1) \setminus \{n_4\}} P_1^j u_{1,j} \right]_{|x_1=0} = \delta_{1,M} g - B_1 u_{1,n_4} - B_1 \left[ \sum_{j \in \Phi(n_1) \setminus \{n_4\}} (I - P_1^j) u_{1,j} \right]_{|x_1=0}. \quad (45)$$

Thanks to Proposition 4.1, we know that  $\Phi(n_1) \setminus \{n_4\}$  is included in  $\mathcal{S}_{io} \cup \mathcal{S}_{ii}$ , so the left hand side term is in  $B_1 E_1^s(i_{\underline{1}}, \bar{\xi}_2)$ . By definition of  $b$  (see Definition 2.7) composing by  $b$  in the left makes this term vanish. We thus have :

$$b \cdot B_1 \left[ \sum_{j \in \Phi(n_1) \setminus \{n_4\}} (I - P_1^j) u_{1,j} \right]_{|x_1=0} = \delta_{1,M} \tilde{g} - b \cdot B_1 u_{1,n_4}, \quad (46)$$

where  $\tilde{g} := b \cdot g$ .

To make this equation more explicit in term of  $\mu_0$ , we use the cascade of equation (19) written for  $n = 0$ . Composing by the partial inverse  $R_1^k$  defined in Definition 4.5, lead us to the relation :

$$\forall j \in \Phi(n_1) \setminus \{n_4\}, (I - P_1^j) u_{1,j} = i R_1^j L(\partial) u_{0,j} = i R_1^j L(\partial) P_1^j u_{0,j},$$

where we used the polarization condition for  $u_{0,j}$ . We thus can write (46) under the form :

$$b \cdot B_1 \left[ \sum_{j \in \Phi(n_1) \setminus \{n_4\}} R_1^j L(\partial) P_1^j u_{0,j} \right]_{|x_1=0} = i \left( -\tilde{g} + b \cdot B_1 u_{1,n_4|_{x_1=0}} \right). \quad (47)$$

Let us recall the following Proposition due to [CG10] :

**Proposition 5.1** ([CG10], Proposition 2) *Let  $P_1^j$ ,  $Q_1^j$  and  $R_1^j$  be defined in Definition 4.5 then we have  $R_1^j A_1 P_1^j = 0$ .*

*Let the vector  $b$  be defined in Definition 2.7. Then there exists a nonzero real number  $\kappa$  such that the following equation holds :*

$$b \cdot B_1 \sum_{j \in \Phi(n_1) \setminus \{n_4\}} R_1^j L(\partial) P_1^j = \kappa \left( \Theta(i_{\underline{1}}, \bar{\xi}_2) \partial_t + \partial_{\xi_2} \Theta(i_{\underline{1}}, \bar{\xi}_2) \partial_2 \right),$$

where  $\Theta$  is defined in Definition 2.5. Moreover,  $\partial_t \Theta(i_{\underline{1}}, \bar{\xi}_2) = 1$  and  $\partial_{\xi_2} \Theta(i_{\underline{1}}, \bar{\xi}_2) = -\frac{\tau}{\xi_2} := \underline{c}$  and  $\kappa := b \cdot B_1 e$ .

**Proof :** We refer to [CG10] for a complete proof. We will here just show that we have the equality  $\partial_{\xi_2} \Theta(i\mathcal{I}, \bar{\xi}_2) = -\frac{\tau}{\xi_2}$ . This equality was already showed in [[CG10], Lemma 7], but since it is important for our purpose, let us give a proof for a sake of completeness.

Necessarily  $\bar{\xi}_2 \neq 0$  otherwise the Kreiss-Lopatinskii condition will break down for  $\gamma > 0$  because it would say that the boundary condition on  $\partial\Omega_1$  does not satisfy the weak Kreiss-Lopatinskii condition. We then use the fact that  $d = 2$  and the relation  $R_1^j A_1 P_1^j = 0$ . First we have :

$$\forall j \in \Phi(n_1) \setminus \{n_4\}, \mathcal{L}(d\varphi_j) P_1^j = \tau P_1^j + \xi_1^j A_1 P_1^j + \bar{\xi}_2 A_2 P_1^j = 0,$$

we compose by  $R_1^j$ , the term  $R_1^j A_1 P_1^j$  disappears. Then we sum over  $j \in \Phi(n_1) \setminus \{n_4\}$  and multiply by  $b \cdot B_1$  to obtain :

$$b \cdot B_1 \sum_{j \in \Phi(n_1) \setminus \{n_4\}} R_1^j A_2 P_1^j = -\frac{\tau}{\xi_2} b \cdot B_1 \sum_{j \in \Phi(n_1) \setminus \{n_4\}} R_1^j P_1^j.$$

□

As a consequence of this Proposition, (47) can be expressed under the form :

$$\left( \partial_t \mu_0 - \frac{\tau}{\xi_2} \mu_0 \right) = \frac{i}{\kappa} \left( -\tilde{g} \delta_{1,M} + b \cdot B_1 P_2^{n_4} u_{1,n_4|_{x_1=0}} + b \cdot B_1 (I - P_2^{n_4}) u_{1,n_4|_{x_1=0}} \right). \quad (48)$$

Thus to determine the unknown trace  $\mu_0$ , we want to solve the same transport equation as in [CG10]. But, in the analysis of Coulombel and Guès the amplitude  $u_{1,n_4}$  which acts like a source term in (46) could be determinated independantly of  $\mu_0$ . Indeed, for the geometry of the half space  $u_{1,n_4}$  is an outgoing amplitude so it satisfies a transport equation which does not require boundary condition for its resolution (see [CG10] for more details).

For the quarter space geometry it is not true anymore because  $u_{1,n_4}$  depends on  $u_{1,n_3}$ , which depends on  $u_{1,n_2}$  and so on. However, equation (46) meets with the intuition that we gave in paragraph 3 because this equation says that the amplification of order zero is "turned on" by the outgoing (for the side  $\partial\Omega_1$ ) mode  $u_{1,n_4}$ . Moreover, we mentionned at the end of the previous paragraph that amplitudes  $u_{0,j}$  for  $j = n_1, \dots, n_4$  can be expressed in term of the unknown trace  $\mu_0$ .

So our purpose is now to express the "source term"  $u_{1,n_4}$  in terms of the unknown trace  $\mu_0$ . In view to do this, we will show in a first time that its unpolarized part  $(I - P_2^{n_4}) u_{1,n_4}$  can be expressed in term of  $\mu_0$ . In a second time we will show that its polarized part  $P_2^{n_4} u_{1,n_4}$  can be expressed in terms of  $\mu_0$  and a new unknown trace  $\mu_1$  (which will be the unknown part of the polarized part of the trace of  $u_{1,n_1}$  on the side  $\partial\Omega_1$ ). The determination of the dependency of  $(I - P_2^{n_4}) u_{1,n_4}$  in term of  $\mu_0$  is made in the following paragraph.

### 5.1.1 Unpolarized part of the terms of order one.

In a classical way, after composition of the second equation of 19 (written for  $n = 0$  and  $k = n_j$ ,  $j = 1, \dots, 4$ ) by the pseudo-inverse  $R_1^{n_j}$  introduced in Definition 4.5, we obtain that :

$$(I - P_1^{n_j}) u_{1,n_j} = i R_1^{n_j} L(\partial) u_{0,n_j}, \text{ if } j \text{ is odd, } (I - P_2^{n_j}) u_{1,n_j} = i R_2^{n_j} L(\partial) u_{0,n_j}, \text{ if } j \text{ is even,} \quad (49)$$

where we used the fact that by definition  $R_1^{n_j} \mathcal{L}(d\varphi_{n_j}) = I - P_1^{n_j}$ .

Let us recall the following lemma, already used in [CG10] :

**Lemma 5.1** *We have the following equalities :*

$$R_1^{n_1} A_1 P_1^{n_1} = R_1^{n_3} A_1 P_1^{n_3} = 0, \text{ and, } R_2^{n_2} A_2 P_2^{n_2} = R_2^{n_4} A_2 P_2^{n_4} = 0.$$

Thanks to this lemma we can compute precisely the values of the unpolarized part of the amplitudes of order one. Indeed for  $(I - P_1^{n_1})u_{1,n_1}$  we thus have :

$$\begin{aligned} (I - P_1^{n_1})u_{1,n_1} &= -R_1^{n_1} [\partial_t + A_1\partial_1 + A_2\partial_2] \mu_0 \left( t - \frac{1}{v_{n_1,1}}x_1, x_2 - \beta_1^{-1}x_1 \right) e, \\ &= -[(\partial_t\mu_0)R_1^{n_1}e + (\partial_2\mu_0)R_1^{n_1}A_2e] \left( t - \frac{1}{v_{n_1,1}}x_1, x_2 - \beta_1^{-1}x_1 \right), \end{aligned} \quad (50)$$

where we used the fact that  $P_1^{n_1}e = e$  and lemma 5.1.

But as  $e \in \ker \mathcal{L}(d\varphi_{n_1})$ , we have  $(\underline{\tau} + \underline{\xi}_1 A_1 + \bar{\xi}_2 A_2)e = 0$ . We compose by  $R_1^{n_1}$ , and we use Lemma 5.1 to show that :

$$R_1^{n_1} A_2 e = -\frac{\underline{\tau}}{\bar{\xi}_2} R_1^{n_1} e. \quad (51)$$

This relation permits to reformulate (50) under the form :

$$(I - P_1^{n_1})u_{1,n_1} = -\left[ \partial_t \mu_0 - \frac{\underline{\tau}}{\bar{\xi}_2} \partial_2 \mu_0 \right] \left( t - \frac{1}{v_{n_1,1}}x_1, x_2 - \beta_1^{-1}x_1 \right) R_{n_1} e. \quad (52)$$

For simplicity let us introduce the notation :

$$\mathcal{T} := \partial_t - \frac{\underline{\tau}}{\bar{\xi}_2} \partial_2. \quad (53)$$

**Remark** The fact that one restricts to a corner problem (1) with only two dimensional variables is used in a non trivial way to establish relation (51) which allows us to reformulate (50) under the form (52). We will see in a moment why this reformulation is so important in the proof.

We do not know if the restriction  $d = 2$  is really necessary, however it has the advantage to make all the following computations much more simpler.

The same computation can be repeated to determine the unpolarized part of the amplitudes  $u_{1,n_j}$ , for  $j = 2, \dots, 4$ . Unfortunately, since our aim is to determine the exact value of the trace  $u_{1,n_4}|_{x_1=0}$ , we also need the exact values of the unpolarized part of the amplitudes  $u_{1,n_j}$ , for  $j = 2, \dots, 4$ . After some computations, we find :

$$(I - P_{n_2})u_{1,n_2} = \left[ \left( 1 + \frac{\underline{\tau}}{v_{n_1,1}\bar{\xi}_1} \right) \partial_t \mu_0 + \frac{\underline{\tau}}{\bar{\xi}_1} \partial_2 \mu_0 \right] (\mathbf{t}_2, \mathbf{x}_2) R_{n_2} \mathcal{S}_{n_2} e, \quad (54)$$

$$(I - P_{n_3})u_{1,n_3} = -\left[ \left( 1 - \frac{\underline{\tau}}{v_{n_2,2}\bar{\xi}_2} \left( 1 - \frac{v_{n_2,1}}{v_{n_1,1}} \right) \right) \partial_t \mu_0 + \frac{\underline{\tau}}{\bar{\xi}_2} \beta_2^{-1} \beta_1^{-1} \partial_2 \mu_0 \right] (\mathbf{t}_3, \mathbf{x}_3) R_{n_3} \mathcal{S}_{n_3} e \quad (55)$$

$$(I - P_{n_4})u_{1,n_4} = \left[ \left( 1 + \frac{\underline{\tau}}{\bar{\xi}_1} A \right) \partial_t \mu_0 + \frac{\underline{\tau}}{\bar{\xi}_1} \prod_{j=1}^3 \beta_j^{-1} \partial_2 \mu_0 \right] (\mathbf{t}_4, \mathbf{x}_4) R_{n_4} \mathcal{S}_{n_4} e, \quad (56)$$

At this step of the proof, the term depending on  $(I - P_2^{n_4})u_{1,n_4}$  which appears in the right hand side of (48) has been express in terms of the function  $\mu_0$ . So we just have to express the term  $P_2^{n_4}u_{1,n_4}$ . In paragraph 5.1.3 we will see how the knowledge of the unpolarized part of the amplitudes of order one enable us to determine the polarized part of the amplitudes of order one. However, before to do this determination, it is useful (and it will a lot simplify the computations) to express the unpolarized part of the amplitude of order one in terms of the transport operator along the boundary. Moreover, as we will see in paragraph 5.1.4, this reformulation will also be essential for the resolution of the initialization equation (48).

### 5.1.2 Reformulation of equations (54)-(55) and (56).

The following lemma which is just an algebraic property based on a simple computation is however fundamental for our analysis. Indeed it will permit to reformulate equation (48) in a much more pleasant form. The fact that we are able to reformulate (48) in a particular form (more precisely under the form  $(I - \tilde{T})(\mathcal{T}\mu_0)$  see (90) for more details) is not anodyne at all when one wants to solve (48) in view to determine  $\mu_0$ , because it will permit to express  $\mathcal{T}\mu_0$  as a sum of iterations of  $\tilde{T}$  corresponding to the number of complete circuits that have been made.

**Lemma 5.2** *If  $d = 2$ , we have the following equalities :*

$$1 + \frac{\tau}{\xi_1 v_{n_1,1}} = -\frac{\bar{\xi}_2}{\xi_1} \beta_1^{-1}, \quad (57)$$

$$\left(1 - \frac{\tau}{v_{n_2,2} \xi_2} \left(1 - \frac{v_{n_2,1}}{v_{n_1,1}}\right)\right) = \frac{\bar{\xi}_2}{\xi_2} \beta_1^{-1} \beta_2^{-1}, \quad (58)$$

$$1 + \frac{\tau}{\xi_1} A = -\frac{\bar{\xi}_2}{\xi_1} \beta_1^{-1} \beta_2^{-1} \beta_3^{-1}, \quad (59)$$

where we recall that the parameter  $A$  appearing in (59) is defined in (38).

**Proof :** We will only demonstrate (59), the proof of the two other equalities are simpler and follow exactly the same kind of computations.

First let us develop :

$$\begin{aligned} 1 + \frac{\tau}{\xi_1} A &= \frac{1}{\xi_1} \left[ \bar{\xi}_1 + \frac{\tau}{v_{n_3,1}} \left(1 - \frac{v_{n_3,2}}{v_{n_2,2}} \left(1 - \frac{v_{n_2,1}}{v_{n_1,1}}\right)\right) \right], \\ &= \frac{1}{v_{n_3,1} v_{n_2,2} v_{n_1,1} \bar{\xi}_1} [v_{n_3,1} v_{n_2,2} v_{n_1,1} \bar{\xi}_1 + v_{n_2,2} v_{n_1,1} \tau - v_{n_3,2} v_{n_1,1} \tau + v_{n_3,2} v_{n_2,1} \tau]. \end{aligned} \quad (60)$$

Let us recall that for all  $(\tau, \xi_1, \xi_2) \in V$ , we have

$$\tau + \lambda(\xi_1, \xi_2) = 0,$$

using the fact that  $\lambda$  is homogenous of degree one, Euler formula implies that :

$$\tau + (\xi_1, \xi_2) \cdot \nabla_{\xi} \lambda(\xi_1, \xi_2) = 0. \quad (61)$$

In particular, applying (61) to  $f^{n_1} = (\tau, \xi_1, \bar{\xi}_2)$ ,  $f^{n_2} = (\tau, \xi_1, \xi_2)$  and  $f^{n_3} = (\tau, \bar{\xi}_1, \xi_2)$  we obtain respectively :

$$\tau + v_{n_1,1} \xi_1 + v_{n_1,2} \bar{\xi}_2 = 0, \quad (62)$$

$$\tau + v_{n_2,1} \xi_1 + v_{n_2,2} \xi_2 = 0, \quad (63)$$

$$\tau + v_{n_3,1} \bar{\xi}_1 + v_{n_3,2} \xi_2 = 0. \quad (64)$$

From (64) we deduce that :

$$v_{n_3,1} v_{n_2,2} v_{n_1,1} \bar{\xi}_1 = -v_{n_2,2} v_{n_1,1} \tau - v_{n_2,2} v_{n_1,1} v_{n_3,2} \xi_2. \quad (65)$$

Using (63) the second term in the right hand side of (65) this equation can be written under the form :

$$v_{n_2,2} v_{n_1,1} v_{n_3,2} \xi_2 = v_{n_1,1} v_{n_3,2} \tau + v_{n_2,1} v_{n_1,1} v_{n_3,2} \bar{\xi}_1. \quad (66)$$

Once again thanks to (62) the second term in the right hand side of (66), it becomes :

$$v_{n_2,1}v_{n_1,1}v_{n_3,2}\bar{\xi}_1 = -v_{n_2,1}v_{n_3,2}\bar{\mathcal{T}} - v_{n_1,2}v_{n_2,1}v_{n_3,2}\bar{\xi}_2. \quad (67)$$

Combining equations (65), (66) and (67) we obtain :

$$v_{n_3,1}v_{n_2,2}v_{n_1,1}\bar{\xi}_1 = -v_{n_2,2}v_{n_1,1}\bar{\mathcal{T}} + v_{n_3,2}v_{n_1,1}\bar{\mathcal{T}} - v_{n_3,2}v_{n_2,1}\bar{\mathcal{T}} - v_{n_1,2}v_{n_2,1}v_{n_3,2}\bar{\xi}_2.$$

Equality (59) follows from (60). □

Thanks to Lemma 5.2 formulas (54)-(55) and (56), which give the unpolarized part of the amplitudes  $u_{1,n_i}$ , for  $i = 2, \dots, 4$ , can be written under the more pleasant form :

$$(I - P_{n_2})u_{1,n_2} = -\frac{\bar{\xi}_2}{\bar{\xi}_1}\beta_1^{-1}(\mathcal{T}\mu_0)(\mathbf{t}_2, \mathbf{x}_2)R_{n_2}\mathcal{S}_{n_2}e, \quad (68)$$

$$(I - P_{n_3})u_{1,n_3} = -\frac{\bar{\xi}_2}{\bar{\xi}_2}\prod_{j=1}^2\beta_j^{-1}(\mathcal{T}\mu_0)(\mathbf{t}_3, \mathbf{x}_3)R_{n_3}\mathcal{S}_{n_3}e, \quad (69)$$

$$(I - P_{n_4})u_{1,n_4} = -\frac{\bar{\xi}_2}{\bar{\xi}_1}\prod_{j=1}^3\beta_j^{-1}(\mathcal{T}\mu_0)(\mathbf{t}_4, \mathbf{x}_4)R_{n_4}\mathcal{S}_{n_4}e, \quad (70)$$

where the transport operator  $\mathcal{T}$  is defined in (53).

We thus remark that the unpolarized part of the amplitudes  $u_{1,n_j}$ ,  $j = 1, \dots, 4$  depend of the same transport operator which is precisely the transport operator applied to  $\mu_0$  in equation (48).

Let us conclude this paragraph by the determination of the unpolarized part of the amplitudes  $u_{1,j}$  for  $\underline{i} \in \Phi(n_1)^* \setminus \{n_4\}$ . The knowledge of these amplitudes will be useful in paragraph 5.1.5. From (44) and the relation (49), one easily obtains, that after the reformulation of  $(I - P_{n_4})u_{1,n_4}$ , we have :

$$(I - P_1^{\underline{i}})u_{1,\underline{i}|_{x_1=0}} = i \left[ (\mathcal{S}^{n_4}e)R_1^{\underline{i}}L(\partial)\mu_0(\mathbf{t}_4, \mathbf{x}_4)P_1^{\underline{i}}\phi_1^{\underline{i}}B_1e_4 \right]_{|_{x_1=0}}, \quad (71)$$

from which we deduce that up to some derivatives and multiplications,  $(I - P_1^{\underline{i}})u_{1,\underline{i}|_{x_1=0}}$  depend on  $\mu_0(\underline{\mathbf{t}}, \underline{\mathbf{x}})$ .

### 5.1.3 Polarized part of the terms of order one.

The knowledge of the unpolarized part of the amplitudes of order one enable us to determine the polarized part of the amplitudes of order one, and will conclude the determination of the right hand side in equation (48).

Indeed, let us consider equation (19) written for  $n = 1$  and  $k = n_j, j = 1, \dots, 4$ . This equation reads :

$$\begin{aligned} i\mathcal{L}(d\varphi_{n_j})u_{2,n_j} + L(\partial)P_1^{n_j}u_{1,n_j} &= -L(\partial)(I - P_1)^{n_j}u_{1,n_j}, \text{ for } j \text{ odd,} \\ i\mathcal{L}(d\varphi_{n_j})u_{2,n_j} + L(\partial)P_2^{n_j}u_{1,n_j} &= -L(\partial)(I - P_2)^{n_j}u_{1,n_j}, \text{ for } j \text{ even.} \end{aligned}$$

Then composing by  $Q_1^{n_1}, Q_1^{n_3}$  in the first equation, and by  $Q_2^{n_2}, Q_2^{n_4}$  in the second one, makes the term depending of  $u_{2,n_j}$  disappears and from Lax lemma we obtain that the polarized parts of order one satisfy the transport equations :

$$(\partial_t + v_{n_j} \cdot \nabla_x)Q_1^{n_j}P_1^{n_j}u_{1,n_j} = -Q_1^{n_j}L(\partial)(I - P_1)^{n_j}u_{1,n_j}, \text{ if } j = 1, 3, \quad (72)$$

$$(\partial_t + v_{n_j} \cdot \nabla_x)Q_2^{n_j}P_2^{n_j}u_{1,n_j} = -Q_2^{n_j}L(\partial)(I - P_2)^{n_j}u_{1,n_j}, \text{ if } j = 2, 4, \quad (73)$$

with initial and boundary conditions given by (23)-(22). A preliminary to obtain the exact values of the solutions of (72)-(73) is thus to determine the source terms. This can be done thanks to the reformulation made in paragraph 5.1.2. Let us start by the term  $Q_1^{n_1} L(\partial)(I - P_1)^{n_1} u_{1,n_1}$ , from equation (52), an explicit computation gives :

$$Q_1^{n_1} L(\partial)(I - P_1)^{n_1} u_{1,n_1} = - \left[ Q_1^{n_1} R_1^{n_1} e \left( \partial_{tt}^2 \mu_0 - \frac{\tau}{\xi_2} \partial_{t_2}^2 \mu_0 \right) + Q_1^{n_1} A_2 R_1^{n_1} e \left( \partial_{t_2}^2 \mu_0 - \frac{\tau}{\xi_2} \partial_{22}^2 \mu_0 \right) \right] \left( t - \frac{1}{v_{n_1,1}} x_1, x_2 - \beta_1^{-1} x_1 \right). \quad (74)$$

Where we used the fact that  $Q_1^{n_1} A_1 R_1^{n_1} e = 0$  (see paragraph 5.1.1 for more details).

To make this equation more explicit in terms of  $\mu_0$ , as it has been done for the determination of the unpolarized parts of order one, we are looking for a relation linking  $Q_1^{n_1} R_1^{n_1} e$  and  $Q_1^{n_1} A_2 R_1^{n_1} e$ . Let us recall that  $\ker Q_1^{n_1} = \text{Ran } \mathcal{L}(d\varphi_{n_1})$  so for all  $X \in \mathbb{C}^N$  we have  $Q_1^{n_1} \mathcal{L}(d\varphi_{n_1})X = 0$ . In particular, for  $X = R_1^{n_1} e$  and we obtain :

$$\bar{\tau} Q_1^{n_1} R_1^{n_1} e + \bar{\xi}_1 Q_1^{n_1} A_1 R_1^{n_1} e + \bar{\xi}_2 Q_1^{n_1} A_2 R_1^{n_1} e = 0 \implies Q_1^{n_1} A_2 R_1^{n_1} e = -\frac{\tau}{\xi_2} Q_1^{n_1} R_1^{n_1} e. \quad (75)$$

Using this relation in (74) thus gives :

$$\begin{aligned} Q_1^{n_1} L(\partial)(I - P_1)^{n_1} u_{1,n_1} &= - \left[ Q_1^{n_1} R_1^{n_1} e \left( \partial_{tt}^2 \mu_0 - 2\frac{\tau}{\xi_2} \partial_{t_2}^2 \mu_0 + \left( \frac{\tau}{\xi_2} \right)^2 \partial_{22}^2 \mu_0 \right) \right] \left( t - \frac{1}{v_{n_1,1}} x_1, x_2 - \beta_1^{-1} x_1 \right), \\ &= - [Q_1^{n_1} R_1^{n_1} e \mathcal{T}^2 \mu_0] \left( t - \frac{1}{v_{n_1,1}} x_1, x_2 - \beta_1^{-1} x_1 \right), \end{aligned} \quad (76)$$

where we recall that the transport operator  $\mathcal{T}$  is defined in (53).

Let us remark that, exactly as for the unpolarized part of amplitude of order one, the source term in (72) expresses in terms of the transport operator  $\mathcal{T}$ .

We then repeat the same kind of computations for the terms  $Q_2^{n_2} L(\partial)(I - P_2)^{n_2} u_{1,n_2}$ ,  $Q_1^{n_3} L(\partial)(I - P_1)^{n_3} u_{1,n_3}$  and  $Q_2^{n_4} L(\partial)(I - P_2)^{n_4} u_{2,n_4}$ . Using analogous relations as (75), more precisely :

$$Q_2^{n_2} A_1 R_2^{n_2} \mathcal{S}^{n_2} e = -\frac{\tau}{\xi_1} Q_2^{n_2} R_2^{n_2} \mathcal{S}^{n_2} e, \quad (77)$$

$$Q_1^{n_3} A_2 R_1^{n_3} \mathcal{S}^{n_3} e = -\frac{\tau}{\xi_2} Q_1^{n_3} R_1^{n_3} \mathcal{S}^{n_3} e, \quad (78)$$

$$Q_2^{n_4} A_1 R_2^{n_4} \mathcal{S}^{n_4} e = -\frac{\tau}{\xi_1} Q_2^{n_4} R_2^{n_4} \mathcal{S}^{n_4} e, \quad (79)$$

and Lemma 5.2 in equations (68)-(69) and (70), some tedious (but explicit) computations give :

$$Q_2^{n_2} L(\partial)(I - P_2^{n_2}) u_{1,n_2} = \beta_1^{-2} \left( \frac{\bar{\xi}_2}{\bar{\xi}_1} \right)^2 [Q_2^{n_2} R_2^{n_2} \mathcal{S}^{n_2} e \mathcal{T}^2 \mu_0] (\underline{\mathbf{t}}_2, \underline{\mathbf{x}}_2), \quad (80)$$

$$Q_1^{n_3} L(\partial)(I - P_1^{n_3}) u_{1,n_3} = -\beta_1^{-2} \beta_2^{-2} \left( \frac{\bar{\xi}_2}{\bar{\xi}_2} \right)^2 [Q_1^{n_3} R_1^{n_3} \mathcal{S}^{n_3} e \mathcal{T}^2 \mu_0] (\underline{\mathbf{t}}_3, \underline{\mathbf{x}}_3), \quad (81)$$

$$Q_2^{n_4} L(\partial)(I - P_2^{n_4}) u_{1,n_4} = \prod_{j=1}^3 \beta_j^{-2} \left( \frac{\bar{\xi}_2}{\bar{\xi}_1} \right)^2 [Q_2^{n_4} R_2^{n_4} \mathcal{S}^{n_4} e \mathcal{T}^2 \mu_0] (\underline{\mathbf{t}}_4, \underline{\mathbf{x}}_4). \quad (82)$$

More details about those computations can be find in Appendix 7.2.

Now that the source term in equation (72) is express in term of  $\mathcal{T}^2\mu_0$ , we can solve this equation in terms of  $\mathcal{T}^2\mu_0$ . Using the fact that, from the strict hyperbolicity assumption,  $\ker B_1 = E_1^s(i_{\underline{t}}, \bar{\xi}_2) = \text{vect}\{e\}$ , we write as in the beggining of paragraph 5.1 :

$$P_1^{n_1}u_{1,n_1}(t, x) := \nu_{1,n_1}(t, x)e.$$

Let us also define,

$$\nu_{1,n_1|_{x_1=0}}(t, x_2) := \mu_1(t, x_2). \quad (83)$$

The transport equation (72) becomes :

$$\begin{cases} (\partial_t + v_{n_1} \cdot \nabla_x)\nu_{1,n_1} = -\widetilde{Q_1^{n_1}}Q_1^{n_1}L(\partial)(I - P_1^{n_j})u_{1,n_1}, \\ \nu_{1,n_1|_{x_1=0}} = \mu_1(t, x_2) \\ \nu_{1,n_1|_{t \leq 0}} = 0, \end{cases} \quad (84)$$

where  $\widetilde{Q_1^{n_1}}$  denotes the inverse of the restriction of  $Q_1^{n_1}$  to  $\text{Ran}P_1^{n_1}$  and where the trace function  $\mu_{1,n_1}$  is a unknown. Then integrating along the characteristics give the exact value of  $\nu_{1,n_1}$  (and thus  $P_1^{n_1}u_{1,n_1}$ ) in terms of the unknown traces  $\mu_0, \mu_1$ . More precisely, we have to study two separates cases :

- $t - \frac{1}{v_{n_1,1}x_1} < 0$ . Then the transported information is above the characteristic, the transported condition is the initial condition. Moreover, as the function  $\mu_0$  is assumed to satisfy  $\mu_{0|_{t \leq 0}}$ , one can checks on (76) that the transport associated to the source term in the interior is zero. Thus the associated solution is zero.

- $t - \frac{1}{v_{n_1,1}x_1} > 0$ . Then the transported information is below the characteristic, the transported condition is the boundary condition and this time the transport associated to the source term in the interior does not necessary vanish. Integrate along the characteristics gives the explicit value :

$$\begin{aligned} \nu_{1,n_1} &= \mu_1\left(t - \frac{1}{v_{n_1,1}}x_1; x_2 - \beta_1^{-1}x_1\right) \\ &+ c \int_0^{x_1} \mathcal{T}^2\mu_0\left(t - \frac{s}{v_{n_1,1}} - \frac{s}{v_{n_1,1}}(x_1 - s), x_2 - \beta_1^{-1}s - \beta_1^{-1}(x_1 - s)\right) ds \\ &= \mu_1\left(t - \frac{1}{v_{n_1,1}}x_1; x_2 - \beta_1^{-1}x_1\right) + cx_1 \mathcal{T}^2\mu_0\left(t - \frac{s}{v_{n_1,1}}x_1, x_2 - \beta_1^{-1}x_1\right), \end{aligned} \quad (85)$$

indeed, from the explicit value of the source term given in (76), one can easily checks that the source term lies along the characteristics, so the integral term is just a multiplication by the length of the characteristic.

From this formula, and the formula giving the unpolarized part of  $u_{1,n_1}$ , we can thus give the following value for  $u_{1,n_1}$  :

$$\begin{aligned} u_{1,n_1} &= P_1^{n_1}u_{1,n_1} + (I - P_1^{n_1})u_{1,n_1} \\ &= [\mu_1 + cx_1 \mathcal{T}^2\mu_0 e - \mathcal{T}\mu_0 R_{n_1} e] \left(t - \frac{1}{v_{n_1,1}}x_1, x_2 - \beta_1^{-1}x_1\right), \end{aligned}$$

from which, we deduce the value of  $u_{1,n_1|_{x_2=0}}$  in terms of  $\mathcal{T}\mu_0$  and  $\mu_1$  :

$$u_{1,n_1|_{x_2=0}} = [\mu_1 + cx_1 \mathcal{T}^2\mu_0 e - \mathcal{T}\mu_0 R_{n_1} e] \left(t - \frac{1}{v_{n_1,1}}x_1, -\beta_1^{-1}x_1\right). \quad (86)$$



where  $c$  stands for an explicitly computable constant.

This trace then enable us to determine the amplitude  $u_{1,n_2}$ , then the amplitude  $u_{1,n_3}$  and finally the amplitude  $u_{1,n_4}$  and more precisely its trace on  $\{x_1 = 0\}$  which appears in equation (48). Indeed, to fully determine  $u_{1,n_j}$ ,  $j = 2, \dots, 4$  we just have to determine their polarized parts, thanks to paragraph 5.1.1. Let

$$P_2^{n_j} u_{1,n_j}(t, x) = \nu_{1,n_j}(t, x) e_j, \text{ for } j = 2, 4, P_1^{n_j} u_{1,n_j}(t, x) = \nu_{1,n_j}(t, x) e_j \text{ for } j = 3,$$

then, thanks to the uniform Kreiss-Lopatinskii condition and the fact that  $n_2 \in \mathcal{I}_{oi}$ ,  $\nu_{1,n_2}$  satisfy the transport equation :

$$\begin{cases} (\partial_t + v_{n_2} \cdot \nabla_x) \nu_{1,n_2} = -\widetilde{Q}_2^{n_2} Q_2^{n_2} L(\partial)(I - P_2)^{n_2} u_{1,n_2}, \\ \nu_{1,n_2}|_{x_2=0} = -\mathcal{I}^{n_2} u_{1,n_1}|_{x_2=0} - \mathcal{I}^{n_2} (I - P_2^{n_2}) u_{1,n_2}|_{x_2=0}, \\ \nu_{1,n_2}|_{t \leq 0} = 0, \end{cases} \quad (87)$$

where the sources terms are given by (80), (86) and (68). We integrate along the characteristics, once again there are two cases to separate :

- $t - \frac{1}{v_{n_2,2x_2}} \leq 0$  ; The condition transported is the initial condition, and noticing that the source term in the interior is evaluated in  $\mathbf{t}_2(t, x) \leq t - \frac{1}{v_{n_2,2x_2}}$  (we recall that  $\mathbf{t}_2$  is defined in (32)) we deduce that the transport of the source term in the interior is also zero.

- $t - \frac{1}{v_{n_2,2x_2}} > 0$  ; The condition transported is boundary condition. Integrate along the characteristics gives :

$$\nu_{1,n_2} = -\mathcal{I}^{n_2} [\mu_1 + cx_1 \mathcal{I}^2 \mu_0 e - \mathcal{I} \mu_0 R_{n_1} e] (\mathbf{t}_2, \mathbf{x}_2) + c_2 x_2 \mathcal{I}^2 \mu_0 (\mathbf{t}_2, \mathbf{x}_2),$$

where  $c_2 := \beta_1^{-2} \left( \frac{\bar{\xi}_2}{\bar{\xi}_1} \right)^2 \widetilde{Q}_2 Q_2^{n_2} R_2^{n_2} \mathcal{I}^{n_2} e$ .

So at this step, we obtain the polarized part of the amplitude  $u_{n_2,1}$ , from which we deduce the value of its trace on  $\{x_1 = 0\}$ . More precisely we have :

$$\begin{aligned} u_{n_2,1} = & - \left[ \mathcal{I}^{n_2} [\mu_1 + cx_1 \mathcal{I}^2 \mu_0 e - \mathcal{I} \mu_0 R_{n_1} e] (\mathbf{t}_2, \mathbf{x}_2) + c_2 x_2 \mathcal{I}^2 \mu_0 (\mathbf{t}_2, \mathbf{x}_2) \right. \\ & - \left. \frac{\bar{\xi}_2}{\bar{\xi}_1} \beta_1^{-1} (\mathcal{I} \mu_0) (\mathbf{t}_2, \mathbf{x}_2) R_{n_2} \mathcal{I}^{n_2} e \right] e_2 \\ & - \frac{\bar{\xi}_2}{\bar{\xi}_1} \beta_1^{-1} (\mathcal{I} \mu_0) (\mathbf{t}_2, \mathbf{x}_2) R_{n_2} \mathcal{I}^{n_2} e. \end{aligned} \quad (88)$$

We then use this trace in the transport equation determining  $\nu_{1,n_3}$ . Integrating along the characteristics we obtain the trace of  $P_1^{n_3} u_{n_3,1}$ . The important fact is that (as in the resolution of the resolution of (84) and (87)), the source term in the interior already lies along the characteristics so its contribution to the transport is just a multiplication by  $x_1$  of itself. Moreover equations (88), (69) and (81) tell us that the source terms in the transport equation determining  $\nu_{1,n_3}$  depends on  $\mu_1$ ,  $\mathcal{I} \mu_0$  and  $\mathcal{I}^2 \mu_0$ , all evaluated in  $(\mathbf{t}_2, \mathbf{x}_2)$ . As a consequence  $P_1^{n_3} u_{1,n_3}$  depends on  $\mu_1$ ,  $\mathcal{I} \mu_0$  and  $\mathcal{I}^2 \mu_0$ , all evaluated in  $(\mathbf{t}_3, \mathbf{x}_3)$ , then from equation (69) so do  $u_{1,n_3}$ .

To conclude, we compute the trace of  $u_{1,n_3}$  on the set  $\{x_2 = 0\}$  and we use it as a source term in the transport equation determining  $\nu_{1,n_4}$ . Repeating exactly the same arguments we show that the  $P_2^{n_4} u_{1,n_4}$  depends on  $\mu_1$ ,  $\mathcal{I} \mu_0$  and  $\mathcal{I}^2 \mu_0$ , all evaluated in  $(\mathbf{t}_4, \mathbf{x}_4)$ . This fact implies that  $P_2^{n_4} u_{1,n_4}|_{x_1=0}$  and from (70)  $P_2^{n_4} u_{1,n_4}|_{x_1=0}$  depend on  $\mu_1$ ,  $\mathcal{I} \mu_0$  and  $\mathcal{I}^2 \mu_0$ , all evaluated in  $(\mathbf{t}, \mathbf{x})$ .

As a consequence the right hand side of (48) has been expressed in terms of  $\mu_1$ ,  $\mathcal{T}\mu_0$  and  $\mathcal{T}^2\mu_0$ , all evaluated in  $(\underline{\mathbf{t}}, \underline{\mathbf{x}})$ , more precisely (48) reads:

$$(\mathcal{T}\mu_0)(t, x_2) - (\mathbb{T}\mu_0)(t, x_2) = \frac{i}{\kappa} \left( -\tilde{g}\delta_{1,M} + \mathcal{S}^{n_4}e\mu_1(\underline{\mathbf{t}}, \underline{\mathbf{x}}) \right),$$

where  $\mathbb{T}$  is up to multiplications by some (possibly complicated functions) the sum of  $\mathcal{T}\mu_0$  and  $\mathcal{T}^2\mu_0$  followed by the evaluation in  $(\underline{\mathbf{t}}, \underline{\mathbf{x}})$ . This reformulation is the keystone of our construction because it permits to factorize the transport phenomenon along the boundary in the left hand side to rewrite (48) under the form :

$$[(I - \mathbb{T})\mathcal{T}\mu_0](t, x_2) = \delta_{1,M}\tilde{g} + \mu_1(\underline{\mathbf{t}}, \underline{\mathbf{x}}),$$

where we set  $\tilde{\mu}_1 := \frac{i}{\kappa}\mathcal{S}^{n_4}e\mu_1$  and then drop the tilda and where we did the same operation (up to the sign) for  $\tilde{g}$ <sup>7</sup>.

In the following paragraph, we explain how this particular structure for equation (48) permit us to determine the unknown  $\mu_0$ .

#### 5.1.4 Resolution of equation (48), preliminary study.

The resolution of equation (48) is based upon the study of the influence of the change of variables  $(\underline{\mathbf{t}}, \underline{\mathbf{x}})$  on the profile spaces. But these spaces are not defined in the same way depending on the value of the dilatation parameter  $\beta$ . That is why the resolution of equation (48) needs to be discussed in two distinct frameworks.

#### The case $\beta \leq 1$ , the information get closer of the corner or admits a periodic pattern.

Before to solve equation (48) we give a useful property of the spaces of profiles  $\mathcal{P}_{b,\leq}^{M-n}$  :

**Proposition 5.2** *If  $\mu \in \mathcal{P}_{b,\leq}^{M-n}$  then  $\mu(\underline{\mathbf{t}}, \underline{\mathbf{x}}) \in \mathcal{P}_{b,\leq}^{M-n+1}$ .*

**Proof :** We assume that  $M - n + 1 > 0$  and  $M - n > 0$ , the other cases are similar and will not be demonstrate here.

In a first time we show that  $\mu(\underline{\mathbf{t}}, \underline{\mathbf{x}})$  is zero for  $t \in ]-\infty, \underline{T}_{M-n+1}[$ . If  $t < \underline{T}_{M-n}$  then so do  $\underline{\mathbf{t}} < \underline{T}_{M-n}$  and it is automatic. So let us assume that  $\underline{T}_{M-n} < t < \underline{T}_{M-n+1}$  and  $\underline{\mathbf{t}} < \underline{T}_{M-n+1}$ . By definition of  $\mathcal{P}_{b,\leq}^{M-n}$ ,  $\mu(\underline{\mathbf{t}}, x)$  is zero if  $x < \beta^{M-n}\underline{y}_0$ . Thus we restrict our attention to the case  $x \geq \beta^{M-n}\underline{y}_0$ . We thus obtain that :

$$\underline{\mathbf{t}} < \underline{T}_{M-n+1} - \alpha\beta^{M-n}\underline{y}_0 = \underline{T}_{M-n},$$

from which it follows that  $\mu(\underline{\mathbf{t}}, \cdot)$  is zero.

Now, let  $k$  be such that  $M - n + 1 \leq k \leq M$  and fix  $t$  in  $[\underline{T}_k, \min(\underline{T}_{k+1}, T)[$ , we will distinguish three cases depending of the value of  $\underline{\mathbf{t}}$  :

*i)* If  $\underline{\mathbf{t}} < \underline{T}_{k-1}$  then it follows that :

$$\underline{T}_k < t < \alpha\beta^{-1}x + \underline{T}_{k-1}, \tag{89}$$

from which we deduce that  $x > \beta^k\underline{y}_0$ . Consequently it is, in fact, not useful to study this case to show that  $\mu(\underline{\mathbf{t}}, \underline{\mathbf{x}})$  is zero for  $x < \beta^k\underline{y}_0$ .

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<sup>7</sup>Let us stress that the operator  $\mathbb{T}$  of this paper has nothing in common with the operator  $\mathbb{T}$  of [Ben]. Indeed, the operator  $\mathbb{T}$  of [Ben] is of order zero and necessitates to be well-defined the uniform Kreiss-Lopatinskii condition.

ii) If  $\underline{T}_{k-1} < \underline{t} < \underline{T}_k$  then by definition of  $\mathcal{P}_{b,\leq}^{M-n}$ ,  $\mu(\underline{t}, x)$  is zero for  $x < \beta^{k-1}\underline{y}_0$  and as a consequence  $\mu(\underline{t}, \underline{x})$  is zero for  $x < \beta^k\underline{y}_0$ .

iii) If  $\underline{T}_k < \underline{t}$  then we can repeat the argument applied in equation (89) to show that, this time, necessarily we have  $\underline{y}_0\beta^{k+1} > x$  for which we deduce that  $\mu(\underline{t}, \underline{x})$  is zero.

□

We also give a useful proposition concerning the influence of the change of variables  $(\underline{t}, \underline{x})$  on the source term  $g$ . This proposition shows that even if  $g$  is not in some  $\mathcal{P}_b^k$  the change of variable  $(\underline{t}, \underline{x})$  has good properties on the support of  $g$ . More precisely :

**Proposition 5.3** *Let  $g$  be a smooth function which is zero for negative times and satisfying*

$$\forall t > 0, g(t, x) = 0 \text{ if } x < \underline{y}_0.$$

*Then for all  $l \in \mathbb{N}^*$ ,  $g(\underline{t}^l, \underline{x}^l) \in \mathcal{P}_{b,\leq}^l$ .*

**Proof :** The proof pick up some ideas about the proof of Proposition 5.2. However, we give it for a sake of completeness.

First let us show that  $g(\underline{t}^l, \underline{x}^l) \in \mathcal{P}_{b,\leq}^l$  is zero for  $t < \underline{T}_l$ . If  $\underline{t}^l < 0$  is it trivial so let us assume that  $\underline{t}^l \geq 0$  and that  $\underline{x}^l \geq \underline{y}_0$ , then for  $t < \underline{T}_l$  we have the following bound :

$$\underline{t}^l < \alpha\underline{y}_0 \left( \sum_{j=0}^{l-1} \beta^j - \beta^l \sum_{j=1}^l \beta^{-j} \right) < 0,$$

which is a contradiction.

Now let fix  $k$  such that  $l \leq k \leq M$  and a time  $t \in [\underline{T}_k, \min(T, \underline{T}_{k+1})[$ . Once again let us assume that  $\underline{t}^l > 0$ , so  $g(\underline{t}^l, \underline{x}^l)$  is zero if  $\underline{x}^l < \underline{y}_0$ , from the support property of  $g$ . In other word  $g(\underline{t}^l, \underline{x}^l)$  is zero for  $x < \beta^l\underline{y}_0$ . But using the fact that  $\beta \leq 1$  and that  $l \leq k$  it follows that in particular  $g(\underline{t}^l, \underline{x}^l)$  is zero for  $x < \beta^k\underline{y}_0$

□

Now it is time to study equation (48). At the end of the previous paragraph we explained why (48) could be rewritten under the form :

$$[(I - \mathbb{T})(\mathcal{I}\mu_0)](t, x_2) = \tilde{g}\delta_{M,1} + \mu_1(\underline{t}, \underline{x}), \quad (90)$$

where the operator  $\mathbb{T}$  reads<sup>8</sup> :

$$(\mathbb{T}u)(t, x_2) := c_1u(\underline{t}(t, x_2), \underline{x}(x_2)) + c_2(t, x_2)(\mathcal{I}u)(\underline{t}(t, x_2), \underline{x}(x_2)).$$

The idea of the resolution is to remark that the operator  $\mathbb{T}$  is expressed in the variables  $(\underline{t}, \underline{x})$  and thus "cost" in terms of time of travel the time needed to make one complete circuit around the loop. Thus as we arrange the things in such a way that we can only make  $M$  turns around the loop, one can always invert  $I - \mathbb{T}$  by taking the Neumann serie expansion. Indeed, the terms associated with  $\mathbb{T}^j$  with  $j$  big "cost too much time" to appear and consequently are zero. Then one shows that we can repeat exactly the same reasoning to show that we can express  $\mu_0$  in terms

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<sup>8</sup>Notice that the precise values of the constant  $c_1$  and of the function  $c_2$  can be exactly express from equations (70) and explicit computations in the resolution of the transport equations mentionned in paragraph 5.1.3. However, it is useless for our purpose .

of  $\mu_2$  (the scalar component of the unknown trace  $u_{2,n_1}$ ) and so on to express  $\mu_0$  in terms of some  $\mu_k$  (the scalar component of the unknown trace  $u_{k,n_1}$ ) for  $k$  as big as we want and in terms of the source term  $g$ . However, the trace  $\mu_0$  in fact does not depend on  $\mu_k$  for  $k$  big, because this should say that we have made  $k$  complete circuits around the loop. We thus determine  $\mu_0$  in terms of  $g$  only.

Formally, we can always invert the operator  $(I - \mathbb{T})$  appearing in equation (90) by taking the Neumann serie expansion. This gives the formal value of  $\mathcal{T}\mu_0$  :

$$(\mathcal{T}\mu_0)(t, x) = \delta_{M,1} \sum_{j \geq 0} \mathbb{T}^j \tilde{g}(t, x) + \sum_{j \geq 0} \mathbb{T}^j \mu_1(\underline{\mathbf{t}}, \underline{\mathbf{x}}). \quad (91)$$

Let us remark that the second sum in the right hand side of (91) reads :

$$\sum_{j \geq 1} (F_j(\mu_1))(\underline{\mathbf{t}}^j, \underline{\mathbf{x}}^j), \quad (92)$$

where the operators  $F_j(\mu_1)$  are some explicitly computable operators and where  $\underline{\mathbf{t}}^j$  and  $\underline{\mathbf{x}}^j$  denote that we made the change of variables  $(t, x) \rightarrow (\underline{\mathbf{t}}(t, x), \underline{\mathbf{x}}(x))$   $j$  times that is :

$$\underline{\mathbf{t}}^j = \underline{\mathbf{t}}^j(t, x) := t - \alpha \sum_{l=0}^j \beta^{-l} x \text{ and } \underline{\mathbf{x}}^j = \underline{\mathbf{x}}^j(x) := \beta^{-j} x.$$

Now remember that by definition of the profiles spaces see Definition 5.2 we are looking for  $\mu_1$  to be in  $\mathcal{P}_{b, \leq}^{M-1}$ , so using the fact that the operators  $F_j$  are sums of derivatives (up to some multiplication by known functions), it follows that the  $F_j(\mu_1)$  are in  $\mathcal{P}_{b, \leq}^{M-1}$ . Proposition 5.2 shows that  $F_j(\mu_1)(\underline{\mathbf{t}}^j, \underline{\mathbf{x}}^j)$  is in  $\mu_1 \in \mathcal{P}_{b, \leq}^{M-1+j}$ . In the same way Proposition 5.3 applied to the first sum in the right hand side of (91) shows that this sum contains at most  $M$  terms.

As a consequence, equation (91) in fact reads :

$$(\mathcal{T}\mu_0)(t, x) = \delta_{M,1} \sum_{j=0}^1 \mathbb{T}^j \tilde{g}(\underline{\mathbf{t}}^j, \underline{\mathbf{x}}^j) + \mu_1(\underline{\mathbf{t}}, \underline{\mathbf{x}}), \quad (93)$$

which gives a (rigorous, because the Neumann expansion for  $(I - \mathbb{T})$  is finite) value of  $(\mathcal{T}\mu_0)$  in terms of the (still) unknown function  $\mu_1 \in \mathcal{P}_{b, \leq}^{M-1}$ . In terms of wave packets propagation (93) tell us that  $\mu_0$ , compared to the others terms in the WKB expansion, only depends on  $\mu_1$  which has made a complete circuit around the loop. This fact agrees with the intuition given in paragraph 3.1 that the amplitude  $u_{0,n_1}$  does not depend of the amplitudes  $u_{n,n_1}$  for  $n$  large enough because these amplitudes do not have achieve enough complete circuits around the loop. Let us also stress that the term depending of  $\tilde{g}$  in the right hand side of (93) starts to be evaluated in  $(t, x)$  while the term depending of  $\mu_1$  is evaluated in  $(\underline{\mathbf{t}}, \underline{\mathbf{x}})$ . This will be a crucial point in the following.

Assume that  $\mu_1$  is a known function in  $\mathcal{P}_{b, \leq}^{M-1}$ , then is it easy to compute  $\mu_0$  : indeed by definition of the transport operator  $\mathcal{T}$ ,  $\mu_0$  satisfies the transport equation :

$$\begin{cases} \partial_t \mu_0 - \frac{\tau}{\xi_2} \partial_2 \mu_0 = \delta_{M,1} \sum_{j=0}^1 \mathbb{T}^j \tilde{g}(\underline{\mathbf{t}}^j, \underline{\mathbf{x}}^j) + \mu_1(\underline{\mathbf{t}}, \underline{\mathbf{x}}), \\ \mu_0|_{x_2=0} = 0, \\ \mu_0|_{t \leq 0} = 0, \end{cases} \quad (94)$$

which can be integrated along the characteristics to obtain the value of  $\mu_0$ . Let us denote by  $\mathcal{H}$  the application that to a given source term in the interior for the transport equation (94) associate

the solution of the transport equation (94). We can thus write :

$$\mu_0 = \mathcal{K} \left( \delta_{M,1} \sum_{j=0}^M \mathbb{T}^j \tilde{g}(\underline{\mathbf{t}}^j, \underline{\mathbf{x}}^j) + \mu_1(\underline{\mathbf{t}}, \underline{\mathbf{x}}) \right). \quad (95)$$

Let us remark that as we assumed that  $-\frac{\tau}{\xi_2} > 0$  (see Assumption 4.3), the transport operator  $\mathcal{T}$  "pushes" the information away from the corner. As a consequence, integrating (94) along the characteristics show that applying the operator  $\mathcal{K}$  does not destroy the property to be zero on the strip  $\{t \in \mathbb{R}, 0 < x < Y\}$ , for  $Y > 0$ . This remark shows the following proposition :

**Proposition 5.4** *Let  $\mathcal{K}$  be defined previously. If  $\mu \in \mathcal{P}_{b,\leq}^k$ , for some  $k \in \mathbb{N}$ , then  $\mathcal{K}\mu \in \mathcal{P}_{b,\leq}^k$*

We thus have

$$\mu_0 = \mathcal{K} \left( \delta_{M,1} \sum_{j=0}^1 \mathbb{T}^j \tilde{g}(\underline{\mathbf{t}}^j, \underline{\mathbf{x}}^j) + \mu_1(\underline{\mathbf{t}}, \underline{\mathbf{x}}) \right), \quad (96)$$

so to know  $\mu_0$  we have to determine the value of the right hand side of (91) in term of the amplitude of the source term of the corner problem (14)  $g$  and not in term of the unknown amplitude  $\mu_1$  evaluated in  $(\underline{\mathbf{t}}, \underline{\mathbf{x}})$ .

Let us also remark that equation (96) meets with the intuition, described in paragraph 3.1, that  $u_{0,n_1}$  is "turned on" by the amplitude  $u_{1,n_1}$  which has made one complete circuit around the loop. Moreover, formally, if  $\mu_1$  is in  $\mathcal{P}_{b,\leq}^{M-1}$  then Proposition 5.2 implies that  $\mu_1(\underline{\mathbf{t}}, \underline{\mathbf{x}})$  is in  $\mathcal{P}_{b,\leq}^M$  and using the fact that spaces  $\mathcal{P}_{b,\leq}^k$  are invariants sets for the operator  $\mathcal{K}$  (see Proposition 5.4 for more details) then equation (96) should give, as expected, a trace  $\mu_0$  in  $\mathcal{P}_{b,\leq}^M$ .

In the following paragraph, we show that the unknown trace  $\mu_1$  can be expressed in terms of the unknown scalar part of the trace  $\mu_2$  and  $\mu_0$ . But before that let us give some elements about the resolution of equation (48) in the case  $\beta > 1$ .

**The case  $\beta > 1$ , the information does not approach the corner.** As for the case  $\beta \leq 1$  we start by a study of the influence of the change of variables  $(\underline{\mathbf{t}}, \underline{\mathbf{x}})$  on the set of profiles  $\mathcal{P}_{b,>}^{M-n}$ .

**Proposition 5.5** *If  $\mu \in \mathcal{P}_{b,>}^{M-n}$ , then  $\mu(\underline{\mathbf{t}}, \underline{\mathbf{x}}) \in \mathcal{P}_{b,>}^{M-n+1}$ . In particular,  $\mu(\underline{\mathbf{t}}^j, \underline{\mathbf{x}}^j)$  is zero for  $l$  larger than  $n + 1$ .*

The proof of this proposition is based on the same ideas than the proof of Proposition 5.2. However as this proof is simpler than the proof of Proposition 5.2, will not write this proof here.

Let us also note that thanks to the conditions imposed on the source term  $g$ , Proposition 5.3 is trivial in the framework  $\beta > 1$ .

With Proposition 5.5 in hand it is then easy to show that the Neumann serie expansion associated with equation (90) contains a finite number of nonzero terms. One can thus reiterate the arguments described in the framework  $\beta \leq 1$ , to show that (96) holds even for  $\beta > 1$ .

### 5.1.5 The equation on $\mu_n$ , $n > 0$ .

In this paragraph we give an equation determining the unknown scalar part of the trace  $u_{n,n_1}|_{x_1=0}$  for all  $n \in \mathbb{N}^*$ . As the reader may notice, this equation will look like equation (48) determining  $\mu_0$ , up to some extra terms due to the fact that the amplitudes  $u_{n,n_1}$  are not polarized for  $n \in \mathbb{N}^*$ .

As for the unknown  $\mu_0$ , to obtain this new equation we study the boundary condition (22) written for  $j = n_1$  :

$$B_1 \left[ \sum_{j \in \Phi(n_1)} u_{n,j} \right]_{|x_1=0} = \delta_{n,M+1} g.$$

And we write this equation under the form :

$$B_1 \left[ \sum_{j \in \Phi(n_1) \setminus \{n_4\}} P_1^j u_{n,j} \right]_{|x_1=0} + B_1 \left[ \sum_{j \in \Phi(n_1) \setminus \{n_4\}} (I - P_1^j) u_{n,j} \right]_{|x_1=0} = \delta_{n,M+1} g - B_1 u_{n,n_4}|_{x_1=0}. \quad (97)$$

According to Proposition 4.1, the first term in the left hand side of (97) is in  $B_1 E_1^s(i\bar{\tau}, \bar{\xi}_2)$ . So taking the inner product of (97) by the vector  $b$  defined in Definition 2.7 gives :

$$b \cdot B_1 \left[ \sum_{j \in \Phi(n_1) \setminus \{n_4\}} (I - P_1^j) u_{n,j} \right]_{|x_1=0} = \delta_{n,M+1} b \cdot g - b \cdot B_1 u_{n,n_4}|_{x_1=0}, \quad (98)$$

where we used the cascade of equations (19). But now, let us recall that from the cascade of equations (19) we know that :

$$(I - P_1^j) u_{n,j} = i R_1^j L(\partial) u_{n-1,j} = i R_1^j L(\partial) \left( P_1^j u_{n-1,j} + (I - P_1^j) u_{n-1,j} \right),$$

and using this relation in equation (98) lead us to the following relation :

$$\begin{aligned} ib \cdot B_1 \left[ \sum_{j \in \Phi(n_1) \setminus \{n_4\}} R_1^j L(\partial) P_1^j u_{n-1,j} \right]_{|x_1=0} &= b \cdot \delta_{n,M+1} g - b \cdot B_1 u_{n,n_4}|_{x_1=0} \\ &- ib \cdot B_1 \left[ \sum_{j \in \Phi(n_1) \setminus \{n_4\}} R_1^j L(\partial) (I - P_1^j) u_{n-1,j} \right]_{|x_1=0} \end{aligned} \quad (99)$$

We thus apply Proposition 5.1 to obtain that :

$$\mathcal{T} \mu_{n-1} = \delta_{n,M+1} b \cdot g - b \cdot B_1 u_{n,n_4}|_{x_1=0} - ib \cdot B_1 \left[ \sum_{j \in \Phi(n_1) \setminus \{n_4\}} R_1^j L(\partial) (I - P_1^j) u_{n-1,j} \right]_{|x_1=0}, \quad (100)$$

which tells us that the scalar unknown part of the amplitude  $u_{n-1,n_1}$ , namely  $\mu_{n-1}$ , satisfies the same transport equation as  $\mu_0$  up to the extra "source term"  $g$ , in the particular setting  $n = M + 1$ , and up to the extra term  $-ib \cdot B_1 \left[ \sum_{j \in \Phi(n_1) \setminus \{n_4\}} R_1^j L(\partial) (I - P_1^j) u_{n-1,j} \right]_{|x_1=0}$ . We can reiterate the computations made in paragraphs 5.1.1 and 5.1.3 to make the terms  $b \cdot B_1 u_{n,n_4}|_{x_1=0}$  explicit in terms of  $\mu_{n-1}$  and  $\mu_n$ . We thus obtain :

$$\begin{aligned} [(I - \mathbb{T}) \mathcal{T} \mu_{n-1}](t, x_2) &= \delta_{n,M+1} b \cdot g(t, x_2) + \mu_n(\underline{t}, \underline{x}) \\ &- ib \cdot B_1 \left[ \sum_{j \in \Phi(n_1) \setminus \{n_4\}} R_1^j L(\partial) (I - P_1^j) u_{n-1,j} \right]_{|x_1=0}(t, x_2). \end{aligned} \quad (101)$$

Then we reiterate the computations of paragraph 5.1.1 to treat the last term in the right hand side of (101). First of all let us remark that the terms  $(I - P_1^j) u_{n-1,j}$  for  $j \neq n_1, n_4$  are given by

equation (71) for  $n = 2$  but which is totally similar for  $n > 2$ . So after having applied the operator  $R_1^j L(\partial)$  and taking the trace on  $\{x_1 = 0\}$  one can show that the sum on  $j \in \Phi^*(n_1) \setminus n_4$  appearing in the right hand side of (101) depends on  $\mu_{n-2}(\underline{\mathbf{t}}, \underline{\mathbf{x}})$  (and possibly of the source term  $g$ ). That is why we write

$$\left[ \sum_{j \in \Phi^*(n_1) \setminus \{n_4\}} R_1^j L(\partial)(I - P_1^j)u_{n-1,j} \right]_{|x_1=0} = (\Lambda_1 \mu_{n-2})(\underline{\mathbf{t}}, \underline{\mathbf{x}}),$$

where  $\Lambda_1$  is some explicitly computable operator which is a sum of derivatives up to some multiplications. However its exact expression is not interesting for our purpose.

The same argument for  $(I - P_1^{n_1})u_{n-1,n_1}$  (which is given by (52) in the case  $n = 2$ , but which is totally analogous when  $n > 2$ ) then show that we can write :

$$[R_1^{n_1} L(\partial)(I - P_1^{n_1})u_{n-1,n_1}]_{|x_1=0} = (\Lambda_0 \mu_{n-2})(t, x),$$

where  $\Lambda_0$  is an operator like  $\Lambda_1$ .

We can thus write (101) under its final form that is :

$$[(I - \mathbb{T})\mathcal{T}\mu_{n-1}](t, x_2) = \delta_{n,M+1}\tilde{g}(t, x_2) + \mu_n(\underline{\mathbf{t}}, \underline{\mathbf{x}}) + (\Lambda\mu_{n-2})(t, x_2), \quad (102)$$

where we set  $\Lambda u := (\Lambda_0 u)(t, x) + (\Lambda_1 u)(\underline{\mathbf{t}}, \underline{\mathbf{x}})$ . The only point to keep in mind about  $\Lambda$  is that it is a sum (up to some multiplications) of derivatives. As a consequence, the spaces  $\mathcal{P}_{b,\leq}^k$  and  $\mathcal{P}_{b,>}^k$  are invariant sets for the operator  $\Lambda$ .

### 5.1.6 End of the resolution of equation (48).

In this paragraph we describe the end of the resolution of equation (48) in the case  $\beta \leq 1$ . Indeed as we will see this resolution does not take into account the precise value of  $\beta$  but only need Proposition 5.2. But Proposition 5.2 has its analogous in the case  $\beta > 1$  (that is to say Proposition 5.5). That is why we feel free to denote the profile space  $\mathcal{P}_b^{M-n}$  in place of  $\mathcal{P}_{b,\leq}^{M-n}$ , in view to shorten the notations.

Let us recall that at the end of paragraph 5.1.4 we write equation (48) under the form :

$$\mu_0 = \mathcal{K}(\mu_1(\underline{\mathbf{t}}, \underline{\mathbf{x}})),$$

from which we deduced that if  $\mu_1 \in \mathcal{P}_b^{M-1}$  then  $\mu_0 \in \mathcal{P}_b^M$ . Because of the definition of the profile spaces (see Definition 5.2) we assume that the for all  $1 \leq n \leq M+1$ ,  $\mu_n \in \mathcal{P}_b^{M-n}$ . Our aim is then to show that equation (96) defines a unique  $\mu_0 \in \mathcal{P}_b^M$  and to express this solution  $\mu_0$  as a function of the source term.

Using the analysis described in the previous paragraph we are now able to give more informations about the unknown  $\mu_1$ . Indeed, (102) written for  $n = 2$  reads :

$$[(I - \mathbb{T})\mathcal{T}\mu_1](t, x_2) = \delta_{M,2}\tilde{g}(t, x_2) + \mu_2(\underline{\mathbf{t}}, \underline{\mathbf{x}}) + (\Lambda\mu_0)(t, x_2). \quad (103)$$

Using the fact that  $\mu_0$  is in  $\mathcal{P}_b^M$  and the fact that  $\Lambda$  keep  $\mathcal{P}_b^M$  we obtain that the term  $(\Lambda\mu_0)(t, x)$  appearing in (103) is in  $\mathcal{P}_b^M$ . As in paragraph 5.1.4, we write the Neumann serie expansion associated to (103). Proposition 5.2 shows that this expansion contains only two terms, and reads :

$$(\mathcal{T}\mu_1)(t, x_2) := \delta_{M,1}(\tilde{g}(t, x_2) + \mathbb{T}\tilde{g}(\underline{\mathbf{t}}, \underline{\mathbf{x}})) + (\mu_2(\underline{\mathbf{t}}, \underline{\mathbf{x}}) + \mathbb{T}\mu_2(\underline{\mathbf{t}}^2, \underline{\mathbf{x}}^2)) + \Lambda\mu_0(t, x_2). \quad (104)$$

Up to the source term in the interior,  $\mu_0$  and  $\mu_1$  solve the same transport equation and we can write :

$$\mu_1(t, x) = \mathcal{K} \left( \delta_{M,1} (\tilde{g}(t, x_2) + \mathbb{T}\tilde{g}(\underline{\mathbf{t}}, \underline{\mathbf{x}})) + (\mu_2(\underline{\mathbf{t}}, \underline{\mathbf{x}}) + \mathbb{T}\mu_2(\underline{\mathbf{t}}^2, \underline{\mathbf{x}}^2)) + \Lambda\mu_0(t, x_2) \right). \quad (105)$$

where we recall that  $\mathcal{K}$  is the operator that to a source term  $f$  associates the solution of the transport equation  $\mathcal{T}u = f$ . Then when one evaluates (105) in  $(\underline{\mathbf{t}}, \underline{\mathbf{x}})$  he obtains, using Propositions 5.2 and 5.3, that :

$$\mu_1(\underline{\mathbf{t}}, \underline{\mathbf{x}}) = \mathcal{K} \left( \delta_{M,1}\tilde{g}(\underline{\mathbf{t}}, \underline{\mathbf{x}}) + \mu_2(\underline{\mathbf{t}}^2, \underline{\mathbf{x}}^2) \right). \quad (106)$$

Let us stress that (106) written in this form is not true. Indeed in this formulation, we used the fact that  $\mathcal{K}$  and the evaluation  $(\underline{\mathbf{t}}, \underline{\mathbf{x}})$  commute which is clearly false. However we are in this purpose only interested in the profile spaces in which the terms in the right hand side of (106) lies and we are not really interested in their precise values. As from Proposition 5.4,  $\mathcal{K}$  keeps the spaces  $\mathcal{P}_b^{M-n}$  invariant, our abuse of notations is not so important and we will do it until the end of the proof in view to simplify as much as possible the formulas. But the reader has to keep in mind that if he really wants to compute the WKB expansion it is necessary to apply  $\mathcal{K}$  and then make the evaluation  $(\underline{\mathbf{t}}, \underline{\mathbf{x}})$ .

From (96) and (106) we deduce that

$$\mu_0(t, x) = \mathcal{K} \mu_1(\underline{\mathbf{t}}, \underline{\mathbf{x}}) = \mathcal{K}^2 \left( \delta_{M,1}\tilde{g}(\underline{\mathbf{t}}, \underline{\mathbf{x}}) + \mu_2(\underline{\mathbf{t}}^2, \underline{\mathbf{x}}^2) \right), \quad (107)$$

equation which tell us that  $\mu_0$  can be express in terms of  $\mu_2$  evaluated in a time corresponding to two complete circuits around the loop.

More generally, we can apply the same arguments as those used for  $n = 1$  in (102) to obtain the formal formula : for all  $n > 0$  :

$$\mu_n(t, x_2) = \mathcal{K} \left( \delta_{n,M} \sum_{k=0}^M \mathbb{T}^k \tilde{g}(\underline{\mathbf{t}}^k, \underline{\mathbf{x}}^k) + \sum_{k=1}^{\min(n+1, M)} \mathbb{T}^{k-1} \mu_{n+1}(\underline{\mathbf{t}}^k, \underline{\mathbf{x}}^k) + \sum_{k=0}^{\min(n-1, M)} \mathbb{T}^k \Lambda \mu_{n-1}(\underline{\mathbf{t}}^k, \underline{\mathbf{x}}^k) \right). \quad (108)$$

If for all  $0 < n \leq M + 1$ ,  $\mu_n \in \mathcal{P}_b^{M-n}$ , these formulas make sense for  $0 < n \leq M$  (because once again, in this framework, we can ensure that the Neumann serie expansion contains a finite number of terms). We deduce from (108) that for all  $l \leq M$  :

$$\mu_n(\underline{\mathbf{t}}^l, \underline{\mathbf{x}}^l) = \mathcal{K} \left( \delta_{n,M} \sum_{k=l}^M \mathbb{T}^k \tilde{g}(\underline{\mathbf{t}}^k, \underline{\mathbf{x}}^k) + \sum_{k=l+1}^{\min(n+1, M)} \mathbb{T}^{k-1} \mu_{n+1}(\underline{\mathbf{t}}^k, \underline{\mathbf{x}}^k) + \sum_{k=l}^{\min(n-1, M)} \mathbb{T}^k \Lambda \mu_{n-1}(\underline{\mathbf{t}}^k, \underline{\mathbf{x}}^k) \right), \quad (109)$$

from which it follows that for all  $n < M$ , each  $\mu_n(\underline{\mathbf{t}}^n, \underline{\mathbf{x}}^n)$  is equal to  $\mu_{n+1}(\underline{\mathbf{t}}^{n+1}, \underline{\mathbf{x}}^{n+1})$ , and that for  $n = M$ ,  $\mu_n(\underline{\mathbf{t}}^n, \underline{\mathbf{x}}^n)$  is equal to  $g(\underline{\mathbf{t}}^M, \underline{\mathbf{x}}^M)$ . A simple recurrence in (107) then shows that :

$$\mu_0(t, x_2) = \mathcal{K}^M \tilde{g}(\underline{\mathbf{t}}^M, \underline{\mathbf{x}}^M), \quad (110)$$

equation determining in a unique way  $\mu_0 \in \mathcal{P}_b^M$  in terms of the known source term  $g$ . This conclude the resolution of equation (48).

Let us notice that (110) meet with the intuition described in 3.1 that  $\mu_0$  depends of the information that was present at time zero which has made  $M$  complete circuits around the loop.



## 5.2 Determination of the others amplitudes in the WKB expansion.

Once the amplitudes of the loop indeces are constructed it is easy to construct the amplitudes for the other indeces in the WKB expansion. The construction and, in particular, the order of resolution is exactly the same as the order used in [Ben] so we will not give all the details here. We refer to [Ben] for a complete proof.

### 5.2.1 Determination of the oscillating amplitudes.

Let us recall that thanks to Proposition 4.3 we know that the set of indeces  $\mathcal{S}_{os} \setminus \{n_j\}$  can be expressed as a partition (see (12) for more details). We will here describe the determination of an arbitrary oscillating amplitude associated to an index  $\underline{i}$  in one of the trees  $A_{\ell_1}$ . The determination of the amplitudes in the other sets composing (12) is similar and will not be discussed here.

Let us denote by  $\ell_{\underline{i}} := (\ell_1, \ell_2, \dots, \ell_p)$  the type  $V$  sequence linking  $n_1$  to  $\underline{i}$  (see Definition 4.3). Then before to determine the amplitude associated to  $\underline{i}$  we will determine all the amplitudes in the sequence  $\ell$ . First thanks to Proposition 4.4,  $\ell_1 \in \mathcal{S}_{oi}$  and we know from Lax Lemma that the amplitude  $u_{0, \ell_1}$  satisfies the transport equation :

$$\begin{cases} (\partial_t + v_{\ell_1} \cdot \nabla_x) P_2^{\ell_1} u_{0, \ell_1} = 0, \\ P_2^{\ell_1} u_{0, \ell_1}|_{x_2=0} = P_2^{\ell_1} B_2 u_{0, n_1}|_{x_2=0}, \\ u_{0, \ell_1}|_{t \leq 0} = 0, \end{cases}$$

with a known source term on the boundary since  $u_{0, n_1}$  and consequently  $u_{0, n_1}|_{x_2=0}$  as already been determined. So we can integrate this transport equation along the characteristics to determine  $u_{0, \ell_1}$ . Notice that the amplification of order  $M+1$  then spreads after a time strictly more than  $\underline{T}_M$  to  $u_{0, \ell_1}$ .

Then we construct  $u_{0, \ell_2}$ . Applying Proposition 4.4,  $\ell_2 \in \mathcal{S}_{io}$  and in view to determine  $u_{0, \ell_2}$  we have to solve the transport equation :

$$\begin{cases} (\partial_t + v_{\ell_2} \cdot \nabla_x) P_1^{\ell_2} u_{0, \ell_2} = 0, \\ P_1^{\ell_2} u_{0, \ell_2}|_{x_1=0} = P_1^{\ell_2} B_1 u_{0, \ell_1}|_{x_1=0}, \\ u_{0, \ell_2}|_{t \leq 0} = 0, \end{cases}$$

which is easy because the source term have already been constructed.

Then we can repeat this process until we reach  $\underline{i}$ . Notice that in the particular case  $\underline{i} \in \mathcal{S}_{ii}$  we need to solve the transport equation :

$$\begin{cases} (\partial_t + v_{\underline{i}} \cdot \nabla_x) P_1^{\underline{i}} u_{0, \underline{i}} = 0, \\ P_1^{\underline{i}} u_{0, \underline{i}}|_{x_1=0} = -S_1^{\underline{i}} B_1 u_{0, \ell_p}|_{x_1=0}, \\ P_1^{\underline{i}} u_{0, \underline{i}}|_{x_2=0} = 0, \\ u_{0, \underline{i}}|_{t \leq 0} = 0, \end{cases} \quad , \quad \left( \text{resp} \begin{cases} (\partial_t + v_{\underline{i}} \cdot \nabla_x) P_2^{\underline{i}} u_{0, \underline{i}} = 0, \\ P_2^{\underline{i}} u_{0, \underline{i}}|_{x_1=0} = 0, \\ P_2^{\underline{i}} u_{0, \underline{i}}|_{x_2=0} = -S_2^{\underline{i}} B_2 u_{0, \ell_p}|_{x_2=0}, \\ u_{0, \underline{i}}|_{t \leq 0} = 0, \end{cases} \right)$$

if  $p$  is odd (resp. even) and we use the fact that the trace of  $u_{0, \ell_p}$  on  $\{x_1 = 0\}$  (resp.  $\{x_2 = 0\}$ ) is zero near the corner (see [Ben] for more details) to ensure that  $u_{0, \underline{i}}$  is regular enough.

### 5.2.2 Determination of the evanescent amplitudes.

In this paragraph we conclude the construction of the leading order of the geometric optics expansion by giving elements of proof to construct an arbitrary evanescent amplitude lying in one of the trees  $A_{\ell_1}$ . Without loss of generality let us assume that this amplitude is associated to an index  $\underline{i} \in \mathfrak{R}_1$ . Proposition 4.4 then implies that the type  $V$  sequence  $\ell_{\underline{i}} := (\ell_1, \ell_2, \dots, \ell_p)$  linking  $n_1$  to  $\underline{i}$  has an odd number of terms. Moreover, we can assume that the amplitude associated to the index  $\ell_j$  have already been constructed. We then recall the following lemma due to [Les07] which states that the evanescent equations in the cascade (19) can be solved in the profile space  $\mathbb{P}_{ev,1}$  :

**Lemma 5.3** *For  $j = 1, 2$ , and  $\underline{k} \in \mathfrak{R}_i$ , let*

$$\begin{aligned} \mathbb{P}_{ev,j}^{\underline{k}} U(X_j) &:= e^{X_j \mathcal{A}_j(\underline{\tau}, \xi_{3-j}^{\underline{k}})} P_{s,j}^{\underline{k}} U(0), \\ \mathbb{Q}_{ev,j}^{\underline{k}} F(X_j) &:= \int_0^{X_j} e^{(X_j-s) \mathcal{A}_j(\underline{\tau}, \xi_{3-j}^{\underline{k}})} P_{s,j}^{\underline{k}} A_j^{-1} F(s) ds - \int_{X_j}^{+\infty} e^{(X_j-s) \mathcal{A}_j(\underline{\tau}, \xi_{3-j}^{\underline{k}})} P_{u,j}^{\underline{k}} A_i^{-1} F(s) ds \end{aligned} \quad (111)$$

Then, for all  $F \in P_{ev,j}$  the equation :

$$L_{\underline{k}}(\partial_{X_j})U = F,$$

admits a solution in  $P_{ev,i}$ . Moreover, this solution reads :

$$U = \mathbb{P}_{ev,i}^{\underline{k}} U + \mathbb{Q}_{ev,i}^{\underline{k}} F.$$

This lemma tell us that to construct any evanescent amplitude  $f$  on the side  $\partial\Omega_1$  it is in fact sufficient to know the value of the trace the solution on  $\{X_1 = 0\}$ . As already mentionned in [Les07], to determine this trace we study the boundary condition (22) written for  $k = \underline{i}$  and  $n = 0$  from which we deduce the value of the "double" trace on  $\{X_1 = x_1 = 0\}$ . Indeed, from the uniform Kreiss-Lopatinskii condition we obtain :

$$U_{0,\underline{i},k|_{X_1=x_1=0}} = -S_{ev,1}^{\underline{i}} B_1 u_{0,\ell_p|_{x_1=0}},$$

where we recall that the right hand side is supposed to be known.

Then we are free to straighten the "double" trace into a single one by setting (for example) :

$$U_{0,\underline{i},k|_{X_1=0}} = -\chi(x_1) S_{ev,1}^{\underline{i}} B_1 u_{0,\ell_p|_{x_1=0}},$$

where  $\chi$  is some function in  $\mathcal{D}([-1, +\infty[))$  satisfying  $\chi(0) = 1$ .

It is interesting to remark that the evanescent amplitudes suffer the same amplification as the oscillating ones and this even if the degeneracy of the uniform Kreiss-Lopatinskii condition is in the hyperbolic region. Such a behaviour was not observable for the hyperbolic boundary value problem in the half space. As for the oscillating amplitudes, one can show that evanescent amplitudes of order zero are zero for  $t$  less than  $\underline{T}_M$ .

### 5.3 Construction of the higher order terms and summary.

As for the construction of the leading order of the WKB expansion, we have to distinguish the case  $\beta \leq 1$  and the case  $\beta > 1$ . But as, once again, we only need 5.2 (resp. 5.5) to conclude if  $\beta \leq 1$  (resp.  $\beta > 1$ ) we will only describe the construction of higher order terms for  $\beta \leq 1$ . We will thus continue to note  $\mathcal{P}_b^{M-n}$  for  $\mathcal{P}_{b,\leq}^{M-n}$ .

### 5.3.1 The term of order one.

Once the amplitudes of order zero, and more precisely the keystone  $\mu_0$ , are determined we can repeat our method of construction to determine the amplitudes of order one. As for the leading order, we start by the determination of the indexes of the loop, that is the  $n_j$  for  $j = 1, \dots, 4$ . Let us remark that from equations (52)-(68)-(69) and (70), we know the unpolarized parts of the  $u_{1,n_j}$ ,  $j = 1, \dots, 4$  in terms of  $\mu_0$  (which at this step of the proof is a known function). Similarly we also know the values of the polarized parts in terms of  $\mu_1$ .

So we just have to determine the unknown trace  $\mu_1$  to construct the amplitudes of order one for the loop elements. Then noticing that we know the unpolarized part of the indexes outside of the loop (because they only depend of the associated amplitude of order zero) and reiterating the same computations as in paragraph 5.2 permit to determine their polarized part in term of  $\mu_1$ . As a consequence we just have to determine  $\mu_1$  to construct all the amplitudes of order one.

To do this, let us recall that, from paragraph 5.1.4,  $\mu_1$  satisfies equation (105), that is

$$(\mathcal{T}\mu_1)(t, x_2) := \delta_{M,2} (\tilde{g}(t, x_2) + \mathbb{T}\tilde{g}(\underline{\mathbf{t}}, \underline{\mathbf{x}}) + \mathbb{T}^2\tilde{g}(\underline{\mathbf{t}}^2, \underline{\mathbf{x}}^2)) + (\mu_2(\underline{\mathbf{t}}, \underline{\mathbf{x}}) + \mathbb{T}\mu_2(\underline{\mathbf{t}}^2, \underline{\mathbf{x}}^2)) + \Lambda\mu_0(t, x_2). \quad (113)$$

where  $\mu_2$ , appearing in the right hand side, is assumed to be  $\mathcal{P}_b^{M-2}$  and where we used the same notations of paragraph 5.1.4. From (113), after integration along the characteristics, we deduce that :

$$\mu_1(t, x) = \mathcal{K} \left( \delta_{M,1} \sum_{k=0}^1 \mathbb{T}^k \tilde{g}(\underline{\mathbf{t}}^k, \underline{\mathbf{x}}^k) + \Lambda\mu_0(t, x) \right) + \delta_{M \geq 2} \mathcal{K} \left( \sum_{k=1}^{\min(M,2)} \mathbb{T}^{k-1} \mu_2(\underline{\mathbf{t}}^k, \underline{\mathbf{x}}^k) \right). \quad (114)$$

In this equation the first term in the right hand side is a known function which only depends on  $\tilde{g}$ . So to express  $\mu_1$  in terms of  $\tilde{g}$  we just have to express  $\mu_2(\underline{\mathbf{t}}, \underline{\mathbf{x}})$  (and possibly  $\mu_2(\underline{\mathbf{t}}^2, \underline{\mathbf{x}}^2)$ ) in terms of  $\tilde{g}$ . Reiterating the same arguments, we use equation (102) written for  $n = 3$  to obtain :

$$\mu_2(t, x) = \mathcal{K} \left( \delta_{M,2} \sum_{k=0}^2 \mathbb{T}^k \tilde{g}(\underline{\mathbf{t}}^k, \underline{\mathbf{x}}^k) + \sum_{k=1}^{\min(M,3)} \mathbb{T}^{k-1} \mu_3(\underline{\mathbf{t}}^k, \underline{\mathbf{x}}^k) \right) + \mathcal{K} (\Lambda\mu_1(t, x) + \mathbb{T}\Lambda\mu_1(\underline{\mathbf{t}}, \underline{\mathbf{x}})), \quad (115)$$

and we see to appear a new difficulty in (115) compared to (96). Indeed in (115) there are two terms which depends on  $\mu_1$  each of them lying in  $\mathcal{P}_b^M$  and in  $\mathcal{P}_b^{M-1}$  so *a priori* they are not zero. However in (114) we are not interested in  $\mu_2$  but we are interested in  $\mu_2(\underline{\mathbf{t}}, \underline{\mathbf{x}})$  and possibly of  $\mu_2(\underline{\mathbf{t}}^2, \underline{\mathbf{x}}^2)$ . A simple change of variables in (115) shows that these quantities are given by :

$$\begin{aligned} \mu_2(\underline{\mathbf{t}}, \underline{\mathbf{x}}) &= \mathcal{K} \left( \delta_{M,2} \sum_{k=1}^2 \mathbb{T}^{k-1} \tilde{g}(\underline{\mathbf{t}}^k, \underline{\mathbf{x}}^k) + \sum_{k=2}^{\min(M,3)} \mathbb{T}^{k-2} \mu_3(\underline{\mathbf{t}}^k, \underline{\mathbf{x}}^k) \right) + \mathcal{K} \Lambda\mu_1(\underline{\mathbf{t}}, \underline{\mathbf{x}}), \quad (116) \\ \mu_2(\underline{\mathbf{t}}^2, \underline{\mathbf{x}}^2) &= \mathcal{K} (\delta_{M,2} \tilde{g}(\underline{\mathbf{t}}^2, \underline{\mathbf{x}}^2) + \delta_{M \geq 3} \mu_3(\underline{\mathbf{t}}^3, \underline{\mathbf{x}}^3)), \end{aligned}$$

where for  $I \subset \mathbb{R}$ ,  $\delta_I$  is the characteristic function of  $I$ . Now let us make the change of variables  $(\underline{\mathbf{t}}, \underline{\mathbf{x}})$  in (114), it follows that :

$$\mu_1(\underline{\mathbf{t}}, \underline{\mathbf{x}}) = \delta_{M,1} \mathcal{K} (\tilde{g}(\underline{\mathbf{t}}, \underline{\mathbf{x}})) + \delta_{M \geq 2} \mathcal{K} \mu_2(\underline{\mathbf{t}}^2, \underline{\mathbf{x}}^2).$$

that is to say :

$$\mathcal{K} \Lambda\mu_1(\underline{\mathbf{t}}, \underline{\mathbf{x}}) = \begin{cases} \mathcal{K} \Lambda \mathcal{K} \tilde{g}(\underline{\mathbf{t}}, \underline{\mathbf{x}}), & \text{if } M = 1, \\ \mathcal{K} \Lambda \mathcal{K}^2 (\delta_{M,2} \tilde{g}(\underline{\mathbf{t}}^2, \underline{\mathbf{x}}^2) + \delta_{M \geq 3} \mu_3(\underline{\mathbf{t}}^3, \underline{\mathbf{x}}^3)), & \text{if } M > 1, \end{cases} \quad (117)$$

and we are free to reinject (117) in (116) and then to reinject (116) in (114) to obtain that :

$$\mu_1(t, x) = \begin{cases} \mathcal{K} (\tilde{g}(t, x) + \mathbb{T}\tilde{g}(\underline{\mathbf{t}}, \underline{\mathbf{x}}) + \Lambda\mu_0(t, x)) + \mathcal{K}^2\Lambda\mathcal{K}\tilde{g}(\underline{\mathbf{t}}, \underline{\mathbf{x}}), & \text{if } M = 1, \\ \mathcal{K}\Lambda\mu_0(t, x) \\ + \mathcal{K}^2 \left( \delta_{M,2} \sum_{k=1}^2 \mathbb{T}^{k-1} \tilde{g}(\underline{\mathbf{t}}^k, \underline{\mathbf{x}}^k) + \sum_{k=2}^{\min(M,3)} \mathbb{T}^{k-2} \mu_3(\underline{\mathbf{t}}^k, \underline{\mathbf{x}}^k) \right) & \text{if } M > 1. \\ + \mathcal{K}^2\Lambda\mathcal{K}^2 (\delta_{M,2}\tilde{g}(\underline{\mathbf{t}}^2, \underline{\mathbf{x}}^2) + \delta_{M \geq 3}\mu_3(\underline{\mathbf{t}}^3, \underline{\mathbf{x}}^3)) \\ + \mathcal{K}\mathbb{T}\mathcal{K} (\delta_{M,2}\tilde{g}(\underline{\mathbf{t}}^2, \underline{\mathbf{x}}^2) + \delta_{M \geq 3}\mu_3(\underline{\mathbf{t}}^3, \underline{\mathbf{x}}^3)) , \end{cases} \quad (118)$$

Equation (118) ends the discussion in the particular case  $M = 1$ . To treat the case  $M > 1$ , let us remark that in (118) the unknown part of the right hand side of equation depends on  $\mu_3(\underline{\mathbf{t}}^2, \underline{\mathbf{x}}^2)$  and possibly on  $\mu_3(\underline{\mathbf{t}}^3, \underline{\mathbf{x}}^3)$ . So we are exactly in the same situation as in equation (114) with  $\mu_3(\underline{\mathbf{t}}^2, \underline{\mathbf{x}}^2)$ ,  $\mu_3(\underline{\mathbf{t}}^3, \underline{\mathbf{x}}^3)$  in place of  $\mu_2(\underline{\mathbf{t}}, \underline{\mathbf{x}})$ ,  $\mu_2(\underline{\mathbf{t}}^2, \underline{\mathbf{x}}^2)$  and we can thus repeat the same computations to express  $\mu_1$  in terms of  $\mu_4(\underline{\mathbf{t}}^3, \underline{\mathbf{x}}^3)$  and  $\mu_4(\underline{\mathbf{t}}^4, \underline{\mathbf{x}}^4)$  and so on.

A tedious (but not difficult) reiterative process shows that for all  $M > 1$ ,  $\mu_1$  is given by :

$$\begin{aligned} \mu_1(t, x_2) &= \mathcal{K}\Lambda\mu_0(t, x_2) + \mathcal{K}^M (\tilde{g}(\underline{\mathbf{t}}^{M-1}, \underline{\mathbf{x}}^{M-1}) + \mathbb{T}\tilde{g}(\underline{\mathbf{t}}^M, \underline{\mathbf{x}}^M)) + \sum_{k=0}^{M-1} \mathcal{K}^{M+1-k}\Lambda\mathcal{K}^{k+1}\tilde{g}(\underline{\mathbf{t}}^M, \underline{\mathbf{x}}^M) \\ &+ \sum_{k=1}^{M-1} \mathcal{K}^k\mathbb{T}\mathcal{K}^{M-k}\tilde{g}(\underline{\mathbf{t}}^M, \underline{\mathbf{x}}^M), \end{aligned} \quad (119)$$

equation which determines in a unique way  $\mu_1$  in terms of  $\tilde{g}$  and the known operators  $\Lambda$ ,  $\mathbb{T}$ . This completes the construction of  $\mu_1$  and more generally of the amplitudes of order one in the  $WKB$  expansion.

Moreover, all the terms composing (119) are in  $\mathcal{P}_b^M$ , except  $\tilde{g}(\underline{\mathbf{t}}^{M-1}, \underline{\mathbf{x}}^{M-1})$  which is in  $\mathcal{P}_b^{M-1}$ . Thus  $\mu_1$  defined by (119) is an element of  $\mathcal{P}_b^{M-1}$ , so it is in the good profile space. To conclude this discussion let us remark that before the time  $\underline{T}_M$ ,  $\mu_1$  (and consequently the  $u_{1,n_j}$ ) only depend on the information that was initially present and which has made  $M - 1$  complete circuits around the loop.

Once the amplitudes for indices associated to the loop are known, the construction of the amplitudes which do not lie on the loop follows the same kind of arguments as those given in paragraph 5.2. A precise construction will not be given here and we refer to [Ben] for more details.

### 5.3.2 Summary : the construction of higher order terms.

In this paragraph we sketch some elements about the construction of the amplitudes of higher order in view to give a summary of the previous construction. Let us assume that the amplitudes of order less than  $n - 1$  have already been determined, our aim is to construct the amplitude of order  $n$ .

*i)* First of all, from the cascade of equations (19), we know that the unpolarized part of the oscillating amplitudes of order  $n$  is known. So we only have to determine the polarized part of the oscillating amplitudes. Moreover, from Lemma 5.3 and the definition of the operator  $\mathbb{Q}_{ev1}$  (resp.  $\mathbb{Q}_{ev2}$ ), see (112), concerning the evanescent amplitudes, we will only have to determine  $\mathbb{P}_{ev1}$  (resp.  $\mathbb{P}_{ev2}$ ).

*ii)* To determine the polarized part of the oscillating amplitudes of order  $n$ , we start, as it has been done in paragraphs 5.1 and 5.3.1 by the determination of the loop's indices. We thus reiterate the computations of paragraph 5.1 to show that to construct these polarized parts it is necessary and sufficient to construct the scalar component of the trace on  $\partial\Omega_1$  of the amplitude  $u_{n,n_1}$ , that is

to say,  $\mu_n$ .

iii) The analysis of paragraphs 5.1.5 and 5.1.6 tells us that this trace has to satisfy equation (108). In other words,  $\mu_n$  depends on the unknown  $\mu_{n+1}$  and the known functions  $\mu_{n-1}$  (and possibly of  $\tilde{g}$ ). More precisely  $\mu_n$  depends on the  $\mu_{n+1}(\underline{\mathbf{t}}^k, \underline{\mathbf{x}}^k)$ , for  $k = 1, \dots, \min(n+1, M)$ . Then applying again equation (108) but for  $\mu_{n+1}$  we obtain that  $\mu_{n+1}$  depends on  $\mu_n$  and  $\mu_{n+2}$ . With more details, the  $\mu_{n+1}(\underline{\mathbf{t}}^k, \underline{\mathbf{x}}^k)$ , for  $k = 1, \dots, \min(n+1, M)$  admit the following dependencies :

$$\begin{array}{c|c|c} \mu_{n+1}(\underline{\mathbf{t}}, \underline{\mathbf{x}}) & \mu_n(\underline{\mathbf{t}}^k, \underline{\mathbf{x}}^k) & \mu_{n+2}(\underline{\mathbf{t}}^k, \underline{\mathbf{x}}^k) \\ \mu_{n+1}(\underline{\mathbf{t}}^2, \underline{\mathbf{x}}^2) & k = 1, \dots, \underline{\min} & k = 2, \dots, \overline{\min} \\ \vdots & k = 2, \dots, \underline{\min} & k = 3, \dots, \overline{\min} \\ \mu_{n+1}(\underline{\mathbf{t}}^{\min}, \underline{\mathbf{x}}^{\min}) & \vdots & \vdots \\ & k \in \emptyset & k \in \emptyset \text{ or } k = M \end{array} \quad (120)$$

where we denoted  $\underline{\min} := \min(n, M)$ ,  $\min := \min(n+1, M)$  and  $\overline{\min} = \min(n+2, M)$ . Let us study the dependency on  $\mu_n$ . The worst term is  $\mu_n(\underline{\mathbf{t}}, \underline{\mathbf{x}})$ , indeed the other terms will be eliminate before  $\mu_n(\underline{\mathbf{t}}, \underline{\mathbf{x}})$  (by the same arguments) and are harmless. Using equation (109), we obtain that  $\mu_n(\underline{\mathbf{t}}, \underline{\mathbf{x}})$  in fact depends on  $\mu_{n+1}(\underline{\mathbf{t}}^k, \underline{\mathbf{x}}^k)$ , for  $k = 2, \dots, \underline{\min}$ . But the tabular (120) tells us that these traces depends, in fact, on  $\mu_n(\underline{\mathbf{t}}^k, \underline{\mathbf{x}}^k)$  for  $k = 2, \dots, \underline{\min}$ . As a consequence,  $\mu_{n+1}(\underline{\mathbf{t}}, \underline{\mathbf{x}})$  can be express in terms of the  $\mu_n(\underline{\mathbf{t}}^k, \underline{\mathbf{x}}^k)$  for  $k = 2, \dots, \underline{\min}$ . Repeating the same argument we obtain that  $\mu_{n+1}(\underline{\mathbf{t}}, \underline{\mathbf{x}})$  can be express in terms of the  $\mu_n(\underline{\mathbf{t}}^k, \underline{\mathbf{x}}^k)$  for  $k = 3, \dots, \underline{\min}$ . Then if one repeats  $\underline{\min}$  times this argument, he obtains that the  $\mu_{n+1}(\underline{\mathbf{t}}^k, \underline{\mathbf{x}}^k)$  for  $k = 1, \dots, \underline{\min}$  (and consequently  $\mu_n$ ) can be express in terms of  $\mu_{n+2}(\underline{\mathbf{t}}^k, \underline{\mathbf{x}}^k)$  for  $k > 2$  only.

We thus repeat exactly the same reasoning for  $\mu_{n+2}(\underline{\mathbf{t}}^k, \underline{\mathbf{x}}^k)$  to show that it can be express in terms of the  $\mu_{n+3}(\underline{\mathbf{t}}^k, \underline{\mathbf{x}}^k)$  for  $k > 3$  and so one to determine  $\mu_n$  in terms of  $\tilde{g}$  and  $\mu_{n-1}$  (up to some compositions by the operators  $\mathbb{T}$  and  $\Lambda$ ) only, as it has already been made in paragraphs 5.1.6 and 5.3.1. This conclude the construction of the trace  $\mu_n$  and as a consequence, the construction of the amplitudes linked to the loop's indeces of order  $n$ .

iv) The construction of the other polarized part of the oscillating (or equivalently of the  $\mathbb{P}_{ev1}U$  (resp.  $\mathbb{P}_{ev2}U$ ) for the evanescent amplitudes for the side  $\partial\Omega_1$  (resp.  $\partial\Omega_2$ )) is then easy. Indeed these amplitudes are linked to frequencies for which the uniform Kreiss-Lopatinskii condition holds and we can use the "tree" structure of the frequency set to conclude (see [Ben] for a precise construction).

## 6 Proof of the main result.

With Theorem 5.1 in hand, it is now easy to show Theorem 3.1. We argue by contradiction. We thus assume that the corner problem (1) is weakly (or strongly) well-posed in the sense that there exists a positive integer  $K$ , such that the corner problem (1) admits a solution satisfying the energy estimate :

$$\|u\|_{L^2(\Omega_T)}^2 + \|u|_{x_1=0}\|_{L^2(\partial\Omega_{1,T})}^2 + \|u|_{x_2=0}\|_{L^2(\partial\Omega_{2,T})}^2 \leq C_T \left( \|f\|_{L^2(\Omega_T)}^2 + \|g_1\|_{H^K(\partial\Omega_{1,T})}^2 + \|g_2\|_{L^2(\partial\Omega_{2,T})}^2 \right), \quad (121)$$

for  $T > 0$  if  $\beta \geq 1$  and  $0 < T < T_{max}$  if  $\beta < 1$ . According to Theorem 5.1, for any  $M \in \mathbb{N}$  one can always construct a truncated geometric optics expansions for the corner problem (14). It is given

by :

$$\begin{aligned}
u_{app}^\varepsilon &:= \sum_{n=0}^1 \sum_{k \in \mathcal{J}_{os}} \varepsilon^n e^{i\varphi_k(t,x)} u_{n,k}(t,x) \\
&+ \sum_{n=0}^1 \sum_{k \in \mathfrak{X}_1} \varepsilon^n e^{i\psi_{k,1}(t,x_2)} U_{n,k,1} \left( t, x, \frac{x_1}{\varepsilon} \right) + \sum_{n=0}^1 \sum_{k \in \mathfrak{X}_2} \varepsilon^n e^{i\psi_{k,2}(t,x_1)} U_{n,k,2} \left( t, x, \frac{x_2}{\varepsilon} \right).
\end{aligned} \tag{122}$$

Moreover one can always choose  $g$  in such a way that the leading order of  $u_{app}^\varepsilon$  is not identically zero.

Let  $u^\varepsilon$  be a solution of (14). The error  $u^\varepsilon - u_{app}^\varepsilon$  satisfies the corner problem :

$$\begin{cases} L(\partial)(u^\varepsilon - u_{app}^\varepsilon) = f^\varepsilon, & \text{in } \Omega_T, \\ B_1(u^\varepsilon - u_{app}^\varepsilon)|_{x_1=0} = 0, & \text{on } \partial\Omega_{1,T}, \\ B_2(u^\varepsilon - u_{app}^\varepsilon)|_{x_2=0} = 0, & \text{on } \partial\Omega_{2,T} \\ (u^\varepsilon - u_{app}^\varepsilon)|_{t \leq 0} = 0, & \text{on } \Omega, \end{cases}$$

where :

$$f^\varepsilon := \varepsilon \left[ \sum_{k \in \mathcal{J}_{os}} e^{i\frac{\varphi_k}{\varepsilon}} L(\partial)u_{1,k} + \sum_{k \in \mathfrak{X}_1} e^{i\frac{\psi_{k,1}}{\varepsilon}} (L(\partial)U_{1,k,1})|_{X_1=\frac{x_1}{\varepsilon}} + \sum_{k \in \mathfrak{X}_2} e^{i\frac{\psi_{k,2}}{\varepsilon}} (L(\partial)U_{1,k,2})|_{X_2=\frac{x_2}{\varepsilon}} \right].$$

It follows from the energy estimate (121) that :

$$\|u^\varepsilon - u_{app}^\varepsilon\|_{L^2(\Omega_T)} \leq \|f^\varepsilon\|_{L^2(\Omega_T)}. \tag{123}$$

It is then easy to see from its expression that  $f^\varepsilon$  is  $O(\varepsilon)$  in  $L^2(\Omega_T)$ . Using the fact that  $u^\varepsilon$  is a solution of (14) we obtain that

$$\|u^\varepsilon\|_{L^2(\Omega_T)} \leq \|g^\varepsilon\|_{H^K(\Omega_T)}, \tag{124}$$

from which we deduce that  $u^\varepsilon$  is  $O(\varepsilon^{M-K})$  in  $L^2(\Omega_T)$ . Let us choose  $M > K$ , by the triangle inequality and inequalities (123) and (124), it follows that  $u_{app}^\varepsilon$  is at least  $O(\varepsilon)$  which is a contradiction with the fact that  $u_{0,n_1}$  is nonzero.

## 7 Examples, conclusion and conjectures.

### 7.1 Examples.

We consider the following corner problem :

$$\begin{cases} \partial_t u + A_1 \partial_1 u + A_2 \partial_2 u = 0, & \text{for } x_1, x_2 > 0 \\ B_1 u|_{x_1=0} = g^\varepsilon, & \text{for } x_2 > 0, \\ B_2 u|_{x_2=0} = 0, & \text{for } x_1 > 0, \\ u|_{t \leq 0} = 0, \end{cases} \tag{125}$$

with

$$A_1 := \begin{bmatrix} 0 & \sqrt{5} & 0 \\ \sqrt{5} & -4 & 0 \\ 0 & 0 & -\frac{5}{7} \end{bmatrix}, \text{ and } A_2 := \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

One can see that this example is just a modification of the example in [Ben15] to make sure that the transport velocity along the boundary has the good sign.

The section of the characteristic variety  $V$  with  $\{\underline{t} = 1\}$  is given by :

$$V_{\{\underline{t}=1\}} = \left\{ (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \left( 1 - \frac{5}{7}\xi_1 + \xi_2 \right) (1 - 4\xi_1 + 4\xi_1\xi_2 - 5\xi_1^2 - \xi_2^2) = 0 \right\},$$

and is depicted in Figure 3 as well as a part of the phase generation process associated to the source term :

$$g^\varepsilon(t, x_2) := e^{\frac{i}{\varepsilon}(t-8x_2)}g(t, x_2),$$

In particular, let us remark that we have a loop with phases  $\varphi_1, \dots, \varphi_4$  defined by :

$$\begin{aligned} \varphi_1(t, x) &:= t - \frac{21}{5}x_1 - 8x_2, \quad v_1 := \begin{bmatrix} 3 \\ \frac{2}{5} \end{bmatrix}, \\ \varphi_2(t, x) &:= t - \frac{21}{5}x_1 - 4x_2, \quad v_2 := \begin{bmatrix} -\frac{5}{7} \\ 1 \end{bmatrix}, \\ \varphi_3(t, x) &:= t - 3x_1 - 4x_2, \quad v_3 := \begin{bmatrix} 5 \\ -2 \end{bmatrix}, \\ \varphi_4(t, x) &:= t - 3x_1 - 8x_2, \quad v_4 := \begin{bmatrix} 5 \\ -2 \end{bmatrix}, \end{aligned}$$

where the  $v_j$  denote the group velocities. Thus one can see that the group velocities  $v_1, v_3$  are incoming-outgoing while the group velocities  $v_2, v_4$  are outgoing-incoming. One can also shows that the stable subspace  $E_1^s(i, -8) := \text{vect}(e)$  where the generator vector  $e$  is given by :

$$e := \begin{bmatrix} \frac{7}{3} \\ \sqrt{5} \\ 0 \end{bmatrix}.$$

Then if one chooses for boundary condition on the side  $\partial\Omega_1$  : for fixed  $\delta \in \mathbb{R}$

$$B_1 := \begin{bmatrix} -\sqrt{5} & \frac{7}{3} & \delta \end{bmatrix},$$

he can also easily see that  $B_1 e_1 = 0$ , and thus the uniform Kreiss-Lopatinskii condition is not satisfied for the side  $\partial\Omega_1$ . Some extra computations allow to show that such a  $B_1$  gives a problem in the  $WR$  class for the side  $\partial\Omega_1$ . Then one choose  $B_2$  such that the associated boundary condition is for example strictly dissipative. The velocity of the transport along the side  $\partial\Omega_1$  is then given by  $\frac{1}{8}$  and we can apply Theorem 3.1 to show that the corner problem (125) is ill-posed.

Let us remark that apply Theorem 3.1 to the corner problem (125) may seem to be a bit abusive. Indeed, Theorem 3.1 needs strict hyperbolicity while the corner problem (125) is clearly not strictly hyperbolic. However, in the proof of Theorem 3.1, the only points where we used the hyperbolicity assumption were to establish the block structure and to use Lax Lemma. It can be show that these points are still true for geometrically regular hyperbolic systems (see [MZ05] for a precise definition) as soon as we are away from the crossing points, which is the case for (125) under this choice of the source term.

As a consequence if one wants to give an example for which Theorem 3.1 apply, that is to say that a strictly hyperbolic operator is needed, the simplest way to do that is probably to use the example of [SS75, paragraph 7]. In this example, the authors construct a system whose characteristic variety is composed of two intersecting ellipses chosen in such a configuration that a loop exists. Without loss of generality, we can assume that the loop is include in the half space  $\{\xi_2 < 0\}$ , to make sure

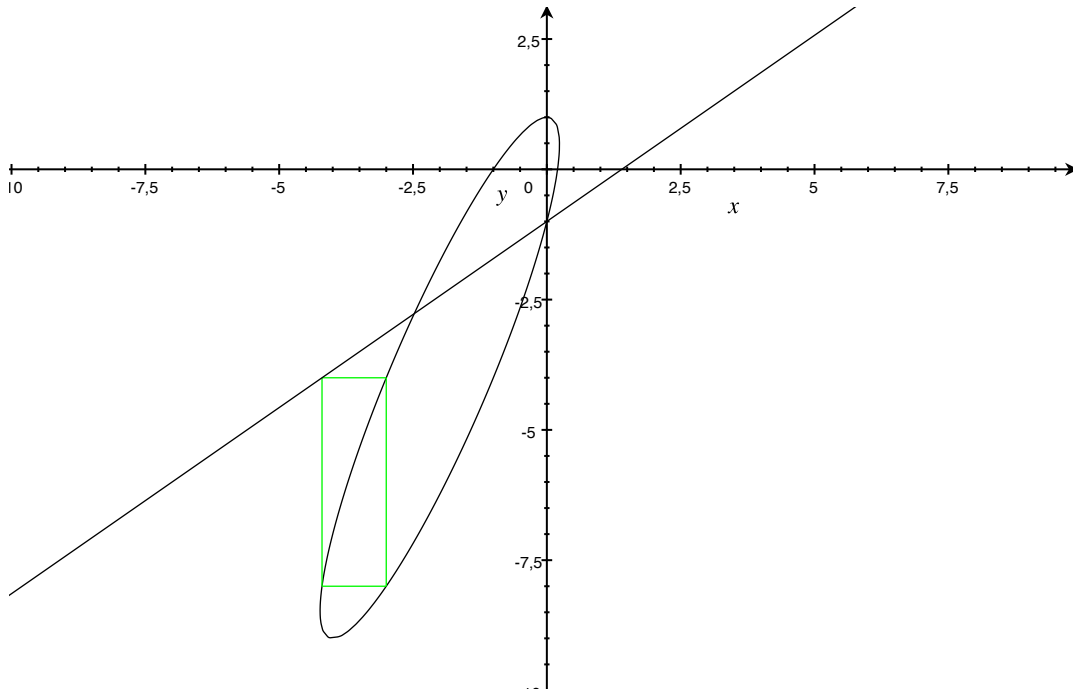


Figure 3: The characteristic variety for the corner problem (125).

that the velocity along the boundary has the good sign. Then one chooses a boundary condition to make sure that the associated corner problem is in the  $WR$  class. This point can be easily done because at this stage of the construction the system is composed of two decoupled parts.

To obtain a strictly hyperbolic system, it is sufficient to make the perturbation described in [SS75, paragraph 9]. One can conclude that the associated perturbed operator is strictly hyperbolic and that it is still in  $WR$  class, because this class is stable by small perturbations (see [BGRSZ02]).

## 7.2 Conclusion and conjectures.

In this article, we built the rigorous geometric optics expansion for a corner problem for which the uniform Kreiss-Lopatinskii condition breaks down on a selfinteraction frequency. We have shown that it was always possible to choose the source term of the corner problem in such a way that the associated geometric optics expansion suffers  $M + 1$  amplifications, before a fixed time  $T$ , compared to the source term, where  $M$  can be arbitrarily big. This leads us to the fact that such a corner problem can not be weakly well-posed because it suffers an arbitrarily big number of loss of derivative on the side  $\partial\Omega_1$  and thus present an Hadamard instability.

As a consequence we show that for hyperbolic corner problem a weak instability can be repeated an arbitrary number of times to cause a violent instability which differs from the degeneracy of the weak Kreiss-Lopatinskii condition. In terms of well-posedness it is the worst case possible.

However there are other kinds of degeneracy of the uniform Kreiss-Lopatinskii condition. And in these cases, one should be more optimistic. Let us formulate the following reasonable conjectures.

If the uniform Kreiss-Lopatinskii condition breaks down in the elliptic area, using the fact that evanescent modes for the side  $\partial\Omega_1$  are not reflected on the side  $\partial\Omega_2$ , the amplification observed



in the high frequency expansion for the boundary value problem in the half space should not be improved. So we believe that the leading order in the WKB expansion should be of order  $\varepsilon^0$  with a source term in the interior of order  $\varepsilon$  and a source term on the side  $\partial\Omega_1$  of order  $\varepsilon^0$ . The associated corner problem should be weakly well-posed with an energy estimate reading :

$$\|u\|_{L^2(\Omega_T)}^2 + \|u|_{x_1=0}\|_{L^2(\partial\Omega_{1,T})}^2 + \|u|_{x_2=0}\|_{L^2(\partial\Omega_{2,T})}^2 \leq C_T \left( \|f\|_{L^2(\Omega_T)}^2 + \|g_1\|_{H^1(\partial\Omega_{1,T})}^2 + \|g_2\|_{L^2(\partial\Omega_{2,T})}^2 \right).$$

When the uniform Kreiss-Lopatinskii condition is violated in the mixed area in such a way that  $\ker B_1 \cap E_1^s(\zeta) = \ker B_1 \cap E_1^{s,e}(\zeta) = \text{span}\{e\}$ , where  $E_1^{s,e}(\zeta)$  denotes the "elliptic" component of  $E_1^s(\zeta)$  (see [Ben14] and [Les07]), then the same argument should apply. The conjecture is then that we have the same amplifications as for the boundary value problem in the half space. The expected energy estimate is :

$$\|u\|_{L^2(\Omega_T)}^2 + \|u|_{x_1=0}\|_{L^2(\partial\Omega_{1,T})}^2 + \|u|_{x_2=0}\|_{L^2(\partial\Omega_{2,T})}^2 \leq C_T \left( \|f\|_{H^{\frac{1}{2}}(\Omega_T)}^2 + \|g_1\|_{H^1(\partial\Omega_{1,T})}^2 + \|g_2\|_{L^2(\partial\Omega_{2,T})}^2 \right),$$

or

$$\|u\|_{L^2(\Omega_T)}^2 + \|u|_{x_1=0}\|_{L^2(\partial\Omega_{1,T})}^2 + \|u|_{x_2=0}\|_{L^2(\partial\Omega_{2,T})}^2 \leq C_T \left( \|f\|_{L^2(\Omega_T)}^2 + \|g_1\|_{H^1(\partial\Omega_{1,T})}^2 + \|g_2\|_{L^2(\partial\Omega_{2,T})}^2 \right),$$

depending of a technical assumption already discussed in [Ben14].

Finally when the uniform Kreiss-Lopatinskii condition fails in the hyperbolic region but for non-selfinteracting frequencies (or equivalently in the mixed region with  $\ker B_1 \cap E_1^s(\zeta) = \ker B_1 \cap E_1^{s,h}(\zeta) = \text{span}\{e\}$ , where  $E_1^{s,h}(\zeta)$  denotes the "hyperbolic" component of  $E_1^s(\zeta)$ ) the conjecture is that the leading order in the geometric optics expansion is of order  $\varepsilon^0$  for source terms of order  $\varepsilon^M$ , where  $M$  denotes the number of time that a ray has been amplified. More precisely  $M$  is defined by :

$$M := \max_{i \in \mathcal{I}_{os}} \#(\Upsilon \cap L_i),$$

where  $L_i$  is the set containing the values of the type  $V$  sequence linking  $i$  to the first generated index in the phase generation process. Indeed, an amplified ray should be amplified again if it contains in its reflections a phase associated to a frequency of degeneracy of the uniform Kreiss-Lopatinskii condition. The conjectured energy estimate is thus given by :

$$\begin{aligned} \|u\|_{L^2(\Omega_T)}^2 + \|u|_{x_1=0}\|_{L^2(\partial\Omega_{1,T})}^2 + \|u|_{x_2=0}\|_{L^2(\partial\Omega_{2,T})}^2 &\leq C_T \left( \|f\|_{L_{\tilde{x}_1}^2(H^M(\partial\Omega_{1,T}))}^2 \right. \\ &\quad \left. + \|g_1\|_{H^M(\partial\Omega_{1,T})}^2 + \|g_2\|_{L^2(\partial\Omega_{2,T})}^2 \right). \end{aligned} \quad (126)$$

All the previous conjectures are made under the assumption that the transport along the boundary spreads the information away from the corner. The energy estimates and the amplifications in the geometric optics expansions when the transport along the boundary send the information to the corner is left for future studies.

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## Appendix

### Details of the computations to establish equations (80)-(81) and (82).

Let us develop :

$$\begin{aligned} Q_1^{n_3} L(\partial)(I - P_1^{n_3})u_{1,n_3} &= -\frac{\bar{\xi}_2}{\xi_2} \beta_1^{-1} \beta_2^{-1} \left[ Q_1^{n_3} R_1^{n_3} \mathcal{S}^{n_3} e \left( \partial_{tt}^2 \mu_0 - \frac{\tau}{\xi_2} \partial_{t_2}^2 \mu_0 \right) \right. \\ &+ Q_1^{n_3} A_2 R_1^{n_3} \mathcal{S}^{n_3} e \left( \left( -\frac{1}{v_{n_2,2}} + \frac{\beta_2^{-1}}{v_{n_1,1}} \right) \partial_{tt}^2 \mu_0 + \beta_1^{-1} \beta_2^{-1} \partial_{t_2}^2 \mu_0 \right. \\ &\left. \left. - \frac{\tau}{\xi_2} \left( \left( -\frac{1}{v_{n_2,2}} + \frac{\beta_2^{-1}}{v_{n_1,1}} \right) \partial_{t_2}^2 \mu_0 + \beta_1^{-1} \beta_2^{-1} \partial_{22}^2 \mu_0 \right) \right) \right] (\mathbf{t}_3, \mathbf{x}_3), \end{aligned}$$

where we used the fact that  $Q_1^{n_3} A_1 R_1^{n_3} = 0$ . We then use the relation between  $Q_1^{n_3} R_1^{n_3} \mathcal{S}^{n_3} e$  and  $Q_1^{n_3} A_2 R_1^{n_3} \mathcal{S}^{n_3} e$  given in (78) to express  $Q_1^{n_3} L(\partial)(I - P_1^{n_3})u_{1,n_3}$  in terms of  $Q_1^{n_3} R_1^{n_3} \mathcal{S}^{n_3} e$  only :

$$\begin{aligned} Q_1^{n_3} L(\partial)(I - P_1^{n_3})u_{1,n_3} &= -\frac{\bar{\xi}_2}{\xi_2} \beta_1^{-1} \beta_2^{-1} Q_1^{n_3} R_1^{n_3} \mathcal{S}^{n_3} e \left[ \partial_{tt}^2 \mu_0 \left( 1 + \frac{\tau}{\xi_2} \left( \frac{1}{v_{n_2,2}} - \frac{\beta_2^{-1}}{v_{n_1,1}} \right) \right) \right. \\ &\left. - C_3 \partial_{t_2}^2 \mu_0 + \partial_{22}^2 \mu_0 \beta_1^{-1} \beta_2^{-1} \frac{\tau^2}{\xi_2 \xi_2} \right] (\mathbf{t}_3, \mathbf{x}_3), \end{aligned} \quad (127)$$

where

$$C_3 := \frac{\tau}{\xi_2} \left[ 1 + \frac{\tau}{\xi_2} \left( \frac{1}{v_{n_2,2}} - \frac{\beta_2^{-1}}{v_{n_1,1}} \right) \right] + \frac{\tau}{\xi_2} \beta_1^{-1} \beta_2^{-1}.$$

From Lemma 5.2, we know that the first term in  $C_3$  is  $\frac{\bar{\xi}_2}{\xi_2} \beta_1^{-1} \beta_2^{-1}$ . We thus obtain that  $C_3 = 2 \frac{\tau}{\xi_2} \beta_1^{-1} \beta_2^{-1}$ , and we can factorize (127) to obtain (81).

Using the fact that  $Q_2^{n_4} A_2 R_2^{n_4} = 0$  we develop  $Q_2^{n_4} L(\partial)(I - P_2^{n_4})u_{1,n_4}$  to obtain :

$$\begin{aligned} Q_2^{n_4} L(\partial)(I - P_2^{n_4})u_{1,n_4} &= -\frac{\bar{\xi}_2}{\xi_1} \prod_{j=1}^3 \beta_j^{-1} \left[ Q_2^{n_4} R_2^{n_4} \mathcal{S}^{n_4} e \left( \partial_{tt}^2 \mu_0 - \frac{\tau}{\xi_2} \partial_{t_2}^2 \mu_0 \right) \right. \\ &+ Q_2^{n_4} A_1 R_2^{n_4} \mathcal{S}^{n_4} e \left( -A \partial_{tt}^2 \mu_0 - \prod_{j=1}^3 \beta_j^{-1} \partial_{t_2}^2 \mu_0 \right. \\ &\left. \left. - \frac{\tau}{\xi_2} \left( -A \partial_{t_2}^2 \mu_0 - \prod_{j=1}^3 \beta_j^{-1} \partial_{22}^2 \mu_0 \right) \right) \right] (\mathbf{t}_4, \mathbf{x}_4), \end{aligned}$$

where we recall that  $A$  is defined in (38). We then use (79) to express this equation in terms of  $Q_2^{n_4} R_2^{n_4} \mathcal{S}^{n_4} e$  only :

$$Q_2^{n_4} L(\partial)(I - P_2^{n_4})u_{1,n_4} = -\frac{\bar{\xi}_2}{\xi_1} \prod_{j=1}^3 \beta_j^{-1} Q_2^{n_4} R_2^{n_4} \mathcal{S}^{n_4} e \left[ \partial_{tt}^2 \mu_0 \left( 1 + A \frac{\tau}{\xi_1} \right) + C_4 \partial_{t_2}^2 \mu_0 + \frac{\tau^2}{\xi_1 \bar{\xi}_2} \prod_{j=1}^3 \beta_j^{-1} \partial_{22}^2 \mu_0 \right] (\mathbf{t}_4, \mathbf{x}_4),$$

where  $C_4$  is defined by :

$$C_4 := -\frac{\tau}{\xi_2} \left( 1 + A \frac{\tau}{\xi_1} \right) + \frac{\tau}{\xi_1} \prod_{j=1}^3 \beta_j^{-1}.$$

Once again thanks to Lemma 5.2, we obtain that  $C_4$  in fact reads  $C_4 = 2 \frac{\tau}{\xi_1} \prod_{j=1}^3 \beta_j^{-1}$ . As a consequence we obtain (82).

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