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► **To cite this version:**

Thomas Blanc, Mihai Bostan, Franck Boyer. Asymptotic analysis of parabolic equations with stiff transport terms by a multi-scale approach. *Discrete and Continuous Dynamical Systems - Series A*, 2017, 37 (9), pp.4637-4676. 10.3934/dcds.2017200 . hal-01242679v3

HAL Id: hal-01242679

<https://hal.science/hal-01242679v3>

Submitted on 19 Jan 2017

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Asymptotic analysis of parabolic equations with stiff transport terms by a multi-scale approach

Thomas BLANC ^{*}, Mihai BOSTAN [†], Franck BOYER [‡]

(January 19, 2017)

Abstract

We perform the asymptotic analysis of parabolic equations with stiff transport terms. This kind of problem occurs, for example, in collisional gyrokinetic theory for tokamak plasmas, where the velocity diffusion of the collision mechanism is dominated by the velocity advection along the Laplace force corresponding to a strong magnetic field. This work appeal to the filtering techniques. Removing the fast oscillations associated to the singular transport operator, leads to a stable family of profiles. The limit profile comes by averaging with respect to the fast time variable, and still satisfies a parabolic model, whose diffusion matrix is completely characterized in terms of the original diffusion matrix and the stiff transport operator. Introducing first order correctors allows us to obtain strong convergence results, for general initial conditions (not necessarily well prepared).

Keywords: Average operators, Ergodic means, Unitary groups, Multiple scales, Homogenization.

AMS classification: 35Q75, 78A35

1 Introduction

General framework. In many applications we deal with partial differential equations with disparate scales. The solutions of the problems in hand fluctuate at very different scales and for the moment, solving numerically for both slow and fast scales seems out of reach. Depending on the particular regimes we are interested in, it could be worth to solve an averaged problem with respect to the slow variable, after smoothing out the fast oscillations. In this work we focus on parabolic models perturbed by stiff transport operators

$$\begin{cases} \partial_t u^\varepsilon - \operatorname{div}_y(D(y)\nabla_y u^\varepsilon) + \frac{1}{\varepsilon} b(y) \cdot \nabla_y u^\varepsilon = 0, & (t, y) \in \mathbb{R}_+ \times \mathbb{R}^m \\ u^\varepsilon(0, y) = u^{\text{in}}(y), & y \in \mathbb{R}^m. \end{cases} \quad (1)$$

Here $b : \mathbb{R}^m \rightarrow \mathbb{R}^m$ and $D : \mathbb{R}^m \rightarrow \mathcal{M}_m(\mathbb{R})$ are given fields of vectors and symmetric positive definite matrices, and $\varepsilon > 0$ is a small parameter destined to converge to 0. If the vector

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field b is divergence free, the energy balance writes

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^m} (u^\varepsilon(t, y))^2 dy + \int_{\mathbb{R}^m} D(y) \nabla_y u^\varepsilon \cdot \nabla_y u^\varepsilon dy = 0.$$

Therefore, since the matrices $D(y)$ are positive definite, the L^2 norms of the solutions $(u^\varepsilon)_{\varepsilon>0}$ decrease in time, and we expect that the limit model still behaves like a parabolic one, whose diffusion matrix field needs to be determined.

Motivating example. This work is motivated by the study of collisional models for the gyrokinetic theory in tokamak plasmas. The fluctuations of the presence density of charged particles are due to the transport in space and velocity (under the action of electro-magnetic fields), but also to the collision mechanisms. In the framework of the magnetic confinement fusion, the external magnetic fields are very large, leading to a stiff velocity advection, due to the magnetic force $qv \wedge B^\varepsilon = qv \wedge \frac{B}{\varepsilon}$. Here q stands for the particle charge and $B^\varepsilon = \frac{B}{\varepsilon}$ represents a strong magnetic field, when ε goes to 0. Using a Fokker-Planck operator for taking into account the collisions between particles, we are led to the Fokker-Planck equation

$$\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon + \frac{q}{m} \left(E + v \wedge \frac{B}{\varepsilon} \right) \cdot \nabla_v f^\varepsilon = \nu \operatorname{div}_v (\Theta \nabla_v f^\varepsilon + v f^\varepsilon), \quad (t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3 \quad (2)$$

where E is the electric field, m is the particle mass, ν is the collision frequency and Θ is the temperature. Notice that in the Fokker-Planck equation, the diffusion occurs only in the velocity space, and therefore (2) is different from (1). Nevertheless, we expect that the main arguments used for the treatment of (1) still apply when investigating the asymptotic behavior of (2), see Section 3.2. The asymptotic analysis of (2), when neglecting the collisions is now well understood [8, 25, 26, 28]. It can be handled by averaging the perturbed model along the characteristic flow associated to the dominant transport operator. Recently, models including collisions have been analyzed formally by using the averaging method [9, 10]. In particular, it was emphasized that, averaging with respect to the fast cyclotronic motion leads to diffusion not only in velocity, but also with respect to the perpendicular space directions, see (32). At least at the formal level, the theoretical study presented in this work allows to perform the asymptotic analysis of the Fokker-Planck equation, under strong magnetic fields, see Section 3.2. The study of the averaged diffusion matrix field is crucial when determining the equilibria of the limit Fokker-Planck equation (2), when ε goes to zero. Numerical results concerning strongly anisotropic elliptic and parabolic problems were obtained in [21, 24, 18].

The advection-diffusion problems with large drift in periodic setting have been extensively studied by many authors [23, 22, 33, 5, 34, 29]. For the homogenization of the associated eigenvalue problems, we refer to [6, 30]. The aim of this paper is to obtain an homogenization result for the problem (1) without periodicity assumptions on the advection field and for initial data not necessarily well prepared. The effective limit problem comes by appealing to an ergodic theory result. It is a parabolic problem, whose diffusion matrix field appears as the average of the original diffusion matrix field, along a flow. We specify sufficient assumptions which guarantee the well definition of such an average matrix field. Moreover, we study the behavior of the solutions of (1), when ε goes to zero, and estimate the convergence rate in L^2 , by constructing a suitable corrector. The asymptotic analysis of the system (1) was provided also in [29] in the new framework of weak Σ -convergence along a flow. This approach leads to the same effective limit problem as our, but with a different interpretation based on the theory of ergodic algebra with mean value.

Our paper is organized as follows. The main results are introduced in Section 2. We indicate the main lines of our arguments by performing formal computations. In Section 3

we present a brief overview on the construction of the average operators for matrix fields through the ergodic theory. Section 4 is devoted to obtaining uniform estimates for solutions of (1), in view of convergence results. In Section 5 we establish two-scale convergence results, in the ergodic setting, which allows us to handle situations with non periodic fast variables. Up to our knowledge, these results have not been reported yet. The proofs of the main theorems are detailed in Section 6. Some technical arguments are presented in Appendix A.

2 Presentation of the main results and formal approach

The subject matter of this paper concentrates on the asymptotic analysis of (1), when ε becomes small. Obviously, the fast time oscillations come from the large advection field $\frac{b(y)}{\varepsilon}$. Indeed, when neglecting the diffusion operator, the problem (1) reduces to a transport model, whose solution writes

$$u^\varepsilon(t, y) = u^{\text{in}}(Y(-t/\varepsilon; y)), \quad (t, y) \in \mathbb{R}_+ \times \mathbb{R}^m. \quad (3)$$

Here $(s, y) \in \mathbb{R} \times \mathbb{R}^m \rightarrow Y(s; y) \in \mathbb{R}^m$ stands for the characteristic flow of $b \cdot \nabla_y$

$$\frac{dY}{ds} = b(Y(s; y)), \quad (s, y) \in \mathbb{R} \times \mathbb{R}^m, \quad Y(0; y) = y, \quad y \in \mathbb{R}^m.$$

This flow is well defined under standard smoothness assumptions

$$\begin{cases} b \in W_{\text{loc}}^{1, \infty}(\mathbb{R}^m), \\ \exists C > 0 \text{ such that } |b(y)| \leq C(1 + |y|), \quad y \in \mathbb{R}^m. \end{cases} \quad (4)$$

Under the above hypotheses the flow Y is global and smooth, $Y \in W_{\text{loc}}^{1, \infty}(\mathbb{R} \times \mathbb{R}^m)$. Moreover, we assume that

$$b \text{ is divergence free,} \quad (5)$$

which guarantees that the transformation $y \in \mathbb{R}^m \rightarrow Y(s; y) \in \mathbb{R}^m$ is measure preserving for any $s \in \mathbb{R}$. Motivated by (3), we introduce the new unknowns

$$v^\varepsilon(t, z) = u^\varepsilon(t, Y(t/\varepsilon; z)), \quad (t, z) \in \mathbb{R}_+ \times \mathbb{R}^m, \quad \varepsilon > 0 \quad (6)$$

and we expect to get stability for the family $(v^\varepsilon)_{\varepsilon > 0}$, when ε goes to 0. In that case we will deduce that, for small $\varepsilon > 0$, u^ε behaves like $v(t, Y(-t/\varepsilon; y))$, for some profile $v = \lim_{\varepsilon \searrow 0} v^\varepsilon$, that is, u^ε appears as the composition product between a stable profile and the fast oscillating flow $Y(-t/\varepsilon; y)$. We prove mainly two strong convergence results for general initial conditions (not necessarily well prepared), whose simplified versions are stated below. For detailed assertions see Theorems 2.2, 2.3.

Theorem *We denote by $(u^\varepsilon)_{\varepsilon > 0}$ the variational solutions of (1) and by $(v^\varepsilon)_{\varepsilon > 0}$ the functions*

$$v^\varepsilon(t, z) = u^\varepsilon(t, Y(t/\varepsilon; z)), \quad (t, z) \in \mathbb{R}_+ \times \mathbb{R}^m, \quad \varepsilon > 0.$$

1. *Under suitable hypotheses on the vector field b , the matrix field D and the initial condition u^{in} , the family $(v^\varepsilon)_{\varepsilon > 0}$ converges strongly in $L_{\text{loc}}^\infty(\mathbb{R}_+; L^2(\mathbb{R}^m))$ to the unique variational solution $v \in L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^m))$ of (26), whose diffusion matrix field $\langle D \rangle$ comes by averaging the matrix field D along the flow of the vector field b (cf. Theorem 2.1), that is*

$$\langle D \rangle = \lim_{S \rightarrow +\infty} \frac{1}{S} \int_0^S \partial Y(-s; Y(s; \cdot)) D(Y(s; \cdot)) {}^t \partial Y(-s; Y(s; \cdot)) ds.$$

2. Under additional regularity hypotheses, we have the error estimate

$$\sup_{t \in [0, T]} \|u^\varepsilon(t, \cdot) - v(t, Y(-t/\varepsilon; \cdot))\|_{L^2(\mathbb{R}^m)} \leq C_T \varepsilon.$$

The problem satisfied by v^ε is obtained by performing the change of variable $y = Y(t/\varepsilon; z)$ in (1). A straightforward computation based on the chain rule leads to

$$\partial_t v^\varepsilon(t, z) = \partial_t u^\varepsilon(t, Y(t/\varepsilon; z)) + \frac{1}{\varepsilon} b(Y(t/\varepsilon; z)) \cdot (\nabla_y u^\varepsilon)(t, Y(t/\varepsilon; z)).$$

To deal with the diffusion term, we need to compute the action of the operator $\operatorname{div}_y(D\nabla_y \cdot)$ when applied to a function of the form $v \circ Y(-s, \cdot)$ for some $s \in \mathbb{R}$. A tedious but straightforward computation shows that we can write, for any integrable function $z \mapsto w(z)$, any matrix field $y \mapsto C(y)$, and any $s \in \mathbb{R}$,

$$\operatorname{div}_y \{C\nabla_y(w \circ Y(-s, \cdot))\} = (\operatorname{div}_z(G(s)C\nabla_z w)) \circ Y(-s, \cdot) \quad \text{in } \mathcal{D}'(\mathbb{R}^m), \quad (7)$$

where the family of matrix fields $(G(s)C)_s$ is defined by

$$\begin{aligned} (G(s)C)(z) &:= \partial Y^{-1}(s; z) C(Y(s; z)) {}^t \partial Y^{-1}(s; z) \\ &= \partial Y(-s; Y(s; z)) C(Y(s; z)) {}^t \partial Y(-s; Y(s; z)), \quad (s, z) \in \mathbb{R} \times \mathbb{R}^m, \end{aligned} \quad (8)$$

Using those notations, the change of variable $y = Y(t/\varepsilon; z)$ in (1) leads to

$$\begin{cases} \partial_t v^\varepsilon - \operatorname{div}_z((G(t/\varepsilon)D)\nabla_z v^\varepsilon) = 0, & (t, z) \in \mathbb{R}_+ \times \mathbb{R}^m \\ v^\varepsilon(0, z) = u^\varepsilon(0, z) = u^{\text{in}}(z), & z \in \mathbb{R}^m, \varepsilon > 0 \end{cases} \quad (9)$$

The new diffusion problem (9) seems simpler than the original problem (1), because the singular term $\frac{1}{\varepsilon} b \cdot \nabla_y$ has disappeared. Nevertheless, the new model depends on a fast time variable $s = t/\varepsilon$, through the diffusion matrix field $G(t/\varepsilon)D$, and a slow time variable t . We deal with a two-scale problem in time. As often in asymptotic analysis of multiple scale problems, a way to understand the behavior of the solutions $(v^\varepsilon)_{\varepsilon > 0}$ when ε goes to 0 and to identify the limit problem is to use a formal development whose terms depend both on the slow and fast time variables

$$v^\varepsilon(t, z) = v(t, t/\varepsilon, z) + \varepsilon v^1(t, t/\varepsilon, z) + \dots \quad (10)$$

This method is used in many frameworks such as periodic homogenization for elliptic and parabolic systems [1, 32], transport equations [13, 19] or kinetic equations [11]. Plugging the Ansatz (10) in (9) and identifying the terms of the same order with respect to ε , lead to the hierarchy of equations

$$\partial_s v = 0 \quad (11)$$

$$\partial_t v - \operatorname{div}_z(G(s)D\nabla_z v) + \partial_s v^1 = 0 \quad (12)$$

⋮

Equation (11) says that the first profile v does not depend on the fast time variable s , that is $v = v(t, z)$. We expect that v is the limit of the family $(v^\varepsilon)_{\varepsilon > 0}$, when ε goes to 0. The slow time evolution of v is given by (12), but we need to eliminate the second profile v^1 . Actually v^1 appears as a Lagrange multiplier which guarantees that at any time t , the profile v satisfies the constraint $\partial_s v = 0$. In the periodic case, we eliminate v^1 by taking the average

over one period. However, without periodicity assumptions, we need to use ergodic averaging techniques, that is to write

$$\begin{cases} \partial_t v - \operatorname{div}_z \left\{ \left(\lim_{S \rightarrow +\infty} \frac{1}{S} \int_0^S G(s) D \, ds \right) \nabla_z v \right\} = 0, & (t, z) \in \mathbb{R}_+ \times \mathbb{R}^m \\ v(0, z) = u^{\text{in}}(z), & z \in \mathbb{R}^m. \end{cases} \quad (13)$$

The key point is that $(G(s))_{s \in \mathbb{R}}$ is a C^0 -group of unitary operators (on some Hilbert space to be determined), and thanks to von Neumann's ergodic mean theorem [35], the limit $\langle D \rangle = \lim_{S \rightarrow +\infty} \frac{1}{S} \int_0^S G(s) D \, ds$ makes sense. The Hilbert space which realizes $(G(s))_{s \in \mathbb{R}}$ as a C^0 -group of unitary operators appears as a L^2 weighted space, with respect to some field of symmetric positive definite matrices. We assume that there is a matrix field P such that

$${}^t P = P, \quad P(y) \xi \cdot \xi > 0, \quad \xi \in \mathbb{R}^m \setminus \{0\}, \quad y \in \mathbb{R}^m, \quad P^{-1}, P \in L_{\text{loc}}^2(\mathbb{R}^m) \quad (14)$$

$$[b, P] = 0, \quad \text{in } \mathcal{D}'(\mathbb{R}^m), \quad (15)$$

where the bracket between a vector field c and a matrix field A is a matrix field formally defined by

$$[c, A] := (c \cdot \nabla_y) A - \partial_y c A - A {}^t \partial_y c. \quad (16)$$

The condition (15) naturally appears in our problem because of the following proposition.

Proposition 2.1 *Consider a vector field $c \in W_{\text{loc}}^{1, \infty}(\mathbb{R}^m)$ (not necessarily divergence free) with at most linear growth at infinity and $A(y) \in L_{\text{loc}}^1(\mathbb{R}^m)$ a matrix field.*

1. *We can express the commutator between the advection operator $c \cdot \nabla_y$ and the diffusion operator $\operatorname{div}_y(A(y) \nabla_y)$ as follows*

$$[c \cdot \nabla_y, \operatorname{div}_y(A(y) \nabla_y)] = \operatorname{div}_y([c, A] \nabla_y) + ({}^t A(y) \nabla_y \operatorname{div}_y c) \cdot \nabla_y. \quad (17)$$

In particular, if $\operatorname{div}_y c = 0$, this commutator is also a diffusion operator with respect to the matrix field $[c, A]$.

2. *The following assertions are equivalent*

- (a) *We have $[c, A] = 0$ in $\mathcal{D}'(\mathbb{R}^m)$*
- (b) *For any $s \in \mathbb{R}$, $y \in \mathbb{R}^m$, we have*

$$A(Y(s; y)) = \partial Y(s; y) A(y) {}^t \partial Y(s; y).$$

where Y stands for the flow associated with the vector field c .

- (c) *For any $s \in \mathbb{R}$, we have*

$$G(s) A = A,$$

where G is defined in (8).

Proof.

1. A straightforward computation first shows that the commutator between $c \cdot \nabla_y$ and div_y is given by

$$[c \cdot \nabla_y, \operatorname{div}_y] \xi = \xi \cdot \nabla_y \operatorname{div}_y c - \operatorname{div}_y(\partial_y c \xi), \quad \xi \in (C^2(\mathbb{R}^m))^m.$$

Using the above formula with $\xi = A(y)\nabla_y u$, for a smooth u , one gets

$$\begin{aligned} c \cdot \nabla_y(\operatorname{div}_y(A(y)\nabla_y u)) - \operatorname{div}_y(c \cdot \nabla_y(A(y)\nabla_y u)) \\ = A(y)\nabla_y u \cdot \nabla_y \operatorname{div}_y c - \operatorname{div}_y(\partial_y c A(y)\nabla_y u). \end{aligned} \quad (18)$$

Taking into account that

$$\begin{aligned} c \cdot \nabla_y(A(y)\nabla_y u) &= (c \cdot \nabla_y A)\nabla_y u + A(y)(\partial^2 u)c(y) \\ &= (c \cdot \nabla_y A)\nabla_y u + A(y)\nabla_y(c \cdot \nabla_y u) - A(y)^t \partial_y c \nabla_y u \end{aligned}$$

we deduce by (18)

$$\begin{aligned} c \cdot \nabla_y(\operatorname{div}_y(A(y)\nabla_y u)) - \operatorname{div}_y(A(y)\nabla_y(c \cdot \nabla_y u)) &= A(y)\nabla_y u \cdot \nabla_y \operatorname{div}_y c \\ &\quad + \operatorname{div}_y((c \cdot \nabla_y A - \partial_y c A(y) - A(y)^t \partial_y c)\nabla_y u), \end{aligned}$$

which is the claimed formula.

2. This characterization is proved in [12, Prop. 3.8]. □

Let us now introduce some useful function spaces. We recall that for any two matrices in $\mathcal{M}_m(\mathbb{R})$, the notation $A : B$ stands for $\operatorname{tr}({}^t AB)$.

Definition 2.1 For any matrix field $M : \mathbb{R}^m \rightarrow \mathcal{M}_m(\mathbb{R}) \in L^2_{\text{loc}}(\mathbb{R}^m)$ made of positive definite symmetric matrices, we introduce the weighted L^2 space

$$H_M = \left\{ A : \mathbb{R}^m \rightarrow \mathcal{M}_m(\mathbb{R}) \text{ measurable} : M^{1/2} A M^{1/2} \in L^2 \right\},$$

which is a Hilbert space for the natural scalar product

$$(A, B)_M := \int_{\mathbb{R}^m} (M^{1/2} A M^{1/2}) : (M^{1/2} B M^{1/2}) \, dy = \int_{\mathbb{R}^m} M A : B M \, dy, \quad \forall A, B \in H_M.$$

The associated norm is denoted by $|A|_M$.

Similarly we introduce the Banach space

$$H_M^\infty = \left\{ A : \mathbb{R}^m \rightarrow \mathcal{M}_m(\mathbb{R}) \text{ measurable} : M^{1/2} A M^{1/2} \in L^\infty \right\},$$

equipped with the norm

$$|A|_{H_M^\infty} := |M^{1/2} A M^{1/2}|_{L^\infty}.$$

Assume that there is a continuous function ψ , which is left invariant by the flow of b , and goes to infinity when $|y|$ goes to infinity

$$\psi \in C(\mathbb{R}^m), \quad \psi \circ Y(s; \cdot) = \psi \quad \text{for any } s \in \mathbb{R}, \quad \lim_{|y| \rightarrow +\infty} \psi(y) = +\infty. \quad (19)$$

Since the sets $\{\psi \leq k\}$, for $k \in \mathbb{N}$ are compact sets invariant by the flow of b , we will be able to perform our analysis in the local spaces

$$H_{M,\text{loc}} = \left\{ A : \mathbb{R}^m \rightarrow \mathcal{M}_m(\mathbb{R}) \text{ measurable} : \mathbf{1}_{\{\psi \leq k\}} A \in H_M \text{ for any } k \in \mathbb{N} \right\}.$$

We say that a family $(A_i)_i \subset H_{M,\text{loc}}$ converges in $H_{M,\text{loc}}$ toward some $A \in H_{M,\text{loc}}$ iff for any $k \in \mathbb{N}$, the family $(\mathbf{1}_{\{\psi \leq k\}} A_i)_i$ converges in H_M toward $\mathbf{1}_{\{\psi \leq k\}} A$.

We define similar spaces for vector fields.

Definition 2.2 *With the same assumption as in the previous definition, we define*

$$X_M := \{c : \mathbb{R}^m \rightarrow \mathbb{R}^m \text{ measurable} : M^{1/2}c \in L^2\}$$

which is a Hilbert space for the scalar product

$$(c, d)_M := \int_{\mathbb{R}^m} M(y) : c(y) \otimes d(y) \, dy, \quad \forall c, d \in X_M,$$

and the Banach space

$$X_M^\infty = \{c : \mathbb{R}^m \rightarrow \mathbb{R}^m \text{ measurable} : M^{1/2}c \in L^\infty(\mathbb{R}^m)\},$$

equipped with the norm $|c|_{X_M^\infty} := |M^{1/2}c|_{L^\infty}$.

The following properties are straightforward.

Proposition 2.2 *We consider the same assumptions as in the Definitions above.*

- *We have the continuous embedding $C_c^0(\mathbb{R}^m, \mathcal{M}_m(\mathbb{R})) \subset H_M$*
- *If $M^{-1} \in L_{loc}^2(\mathbb{R}^m)$ then $H_M \subset L_{loc}^1(\mathbb{R}^m; \mathcal{M}_m(\mathbb{R}))$.*

Moreover the space $H_{M^{-1}}$ can be identified to the dual of H_M through the L^2 pairing that can be writing as

$$\begin{aligned} \langle A, B \rangle_{M, M^{-1}} &:= \int_{\mathbb{R}^m} A(y) : B(y) \, dy = \int_{\mathbb{R}^m} M^{1/2} A M^{1/2} : M^{-1/2} B M^{-1/2} \, dy \\ &\leq |A|_M |B|_{M^{-1}}, \quad \forall A \in H_M, \forall B \in H_{M^{-1}}. \end{aligned}$$

Finally, the maps $A \in H_M \rightarrow M A M \in H_{M^{-1}}$ and $B \in H_{M^{-1}} \rightarrow M^{-1} B M^{-1} \in H_M$ are linear isometries that are reciprocal one from each other.

We can now come back to the problem of giving a sense to the average process in (13).

Given a matrix field P satisfying (14), (15), we set $Q = P^{-1}$ and we shall prove that the family of applications $G(s) : H_Q \rightarrow H_Q, s \in \mathbb{R}$, is a C^0 -group of unitary operators on H_Q (see Proposition 3.1). Thanks to the von Neumann's ergodic theorem (see Theorem 3.1 and [35] for more details), we find that the average of a matrix field $\langle A \rangle := \lim_{S \rightarrow +\infty} \frac{1}{S} \int_0^S G(s) A \, ds$ is well defined and coincides with the orthogonal projection on the invariant subspace $\{B \in H_Q : G(s)B = B \text{ for any } s \in \mathbb{R}\}$ see also [12, 13]. Under the assumption (19), the group $(G(s))_{s \in \mathbb{R}}$ also acts on $H_{Q,loc}$. In particular, any matrix field of $H_Q^\infty \subset H_{Q,loc}$ possesses an average in $H_{Q,loc}$, still denoted by $\langle \cdot \rangle$, as for matrix fields in H_Q .

Theorem 2.1 *Assume that (4), (5), (14), (15), (19) hold true. We denote by L the infinitesimal generator of the group $(G(s))_{s \in \mathbb{R}}$.*

1. *For any matrix field $A \in H_Q$ we have the strong convergence in H_Q*

$$\langle A \rangle := \lim_{S \rightarrow +\infty} \frac{1}{S} \int_r^{r+S} \underbrace{\partial Y(-s; Y(s; \cdot)) A(Y(s; \cdot)) {}^t \partial Y(-s; Y(s; \cdot))}_{=(G(s)A)(y)} \, ds = \text{Proj}_{\ker L} A$$

uniformly with respect to $r \in \mathbb{R}$.

2. *If $A \in H_Q$ is a field of symmetric positive semi-definite matrices, then so is $\langle A \rangle$.*

3. Let $\mathcal{S} \subset \mathbb{R}^m$ be an invariant set of the flow of b , that is $Y(s; \mathcal{S}) = \mathcal{S}$ for any $s \in \mathbb{R}$. If $A \in H_Q$ is such that

$$Q^{1/2}(y)A(y)Q^{1/2}(y) \geq \alpha I_m, \quad y \in \mathcal{S},$$

for some $\alpha > 0$, then we have

$$Q^{1/2}(y) \langle A \rangle (y) Q^{1/2}(y) \geq \alpha I_m, \quad y \in \mathcal{S}$$

and in particular, $\langle A \rangle (y)$ is positive definite for $y \in \mathcal{S}$.

4. If $A \in H_Q \cap H_Q^\infty$, then $\langle A \rangle \in H_Q \cap H_Q^\infty$ and

$$|\langle A \rangle|_Q \leq |A|_Q, \quad |\langle A \rangle|_{H_Q^\infty} \leq |A|_{H_Q^\infty}.$$

5. For any matrix field $A \in H_{Q,\text{loc}}$, the family

$$\left(\frac{1}{S} \int_0^S \partial Y(-s; Y(s; \cdot)) A(Y(s; \cdot)) {}^t \partial Y(-s; Y(s; \cdot)) \, ds \right)_{S>0}$$

converges in $H_{Q,\text{loc}}$, when S goes to infinity. Its limit, denoted by $\langle A \rangle$, satisfies

$$\mathbf{1}_{\{\psi \leq k\}} \langle A \rangle = \langle \mathbf{1}_{\{\psi \leq k\}} A \rangle, \quad \text{for any } k \in \mathbb{N}$$

where the symbol $\langle \cdot \rangle$ in the right hand side stands for the average operator on H_Q . In particular, any matrix field $A \in H_Q^\infty$ has an average in $H_{Q,\text{loc}}$ and $|\langle A \rangle|_{H_Q^\infty} \leq |A|_{H_Q^\infty}$. If $A \in H_{Q,\text{loc}}$ is such that

$$Q^{1/2}(y)A(y)Q^{1/2}(y) \geq \alpha I_m, \quad y \in \mathbb{R}^m,$$

for some $\alpha > 0$, then we have

$$Q^{1/2}(y) \langle A \rangle (y) Q^{1/2}(y) \geq \alpha I_m, \quad y \in \mathbb{R}^m.$$

The construction presented above of the average matrix field relies on the hypotheses (4), (14), (15). However, the asymptotic analysis of the family $(u^\varepsilon)_{\varepsilon>0}$ requires to estimate the derivatives of $(u^\varepsilon)_{\varepsilon>0}$, up to the order three, see Propositions 4.3, 4.4. To obtain those estimates, it is very convenient to use derivatives along vector fields which commute with the derivatives along b , such vector fields are said to be *in involution with respect to b* . In this case, the uniform regularity of $(u^\varepsilon)_{\varepsilon>0}$ comes easily by taking the derivatives of (1) along these vector fields in involution with respect to b . More precisely, we shall assume that there is a matrix field $R(y)$ such that

$$\begin{cases} \det R(y) \neq 0, \quad y \in \mathbb{R}^m, \quad R \in L_{\text{loc}}^1(\mathbb{R}^m) \\ (b \cdot \nabla_y) R + R \partial_y b = 0 \text{ in } \mathcal{D}'(\mathbb{R}^m). \end{cases} \quad (20)$$

The equation satisfied by R in (20) is equivalent to $R(Y(s; y)) \partial Y(s; y) = R(y)$, $(s, y) \in \mathbb{R} \times \mathbb{R}^m$, which also writes

$$\partial Y(s; y) R^{-1}(y) = R^{-1}(Y(s; y)), \quad (s, y) \in \mathbb{R} \times \mathbb{R}^m, \quad (21)$$

and is finally equivalent to $(b \cdot \nabla_y) R^{-1} = \partial_y b R^{-1}$, saying that the columns of R^{-1} are vector fields in involution with b , see [12] Proposition 3.4. for more details. The vector fields in the

columns of R^{-1} are denoted $b_i, 1 \leq i \leq m$. At any point $y \in \mathbb{R}^m$ they form a basis for \mathbb{R}^m cf. (20) and are supposed smooth and sublinear

$$\begin{cases} b_i \in W_{\text{loc}}^{1,\infty}(\mathbb{R}^m), \operatorname{div}_y b_i \in L^\infty(\mathbb{R}^m), \quad 1 \leq i \leq m, \\ \forall i \in \{1, \dots, m\}, \exists C_i > 0 \text{ such that } |b_i(y)| \leq C_i(1 + |y|), \quad y \in \mathbb{R}^m, \end{cases} \quad (22)$$

which guarantees the existence of the global flows $Y_i(s; y) \in W_{\text{loc}}^{1,\infty}(\mathbb{R} \times \mathbb{R}^m), i \in \{1, \dots, m\}$. We claim that the hypotheses (20) imply (14), (15). Clearly $R^{-1} \in L_{\text{loc}}^\infty(\mathbb{R}^m)$, since b_i , which are the columns of R^{-1} , are supposed to be locally bounded on \mathbb{R}^m . Since $y \rightarrow R^{-1}(y)$ is continuous, the function $y \rightarrow \det R^{-1}(y)$ remains away from 0 on any compact set of \mathbb{R}^m , implying that $R = (R^{-1})^{-1} \in L_{\text{loc}}^\infty(\mathbb{R}^m)$. In particular ${}^t R R, ({}^t R R)^{-1}$ are locally bounded, and therefore locally square integrable on \mathbb{R}^m . We define $Q = {}^t R R, P = Q^{-1} = R^{-1} {}^t R^{-1}$ and observe that (14), (15) are satisfied. Indeed, $P(y)$ is symmetric, positive definite, locally square integrable, together with its inverse $Q = P^{-1}$ and, thanks to (21), we have

$$\begin{aligned} P(Y(s; y)) &= R^{-1}(Y(s; y)) {}^t R^{-1}(Y(s; y)) \\ &= \partial Y(s; y) R^{-1}(y) {}^t R^{-1}(y) {}^t \partial Y(s; y) \\ &= \partial Y(s; y) P(y) {}^t \partial Y(s; y) \end{aligned}$$

saying that $[b, P] = 0$ in $\mathcal{D}'(\mathbb{R}^m)$ cf. Proposition 2.1. Under the hypotheses (22), the spaces H_Q, H_Q^∞ satisfy

$$\begin{aligned} H_Q &= \{A : \mathbb{R}^m \rightarrow \mathcal{M}_m(\mathbb{R}) \text{ measurable} : R A {}^t R \in L^2(\mathbb{R}^m)\}, \\ H_Q^\infty &= \{A : \mathbb{R}^m \rightarrow \mathcal{M}_m(\mathbb{R}) \text{ measurable} : R A {}^t R \in L^\infty(\mathbb{R}^m)\}. \end{aligned}$$

Given the family $(b_i)_{1 \leq i \leq m}$ of vector fields in involution with respect to b , we construct the following Sobolev-type space on \mathbb{R}^m

$$\begin{aligned} H_R^1 &:= \bigcap_{i=1}^m \operatorname{dom}(b_i \cdot \nabla_y) \\ &= \{u \in L^2(\mathbb{R}^m) : b_i \cdot \nabla_y u \in L^2(\mathbb{R}^m), \forall i \in \{1, \dots, m\}\} \end{aligned} \quad (23)$$

which is a Hilbert space with the scalar product

$$(u, v)_R = \int_{\mathbb{R}^m} u(y)v(y) \, dy + \sum_{i=1}^m \int_{\mathbb{R}^m} (b_i \cdot \nabla_y u)(b_i \cdot \nabla_y v) \, dy, \quad u, v \in H_R^1.$$

The associated norm is denoted by $|\cdot|_R$. The operators $b_i \cdot \nabla_y$ are the infinitesimal generators of the C^0 -groups of linear transformations on $L^2(\mathbb{R}^m)$ given by

$$\tau_i(s)u := u \circ Y_i(s; \cdot), \quad u \in L^2(\mathbb{R}^m), \quad s \in \mathbb{R}, \quad i \in \{1, \dots, m\}.$$

The hypothesis $\operatorname{div}_y b_i \in L^\infty(\mathbb{R}^m)$ plays a crucial role when looking for a bound for the Jacobian determinant of ∂Y_i .

Remark 2.1 Notice that every element of H_R^1 has a gradient in the distribution sense that belongs to $L_{\text{loc}}^2(\mathbb{R}^m)$. More precisely, for any $u \in H_R^1$, we have

$$\nabla_y u = {}^t R {}^t (b_1 \cdot \nabla_y u, \dots, b_m \cdot \nabla_y u),$$

and

$$|u|_R^2 = \|u\|_{L^2(\mathbb{R}^m)}^2 + |\nabla u|_P^2. \quad (24)$$

It will be convenient in the sequel to make use of the following differential operators

$$\nabla_y^R := {}^t R^{-1} \nabla_y = {}^t (b_1 \cdot \nabla_y, \dots, b_m \cdot \nabla_y),$$

as well as its higher order versions

$$\begin{aligned} (\nabla_y^R \otimes \nabla_y^R)_{ij} &:= b_i \cdot \nabla_y (b_j \cdot \nabla_y), \quad i, j \in \{1, \dots, m\}, \\ (\nabla_y^R \otimes \nabla_y^R \otimes \nabla_y^R)_{ijk} &:= b_i \cdot \nabla_y (b_j \cdot \nabla_y (b_k \cdot \nabla_y)), \quad i, j, k \in \{1, \dots, m\}. \end{aligned}$$

The results in Section 3 hold true under the hypotheses (4), (14), (15) but in Sections 4, 6 and Appendix A we shall need the stronger hypotheses (20) instead of (14), (15).

Moreover we assume that

$$\begin{cases} {}^t D = D, \quad D \in H_Q^\infty, \quad b \in X_Q^\infty \\ \exists \alpha > 0 \text{ such that } Q^{1/2}(y)D(y)Q^{1/2}(y) \geq \alpha I_m, \quad y \in \mathbb{R}^m \end{cases} \quad (25)$$

where $Q = {}^t R R$ and the columns of R^{-1} are given by the vector fields b_1, \dots, b_m .

In view of Theorem 2.1, the formal limit problem (13) of the family of systems (9) becomes

$$\begin{cases} \partial_t v - \operatorname{div}_z (\langle D \rangle \nabla_z v) = 0, & (t, z) \in \mathbb{R}_+ \times \mathbb{R}^m \\ v(0, z) = u^{\text{in}}(z), & z \in \mathbb{R}^m. \end{cases} \quad (26)$$

Under some regularity assumptions (see Section 6), we will obtain a strong convergence result for the family $(v^\varepsilon)_{\varepsilon > 0}$ in $L_{\text{loc}}^\infty(\mathbb{R}_+; L^2(\mathbb{R}^m))$, toward the solution v of the problem (26). Coming back to the family $(u^\varepsilon)_{\varepsilon > 0}$, through the change of variable in (6), and thanks to the fact that for any $s \in \mathbb{R}$, $Y(s; \cdot)$ is measure preserving, we justify that at any time $t \in \mathbb{R}_+$, $\varepsilon > 0$, $u^\varepsilon(t, \cdot)$ behaves (in $L^2(\mathbb{R}^m)$) like the composition product between $v(t, \cdot)$ and $Y(-t/\varepsilon; \cdot)$, that is

$$\lim_{\varepsilon \searrow 0} \|u^\varepsilon(t, \cdot) - v(t, Y(-t/\varepsilon; \cdot))\|_{L^2} = \lim_{\varepsilon \searrow 0} \|v^\varepsilon(t, \cdot) - v(t, \cdot)\|_{L^2} = 0,$$

uniformly with respect to $t \in [0, T]$, for any $T \in \mathbb{R}_+$.

Passing to the limit in (9), when $\varepsilon \searrow 0$, requires a two-scale analysis. This can be achieved by standard homogenization arguments, in the setting of periodic fast variables. The aim of this paper is to obtain results without periodicity assumptions, a framework where not so much results are available. The key point is to appeal to ergodic means along C^0 -groups of unitary operators. More precisely, the main difficulty is passing to the limit in the integral term

$$\int_0^{+\infty} \int_{\mathbb{R}^m} G(t/\varepsilon) D \nabla_z v^\varepsilon \cdot \nabla_z \Phi \, dz dt$$

that appears in the variational formulation of (9), for any smooth test function Φ . The argument combines the weak convergence of $(\nabla_z v^\varepsilon)_{\varepsilon > 0}$ as $\varepsilon \searrow 0$ in $L^2([0, T]; X_P)$, $T \in \mathbb{R}_+$, and the strong convergence of $\left(\frac{1}{S} \int_0^S G(s) D \, ds\right)_{S > 0}$, as $S \rightarrow +\infty$, in $H_{Q, \text{loc}}$. More exactly, we prove that for any family of functions $(w^\beta)_{\beta > 0}$, admitting the same modulus of continuity in $C([0, T]; X_P)$ and converging weakly in $L^2([0, T]; X_P)$, as $\beta \searrow 0$, toward some w^0 , we have

$$\lim_{(\beta, \varepsilon) \rightarrow (0, 0)} \int_0^T \int_{\mathbb{R}^m} \theta(t, y) \otimes w^\beta(t, y) : G(t/\varepsilon) D \, dy dt = \int_0^T \int_{\mathbb{R}^m} \theta(t, y) \otimes w^0(t, y) : \langle D \rangle \, dy dt$$

for any $\theta \in L^2([0, T]; X_P)$, see Propositions 5.1, 5.2. The strong convergences of $(\nabla_z v^\varepsilon)_{\varepsilon > 0}$ in $L^2([0, T]; X_P)$ and $(v^\varepsilon)_{\varepsilon > 0}$ in $L^\infty([0, T]; L^2(\mathbb{R}^m))$ come by energy balances. This is a consequence of a general result, see Proposition 5.3. Under the coercivity condition in (25),

we prove that the weak convergence of $(w^\beta)_{\beta>0}$ in $L^2([0, T]; X_P)$ toward w^0 together with the inequality

$$\limsup_{(\beta, \varepsilon) \rightarrow (0, 0)} \int_0^T \int_{\mathbb{R}^m} w^\beta(t, y) \otimes w^\beta(t, y) : G(t/\varepsilon)D \, dy dt \leq \int_0^T \int_{\mathbb{R}^m} w^0(t, y) \otimes w^0(t, y) : \langle D \rangle \, dy dt$$

imply the strong convergence of $(w^\beta)_{\beta>0}$ in $L^2([0, T]; X_P)$ toward w^0 .

Theorem 2.2 *Assume that the hypotheses (4), (5) (19), (22), (25), (35) hold true together with all the regularity conditions in Proposition 4.4. We suppose that $u^{\text{in}} \in H_R^2$ and we denote by $(u^\varepsilon)_{\varepsilon>0}$ the variational solutions of (1) and by $(v^\varepsilon)_{\varepsilon>0}$ the functions*

$$v^\varepsilon(t, z) = u^\varepsilon(t, Y(t/\varepsilon; z)), \quad (t, z) \in \mathbb{R}_+ \times \mathbb{R}^m, \quad \varepsilon > 0.$$

Then the family $(v^\varepsilon)_{\varepsilon>0}$ converges strongly in $L_{\text{loc}}^\infty(\mathbb{R}_+; L^2(\mathbb{R}^m))$ to the unique variational solution $v \in L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^m))$ of (26). The function v has the regularity

$$\partial_t v, \nabla_z^R v, \nabla_z^R \otimes \nabla_z^R v \in L_{\text{loc}}^\infty(\mathbb{R}_+; L^2(\mathbb{R}^m)), \quad \partial_t \nabla_z^R v \in L_{\text{loc}}^2(\mathbb{R}_+; L^2(\mathbb{R}^m))$$

and $(\nabla_z v^\varepsilon)_{\varepsilon>0}$ converges toward $\nabla_z v$ in $L_{\text{loc}}^2(\mathbb{R}_+; X_P)$ when ε goes to 0. Moreover, the strong convergence of $(v^\varepsilon)_{\varepsilon>0}$ in $L_{\text{loc}}^\infty(\mathbb{R}_+; L^2(\mathbb{R}^m))$, when ε goes to 0, holds true for any initial condition $u^{\text{in}} \in L^2(\mathbb{R}^m)$.

It is easily seen that $(u^\varepsilon)_{\varepsilon>0}$ converges weakly in $L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^m))$, as $\varepsilon \searrow 0$, toward $\langle v \rangle$, see also [12]. We deduce that $(u^\varepsilon)_{\varepsilon>0}$ converges strongly in $L_{\text{loc}}^\infty(\mathbb{R}_+; L^2(\mathbb{R}^m))$, as $\varepsilon \searrow 0$, toward v iff $v = \langle v \rangle$, or equivalently, iff the initial condition is well prepared, *i.e.*, $u^{\text{in}} \in \ker \mathcal{T}$. Under additional hypotheses we can justify that $v^\varepsilon = v + \mathcal{O}(\varepsilon)$ in $L_{\text{loc}}^\infty(\mathbb{R}_+; L^2(\mathbb{R}^m))$, as suggested by the formal Ansatz (10).

Theorem 2.3 *Assume that the hypotheses (4), (5), (19), (22), (25) hold true. Moreover, we assume that the solution v of the limit model (26) is smooth enough, that is*

$$\begin{aligned} \nabla_z^R v &\in L_{\text{loc}}^\infty(\mathbb{R}_+; L^2(\mathbb{R}^m)), \quad \nabla_z^R \otimes \nabla_z^R v \in L_{\text{loc}}^\infty(\mathbb{R}_+; L^2(\mathbb{R}^m)) \\ \nabla_z^R \partial_t v &\in L_{\text{loc}}^1(\mathbb{R}_+; L^2(\mathbb{R}^m)), \quad \nabla_z^R \otimes \nabla_z^R \partial_t v \in L_{\text{loc}}^1(\mathbb{R}_+; L^2(\mathbb{R}^m)) \\ \nabla_z^R \otimes \nabla_z^R \otimes \nabla_z^R v &\in L_{\text{loc}}^2(\mathbb{R}_+; L^2(\mathbb{R}^m)), \quad \nabla_z^R \otimes \nabla_z^R \otimes \nabla_z^R \otimes \nabla_z^R v \in L_{\text{loc}}^1(\mathbb{R}_+; L^2(\mathbb{R}^m)) \end{aligned}$$

and that there is a smooth matrix field $C \in H_\infty^Q$, that is

$$\begin{aligned} \text{div}_y(RC), \quad b_k \cdot \nabla_y \text{div}_y(RC), \quad b_l \cdot \nabla_y (b_k \cdot \nabla_y \text{div}_y(RC)) &\in L^\infty(\mathbb{R}^m), \quad k, l \in \{1, \dots, m\} \\ RC \, {}^t R, \quad b_k \cdot \nabla_y (RC \, {}^t R), \quad b_l \cdot \nabla_y (b_k \cdot \nabla_y (RC \, {}^t R)) &\in L^\infty(\mathbb{R}^m), \quad k, l \in \{1, \dots, m\} \end{aligned}$$

such that the following decomposition holds true

$$D = \langle D \rangle + L(C), \quad \langle C \rangle = 0.$$

We denote by $(u^\varepsilon)_{\varepsilon>0}$ the variational solutions of (1). Then for any $T \in \mathbb{R}_+$, there is a constant C_T such that

$$\begin{aligned} \sup_{t \in [0, T]} \|u^\varepsilon(t, \cdot) - v(t, Y(-t/\varepsilon; \cdot))\|_{L^2(\mathbb{R}^m)} &\leq C_T \varepsilon \\ \left(\int_0^T |\nabla_y u^\varepsilon(t, \cdot) - \nabla_y v(t, Y(-t/\varepsilon; \cdot))|_P^2 \, dt \right)^{1/2} &\leq C_T \varepsilon. \end{aligned}$$

3 The average of a matrix field

3.1 Definition and properties

Motivated by the computations leading to (8), we consider the family of linear transformations $(G(s))_{s \in \mathbb{R}}$, acting on matrix fields. It happens that $(G(s))_{s \in \mathbb{R}}$ is a C^0 -group of unitary operators on H_Q . For any function $f = f(y)$, $y \in \mathbb{R}^m$, the notation $f_s = f_s(z)$ stands for the composition product $f_s = f \circ Y(s; \cdot)$.

Proposition 3.1 *Assume that the hypotheses (4), (5), (14), (15), (19) hold true.*

1. *The family of applications*

$$A \rightarrow G(s)A := \partial Y^{-1}(s; \cdot) A_s {}^t \partial Y^{-1}(s; \cdot) = \partial Y(-s; Y(s; \cdot)) A_s {}^t \partial Y(-s; Y(s; \cdot))$$

is a C^0 -group of unitary operators on H_Q .

2. *If A is a field of symmetric matrices, then so is $G(s)A$, for any $s \in \mathbb{R}$.*

3. *If A is a field of positive semi-definite matrices, then so is $G(s)A$, for any $s \in \mathbb{R}$.*

4. *Let $\mathcal{S} \subset \mathbb{R}^m$ be an invariant set of the flow of b , that is $Y(s; \mathcal{S}) = \mathcal{S}$, for any $s \in \mathbb{R}$. If there is $\alpha > 0$ such that $Q^{1/2}(y)A(y)Q^{1/2}(y) \geq \alpha I_m$, $y \in \mathcal{S}$, then for any $s \in \mathbb{R}$ we have $Q^{1/2}(y)(G(s)A)(y)Q^{1/2}(y) \geq \alpha I_m$, $y \in \mathcal{S}$.*

5. *The family of applications $(G(s))_{s \in \mathbb{R}}$ acts on $H_{Q, \text{loc}}$, that is, if $A \in H_{Q, \text{loc}}$, then $G(s)A \in H_{Q, \text{loc}}$ for any $s \in \mathbb{R}$. Moreover, we have*

$$\mathbf{1}_{\{\psi \leq k\}} G(s)A = G(s)(\mathbf{1}_{\{\psi \leq k\}} A), \quad A \in H_{Q, \text{loc}}, \quad s \in \mathbb{R}, \quad k \in \mathbb{N}.$$

Proof.

1. Thanks to the characterization in Proposition 2.1 we know that

$$P_s = \partial Y(s; \cdot) P {}^t \partial Y(s; \cdot), \quad s \in \mathbb{R}. \quad (27)$$

For any $s \in \mathbb{R}$ we consider the matrix field $\mathcal{O}(s; \cdot) = Q_s^{1/2} \partial Y(s; \cdot) Q^{-1/2}$. Observe that $\mathcal{O}(s; \cdot)$ is a field of orthogonal matrices, for any $s \in \mathbb{R}$. Indeed we have, thanks to (27)

$$\begin{aligned} {}^t \mathcal{O}(s; \cdot) \mathcal{O}(s; \cdot) &= Q^{-1/2} {}^t \partial Y(s; \cdot) Q_s^{1/2} Q_s^{1/2} \partial Y(s; \cdot) Q^{-1/2} \\ &= Q^{-1/2} (\partial Y^{-1}(s; \cdot) P_s {}^t \partial Y^{-1}(s; \cdot))^{-1} Q^{-1/2} \\ &= Q^{-1/2} P^{-1} Q^{-1/2} \\ &= I_m \end{aligned}$$

implying that for any matrix field A we have

$$Q^{1/2} G(s) A Q^{1/2} = Q^{1/2} \partial Y^{-1}(s; \cdot) A_s {}^t \partial Y^{-1}(s; \cdot) Q^{1/2} = {}^t \mathcal{O}(s; \cdot) Q_s^{1/2} A_s Q_s^{1/2} \mathcal{O}(s; \cdot). \quad (28)$$

It is easily seen that if $A \in H_Q$, then for any $s \in \mathbb{R}$

$$\begin{aligned} |G(s)A|_Q^2 &= \int_{\mathbb{R}^m} Q^{1/2} G(s) A Q^{1/2} : Q^{1/2} G(s) A Q^{1/2} \, dy \\ &= \int_{\mathbb{R}^m} {}^t \mathcal{O}(s; \cdot) Q_s^{1/2} A_s Q_s^{1/2} \mathcal{O}(s; \cdot) : {}^t \mathcal{O}(s; \cdot) Q_s^{1/2} A_s Q_s^{1/2} \mathcal{O}(s; \cdot) \, dy \\ &= \int_{\mathbb{R}^m} Q_s^{1/2} A_s Q_s^{1/2} : Q_s^{1/2} A_s Q_s^{1/2} \, dy \\ &= \int_{\mathbb{R}^m} Q^{1/2} A Q^{1/2} : Q^{1/2} A Q^{1/2} \, dy = |A|_Q^2 \end{aligned}$$

proving that $G(s)$ is a unitary transformation for any $s \in \mathbb{R}$. The group property of the family $(G(s))_{s \in \mathbb{R}}$ follows easily from the group property of the flow $(Y(s; \cdot))_{s \in \mathbb{R}}$

$$\begin{aligned} G(s)G(t)A &= \partial Y^{-1}(s; \cdot)(G(t)A)_s {}^t \partial Y^{-1}(s; \cdot) \\ &= \partial Y^{-1}(s; \cdot) \partial Y^{-1}(t; Y(s; \cdot))(A_t)_s {}^t \partial Y^{-1}(t; Y(s; \cdot)) {}^t \partial Y^{-1}(s; \cdot) \\ &= \partial Y^{-1}(t+s; \cdot) A_{t+s} {}^t \partial Y^{-1}(t+s; \cdot) = G(t+s)A, \quad A \in H_Q. \end{aligned}$$

The continuity of the group, *i.e.*, $\lim_{s \rightarrow 0} G(s)A = A$ strongly in H_Q , is left to the reader.

2. Notice that $G(s)$ commutes with transposition

$$\begin{aligned} {}^t(G(s)A) &= {}^t(\partial Y^{-1}(s; \cdot) A_s {}^t \partial Y^{-1}(s; \cdot)) \\ &= \partial Y^{-1}(s; \cdot) {}^t A_s {}^t \partial Y^{-1}(s; \cdot) \\ &= G(s) {}^t A. \end{aligned}$$

In particular, if ${}^t A = A$, then ${}^t(G(s)A) = G(s)A$.

3. We use the formula (28). For any $\xi, \eta \in \mathbb{R}^m$, the notation $\xi \otimes \eta$ stands for the matrix whose (i, j) entry is $\xi_i \eta_j$. For any $\xi \in \mathbb{R}^m$ we have

$$\begin{aligned} G(s)A : Q^{1/2} \xi \otimes Q^{1/2} \xi &= Q^{1/2} G(s) A Q^{1/2} : \xi \otimes \xi \\ &= {}^t \mathcal{O}(s; \cdot) Q_s^{1/2} A_s Q_s^{1/2} \mathcal{O}(s; \cdot) : \xi \otimes \xi \\ &= Q_s^{1/2} A_s Q_s^{1/2} : \mathcal{O}(s; \cdot) (\xi \otimes \xi) {}^t \mathcal{O}(s; \cdot) \\ &= Q_s^{1/2} A_s Q_s^{1/2} : (\mathcal{O}(s; \cdot) \xi) \otimes (\mathcal{O}(s; \cdot) \xi) \\ &= A_s : (Q_s^{1/2} \mathcal{O}(s; \cdot) \xi) \otimes (Q_s^{1/2} \mathcal{O}(s; \cdot) \xi). \end{aligned}$$

As A is a field of positive semi-definite matrices, therefore $G(s)A$ is a field of positive semi-definite matrices as well.

4. Assume that there is $\alpha > 0$ such that $Q^{1/2} A Q^{1/2} \geq \alpha I_m$ on \mathcal{S} . As before we write for any $\xi \in \mathbb{R}^m, y \in \mathcal{S}$

$$Q^{1/2} G(s) A Q^{1/2} : \xi \otimes \xi = (Q^{1/2} A Q^{1/2})_s : (\mathcal{O}(s; \cdot) \xi) \otimes (\mathcal{O}(s; \cdot) \xi) \geq \alpha |\mathcal{O}(s; \cdot) \xi|^2 = \alpha |\xi|^2$$

saying that $Q^{1/2} G(s) A Q^{1/2} \geq \alpha I_m$ on \mathcal{S} .

5. Here $G(s)$ stands for the application $A \rightarrow \partial Y(-s; Y(s; \cdot)) A (Y(s; \cdot)) {}^t \partial Y(-s; Y(s; \cdot))$ independently of A being in H_Q or in $H_{Q, \text{loc}}$. As ψ is left invariant by the flow of b , so is $\mathbf{1}_{\{\psi \leq k\}}$, for any $k \in \mathbb{N}$. If A belongs to $H_{Q, \text{loc}}$, we have

$$\mathbf{1}_{\{\psi \leq k\}} G(s) A = G(s) (\mathbf{1}_{\{\psi \leq k\}} A) \in H_Q, \quad k \in \mathbb{N}, \quad s \in \mathbb{R}$$

saying that $G(s)A \in H_{Q, \text{loc}}, s \in \mathbb{R}$. Moreover, the applications $(G(s))_{s \in \mathbb{R}}$ preserve locally the norm of H_Q

$$|\mathbf{1}_{\{\psi \leq k\}} G(s) A|_Q = |G(s) (\mathbf{1}_{\{\psi \leq k\}} A)|_Q = |\mathbf{1}_{\{\psi \leq k\}} A|_Q, \quad k \in \mathbb{N}, \quad s \in \mathbb{R}.$$

□

We denote by L the infinitesimal generator of the group G in H_Q

$$L : \text{dom}(L) \subset H_Q \rightarrow H_Q, \quad \text{dom} L = \left\{ A \in H_Q : \exists \lim_{s \rightarrow 0} \frac{G(s)A - A}{s} \text{ in } H_Q \right\}$$

and we set $L(A) = \lim_{s \rightarrow 0} \frac{G(s)A - A}{s}$ for any $A \in \text{dom}(L)$. Notice that $C_c^1(\mathbb{R}^m) \subset \text{dom}(L)$ and $L(A) = (b \cdot \nabla_y) A - \partial_y b A - A {}^t \partial_y b$, as soon as $A \in C_c^1(\mathbb{R}^m)$ (use the hypothesis $Q \in L_{\text{loc}}^2(\mathbb{R}^m)$ and the dominated convergence theorem). The main properties of the operator L are summarized below (see [12, Prop. 3.13] for details)

Proposition 3.2 *Assume that the hypotheses (4), (14), (15) hold true.*

1. *The domain of L is dense in H_Q and L is closed.*
2. *The matrix field $A \in H_Q$ belongs to $\text{dom}(L)$ iff there is a constant $C > 0$ such that*

$$|G(s)A - A|_Q \leq C|s|, \quad s \in \mathbb{R}.$$

3. *The operator L is skew-adjoint in H_Q and we have the orthogonal decomposition $H_Q = \ker L \oplus \overline{\text{Range } L}$.*

Remark 3.1 *When working on $H_{Q,\text{loc}}$, the generator of $(G(s))_{s \in \mathbb{R}}$, which is still denoted by L , is defined by*

$$A \in \text{dom}(L) \quad \text{iff} \quad \exists \lim_{s \rightarrow 0} \frac{G(s)(\mathbf{1}_{\{\psi \leq k\}}A) - \mathbf{1}_{\{\psi \leq k\}}A}{s} \in H_Q, \quad k \in \mathbb{N}$$

and

$$\mathbf{1}_{\{\psi \leq k\}}L(A) = \lim_{s \rightarrow 0} \frac{G(s)(\mathbf{1}_{\{\psi \leq k\}}A) - \mathbf{1}_{\{\psi \leq k\}}A}{s}, \quad k \in \mathbb{N}.$$

The transformations $(G(s))_{s \in \mathbb{R}}$ also behave nicely in weighted L^∞ spaces. More precisely, for any $s \in \mathbb{R}$, and any $A \in H_Q^\infty$, we have $G(s)A \in H_Q^\infty$ and $|G(s)A|_{H_Q^\infty} = |A|_{H_Q^\infty}$. Indeed, thanks to (28) and to the orthogonality of $\mathcal{O}(s; \cdot)$, we can write

$$\begin{aligned} Q^{1/2}G(s)AQ^{1/2} : Q^{1/2}G(s)AQ^{1/2} &= {}^t\mathcal{O}(s; \cdot)Q_s^{1/2}A_sQ_s^{1/2}\mathcal{O}(s; \cdot) : {}^t\mathcal{O}(s; \cdot)Q_s^{1/2}A_sQ_s^{1/2}\mathcal{O}(s; \cdot) \\ &= (Q^{1/2}AQ^{1/2} : Q^{1/2}AQ^{1/2})_s. \end{aligned}$$

We are now in position to apply the von Neumann's ergodic mean theorem.

Theorem 3.1 (von Neumann's ergodic mean theorem, see [35]) *Let $(\mathcal{G}(s))_{s \in \mathbb{R}}$ be a C^0 -group of unitary operators on an Hilbert space $(\mathcal{H}, (\cdot, \cdot))$ and \mathcal{L} be its infinitesimal generator. Then for any $x \in \mathcal{H}$, we have the strong convergence in \mathcal{H}*

$$\lim_{S \rightarrow +\infty} \frac{1}{S} \int_r^{r+S} \mathcal{G}(s)x \, ds = \text{Proj}_{\text{Ker } \mathcal{L}}x, \quad \text{uniformly with respect to } r \in \mathbb{R}.$$

The proof of Theorem 2.1 comes immediately, by applying Theorem 3.1 to the group in Proposition 3.1.

Proof of Theorem 2.1. The first and second statements are obvious.

3. For any $\xi \in \mathbb{R}^m$, $\psi \in C_c^0(\mathcal{S})$, $\psi \geq 0$ we have $\psi(\cdot)P^{1/2}\xi \otimes P^{1/2}\xi \in H_Q$ and we can write, thanks to (28)

$$\begin{aligned} (G(s)A, \psi(\cdot)P^{1/2}\xi \otimes P^{1/2}\xi)_Q &= \int_{\mathbb{R}^m} \psi(y)Q^{1/2}G(s)AQ^{1/2} : \xi \otimes \xi \, dy \\ &= \int_{\mathbb{R}^m} \psi(y) {}^t\mathcal{O}(s; y)Q_s^{1/2}A_sQ_s^{1/2}\mathcal{O}(s; y)\xi \cdot \xi \, dy \\ &= \int_{\mathbb{R}^m} \psi(y)Q_s^{1/2}A_sQ_s^{1/2} : \mathcal{O}(s; y)\xi \otimes \mathcal{O}(s; y)\xi \, dy \\ &\geq \alpha \int_{\mathbb{R}^m} |\mathcal{O}(s; y)\xi|^2 \psi(y) \, dy \\ &= \alpha|\xi|^2 \int_{\mathbb{R}^m} \psi(y) \, dy. \end{aligned}$$

Taking the average over $[0, S]$ and letting $S \rightarrow +\infty$ yield

$$\int_{\mathbb{R}^m} \psi(y) Q^{1/2} \langle A \rangle Q^{1/2} : \xi \otimes \xi \, dy = (\langle A \rangle, \psi P^{1/2} \xi \otimes P^{1/2} \xi)_Q \geq \int_{\mathbb{R}^m} \alpha |\xi|^2 \psi(y) \, dy$$

implying that

$$Q^{1/2}(y) \langle A \rangle (y) Q^{1/2}(y) \geq \alpha I_m, \quad y \in \mathcal{S}.$$

4. Obviously, for any $A \in H_Q$, we have by the properties of the orthogonal projection on $\ker L$ that $|\langle A \rangle|_Q = |\text{Proj}_{\ker L} A|_Q \leq |A|_Q$. For the last inequality, consider $M \in \mathcal{M}_m(\mathbb{R})$ a fixed matrix, $\psi \in C_c^0(\mathbb{R}^m)$, $\psi \geq 0$ and, as before, observe that $\psi P^{1/2} M P^{1/2} \in H_Q$, which allows us to write

$$\begin{aligned} (G(s)A, \psi P^{1/2} M P^{1/2})_Q &= \int_{\mathbb{R}^m} Q^{1/2} G(s) A Q^{1/2} : \psi M \, dy \\ &= \int_{\mathbb{R}^m} {}^t \mathcal{O}(s; y) Q_s^{1/2} A_s Q_s^{1/2} \mathcal{O}(s; y) : \psi M \, dy \\ &= \int_{\mathbb{R}^m} Q_s^{1/2} A_s Q_s^{1/2} : \mathcal{O}(s; y) M {}^t \mathcal{O}(s; y) \, dy \\ &\leq \int_{\mathbb{R}^m} \sqrt{Q_s^{1/2} A_s Q_s^{1/2} : Q_s^{1/2} A_s Q_s^{1/2}} \sqrt{\mathcal{O}(s; y) M {}^t \mathcal{O}(s; y) : \mathcal{O}(s; y) M {}^t \mathcal{O}(s; y)} \psi \, dy \\ &\leq |A|_{H_Q^\infty} (M : M)^{1/2} \int_{\mathbb{R}^m} \psi(y) \, dy. \end{aligned}$$

Taking the average over $[0, S]$ and letting $S \rightarrow +\infty$, lead to

$$\int_{\mathbb{R}^m} Q^{1/2} \langle A \rangle Q^{1/2} : M \psi(y) \, dy = (\langle A \rangle, \psi P^{1/2} M P^{1/2})_Q \leq |A|_{H_Q^\infty} (M : M)^{1/2} \int_{\mathbb{R}^m} \psi(y) \, dy.$$

We deduce that

$$Q^{1/2}(y) \langle A \rangle (y) Q^{1/2}(y) : M \leq |A|_{H_Q^\infty} (M : M)^{1/2}, \quad y \in \mathbb{R}^m, \quad M \in \mathcal{M}_m(\mathbb{R})$$

saying that

$$|\langle A \rangle|_{H_Q^\infty} = \text{ess sup}_{y \in \mathbb{R}^m} \sqrt{Q^{1/2}(y) \langle A \rangle (y) Q^{1/2}(y) : Q^{1/2}(y) \langle A \rangle (y) Q^{1/2}(y)} \leq |A|_{H_Q^\infty}.$$

5. Let A be a matrix field in $H_{Q, \text{loc}}$. For any $k \in \mathbb{N}$, $\mathbf{1}_{\{\psi \leq k\}} A$ belongs to H_Q , and by the first statement we know that

$$\lim_{S \rightarrow +\infty} \frac{1}{S} \int_0^S G(s) (\mathbf{1}_{\{\psi \leq k\}} A) \, ds = \langle \mathbf{1}_{\{\psi \leq k\}} A \rangle \in H_Q, \quad k \in \mathbb{N}.$$

It is easily seen that for any $k, l \in \mathbb{N}$ we have

$$\lim_{S \rightarrow +\infty} \frac{1}{S} \int_0^S G(s) (\mathbf{1}_{\{\psi \leq k\}} A) \, ds = \lim_{S \rightarrow +\infty} \frac{1}{S} \int_0^S G(s) (\mathbf{1}_{\{\psi \leq l\}} A) \, ds$$

almost everywhere on $\{\psi \leq \min(k, l)\}$, and thus, there is a matrix field denoted by $\langle A \rangle$, whose restriction on $\{\psi \leq k\}$ coincides with $\langle \mathbf{1}_{\{\psi \leq k\}} A \rangle$ for any $k \in \mathbb{N}$. Notice also that for any $k \in \mathbb{N}$ we have $\langle \mathbf{1}_{\{\psi \leq k\}} A \rangle = 0$ almost everywhere on $\{\psi > k\}$ and thus we obtain

$$\mathbf{1}_{\{\psi \leq k\}} \langle A \rangle = \langle \mathbf{1}_{\{\psi \leq k\}} A \rangle, \quad k \in \mathbb{N}.$$

Observe that for any $k \in \mathbb{N}$, we have the convergence in H_Q

$$\lim_{S \rightarrow +\infty} \mathbf{1}_{\{\psi \leq k\}} \frac{1}{S} \int_0^S G(s)(A) \, ds = \lim_{S \rightarrow +\infty} \frac{1}{S} \int_0^S G(s)(\mathbf{1}_{\{\psi \leq k\}} A) \, ds = \langle \mathbf{1}_{\{\psi \leq k\}} A \rangle = \mathbf{1}_{\{\psi \leq k\}} \langle A \rangle$$

saying that $\lim_{S \rightarrow +\infty} \frac{1}{S} \int_0^S G(s)A \, ds = \langle A \rangle$ in $H_{Q,\text{loc}}$. The inclusion $H_Q^\infty \subset H_{Q,\text{loc}}$ follows by the compactness of $\{\psi \leq k\}, k \in \mathbb{N}$. By the fourth statement we have

$$|\langle A \rangle|_{H_Q^\infty} = \sup_{k \in \mathbb{N}} |\mathbf{1}_{\{\psi \leq k\}} \langle A \rangle|_{H_Q^\infty} = \sup_{k \in \mathbb{N}} |\langle \mathbf{1}_{\{\psi \leq k\}} A \rangle|_{H_Q^\infty} \leq \sup_{k \in \mathbb{N}} |\mathbf{1}_{\{\psi \leq k\}} A|_{H_Q^\infty} = |A|_{H_Q^\infty}.$$

Let A be a matrix field of $H_{Q,\text{loc}}$, such that $Q^{1/2}(y)A(y)Q^{1/2}(y) \geq \alpha I_m, y \in \mathbb{R}^m$, for some $\alpha > 0$. For any $k \in \mathbb{N}$ we have $\mathbf{1}_{\{\psi \leq k\}} A \in H_Q$ and

$$Q^{1/2}(y)\mathbf{1}_{\{\psi \leq k\}} A(y)Q^{1/2}(y) \geq \alpha I_m, \quad y \in \{\psi \leq k\}.$$

By the third statement we deduce that for any $k \in \mathbb{N}$

$$Q^{1/2}(y)\mathbf{1}_{\{\psi \leq k\}} \langle A \rangle (y)Q^{1/2}(y) = Q^{1/2}(y) \langle \mathbf{1}_{\{\psi \leq k\}} A \rangle (y)Q^{1/2}(y) \geq \alpha I_m, \quad y \in \{\psi \leq k\}$$

saying that $Q^{1/2}(y) \langle A \rangle (y)Q^{1/2}(y) \geq \alpha I_m, \quad y \in \mathbb{R}^m$. □

3.2 Examples

In this section we explicitly compute the average matrix field in three cases. We deal with periodic and almost-periodic flows.

3.2.1 Periodic flow

Consider the vector field $b(y) = (\gamma y_2, -\beta y_1)$, for any $y = (y_1, y_2) \in \mathbb{R}^2$, with $\beta, \gamma \in \mathbb{R}_+^*$. The function $\psi(y) = \beta y_1^2 + \gamma y_2^2, y \in \mathbb{R}^2$, is a coercive invariant associated to the field b . We denote by $Y(s; y)$ the flow of the vector field b . We intend to determine the average along the flow Y of the matrix field

$$D(y) = \begin{pmatrix} \lambda_1(y) & 0 \\ 0 & \lambda_2(y) \end{pmatrix}, \quad y \in \mathbb{R}^2$$

where λ_1, λ_2 are two given functions. It is easily seen that the flow is $2\pi/\sqrt{\beta\gamma}$ -periodic and writes $Y(s; y) = \mathcal{R}(-s; \beta, \gamma)y, (s, y) \in \mathbb{R} \times \mathbb{R}^2$, with

$$\mathcal{R}(s; \beta, \gamma) = \begin{pmatrix} \cos(\sqrt{\beta\gamma} s) & -\sqrt{\frac{\gamma}{\beta}} \sin(\sqrt{\beta\gamma} s) \\ \sqrt{\frac{\beta}{\gamma}} \sin(\sqrt{\beta\gamma} s) & \cos(\sqrt{\beta\gamma} s) \end{pmatrix}.$$

By Theorem 2.1 we deduce that

$$\begin{aligned} \langle D \rangle &= \frac{\sqrt{\beta\gamma}}{2\pi} \int_0^{2\pi/\sqrt{\beta\gamma}} \partial Y(-s; Y(s; \cdot)) D(Y(s; \cdot)) {}^t \partial Y(-s; Y(s; \cdot)) \, ds \\ &= \frac{\sqrt{\beta\gamma}}{2\pi} \int_0^{2\pi/\sqrt{\beta\gamma}} \mathcal{R}(s; \beta, \gamma) D(Y(s; \cdot)) \mathcal{R}(-s; \gamma, \beta) \, ds \\ &= \begin{pmatrix} \langle D \rangle_{11} & \langle D \rangle_{12} \\ \langle D \rangle_{21} & \langle D \rangle_{22} \end{pmatrix} \end{aligned}$$

where

$$\langle D \rangle_{11} = \frac{1}{2} \left\langle \lambda_1 [1 + \cos(2\sqrt{\beta\gamma} \cdot)] \right\rangle + \frac{\gamma}{2\beta} \left\langle \lambda_2 [1 - \cos(2\sqrt{\beta\gamma} \cdot)] \right\rangle$$

$$\begin{aligned}\langle D \rangle_{12} &= \langle D \rangle_{21} = \frac{\sqrt{\beta}}{2\sqrt{\gamma}} \left\langle \lambda_1 \sin(2\sqrt{\beta\gamma} \cdot) \right\rangle - \frac{\sqrt{\gamma}}{2\sqrt{\beta}} \left\langle \lambda_2 \sin(2\sqrt{\beta\gamma} \cdot) \right\rangle \\ \langle D \rangle_{22} &= \frac{\beta}{2\gamma} \left\langle \lambda_1 [1 - \cos(2\sqrt{\beta\gamma} \cdot)] \right\rangle + \frac{1}{2} \left\langle \lambda_2 [1 - \cos(2\sqrt{\beta\gamma} \cdot)] \right\rangle\end{aligned}$$

and for any function h the notation $\langle \lambda_i h(\cdot) \rangle$ stands for $\frac{\sqrt{\beta\gamma}}{2\pi} \int_0^{2\pi/\sqrt{\beta\gamma}} \lambda_i(Y(s; y)) h(s) ds$, $i \in \{1, 2\}$. Notice that when λ_1, λ_2 are constant functions along the flow $Y(s; \cdot)$ (that is, when λ_1, λ_2 depend only on $\beta(y_1)^2 + \gamma(y_2)^2$), the expression for $\langle D \rangle$ reduces to

$$\langle D \rangle = \begin{pmatrix} \frac{1}{2}\lambda_1 + \frac{\gamma}{2\beta}\lambda_2 & 0 \\ 0 & \frac{\beta}{2\gamma}\lambda_1 + \frac{1}{2}\lambda_2 \end{pmatrix}.$$

Observe that even if $\lambda_2 = 0$ everywhere (in which case $D(y)$ is nowhere definite positive), then the averaged diffusion matrix $\langle D \rangle$ can still be definite positive everywhere.

3.2.2 Almost-periodic flow

Consider the vector field $b(y) = (y_2, -\omega_1^2 y_1, y_4, -\omega_2^2 y_3)$, $y = (y_1, y_2, y_3, y_4) \in \mathbb{R}^4$, with $\omega_1, \omega_2 \in \mathbb{R}^*$ incommensurable, i.e $\omega_1/\omega_2 \notin \mathbb{Q}$. The function $\psi(y) = \omega_1^2 y_1^2 + y_2^2 + \omega_2^2 y_3^2 + y_4^2$, with $y \in \mathbb{R}^4$, is a coercive invariant associated to the field b . We denote by $Y(s; y)$ the flow of the vector field b . We consider the matrix field

$$D(y) = \begin{pmatrix} \lambda_1(y) & 0 & 0 & 0 \\ 0 & \lambda_2(y) & 0 & 0 \\ 0 & 0 & \lambda_3(y) & 0 \\ 0 & 0 & 0 & \lambda_4(y) \end{pmatrix}, \quad y \in \mathbb{R}^4$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are four given functions, which are constant along the flow $Y(s; \cdot)$. The flow writes $Y(s; y) = \mathcal{R}(-s; \omega_1, \omega_2)y$, $(s, y) \in \mathbb{R} \times \mathbb{R}^4$, with

$$\mathcal{R}(s; \omega_1, \omega_2) = \begin{pmatrix} \cos(s\omega_1) & -\frac{1}{\omega_1} \sin(s\omega_1) & 0 & 0 \\ \omega_1 \sin(s\omega_1) & \cos(s\omega_1) & 0 & 0 \\ 0 & 0 & \cos(s\omega_2) & -\frac{1}{\omega_2} \sin(s\omega_2) \\ 0 & 0 & \omega_2 \sin(s\omega_2) & \cos(s\omega_2) \end{pmatrix}.$$

The incommensurability condition ensures that the flow is not periodic with respect to the variable s . Nevertheless it is almost periodic with respect to s . By Theorem 2.1 we deduce that

$$\begin{aligned}\langle D \rangle &= \lim_{S \rightarrow +\infty} \frac{1}{S} \int_0^S \partial Y(-s; Y(s; \cdot)) D(Y(s; \cdot))^t \partial Y(-s; Y(s; \cdot)) ds \\ &= \lim_{S \rightarrow +\infty} \frac{1}{S} \int_0^S \mathcal{R}(s; \omega_1, \omega_2) D(Y(s; \cdot))^t \mathcal{R}(s; \omega_1, \omega_2) ds \\ &= \begin{pmatrix} \frac{1}{2}\lambda_1(y) + \frac{1}{2\omega_1^2}\lambda_2(y) & 0 & 0 & 0 \\ 0 & \frac{\omega_1^2}{2}\lambda_1(y) + \frac{1}{2}\lambda_2(y) & 0 & 0 \\ 0 & 0 & \frac{1}{2}\lambda_3(y) + \frac{1}{2\omega_2^2}\lambda_4(y) & 0 \\ 0 & 0 & 0 & \frac{\omega_2^2}{2}\lambda_3(y) + \frac{1}{2}\lambda_4(y) \end{pmatrix}.\end{aligned}$$

3.3 The Fokker-Planck equation

We inquire now about the Fokker-Planck equation. For simplicity, we assume that the magnetic field is uniform $B^\varepsilon = (0, 0, B/\varepsilon)$, $x \in \mathbb{R}^3$. In the finite Larmor radius regime (*i.e.*, the typical length in the orthogonal directions is much smaller than the typical length in the parallel direction), the presence density of charged particles f^ε satisfies

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon}(v_1 \partial_{x_1} + v_2 \partial_{x_2}) f^\varepsilon + v_3 \partial_{x_3} f^\varepsilon + \frac{q}{m} E \cdot \nabla_v f^\varepsilon + \frac{qB}{m\varepsilon}(v_2 \partial_{v_1} - v_1 \partial_{v_2}) f^\varepsilon = \nu \operatorname{div}_v \{\Theta \nabla_v f^\varepsilon + v f^\varepsilon\}. \quad (29)$$

In this case, the flow

$$(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3 \mapsto Y(s; x, v) = (X(s; x, v), V(s; x, v)) \in \mathbb{R}^3 \times \mathbb{R}^3,$$

to be considered corresponds to the vector field

$$b(x, v) \cdot \nabla_{x,v} = v_1 \partial_{x_1} + v_2 \partial_{x_2} + \omega_c (v_2 \partial_{v_1} - v_1 \partial_{v_2}), \quad \omega_c = \frac{qB}{m}, \quad (x, v) \in \mathbb{R}^6.$$

We introduce the notation $c(x, v) \cdot \nabla_{x,v} = v_3 \partial_{x_3} + \frac{q}{m} E \cdot \nabla_v$. It is easily seen that

$$\bar{X}(s; \bar{x}, \bar{v}) = \bar{x} + \frac{\perp \bar{v}}{\omega_c} - \frac{\mathcal{R}(-\omega_c s)}{\omega_c} \perp \bar{v}, \quad X_3(s; x_3) = x_3, \quad \bar{V}(s; \bar{v}) = \mathcal{R}(-\omega_c s) \bar{v}, \quad V_3(s; v_3) = v_3$$

where we have used the notations $\bar{x} = (x_1, x_2)$, $\bar{v} = (v_1, v_2)$, $\perp \bar{v} = (v_2, -v_1)$ and $\mathcal{R}(\theta)$ stands for the rotation in \mathbb{R}^2 of angle $\theta \in \mathbb{R}$. The Jacobian matrix of the flow writes

$$\partial_{x,v} Y(s; x, v) = \begin{pmatrix} I_2 & 0_{2 \times 1} & \frac{I_2 - \mathcal{R}(-\omega_c s)}{\omega_c} \mathcal{E} & 0_{2 \times 1} \\ 0_{1 \times 2} & 1 & 0_{1 \times 2} & 0 \\ 0_{2 \times 2} & 0_{2 \times 1} & \mathcal{R}(-\omega_c s) & 0_{2 \times 1} \\ 0_{1 \times 2} & 0 & 0_{1 \times 2} & 1 \end{pmatrix}$$

where $0_{m \times n}$ stands for the null matrix with m lines and n columns, and $\mathcal{E} = \mathcal{R}(-\pi/2)$. We indicate the main lines for the asymptotic analysis of the Fokker-Planck equation (29). We appeal to the weak formulation of (29), written for the test function $(t, y) \rightarrow \varphi(t, Y(-t/\varepsilon; y))$, $y = (x, v)$, $Y = (X, V)$, $\varphi \in C_c^1(\mathbb{R}_+ \times \mathbb{R}^6)$. Denoting by f_{in} the initial density, we obtain

$$\begin{aligned} & - \int_{\mathbb{R}^6} f_{\text{in}}(y) \varphi(0, y) \, dy - \int_0^{+\infty} \int_{\mathbb{R}^6} f^\varepsilon(t, y) \left\{ \partial_t \varphi(t) - \frac{1}{\varepsilon} b \cdot (\nabla_z \varphi)(t) \right\} (Y(-t/\varepsilon; y)) \, dy dt \\ & - \int_0^{+\infty} \int_{\mathbb{R}^6} f^\varepsilon(t, y) c(t, y) \cdot {}^t \partial_y Y(-t/\varepsilon; y) (\nabla_z \varphi)(t, Y(-t/\varepsilon; y)) \, dy dt \\ & - \int_0^{+\infty} \int_{\mathbb{R}^6} f^\varepsilon(t, y) \frac{b(y)}{\varepsilon} \cdot {}^t \partial_y Y(-t/\varepsilon; y) (\nabla_z \varphi)(t, Y(-t/\varepsilon; y)) \, dy dt \\ & = -\nu \int_0^{+\infty} \int_{\mathbb{R}^6} (\Theta \nabla_v f^\varepsilon + v f^\varepsilon) \cdot {}^t \partial_v Y(-t/\varepsilon; y) (\nabla_z \varphi)(t, Y(-t/\varepsilon; y)) \, dy dt. \end{aligned}$$

We introduce the new densities $(g^\varepsilon)_\varepsilon$ given by

$$f^\varepsilon(t, y) = g^\varepsilon(t, Y(-t/\varepsilon; y)), \quad (t, y) \in \mathbb{R}_+ \times \mathbb{R}^6$$

and we use the identity

$$b(Y(-t/\varepsilon; y)) - \partial_y Y(-t/\varepsilon; y) b(y) = 0.$$

After performing the change of variables $z = Y(-t/\varepsilon; y)$, the above formulation reduces to

$$\begin{aligned}
& - \int_{\mathbb{R}^6} f_{\text{in}}(z) \varphi(0, z) \, dz - \int_0^{+\infty} \int_{\mathbb{R}^6} g^\varepsilon(t, z) \partial_t \varphi(t, z) \, dz dt \\
& \quad - \int_0^{+\infty} \int_{\mathbb{R}^m} g^\varepsilon \partial_y Y(-t/\varepsilon; Y(t/\varepsilon; z)) c(t, Y(t/\varepsilon; z)) \cdot \nabla_z \varphi(t, z) \, dz dt \\
& = -\nu \int_0^{+\infty} \int_{\mathbb{R}^6} \Theta \partial_v Y(-t/\varepsilon; Y(t/\varepsilon; z)) {}^t \partial_v Y(-t/\varepsilon; Y(t/\varepsilon; z)) \nabla_z g^\varepsilon \cdot \nabla_z \varphi \, dz dt \\
& \quad - \nu \int_0^{+\infty} \int_{\mathbb{R}^6} g^\varepsilon(t, z) \partial_v Y(-t/\varepsilon; Y(t/\varepsilon; z)) V(t/\varepsilon; z) g^\varepsilon(t, z) \cdot \nabla_z \varphi \, dz dt.
\end{aligned}$$

Motivated by Theorems 2.2, 2.3, we expect that $(g^\varepsilon)_\varepsilon$ converges in $L_{\text{loc}}^\infty(\mathbb{R}_+; L^2(\mathbb{R}^6))$ to the solution of the problem

$$\begin{cases} \partial_t g + \mathcal{C}(t, z) \cdot \nabla_z g = \nu \operatorname{div}_z \{ \Theta \mathcal{D}(z) \nabla_z g + \mathcal{V}(z) g \}, & (t, z) \in \mathbb{R}_+ \times \mathbb{R}^6 \\ g(0, z) = f_{\text{in}}(z), & z \in \mathbb{R}^6 \end{cases}$$

where the matrix field \mathcal{D} and the vector fields \mathcal{C}, \mathcal{V} are given by ergodic averages

$$\mathcal{D}(z) = \lim_{S \rightarrow +\infty} \frac{1}{S} \int_0^S \partial_v Y(-s; Y(s; z)) {}^t \partial_v Y(-s; Y(s; z)) \, ds$$

$$\mathcal{C}(t, z) = \lim_{S \rightarrow +\infty} \frac{1}{S} \int_0^S \partial_y Y(-s; Y(s; z)) c(t, Y(s; z)) \, ds \tag{30}$$

$$\begin{aligned} \mathcal{V}(z) &= \lim_{S \rightarrow +\infty} \frac{1}{S} \int_0^S \partial_v Y(-s; Y(s; z)) V(s; z) \, ds \\ &= \lim_{S \rightarrow +\infty} \frac{1}{S} \int_0^S \partial_y Y(-s; Y(s; z)) {}^t(0, V(s; z)) \, ds. \end{aligned} \tag{31}$$

Observe that \mathcal{D} is the average of the diffusion matrix D entering the Fokker-Planck equation (29) cf. Theorem 2.1

$$D = \sum_{i=1}^3 e_{v_i} \otimes e_{v_i} = \begin{pmatrix} 0_{3 \times 3} & 0_{3 \times 3} \\ 0_{3 \times 3} & I_3 \end{pmatrix}.$$

After direct computations we obtain (thanks to the periodicity of the flow)

$$\begin{aligned} \mathcal{D}(z) &= \lim_{S \rightarrow +\infty} \frac{1}{S} \int_0^S \partial_y Y(-s; Y(s; z)) D {}^t \partial_y Y(-s; Y(s; z)) \, ds \\ &= \frac{\omega_c}{2\pi} \int_0^{2\pi/\omega_c} \partial_y Y(-s; Y(s; z)) \begin{pmatrix} 0_{3 \times 3} & 0_{3 \times 3} \\ 0_{3 \times 3} & I_3 \end{pmatrix} {}^t \partial_y Y(-s; Y(s; z)) \, ds \\ &= \begin{pmatrix} \frac{2I_2}{\omega_c^2} & 0_{2 \times 1} & -\frac{\mathcal{E}}{\omega_c} & 0_{2 \times 1} \\ 0_{1 \times 2} & 0 & 0_{1 \times 2} & 0 \\ \frac{\mathcal{E}}{\omega_c} & 0_{2 \times 1} & I_2 & 0_{2 \times 1} \\ 0_{1 \times 2} & 0 & 0_{1 \times 2} & 1 \end{pmatrix}. \end{aligned} \tag{32}$$

Notice that the averaged Fokker-Planck kernel \mathcal{D} contains diffusion terms not only in velocity variables (as in initial the Fokker-Planck kernel D) but also in space variables (orthogonal to the magnetic lines). This was actually observed in gyrokinetic theory and numerical simulations [15, 16, 17, 27, 36].

It remains to clarify the existence of the ergodic average for vector fields. This comes from the fact that the family of maps

$$c \in X_Q \rightarrow \partial_y Y(-s; Y(s; \cdot))c(Y(s; \cdot)) \in X_Q, \quad s \in \mathbb{R}$$

forms a C^0 -group of unitary operators in X_Q , see also Proposition 3.1. Consequently, thanks to the von Neumann's theorem, we deduce that the vector fields \mathcal{C} and \mathcal{V} introduced in (30), (31) are well defined (see Theorem 2.1).

4 Well posedness for the perturbed problem and uniform estimates

For solving (1), we appeal to variational methods. We use the continuous embedding $H_R^1 \hookrightarrow L^2(\mathbb{R}^m)$, with dense image (since $C_c^1(\mathbb{R}^m) \subset H_R^1$).

Proposition 4.1 *Assume that b satisfies (4), (5), (19), that R and R^{-1} satisfy (20) and (22), and that D satisfies (25).*

1. For any $\varepsilon > 0$, the bilinear form $a^\varepsilon : H_R^1 \times H_R^1 \rightarrow \mathbb{R}$ defined by

$$a^\varepsilon(u, v) := \int_{\mathbb{R}^m} D(y) \nabla_y u \cdot \nabla_y v \, dy + \frac{1}{\varepsilon} \int_{\mathbb{R}^m} (b \cdot \nabla_y u) v(y) \, dy, \quad u, v \in H_R^1,$$

is well defined, continuous and coercive on H_R^1 with respect to $L^2(\mathbb{R}^m)$.

2. The bilinear form $\langle a \rangle : H_R^1 \times H_R^1 \rightarrow \mathbb{R}$ defined by

$$\langle a \rangle(u, v) := \int_{\mathbb{R}^m} \langle D \rangle(y) \nabla_y u \cdot \nabla_y v \, dy, \quad u, v \in H_R^1,$$

is well defined, continuous and coercive on H_R^1 with respect to $L^2(\mathbb{R}^m)$.

Proof.

1. For any $u, v \in H_R^1$ we have

$$\begin{aligned} |D(y) \nabla_y u \cdot \nabla_y v| &= |Q^{1/2}(y) D(y) Q^{1/2}(y) : (P^{1/2} \nabla_y v) \otimes (P^{1/2} \nabla_y u)| \\ &\leq |D|_{H_Q^\infty} |P^{1/2}(y) \nabla_y v| |P^{1/2}(y) \nabla_y u|, \quad y \in \mathbb{R}^m \end{aligned}$$

and

$$|b(y) \cdot \nabla_y u v(y)| = |Q^{1/2}(y) b(y) \cdot P^{1/2}(y) \nabla_y u v(y)| \leq |b|_{X_Q^\infty} |P^{1/2}(y) \nabla_y u| |v(y)|, \quad y \in \mathbb{R}^m.$$

Therefore, it is easily seen, thanks to (24), that

$$\begin{aligned} |a^\varepsilon(u, v)| &\leq |D|_{H_Q^\infty} |\nabla_y u|_P |\nabla_y v|_P + \frac{1}{\varepsilon} |b|_{X_Q^\infty} |\nabla_y u|_P \|v\|_{L^2(\mathbb{R}^m)} \\ &\leq \left(|D|_{H_Q^\infty} + \frac{1}{\varepsilon} |b|_{X_Q^\infty} \right) |u|_R |v|_R \end{aligned}$$

showing that the bilinear application $a^\varepsilon(\cdot, \cdot)$ is well defined and continuous. We inquire now about the coercivity of a^ε on H_R^1 , with respect to $L^2(\mathbb{R}^m)$. For any $u \in H_R^1$ we have, thanks

to the anti-symmetry of $b \cdot \nabla_y$

$$\begin{aligned}
a^\varepsilon(u, u) + \alpha \|u\|_{L^2(\mathbb{R}^m)}^2 &= \int_{\mathbb{R}^m} D(y) \nabla_y u \cdot \nabla_y u \, dy + \alpha \|u\|_{L^2(\mathbb{R}^m)}^2 \\
&= \int_{\mathbb{R}^m} Q^{1/2}(y) D(y) Q^{1/2}(y) : (P^{1/2}(y) \nabla_y u) \otimes (P^{1/2}(y) \nabla_y u) \, dy + \alpha \|u\|_{L^2(\mathbb{R}^m)}^2 \\
&\geq \alpha \int_{\mathbb{R}^m} |P^{1/2}(y) \nabla_y u|^2 \, dy + \alpha \|u\|_{L^2(\mathbb{R}^m)}^2 \\
&= \alpha \left(|\nabla_y u|_P^2 + \|u\|_{L^2(\mathbb{R}^m)}^2 \right) = \alpha |u|_R^2.
\end{aligned}$$

We emphasize the following inequality, which will be used several times in the sequel

$$D(y) \nabla_y u \cdot \nabla_y u \geq \alpha |\nabla_y^R u|^2, \quad u \in H_R^1. \quad (33)$$

2. We follow the same lines as before since Theorem 2.1 gives that

$$Q^{1/2}(y) \langle D \rangle (y) Q^{1/2}(y) \geq \alpha I_m, \quad y \in \mathbb{R}^m$$

and $|\langle D \rangle|_{H_Q^\infty} \leq |D|_{H_Q^\infty}$. □

Proposition 4.2 *Assume that b satisfies (4), (5), (19), that R and R^{-1} satisfy (20) and (22), and that D satisfies (25).*

There exists u^ε (resp. v) a unique variational solution of (1) (resp. (26)). Moreover, we have

$$\|u^\varepsilon\|_{L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^m))} \leq \|u^{\text{in}}\|_{L^2(\mathbb{R}^m)}, \quad \|\nabla_y u^\varepsilon\|_{L^2(\mathbb{R}_+; X_P)} \leq \frac{\|u^{\text{in}}\|_{L^2(\mathbb{R}^m)}}{\sqrt{2\alpha}}, \quad \varepsilon > 0$$

and

$$\|v\|_{L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^m))} \leq \|u^{\text{in}}\|_{L^2(\mathbb{R}^m)}, \quad \|\nabla_z v\|_{L^2(\mathbb{R}_+; X_P)} \leq \frac{\|u^{\text{in}}\|_{L^2(\mathbb{R}^m)}}{\sqrt{2\alpha}}.$$

Proof. By [20, Theorems 1 and 2, p. 513] (see also [31]) we deduce that for any $u^{\text{in}} \in L^2(\mathbb{R}^m)$, there is a unique variational solution u^ε for the problem (1), that is $u^\varepsilon \in C_b(\mathbb{R}_+; L^2(\mathbb{R}^m)) \cap L^2(\mathbb{R}_+; H_R^1)$, $\partial_t u^\varepsilon \in L^2(\mathbb{R}_+; (H_R^1)')$ and

$$u^\varepsilon(0) = u^{\text{in}}, \quad \frac{d}{dt} \int_{\mathbb{R}^m} u^\varepsilon(t, y) \varphi(y) \, dy + a^\varepsilon(u^\varepsilon(t), \varphi) = 0, \quad \text{in } \mathcal{D}'(\mathbb{R}_+), \quad \text{for any } \varphi \in H_R^1.$$

Similarly, there is a unique variational solution v for the limit model (26), that is $v \in C_b(\mathbb{R}_+; L^2(\mathbb{R}^m)) \cap L^2(\mathbb{R}_+; H_R^1)$, $\partial_t v \in L^2(\mathbb{R}_+; (H_R^1)')$ and

$$v(0) = u^{\text{in}}, \quad \frac{d}{dt} \int_{\mathbb{R}^m} v(t, z) \psi(z) \, dy + \langle a \rangle (v(t), \psi) = 0, \quad \text{in } \mathcal{D}'(\mathbb{R}_+), \quad \text{for any } \psi \in H_R^1.$$

The above estimates come immediately by the energy balance

$$\frac{1}{2} \frac{d}{dt} \|u^\varepsilon(t)\|_{L^2(\mathbb{R}^m)}^2 + a^\varepsilon(u^\varepsilon(t), u^\varepsilon(t)) = 0, \quad \text{in } \mathcal{D}'(\mathbb{R}_+)$$

which implies

$$\frac{1}{2} \|u^\varepsilon(t)\|_{L^2(\mathbb{R}^m)}^2 + \int_0^t a^\varepsilon(u^\varepsilon(\tau), u^\varepsilon(\tau)) \, d\tau = \frac{1}{2} \|u^{\text{in}}\|_{L^2(\mathbb{R}^m)}^2, \quad t \in \mathbb{R}_+.$$

In particular we deduce $\|u^\varepsilon(t)\|_{L^2(\mathbb{R}^m)} \leq \|u^{\text{in}}\|_{L^2(\mathbb{R}^m)}$, for any $t \in \mathbb{R}_+, \varepsilon > 0$, and

$$2\alpha \int_0^t |\nabla_y u^\varepsilon(\tau)|_P^2 \, d\tau \leq 2 \int_0^t a^\varepsilon(u^\varepsilon(\tau), u^\varepsilon(\tau)) \, d\tau \leq \|u^{\text{in}}\|_{L^2(\mathbb{R}^m)}^2, \quad t \in \mathbb{R}_+, \quad \varepsilon > 0$$

saying that $\|\nabla_y u^\varepsilon\|_{L^2(\mathbb{R}_+; X_P)} \leq \frac{\|u^{\text{in}}\|_{L^2(\mathbb{R}^m)}}{\sqrt{2\alpha}}$, for any $\varepsilon > 0$. The estimates for v follow similarly. □

Remark 4.1 The family $(v^\varepsilon(t, \cdot))_{\varepsilon>0} = (u^\varepsilon(t, Y(t/\varepsilon; \cdot)))_{\varepsilon>0}$ satisfies the same estimates as the family $(u^\varepsilon)_{\varepsilon>0}$.

Indeed, performing the change of variable $z \mapsto y = Y(t/\varepsilon; z)$, which is measure preserving, one gets

$$\|v^\varepsilon(t)\|_{L^2(\mathbb{R}^m)}^2 = \|u^\varepsilon(t)\|_{L^2(\mathbb{R}^m)}^2 \leq \|u^{\text{in}}\|_{L^2(\mathbb{R}^m)}^2, \quad t \in \mathbb{R}_+, \quad \varepsilon > 0.$$

Moreover, thanks to (21), we have

$$\begin{aligned} \nabla_z^R v^\varepsilon(t, z) &= {}^t R^{-1}(z) \nabla_z v^\varepsilon(t, z) = {}^t R^{-1}(z) {}^t \partial Y(t/\varepsilon; z) \nabla_y u^\varepsilon(t, Y(t/\varepsilon; z)) \\ &= {}^t (\partial Y(t/\varepsilon; z) R^{-1}(z)) \nabla_y u^\varepsilon(t, Y(t/\varepsilon; z)) \\ &= {}^t R^{-1}(z) (Y(t/\varepsilon; z)) \nabla_y u^\varepsilon(t, Y(t/\varepsilon; z)) = (\nabla_y^R u^\varepsilon)(t, Y(t/\varepsilon; z)) \end{aligned}$$

and therefore

$$\begin{aligned} \|\nabla_z v^\varepsilon\|_{L^2(\mathbb{R}_+; X_P)}^2 &= \int_0^{+\infty} |\nabla_z v^\varepsilon(\tau)|_P^2 d\tau = \int_0^{+\infty} \|\nabla_z^R v^\varepsilon(\tau)\|_{L^2(\mathbb{R}^m)}^2 d\tau \\ &= \int_0^{+\infty} \|\nabla_y^R u^\varepsilon(\tau)\|_{L^2(\mathbb{R}^m)}^2 d\tau = \int_0^{+\infty} |\nabla_y u^\varepsilon(\tau)|_P^2 d\tau = \|\nabla_y u^\varepsilon\|_{L^2(\mathbb{R}_+; X_P)}^2 \\ &\leq \frac{\|u^{\text{in}}\|_{L^2(\mathbb{R}^m)}^2}{2\alpha}. \end{aligned}$$

Using twice the formula

$$b_i \cdot \nabla_z v^\varepsilon(t) = b_i \cdot \nabla_z (u^\varepsilon(t) \circ Y(t/\varepsilon; \cdot)) = (b_i \cdot \nabla_y u^\varepsilon(t)) \circ Y(t/\varepsilon; \cdot)$$

we deduce that

$$b_j \cdot \nabla_z (b_i \cdot \nabla_z v^\varepsilon(t)) = b_j \cdot \nabla_z [(b_i \cdot \nabla_y u^\varepsilon(t)) \circ Y(t/\varepsilon; \cdot)] = [b_j \cdot \nabla_y (b_i \cdot \nabla_y u^\varepsilon(t))] \circ Y(t/\varepsilon; \cdot).$$

Therefore $\|b_j \cdot \nabla_z (b_i \cdot \nabla_z v^\varepsilon(t))\|_{L^2(\mathbb{R}^m)} = \|b_j \cdot \nabla_y (b_i \cdot \nabla_y u^\varepsilon(t))\|_{L^2(\mathbb{R}^m)}$, $i, j \in \{1, \dots, m\}$ as soon as those norms are finite.

Up to now, we have considered solutions with initial condition $u^{\text{in}} \in L^2(\mathbb{R}^m)$. In order to study the stability of the family $(v^\varepsilon)_{\varepsilon>0}$ when ε goes to 0, we need more regularity. This will be the object of the next propositions, in which we analyze how the regularity of the initial condition propagates in time. The idea is to take the directional derivative $b_i \cdot \nabla_y$ of (1), leading to

$$\partial_t (b_i \cdot \nabla_y u^\varepsilon) - \text{div}_y (D(y) \nabla_y (b_i \cdot \nabla_y u^\varepsilon)) + \frac{1}{\varepsilon} b \cdot \nabla_y (b_i \cdot \nabla_y u^\varepsilon) = [b_i \cdot \nabla_y, \text{div}_y (D \nabla_y)] u^\varepsilon. \quad (34)$$

Notice that the key point was to take advantage of the involution between b_i and b , for any $i \in \{1, \dots, m\}$, which guarantees that there is no commutator between the first order operators $b_i \cdot \nabla_y$ and $b \cdot \nabla_y$. More generally, if we apply the directional derivative $c \cdot \nabla_y$ in (1), the right hand side of the corresponding equation in (34) will contain the extra term $\frac{1}{\varepsilon} [b \cdot \nabla_y, c \cdot \nabla_y] u^\varepsilon$, which is clearly unstable, when ε goes to 0, if b and c are not in involution. The estimate for $b_i \cdot \nabla_y u^\varepsilon$ follows by using the energy balance of (34), observing that, thanks to the anti-symmetry of $b \cdot \nabla_y$, we get rid of the term of order $1/\varepsilon$. We assume that for any $i, j \in \{1, \dots, m\}$, the coordinates of the Poisson bracket $[b_i, b_j]$ in the basis $(b_k)_{1 \leq k \leq m}$ are bounded

$$[b_i, b_j] = \sum_{k=1}^m \alpha_{ij}^k b_k, \quad \alpha_{ij}^k \in L^\infty(\mathbb{R}^m), \quad i, j, k \in \{1, \dots, m\}. \quad (35)$$

Proposition 4.3 *Assume that the hypotheses (4), (5), (22), (25), (35) hold true. Moreover we assume that for any $i, j \in \{1, \dots, m\}$*

$$b_i \cdot \nabla_y \operatorname{div}_y b_j \in L^\infty(\mathbb{R}^m), \quad \operatorname{div}_y(RD) \in L^\infty(\mathbb{R}^m)$$

$$R[b_i, D] {}^t R \in L^\infty(\mathbb{R}^m), \quad \sum_{i=1}^m b_i \cdot \nabla_y(R[b_i, D] {}^t R) \in L^\infty(\mathbb{R}^m).$$

If the initial condition belongs to H_R^1 , then we have for any $T \in \mathbb{R}_+$

$$\begin{aligned} \sup_{\varepsilon > 0} \|\nabla_y^R u^\varepsilon\|_{L^\infty([0, T]; L^2(\mathbb{R}^m))} &= \sup_{\varepsilon > 0} \|\nabla_z^R v^\varepsilon\|_{L^\infty([0, T]; L^2(\mathbb{R}^m))} < +\infty \\ \sup_{\varepsilon > 0} \|\nabla_y^R \otimes \nabla_y^R u^\varepsilon\|_{L^2([0, T]; L^2(\mathbb{R}^m))} &= \sup_{\varepsilon > 0} \|\nabla_z^R \otimes \nabla_z^R v^\varepsilon\|_{L^2([0, T]; L^2(\mathbb{R}^m))} < +\infty \\ \sup_{\varepsilon > 0} \|\partial_t v^\varepsilon\|_{L^2([0, T]; L^2(\mathbb{R}^m))} &< +\infty. \end{aligned}$$

Proof. We want to estimate the L^2 norms of $b_i \cdot \nabla_y u^\varepsilon$, $i \in \{1, \dots, m\}$, $\varepsilon > 0$. This can be done by analyzing the translations along the flows Y_i and estimating the L^2 norms of $(u^\varepsilon(t, Y_i(h; y)) - u^\varepsilon(t, y))/h$ uniformly with respect to $h \in \mathbb{R}^*$ and $\varepsilon > 0$. For simplicity, we justify the estimates only for smooth solutions and coefficients (and therefore we use classical derivatives).

We use (17) to express the commutator in the right-hand side of (34), then we multiply this equation by $b_i \cdot \nabla_y u^\varepsilon$. Integrating with respect to y over \mathbb{R}^m and observing that the contribution of the singular term $\frac{1}{\varepsilon} b \cdot \nabla_y(b_i \cdot \nabla_y u^\varepsilon)$ cancels by the anti-symmetry of the operator $b \cdot \nabla_y$, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^m} (b_i \cdot \nabla_y u^\varepsilon(t))^2 dy &+ \int_{\mathbb{R}^m} D(y) \nabla_y (b_i \cdot \nabla_y u^\varepsilon(t)) \cdot \nabla_y (b_i \cdot \nabla_y u^\varepsilon(t)) dy \\ &= - \int_{\mathbb{R}^m} [b_i, D] \nabla_y u^\varepsilon(t) \cdot \nabla_y (b_i \cdot \nabla_y u^\varepsilon(t)) dy \\ &+ \int_{\mathbb{R}^m} D(y) \nabla_y u^\varepsilon(t) \cdot \nabla_y \operatorname{div}_y b_i (b_i \cdot \nabla_y u^\varepsilon(t)) dy. \end{aligned} \quad (36)$$

By hypothesis (25) we have

$$\begin{aligned} D \nabla_y (b_i \cdot \nabla_y u^\varepsilon(t)) \cdot \nabla_y (b_i \cdot \nabla_y u^\varepsilon(t)) &= Q^{1/2} D Q^{1/2} \\ &: P^{1/2} \nabla_y (b_i \cdot \nabla_y u^\varepsilon(t)) \otimes P^{1/2} \nabla_y (b_i \cdot \nabla_y u^\varepsilon(t)) \\ &\geq \alpha |P^{1/2} \nabla_y (b_i \cdot \nabla_y u^\varepsilon(t))|^2 = \alpha |\nabla_y^R (b_i \cdot \nabla_y u^\varepsilon(t))|^2 \\ &= \alpha \sum_{j=1}^m (b_j \cdot \nabla_y (b_i \cdot \nabla_y u^\varepsilon(t)))^2. \end{aligned} \quad (37)$$

Combining (36), (37) leads to

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla_y^R u^\varepsilon(t)\|_{L^2(\mathbb{R}^m)}^2 &+ \alpha \|\nabla_y^R \otimes \nabla_y^R u^\varepsilon(t)\|_{L^2(\mathbb{R}^m)}^2 \\ &\leq - \sum_{i=1}^m \underbrace{\int_{\mathbb{R}^m} [b_i, D] \nabla_y u^\varepsilon(t) \cdot \nabla_y (b_i \cdot \nabla_y u^\varepsilon(t)) dy}_{:= I_i^1} \\ &+ \sum_{i=1}^m \underbrace{\int_{\mathbb{R}^m} D \nabla_y u^\varepsilon(t) \cdot \nabla_y (\operatorname{div}_y b_i) (b_i \cdot \nabla_y u^\varepsilon(t)) dy}_{:= I_i^2}. \end{aligned} \quad (38)$$

In order to bound the term I_i^1 in the right hand side of (38) we write

$$\begin{aligned} I_i^1 &= \int_{\mathbb{R}^m} R[b_i, D]^t R : {}^t R^{-1} \nabla_y u^\varepsilon(t) \otimes {}^t R^{-1} \nabla_y (b_i \cdot \nabla_y u^\varepsilon(t)) \, dy \\ &= \int_{\mathbb{R}^m} R[b_i, D]^t R : \nabla_y^R u^\varepsilon(t) \otimes \nabla_y^R (b_i \cdot \nabla_y u^\varepsilon(t)) \, dy. \end{aligned} \quad (39)$$

Notice that for any $j \in \{1, \dots, m\}$ we have

$$\begin{aligned} b_j \cdot \nabla_y (b_i \cdot \nabla_y u^\varepsilon(t)) &= b_i \cdot \nabla_y (b_j \cdot \nabla_y u^\varepsilon(t)) - [b_i, b_j] \cdot \nabla_y u^\varepsilon(t) \\ &= b_i \cdot \nabla_y (b_j \cdot \nabla_y u^\varepsilon(t)) - \sum_{k=1}^m \alpha_{ij}^k b_k \cdot \nabla_y u^\varepsilon(t) \end{aligned}$$

and therefore we obtain

$$\nabla_y^R (b_i \cdot \nabla_y u^\varepsilon(t)) = b_i \cdot \nabla_y (\nabla_y^R u^\varepsilon(t)) - \mathcal{A}_i \nabla_y^R u^\varepsilon(t) \quad (40)$$

where the matrix \mathcal{A}_i is the one whose entry (j, k) is α_{ij}^k . Combining (39), (40) yields

$$\begin{aligned} I_i^1 &= \int_{\mathbb{R}^m} R[b_i, D]^t R : \nabla_y^R u^\varepsilon(t) \otimes b_i \cdot \nabla_y (\nabla_y^R u^\varepsilon(t)) \, dy \\ &\quad - \int_{\mathbb{R}^m} R[b_i, D]^t R : \nabla_y^R u^\varepsilon(t) \otimes \mathcal{A}_i \nabla_y^R u^\varepsilon(t) \, dy \\ &=: J_i^1 + J_i^2. \end{aligned}$$

To estimate the term J_i^1 , we use the symmetry of the matrix field D and the formula

$$\begin{aligned} b_i \cdot \nabla_y (R[b_i, D]^t R : \nabla_y^R u^\varepsilon(t) \otimes \nabla_y^R u^\varepsilon(t)) &= b_i \cdot \nabla_y (R[b_i, D]^t R) : \nabla_y^R u^\varepsilon(t) \otimes \nabla_y^R u^\varepsilon(t) \\ &\quad + 2R[b_i, D]^t R : \nabla_y^R u^\varepsilon(t) \otimes b_i \cdot \nabla_y (\nabla_y^R u^\varepsilon(t)). \end{aligned}$$

Integrating by parts leads to

$$\begin{aligned} 2 \left| \sum_{i=1}^m J_i^1 \right| &= \left| - \sum_{i=1}^m \int_{\mathbb{R}^m} \operatorname{div}_y b_i R[b_i, D]^t R : \nabla_y^R u^\varepsilon(t) \otimes \nabla_y^R u^\varepsilon(t) \, dy \right. \\ &\quad \left. - \int_{\mathbb{R}^m} \sum_{i=1}^m b_i \cdot \nabla_y (R[b_i, D]^t R) : \nabla_y^R u^\varepsilon(t) \otimes \nabla_y^R u^\varepsilon(t) \, dy \right| \\ &\leq \left(\sum_{i=1}^m \|\operatorname{div}_y b_i\|_{L^\infty} \|R[b_i, D]^t R\|_{L^\infty} + \left\| \sum_{i=1}^m b_i \cdot \nabla_y (R[b_i, D]^t R) \right\|_{L^\infty} \right) \\ &\quad \times \|\nabla_y^R u^\varepsilon(t)\|_{L^2(\mathbb{R}^m)}^2. \end{aligned}$$

The estimate for the term J_i^2 follows immediately, thanks to the hypothesis (35)

$$|J_i^2| \leq \|R[b_i, D]^t R\|_{L^\infty(\mathbb{R}^m)} \|\mathcal{A}_i\|_{L^\infty(\mathbb{R}^m)} \|\nabla_y^R u^\varepsilon(t)\|_{L^2(\mathbb{R}^m)}^2$$

and finally there is a constant C_1 depending on $\max_{1 \leq i \leq m} \|\operatorname{div}_y b_i\|_{L^\infty}$, $\max_{1 \leq i \leq m} \|\mathcal{A}_i\|_{L^\infty}$, $\max_{1 \leq i \leq m} \|R[b_i, D]^t R\|_{L^\infty}$, $\|\sum_{i=1}^m b_i \cdot \nabla_y (R[b_i, D]^t R)\|_{L^\infty}$ such that

$$\left| \sum_{i=1}^m I_i^1 \right| \leq C_1 \|\nabla_y^R u^\varepsilon(t)\|_{L^2(\mathbb{R}^m)}^2, \quad t \in \mathbb{R}_+, \quad \varepsilon > 0. \quad (41)$$

For the term I_i^2 we can write, using the inequality $(R(y)D(y)^t R(y) : R(y)D(y)^t R(y))^{1/2} \leq |D|_{H_Q^\infty}, y \in \mathbb{R}^m$

$$\begin{aligned} |I_i^2| &= \left| \int_{\mathbb{R}^m} RD^t R^t R^{-1} \nabla_y u^\varepsilon(t) \cdot {}^t R^{-1} \nabla_y (\operatorname{div}_y b_i) (b_i \cdot \nabla_y u^\varepsilon(t)) \, dy \right| \\ &\leq |D|_{H_Q^\infty} \|\nabla_y^R \operatorname{div}_y b_i\|_{L^\infty(\mathbb{R}^m)} \int_{\mathbb{R}^m} |\nabla_y^R u^\varepsilon(t)| |b_i \cdot \nabla_y u^\varepsilon(t)| \, dy \\ &\leq |D|_{H_Q^\infty} \|\nabla_y^R \operatorname{div}_y b_i\|_{L^\infty(\mathbb{R}^m)} \|\nabla_y^R u^\varepsilon(t)\|_{L^2(\mathbb{R}^m)}^2 \end{aligned}$$

implying that there is a constant C_2 depending on $|D|_{H_Q^\infty}, \max_{1 \leq i \leq m} \|\nabla_y^R \operatorname{div}_y b_i\|_{L^\infty(\mathbb{R}^m)}$ such that

$$\left| \sum_{i=1}^m I_i^2 \right| \leq C_2 \|\nabla_y^R u^\varepsilon(t)\|_{L^2(\mathbb{R}^m)}^2, \quad t \in \mathbb{R}_+, \quad \varepsilon > 0. \quad (42)$$

Combining (38), (41), (42) and applying Gronwall's lemma imply

$$\|\nabla_y^R u^\varepsilon(t)\|_{L^2(\mathbb{R}^m)} \leq e^{(C_1+C_2)t} \|\nabla_y^R u^{\text{in}}\|_{L^2(\mathbb{R}^m)}, \quad t \in \mathbb{R}_+, \quad \varepsilon > 0$$

and

$$\|\nabla_y^R \otimes \nabla_y^R u^\varepsilon\|_{L^2([0,T];L^2(\mathbb{R}^m))} \leq \frac{e^{(C_1+C_2)t}}{\sqrt{2\alpha}} \|\nabla_y^R u^{\text{in}}\|_{L^2(\mathbb{R}^m)}, \quad \varepsilon > 0.$$

The first and second conclusions follow thanks to Remark 4.1. For the last one, notice that

$$\begin{aligned} \partial_t v^\varepsilon(t, z) &= \partial_t u^\varepsilon(t, Y(t/\varepsilon; z)) + \frac{1}{\varepsilon} b(Y(t/\varepsilon; z)) \cdot \nabla_y u^\varepsilon(t, Y(t/\varepsilon; z)) \\ &= \operatorname{div}_y (D \nabla_y u^\varepsilon(t))(Y(t/\varepsilon; z)) \end{aligned} \quad (43)$$

which implies

$$\|\partial_t v^\varepsilon(t)\|_{L^2(\mathbb{R}^m)} = \|\operatorname{div}_y (D \nabla_y u^\varepsilon(t))\|_{L^2(\mathbb{R}^m)}.$$

By direct computation we obtain

$$\begin{aligned} \operatorname{div}_y (D \nabla_y u^\varepsilon) &= \operatorname{div}_y (D^t R \nabla_y^R u^\varepsilon) = \operatorname{div}_y (RD) \cdot \nabla_y^R u^\varepsilon + RD^t R : \partial_y (\nabla_y^R u^\varepsilon) R^{-1} \\ &= \operatorname{div}_y (RD) \cdot \nabla_y^R u^\varepsilon + RD^t R : \nabla_y^R \otimes \nabla_y^R u^\varepsilon \end{aligned}$$

and therefore

$$\begin{aligned} \sup_{\varepsilon > 0} \|\partial_t v^\varepsilon\|_{L^2([0,T];L^2(\mathbb{R}^m))} &= \sup_{\varepsilon > 0} \|\operatorname{div}_y (D \nabla_y u^\varepsilon)\|_{L^2([0,T];L^2(\mathbb{R}^m))} \\ &\leq \sqrt{T} \|\operatorname{div}_y (RD)\|_{L^\infty(\mathbb{R}^m)} \sup_{\varepsilon > 0} \|\nabla_y^R u^\varepsilon\|_{L^\infty([0,T];L^2(\mathbb{R}^m))} \\ &\quad + |D|_{H_Q^\infty} \sup_{\varepsilon > 0} \|\nabla_y^R \otimes \nabla_y^R u^\varepsilon\|_{L^2([0,T];L^2(\mathbb{R}^m))}, \quad T \in \mathbb{R}_+. \end{aligned}$$

□

Performing similar computations, we can propagate more regularity. The goal is to obtain a uniform bound for $(\partial_t \nabla_z^R v^\varepsilon)_{\varepsilon > 0}$ in $L_{\text{loc}}^2(\mathbb{R}_+; L^2(\mathbb{R}^m))$ which will be necessary for applying Proposition 5.2. This can be achieved for any initial condition $u^{\text{in}} \in H_R^2$. The proof is postponed to Appendix A.

Proposition 4.4 *Assume that the hypotheses (4), (5), (22), (25), (35) hold true. Moreover we assume that for any $i, j, k \in \{1, \dots, m\}$*

$$\nabla_y^R \alpha_{ij}^k \in L^\infty(\mathbb{R}^m), \quad \nabla_y^R \operatorname{div}_y b_j \in L^\infty(\mathbb{R}^m), \quad \nabla_y^R \otimes \nabla_y^R \operatorname{div}_y b_j \in L^\infty(\mathbb{R}^m),$$

$$\operatorname{div}_y(R[b_i, D]) \in L^\infty(\mathbb{R}^m), \quad R[b_i, D]^t R \in L^\infty(\mathbb{R}^m), \quad \sum_{i=1}^m b_i \cdot \nabla_y(R[b_i, D]^t R) \in L^\infty(\mathbb{R}^m)$$

$$R[b_j, [b_i, D]]^t R \in L^\infty(\mathbb{R}^m), \quad \operatorname{div}_y(R[b_j, [b_i, D]]) \in L^\infty(\mathbb{R}^m) \\ \nabla_y^R(RD^t R) \in L^\infty(\mathbb{R}^m), \quad \nabla_y^R \otimes \nabla_y^R(RD^t R) \in L^\infty(\mathbb{R}^m).$$

If the initial condition u^{in} belongs to H_R^2 , then for any $T \in \mathbb{R}_+$ we have

$$\sup_{\varepsilon > 0} \|\nabla_y^R \otimes \nabla_y^R u^\varepsilon\|_{L^\infty([0, T]; L^2(\mathbb{R}^m))} = \sup_{\varepsilon > 0} \|\nabla_z^R \otimes \nabla_z^R v^\varepsilon\|_{L^\infty([0, T]; L^2(\mathbb{R}^m))} < +\infty$$

$$\sup_{\varepsilon > 0} \|\nabla_y^R \otimes \nabla_y^R \otimes \nabla_y^R u^\varepsilon\|_{L^2([0, T]; L^2(\mathbb{R}^m))} = \sup_{\varepsilon > 0} \|\nabla_z^R \otimes \nabla_z^R \otimes \nabla_z^R v^\varepsilon\|_{L^2([0, T]; L^2(\mathbb{R}^m))} < +\infty$$

and

$$\sup_{\varepsilon > 0} \|\partial_t \nabla_z^R v^\varepsilon\|_{L^2([0, T]; L^2(\mathbb{R}^m))} < +\infty.$$

Here the notation $\nabla^R \otimes \nabla^R \otimes \nabla^R w$ stands for the tensor whose entry (i, j, k) is $b_k \cdot \nabla(b_j \cdot \nabla(b_i \cdot \nabla w))$.

Similar computations allow us to estimate the solution of the limit model (26). The arguments are a little bit tedious and we refer also to Appendix A for details.

Proposition 4.5 *Assume that all the hypotheses of Proposition 4.3 hold true, together with (19). Then we have for any $T \in \mathbb{R}_+$*

$$\nabla_z^R v \in L^\infty([0, T]; L^2(\mathbb{R}^m)), \quad \nabla_z^R \otimes \nabla_z^R v \in L^2([0, T]; L^2(\mathbb{R}^m)), \quad \partial_t v \in L^2([0, T]; L^2(\mathbb{R}^m)).$$

Proposition 4.6 *Assume that all the hypotheses of Proposition 4.4 hold true, together with (19). Then for any $T \in \mathbb{R}_+$, we have*

$$\nabla_z^R \otimes \nabla_z^R v \in L^\infty([0, T]; L^2), \quad \nabla_z^R \otimes \nabla_z^R \otimes \nabla_z^R v \in L^2([0, T]; L^2), \quad \partial_t \nabla_z^R v \in L^2([0, T]; L^2).$$

Proof. Apply exactly the same arguments as in the proof of Proposition 4.4, after observing that the matrix field $\langle D \rangle$ satisfies the same hypotheses as the matrix field D (see the proof of Proposition 4.5). \square

5 Two-scale analysis

We intend to investigate the asymptotic behavior of (1), or equivalently (9). For any smooth, compactly supported function $\psi(t, z)$ we have to pass to the limit, when $\varepsilon \searrow 0$, in the formulation

$$- \int_{\mathbb{R}^m} u^{\text{in}}(z) \psi(0, z) \, dz - \int_0^{+\infty} \int_{\mathbb{R}^m} v^\varepsilon(t, z) \partial_t \psi \, dz dt + \int_0^{+\infty} \int_{\mathbb{R}^m} G(t/\varepsilon) D \nabla_z v^\varepsilon \cdot \nabla_z \psi \, dz dt = 0.$$

Clearly, the main difficulty comes from the last integral, which presents two time scales : a slow time variable t and also a fast time variable $s = t/\varepsilon$ (not necessarily periodic). We detail here a general two-scale convergence result, based on ergodic averages. We use the notations introduced in Definition 2.1 and Proposition 2.2.

Proposition 5.1 *Let T be a positive real number. Consider $C \in L^\infty(\mathbb{R}; H_Q)$, such that the family of averages $\left(\frac{1}{S} \int_{s_0}^{s_0+S} C(s) ds\right)_{S>0}$ converges strongly in H_Q toward some $\bar{C} \in H_Q$, uniformly with respect to $s_0 \in \mathbb{R}$, when $S \rightarrow +\infty$. Let $\mathcal{B}_\omega \subset C([0, T]; H_P)$ be a bounded set in $L^1([0, T]; H_P)$, of functions which admit as modulus of continuity in $C([0, T]; H_P)$ the same function $\omega : [0, T] \rightarrow \mathbb{R}_+$ i.e.,*

$$|B(t) - B(t')|_P \leq \omega(|t - t'|), \quad t, t' \in [0, T], \quad B \in \mathcal{B}_\omega$$

with ω non decreasing and $\lim_{\lambda \searrow 0} \omega(\lambda) = 0$. Then

$$\lim_{\varepsilon \searrow 0} \int_0^T \langle B(t), C(t/\varepsilon) \rangle_{P,Q} dt = \int_0^T \langle B(t), \bar{C} \rangle_{P,Q} dt$$

uniformly with respect to $B \in \mathcal{B}_\omega$.

Proof. For any $\delta > 0$, there is $S_\delta > 0$ such that

$$\left| \frac{1}{S} \int_{s_0}^{s_0+S} C(s) ds - \bar{C} \right|_Q < \delta, \quad \text{for any } S \geq S_\delta \text{ and } s_0 \in \mathbb{R}.$$

Performing the change of variable $s = \frac{t}{\varepsilon}$ in the above integral, leads to

$$\left| \frac{1}{T} \int_{t_0}^{t_0+T} C(t/\varepsilon) dt - \bar{C} \right|_Q < \delta, \quad \text{for any } T \geq \varepsilon S_\delta = T_{\delta, \varepsilon} \text{ and } t_0 \in \mathbb{R}. \quad (44)$$

We split the interval $[0, T[$ into a finite number of intervals of size greater or equal than $T_{\delta, \varepsilon}$. For example let $k_{\delta, \varepsilon}$ be $\left\lceil \frac{T}{T_{\delta, \varepsilon}} \right\rceil$. If $T/T_{\delta, \varepsilon}$ is an integer, that is $T/T_{\delta, \varepsilon} = k_{\delta, \varepsilon}$, we consider the intervals

$$[kT_{\delta, \varepsilon}, (k+1)T_{\delta, \varepsilon}[, \quad 0 \leq k \leq k_{\delta, \varepsilon} - 1$$

and if $T/T_{\delta, \varepsilon}$ is not an integer, we take the intervals

$$[kT_{\delta, \varepsilon}, (k+1)T_{\delta, \varepsilon}[, \quad 0 \leq k \leq k_{\delta, \varepsilon} - 2, \quad \text{and } [(k_{\delta, \varepsilon} - 1)T_{\delta, \varepsilon}, T[.$$

Notice that in both cases we have $k_{\delta, \varepsilon}$ intervals, whose sizes are between $T_{\delta, \varepsilon}$ and $2T_{\delta, \varepsilon}$. We denote by $(t_{k, \delta, \varepsilon})_{0 \leq k \leq k_{\delta, \varepsilon}}$, or simply $(t_k)_{0 \leq k \leq k_{\delta, \varepsilon}}$, the end points of these intervals. The last point is always $t_{k_{\delta, \varepsilon}} = T$. Therefore we can write for any $B \in \mathcal{B}_\omega$

$$\begin{aligned} \left| \int_0^T \langle B(t), C(t/\varepsilon) \rangle_{P,Q} dt - \int_0^T \langle B(t), \bar{C} \rangle_{P,Q} dt \right| &= \left| \int_0^T \langle B(t), C(t/\varepsilon) - \bar{C} \rangle_{P,Q} dt \right| \\ &\leq \sum_{k=0}^{k_{\delta, \varepsilon} - 1} \left| \int_{t_k}^{t_{k+1}} \langle B(t), C(t/\varepsilon) - \bar{C} \rangle_{P,Q} dt \right| \\ &\leq \sum_{k=0}^{k_{\delta, \varepsilon} - 1} \left| \int_{t_k}^{t_{k+1}} \langle B(t) - B(t_k), C(t/\varepsilon) - \bar{C} \rangle_{P,Q} dt \right| \\ &\quad + \sum_{k=0}^{k_{\delta, \varepsilon} - 1} \left| \int_{t_k}^{t_{k+1}} \langle B(t_k), C(t/\varepsilon) - \bar{C} \rangle_{P,Q} dt \right| \\ &=: \Sigma_1 + \Sigma_2. \end{aligned} \quad (45)$$

Since the function $t \in [0, T] \rightarrow B(t) \in H_P$ admits ω as modulus of continuity, we obtain the following estimate for Σ_1

$$\begin{aligned}\Sigma_1 &\leq \sum_{k=0}^{k_{\delta, \varepsilon} - 1} \int_{t_k}^{t_{k+1}} \omega(|t - t_k|) |C(t/\varepsilon) - \bar{C}|_Q dt \\ &\leq \sum_{k=0}^{k_{\delta, \varepsilon} - 1} \omega(2T_{\delta, \varepsilon})(t_{k+1} - t_k) 2\|C\|_{L^\infty(\mathbb{R}; H_Q)} \\ &= 2\|C\|_{L^\infty(\mathbb{R}; H_Q)} \omega(2T_{\delta, \varepsilon})T.\end{aligned}\tag{46}$$

The estimate for Σ_2 comes by using (44)

$$\begin{aligned}\Sigma_2 &= \sum_{k=0}^{k_{\delta, \varepsilon} - 1} \left| \int_{t_k}^{t_{k+1}} (PB(t_k)P, C(t/\varepsilon) - \bar{C})_Q dt \right| \\ &= \sum_{k=0}^{k_{\delta, \varepsilon} - 1} \left| \left(PB(t_k)P, \int_{t_k}^{t_{k+1}} (C(t/\varepsilon) - \bar{C}) dt \right)_Q \right| \\ &= \sum_{k=0}^{k_{\delta, \varepsilon} - 1} \left| \left\langle B(t_k), \int_{t_k}^{t_{k+1}} (C(t/\varepsilon) - \bar{C}) dt \right\rangle_{P, Q} \right| \\ &\leq \sum_{k=0}^{k_{\delta, \varepsilon} - 1} \delta(t_{k+1} - t_k) |B(t_k)|_P \\ &\leq \delta [\|B\|_{L^1([0, T]; H_P)} + \omega(2T_{\delta, \varepsilon})T].\end{aligned}\tag{47}$$

Thanks to (45), (46), (47) we deduce

$$\begin{aligned}\left| \int_0^T \langle B(t), C(t/\varepsilon) \rangle_{P, Q} dt - \int_0^T \langle B(t), \bar{C} \rangle_{P, Q} dt \right| &\leq 2\|C\|_{L^\infty(\mathbb{R}; H_Q)} \omega(2T_{\delta, \varepsilon})T \\ &\quad + \delta [\|B\|_{L^1([0, T]; H_P)} + \omega(2T_{\delta, \varepsilon})T].\end{aligned}$$

Let η be a positive real number and $\delta > 0$ small enough such that $\delta\|B\|_{L^1([0, T]; H_P)} < \eta/2$ uniformly with respect to $B \in \mathcal{B}_\omega$ (which is possible since \mathcal{B}_ω is bounded in $L^1([0, T]; H_P)$). Observing that $\lim_{\varepsilon \searrow 0} T_{\delta, \varepsilon} = \lim_{\varepsilon \searrow 0} \varepsilon S_\delta = 0$, and $\lim_{\varepsilon \searrow 0} \omega(2T_{\delta, \varepsilon}) = 0$, we deduce that there is $\varepsilon = \varepsilon(\eta)$ such that for any $0 < \varepsilon < \varepsilon(\eta)$

$$2\|C\|_{L^\infty(\mathbb{R}; H_Q)} \omega(2T_{\delta, \varepsilon})T + \delta \omega(2T_{\delta, \varepsilon})T < \frac{\eta}{2}.$$

Finally we obtain

$$\left| \int_0^T \langle B(t), C(t/\varepsilon) \rangle_{P, Q} dt - \int_0^T \langle B(t), \bar{C} \rangle_{P, Q} dt \right| \leq \delta\|B\|_{L^1([0, T]; H_P)} + \frac{\eta}{2} < \eta$$

for any $0 < \varepsilon < \varepsilon(\eta)$, uniformly with respect to $B \in \mathcal{B}_\omega$. \square

Remark 5.1 *The conclusion of Proposition 5.1 holds true for any pair $(B, C) \in L^1([0, T]; H_P) \times L^\infty(\mathbb{R}; H_Q)$ such that $\left(\frac{1}{S} \int_{s_0}^{s_0+S} C(s) ds\right)_{S>0}$ converges strongly in H_Q toward some $\bar{C} \in H_Q$, uniformly with respect to $s_0 \in \mathbb{R}$, when $S \rightarrow +\infty$. Indeed, observe that*

$$\left| \int_0^T \langle B(t), C(t/\varepsilon) \rangle_{P, Q} dt - \int_0^T \langle B(t), \bar{C} \rangle_{P, Q} dt \right| \leq 2\|B\|_{L^1([0, T]; H_P)} \|C\|_{L^\infty(\mathbb{R}; H_Q)}$$

and thus, by using the density of $C([0, T]; H_P)$ in $L^1([0, T]; H_P)$, it is enough to consider $B \in C([0, T]; H_P)$. But in this case, the uniform continuity of B allows us to pick a modulus of continuity $\omega : [0, T] \rightarrow \mathbb{R}_+$

$$\omega(\lambda) = \sup_{t, t' \in [0, T], |t-t'| \leq \lambda} |B(t) - B(t')|_P, \quad \lambda \in [0, T]$$

and all the arguments in the proof of Proposition 5.1 apply.

In the sequel, we present some consequences of Proposition 5.1 which will be used when justifying the main result in Theorem 2.2. Everytime that $y \rightarrow A(y) : B(y)$ belongs to $L^1(\mathbb{R}^m)$, by the notation $\langle A, B \rangle_{P, Q}$ we understand $\int_{\mathbb{R}^m} A(y) : B(y) dy$.

Proposition 5.2 *Let T be a positive real number. Consider $D \in H_Q^\infty$ a symmetric matrix field and $\mathcal{W}_\omega \subset C([0, T]; X_P)$ a bounded set in $L^2([0, T]; X_P)$ of functions which admit as modulus of continuity in $C([0, T]; X_P)$ the same function $\omega : [0, T] \rightarrow \mathbb{R}_+$, i.e.,*

$$|w(t) - w(t')|_P \leq \omega(|t - t'|), \quad t, t' \in [0, T], \quad w \in \mathcal{W}_\omega$$

with ω non decreasing and $\lim_{\lambda \searrow 0} \omega(\lambda) = 0$. Then for any family $(w^\beta)_{\beta > 0} \subset \mathcal{W}_\omega$ which converges weakly in $L^2([0, T]; X_P)$ toward w^0 when $\beta \searrow 0$, we have

$$\lim_{(\beta, \varepsilon) \rightarrow (0, 0)} \int_0^T \langle \theta(t) \otimes w^\beta(t), G(t/\varepsilon)D \rangle_{P, Q} dt = \int_0^T \langle \theta(t) \otimes w^0(t), \langle D \rangle \rangle_{P, Q} dt \quad (48)$$

for any $\theta \in L^2([0, T]; X_P)$.

Proof. Notice that for any $\theta, w \in L^2([0, T]; X_P)$ we have

$$\begin{aligned} \int_0^T \langle \theta(t) \otimes w(t), G(t/\varepsilon)D \rangle_{P, Q} dt &= \int_0^T \int_{\mathbb{R}^m} \theta(t, y) \otimes w(t, y) : G(t/\varepsilon)D dy dt \\ &= \int_0^T \int_{\mathbb{R}^m} (P^{1/2}(y)\theta(t, y)) \otimes (P^{1/2}(y)w(t, y)) : Q^{1/2}(y)G(t/\varepsilon)DQ^{1/2}(y) dy dt \\ &\leq \int_0^T |G(t/\varepsilon)D|_{H_Q^\infty} \int_{\mathbb{R}^m} |P^{1/2}(y)\theta(t, y)| |P^{1/2}(y)w(t, y)| dy dt \\ &\leq |D|_{H_Q^\infty} \left(\int_0^T \int_{\mathbb{R}^m} P(y)\theta(t, y) \cdot \theta(t, y) dy dt \right)^{1/2} \left(\int_0^T \int_{\mathbb{R}^m} P(y)w(t, y) \cdot w(t, y) dy dt \right)^{1/2} \\ &= |D|_{H_Q^\infty} \|\theta\|_{L^2([0, T]; X_P)} \|w\|_{L^2([0, T]; X_P)} \end{aligned}$$

and similarly, by using $|\langle D \rangle|_{H_Q^\infty} \leq |D|_{H_Q^\infty}$

$$\int_0^T \langle \theta(t) \otimes w(t), \langle D \rangle \rangle_{P, Q} dt \leq |D|_{H_Q^\infty} \|\theta\|_{L^2([0, T]; X_P)} \|w\|_{L^2([0, T]; X_P)}. \quad (49)$$

As the family $(w^\beta)_{\beta > 0}$ is bounded in $L^2([0, T]; X_P)$, it is enough to check (48) for any θ in a dense subset of $L^2([0, T]; X_P)$, for example for any θ such that $P^{1/2}\theta \in C_c^0([0, T] \times \mathbb{R}^m)$. Take $k \in \mathbb{N}$ large enough, such that $k \geq \psi(y)$, $y \in \text{supp}(P^{1/2}\theta)$. As $D \in H_Q^\infty$ belongs to $H_{Q, \text{loc}}$, the matrix field $D_k = \mathbf{1}_{\{\psi \leq k\}} D$ belongs to $H_Q \cap H_Q^\infty$. We appeal to Proposition 5.1 with $C(s) = G(s)D_k$, $\overline{C} = \langle D_k \rangle = \mathbf{1}_{\{\psi \leq k\}} \langle D \rangle$, cf. Theorem 2.1, and $\mathcal{B} = \{\theta \otimes w : w \in \mathcal{W}_\omega\}$. By Proposition 3.1 we know that $(G(s))_{s \in \mathbb{R}}$ is a C^0 -group of unitary operators on H_Q , implying that $C \in L^\infty(\mathbb{R}; H_Q)$. By Theorem 2.1 we deduce that

$$\lim_{S \rightarrow +\infty} \frac{1}{S} \int_{s_0}^{s_0+S} C(s) ds = \overline{C}, \quad \text{uniformly with respect to } s_0 \in \mathbb{R}.$$

For any $w \in \mathcal{W}_\omega$ we write

$$\begin{aligned} \|\theta \otimes w\|_{L^1([0,T];H_P)} &= \int_0^T \left(\int_{\mathbb{R}^m} (P^{1/2}\theta) \otimes (P^{1/2}w) : (P^{1/2}\theta) \otimes (P^{1/2}w) \, dy \right)^{1/2} dt \\ &\leq \int_0^T |\theta(t)|_{X_P^\infty} |w(t)|_P \, dt \\ &\leq \|P^{1/2}\theta\|_{L^2([0,T];L^\infty(\mathbb{R}^m))} \|w\|_{L^2([0,T];X_P)} \end{aligned}$$

and therefore the boundedness of \mathcal{W}_ω in $L^2([0,T];X_P)$ implies the boundedness of \mathcal{B} in $L^1([0,T];H_P)$ (since $P^{1/2}\theta \in C_c^0([0,T] \times \mathbb{R}^m)$). We search now for a continuity modulus of \mathcal{B} . For any $w \in \mathcal{W}_\omega$, $t, t' \in [0, T]$, we have

$$\begin{aligned} |\theta(t) \otimes w(t) - \theta(t') \otimes w(t')|_P &\leq |\theta(t) - \theta(t')|_{X_P^\infty} |w(t)|_P + |\theta(t')|_{X_P^\infty} |w(t) - w(t')|_P \\ &\leq \|P^{1/2}\theta(t) - P^{1/2}\theta(t')\|_{L^\infty(\mathbb{R}^m)} |w(t)|_P + \|P^{1/2}\theta(t')\|_{L^\infty(\mathbb{R}^m)} \omega(|t - t'|) \\ &\leq \omega_\theta(|t - t'|) \|w\|_{C([0,T];X_P)} + \omega(|t - t'|) \|P^{1/2}\theta\|_{C^0([0,T] \times \mathbb{R}^m)} \end{aligned}$$

where ω_θ is a continuity modulus for $P^{1/2}\theta \in C_c^0([0, T] \times \mathbb{R}^m)$. The claim follows if we manage to show that \mathcal{W}_ω is also bounded in $C([0, T]; X_P)$. This comes easily by noticing that for any $t, t' \in [0, T], w \in \mathcal{W}_\omega$ we have

$$|w(t)|_P^2 \leq (|w(t')|_P + \omega(|t - t'|))^2 \leq 2|w(t')|_P^2 + 2\omega^2(T).$$

Integrating with respect to $t' \in [0, T]$ one gets for any $t \in [0, T]$

$$|w(t)|_P^2 \leq \frac{2}{T} \|w\|_{L^2([0,T];X_P)}^2 + 2\omega^2(T)$$

saying that \mathcal{W}_ω is bounded in $C([0, T]; X_P)$. By Proposition 5.1, for any $\eta > 0$, there is $\varepsilon(\eta) > 0$ such that for any $0 < \varepsilon < \varepsilon(\eta), \beta > 0$

$$\left| \int_0^T \left\langle \theta(t) \otimes w^\beta(t), \mathbf{1}_{\{\psi \leq k\}} G(t/\varepsilon) D \right\rangle_{P,Q} dt - \int_0^T \left\langle \theta(t) \otimes w^\beta(t), \mathbf{1}_{\{\psi \leq k\}} \langle D \rangle \right\rangle_{P,Q} dt \right| < \frac{\eta}{2}.$$

As $\text{supp}(P^{1/2}\theta) \subset \{\psi \leq k\}$, the above inequality also writes

$$\left| \int_0^T \left\langle \theta(t) \otimes w^\beta(t), G(t/\varepsilon) D \right\rangle_{P,Q} dt - \int_0^T \left\langle \theta(t) \otimes w^\beta(t), \langle D \rangle \right\rangle_{P,Q} dt \right| < \frac{\eta}{2}, 0 < \varepsilon < \varepsilon(\eta), \beta > 0.$$

By (49) we know that $w \rightarrow \int_0^T \langle \theta(t) \otimes w(t), \langle D \rangle \rangle_{P,Q} dt$ is a linear continuous application on $L^2([0, T]; X_P)$, and since $(w^\beta)_{\beta > 0}$ converges weakly in $L^2([0, T]; X_P)$, toward w^0 , when $\beta \searrow 0$, there is $\beta(\eta) > 0$ such that for any $0 < \beta < \beta(\eta)$

$$\left| \int_0^T \left\langle \theta(t) \otimes w^\beta(t), \langle D \rangle \right\rangle_{P,Q} dt - \int_0^T \left\langle \theta(t) \otimes w^0, \langle D \rangle \right\rangle_{P,Q} dt \right| < \frac{\eta}{2}$$

Therefore, for any $\eta > 0$, there is $\beta(\eta) > 0, \varepsilon(\eta) > 0$ such that for any $0 < \beta < \beta(\eta), 0 < \varepsilon < \varepsilon(\eta)$

$$\left| \int_0^T \left\langle \theta(t) \otimes w^\beta(t), G(t/\varepsilon) D \right\rangle_{P,Q} dt - \int_0^T \left\langle \theta(t) \otimes w^0, \langle D \rangle \right\rangle_{P,Q} dt \right| < \eta.$$

□

Remark 5.2 *The previous arguments show that if $D \in H_Q^\infty$, then*

$$\lim_{\varepsilon \searrow 0} \int_0^T \langle w(t) \otimes w(t), G(t/\varepsilon)D \rangle_{P,Q} dt = \int_0^T \langle w(t) \otimes w(t), \langle D \rangle \rangle_{P,Q} dt \quad (50)$$

for any $w \in L^2([0, T]; X_P)$. Indeed, taking into account that the bilinear application

$$(\theta, w) \in L^2([0, T]; X_P) \times L^2([0, T]; X_P) \rightarrow \int_0^T \langle \theta(t) \otimes w(t), \langle D \rangle \rangle_{P,Q} dt \in \mathbb{R}$$

is continuous, it is enough to establish (50) for w in the set $\{\theta \in L^2([0, T]; X_P) : P^{1/2}\theta \in C_c^0([0, T] \times \mathbb{R}^m)\}$, which is dense in $L^2([0, T]; X_P)$. And this is a direct consequence of Remark 5.1 applied with $D_k = \mathbf{1}_{\{\psi \leq k\}}D$, $k \geq \psi(y)$, $y \in \text{supp}(P^{1/2}\theta)$, since for any $\theta \in L^2([0, T]; X_P)$ such that $P^{1/2}\theta \in C_c^0([0, T] \times \mathbb{R}^m)$, we have

$$\begin{aligned} \|\theta \otimes \theta\|_{L^1([0, T]; H_P)} &= \int_0^T \left(\int_{\mathbb{R}^m} |P^{1/2}\theta|^4 dy \right)^{1/2} dt \leq \int_0^T \|P^{1/2}\theta(t)\|_{C^0(\mathbb{R}^m)} \|P^{1/2}\theta(t)\|_{L^2(\mathbb{R}^m)} dt \\ &\leq \|\theta\|_{L^2([0, T]; X_P)} \|\theta\|_{L^2([0, T]; X_P)} < +\infty. \end{aligned}$$

When the matrix field D is positive definite, the behavior of the upper limit with respect to (β, ε) for the quadratic term $\int_0^T \langle w^\beta(t) \otimes w^\beta(t), G(t/\varepsilon)D \rangle_{P,Q} dt$ characterizes the strong convergence of the family $(w^\beta)_{\beta > 0}$ as shown in the following result.

Proposition 5.3 *Assume the same hypotheses as in Proposition 5.2.*

1. *If the matrix field D is positive semi-definite, then we have*

$$\int_0^T \langle w^0(t) \otimes w^0(t), \langle D \rangle \rangle_{P,Q} dt \leq \liminf_{(\beta, \varepsilon) \rightarrow (0, 0)} \int_0^T \langle w^\beta(t) \otimes w^\beta(t), G(t/\varepsilon)D \rangle_{P,Q} dt.$$

2. *If $(w^\beta)_{\beta > 0}$ converges strongly in $L^2([0, T]; X_P)$ toward w^0 when $\beta \searrow 0$ (the existence of a modulus of continuity ω in $C([0, T]; X_P)$ for the family $(w^\beta)_{\beta > 0}$ is not necessary here), then we have*

$$\int_0^T \langle w^0(t) \otimes w^0(t), \langle D \rangle \rangle_{P,Q} dt = \lim_{(\beta, \varepsilon) \rightarrow (0, 0)} \int_0^T \langle w^\beta(t) \otimes w^\beta(t), G(t/\varepsilon)D \rangle_{P,Q} dt.$$

3. *If there is $\alpha > 0$ such that $Q^{1/2}DQ^{1/2} \geq \alpha I_m$, and*

$$\limsup_{(\beta, \varepsilon) \rightarrow (0, 0)} \int_0^T \langle w^\beta(t) \otimes w^\beta(t), G(t/\varepsilon)D \rangle_{P,Q} dt \leq \int_0^T \langle w^0(t) \otimes w^0(t), \langle D \rangle \rangle_{P,Q} dt$$

then the family $(w^\beta)_{\beta > 0}$ converges strongly in $L^2([0, T]; X_P)$ toward w^0 when $\beta \searrow 0$.

Proof.

1. As the matrix field D is symmetric and positive semi-definite, so is the matrix field $G(t/\varepsilon)D$ for any $t \in [0, T]$ and $\varepsilon > 0$, and thus

$$\begin{aligned} \int_0^T \langle w^0(t) \otimes w^\beta(t), G(t/\varepsilon)D \rangle_{P,Q} dt &= \int_0^T \int_{\mathbb{R}^m} w^0(t, y) \otimes w^\beta(t, y) : G(t/\varepsilon)D dy dt \\ &\leq \int_0^T \int_{\mathbb{R}^m} (w^0(t, y) \otimes w^0(t, y) : G(t/\varepsilon)D)^{1/2} (w^\beta(t, y) \otimes w^\beta(t, y) : G(t/\varepsilon)D)^{1/2} dy dt \\ &\leq \left(\int_0^T \langle w^0(t) \otimes w^0(t), G(t/\varepsilon)D \rangle_{P,Q} dt \right)^{1/2} \left(\int_0^T \langle w^\beta(t) \otimes w^\beta(t), G(t/\varepsilon)D \rangle_{P,Q} dt \right)^{1/2}. \end{aligned}$$

Passing to the lower limit with respect to (β, ε) yields, thanks to Proposition 5.2

$$\int_0^T \langle w^0(t) \otimes w^0(t), \langle D \rangle \rangle_{P,Q} dt \leq \liminf_{(\beta, \varepsilon) \rightarrow (0,0)} \left\{ \left(\int_0^T \langle w^0(t) \otimes w^0(t), G(t/\varepsilon)D \rangle_{P,Q} dt \right)^{1/2} \times \left(\int_0^T \langle w^\beta(t) \otimes w^\beta(t), G(t/\varepsilon)D \rangle_{P,Q} dt \right)^{1/2} \right\}. \quad (51)$$

Thanks to Remark 5.2, we know that

$$\lim_{\varepsilon \searrow 0} \int_0^T \langle w^0(t) \otimes w^0(t), G(t/\varepsilon)D \rangle_{P,Q} dt = \int_0^T \langle w^0(t) \otimes w^0(t), \langle D \rangle \rangle_{P,Q} dt. \quad (52)$$

Using the equality (52) in the inequality (51) leads to our claim.

2. Pick η a positive real number. By Remark 5.2, there is $\varepsilon(\eta)$ such that for any $0 < \varepsilon < \varepsilon(\eta)$

$$\left| \int_0^T \langle w^0(t) \otimes w^0(t), G(t/\varepsilon)D \rangle_{P,Q} dt - \int_0^T \langle w^0(t) \otimes w^0(t), \langle D \rangle \rangle_{P,Q} dt \right| < \frac{\eta}{2}.$$

It is easily seen, thanks to the strong convergence of $(w^\beta)_{\beta>0}$ in $L^2([0, T]; X_P)$ toward w^0 , that there is $\beta(\eta) > 0$ such that for any $0 < \beta < \beta(\eta)$, and any $\varepsilon > 0$

$$\begin{aligned} & \left| \int_0^T \langle w^\beta(t) \otimes w^\beta(t), G(t/\varepsilon)D \rangle_{P,Q} dt - \int_0^T \langle w^0(t) \otimes w^0(t), G(t/\varepsilon)D \rangle_{P,Q} dt \right| \\ & \leq |D|_{H_Q^\infty} \|w^\beta - w^0\|_{L^2([0, T]; X_P)} \left(\|w^\beta\|_{L^2([0, T]; X_P)} + \|w^0\|_{L^2([0, T]; X_P)} \right) < \frac{\eta}{2}. \end{aligned}$$

Therefore the second assertion holds true.

3. We know by Proposition 3.1 that $Q^{1/2}G(t/\varepsilon)DQ^{1/2} \geq \alpha I_m$, for any $t \in \mathbb{R}_+, \varepsilon > 0$ and therefore

$$\begin{aligned} \alpha \|w^\beta - w^0\|_{L^2([0, T]; X_P)}^2 & \leq \int_0^T \langle [w^\beta(t) - w^0(t)] \otimes [w^\beta(t) - w^0(t)], G(t/\varepsilon)D \rangle_{P,Q} dt \\ & = \int_0^T \langle w^\beta(t) \otimes w^\beta(t), G(t/\varepsilon)D \rangle_{P,Q} dt + \int_0^T \langle w^0(t) \otimes w^0(t), G(t/\varepsilon)D \rangle_{P,Q} dt \\ & \quad - \int_0^T \langle w^\beta(t) \otimes w^0(t), G(t/\varepsilon)D \rangle_{P,Q} dt - \int_0^T \langle w^0(t) \otimes w^\beta(t), G(t/\varepsilon)D \rangle_{P,Q} dt. \end{aligned}$$

By Proposition 5.2 we know that

$$\begin{aligned} \lim_{(\beta, \varepsilon) \rightarrow (0,0)} \int_0^T \langle w^\beta(t) \otimes w^0(t), G(t/\varepsilon)D \rangle_{P,Q} dt & = \lim_{(\beta, \varepsilon) \rightarrow (0,0)} \int_0^T \langle w^0(t) \otimes w^\beta(t), G(t/\varepsilon)D \rangle_{P,Q} dt \\ & = \int_0^T \langle w^0(t) \otimes w^0(t), \langle D \rangle \rangle_{P,Q} dt \end{aligned}$$

and by Remark 5.2 we have

$$\lim_{\varepsilon \searrow 0} \int_0^T \langle w^0(t) \otimes w^0(t), G(t/\varepsilon)D \rangle_{P,Q} dt = \int_0^T \langle w^0(t) \otimes w^0(t), \langle D \rangle \rangle_{P,Q} dt.$$

Finally we obtain

$$\begin{aligned} \alpha \limsup_{\beta \searrow 0} \|w^\beta - w^0\|_{L^2([0, T]; X_P)}^2 & \leq \limsup_{(\beta, \varepsilon) \rightarrow (0,0)} \int_0^T \langle w^\beta(t) \otimes w^\beta(t), G(t/\varepsilon)D \rangle_{P,Q} dt \\ & \quad - \int_0^T \langle w^0(t) \otimes w^0(t), \langle D \rangle \rangle_{P,Q} dt \leq 0 \end{aligned}$$

which proves that $(w^\beta)_{\beta>0}$ converges strongly in $L^2([0, T]; X_P)$ toward w^0 when $\beta \searrow 0$. \square

6 Proofs of the main theorems

We establish two convergence results. In Theorem 2.2 we state strong convergence results for the families $(v^\varepsilon)_{\varepsilon>0}$ in $L_{\text{loc}}^\infty(\mathbb{R}_+; L^2(\mathbb{R}^m))$ and $(\nabla_z v^\varepsilon)_{\varepsilon>0}$ in $L_{\text{loc}}^2(\mathbb{R}_+; X_P)$. In Theorem 2.3 we investigate the rate of the above convergences, by introducing a corrector, that is, we justify the leading term in the development (10).

Proof of Theorem 2.2. Assume that u^{in} belongs to H_R^2 , which is the space defined by

$$H_R^2 := \{u \in H_R^1 : \nabla_y^R \otimes \nabla_y^R u \in L^2(\mathbb{R}^m)\}. \quad (53)$$

As u^ε is the variational solution of (1), we have for any $\Phi \in C_c^1(\mathbb{R}_+ \times \mathbb{R}^m)$

$$\begin{aligned} - \int_0^{+\infty} \int_{\mathbb{R}^m} u^\varepsilon(t, y) \partial_t \Phi \, dy dt - \int_{\mathbb{R}^m} u^{\text{in}}(y) \Phi(0, y) \, dy + \int_0^{+\infty} \int_{\mathbb{R}^m} D(y) \nabla_y u^\varepsilon \cdot \nabla_y \Phi \, dy dt \\ - \frac{1}{\varepsilon} \int_0^{+\infty} \int_{\mathbb{R}^m} u^\varepsilon(t, y) b(y) \cdot \nabla_y \Phi \, dy dt = 0. \end{aligned} \quad (54)$$

Actually the above formulation holds true for any compactly supported function in $\mathbb{R}_+ \times \mathbb{R}^m$, which belongs to $W^{1,\infty}(\mathbb{R}_+ \times \mathbb{R}^m)$. Pick a test function $\psi \in C_c^1(\mathbb{R}_+ \times \mathbb{R}^m)$ and let us introduce the function $\Phi^\varepsilon(t, y) = \psi(t, Y(-t/\varepsilon; y))$, $(t, y) \in \mathbb{R}_+ \times \mathbb{R}^m$. Thanks to the hypotheses (4), the function Φ^ε is compactly supported in $\mathbb{R}_+ \times \mathbb{R}^m$, belongs to $W^{1,\infty}(\mathbb{R}_+ \times \mathbb{R}^m)$ and thus satisfies (54). We perform the change of variable $z \mapsto y = Y(t/\varepsilon; z)$. Taking the time and space derivatives of the equalities $\psi(t, z) = \Phi^\varepsilon(t, Y(t/\varepsilon; z))$ and $v^\varepsilon(t, z) = u^\varepsilon(t, Y(t/\varepsilon; z))$ gives

$$\partial_t \psi(t, z) = \partial_t \Phi^\varepsilon(t, Y(t/\varepsilon; z)) + \frac{1}{\varepsilon} b(Y(t/\varepsilon; z)) \cdot \nabla_y \Phi^\varepsilon(t, Y(t/\varepsilon; z))$$

$$\nabla_z \psi(t, z) = {}^t \partial Y(t/\varepsilon; z) \nabla_y \Phi^\varepsilon(t, Y(t/\varepsilon; z)), \quad \nabla_z v^\varepsilon(t, z) = {}^t \partial Y(t/\varepsilon; z) \nabla_y u^\varepsilon(t, Y(t/\varepsilon; z))$$

and the weak formulation (54), written with the test function $\Phi^\varepsilon(t, y)$ becomes

$$\begin{aligned} - \int_0^{+\infty} \int_{\mathbb{R}^m} v^\varepsilon(t, z) \partial_t \psi \, dz dt - \int_{\mathbb{R}^m} u^{\text{in}}(z) \psi(0, z) \, dz \\ + \int_0^{+\infty} \int_{\mathbb{R}^m} \partial Y^{-1}(t/\varepsilon; z) D(Y(t/\varepsilon; z)) {}^t \partial Y^{-1}(t/\varepsilon; z) \nabla_z v^\varepsilon \cdot \nabla_z \psi \, dz dt = 0. \end{aligned}$$

Therefore v^ε is the variational solution of (9). By Propositions 4.3, 4.4 we have, for any $T \in \mathbb{R}_+$

$$\sup_{\varepsilon>0} \{ \|v^\varepsilon\|_{L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^m))} + \|\nabla_z^R v^\varepsilon\|_{L^\infty([0, T]; L^2(\mathbb{R}^m))} + \|\nabla_z^R \otimes \nabla_z^R v^\varepsilon\|_{L^\infty([0, T]; L^2(\mathbb{R}^m))} \} < +\infty$$

$$\sup_{\varepsilon>0} \{ \|\partial_t v^\varepsilon\|_{L^2([0, T]; L^2(\mathbb{R}^m))} + \|\partial_t \nabla_z^R v^\varepsilon\|_{L^2([0, T]; L^2(\mathbb{R}^m))} \} < +\infty.$$

Let us consider a sequence $(\varepsilon_k)_k$ converging to 0 such that

$$\lim_{k \rightarrow +\infty} v^{\varepsilon_k} = v^0 \text{ weakly } \star \text{ in } L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^m)) \quad (55)$$

$$\lim_{k \rightarrow +\infty} \nabla_z v^{\varepsilon_k} = \nabla_z v^0 \text{ weakly } \star \text{ in } L^\infty([0, T]; X_P), \quad T \in \mathbb{R}_+. \quad (56)$$

We claim that v^0 is the variational solution of (26). For any $\eta \in C_c^1(\mathbb{R}_+)$ and $\Phi \in H_R^1$, the variational formulation of (9) yields

$$\begin{aligned} - \int_0^{+\infty} \int_{\mathbb{R}^m} v^{\varepsilon_k}(t, z) \eta'(t) \Phi(z) \, dz dt - \int_{\mathbb{R}^m} u^{\text{in}}(z) \eta(0) \Phi(z) \, dz \\ + \int_0^{+\infty} \int_{\mathbb{R}^m} G(t/\varepsilon_k) D \nabla_z v^{\varepsilon_k} \cdot \eta(t) \nabla_z \Phi \, dz dt = 0. \end{aligned}$$

As $\eta' \Phi$ belongs to $L^1(\mathbb{R}_+; L^2(\mathbb{R}^m))$, the weak \star convergence in $L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^m))$ of $(v^{\varepsilon_k})_k$ gives

$$\int_0^{+\infty} \int_{\mathbb{R}^m} v^{\varepsilon_k}(t, z) \eta'(t) \Phi(z) \, dz dt \xrightarrow{k \rightarrow +\infty} \int_0^{+\infty} \int_{\mathbb{R}^m} v^0(t, z) \eta'(t) \Phi(z) \, dz dt.$$

We use now Proposition 5.2 with $T > 0$ such that $\text{supp } \eta \subset [0, T[$, and $\mathcal{W}_\omega = \{w^k = \nabla_z v^{\varepsilon_k}|_{[0, T] \times \mathbb{R}^m} : k \in \mathbb{N}\}$. Obviously, \mathcal{W}_ω is bounded in $L^2([0, T]; X_P)$ and for any $k \in \mathbb{N}$, $t, t' \in [0, T]$, we can write

$$|\nabla_z v^{\varepsilon_k}(t) - \nabla_z v^{\varepsilon_k}(t')|_P = \|\nabla_z^R v^{\varepsilon_k}(t) - \nabla_z^R v^{\varepsilon_k}(t')\|_{L^2} \leq \sqrt{|t - t'|} \|\partial_t \nabla_z^R v^{\varepsilon_k}\|_{L^2([0, T]; L^2(\mathbb{R}^m))}.$$

Therefore \mathcal{W}_ω is contained in $C([0, T]; X_P)$ and admits the continuity modulus

$$\omega(\lambda) = \sqrt{\lambda} \sup_{\varepsilon > 0} \|\partial_t \nabla_z^R v^\varepsilon\|_{L^2([0, T]; L^2(\mathbb{R}^m))}.$$

Applying Proposition 5.2 with $\theta(t, z) = \eta(t) \nabla_z \Phi(z) \in L^2([0, T]; X_P)$ we deduce that

$$\begin{aligned} \int_0^{+\infty} \int_{\mathbb{R}^m} G(t/\varepsilon_k) D \nabla_z v^{\varepsilon_k} \cdot \eta(t) \nabla_z \Phi \, dz dt &= \int_0^T \langle \eta(t) \nabla_z \Phi \otimes \nabla_z v^{\varepsilon_k}(t), G(t/\varepsilon_k) D \rangle_{P, Q} \, dt \\ &\xrightarrow{k \rightarrow +\infty} \int_0^T \langle \eta(t) \nabla_z \Phi \otimes \nabla_z v^0(t), \langle D \rangle \rangle_{P, Q} \, dt \\ &= \int_0^{+\infty} \int_{\mathbb{R}^m} \langle D \rangle \nabla_z v^0 \cdot \eta(t) \nabla_z \Phi \, dz dt. \end{aligned}$$

Therefore, passing to the limit, when $k \rightarrow +\infty$, in the variational formulation of v^{ε_k} , implies

$$\begin{aligned} - \int_0^{+\infty} \int_{\mathbb{R}^m} v^0(t, z) \eta'(t) \Phi(z) \, dz dt - \int_{\mathbb{R}^m} u^{\text{in}}(z) \eta(0) \Phi(z) \, dz \\ + \int_0^{+\infty} \int_{\mathbb{R}^m} \langle D \rangle \nabla_z v^0 \cdot \eta(t) \nabla_z \Phi \, dz dt = 0 \end{aligned}$$

and thus v^0 is the variational solution of (26) ($v^0 = v$). By the uniqueness of the solution for the limit model (26), we deduce that the convergences in (55), (56) actually hold for the whole family $(v^\varepsilon)_\varepsilon$

$$\lim_{\varepsilon \searrow 0} v^\varepsilon = v \text{ weakly } \star \text{ in } L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^m)), \quad \lim_{\varepsilon \searrow 0} \nabla_z v^\varepsilon = \nabla_z v \text{ weakly } \star \text{ in } L_{\text{loc}}^\infty(\mathbb{R}_+; X_P).$$

The regularity of v follows by Propositions 4.5, 4.6, in particular $\partial_t v \in L_{\text{loc}}^2(\mathbb{R}_+; L^2(\mathbb{R}^m))$. Actually the time derivative $\partial_t v$ belongs to $L_{\text{loc}}^\infty(\mathbb{R}_+; L^2(\mathbb{R}^m))$. This comes immediately by the regularity of $\langle D \rangle$. Indeed, by the proofs of Propositions 4.5, 4.6 we know that $\text{div}_z(R \langle D \rangle) \in L^\infty(\mathbb{R}^m)$, $R \langle D \rangle \, {}^t R \in L^\infty(\mathbb{R}^m)$ and we obtain

$$\begin{aligned} \partial_t v &= \text{div}_z(\langle D \rangle \nabla_z v) = \text{div}_z(\langle D \rangle \, {}^t R \nabla_z^R v) = \text{div}_z(R \langle D \rangle) \cdot \nabla_z^R v + R \langle D \rangle : \partial \nabla_z^R v \\ &= \text{div}_z(R \langle D \rangle) \cdot \nabla_z^R v + R \langle D \rangle \, {}^t R : \nabla_z^R \otimes \nabla_z^R v \in L_{\text{loc}}^\infty(\mathbb{R}_+; L^2(\mathbb{R}^m)). \end{aligned}$$

We concentrate now on the strong convergence of $(v^\varepsilon)_{\varepsilon > 0}$ in $L_{\text{loc}}^\infty(\mathbb{R}_+; L^2(\mathbb{R}^m))$ and $(\nabla_z v^\varepsilon)_{\varepsilon > 0}$ in $L_{\text{loc}}^2(\mathbb{R}_+; X_P)$. By the energy balance associated with (9) we deduce

$$\|v^\varepsilon(t)\|_{L^2(\mathbb{R}^m)}^2 + 2 \int_0^t \langle \nabla_z v^\varepsilon(\tau) \otimes \nabla_z v^\varepsilon(\tau), G(\tau/\varepsilon) D \rangle_{P, Q} \, d\tau = \|u^{\text{in}}\|_{L^2(\mathbb{R}^m)}^2, \quad t \in \mathbb{R}_+. \quad (57)$$

Similarly, the energy balance associated with (26) gives

$$\|v(t)\|_{L^2(\mathbb{R}^m)}^2 + 2 \int_0^t \langle \nabla_z v(\tau) \otimes \nabla_z v(\tau), \langle D \rangle \rangle_{P, Q} \, d\tau = \|u^{\text{in}}\|_{L^2(\mathbb{R}^m)}^2, \quad t \in \mathbb{R}_+. \quad (58)$$

By the first statement in Proposition 5.3 we know that

$$\int_0^t \langle \nabla_z v(\tau) \otimes \nabla_z v(\tau), \langle D \rangle \rangle_{P,Q} d\tau \leq \liminf_{\varepsilon \searrow 0} \int_0^t \langle \nabla_z v^\varepsilon(\tau) \otimes \nabla_z v^\varepsilon(\tau), G(\tau/\varepsilon)D \rangle_{P,Q} d\tau. \quad (59)$$

Combining (57), (58), (59) one gets

$$\begin{aligned} \frac{1}{2} \limsup_{\varepsilon \searrow 0} \{ \|v^\varepsilon(t)\|_{L^2(\mathbb{R}^m)}^2 - \|v(t)\|_{L^2(\mathbb{R}^m)}^2 \} &= \limsup_{\varepsilon \searrow 0} \left\{ \int_0^t \langle \nabla_z v(\tau) \otimes \nabla_z v(\tau), \langle D \rangle \rangle_{P,Q} d\tau \right. \\ &\quad \left. - \int_0^t \langle \nabla_z v^\varepsilon(\tau) \otimes \nabla_z v^\varepsilon(\tau), G(\tau/\varepsilon)D \rangle_{P,Q} d\tau \right\} \\ &= \int_0^t \langle \nabla_z v(\tau) \otimes \nabla_z v(\tau), \langle D \rangle \rangle_{P,Q} d\tau \\ &\quad - \liminf_{\varepsilon \searrow 0} \int_0^t \langle \nabla_z v^\varepsilon(\tau) \otimes \nabla_z v^\varepsilon(\tau), G(\tau/\varepsilon)D \rangle_{P,Q} d\tau \leq 0 \end{aligned}$$

proving that at any time $t \in \mathbb{R}_+$ we have

$$\limsup_{\varepsilon \searrow 0} \|v^\varepsilon(t)\|_{L^2(\mathbb{R}^m)}^2 \leq \|v(t)\|_{L^2(\mathbb{R}^m)}^2. \quad (60)$$

Applying Fatou lemma to the family of non negative functions $t \rightarrow \|u^{\text{in}}\|_{L^2(\mathbb{R}^m)}^2 - \|v^\varepsilon(t)\|_{L^2(\mathbb{R}^m)}^2$ we deduce that

$$\int_0^T \liminf_{\varepsilon \searrow 0} \{ \|u^{\text{in}}\|_{L^2(\mathbb{R}^m)}^2 - \|v^\varepsilon(t)\|_{L^2(\mathbb{R}^m)}^2 \} dt \leq \liminf_{\varepsilon \searrow 0} \int_0^T \{ \|u^{\text{in}}\|_{L^2(\mathbb{R}^m)}^2 - \|v^\varepsilon(t)\|_{L^2(\mathbb{R}^m)}^2 \} dt$$

or equivalently

$$\limsup_{\varepsilon \searrow 0} \int_0^T \|v^\varepsilon(t)\|_{L^2(\mathbb{R}^m)}^2 dt \leq \int_0^T \limsup_{\varepsilon \searrow 0} \|v^\varepsilon(t)\|_{L^2(\mathbb{R}^m)}^2 dt.$$

Therefore, the above inequality, together with the weak convergence of the family $(v^\varepsilon)_{\varepsilon>0}$ in $L^2([0, T]; L^2(\mathbb{R}^m))$ toward v and (60) imply

$$\limsup_{\varepsilon \searrow 0} \int_0^T \|v^\varepsilon(t)\|_{L^2(\mathbb{R}^m)}^2 dt \leq \int_0^T \|v(t)\|_{L^2(\mathbb{R}^m)}^2 dt$$

saying that $(v^\varepsilon)_{\varepsilon>0}$ converges strongly in $L^2([0, T]; L^2(\mathbb{R}^m))$ toward v for any $T \in \mathbb{R}_+$

$$\lim_{\varepsilon \searrow 0} \int_0^T \|v^\varepsilon(t) - v(t)\|_{L^2(\mathbb{R}^m)}^2 dt = 0.$$

We deduce that there is a sequence $(\tilde{\varepsilon}_k)_k$ converging to 0 such that

$$\lim_{k \rightarrow +\infty} \|v^{\tilde{\varepsilon}_k}(t) - v(t)\|_{L^2(\mathbb{R}^m)}^2 = 0, \quad \text{for a.a. } t \in [0, T]. \quad (61)$$

As $\partial_t v \in L^2([0, T]; L^2(\mathbb{R}^m))$ and $\sup_{\varepsilon>0} \|\partial_t v^\varepsilon\|_{L^2([0, T]; L^2(\mathbb{R}^m))} < +\infty$, it is easily seen that (61) holds true for any $t \in [0, T]$, $T \in \mathbb{R}_+$, and thus for any $t \in \mathbb{R}_+$. Actually we have

$$\lim_{\varepsilon \searrow 0} \|v^\varepsilon(T) - v(T)\|_{L^2(\mathbb{R}^m)}^2 = 0, \quad T \in \mathbb{R}_+$$

which implies, thanks to (57), (58)

$$\begin{aligned}
\limsup_{\varepsilon \searrow 0} \int_0^T \langle \nabla_z v^\varepsilon(t) \otimes \nabla_z v^\varepsilon(t), G(t/\varepsilon)D \rangle_{P,Q} dt &= \frac{1}{2} \|u^{\text{in}}\|_{L^2(\mathbb{R}^m)}^2 - \frac{1}{2} \lim_{\varepsilon \searrow 0} \|v^\varepsilon(T)\|_{L^2(\mathbb{R}^m)}^2 \\
&= \frac{1}{2} \|u^{\text{in}}\|_{L^2(\mathbb{R}^m)}^2 - \frac{1}{2} \|v(T)\|_{L^2(\mathbb{R}^m)}^2 \\
&= \int_0^T \langle \nabla_z v(t) \otimes \nabla_z v(t), \langle D \rangle \rangle_{P,Q} dt.
\end{aligned}$$

By the third statement of Proposition 5.3 we deduce that $(\nabla_z v^\varepsilon)_{\varepsilon>0}$ converges strongly in $L^2([0, T]; X_P)$ toward $\nabla_z v$, for any $T \in \mathbb{R}_+$. Finally, in order to prove the convergence of $(v^\varepsilon)_{\varepsilon>0}$ in $L_{\text{loc}}^\infty(\mathbb{R}_+; L^2(\mathbb{R}^m))$ toward v we take the difference between the equations (9) and (26)

$$\partial_t(v^\varepsilon - v) - \text{div}_z \{G(t/\varepsilon)D \nabla_z v^\varepsilon - \langle D \rangle \nabla_z v\} = 0, \quad (t, z) \in \mathbb{R}_+ \times \mathbb{R}^m.$$

Writing the energy balance, we obtain for any $t \in \mathbb{R}_+$

$$\begin{aligned}
\frac{1}{2} \|v^\varepsilon(t) - v(t)\|_{L^2(\mathbb{R}^m)}^2 + \int_0^t \langle [\nabla_z v^\varepsilon(\tau) - \nabla_z v(\tau)] \otimes \nabla_z v^\varepsilon(\tau), G(\tau/\varepsilon)D \rangle_{P,Q} d\tau \\
- \int_0^t \langle [\nabla_z v^\varepsilon(\tau) - \nabla_z v(\tau)] \otimes \nabla_z v(\tau), \langle D \rangle \rangle_{P,Q} d\tau = 0.
\end{aligned}$$

As in the proof of Proposition 5.2 we have

$$\left| \langle [\nabla_z v^\varepsilon(\tau) - \nabla_z v(\tau)] \otimes \nabla_z v^\varepsilon(\tau), G(\tau/\varepsilon)D \rangle_{P,Q} \right| \leq |D|_{H_Q^\infty} |\nabla_z v^\varepsilon(\tau) - \nabla_z v(\tau)|_P |\nabla_z v^\varepsilon(\tau)|_P$$

and

$$\left| \langle [\nabla_z v^\varepsilon(\tau) - \nabla_z v(\tau)] \otimes \nabla_z v(\tau), \langle D \rangle \rangle_{P,Q} \right| \leq |D|_{H_Q^\infty} |\nabla_z v^\varepsilon(\tau) - \nabla_z v(\tau)|_P |\nabla_z v(\tau)|_P$$

and we deduce that for any $t \in [0, T]$ we have

$$\|(v^\varepsilon - v)(t)\|_{L^2}^2 \leq 2|D|_{H_Q^\infty} \|\nabla_z v^\varepsilon - \nabla_z v\|_{L^2([0, T]; X_P)} (\|\nabla_z v^\varepsilon\|_{L^2([0, T]; X_P)} + \|\nabla_z v\|_{L^2([0, T]; X_P)}).$$

The strong convergence of $(v^\varepsilon)_{\varepsilon>0}$ in $L^\infty([0, T]; L^2(\mathbb{R}^m))$ toward v comes from the strong convergence of $(\nabla_z v^\varepsilon)_{\varepsilon>0}$ in $L^2([0, T]; X_P)$ toward $\nabla_z v$, when $\varepsilon \searrow 0$.

It remains to show that our convergence result still holds for any $u^{\text{in}} \in L^2(\mathbb{R}^m)$. Indeed, for any $\delta > 0$, by density we can consider $u_\delta^{\text{in}} \in H_R^2$ such that $\|u^{\text{in}} - u_\delta^{\text{in}}\|_{L^2(\mathbb{R}^m)} \leq \delta/2$. We denote by v_δ^ε (resp. v_δ) the variational solution of (9) (resp. (26)) with the initial condition u_δ^{in} . Thanks to the energy balance we obtain easily

$$\begin{aligned}
\|v^\varepsilon - v\|_{L^\infty([0, T]; L^2(\mathbb{R}^m))} &\leq \|v^\varepsilon - v_\delta^\varepsilon\|_{L^\infty([0, T]; L^2(\mathbb{R}^m))} + \|v_\delta^\varepsilon - v_\delta\|_{L^\infty([0, T]; L^2(\mathbb{R}^m))} \\
&\quad + \|v_\delta - v\|_{L^\infty([0, T]; L^2(\mathbb{R}^m))} \\
&\leq 2\|u^{\text{in}} - u_\delta^{\text{in}}\|_{L^2(\mathbb{R}^m)} + \|v_\delta^\varepsilon - v_\delta\|_{L^\infty([0, T]; L^2(\mathbb{R}^m))}.
\end{aligned}$$

By the first part of the proof, since $u_\delta^{\text{in}} \in H_R^2$, we know that for any $\delta > 0$,

$$\lim_{\varepsilon \searrow 0} \|v_\delta^\varepsilon - v_\delta\|_{L^\infty([0, T]; L^2(\mathbb{R}^m))} = 0$$

and therefore

$$\limsup_{\varepsilon \searrow 0} \|v^\varepsilon - v\|_{L^\infty([0, T]; L^2(\mathbb{R}^m))} \leq \delta, \quad \delta > 0,$$

which gives that $\lim_{\varepsilon \searrow 0} \|v^\varepsilon - v\|_{L^\infty([0, T]; L^2(\mathbb{R}^m))} = 0$, for any $T \in \mathbb{R}_+$. \square

Remark 6.1 Under the assumptions of Theorem 2.2, the family $(v^\varepsilon)_{\varepsilon>0}$ converges strongly in $L_{\text{loc}}^\infty(\mathbb{R}_+; L^2(\mathbb{R}^m))$, which is a much better situation compared with the usual homogenization results for parabolic equations [14]. The above convergence in $L_{\text{loc}}^\infty(\mathbb{R}_+; L^2(\mathbb{R}^m))$ relies on energy balances thanks to Proposition 5.3. The uniform estimates of $(\partial_t v^\varepsilon)_{\varepsilon>0}, (\partial_t \nabla_z^R v^\varepsilon)_{\varepsilon>0}$ in $L_{\text{loc}}^2(\mathbb{R}_+; L^2(\mathbb{R}^m))$ (which are available for initial conditions $u^{\text{in}} \in H_R^2$ cf. Proposition 4.4) play a crucial role, in order to apply Propositions 5.2, 5.3.

The above considerations show that $v^\varepsilon = v + o(1)$ in $L_{\text{loc}}^\infty(\mathbb{R}_+; L^2(\mathbb{R}^m))$, when $\varepsilon \searrow 0$. As suggested by (10), we expect a convergence rate in $\mathcal{O}(\varepsilon)$. This can be achieved assuming that the limit solution v is smooth enough and that there is a smooth matrix field C such that

$$D = \langle D \rangle + L(C) \quad (62)$$

that is, cf. Remark 3.1

$$\mathbf{1}_{\{\psi \leq k\}} D - \langle \mathbf{1}_{\{\psi \leq k\}} D \rangle = \frac{d}{ds} \Big|_{s=0} G(s) (\mathbf{1}_{\{\psi \leq k\}} C), \quad k \in \mathbb{N}.$$

The existence of the matrix field C is essential when constructing the corrector term u^1 , see (65).

Proof of Theorem 2.3. We introduce the functions $\tilde{u}^\varepsilon(t, y) = v(t, Y(-t/\varepsilon; y)), (t, y) \in \mathbb{R}_+ \times \mathbb{R}^m, \varepsilon > 0$. As in the proof of Theorem 2.2 we check that \tilde{u}^ε is the variational solution of the problem

$$\begin{cases} \partial_t \tilde{u}^\varepsilon - \text{div}_y \{G(t/\varepsilon) \langle D \rangle \nabla_y \tilde{u}^\varepsilon\} + \frac{1}{\varepsilon} b(y) \cdot \nabla_y \tilde{u}^\varepsilon = 0, & (t, y) \in \mathbb{R}_+ \times \mathbb{R}^m \\ \tilde{u}^\varepsilon(0, y) = u^{\text{in}}(y), & y \in \mathbb{R}^m. \end{cases}$$

By construction, the average matrix field $\langle D \rangle$ belongs to H_Q^∞ and verifies $G(t/\varepsilon) \langle D \rangle = \langle D \rangle$. Therefore the functions $(\tilde{u}^\varepsilon)_{\varepsilon>0}$ solve the problems

$$\begin{cases} \partial_t \tilde{u}^\varepsilon - \text{div}_y \{\langle D \rangle \nabla_y \tilde{u}^\varepsilon\} + \frac{1}{\varepsilon} b(y) \cdot \nabla_y \tilde{u}^\varepsilon = 0, & (t, y) \in \mathbb{R}_+ \times \mathbb{R}^m \\ \tilde{u}^\varepsilon(0, y) = u^{\text{in}}(y), & y \in \mathbb{R}^m. \end{cases} \quad (63)$$

Recall that the functions $(u^\varepsilon)_{\varepsilon>0}$ satisfy

$$\begin{cases} \partial_t u^\varepsilon - \text{div}_y \{D \nabla_y u^\varepsilon\} + \frac{1}{\varepsilon} b(y) \cdot \nabla_y u^\varepsilon = 0, & (t, y) \in \mathbb{R}_+ \times \mathbb{R}^m \\ u^\varepsilon(0, y) = u^{\text{in}}(y), & y \in \mathbb{R}^m. \end{cases} \quad (64)$$

Notice that both families $(\tilde{u}^\varepsilon)_{\varepsilon>0}, (u^\varepsilon)_{\varepsilon>0}$ verify the same initial condition. The key point for obtaining a convergence rate is to introduce a corrector term. We consider the function

$$\begin{aligned} u^1(t, s, y) &= -\text{div}_z (C \nabla_z v(t))(Y(-s; y)) + \text{div}_y \{C(y) \nabla_y v(t, Y(-s; y))\} \\ &= -\tau(-s) \text{div}_z (C \nabla_z v(t)) + \text{div}_y \{C(y) \nabla_y [\tau(-s) v(t)]\}, \quad (t, s, y) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^m \end{aligned} \quad (65)$$

where we use the notation $\tau(s)f = f \circ Y(s; \cdot)$ for any function f . By (7) we have

$$u^1(t, s, Y(s; z)) = \text{div}_z (G(s) C \nabla_z v(t)) - \text{div}_z (C \nabla_z v(t)) \quad (66)$$

and taking the derivative with respect to s (here L is the infinitesimal generator of the group G) leads to

$$\begin{aligned} \partial_s u^1(t, s, Y(s; z)) + b(Y(s; z)) \cdot \nabla_y u^1(t, s, Y(s; z)) &= \text{div}_z \left\{ \frac{d}{ds} G(s) C \nabla_z v(t) \right\} \\ &= \text{div}_z \{G(s) L(C) \nabla_z v(t)\} \\ &= \{\text{div}_y [L(C) \nabla_y \tau(-s) v(t)]\} (Y(s; z)). \end{aligned}$$

Notice that for the last equality we have used again (7). Therefore the corrector u^1 verifies

$$(\partial_s + b(y) \cdot \nabla_y)u^1(t, s, y) - \operatorname{div}_y\{L(C)\nabla_y v(t, Y(-s; \cdot))\}(y) = 0, \quad (t, s, y) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^m \quad (67)$$

and by definition $u^1(t, 0, y) = 0$, $(t, y) \in \mathbb{R}_+ \times \mathbb{R}^m$. The equation (67) is exactly the equality coming out at the leading order when plugging the Ansatz $u^\varepsilon(t, y) = v(t, Y(-t/\varepsilon; y)) + \varepsilon u^1(t, t/\varepsilon, y) + \dots$ into (64). Indeed, the above Ansatz also writes

$$u^\varepsilon(t, Y(t/\varepsilon; z)) = v(t, z) + \varepsilon u^1(t, t/\varepsilon, Y(t/\varepsilon; z)) + \dots$$

and by observing that

$$\begin{aligned} \frac{d}{dt}u^\varepsilon(t, Y(t/\varepsilon; z)) &= \partial_t u^\varepsilon(t, Y(t/\varepsilon; z)) + \frac{1}{\varepsilon}b(Y(t/\varepsilon; z)) \cdot \nabla_y u^\varepsilon(t, Y(t/\varepsilon; z)) \\ &= [\operatorname{div}_y(D\nabla_y u^\varepsilon(t))](Y(t/\varepsilon; z)) = \operatorname{div}_z[G(t/\varepsilon)D\nabla_z u^\varepsilon(t, Y(t/\varepsilon; z))] \end{aligned}$$

we obtain

$$\begin{aligned} \partial_t v(t, z) + \varepsilon \partial_t u^1(t, t/\varepsilon, Y(t/\varepsilon; z)) + \partial_s u^1(t, t/\varepsilon, Y(t/\varepsilon; z)) & \quad (68) \\ + b(Y(t/\varepsilon; z)) \cdot \nabla_y u^1(t, t/\varepsilon, Y(t/\varepsilon; z)) + \dots &= \operatorname{div}_z(G(t/\varepsilon)D\nabla_z v) \\ + \operatorname{div}_z[G(t/\varepsilon)D\nabla_z(\varepsilon u^1(t, t/\varepsilon, Y(t/\varepsilon; z)))] + \dots & \end{aligned}$$

Taking into account that $\partial_t v = \operatorname{div}_z(\langle D \rangle \nabla_z v)$, we deduce from (68), thanks to (7), that

$$\begin{aligned} (\partial_s + b(y) \cdot \nabla_y)u^1(t, s, y) &= \tau(-s)\operatorname{div}_z[G(s)D\nabla_z v(t)] - \tau(-s)\operatorname{div}_z[\langle D \rangle \nabla_z v(t)] \\ &= \operatorname{div}_y[(D - \langle D \rangle)\nabla_y \tau(-s)v(t)] \\ &= \operatorname{div}_y[L(C)\nabla_y \tau(-s)v(t)] \end{aligned}$$

which corresponds to (67). In particular, for $s = t/\varepsilon$, one gets

$$(\partial_s + b(y) \cdot \nabla_y)u^1(t, t/\varepsilon, y) - \operatorname{div}_y(L(C)\nabla_y \tilde{u}^\varepsilon(t))(y) = 0, \quad (t, y) \in \mathbb{R}_+ \times \mathbb{R}^m, \varepsilon > 0$$

and we obtain the following equation for $\tilde{u}_\varepsilon^1 := u^1(t, t/\varepsilon, y)$

$$\partial_t(\varepsilon \tilde{u}_\varepsilon^1)(t, y) - \operatorname{div}_y(L(C)\nabla_y \tilde{u}^\varepsilon(t)) + \frac{1}{\varepsilon}b(y) \cdot \nabla_y(\varepsilon \tilde{u}_\varepsilon^1)(t, y) = \varepsilon \partial_t u^1(t, t/\varepsilon, y). \quad (69)$$

Summing (63) and (69) yields

$$\partial_t(\tilde{u}^\varepsilon + \varepsilon \tilde{u}_\varepsilon^1) - \operatorname{div}_y[\langle D \rangle + L(C)]\nabla_y \tilde{u}^\varepsilon + \frac{1}{\varepsilon}b(y) \cdot \nabla_y(\tilde{u}^\varepsilon + \varepsilon \tilde{u}_\varepsilon^1) = \varepsilon \partial_t u^1(t, t/\varepsilon, y) \quad (70)$$

which also writes, thanks to (62)

$$\partial_t(\tilde{u}^\varepsilon + \varepsilon \tilde{u}_\varepsilon^1) - \operatorname{div}_y[D\nabla_y(\tilde{u}^\varepsilon + \varepsilon \tilde{u}_\varepsilon^1)] + \frac{1}{\varepsilon}b(y) \cdot \nabla_y(\tilde{u}^\varepsilon + \varepsilon \tilde{u}_\varepsilon^1) = \varepsilon[\partial_t u^1 - \operatorname{div}_y(D\nabla_y u^1)](t, t/\varepsilon, y).$$

Combining (64) and (70), it is easily seen that

$$\begin{aligned} \partial_t(u^\varepsilon - \tilde{u}^\varepsilon - \varepsilon \tilde{u}_\varepsilon^1) - \operatorname{div}_y[D\nabla_y(u^\varepsilon - \tilde{u}^\varepsilon - \varepsilon \tilde{u}_\varepsilon^1)] + \frac{1}{\varepsilon}b(y) \cdot \nabla_y(u^\varepsilon - \tilde{u}^\varepsilon - \varepsilon \tilde{u}_\varepsilon^1) \\ = -\varepsilon[\partial_t u^1 - \operatorname{div}_y(D\nabla_y u^1)](t, t/\varepsilon, y). \end{aligned}$$

Using the energy balance together with the hypothesis $Q^{1/2}DQ^{1/2} \geq \alpha I_m$ we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u^\varepsilon - \tilde{u}^\varepsilon - \varepsilon \tilde{u}_\varepsilon^1\|_{L^2(\mathbb{R}^m)}^2 + \alpha \|\nabla_y(u^\varepsilon - \tilde{u}^\varepsilon - \varepsilon \tilde{u}_\varepsilon^1)\|_P^2 &\leq \varepsilon \|u^\varepsilon - \tilde{u}^\varepsilon - \varepsilon \tilde{u}_\varepsilon^1\|_{L^2(\mathbb{R}^m)} \\ &\times \|\partial_t u^1(t, t/\varepsilon, \cdot) - \operatorname{div}_y(D\nabla_y u^1(t, t/\varepsilon, \cdot))\|_{L^2(\mathbb{R}^m)}, \quad t \in \mathbb{R}_+. \end{aligned}$$

Notice that $(u^\varepsilon - \tilde{u}^\varepsilon - \varepsilon \tilde{u}_\varepsilon^1)|_{t=0} = u^{\text{in}} - u^{\text{in}} - 0 = 0$ and therefore, after integration with respect to $t \in [0, T]$, one gets

$$\|u^\varepsilon - \tilde{u}^\varepsilon - \varepsilon \tilde{u}_\varepsilon^1\|_{L^\infty([0, T]; L^2(\mathbb{R}^m))} \leq \varepsilon \int_0^T \|\partial_t u^1(t, t/\varepsilon, \cdot) - \operatorname{div}_y(D\nabla_y u^1(t, t/\varepsilon, \cdot))\|_{L^2(\mathbb{R}^m)} dt$$

and

$$\begin{aligned} \alpha \int_0^T |\nabla_y(u^\varepsilon - \tilde{u}^\varepsilon - \varepsilon \tilde{u}_\varepsilon^1)|_P^2 dt &\leq \varepsilon \|u^\varepsilon - \tilde{u}^\varepsilon - \varepsilon \tilde{u}_\varepsilon^1\|_{L^\infty([0, T]; L^2(\mathbb{R}^m))} \\ &\quad \times \int_0^T \|\partial_t u^1(t, t/\varepsilon, \cdot) - \operatorname{div}_y(D\nabla_y u^1(t, t/\varepsilon, \cdot))\|_{L^2(\mathbb{R}^m)} dt \\ &\leq \varepsilon^2 \left(\int_0^T \|\partial_t u^1(t, t/\varepsilon, \cdot) - \operatorname{div}_y(D\nabla_y u^1(t, t/\varepsilon, \cdot))\|_{L^2(\mathbb{R}^m)} dt \right)^2. \end{aligned}$$

We are done if the corrector $u^1(t, s, y)$ satisfies uniform estimates with respect to the fast variable s

$$u^1 \in L^\infty([0, T]; L^\infty(\mathbb{R}_s; L^2(\mathbb{R}^m))), \quad \partial_t u^1 \in L^1([0, T]; L^\infty(\mathbb{R}_s; L^2(\mathbb{R}^m)))$$

$$\operatorname{div}_y(D\nabla_y u^1) \in L^1([0, T]; L^\infty(\mathbb{R}_s; L^2(\mathbb{R}^m))), \quad \nabla_y u^1 \in L^2([0, T]; L^\infty(\mathbb{R}_s; X_P)).$$

Let us estimate the $L^2(\mathbb{R}^m)$ norm of u^1 , uniformly with respect to $(t, s) \in [0, T] \times \mathbb{R}$. Thanks to (66) we have

$$\begin{aligned} \|u^1(t, s, \cdot)\|_{L^2(\mathbb{R}^m)} &\leq \|\operatorname{div}_z(G(s)C\nabla_z v(t)) - \operatorname{div}_z(C\nabla_z v(t))\|_{L^2(\mathbb{R}^m)} \\ &\leq 2 \sup_{s \in \mathbb{R}} \|\operatorname{div}_z(G(s)C\nabla_z v(t))\|_{L^2(\mathbb{R}^m)}. \end{aligned}$$

For any $s \in \mathbb{R}$ we can write, using the formula $\operatorname{div}_z(X\xi) = \operatorname{div}_z {}^t X \cdot \xi + {}^t X : \partial_z \xi$, for any smooth matrix field X and vector field ξ

$$\begin{aligned} \operatorname{div}_z(G(s)C\nabla_z v(t)) &= \operatorname{div}_z(G(s)C {}^t R \nabla_z^R v(t)) \tag{71} \\ &= \operatorname{div}_z(RG(s)C) \cdot \nabla_z^R v(t) + RG(s)C {}^t R : \partial_z \nabla_z^R v(t) R^{-1} \\ &= \operatorname{div}_z(RG(s)C) \cdot \nabla_z^R v(t) + RG(s)C {}^t R : \nabla_z^R \otimes \nabla_z^R v(t). \end{aligned}$$

We claim that $\operatorname{div}_z(RG(s)C) = \tau(s) \operatorname{div}_y(RC)$. Indeed, for any smooth compactly supported vector field $\Phi = \Phi(y)$ we have, thanks to (21)

$$\begin{aligned} \int_{\mathbb{R}^m} \operatorname{div}_z(RG(s)C) \cdot \Phi(Y(s; z)) dz &= - \int_{\mathbb{R}^m} RG(s)C : \partial_z \{\Phi(Y(s; z))\} dz \tag{72} \\ &= - \int_{\mathbb{R}^m} RG(s)C {}^t R : (\partial_y \Phi)(Y(s; z)) \partial Y(s; z) R^{-1} dz \\ &= - \int_{\mathbb{R}^m} (RC {}^t R)(Y(s; z)) : (\partial_y \Phi R^{-1})(Y(s; z)) dz \\ &= - \int_{\mathbb{R}^m} RC {}^t R : \partial_y \Phi R^{-1} dy \\ &= - \int_{\mathbb{R}^m} RC : \partial_y \Phi dy \\ &= \int_{\mathbb{R}^m} \operatorname{div}_y(RC) \cdot \Phi(y) dy \\ &= \int_{\mathbb{R}^m} \tau(s) [\operatorname{div}_y(RC)] \cdot \Phi(Y(s; z)) dz. \end{aligned}$$

Coming back to (71) we obtain

$$\operatorname{div}_z(G(s)C\nabla_z v(t)) = \tau(s)[\operatorname{div}_y(RC)] \cdot \nabla_z^R v(t) + \tau(s)(RC^t R) : \nabla_z^R \otimes \nabla_z^R v(t) \quad (73)$$

and therefore

$$\|\operatorname{div}_z(G(s)C\nabla_z v(t))\|_{L^2(\mathbb{R}^m)} \leq \|\operatorname{div}_y(RC)\|_{L^\infty(\mathbb{R}^m)} \|\nabla_z^R v(t)\|_{L^2(\mathbb{R}^m)} + |C|_{H_Q^\infty} \|\nabla_z^R \otimes \nabla_z^R v(t)\|_{L^2(\mathbb{R}^m)}$$

saying that

$$\begin{aligned} \|u^1\|_{L^\infty([0,T];L^\infty(\mathbb{R}_s;L^2(\mathbb{R}^m)))} &\leq 2\|\operatorname{div}_y(RC)\|_{L^\infty(\mathbb{R}^m)} \|\nabla_z^R v\|_{L^\infty([0,T];L^2(\mathbb{R}^m))} \\ &\quad + 2|C|_{H_Q^\infty} \|\nabla_z^R \otimes \nabla_z^R v(t)\|_{L^\infty([0,T];L^2(\mathbb{R}^m))}. \end{aligned}$$

Similarly, taking the derivative of (66) with respect to t yields

$$\begin{aligned} \|\partial_t u^1\|_{L^1([0,T];L^\infty(\mathbb{R}_s;L^2(\mathbb{R}^m)))} &\leq 2\|\operatorname{div}_y(RC)\|_{L^\infty(\mathbb{R}^m)} \|\nabla_z^R \partial_t v\|_{L^1([0,T];L^2(\mathbb{R}^m))} \\ &\quad + 2|C|_{H_Q^\infty} \|\nabla_z^R \otimes \nabla_z^R \partial_t v(t)\|_{L^1([0,T];L^2(\mathbb{R}^m))}. \end{aligned}$$

It remains to estimate the space derivatives of u^1 . The key point is that ∇^R commutes with $\tau(s)$, i.e.

$$\nabla_z^R(\tau(s)f) = \nabla_z^R\{f(Y(s;\cdot))\} = (\nabla_y^R f)(Y(s;\cdot)) = \tau(s)(\nabla_y^R f)$$

for any smooth function $f = f(y)$. Indeed, for any $i \in \{1, \dots, m\}$ we have

$$\begin{aligned} b_i \cdot \nabla_z(\tau(s)f)(z) &= \lim_{h \rightarrow 0} \frac{f(Y(s; Y_i(h; z))) - f(Y(s; z))}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(Y_i(h; Y(s; z))) - f(Y(s; z))}{h} \\ &= b_i(Y(s; z)) \cdot (\nabla_y f)(Y(s; z)) = \tau(s)(b_i \cdot \nabla_y f)(z). \end{aligned}$$

Applying the operator ∇^R in (66) and using (71), (73) lead to

$$\begin{aligned} (\nabla_y^R u^1(t, s, \cdot))(Y(s; \cdot)) &= \nabla_z^R u^1(t, s, Y(s; \cdot)) = \nabla_z^R[\operatorname{div}_z(G(s)C\nabla_z v(t)) - \operatorname{div}_z(C\nabla_z v(t))] \\ &= \nabla_z^R[\tau(s)(\operatorname{div}_y(RC)) \cdot \nabla_z^R v(t) + \tau(s)(RC^t R) : \nabla_z^R \otimes \nabla_z^R v(t)] \\ &\quad - \nabla_z^R[\operatorname{div}_z(RC) \cdot \nabla_z^R v(t) + (RC^t R) : \nabla_z^R \otimes \nabla_z^R v(t)]. \end{aligned} \quad (74)$$

Appealing one more time to the commutation between $\tau(s)$ and ∇^R we deduce that for any $k \in \{1, \dots, m\}$

$$\begin{aligned} b_k \cdot \nabla_z[\tau(s)(\operatorname{div}_y(RC)) \cdot \nabla_z^R v(t) + \tau(s)(RC^t R) : \nabla_z^R \otimes \nabla_z^R v(t)] &\quad (75) \\ &= \tau(s)(b_k \cdot \nabla_y \operatorname{div}_y(RC)) \cdot \nabla_z^R v(t) + \tau(s) \operatorname{div}_y(RC) \cdot (b_k \cdot \nabla_z \nabla_z^R v(t)) \\ &\quad + \tau(s)(b_k \cdot \nabla_y(RC^t R)) : \nabla_z^R \otimes \nabla_z^R v(t) + \tau(s)(RC^t R) : b_k \cdot \nabla_z(\nabla_z^R \otimes \nabla_z^R v(t)). \end{aligned}$$

Therefore there is a constant K depending on $\|\operatorname{div}_y(RC)\|_{L^\infty(\mathbb{R}^m)} + \|RC^t R\|_{L^\infty(\mathbb{R}^m)} + \sum_{k=1}^m \|b_k \cdot \nabla_y \operatorname{div}_y(RC)\|_{L^\infty(\mathbb{R}^m)} + \sum_{k=1}^m \|b_k \cdot \nabla_y(RC^t R)\|_{L^\infty(\mathbb{R}^m)}$ such that

$$\begin{aligned} \|\nabla_y^R u^1(t, \cdot, \cdot)\|_{L^\infty(\mathbb{R}_s;L^2(\mathbb{R}^m))} &\leq K\{\|\nabla_z^R v(t)\|_{L^2(\mathbb{R}^m)} + \|\nabla_z^R \otimes \nabla_z^R v(t)\|_{L^2(\mathbb{R}^m)} \\ &\quad + \|\nabla_z^R \otimes \nabla_z^R \otimes \nabla_z^R v(t)\|_{L^2(\mathbb{R}^m)}\}. \end{aligned}$$

We deduce that

$$\begin{aligned} \|\nabla_y u^1\|_{L^2([0,T];L^\infty(\mathbb{R}_s;X_P))} &\leq K\{\|\nabla_z^R v\|_{L^2([0,T];L^2(\mathbb{R}^m))} + \|\nabla_z^R \otimes \nabla_z^R v\|_{L^2([0,T];L^2(\mathbb{R}^m))} \\ &\quad + \|\nabla_z^R \otimes \nabla_z^R \otimes \nabla_z^R v\|_{L^2([0,T];L^2(\mathbb{R}^m))}\}. \end{aligned}$$

For the second space derivatives of u^1 , we write as before

$$\operatorname{div}_y(D\nabla_y u^1) = \operatorname{div}_y(D^t R \nabla_y^R u^1) = \operatorname{div}_y(RD) \cdot \nabla_y^R u^1 + RD^t R : \nabla_y^R \otimes \nabla_y^R u^1.$$

Notice that $\operatorname{div}_y(RD) \cdot \nabla_y^R u^1$ belongs to $L^1([0, T]; L^\infty(\mathbb{R}_s; L^2(\mathbb{R}^m)))$ since by hypotheses $\operatorname{div}_y(RD) \in L^\infty(\mathbb{R}^m)$ and we already know that $\nabla_y u^1 \in L^2([0, T]; L^\infty(\mathbb{R}_s; X_P))$. As the matrix field $RD^t R$ belongs to $L^\infty(\mathbb{R}^m)$, it remains to check that $\nabla_y^R \otimes \nabla_y^R u^1$ belongs to $L^1([0, T]; L^\infty(\mathbb{R}_s; L^2(\mathbb{R}^m)))$. For doing that, we apply one more time the operator ∇_z^R in (74), or equivalently the operator $b_l \cdot \nabla_z$ in (75). Using again the commutation between $\tau(s)$ and $b_l \cdot \nabla_z$ we obtain

$$\begin{aligned} & b_l \cdot \nabla_z \{ b_k \cdot \nabla_z [\tau(s)(\operatorname{div}_y(RC)) \cdot \nabla_z^R v(t) + \tau(s)(RC^t R) : \nabla_z^R \otimes \nabla_z^R v(t)] \} \\ &= [b_l \cdot \nabla_y (b_k \cdot \nabla_y (\operatorname{div}_y(RC)))]_s \cdot \nabla_z^R v(t) + [b_k \cdot \nabla_y (\operatorname{div}_y(RC))]_s \cdot [b_l \cdot \nabla_z (\nabla_z^R v(t))] \\ &+ [b_l \cdot \nabla_y (\operatorname{div}_y(RC))]_s \cdot (b_k \cdot \nabla_z (\nabla_z^R v(t))) + (\operatorname{div}_y(RC))_s \cdot \{ b_l \cdot \nabla_z [b_k \cdot \nabla_z (\nabla_z^R v(t))] \} \\ &+ [b_l \cdot \nabla_y (b_k \cdot \nabla_y (RC^t R))]_s : \nabla_z^R \otimes \nabla_z^R v(t) + [b_k \cdot \nabla_y (RC^t R)]_s : b_l \cdot \nabla_z (\nabla_z^R \otimes \nabla_z^R v(t)) \\ &+ [b_l \cdot \nabla_y (RC^t R)]_s : b_k \cdot \nabla_z (\nabla_z^R \otimes \nabla_z^R v(t)) + (RC^t R)_s : b_k \cdot \nabla_z (b_l \cdot \nabla_z (\nabla_z^R \otimes \nabla_z^R v(t))) \end{aligned}$$

which belongs to $L^1([0, T]; L^\infty(\mathbb{R}_s; L^2(\mathbb{R}^m)))$, thanks to the hypotheses on the matrix field C and the solution v . \square

A Proofs of Propositions 4.4, 4.5

Proof of Proposition 4.4. For any $i, j, k \in \{1, \dots, m\}$ we introduce the notations $u_i^\varepsilon = b_i \cdot \nabla_y u^\varepsilon$, $v_i^\varepsilon = b_i \cdot \nabla_z v^\varepsilon$, $u_{ij}^\varepsilon = b_j \cdot \nabla_y (b_i \cdot \nabla_y u^\varepsilon)$, $v_{ij}^\varepsilon = b_j \cdot \nabla_z (b_i \cdot \nabla_z v^\varepsilon)$, $u_{ijk}^\varepsilon = b_k \cdot \nabla_y (b_j \cdot \nabla_y (b_i \cdot \nabla_y u^\varepsilon))$ and $v_{ijk}^\varepsilon = b_k \cdot \nabla_z (b_j \cdot \nabla_z (b_i \cdot \nabla_z v^\varepsilon))$. With these notations, the equation (34) becomes

$$\partial_t u_i^\varepsilon - \operatorname{div}_y(D(y)\nabla_y u_i^\varepsilon) + \frac{1}{\varepsilon} b \cdot \nabla_y u_i^\varepsilon = \operatorname{div}_y([b_i, D]^t R \nabla_y^R u_i^\varepsilon) + D^t R \nabla_y^R u_i^\varepsilon \cdot \nabla_y \operatorname{div}_y b_i.$$

Taking now the directional derivative $b_j \cdot \nabla_y$, yields

$$\begin{aligned} \partial_t u_{ij}^\varepsilon - \operatorname{div}_y(D(y)\nabla_y u_{ij}^\varepsilon) + \frac{1}{\varepsilon} b \cdot \nabla_y u_{ij}^\varepsilon &= \operatorname{div}_y([b_j, D]^t R \nabla_y^R u_i^\varepsilon) + RD^t R \nabla_y^R u_i^\varepsilon \cdot \nabla_y^R \operatorname{div}_y b_j \\ &+ b_j \cdot \nabla_y \operatorname{div}_y([b_i, D] \nabla_y u^\varepsilon) + b_j \cdot \nabla_y (D^t R \nabla_y^R u^\varepsilon \cdot \nabla_y \operatorname{div}_y b_i). \end{aligned} \quad (76)$$

Thanks to the commutation formula (17), we have

$$\begin{aligned} b_j \cdot \nabla_y \operatorname{div}_y([b_i, D] \nabla_y u^\varepsilon) &= [b_j \cdot \nabla_y, \operatorname{div}_y([b_i, D] \nabla_y)] u^\varepsilon + \operatorname{div}_y([b_i, D] \nabla_y (b_j \cdot \nabla_y u^\varepsilon)) \\ &= \operatorname{div}_y([b_j, [b_i, D]] \nabla_y u^\varepsilon) + [b_i, D] \nabla_y u^\varepsilon \cdot \nabla_y \operatorname{div}_y b_j + \operatorname{div}_y([b_i, D] \nabla_y (b_j \cdot \nabla_y u^\varepsilon)) \\ &= \operatorname{div}_y([b_j, [b_i, D]]^t R \nabla_y^R u^\varepsilon) + R[b_i, D]^t R \nabla_y^R u^\varepsilon \cdot \nabla_y^R \operatorname{div}_y b_j + \operatorname{div}_y([b_i, D]^t R \nabla_y^R u_j^\varepsilon). \end{aligned} \quad (77)$$

Combining (76), (77) we obtain

$$\begin{aligned} \partial_t u_{ij}^\varepsilon - \operatorname{div}_y(D(y)\nabla_y u_{ij}^\varepsilon) + \frac{1}{\varepsilon} b \cdot \nabla_y u_{ij}^\varepsilon &= \operatorname{div}_y([b_j, D]^t R \nabla_y^R u_i^\varepsilon) + RD^t R \nabla_y^R u_i^\varepsilon \cdot \nabla_y^R \operatorname{div}_y b_j \\ &+ \operatorname{div}_y([b_j, [b_i, D]]^t R \nabla_y^R u^\varepsilon) + R[b_i, D]^t R \nabla_y^R u^\varepsilon \cdot \nabla_y^R \operatorname{div}_y b_j \\ &+ \operatorname{div}_y([b_i, D]^t R \nabla_y^R u_j^\varepsilon) + b_j \cdot \nabla_y (D^t R \nabla_y^R u^\varepsilon \cdot \nabla_y \operatorname{div}_y b_i). \end{aligned}$$

Multiplying by u_{ij}^ε and integrating on \mathbb{R}^m lead to

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^m} (u_{ij}^\varepsilon)^2 dy + \int_{\mathbb{R}^m} D \nabla_y u_{ij}^\varepsilon \cdot \nabla_y u_{ij}^\varepsilon dy &= - \int_{\mathbb{R}^m} R[b_j, D]^t R \nabla_y^R u_i^\varepsilon \cdot \nabla_y^R u_{ij}^\varepsilon dy \\
&+ \int_{\mathbb{R}^m} R D^t R \nabla_y^R u_i^\varepsilon \cdot \nabla_y^R \operatorname{div}_y b_j u_{ij}^\varepsilon dy + \int_{\mathbb{R}^m} \operatorname{div}_y ([b_j, [b_i, D]]^t R \nabla_y^R u^\varepsilon) u_{ij}^\varepsilon dy \\
&+ \int_{\mathbb{R}^m} R[b_i, D]^t R \nabla_y^R u^\varepsilon \cdot \nabla_y^R (\operatorname{div}_y b_j) u_{ij}^\varepsilon dy + \int_{\mathbb{R}^m} u_{ij}^\varepsilon \operatorname{div}_y ([b_i, D]^t R \nabla_y^R u_j^\varepsilon) dy \\
&+ \int_{\mathbb{R}^m} u_{ij}^\varepsilon b_j \cdot \nabla_y (D^t R \nabla_y^R u^\varepsilon \cdot \nabla_y (\operatorname{div}_y b_i)) dy \\
&=: K_{ij}^1 + K_{ij}^2 + K_{ij}^3 + K_{ij}^4 + K_{ij}^5 + K_{ij}^6.
\end{aligned} \tag{78}$$

By hypothesis (25) we have, cf. (33)

$$D \nabla_y u_{ij}^\varepsilon \cdot \nabla_y u_{ij}^\varepsilon \geq \alpha |P^{1/2} \nabla_y u_{ij}^\varepsilon|^2 = \alpha |\nabla_y^R u_{ij}^\varepsilon|^2. \tag{79}$$

Exactly as before we obtain

$$\nabla_y^R (u_{ij}^\varepsilon) = b_j \cdot \nabla_y (\nabla_y^R u_i^\varepsilon) - \mathcal{A}_j \nabla_y^R u_i^\varepsilon$$

which allows us to replace K_{ij}^1 by

$$\begin{aligned}
K_{ij}^1 &= \int_{\mathbb{R}^m} R[b_j, D]^t R \nabla_y^R u_i^\varepsilon(t) \cdot \mathcal{A}_j \nabla_y^R u_i^\varepsilon(t) dy \\
&+ \frac{1}{2} \int_{\mathbb{R}^m} b_j \cdot \nabla_y (R[b_j, D]^t R) : \nabla_y^R u_i^\varepsilon(t) \otimes \nabla_y^R u_i^\varepsilon(t) dy \\
&+ \frac{1}{2} \int_{\mathbb{R}^m} (\operatorname{div}_y b_j) R[b_j, D]^t R : \nabla_y^R u_i^\varepsilon(t) \otimes \nabla_y^R u_i^\varepsilon(t) dy.
\end{aligned}$$

Thanks to our hypotheses, there is a constant C_3 (not depending on ε or t) such that

$$\sum_{j=1}^m \sum_{i=1}^m |K_{ij}^1| \leq C_3 \|\nabla_y^R \otimes \nabla_y^R u^\varepsilon(t)\|_{L^2(\mathbb{R}^m)}^2, \quad t \in \mathbb{R}_+, \quad \varepsilon > 0.$$

Obviously, there is a constant C_4 (not depending on ε or t) such that

$$\sum_{j=1}^m \sum_{i=1}^m |K_{ij}^2| \leq C_4 \|\nabla_y^R \otimes \nabla_y^R u^\varepsilon(t)\|_{L^2(\mathbb{R}^m)}^2, \quad t \in \mathbb{R}_+, \quad \varepsilon > 0.$$

We consider now the term K_{ij}^3 , which writes

$$\begin{aligned}
K_{ij}^3 &= \int_{\mathbb{R}^m} u_{ij}^\varepsilon(t) \operatorname{div}_y (R[b_j, [b_i, D]]) \cdot \nabla_y^R u^\varepsilon(t) dy \\
&+ \int_{\mathbb{R}^m} u_{ij}^\varepsilon(t) R[b_j, [b_i, D]]^t R : \nabla_y^R \otimes \nabla_y^R u^\varepsilon(t) dy.
\end{aligned}$$

It is easily seen that there is a constant C_5 (not depending on ε or t) such that

$$\begin{aligned}
\sum_{j=1}^m \sum_{i=1}^m (|K_{ij}^3| + |K_{ij}^4| + |K_{ij}^6|) &\leq C_5 \left(\|\nabla_y^R \otimes \nabla_y^R u^\varepsilon(t)\|_{L^2(\mathbb{R}^m)} \|\nabla_y^R u^\varepsilon(t)\|_{L^2(\mathbb{R}^m)} \right. \\
&\quad \left. + \|\nabla_y^R \otimes \nabla_y^R u^\varepsilon(t)\|_{L^2(\mathbb{R}^m)}^2 \right), \quad t \in \mathbb{R}_+, \quad \varepsilon > 0.
\end{aligned}$$

It remains to estimate the term K_{ij}^5 . For any $i, j \in \{1, \dots, m\}$ we have

$$u_{ij}^\varepsilon(t) = u_{ji}^\varepsilon(t) - \sum_{k=1}^m \alpha_{ij}^k u_k^\varepsilon(t)$$

and therefore K_{ij}^5 writes

$$\begin{aligned} K_{ij}^5 &= - \sum_{k=1}^m \int_{\mathbb{R}^m} \alpha_{ij}^k u_k^\varepsilon(t) \operatorname{div}_y ([b_i, D]^t R \nabla_y^R u_j^\varepsilon(t)) \, dy \\ &\quad + \int_{\mathbb{R}^m} u_{ji}^\varepsilon(t) \operatorname{div}_y ([b_i, D]^t R \nabla_y^R u_j^\varepsilon(t)) \, dy \\ &= \sum_{k=1}^m \int_{\mathbb{R}^m} \nabla_y^R (\alpha_{ij}^k u_k^\varepsilon(t)) \cdot R [b_i, D]^t R \nabla_y^R u_j^\varepsilon(t) \, dy \\ &\quad - \int_{\mathbb{R}^m} \nabla_y^R (b_i \cdot \nabla_y u_j^\varepsilon(t)) \cdot R [b_i, D]^t R \nabla_y^R u_j^\varepsilon(t) \, dy =: K_{ij}^7 + K_{ij}^8. \end{aligned}$$

Clearly, there is a constant C_6 (not depending on ε or t) such that

$$\begin{aligned} \sum_{j=1}^m \sum_{i=1}^m |K_{ij}^7| &\leq C_6 (\|\nabla_y^R \otimes \nabla_y^R u^\varepsilon(t)\|_{L^2(\mathbb{R}^m)} \|\nabla_y^R u^\varepsilon(t)\|_{L^2(\mathbb{R}^m)} \\ &\quad + \|\nabla_y^R \otimes \nabla_y^R u^\varepsilon(t)\|_{L^2(\mathbb{R}^m)}^2), \quad t \in \mathbb{R}_+, \quad \varepsilon > 0. \end{aligned}$$

For the last term K_{ij}^8 we use (40) and we get as before

$$\begin{aligned} K_{ij}^8 &= \int_{\mathbb{R}^m} \mathcal{A}_i \nabla_y^R u_j^\varepsilon(t) \cdot R [b_i, D]^t R \nabla_y^R u_j^\varepsilon(t) \, dy \\ &\quad - \int_{\mathbb{R}^m} b_i \cdot \nabla_y (\nabla_y^R u_j^\varepsilon(t)) \cdot R [b_i, D]^t R \nabla_y^R u_j^\varepsilon(t) \, dy \\ &= \int_{\mathbb{R}^m} \mathcal{A}_i \nabla_y^R u_j^\varepsilon(t) \cdot R [b_i, D]^t R \nabla_y^R u_j^\varepsilon(t) \, dy \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^m} (\operatorname{div}_y b_i) R [b_i, D]^t R : \nabla_y^R u_j^\varepsilon(t) \otimes \nabla_y^R u_j^\varepsilon(t) \, dy \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^m} b_i \cdot \nabla_y (R [b_i, D]^t R) : \nabla_y^R u_j^\varepsilon(t) \otimes \nabla_y^R u_j^\varepsilon(t) \, dy \end{aligned}$$

implying that there is a constant C_7 (not depending on ε or t) such that

$$\sum_{j=1}^m \sum_{i=1}^m |K_{ij}^8| \leq C_7 \|\nabla_y^R \otimes \nabla_y^R u^\varepsilon(t)\|_{L^2(\mathbb{R}^m)}^2, \quad t \in \mathbb{R}_+, \quad \varepsilon > 0.$$

Putting together (78), (79) and the estimates for all the terms $K_{ij}^r, i, j \in \{1, \dots, m\}, r \in \{1, \dots, 8\}$ we deduce that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla_y^R \otimes \nabla_y^R u^\varepsilon\|_{L^2(\mathbb{R}^m)}^2 + \alpha \|\nabla_y^R \otimes \nabla_y^R \otimes \nabla_y^R u^\varepsilon\|_{L^2(\mathbb{R}^m)}^2 &\leq C \|\nabla_y^R \otimes \nabla_y^R u^\varepsilon\|_{L^2(\mathbb{R}^m)} \\ &\times (\|\nabla_y^R \otimes \nabla_y^R u^\varepsilon\|_{L^2(\mathbb{R}^m)} + \|\nabla_y^R u^\varepsilon\|_{L^2(\mathbb{R}^m)}), \quad C = \sum_{r=3}^7 C_r. \end{aligned}$$

Applying Gronwall's lemma yields, for some constant C_T depending only on T and the coefficients α_{ij}^k , the vector fields b_i and the matrix field D

$$\begin{aligned} \|\nabla_y^R \otimes \nabla_y^R u^\varepsilon\|_{L^\infty([0,T];L^2(\mathbb{R}^m))} &= \|\nabla_z^R \otimes \nabla_z^R v^\varepsilon\|_{L^\infty([0,T];L^2(\mathbb{R}^m))} \\ &\leq C_T (\|\nabla_y^R u^{\text{in}}\|_{L^2(\mathbb{R}^m)} + \|\nabla_y^R \otimes \nabla_y^R u^{\text{in}}\|_{L^2(\mathbb{R}^m)}), \quad \varepsilon > 0 \end{aligned}$$

and

$$\begin{aligned} \|\nabla_y^R \otimes \nabla_y^R \otimes \nabla_y^R u^\varepsilon\|_{L^2([0,T];L^2(\mathbb{R}^m))} &= \|\nabla_z^R \otimes \nabla_z^R \otimes \nabla_z^R v^\varepsilon\|_{L^2([0,T];L^2(\mathbb{R}^m))} \\ &\leq C_T (\|\nabla_y^R u^{\text{in}}\|_{L^2(\mathbb{R}^m)} + \|\nabla_y^R \otimes \nabla_y^R u^{\text{in}}\|_{L^2(\mathbb{R}^m)}), \quad \varepsilon > 0. \end{aligned}$$

For estimating $\partial_t \nabla_z^R v^\varepsilon$ we take the directional derivative $b_i \cdot \nabla_z$ in (43). As the vector fields b_i, b are in involution, that is $[b_i, b] = 0$, the flows Y_i, Y are commuting [2, 3], and therefore the derivative along b_i commutes with the translation along the flow of b (take the derivative with respect to h , at $h = 0$, of the equality $f(Y(s; Y_i(h; \cdot))) = f(Y_i(h; Y(s; \cdot)))$). We deduce that

$$\partial_t (b_i \cdot \nabla_z v^\varepsilon)(t, z) = [b_i \cdot \nabla_y \text{div}_y (D \nabla_y u^\varepsilon)](Y(t/\varepsilon; z))$$

which implies

$$\begin{aligned} \|\partial_t \nabla_z^R v^\varepsilon(t)\|_{L^2(\mathbb{R}^m)} &= \|\nabla_y^R \text{div}_y (D \nabla_y u^\varepsilon(t))\|_{L^2(\mathbb{R}^m)} \\ &= \|\nabla_y^R \text{div}_y (D^t R \nabla_y^R u^\varepsilon(t))\|_{L^2(\mathbb{R}^m)} \\ &= \|\nabla_y^R (\text{div}_y (RD) \cdot \nabla_y^R u^\varepsilon(t)) + \nabla_y^R (RD^t R : \nabla_y^R \otimes \nabla_y^R u^\varepsilon(t))\|_{L^2}. \end{aligned}$$

We claim that for any $i \in \{1, \dots, m\}$ we have the equality

$$\text{div}_y (RD)_i = \sum_{j=1}^m b_j \cdot \nabla_y (RD^t R)_{ij} + \sum_{j=1}^m (RD^t R)_{ij} \text{div}_y b_j. \quad (80)$$

Indeed, for any $i \in \{1, \dots, m\}$ we can write (here $(e_k)_{1 \leq k \leq m}$ stands for the canonical basis of \mathbb{R}^m)

$$\begin{aligned} (\text{div}_y RD)_i &= \text{div}_y (RD^t R^t R^{-1})_i = \sum_{k=1}^m \partial_{y_k} \sum_{j=1}^m (RD^t R)_{ij} R_{kj}^{-1} \\ &= \sum_{j=1}^m \sum_{k=1}^m \partial_{y_k} [(RD^t R)_{ij} (b_j \cdot e_k)] \\ &= \sum_{j=1}^m \text{div}_y [(RD^t R)_{ij} b_j]. \end{aligned}$$

Thanks to our hypotheses and formula (80), it is easily seen that there is a constant depending only on the coefficients α_{ij}^k , the vector fields b_i and the matrix field D such that

$$\begin{aligned} \|\partial_t \nabla_z^R v^\varepsilon\|_{L^2(\mathbb{R}^m)} &\leq C (\|\nabla_y^R u^\varepsilon(t)\|_{L^2(\mathbb{R}^m)} + \|\nabla_y^R \otimes \nabla_y^R u^\varepsilon(t)\|_{L^2(\mathbb{R}^m)} \\ &\quad + \|\nabla_y^R \otimes \nabla_y^R \otimes \nabla_y^R u^\varepsilon(t)\|_{L^2(\mathbb{R}^m)}). \end{aligned}$$

Thanks to the uniform estimates satisfied by $\nabla_y^R u^\varepsilon, \nabla_y^R \otimes \nabla_y^R u^\varepsilon$ in $L^\infty([0, T]; L^2(\mathbb{R}^m))$, and by $\nabla_y^R \otimes \nabla_y^R \otimes \nabla_y^R u^\varepsilon$ in $L^2([0, T]; L^2(\mathbb{R}^m))$, we obtain that, for any $T \in \mathbb{R}_+$, there is a constant \tilde{C}_T (depending only on T and $\alpha, \alpha_{ij}^k, b_i, D$) such that

$$\|\partial_t \nabla_z^R v^\varepsilon\|_{L^2([0,T];L^2(\mathbb{R}^m))} \leq \tilde{C}_T (\|\nabla_y^R u^{\text{in}}\|_{L^2(\mathbb{R}^m)} + \|\nabla_y^R \otimes \nabla_y^R u^{\text{in}}\|_{L^2(\mathbb{R}^m)}), \quad \varepsilon > 0.$$

□

Proof of Proposition 4.5. We perform exactly the same computations as in the proof of Proposition 4.3 (it does not matter that (26) has no the term $\frac{1}{\varepsilon}b \cdot \nabla_z v$). Nevertheless, we have to check that all the hypotheses on the matrix field D in Proposition 4.3 are also satisfied by the matrix field $\langle D \rangle$. By Theorem 2.1 we deduce that ${}^t \langle D \rangle = \langle D \rangle$, $\langle D \rangle \in H_Q^\infty$, $|\langle D \rangle|_{H_Q^\infty} \leq |D|_{H_Q^\infty}$, $Q^{1/2} \langle D \rangle Q^{1/2} \geq \alpha I_m$ and therefore the hypotheses (25) corresponding to the matrix field $\langle D \rangle$ hold true. We also need to show that $R[b_i, \langle D \rangle] {}^t R \in L^\infty(\mathbb{R}^m)$, $\operatorname{div}_y(R \langle D \rangle) \in L^\infty(\mathbb{R}^m)$, $\sum_{i=1}^m b_i \cdot \nabla_y(R[b_i, \langle D \rangle] {}^t R) \in L^\infty(\mathbb{R}^m)$, provided that the same conditions are satisfied by the matrix field D . The key point is that for any $i \in \{1, \dots, m\}$, the groups $(G_i(h))_{h \in \mathbb{R}}$, $(G(s))_{s \in \mathbb{R}}$ are commuting, where $G_i(h)$ is defined by

$$G_i(h)A = \partial Y_i^{-1}(h; \cdot)A(Y_i(h; \cdot)) {}^t \partial Y_i^{-1}(h; \cdot).$$

It is easily seen that for any $s, h \in \mathbb{R}$

$$\begin{aligned} (G(s) \circ G_i(h))A &= G(s)(G_i(h)A) = \partial Y^{-1}(s; \cdot)(G_i(h)A)(Y(s; \cdot)) {}^t \partial Y^{-1}(s; \cdot) \\ &= \partial Y^{-1}(s; \cdot) \partial Y_i^{-1}(h; Y(s; \cdot))A(Y_i(h; Y(s; \cdot))) {}^t \partial Y_i^{-1}(h; Y(s; \cdot)) {}^t \partial Y^{-1}(s; \cdot) \end{aligned} \quad (81)$$

and

$$\begin{aligned} (G_i(h) \circ G(s))A &= G_i(h)(G(s)A) = \partial Y_i^{-1}(h; \cdot)(G(s)A)(Y_i(h; \cdot)) {}^t \partial Y_i^{-1}(h; \cdot) \\ &= \partial Y_i^{-1}(h; \cdot) \partial Y^{-1}(s; Y_i(h; \cdot))A(Y(s; Y_i(h; \cdot))) {}^t \partial Y^{-1}(s; Y_i(h; \cdot)) {}^t \partial Y_i^{-1}(h; \cdot). \end{aligned} \quad (82)$$

By the involution between b and b_i , we know that

$$Y_i(h; Y(s; \cdot)) = Y(s; Y_i(h; \cdot)) \quad (83)$$

and by differentiation one gets

$$\partial Y_i(h; Y(s; \cdot)) \partial Y(s; \cdot) = \partial Y(s; Y_i(h; \cdot)) \partial Y_i(h; \cdot)$$

which also writes

$$\partial Y^{-1}(s; \cdot) \partial Y_i^{-1}(h; Y(s; \cdot)) = \partial Y_i^{-1}(h; \cdot) \partial Y^{-1}(s; Y_i(h; \cdot)). \quad (84)$$

Combining (81), (82), (83), (84) we obtain the commutation property between the groups $(G_i(h))_{h \in \mathbb{R}}$, $(G(s))_{s \in \mathbb{R}}$, for any $i \in \{1, \dots, m\}$. Notice that the hypothesis $R[b_i, D] {}^t R \in L^\infty(\mathbb{R}^m)$, or equivalently $[b_i, D] \in H_Q^\infty$, should be understood in $\mathcal{D}'(\mathbb{R}^m)$, that is, there is a matrix field, denoted $[b_i, D]$, which belongs to H_Q^∞ , such that for any $A \in C_c^1(\mathbb{R}^m)$ we have

$$\int_{\mathbb{R}^m} D : (-b_i \cdot \nabla_y A - (\operatorname{div}_y b_i)A - {}^t \partial b_i A - A \partial b_i) dy = \int_{\mathbb{R}^m} [b_i, D] : A dy. \quad (85)$$

We introduce the operator $L_i(D) = [b_i, D]$ and its formal adjoint

$$L_i^*(A) = -b_i \cdot \nabla_y A - (\operatorname{div}_y b_i)A - {}^t \partial b_i A - A \partial b_i, \quad A \in C_c^1(\mathbb{R}^m).$$

Using these notations, (85) becomes

$$\int_{\mathbb{R}^m} D : L_i^*(A) dy = \int_{\mathbb{R}^m} L_i(D) : A dy, \quad A \in C_c^1(\mathbb{R}^m).$$

Notice that H_Q^∞ is the topological dual of the space

$$H_P^1 = \left\{ A : \mathbb{R}^m \rightarrow \mathcal{M}_m(\mathbb{R}) \text{ measurable} : \int_{\mathbb{R}^m} (P(y)A(y) : A(y)P(y))^{1/2} dy < +\infty \right\}$$

and thus $[b_i, D] \in H_Q^\infty$ iff there is a constant C_i such that $\int_{\mathbb{R}^m} D : L_i^*(A) \, dy \leq C_i |A|_{H_P^1}$ for any $A \in C_c^1(\mathbb{R}^m)$. A straightforward computation shows that if $B \in H_Q^\infty$ is such that $G_i(h)B \in H_Q^\infty$ for any $h \in \mathbb{R}$ and $h \in \mathbb{R}^* \rightarrow (G_i(h)B - B)/h$ is bounded in H_Q^∞ , then $(G_i(h)B - B)/h$ converges toward $L_i(B)$ weakly \star in H_Q^∞ , when $h \rightarrow 0$. Indeed, for any $A \in C_c^1(\mathbb{R}^m) \subset H_P^1$, we have

$$\begin{aligned}
& \int_{\mathbb{R}^m} \frac{G_i(h)B - B}{h} : A \, dy = \int_{\mathbb{R}^m} \frac{\partial Y_i^{-1}(h; y) B(Y_i(h; y))^t \partial Y_i^{-1}(h; y) - B(y)}{h} : A(y) \, dy \\
& = \int_{\mathbb{R}^m} \frac{\partial Y_i(-h; Y_i(h; y)) B(Y_i(h; y))^t \partial Y_i(-h; Y_i(h; y)) - B(y)}{h} : A(y) \, dy \\
& = \frac{1}{h} \int_{\mathbb{R}^m} \left\{ \det(\partial Y_i(-h; z)) \partial Y_i(-h; z) B(z)^t \partial Y_i(-h; z) : A(Y_i(-h; z)) - B(z) : A(z) \right\} dz \\
& = \int_{\mathbb{R}^m} \frac{\det(\partial Y_i(-h; z)) - 1}{h} \partial Y_i(-h; z) B(z)^t \partial Y_i(-h; z) : A(Y_i(-h; z)) \, dz \\
& + \int_{\mathbb{R}^m} B(z) : \frac{^t \partial Y_i(-h; z) A(Y_i(-h; z)) \partial Y_i(-h; z) - A(z)}{h} \, dz \\
& \xrightarrow{h \rightarrow 0} - \int_{\mathbb{R}^m} \operatorname{div}_z b_i B(z) : A(z) \, dz - \int_{\mathbb{R}^m} B(z) : (b_i \cdot \nabla_z A + ^t \partial b_i A + A \partial b_i) \, dz \\
& = \int_{\mathbb{R}^m} B(z) : L_i^*(A) \, dz.
\end{aligned}$$

We deduce that any weak \star limit point in H_Q^∞ satisfies

$$\int_{\mathbb{R}^m} \operatorname{w} \star \lim_{h_k \rightarrow 0} \frac{G_i(h_k)B - B}{h_k} : A \, dy = \int_{\mathbb{R}^m} B : L_i^*(A) \, dy$$

for any $A \in C_c^1(\mathbb{R}^m)$. Therefore all the family $(G_i(h)B - B)/h$ converges weakly \star in H_Q^∞ , as $h \rightarrow 0$, and $L_i(B) = \lim_{h \rightarrow 0} \frac{G_i(h)B - B}{h}$, weakly \star in H_Q^∞ .

We claim that for any $s \in \mathbb{R}$, we have $L_i(G(s)D) = G(s)(L_i(D))$, that is

$$\int_{\mathbb{R}^m} G(s)D : L_i^*(A) \, dy = \int_{\mathbb{R}^m} G(s)L_i(D) : A \, dy, \quad A \in C_c^1(\mathbb{R}^m). \quad (86)$$

By density arguments (notice that $B_n \rightharpoonup B$ weakly \star in H_Q^∞ , implies $G(s)B_n \rightharpoonup G(s)B$ weakly \star in H_Q^∞ , for any $s \in \mathbb{R}$) it is enough to show that $L_i(G(s)B) = G(s)L_i(B)$ for any smooth, compactly supported matrix field B . Let us consider a smooth, compactly supported matrix field B . Obviously, for any $h \in \mathbb{R}^*$ we have $G_i(h)B \in H_Q^\infty$ and $\frac{G_i(h)B - B}{h} \rightharpoonup L_i(B)$ weakly \star in H_Q^∞ , when $h \rightarrow 0$. We deduce that $G_i(h)G(s)B = G(s)G_i(h)B \in H_Q^\infty$ for any $h \in \mathbb{R}$ and

$$\frac{G_i(h)G(s)B - G(s)B}{h} = \frac{G(s)G_i(h)B - G(s)B}{h} = G(s) \frac{G_i(h)B - B}{h} \xrightarrow{h \rightarrow 0} G(s)L_i(B)$$

weakly \star in H_Q^∞ . By the previous remark, we obtain $L_i(G(s)B) = G(s)L_i(B)$ for any $s \in \mathbb{R}$. Now it is easily seen that $L_i(\langle D \rangle) = [b_i, \langle D \rangle] \in H_Q^\infty$, if $L_i(D) = [b_i, D] \in H_Q^\infty$. Indeed, averaging (86) with respect to s one gets

$$\int_{\mathbb{R}^m} \frac{1}{S} \int_0^S G(s)D \, ds : L_i^*(A) \, dy = \int_{\mathbb{R}^m} \frac{1}{S} \int_0^S G(s)L_i(D) \, ds : A \, dy, \quad A \in C_c^1(\mathbb{R}^m).$$

Taking into account that $\frac{1}{S} \int_0^S G(s)D \, ds \rightarrow \langle D \rangle$ in $H_{Q, \text{loc}}$, when $S \rightarrow +\infty$, that $PL_i^*(A)P \in H_Q$ and noticing that

$$\left| \frac{1}{S} \int_0^S G(s)L_i(D) \, ds \right|_{H_Q^\infty} \leq \frac{1}{S} \int_0^S |G(s)L_i(D)|_{H_Q^\infty} \, ds = |L_i(D)|_{H_Q^\infty}$$

we deduce that any weak \star limit point in H_Q^∞ of $\left(\frac{1}{S} \int_0^S G(s)L_i(D) ds\right)_S$ satisfies

$$\int_{\mathbb{R}^m} \langle D \rangle : L_i^*(A) dy = \int_{\mathbb{R}^m} w \star \lim_{S \rightarrow +\infty} \frac{1}{S} \int_0^S G(s)L_i(D) ds : A dy$$

for any $A \in C_c^1(\mathbb{R}^m) \subset H_P^1$, saying that $L_i(\langle D \rangle) = \lim_{S \rightarrow +\infty} \frac{1}{S} \int_0^S G(s)L_i(D) ds$, weakly \star in H_Q^∞ . In particular $|L_i(\langle D \rangle)|_{H_Q^\infty} \leq |L_i(D)|_{H_Q^\infty}$.

We concentrate now on the hypothesis $\sum_{i=1}^m b_i \cdot \nabla_y (R[b_i, D]^t R) \in L^\infty(\mathbb{R}^m)$. For any $s \in \mathbb{R}$ we have

$$RG(s)[b_i, D]^t R = R \partial Y^{-1}(s; \cdot)[b_i, D](Y(s; \cdot))^t \partial Y^{-1}(s; \cdot)^t R = (R[b_i, D]^t R)(Y(s; \cdot))$$

and since $[b_i, b] = 0$, we obtain

$$b_i \cdot \nabla_z (RG(s)[b_i, D]^t R) = b_i \cdot \nabla_z ((R[b_i, D]^t R) \circ Y(s; \cdot)) = (b_i \cdot \nabla_y (R[b_i, D]^t R)) \circ Y(s; \cdot).$$

Multiplying by a smooth compactly supported matrix field $A \in C_c^1(\mathbb{R}^m)$ and averaging with respect to s one gets

$$\begin{aligned} - \int_{\mathbb{R}^m} \frac{\operatorname{div}_z b_i}{S} \int_0^S G(s)L_i(D) ds : {}^t R A R dz - \int_{\mathbb{R}^m} \frac{1}{S} \int_0^S G(s)L_i(D) ds : {}^t R (b_i \cdot \nabla_z A) R dz \\ = \int_{\mathbb{R}^m} \frac{1}{S} \int_0^S (b_i \cdot \nabla_y (R L_i(D)^t R))(Y(s; z)) ds : A(z) dz. \end{aligned}$$

We use now the weak \star convergence in H_Q^∞

$$\lim_{S \rightarrow +\infty} \frac{1}{S} \int_0^S G(s)L_i(D) ds = L_i(\langle D \rangle)$$

and the facts that ${}^t R A R \in H_P^1$, $\operatorname{div}_y b_i \in L^\infty(\mathbb{R}^m)$ implying that

$$\begin{aligned} - \int_{\mathbb{R}^m} \sum_{i=1}^m \operatorname{div}_z b_i R L_i(\langle D \rangle)^t R : A dz - \int_{\mathbb{R}^m} \sum_{i=1}^m R L_i(\langle D \rangle)^t R : b_i \cdot \nabla_z A dz \\ = \int_{\mathbb{R}^m} \lim_{S \rightarrow +\infty} \frac{1}{S} \int_0^S \sum_{i=1}^m (b_i \cdot \nabla_y (R L_i(D)^t R))(Y(s; z)) ds : A(z) dz. \end{aligned} \quad (87)$$

For passing to the limit in the last integral we use the weak \star convergence in $L^\infty(\mathbb{R}^m)$, since the family $\left(\sum_{i=1}^m (b_i \cdot \nabla_y (R L_i(D)^t R)) \circ Y(s; \cdot)\right)_{s \in \mathbb{R}}$ is bounded in $L^\infty(\mathbb{R}^m)$. We deduce that

$$\sum_{i=1}^m b_i \cdot \nabla_y (R L_i(\langle D \rangle)^t R) = w \star \lim_{S \rightarrow +\infty} \frac{1}{S} \int_0^S \sum_{i=1}^m (b_i \cdot \nabla_y (R L_i(D)^t R)) \circ Y(s; \cdot) ds \in L^\infty(\mathbb{R}^m).$$

It remains to observe that for any $s \in \mathbb{R}$, $\operatorname{div}_z (RG(s)D) = \operatorname{div}_y (RD) \circ Y(s; \cdot)$ (see (72) for details), which implies

$$\operatorname{div}_z (R \langle D \rangle) = w \star \lim_{S \rightarrow +\infty} \frac{1}{S} \int_0^S \operatorname{div}_y (RD) \circ Y(s; \cdot) ds \in L^\infty(\mathbb{R}^m)$$

where, as before, the last limit should be understood in the weak \star $L^\infty(\mathbb{R}^m)$ sense. \square

Acknowledgments

This work has been carried out within the framework of the EUROfusion Consortium and has received funding from the Euratom research and training programme 2014-2018 under grant agreement No 633053. The views and opinions expressed herein do not necessarily reflect those of the European Commission.

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