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What has Instrumental Variable method to offer for system identification?

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Abstract: This paper gathers several experiences of using instrumental variable method in different contexts: closed-loop system identification, LPV model, frequency domain framework.

Keywords: instrumental variable, system identification, closed-loop, LPV model, frequency domain

1. INTRODUCTION

Mathematical models of dynamic systems are required in most area of scientific enquiry and take various forms, such as differential equations, difference equations, state-space equations and transfer functions. The most widely used approach to mathematical modeling involves the construction of mathematical equations based on physical laws that are known to govern the behaviour of the system. While the advantage of these methods relies on the deep physical insight of the resulting model, their main drawback is the complexity of the model that makes them difficult to be used in applications such as control system design, prediction or decision making.

An alternative to physically-based mathematical modeling is the so-called data-based system identification, which can be applied to any system where experimental data are available. A large scope of system identification approaches has been developed over the past decades. Amongst these, we can cite the prediction error and maximum-likelihood frameworks (see *e.g.* Ljung (1999); Söderström and Stoica (1989); Young (2011)), the subspace-based identification (see *e.g.* Van Overschee and De Moor (1996); Katayama (2005)), the frequency-domain identification (see *e.g.* Pintelon and Schoukens (2001); McKelvey (2002)), the closed-loop identification case (see *e.g.* Van den Hof (1998); Forssell and Ljung (1999); Ninness and Hjalmarsson (2005); Gilson and Van den Hof (2005)).

Most physical systems are continuous-time (CT) whereas, mainly due to the advent of digital computers, research on system identification has concentrated on discrete-time (DT) models from underlying CT systems input/output samples. Recently, interest in identification of CT systems from DT data has arisen (see *e.g.* Sinha and Rao (1991); Unbehauen and Rao (1987); Garnier and Wang (2008) and references herein) and offer a clever solution in many cases such as irregularly sampled data.

Moreover, systems encountered in practice are often nonlinear or present a time-varying nature. Unlike linearity, non-linearity is a non-property and therefore, non-linearity cannot be defined in a general way. A common framework for the identification of nonlinear models has nevertheless been presented in Sjöberg et al. (1995) and Juditsky et al. (1995). Usually, nonlinear models are classified into two classes: non-parametric models and parametric models. However, another type of models

has more recently arose the attention of the system identification community and form an intermediate step between Linear Time-Invariant (LTI) systems and nonlinear/time-varying plants: the model class of Linear Parameter-Varying (LPV) systems (Bamieh and Giarré (2002); Tóth (2010)).

When considering methods that can be used to identify (linear or non linear, CT or DT) models of systems operating in open or closed-loop, instrumental variable (IV) techniques are rather attractive since they are normally simple or iterative modifications of the linear regression algorithm. For instance, when dealing with complex processes, it can be attractive to rely on methods, such as these, that do not require non-convex optimization algorithms. In addition to this computationally attractive property, IV methods also have the potential advantage that they can yield consistent and asymptotically unbiased estimates of the plant model parameters if the noise does not have rational spectral density or if the noise model is mis-specified; or even if the control system is non-linear and/or time-varying, in the closed-loop framework (Gilson and Van den Hof (2005); Gilson et al. (2011)). Even if several works arise these last ten years (*e.g.* Young (2011); Dankers et al. (2014); Van Herpen et al. (2014); Laurain et al. (2010); Douma (2006)), IV methods have not yet really received the attention that it deserves.

This paper is dedicated to the use of IV methods in several cases of system identification. After an introduction of the IV principles in Section 2, the focus is made on closed-loop system in Section 3, on LPV models with an application on rainfall-flow modeling in Section 4 and on frequency domain framework in Section 5.

2. INSTRUMENTAL VARIABLE METHOD

System identification is based on three main ingredients: data (experiment design), model set selection, identification criterion, which are used to estimate a model of a given system. In this paper, we will mainly focus on the identification criterion named Instrumental Variable (IV).

IV is a criterion aiming at minimizing the prediction error. Consider a stable, linear, Single Input Single Output (SISO) data-generating system assumed to be described as

$$S : y(t_k) = G_0(q)u(t_k) + H_0(q)e(t_k) \quad (1)$$

The plant is denoted by $G_0(q) = B_0(q^{-1})/A_0(q^{-1})$ with the numerator and denominator degree equals to n_0 , q^{-1} is the

delay operator with $q^{-i}x(t_k) = x(t_{k-i})$. u describes the plant input signal, y the plant output signal. A colored disturbance $\xi_0(t_k) = H_0(q)e_0(t_k)$ is assumed to affect the system, where e_0 is a white noise, with zero mean and variance $\sigma_{e_0}^2$.

The following general model structure and parameter plant model are chosen to model the system

$$\mathcal{M}: y(t_k) = G(q, \theta)u(t_k) + H(q, \theta)\varepsilon(t_k) \quad (2)$$

$$\mathcal{G}: G(q, \theta) = \frac{B(q^{-1}, \theta)}{A(q^{-1}, \theta)} \quad (3)$$

In the prediction error method (PEM), the parameters are computed by minimizing the criterion function (see Ljung (1999))

$$V(q, \theta) = \frac{1}{N} \sum_{k=1}^N [\varepsilon(t_k, \theta)]^2 \quad (4)$$

where $\varepsilon(t_k, \theta) = y(t_k) - \hat{y}(t_k, \theta)$ is the prediction error. Therefore, the parameters are given as

$$\hat{\theta} = \arg \min_{\theta} \frac{1}{N} \sum_{k=1}^N (y(t_k) - \hat{y}(t_k, \theta))^2 \quad (5)$$

It has to be noted that the estimation of $\hat{\theta}$ might be a non convex optimization problem for a general nonlinear one-step-ahead predictor. However, the problem (5) can be simplified, for *e.g.*, by choosing an adequate model structure. As a result, for a linear regression, $\hat{\theta}$ is provided by solving the LS solution where

$$\hat{\theta}_{ls} = \arg \min_{\theta} \sum_{k=1}^N (y(t_k) - \varphi^T(t_k)\theta)^2 \quad (6)$$

with $\varphi(t_k)$ denotes the regressor.

The other solution is to use the IV criterion where its basic version aims at computing the estimate $\hat{\theta}$ by solving (Söderström and Stoica (1983))

$$\hat{\theta}_{biv} = \text{sol} \left\{ \frac{1}{N} \sum_{k=1}^N \zeta(t_k) (y(t_k) - \varphi^T(t_k)\theta) = 0 \right\} \quad (7)$$

where $\zeta(t_k)$ is the so-called instrument. There is a large amount of freedom in the choice of the instrument. It should be correlated with the data but uncorrelated with the noise. This idea has been generalized to the extended IV framework where

$$\hat{\theta}_{xiv} = \arg \min_{\theta} \left\| \left[\frac{1}{N} \sum_{k=1}^N L(q)\zeta(t_k)L(q)\varphi^T(t_k) \right] \theta - \left[\frac{1}{N} \sum_{k=1}^N L(q)\zeta(t_k)L(q)y(t_k) \right] \right\|_W^2, \quad (8)$$

where $\zeta(t) \in \mathbb{R}^{n_\zeta}$ with $n_\zeta \geq 2n$, $\|x\|_W^2 = x^T W x$, with W a positive definite weighting matrix and $L(q)$ a stable prefilter.

By definition, when $G_0 \in \mathcal{G}$, the extended-IV estimate is consistent under the following two conditions¹

- $\mathbb{E}L(q)\zeta(t_k)L(q)\varphi^T(t_k)$ is full column rank,
- $\mathbb{E}L(q)\zeta(t_k)L(q)v_0(t_k) = 0$.

The interesting property of the IV methods is that they provide asymptotically unbiased estimates even if the noise is misspecified. However, the choice of $\zeta(t_k)$, n_ζ , W and the prefilter $L(q)$ may have a considerable effect on the covariance matrix.

¹ The notation $\mathbb{E}[\cdot] = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N \mathbb{E}[\cdot]$ is adopted from the prediction error framework of Ljung (1999).

The optimal IV algorithm providing the minimum value of the covariance matrix is known to be obtained for (see Söderström and Stoica (1983); Young (2011, 2014))

$$\hat{\varphi}_f(t_k) = L^{opt}(q)\hat{\varphi}(t_k), \quad (9)$$

$$L^{opt}(q) = \frac{1}{A_0(q^{-1})H_0(q)}, \text{ and } \zeta(t_k) = \hat{\varphi}(t_k). \quad (10)$$

where $\hat{\varphi}(t_k)$ is the noise-free part of $\varphi(t_k)$. Using equations (8) and (9)-(10), the following IV estimate is optimal

$$\hat{\theta}_{iv}^{opt}(N) = \left(\sum_{t=1}^N \zeta_f(t_k)\varphi_f^T(t_k) \right)^{-1} \left(\sum_{t=1}^N \zeta_f(t)y_f(t) \right) \quad (11)$$

and where the regressor $\varphi_f(t_k) = L^{opt}(q)\varphi(t_k)$, the output $y_f(t_k) = L^{opt}(q)y(t_k)$ and the instrument vector $\zeta_f(t_k) = L^{opt}(q)\zeta(t_k)$ are filtered by $L^{opt}(q)$ (10).

It has to be noted that in this IV estimator, the optimal choice of instruments and prefilter is dependent on unknown system properties which has to be taken care of with an iterative procedure.

3. CLOSED-LOOP SYSTEM IDENTIFICATION

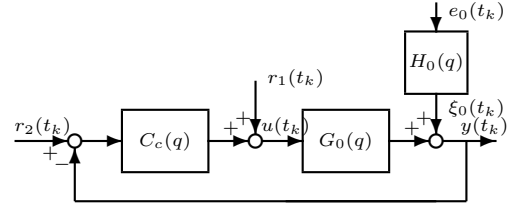


Fig. 1. Closed-loop system configuration

The basic difference between open-loop and closed-loop (CL) system identification is due to the correlation between the input $u(t_k)$ and the noise which conduces the usual open-loop system identification procedure to bias results in the closed-loop context. Therefore several closed-loop methods have been dealt with in the literature and this paper focuses on the IV solution.

Consider a stable, linear, SISO, closed-loop system of the form shown in Figure 1. The data generating system is assumed to be given by the following relations

$$\mathcal{S}: \begin{cases} y(t_k) = G_0(q)u(t_k) + H_0(q)e_0(t_k) \\ u(t_k) = r(t_k) - C_c(q)y(t_k), \end{cases} \quad (12)$$

$$\text{where } r(t_k) = r_1(t_k) + C_c(q)r_2(t_k). \quad (13)$$

The plant is denoted by $G_0(q) = B_0(q^{-1})/A_0(q^{-1})$ with the numerator and denominator degree equals to n_0 , the controller is denoted by $C_c(q)$. The general model structure and parameterized plant model are chosen respectively as

$$\mathcal{M}: y(t_k) = G(q, \theta)u(t_k) + H(q, \theta)\varepsilon(t_k, \theta), \quad (14)$$

$$\mathcal{G}: G(q, \rho) = \frac{B(q^{-1}, \theta)}{A(q^{-1}, \theta)} \quad (15)$$

where n denotes the plant model order and with the pair (B, A) assumed to be coprime.

As for the open-loop situation, the choice of the design variables as the instrument $\zeta(t)$ and the prefilter $L(q)$ have a considerable effect on the covariance matrix produced by the IV estimation algorithm. The covariance properties of the closed-loop IV methods have been investigated in Gilson and Van den Hof (2005) and further insights about the choice of these design

variables are dealt with in Gilson et al. (2011). It has been shown that the optimal solution is given by (11), however, since not only the output but also the input are corrupted by the noise, the computation of the noise-free regressor (9) has to take into account the noise-free part of y and u .

As a result, the closed-loop optimal IV estimator is dependent on unknown system properties, *i.e.* the plant as well as the noise dynamics. Whereas dependency of plant dynamics could be taken care of by an iterative procedure where the instrument and prefilter are constructed on the basis of a previous plant model estimate $\hat{\theta}^{i-1}$, knowledge of the noise dynamics is generally missing in an extended IV estimator as it is not particularly estimated.

Therefore the next step to an optimal IV method is to extend the estimator (8) with a procedure to estimate an appropriate noise model, to be used as a basis for constructing the optimal prefilter $L^{opt}(q)$ given in (10).

Then several solutions have been proposed in a unified way, depending on the structure of the process and noise models (see Gilson et al. (2011)): ARX, ARARX, OE, BJ models are analyzed to develop the *refined IV* method dedicated to the closed-loop framework.

Simulation examples

The simulation model is based on the relations (12), where

$$G_0(q) = \frac{0.0997q^{-1} - 0.0902q^{-2}}{1 - 1.8858q^{-1} + 0.9048q^{-2}}, \quad n = 2$$

$$C_c(q) = \frac{10.75 - 9.25q^{-1}}{1 - q^{-1}},$$

The excitation signal $r(t)$ is a pseudo random binary signal of maximal length, with the number of stages for the shift register set to 9 and the clock period set to 8; $e_0(t)$ is a white noise uncorrelated with $r(t)$. The case of colored noise acting on the loop is illustrated on this example, with

$$H_0(q) = \frac{1 + 0.5q^{-1}}{1 - 0.85q^{-1}}.$$

The following algorithms are used to estimate this model Gilson et al. (2011):

- *clivr* method (with $G_0 \in \mathcal{G}$, $S \notin \mathcal{M}$), the 1st closed-loop IV method, where $\zeta(t_k)$ = delayed version of the reference signal $r(t_k)$, method initially developed by Söderström et al. (1987).
- *cliv_arx* method (with $G_0 \in \mathcal{G}$, $S \notin \mathcal{M}$), optimal IV, using ARX structure, controller known or not.
- *cliv_bj* method (with $S \in \mathcal{M}$), optimal IV, using BJ structure, controller known or not.
- *pem* algorithm (with $S \in \mathcal{M}$), applied to the closed-loop data ($u(t_k)$ and $y(t_k)$) known to be theoretically efficient in the $S \in \mathcal{M}$ case (see Forsell and Ljung (1999)).

The plant parameters are estimated on the basis of closed-loop data of length $N = 4088$. A Monte-Carlo simulation of 100 runs is performed for a signal to noise ratio $\text{SNR} = 25\text{dB}$ (with new noise at each run).

It can be seen from figure 2, that even with a wrong assumption on the noise, the two IV methods *clivr* and *cliv_arx* provide unbiased results with a lower variance thanks to the use of *cliv_arx*. In the case where $S \in \mathcal{M}$, optimal estimates (no bias, minimum variance) are provided by the *cliv_bj* method whereas some initialization problems may occur with *pem*.

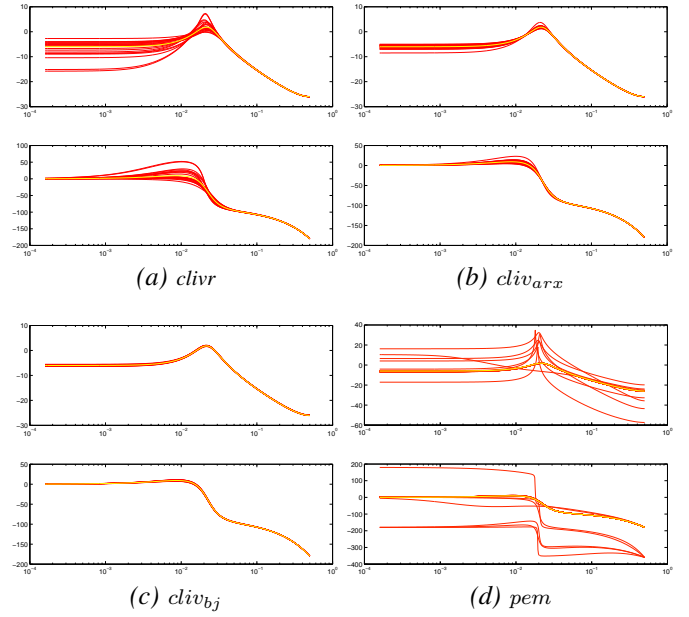


Fig. 2. True (yellow) and estimated (red) Bode diagrams (gain and phase (degree)) of the plant model $G(q, \theta)$ over the 100 MCS, colored noise

4. LPV MODEL IDENTIFICATION, APPLICATION TO RAINFALL/FLOW MODELING

When dealing with NL model, the first question to address is which kind of non linearity is to be used. In this paper, a focus is made on the use of NL model for environmental data: rainfall and flow in a rural catchment. The most common modeling for rainfall/flow relationship is the parametric block representation named Hammerstein structure (see Figure 3). In this case, the system behavior can be described in the following way : the catchment reacts as a sponge which retains the water until it reaches its full retaining capacity and the runoff starts following a linear tank model. Consequently in the Hammerstein case, the static non-linearity represents the loss of rain which does not reach the outlet of the catchment. Nonetheless, the sponge effect can be translated into a dynamic

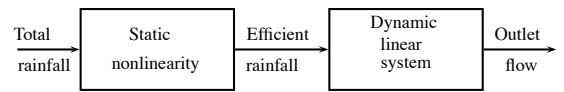


Fig. 3. Hammerstein model

change more than in a static non-linearity. Consequently, aside from the block models, another type of models drew the attention of the system identification community lately: the model class of Linear Parameter-Varying (LPV) systems (see Figure 4). In LPV systems the signal relations are considered to be linear just as in the LTI case, but the parameters are assumed to be function of a measurable time-varying signal, the so-called scheduling variable.

The main advantage of the LPV model is to represent a trade-off between LTI models and NL models. As a result, they present a wide range of behavior representation capability to be used in many practical applications. A LPV model is given as

$$\mathcal{M} \begin{cases} A(p_k, q^{-1})\chi(t_k) = B(p_k, q^{-1})u(t_k) \\ y(t_k) = \chi(t_k) + H(q)e(t_k) \end{cases} \quad (16)$$

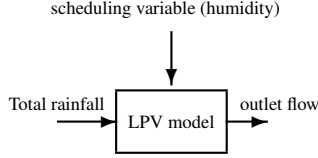


Fig. 4. LPV model

where χ is the noise-free output, u is the input, e is the additive noise with bounded spectral density, y is the noisy output of the system. p are the so-called scheduling variables. For LTI systems, the coefficients of A and B are constant in time while they are time-varying, depending on p , for LPV models. $A(p_k, q^{-1})$ and $B(p_k, q^{-1})$ are polynomials in q^{-1} of degree n_a and n_b , respectively:

$$A(p_k, q^{-1}) = 1 + \sum_{i=1}^{n_a} a_i(p_k)q^{-i} \quad (17)$$

$$B(p_k, q^{-1}) = \sum_{j=0}^{n_b} b_j(p_k)q^{-j} \quad (18)$$

with

$$a_i(p_k) = a_{i,0} + \sum_{l=1}^{n_\alpha} a_{i,l}f_l(p_k) \quad i = 1, \dots, n_a \quad (19)$$

$$b_j(p_k) = b_{j,0} + \sum_{l=1}^{n_\beta} b_{j,l}g_l(p_k) \quad j = 0, \dots, n_b. \quad (20)$$

In this parametrization, $\{f_l\}_{l=1}^{n_\alpha}$ and $\{g_l\}_{l=1}^{n_\beta}$ are functions of p , with static dependence, allowing the identifiability of the model (pairwise orthogonal functions for example). It can be noticed that the knowledge of $\{a_{i,l}\}_{i=1, l=1}^{n_a, n_\alpha}$ and $\{b_{j,l}\}_{j=0, l=0}^{n_b, n_\beta}$ ensures the knowledge of the full model. Therefore, these model parameters are stacked column-wise in the parameter vector θ (with $n_\theta = n_a(n_\alpha + 1) + (n_b + 1)(n_\beta + 1)$),

$$\theta = [a_1 \dots a_{n_a} \ b_0 \dots b_{n_b}]^\top \in \mathbb{R}^{n_\theta},$$

with $a_i = [a_{i,0} \ a_{i,1} \ \dots \ a_{i,n_\alpha}] \in \mathbb{R}^{n_\alpha+1}$

$$b_j = [b_{j,0} \ b_{j,1} \ \dots \ b_{j,n_\beta}] \in \mathbb{R}^{n_\beta+1}.$$

4.1 IV estimator for LPV model

A parametrized model may then be chosen as

$$A(p_k, q^{-1}, \theta)\chi(t_k) = B(p_k, q^{-1}, \theta)u(t_k)$$

$$A(p_k, q^{-1}, \theta)y(t_k) = B(p_k, q^{-1}, \theta)u(t_k) + A(p_k, q^{-1}, \theta)v(t_k)$$

Then, if $S \in \mathcal{M}$, it can be written in terms of the following linear regression

$$y(t_k) = \varphi^\top(t_k, p_k)\theta + A(p_k, q^{-1}, \theta)v(t_k), \quad (21)$$

$$\varphi(t_k, p_k) = [-y(t_{k-1}, p_k) \ \dots \ -y(t_{k-n_a}, p_k) \ u_{0,0}(t_k, p_k) \ \dots \ u_{n_b, n_\beta}(t_k, p_k)]^\top \quad (22)$$

with $v(t_k) = H(q^{-1}, \theta)e(t_k)$. The estimation by optimal IV requires to filter the data by the inverse of the noise model $A(p_k, q^{-1}, \theta)H(q^{-1}, \theta)$. Since the noise model involves an LPV filter, the solution is not strait-forward and the minimization of the prediction error on this model cannot be directly applied. A solution has therefore been proposed to tackle this problem by rewriting the LPV SISO model into a LTI MISO model with unknown outputs $\chi(t_k)$ as (see Laurain et al. (2010))

$$y(t_k) = \varphi^\top(t_k, p_k)\theta + F(q^{-1}, \theta)v(t_k) \quad \text{with} \quad (23)$$

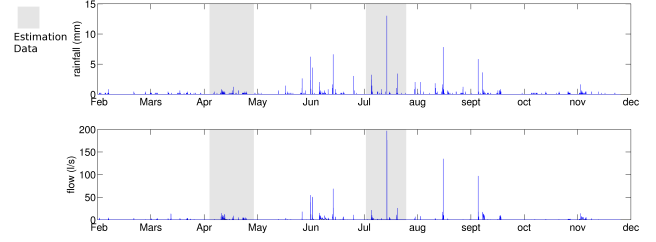


Fig. 5. Rainfall and flow data for the Hohrain catchment (Alsace) during 2008

$$\varphi(t_k, p_k) = [-y(t_{k-1}) \ \dots \ -y(t_{k-n_a}) \ -\chi_{1,1}(t_k, p_k) \ \dots \ -\chi_{n_a, n_\alpha}(t_k, p_k) \ u_{0,0}(t_k, p_k) \ \dots \ u_{n_b, n_\beta}(t_k, p_k)]^\top \quad (24)$$

Since the filter $F(q^{-1}, \theta)$ in (23) is no longer dependent on the scheduling variable p_k , an IV estimator can now be applied to this modified model resulting in

$$\hat{\theta}_{iv} = \left[\sum_{k=1}^N \zeta_f(t_k)\varphi_f^\top(t_k) \right]^{-1} \sum_{k=1}^N \zeta_f(t_k)y_f(t_k), \quad (25)$$

where the instrument is made up of the noise-free version of the regressor and the filter as the inverse of the noise model.

It can be pointed out that, as previously, the IV solution depends on unknown system properties and it is therefore necessary to estimate the deterministic noise-free terms $\chi_{i,l}$ in order to get the IV estimates. This is handled by using an iterative scheme named the *Refined Instrumental Variable* (RIV) approach dedicated to the LPV model estimation (see Laurain et al. (2010)).

4.2 Data-based catchment modeling

The identification of rainfall/runoff relationship is a challenging issue, mainly because of the complexity to find a suitable model for a whole given catchment (Beven (2000)). The Hohrain catchment area studied in this paper is located in the Alsatian vineyard (Eastern part of France). The problem is : given the total rainfall data u and the outlet flow data y sampled at times t_k , $k = 1, \dots, N$, the goal is to estimate the rainfall/runoff relationship. In the given case, the sample time is 6 minutes, the flow unit is l/s and the rainfall is expressed in mm . The data are measured during one year and over the 70000 samples acquired, only 5000 are relevant for the identification process. Three methods have been used on this data set:

- refined IV method with a linear model,
- proposed IV method in the LPV model context,
- the LS method in the LPV model context.

The results are illustrated on two different rainfall events: an important and a small one. Fig. 6(a) and 6(b) provide the results for the IV solutions, whereas Fig. 7(a) and 7(b) illustrate the results for the LS estimation. It can be noticed in these figures that, as expected, the linear model cannot cope with the nonlinearities present in the catchment and the resulting model is not satisfactory neither for a small rainfall event, nor for an important one. On the contrary, the LPV model solutions provide more accurate results and the proposed refined IV method offers a really efficient solution to this rainfall/flow estimation problem. The scheduling variable used here is an estimation of the catchment humidity and making the model dependent on it provides an accurate modeling, whatever the intensity of the rainfall event.

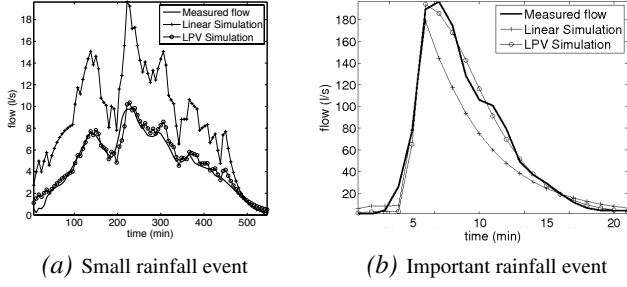


Fig. 6. Linear and LPV models, IV methods

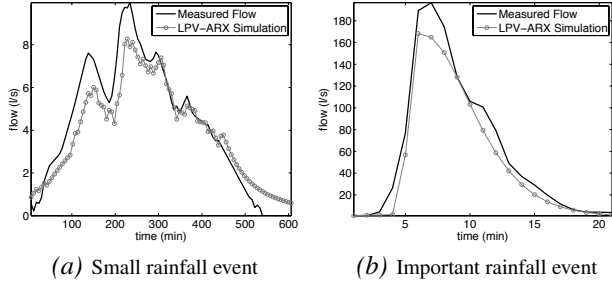


Fig. 7. LPV model, LS method

5. FREQUENCY DOMAIN IDENTIFICATION

We consider here a linear time invariant, input output system given by a rational transfer function

$$\mathcal{S} : G_0(s) = \frac{B_0(s)}{A_0(s)}. \quad (26)$$

where s denotes the Laplace transform variable. The frequency domain identification problem is to determine an estimate of this system from a measured frequency response

$$G(j\omega_k) = G_0(j\omega_k) + V(j\omega_k), \quad k = 1, \dots, N$$

which is available in a particular frequency grid denoted ω_k , $k = 1, \dots, N$ and where $G(j\omega)$ denotes the measurement of the system frequency function $G_0(j\omega)$, corrupted with the measurement noise $V(j\omega)$. We will assume in this paper that we have access to the system measurements in the frequency domain.

The system is modeled by its transfer function $G(s)$ and parametrized by a parameter vector θ , as follows

$$\mathcal{G} : G(s, \theta) = \frac{B(s, \theta)}{A(s, \theta)} \quad (27)$$

where

$$B(s, \theta) = b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0 \quad (28)$$

$$A(s, \theta) = s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0 \quad (29)$$

$$\text{with } \theta = [a_0 \dots a_{n-1} \ b_0 \dots b_m] \in \mathbb{R}^{n+m+1}. \quad (30)$$

If the plant G_0 is included in the chosen model set \mathcal{G} (i.e. $G_0 \in \mathcal{G}$), the output $G(j\omega)$ can be written as

$$\mathcal{G} : G(j\omega_k) = \Phi^*(j\omega_k)\theta_0 + A_0(j\omega_k)V(j\omega_k) \quad (31)$$

with the regressor

$$\Phi^*(j\omega_k) = [-G(j\omega_k) \ \dots \ -(j\omega_k)^{n-1}G(j\omega_k) \ 1 \ \dots \ (j\omega_k)^m]$$

5.1 IV method in the frequency domain

The determination of the parameter vector θ on the basis of the measured frequency domain data may be achieved by minimising

$$J(\theta) = \frac{1}{N} \sum_{k=1}^N |\varepsilon(j\omega_k, \theta)|^2, \quad (32)$$

$$\text{with } \varepsilon(j\omega_k, \theta) = G(j\omega_k) - G(j\omega_k, \theta) \quad (33)$$

The Least-Square (LS) solution to this problem is known to give biased results since $\Phi(j\omega_k)$ and the noise are correlated, whereas the IV method

$$\hat{\theta}_{iv} = \arg \min_{\theta} \left\| \sum_{k=1}^N \zeta(j\omega_k) \left[\frac{1}{A_0(j\omega_k)} G(j\omega_k) - \frac{1}{A_0(j\omega_k)} \Phi^*(j\omega_k)\theta \right] \right\|^2 \quad (34)$$

provides unbiased result whatever the noise is, under the following two conditions

- $\frac{1}{N} \sum_{k=1}^N \zeta(j\omega_k) \frac{1}{A_0(j\omega_k)} \Phi^*(j\omega_k)$ is full column rank
- $\frac{1}{N} \sum_{k=1}^N \zeta(j\omega_k) W(j\omega_k) = 0$,

and with

$$\zeta(j\omega_k) = [-G_0(j\omega_k) \ -(j\omega_k)^{n-1}G_0(j\omega_k) \ 1 \ \dots \ (j\omega_k)^m], \quad k = 1, \dots, N, \text{ therefore } \zeta(j\omega_k) \text{ corresponds to noise-free part of the regressor (see Gilson et al. (2013)).}$$

As previously, the IV solution is dependent on unknown system property that has to be taken care of by an iterative procedure where the instruments and prefilter are constructed from previous model estimate. A refined IV dedicated to the frequency domain has been developed in that sense (Gilson et al. (2013)). Even though the IV algorithm offers a nice solution to remove the bias of the least squares method in case of noisy measurements, it still relies on the use of the normal matrix. A poor conditioning of this matrix results in poor or erroneous estimates of the system parameters. This problem especially occurs when identifying systems with a large dynamic range, as it is often the case in the frequency domain.

A solution has then been proposed to improve the IV method in this case. It relies on the technique developed in Welsh and Goodwin (2003), based on a particular set of basis functions, which is aimed specifically at improving the numerical properties of the normal matrix in a rational function estimation over a large dynamic range. A key point in this approach is that the method restricts the dynamic range over which each coefficient is estimated by the use of frequency localizing basis functions (FLBF) which span a desired frequency region:

$$F_k(s) = \prod_{l=1}^k \frac{s^{k-1} p_k}{s + p_l}, \quad k = 1, \dots, n$$

These functions allow the normal matrix to take on a near block diagonal form and hence improve its conditioning. Moreover, the filters used in these functions are bandpass and hence easy to implement. In Gilson et al. (2013), a method introducing basis functions into an IV procedure has been proposed to consistently handle the frequency domain identification case.

5.2 Data-based resonant beam modeling

This IV method associated with the frequency localizing basis functions has been used to estimate the real frequency response data collected from a resonant beam. The experimental data spans approximately 2 decades and is shown in Figure 8 by the blue dots. Two methods have been used to estimate this process:

- flbf-ls: least-square estimation technique associated with the FLBF (see Welsh and Goodwin (2003))

- `flbf-iv`: refined IV estimation associated with the FLBF (see Gilson et al. (2013)).

The resulting estimates are shown in figure 8 (`flbf-ls` in magenta and `flbf-iv` in green), both provide reasonably accurate results with slightly better accuracy obtained from the IV solution, especially in the high frequencies. Again it would seem that an excellent fit has been obtained with this easy to tune method.

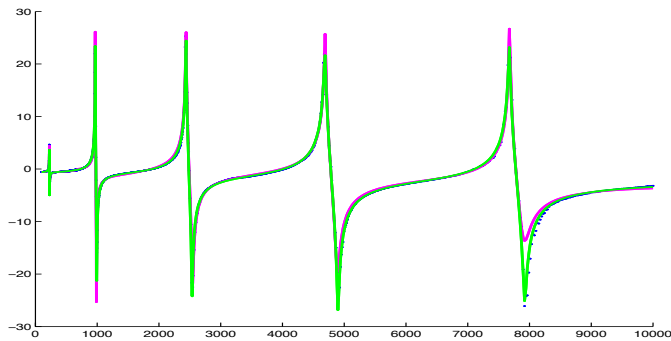


Fig. 8. Resonant Beam Data. Dots - Experimental Data, Solid - Estimated Model using `flbf-ls` (magenta) and `flbf-iv` (green) methods

6. CONCLUSION

In conclusion, this paper has gathered several experiences of using instrumental variable technique in different contexts. It has been illustrated that IV method has a lot to offer for system identification: computationally attractive property, yields consistent estimates even in the case of mis-specified noise model, may potentially be used in many different practical situations: linear, non linear model, open-loop, closed-loop...

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REFERENCES

- Bamieh, B. and Giarré, L. (2002). Identification of linear parameter-varying models. *International Journal of Robust Nonlinear Control*, 12, 841–853.
- Beven, K.J. (2000). *Rainfall-Runoff Modelling: The Primer*. John Wiley, Hoboken, NJ.
- Dankers, A., Van den Hof, P., Bombois, X., and Heuberger, P. (2014). Errors-in-variables identification in dynamic networks. In *19th IFAC World Congress*. Cape Town, South Africa.
- Douma, S. (2006). *From data to performance - System identification uncertainty and robust control design*. Ph.D. thesis, Delft University of Technology, Delft, The Netherlands.
- Forsell, U. and Ljung, L. (1999). Closed-loop identification revisited. *Automatica*, 35(7), 1215–1241.
- Garnier, H. and Wang, L. (eds.) (2008). *Identification of continuous-time models from sampled data*. Springer Verlag, London.
- Gilson, M., Garnier, H., Young, P.C., and Van den Hof, P. (2011). Optimal instrumental variable method for closed-loop identification. *IET Control Theory and Applications*, 5(10), 1147–1154.
- Gilson, M. and Van den Hof, P. (2005). Instrumental variable methods for closed-loop system identification. *Automatica*, 41(2), 241–249.
- Gilson, M., Welsh, J., and Garnier, H. (2013). Frequency-domain instrumental variable based method for wide band system identification. In *IEEE American Control Conference*. Washington – USA.
- Juditsky, A., Hjalmarsson, H., Benveniste, A., Delyon, B., Ljung, L., Sjöberg, J., and Zhang, Q. (1995). Nonlinear black-box modeling in system identification: mathematical foundations. *Automatica*, 31(12), 1725–1750.
- Katayama, T. (2005). *Subspace methods for system identification*. Springer.
- Laurain, V., Gilson, M., Tóth, R., and Garnier, H. (2010). Refined instrumental variable methods for identification of LPV Box-Jenkins models. *Automatica*, 46(6), 959–967.
- Ljung, L. (1999). *System Identification : Theory for the User - Second Edition*. Prentice-Hall.
- McKelvey, T. (2002). Frequency domain identification methods. *Circuits Systems Signal Processing*, 39–55.
- Ninness, B. and Hjalmarsson, H. (2005). On the frequency domain accuracy of closed-loop estimates. *Automatica*, 41(7), 1109–1122.
- Pintelon, R. and Schoukens, J. (2001). *System identification : a frequency domain approach*. IEEE Press, New York.
- Sinha, N. and Rao, G. (1991). *Identification of continuous-time systems. Methodology and computer implementation*. Kluwer Academic Press, Dordrecht.
- Sjöberg, J., Zhang, Q., Ljung, L., Benveniste, A., Delyon, B., Glorennec, P.Y., Hjalmarsson, H., and Juditsky, A. (1995). Nonlinear black-box modeling in system identification: a unified overview. *Automatica*, 31(12), 1691–1724.
- Söderström, T. and Stoica, P. (1983). *Instrumental Variable Methods for System Identification*. Springer-Verlag.
- Söderström, T. and Stoica, P. (1989). *System identification*. Prentice-Hall.
- Söderström, T., Stoica, P., and Trulsson, E. (1987). Instrumental variable methods for closed-loop systems. In *10th World IFAC Congress*, 363–368. Munich - Germany.
- Tóth, R. (ed.) (2010). *Modeling and identification of linear parameter-varying systems. Lecture Notes in Control and Information Sciences*. Springer, Heidelberg.
- Unbehauen, H. and Rao, G. (1987). *Identification of continuous systems*. Systems and Control Series, Amsterdam.
- Van den Hof, P. (1998). Closed-loop issues in system identification. *Annual Reviews in Control*, 22, 173–186.
- Van Herpen, R., Oomen, T., and Steinbuch, M. (2014). Optimally conditioned instrumental variable approach for frequency-domain system identification. *Automatica*, 50(9), 2281–2293.
- Van Overschee, P. and De Moor, B. (1996). *Subspace identification for linear systems. Theory, Implementation, Applications*. Kluwer Academic Press, Dordrecht.
- Welsh, J. and Goodwin, G. (2003). Frequency localising basis functions for wide-band identification. In *IEEE (ed.), European Control Conference*. Cambridge - UK.
- Young, P.C. (2014). Refined instrumental variable estimation: Maximum likelihood optimization of a unified box-jenkins model. *Automatica*, 52, 35–46.
- Young, P. (2011). *Recursive Estimation and Time-Series Analysis*. Springer Verlag, London, second edition.