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Dedicated to R.J. Baxter, for his 75th birthday.

Abstract.
We recall that diagonals of rational functions naturally occur in lattice statistical mechanics and enumerative combinatorics. In all the examples emerging from physics, the minimal linear differential operators annihilating these diagonals of rational functions have been shown to actually possess orthogonal or symplectic differential Galois groups. In order to understand the emergence of such orthogonal or symplectic groups, we analyze exhaustively three sets of diagonals of rational functions, corresponding respectively to rational functions of three variables, four variables and six variables. We impose the constraints that the degree of the denominators in each variable is at most one, and the coefficients of the monomials are 0 or ± 1, so that the analysis can be exhaustive. We find the minimal linear differential operators annihilating the diagonals of these rational functions of three, four, five and six variables. We find that, even for these sets of examples which, at first sight, have no relation with physics, their differential Galois groups are always orthogonal or symplectic groups. We discuss the conditions on the rational functions such that the operators annihilating their diagonals do not correspond to orthogonal or symplectic differential Galois groups, but rather to generic special linear groups.

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1. Introduction

In previous papers [1, 2] it has been shown that the n-fold integrals $\chi^{(n)}$ corresponding to the n-particle contributions of the magnetic susceptibility of the Ising model [3, 4, 5], as well as various other n-fold integrals of the "Ising class" [6, 7], or n-fold integrals from enumerative combinatorics, like lattice Green functions [8, 9, 10, 11], correspond to a distinguished class of functions *generalizing algebraic functions*: they are actually *diagonals of rational functions* [12, 13, 14, 15, 16]. As a consequence, the power series expansions of the analytic solutions at $x = 0$ of these linear differential equations are "Derived From Geometry" [17] and are *globally bounded* [18], which means that, after a rescaling of the expansion variable, they can be cast into series expansions with *integer coefficients* [1, 2].

In another paper [10] revisiting miscellaneous linear differential operators mostly associated with lattice Green functions in arbitrary dimensions, but also Calabi-Yau operators and order-seven linear differential operators corresponding to exceptional differential Galois groups, we showed that these irreducible operators are not only *globally nilpotent* [17], but are also *homomorphic to their* (formal) *adjoints*. Considering these linear differential operators, or, sometimes, equivalent operators†, we showed that, either their symmetric square or their exterior square, *have a rational solution* [10]. This is a general result: an irreducible♯ linear differential operator homomorphic to its (formal) adjoint is necessarily such that either its *symmetric square*, or its *exterior square* has a *rational solution*, and this situation corresponds to the occurrence of a special differential Galois group [10]. We thus defined the notion of being "Special Geometry" for a linear differential operator if it is irreducible, globally nilpotent, and such that it is homomorphic to its (formal) adjoint [20]. Since many "Derived From Geometry" [17] n-fold integrals ("Periods") occurring in physics are seen to be *diagonals of rational functions* [1, 2], we address several examples of (minimal order) operators annihilating diagonals of rational functions, and remark that they also seem to be, almost systematically, associated with irreducible factors *homomorphic to their adjoints* [10].

Finally in a last paper [21] revisiting an order-six linear differential operator, already introduced in [10], having a solution which is the diagonal of a rational function of three variables, we saw that the corresponding linear differential operator is such that its exterior square has a rational solution, indicating that it has a selected differential Galois group, and that it is actually *homomorphic to its adjoint*. We obtained the two corresponding intertwiners giving this homomorphism to the adjoint relation. We showed that these intertwiners are also homomorphic to their adjoints [21] and have a simple decomposition, already underlined in a previous paper [10], in terms of order-two self-adjoint operators. From these results, we deduced a new form of decomposition of operators for this selected order-six linear differential operator in terms of three order-two self-adjoint operators. We generalized this decomposition to decompositions in terms of arbitrary self-adjoint operators of arbitrary orders, provided the orders have the same parity [21].

This natural emergence in physics of n-fold integrals that are diagonals of rational functions, such that their associated linear differential operators correspond to selected differential Galois groups, $SO(n, \mathbb{C})$ or $Sp(n, \mathbb{C})$ (or subgroups, like exceptional groups [10]), was illustrated on important problems of lattice statistical mechanics like the n-fold integrals $\chi^{(5)}$ and $\chi^{(6)}$ of the square Ising model [20], or non-trivial lattice Green functions examples [11].

The occurrence of diagonals of rational functions necessarily yields linear

† In the sense of the equivalence of linear differential operators, see [19].

♯ Minimal order operators annihilating diagonals of rational functions may be irreducible, in general. In that case, all their irreducible factors have the above property.

differential operators that are [17] *globally nilpotent*††, with series-solutions that are *globally bounded*♯ [18], and such that these series identify *modulo prime*, or even *powers of primes*§ to series expansions of *algebraic functions* [1, 2]. We may call these (generally transcendental) holonomic functions "quasi-algebraic" transcendental functions.

If the natural emergence in physics of diagonals of rational functions also corresponds to highly selected linear differential operators, the fact, that we observe on all our (large number of quite non-trivial) examples of lattice statistical mechanics [4, 5, 20, 22, 23, 24, 25], enumerative combinatorics [10, 11], that such selected linear differential operators have *systematically special differential Galois groups*, is *not well understood*. The occurrence of selected differential Galois groups ($SO(n, \mathbb{C})$, $Sp(n, \mathbb{C})$, or subgroups of $SO(n, \mathbb{C})$, $Sp(n, \mathbb{C})$, like exceptional groups such as G_2 see [21]) is now understood, after [10], as the fact that the corresponding linear differential operators are *non-trivially homomorphic to their adjoints*. However the "quasi-algebraic" character of the diagonal of rational functions (reduction to algebraic functions modulo primes or powers of primes) is *not enough* to yield the property, for the corresponding linear differential operator, to be homomorphic to its adjoint.

We have accumulated a very large number of examples‖ of diagonals of rational functions for which, quite systematically, the corresponding linear differential operators are homomorphic to their adjoints, thus having selected differential Galois groups. The set of diagonals of rational functions is an extremely large set: an accumulation of examples, that one can hope to be representative of the generic case, is just a way to build educated guess, or intuition. Among such results the exhaustive analysis of certain sets is worth mentioning. For instance, a set of 210 explicit linear differential operators annihilating periods, which are *actually diagonals of rational functions*, arising from *mirror symmetries*‡ (associated with reflexive 4-polytopes defining 68 topologically different Calabi-Yau 3-folds, see [27, 28]) obtained by P. Lairez [28], has been analyzed in [21]. Among these 210 operators many correspond to "standard" Calabi-Yau ODEs, already analyzed in various papers [7]†. However, remarkably, the other linear differential operators are (non classical Calabi-Yau) *higher order* operators of even orders $N = 6, 8, 10, \cdots, 24$. It was found that all these linear differential operators have *symplectic* differential Galois groups, $Sp(N, \mathbb{C})$, with a remarkable canonical decomposition [21], in terms of self-adjoint order-two operators, except the self-adjoint "rightmost operator" which is systematically of *order four*.

After these accumulations of examples of diagonals of rational functions yielding systematically selected differential Galois groups, one might be led to conjecture that the linear differential operators annihilating diagonals of rational functions are necessarily homomorphic to their adjoints. Such a conjecture is in fact trivially false, as can be seen with the simple ${}_3F_2$ hypergeometric example

†† And, thus, rational number exponents, for all the singularities of these Fuchsian equations, wronskians of these linear differential operators that are N-th roots of rational functions, etc ...

♯ They can be recast into series with integer coefficients [1, 2].

§ And, in fact, modulo any integer.

‖ Unpublished results.

‡ Using a smoothing criterion of Namikawa [26], Batyrev and Kreuzer found [27] 30241 reflexive 4-polytopes such that the corresponding Calabi-Yau hypersurfaces are smoothable by a flat deformation. In particular, they found 210 reflexive 4-polytopes defining 68 topologically different Calabi-Yau 3-folds with $h_{11} = 1$.

† They are *order-four* irreducible operators satisfying the "*Calabi-Yau condition*: they are, up to a conjugation by a function, irreducible *order-four self-adjoint* operators [10]. This amounts to saying that the exterior square of these order-four operators is of *order five*.

${}_3F_2([1/3,\, 1/3,\, 1/3],\, [1,\, 1],\, 3^6\, x)$ which is the *Hadamard product* [1, 2, 29, 30] of three times the simple algebraic function $(1 - 3^2\, x)^{-1/3}$ (Hadamard cube), and is, thus, the diagonal of a rational function [1, 2], the corresponding order-three linear differential operator having a $SL(3,\, \mathbb{C})$ differential Galois group. In other words this operator, associated with the diagonal of a rational function, *cannot be homomorphic to its adjoint* (even with an algebraic extension). One can find many other similar counter-examples of diagonals of a rational function with a $SL(3,\, \mathbb{C})$ differential Galois group. For instance, ${}_3F_2([1/3,\, 1/3,\, 1/5],\, [1,\, 1],\, 3^4\, 5^2\, x)$ which is the *Hadamard product* of two times $(1 - 3^2\, x)^{-1/3}$ with $(1 - 5^2\, x)^{-1/5}$, or ${}_3F_2([1/2,\, 1/2,\, 1/3],\, [1,\, 1],\, 2^4\, 3^2\, x)$ which is the *Hadamard product* of two times the algebraic function $(1 - 2^2\, x)^{-1/2}$ with $(1 - 3^2\, x)^{-1/3}$, are two such examples.

The property, for a linear differential operator, to be homomorphic to its adjoint, can be seen to be "some kind"†† of *Poincaré duality* [32, 31]. Actually, considering a diagonal of a rational function, amounts to considering an algebraic variety. If this algebraic variety is "smooth enough" (not too singular), then one will have a Poincaré duality†, which, in some abstract $\mathcal{D}$-module perspective [34], should correspond to the previous "homomorphism to the adjoint" property. A simpler occurence of this phenomenon is shown by Bogner in proposition 3.4, page 5 of [33]. In simpler words, the emergence of a selected differential Galois group should be natural for the diagonals of a rational function, when the rational function is not "too singular".

For diagonals of rational functions that we know to occur naturally in theoretical physics‡, understanding the emergence of such a duality yielding selected differential Galois groups, at least in a physicist's perspective¶ requires the analysis of other well-defined sets of diagonals of rational functions, along the line of the previously mentioned exhaustive analysis [1, 2] of the 210 explicit linear differential operators arising from mirror symmetries.

This paper will provide such an analysis, introducing well-defined sets of diagonals of rational functions of three, four and six variables, showing that all these examples yield selected differential Galois groups, namely *orthogonal or symplectic* groups. We will first analyze a large set of rational functions of *three* variables, namely rational functions with denominators with degree bounded by 1. It turns out that all these diagonals of rational functions of *three* variables correspond to *modular forms*, so that all can, in principle, be written as ${}_2F_1$ hypergeometric functions with *two pull-backs* [35].

We will finally provide miscellaneous examples that are worth keeping in mind when one tries to understand in which case selected differential Galois groups do not occur for diagonals of rational functions, so that one could see if such "exceptional" cases can also occur in a physics framework.

†† To be more explicit would require to write a mathematical paper, calling out a Deligne-Steenbrink-Zucker theorem saying that Gauss-Manin connections are "variations of polarized mixed Hodge structures", the associated graded modules being Gauss-Manin connections of *smooth* projective varieties, and that they are *self-adjoint by Poincaré duality.* This is far beyond the scope of this "learn-by-example" paper (see for instance [31]).

† The Poincaré duality theorem is a result on the structure of the homology and cohomology groups of manifolds (see chapter 4, page 53 of [32]).

‡ And we know why, see [1, 2].

¶ Who wants to understand why orthogonal, symplectic or exceptional groups occur in his problems.

2. Diagonals of rational functions: preliminary comments

Let us recall the definition of the diagonal of a rational function of n variables $\mathcal{R}(x_1, \ldots, x_n) \, = \, \mathcal{P}(x_1, \ldots, x_n)/\mathcal{Q}(x_1, \ldots, x_n)$ where $\mathcal{P}$ and $\mathcal{Q}$ are polynomials of $x_1, \, \cdots \, , \, x_n$ with *rational coefficients* such that $\mathcal{Q}(0, \ldots, 0) \neq 0$. The diagonal of $\mathcal{R}$ is defined from its multi-Taylor expansion

$$\mathcal{R}\Big(x_1, \, x_2, \, \ldots, \, x_n\Big) \,\,\, = \,\,\, \sum_{m_1\,=\,0}^{\infty} \, \cdots \, \sum_{m_n\,=\,0}^{\infty} \, R_{m_1, \, \ldots, \, m_n} \cdot x_1^{m_1} \, \cdots \, x_n^{m_n}, \qquad (1)$$

as the series of *one variable*

$$Diag\Big(\mathcal{R}\Big(x_1, \, x_2, \, \ldots, \, x_n\Big)\Big) \,\,\, = \,\,\, \sum_{m\,=\,0}^{\infty} \, R_{m, \, m, \, \ldots, \, m} \cdot x^m. \qquad (2)$$

In the generic case, denoting d the degree of the denominator of a rational function of n variables, and Ω the order of the minimal order linear differential operator annihilating the diagonal of this rational function, the order Ω grows with n and d like $\simeq d^n$ (see e.g. [36]):

$$\Omega \,\, = \,\, \frac{1}{d} \cdot \Big((d-1)^{n+1} \, + \, (-1)^{n+1} \cdot (d-1)\Big) \,\, < \,\, d^n. \qquad (3)$$

From a *theory of singularity* perspective [37], one can imagine that the analysis of such diagonals of rational functions depends "essentially" on the denominator of the rational function. For that reason we will restrict most of the examples of this paper to rational functions with a numerator normalised to 1, *so that the number of cases to analyze will be reasonable*. Note, however, that, even in a theory of singularity perspective, this restriction to numerators equal to a constant, is not innocent, as will be seen below (see the Remark in section (7.1)). We impose this restriction, for the simplicity of the calculations, and in order to perform exhaustive analysis of certain sets of examples.

Furthermore, in order to be able to find a linear differential operator for the diagonal, using reasonable computer resources, we will, for a given number of variables n, restrict to denominators of the lowest possible degree, imposing, for instance, that the degree, in each variable x_i, of the monomials $x_1^{d_1} \, x_2^{d_2} \, \cdots \, x_n^{d_n}$ of the denominator of the rational function is *at most* 1. Formula (3) gives the upper bound for the order of the linear differential operator, which is actually reached when the polynomial at the denominator, has *all its monomials* (no monomial has a zero coefficient). In practice, in the denominator of the rational function examples emerging from physics, the polynomial at the denominator is (fortunately) sparse, being the sum of a *quite small set of monomials*. In such (physical) cases, the order of the corresponding linear differential operator is less than the one given by (3) which grows like $\simeq d^n$ and thus becomes quickly too large for any formal calculation.

Imposing a constraint on the degree in each variable, instead of a constraint on the degree of the denominator $d \, = \, d_1 + d_2 \, + \, \cdots \, + d_n$, reduces the number of monomials and, consequently, reduces quite drastically, the order of the corresponding linear differential operator.

Furthermore, it is clear (by definition of the diagonal of a function) that scaling the variables $x_i \, \rightarrow \, \lambda_i \cdot x_i$, amounts to performing a simple scaling on the diagonal: $\Delta(x) \, \rightarrow \, \Delta(\lambda \cdot x)$ where $\lambda \, = \, \lambda_1 \cdot \lambda_2 \cdots \lambda_n$. Consequently we will, in this paper, often restrict the coefficients of the monomials to a narrow set of *small* integer values, namely $0, 1$, or $0, \pm 1$, in order to have a reasonably small number of cases to analyze to avoid any explosion of the combinatorics. It is, of course, clear that the coefficients of rational functions cannot, in general, be reduced to $0, \pm 1$.

Diagonals of rational functions of two variables necessarily yield *algebraic functions* [1, 2, 38, 39, 40, 41]. Therefore, in the next sections, we will consider diagonal of rational functions, of *more than two variables*.

3. Diagonals of rational functions of three variables

Let us, first, consider diagonals of rational functions of the form $1/(1 - P(x, y, z))$ where $P(x, y, z)$ is a polynomial of *three* variables x, y, z, sum of monomials $x^m y^n z^p$ where the degrees m, n, p are 0 or 1, and where the coefficients in front of these monomials are restricted to take two values $0, 1$. We will say that two rational functions of the form $1/(1 - P)$ are in the same class if they have the same diagonal and hence the same linear differential operator annihilating this diagonal. With these two constraints (on the degrees and values of the coefficients), one finds only 20 classes of rational functions (or diagonals). For all these cases, the linear differential operator is of order two, the diagonal being a ${}_2F_1$ hypergeometric function, or a HeunG function [35] that can be rewritten as a ${}_2F_1$ hypergeometric function with *two possible pullbacks*, which means that this is, in fact, a *modular form* [35].

With the same constraint on the degrees m, n, p to be 0 or 1, restricting the coefficients in front of these monomials to take three values $0, \pm 1$, one obtains 85 classes of linear differential operators.

Let us just give a few examples, the exhaustive list of results being given in a web page of supplementary material [42].

• For the polynomial $P(x, y, z) = x + y + z - xz - yz + xyz$, the diagonal of $1/(1 - P(x, y, z))$, which corresponds to the sequence $[1, 3, 13, 63, 321, 1683, \cdots]$ of Central Delannoy numbers (see Sloane's on-line encyclopedia [43] of integer sequences: A001850), is a simple *algebraic function* $(1 - 6x + x^2)^{-1/4}$, annihilated by an order-one operator.

• For the polynomial $P(x, y, z) = x + y + z - xz - xyz$, the diagonal of $1/(1 - P(x, y, z))$ is the pullbacked ${}_2F_1$ hypergeometric function

$$(1 - 12x + 14x^2 + 12x^3 + x^4)^{-1/4} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], P_1(x)\Big)$$

$$\text{where:} \qquad P_1(x) = 1728 \frac{x^5 \cdot (1 - 11x - x^2)}{(1 - 12x + 14x^2 + 12x^3 + x^4)^3}. \qquad (4)$$

This *Hauptmodul* [35] $P_1(x)$ corresponds to the Hauptmodul $12^3/j_5'$ (see Table 5 in [46])

$$P_1(x) = \frac{12^3 \cdot z^5}{(z^2 + 250z + 3125)^3} \qquad \text{with:} \qquad z = \frac{5^3 \cdot x}{1 - 11x - x^2}. \qquad (5)$$

This diagonal corresponds to the sequence $[1, 3, 19, 147, 1251, 11253, \cdots]$ of *Apery numbers* (see Sloane's on-line encyclopedia [43] of integer sequences: A005258). The pullbacked ${}_2F_1$ hypergeometric function (4) can also be written *with another pullback*

$$(1 + 228x + 494x^2 - 228x^3 + x^4)^{-1/4} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], P_2(x)\Big)$$

$$\text{where:} \qquad P_2(x) = 1728 \frac{x \cdot (1 - 11x - x^2)^5}{(1 + 228x + 494x^2 - 228x^3 + x^4)^3}. \qquad (6)$$

This Hauptmodul [35] $P_2(x)$ corresponds to the Hauptmodul $12^3/j_5$ (see Table 4 in [46])

$$P_2(x) \; = \; \frac{12^3 \cdot z}{(z^2 \, + 10\, z \, + 5)^3} \qquad \hbox{with:} \qquad z \; = \; \frac{5^3 \cdot x}{1 \, - 11\, x \, - x^2}. \tag{7}$$

Changing $P_1(x)$ into $P_2(x)$ amounts to changing $z \leftrightarrow \, 5^3/z$.

These two pullbacks, $Y \, = \, P_1(x)$ and $Z \, = \, P_2(x)$, are related by an algebraic curve, namely the (genus zero) *modular curve*:

$$\begin{aligned}
& 2^{54} \cdot 5^3 \cdot 11^9 \cdot Y^6 Z^6 \, + 110949 \cdot 2^{47} \cdot 11^6 \cdot 5^3 \cdot Y^5 Z^5 \cdot (Y+Z) \\
& + 3 \cdot 2^{36} \cdot 11^3 \cdot Y^4 Z^4 \cdot (2735484611275\, (Y^2 \, + \, Z^2) \, - 107937074856652\, YZ) \\
& + 2^{29} \cdot 5^2 \cdot Y^3 Z^3 \cdot (Y+Z) \cdot (4046657341273198\, (Y^2 \, + \, Z^2) \\
& \qquad \qquad + 523793662474799327 \cdot YZ) \\
& + 3 \cdot 2^{18} \cdot 5 \cdot \, Y^2 Z^2 \cdot (13^2 \cdot 647451979 \cdot (Y^4 + \, Z^4) \\
& + 5^3 \cdot 3482348755357972227 \cdot Y^2 Z^2 \, - 16391442082714013450 \cdot (YZ^3 \, + Y^3 Z)) \\
& + 30720 \cdot YZ \cdot (Y+Z) \cdot (36983 \cdot (Y^4 \, + \, Z^4) \, + 2421471845930417 \cdot (YZ^3 \, + Y^3 Z) \\
& \qquad \qquad + 506711612589929401008\, Y^2 Z^2) \\
& + (Y^6 \, + Z^6) \, - 246683410950 \cdot (YZ^5 \, + Y^5 Z) \, - 441206965512914835246100 \cdot Y^3 Z^3 \\
& \qquad + 383083609779811215375 \cdot (Y^4 Z^2 \, + \, Z^4 Y^2) \\
& \quad + 2^{11} \cdot 3^3 \cdot 5^2 \cdot YZ \cdot (Y+Z) \cdot (2535689 \cdot (Y^2 \, + \, Z^2) \, + 134848657695982\, YZ) \\
& \quad - 59719680 \cdot YZ \cdot (227547 \cdot (Y^2 \, + \, Z^2) - 83299968230 \cdot YZ) \\
& \qquad + 2^{21} \cdot 3^{10} \cdot 5 \cdot 31 \cdot YZ \cdot (Y+Z) \, - 2^{24} \cdot 3^{12} \cdot YZ \; = \; 0.
\end{aligned} \tag{8}$$

The pullbacked ${}_2F_1$ hypergeometric function (4), or equivalently (6), is a *modular form* (see the J_5 of Maier [46]).

• For polynomial $P(x, \, y, \, z) \; = \; x \, + \, y \, + \, x\,z \, + \, y\,z \, + \, x\,y\,z$, the diagonal of $1/(1 - P(x, \, y, \, z))$ is the pullbacked ${}_2F_1$ hypergeometric function

$$(1 \, - 20\, x \, + 54\, x^2 \, - 20\, x^3 \, + x^4)^{-1/4} \cdot \, {}_2F_1\Big([\frac{1}{12}, \, \frac{5}{12}], \, [1], \, P_1(x)\Big)$$

$$\hbox{where:} \qquad P_1(x) \; = \; 1728\, \frac{x^4 \cdot (1 - 18\, x \, + x^2)\, (x-1)^2}{(1 \, - 20\, x \, + 54\, x^2 \, - 20\, x^3 \, + x^4)^3}. \tag{9}$$

This Hauptmodul $P_1(x)$ corresponds to the Hauptmodul $12^3/j_2'$ (see Table 5 in [46])

$$P_1(x) \; = \; \frac{12^3 \cdot z^2}{(z \, + \, 256)^3} \qquad \hbox{with:} \qquad z \; = \; \frac{2^{12} \cdot x^2}{(1 \, - x)^2 \, (1 \, - 18\, x \, + x^2)}. \tag{10}$$

This diagonal corresponds to the sequence $[1, 5, 49, 605, 8281, 120125 \cdots]$ (see Sloane's on-line encyclopedia of integer sequences: A243945). Similarly, the pullbacked ${}_2F_1$ hypergeometric function (9) can also be written

$$(1 - 20\, x + 294\, x^2 - 20\, x^3 + x^4)^{-1/4} \cdot \, {}_2F_1\Big([\frac{1}{12}, \, \frac{5}{12}], \, [1], \, P_2(x)\Big)$$

$$\hbox{where:} \qquad P_2(x) \; = \; 1728\, \frac{x^2 \cdot (1 - 18\, x \, + x^2)^2 \cdot (1-x)^4}{(1 \, - 20\, x \, + 294\, x^2 \, - 20\, x^3 \, + x^4)^3}. \tag{11}$$

This Hauptmodul $P_2(x)$ corresponds to the Hauptmodul $12^3/j_2$ (see Table 4 in [46])

$$P_2(x) \; = \; \frac{12^3 \cdot z}{(z \, + 16)^3} \qquad \hbox{with:} \qquad z \; = \; \frac{2^{12} \cdot x^2}{(1 \, - x)^2 \, (1 \, - 18\, x \, + x^2)}. \tag{12}$$

Changing $P_1(x)$ into $P_2(x)$ amounts to changing $z \leftrightarrow 2^{12}/z$. These two pullbacks, $Y = P_1(x)$ and $Z = P_2(x)$, are related by an algebraic curve, namely the (genus zero) *modular curve*:

$$5^9\, Y^3 Z^3 \, - 5^6 \cdot 12\, Y^2 Z^2 \cdot (Y+Z) \, + 375 \cdot Y\, Z \cdot (16\, Y^2 \, - 4027 \cdot Y Z \, + 16\, Z^2)$$
$$- 64 \cdot (Y+Z) \cdot (Y^2 + 1487\, YZ + Z^2) \, + 2^{12} \cdot 3^3 \cdot YZ \; = \; 0. \qquad (13)$$

The pullbacked ${}_2F_1$ hypergeometric function (9), or equivalently (11), is a *modular form* (see the J_2 of Maier [46], or even J_4 of Maier [46] but with $z = 2^{12}\, x \cdot (1-x)^2/(1-18\, x + x^2)^2$).

• For the polynomial $P(x, y, z) = xyz + xy + xz + yz + x + y + z$, the diagonal of $1/(1 - P(x, y, z))$ is the pullbacked¶ ${}_2F_1$ hypergeometric function

$$\frac{1}{1-x} \cdot \, {}_2F_1\Big([\frac{1}{3}, \frac{2}{3}], [1], P(x)\Big) \qquad \text{where:} \qquad P(x) \; = \; \frac{54\, x}{(1-x)^3}. \qquad (14)$$

This diagonal corresponds to the sequence $[1, 13, 409, 16081, 699121, 32193253 \cdots]$ (see Sloane's on-line encyclopedia of integer sequences: A126086). This series can also be written as the pullbacked ${}_2F_1$ hypergeometric function

$$(1-x)^{-1/4} \cdot (1 + 429\, x + 3\, x^2 - x^3)^{-1/4} \cdot \, {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], P_1(x)\Big)$$
$$\text{where:} \qquad P_1(x) \; = \; 3456\, \frac{x \cdot (1 - 57\, x + 3\, x^2 - x^3)^3}{(1-x)^3\, (1 + 429\, x + 3\, x^2 - x^3)^3}, \qquad (15)$$

or as the pullbacked ${}_2F_1$ hypergeometric function

$$(1-x)^{-1/4} \cdot (1 - 51\, x + 3\, x^2 - x^3)^{-1/4} \cdot \, {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], P_2(x)\Big)$$
$$\text{where:} \qquad P_2(x) \; = \; 13824\, \frac{x^3 \cdot (1 - 57\, x + 3\, x^2 - x^3)}{(1-x)^3\, (1 - 51\, x + 3\, x^2 - x^3)^3}. \qquad (16)$$

These two pullbacks correspond respectively to the Hauptmodul [35] $12^3/j_3$ of Table 4 in [46] and $12^3/j_3'$ of Table 5 in [46]:

$$P_1(x) \; = \; \frac{12^3 \cdot z}{(z+27) \cdot (z+3)^3}, \qquad P_2(x) \; = \; \frac{12^3 \cdot z^3}{(z+27) \cdot (z+243)^3}$$
$$\text{with:} \qquad z \; = \; 2 \cdot \frac{9^3 \cdot x}{1 - 57\, x + 3\, x^2 - x^3}. \qquad (17)$$

Changing $P_1(x)$ into $P_2(x)$ amounts to changing $z \leftrightarrow 3^6/z$. Again, these two pullbacks are related by a (genus zero) *modular curve*:

$$2^{27} \cdot 5^9 \cdot Y^3 Z^3 \cdot (Y+Z) \, + 2^{18} \cdot 5^6 \cdot Y^2 Z^2 \cdot (27\, Y^2 - 45946\, YZ + 27\, Z^2)$$
$$+ 2^9 \cdot 5^3 \cdot 3^5 \cdot \, YZ \cdot (Y+Z) \cdot (Y^2 + 241433\, YZ + Z^2)$$
$$+ 729 \cdot (Y^4 \, + Z^4) \, - 779997924 \cdot (YZ^3) \, + Y^3 Z \, + 31949606 \cdot 3^{10} \cdot Y^2 Z^2$$
$$+ 2^9 \cdot 3^{11} \cdot 31 \cdot Y\, Z \cdot (Y+Z) - 2^{12} \cdot 3^{12} \cdot YZ \; = \; 0.$$

The pullbacked ${}_2F_1$ hypergeometric function (16) is a *modular form* (see the J_3 of Maier [46]).

• For the polynomial $P(x, y, z) = xyz + xy + xz + yz$, the diagonal of $1/(1 - P(x, y, z))$ is the pullbacked ${}_2F_1$ hypergeometric function

$$\frac{1}{1-x} \cdot \, {}_2F_1\Big([\frac{1}{3}, \frac{2}{3}], [1], P(x)\Big) \qquad \text{where:} \qquad P(x) \; = \; \frac{27\, x^2}{(1-x)^3}. \qquad (18)$$

¶ Note that this pullback can be obtained using the Maple program "hypergeomdeg3" of V. J. Kunwar and M. van Hoeij [44, 45].

This diagonal corresponds to the sequence $[1, 1, 7, 25, 151, 751 \cdots]$ (see Sloane's on-line encyclopedia of integer sequences: A208425).

In Appendix A we give a set of diagonals of $1/(1 - P(x, y, z))$ which can be seen to be *modular forms*: they can be written as ${}_2F_1$ hypergeometric functions with two different pullbacks.

Most of the other examples of diagonals, given in a web page as well as in the supplementary material [42], are not yet in Sloane's on-line encyclopedia of integer sequences. For instance the sequence $[1, 4, 42, 520, 7090, 102144, \cdots]$, which corresponds to the diagonal of the rational function $1/(1 - x - y - x\,y - x\,z - y\,z)$, can be written as a ${}_2F_1$ hypergeometric function with a rational pullback:

$$\Bigl(\frac{1}{1 - 16\,x - 8\,x^2}\Bigr)^{1/4} \cdot {}_2F_1\Bigl([\frac{1}{12}, \frac{5}{12}], [1], 1728 \cdot \frac{x^4 \cdot (1+x)^2 (2 - 34\,x - 27\,x^2)}{(1 - 16\,x - 8\,x^2)^3}\Bigr),$$

All these results show a close relation between diagonals of our simple examples of rational functions of three variables, and *modular forms*. This suggests, quite naturally, to perform similar calculations, but, now, with *four* variables.

4. Diagonals of rational functions of four variables

Let us now consider diagonal of rational functions of the form $1/(1 - P(x, y, z, w))$ where $P(x, y, z, w)$ is a polynomial of four variables x, y, z, w, sum of monomials $x^m\, y^n\, z^p\, w^q$ where the degrees m, n, p, q are 0 or 1, and where the coefficients in front of these monomials are restricted to take *only two values* 0, 1. With these two constraints (on the degrees and on the values of the coefficients), one finds an exhaustive list of only 879 cases¶.

Remark 1: We have used the package "HolonomicFunctions" written by C. Koutschan [47, 48, 49], based on the method of *creative telescoping* [50, 51], which enables to obtain directly, and very efficiently, the linear differential operator annihilating the diagonal of a given rational function, *without calculating the series expansion* of the corresponding diagonal‡. From time to time Koutschan's algorithms do not provide the *minimal order* linear differential operator. It is thus necessary to systematically check whether the linear differential operator obtained by creative telescoping is minimal, and if not, to find the minimal one.

Remark 2: The command "FindCreativeTelescoping", described in [51], is (usually) *extremely fast*, but it uses some heuristics, which means that sometimes it can return a non-minimal result or run forever: it is not an "algorithm" in the strict sense†.

Among these 879 cases, the linear differential operators annihilating the diagonals have minimal orders running from 1 to 10, as given in Table 1.

¶ When different rational functions $1/(1 - P(x, y, z, w))$ yield the *same diagonal* (in practice the same first ten coefficients of the series) we select one rational function to represent the diagonal. This is the way we define these 879 different classes of rational functions.
‡ One does not obtain the linear differential operator from a "guessing procedure" on the series of the diagonal. Furthermore this algorithm is "certified": we are sure that the operator annihilates the diagonal.
† In contrast, Chyzak's algorithm [52] is designed such that it finds the *minimal order operator*, but it is often much slower than Koutschan's heuristics (in particular for the very large operators emerging in physics [4, 5, 53, 54]). In Koutschan's package, Chyzak's algorithm is implemented in the command CreativeTelescoping (same input/output specification as FindCreativeTelescoping).

Table 1. Number of operators corresponding to the various orders.

Order of Oper.	1	2	3	4	5	6	7	8	9	10
Number of Oper.	1	2	20	128	240	231	155	54	41	7

The order-one linear differential operator corresponds to the diagonal of the rational function $1/(1 - z - wy - xy)$. It is easily seen that its diagonal is equal to 1, so the order-one operator is just D_x.

For the order-two linear differential operators, the corresponding diagonals are two pullbacked ${}_2F_1$ hypergeometric functions, respectively

$$(1 - 32\,x + 16\,x^2)^{-1/4} \cdot \, {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], P(x)\Big)$$

$$\text{where:} \qquad P(x) \, = \, 1728\,\frac{x^3 \cdot (2 - 71\,x + 16\,x^2)}{(1 - 32\,x + 16\,x^2)^3}, \qquad (19)$$

for the diagonal of $1/(1-(y + z + x\,z + x\,w + x\,y\,w))$ which reads

$$1 + 8\,x + 156\,x^2 + 3800\,x^3 + 102340\,x^4 + 2919168\,x^5 + 86427264\,x^6 + 2626557648\,x^7 + 81380484900\,x^8 + 2559296511200\,x^9 + \cdots \qquad (20)$$

and

$$(1 - 40\,x + 16\,x^2)^{-1/4} \cdot \, {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], P(x)\Big) \qquad (21)$$

$$\text{where:} \qquad P(x) \, = \, 6912\,\frac{x^3 \cdot (1 - 44\,x - 16\,x^2)}{(1 - 40\,x + 16\,x^2)^3}.$$

for† the diagonal of $1/(1-(y + z + x\,w + x\,z\,w + x\,y\,w))$ which reads:

$$1 + 10\,x + 246\,x^2 + 7540\,x^3 + 255430\,x^4 + 9163980\,x^5 + 341237820\,x^6 + 13042646760\,x^7 + 508236930630\,x^8 + 20101587623260\,x^9 + \cdots \qquad (22)$$

Note, however, that the series (20) of the diagonal of $1/(1-(y+z+x\,z+x\,w+x\,y\,w))$ actually identifies with the diagonal of a rational function of just *three* variables $1/(1 - x - y - z - x\,y)$, already found among the previous 20 cases of section (3). Similarly the series (22) of the diagonal of $1/(1-(y+z+x\,w+x\,z\,w+x\,y\,w))$ also identifies with the diagonal of a rational function of three variables $1/(1 - x - y - z - x\,y - y\,z)$.

The results for all these 879 cases are given exhaustively in our web page of supplementary material [42]. Let us summarize these results in the following.

For all the twenty cases, corresponding to order-three linear differential operators, we have $SO(3, \mathbb{C})$ differential Galois groups. As a consequence [10], *all* these linear differential operators are actually *symmetric squares of order-two operators*, some with very simple ${}_2F_1$ hypergeometric functions, namely ${}_2F_1([3/8, 1/8], [1], 256\,x^3)$ or ${}_2F_1([1/3, 1/6], [1], 108\,x^3)$, some with, at first sight, more involved HeunG function solutions [35] which turn out to be pullbacked ${}_2F_1$ hypergeometric functions, with *two possible pullbacks*, and, in fact, *modular forms* [35].

The 128 order-four linear differential operators are (non-trivially) homomorphic to their adjoints. They have $SO(4, \mathbb{C})$ differential Galois groups and have a canonical

† These two series (20) and (22) are not in Sloane's on-line encyclopedia http://oeis.org.

decomposition [21] of the form $(A_1 B_3 + 1) \cdot r(x)$, where A_1 and B_3 are respectively order-one and *order-three self adjoint* operators, $r(x)$ being a rational function.

Similarly, the 240 order-five linear differential operators are (non-trivially) homomorphic to their adjoints. They have $SO(5, \mathbb{C})$ differential Galois groups and have a canonical decomposition [21] of the form $(A_1 B_1 C_3 + A_1 + C_3) \cdot r(x)$, where A_1, B_1 and C_3 are respectively two order-one and one *order-three self adjoint* operators, $r(x)$ being a rational function.

The 231 order-six linear differential operators are (non-trivially) homomorphic to their adjoints. They have $SO(6, \mathbb{C})$ differential Galois groups and have a canonical decomposition [21] of the form $(A_1 B_1 C_1 D_3 + A_1 B_1 + A_1 D_3 + C_1 D_3 + 1) \cdot r(x)$, where the A_1, B_1, C_1, operators are order-one self-adjoint operators, the rightmost operator D_3 being an *order-three self-adjoint* operator.

The following 155 order-seven linear differential operators are (non-trivially) homomorphic to their adjoints. They have $SO(7, \mathbb{C})$ differential Galois groups and have a canonical decomposition [21] of the form $(A_1 B_1 C_1 D_1 E_3 + A_1 B_1 E_3 + A_1 D_1 E_3 + A_1 B_1 C_1 + C_1 D_1 E_3 + A_1 + C_1 + E_3) \cdot r(x)$, where the A_1, B_1, C_1, D_1 operators are order-one self-adjoint operators, the rightmost operator E_3 being an order-three self-adjoint operator.

The 54 order-eight linear differential operators are (non-trivially) homomorphic to their adjoints. They have $SO(8, \mathbb{C})$ differential Galois groups and have a canonical decomposition described in [21], generalization of the previous ones, with, again, five order-one self-adjoint operators, and a rightmost *order-three self-adjoint* operator.

The 41 order-nine linear differential operators are (non-trivially) homomorphic to their adjoints. They have $SO(9, \mathbb{C})$ differential Galois groups and have a canonical decomposition described in [21], generalization of the previous ones, with, six order-one self-adjoint operators, and a rightmost *order-three self-adjoint* operator.

Finally, the seven order-ten linear differential operators are (non-trivially) homomorphic to their adjoints. They have $SO(10, \mathbb{C})$ differential Galois groups and have a canonical decomposition described in [21], generalization of the previous ones, with, seven order-one self-adjoint operators, and a rightmost *order-three self-adjoint* operator.

These results are reminiscent of the results obtained on a set of 210 explicit linear differential operators annihilating diagonals of rational functions, arising from mirror symmetries and corresponding to reflexive 4-polytopes [28], recalled in the introduction. One notes, however, that the *symplectic* $Sp(n, \mathbb{C})$ differential Galois groups with a canonical decomposition in order-two self-adjoint operators and a rightmost order-four self-adjoint operator, encountered with these reflexive 4-polytopes examples, is now replaced by *orthogonal* $SO(n, \mathbb{C})$ differential Galois groups with a canonical decomposition in order-one self-adjoint operators and a rightmost *order-three self-adjoint* operator.

The calculations performed here, in order to see that these (quite large) linear differential operators are (non trivially) homomorphic to their adjoints and to find their canonical decompositions [21], are similar to the ones described in [21] for the reflexive 4-polytopes examples: in order to find the intertwiner, we introduced a specialized algorithm because the Maple command $Homomorphisms(adjoint(L), L)$ never terminates on these large operators [21]. We use a fuchsian linear differential system associated to L, the *theta-system* [21], a slight generalization of the companion system which has simple poles at each finite singularity. One then finds a rational solution of an associated system with similar coefficients (its second symmetric/exterior power), which gives the intertwiner. An inversion of this intertwiner modulo L gives the intertwiner corresponding to the Maple command $Homomorphisms(L,\, adjoint(L))$. Finally, one obtains the canonical decomposition from simple euclidean divisions [21].

For the last seven order-ten operators even the theta-system calculations were quite massive, and required up to two weeks of CPU time and up to 80 Gigaoctets of memory for one linear differential operator.

5. Diagonals of rational functions associated with orthogonal as well as symplectic groups

It is tempting to simply, and straightforwardly, generalize† these two sets of results for diagonals of well-defined finite sets of rational functions, namely the symplectic $Sp(n,\, \mathbb{C})$ differential Galois groups [28] (with an order-four rightmost self-adjoint operator as for reflexive 4-polytopes [28]), and the orthogonal $SO(n,\, \mathbb{C})$ differential Galois groups (with a rightmost order-three self-adjoint operator). The situation can be slightly more involved (and richer) than a straightforward generalization of the previous results. In fact, *diagonals are not necessarily* solutions of an *irreducible* linear differential operator.

To see this let us consider the diagonal¶ of the rational function of four variables with a factorized denominator $R \, = \, 1/(1-x\,-y\,-u\,-z)/(1-u-z-u\,z)$, which reads:

$$Diag(R) \, = \, 1 \, + 42\,x \, + 4878\,x^2 \, + 748020\,x^3 \, + 130916310\,x^4 \, + 24762428460\,x^5 \\ + 4929691760532\,x^6 \, + 1017691904736936\,x^7 \, + \, \cdots \tag{23}$$

It is solution of an order-seven linear differential operator which factorizes§ into an *order-three* and an *order-four* operator: $L_7 \, = \, L_3 \cdot L_4$. Note that L_4 *does not annihilate* this diagonal.

This means that the series (23) is solution of $L_7 \, = \, L_3 \cdot L_4$, but not of L_4 which has a solution analytic at $x \, = 0$, with a series expansion with *integer coefficients*♯ different from (23):

$$1 \, + 214\,x \, + 97278\,x^2 \, + 53983020\,x^3 \, + 32898451110\,x^4 \, + 21172639875156\,x^5 \\ + 14121624413802444\,x^6 \, + \, \cdots \tag{24}$$

The series $L_4(Diag(R))$, solution of L_3, is *globally bounded* [1, 2]. If one normalizes L_4 to be an operator with polynomial coefficients (instead of being a monic operator), the series $L_4(Diag(R))$, solution of L_3, is a series with *integer coefficients*:

$$17888 + 25769200\,x + 17312032256\,x^2 + 8722773606816\,x^3 + 3775743401539200\,x^4 \\ + 1486619414765913792\,x^5 + 548416028673746513280\,x^6 \, + \, \cdots$$

† In particular after a set of other unpublished results we have obtained.

¶ Note that this diagonal is factorized but is not the Hadamard product of two diagonals (see Appendix B.2).

§ Using the DFactorLCLM command one sees that one does not have a direct sum factorization, just the simple factorization $L_7 \, = \, L_3 \cdot L_4$.

♯ A natural question corresponds to ask if such series (24) with integer coefficients are necessarily a diagonal of rational function. It is true for ODEs of minimum weight for the monodromy filtration [18], but in the general case, it is still a conjecture. It, however, seems that one can prove that such series are “automatic” (i.e. reduce to algebraic functions) modulo powers of primes (G. Christol, private communication).

Note that the order-three operator L_3 has a canonical $SO(3, \mathbb{C})$ orthogonal decomposition [21] $(A_1 B_1 C_1 + A_1 + C_1) \cdot r(x)$, where A_1, B_1, C_1, are order-one self-adjoint operators. In contrast L_4 has the canonical $Sp(4, \mathbb{C})$ symplectic decomposition $(A_2 B_2 + 1) \cdot \rho(x)$, where A_2, B_2 are order-two self-adjoint operators. In other words, the diagonal (23) is associated with both orthogonal *and* symplectic differential Galois groups! This is, in fact, the situation we expect generically: the diagonal of the rational function will be solution of a *non-irreducible* linear differential operator, *each of its factors* corresponding to *orthogonal or symplectic* differential Galois groups.

We have, for instance, the same results for another rational function such that its denominator is factorised. The diagonal of $R = 1/(1-x-y-z-u)/(1-u-z-xz)$, namely

$$1 + 44\,x + 5061\,x^2 + 771000\,x^3 + 134309890\,x^4 + 25316919264\,x^5 + 5026804760628\,x^6 + \cdots \qquad (25)$$

is solution of an order-seven linear differential operator which factorizes into an *order-three* and an *order-four* operator: $L_7 = L_3 \cdot L_4$ (but, again there is no direct sum factorization). This diagonal (25) is solution of L_7 but not of L_4. If one normalizes L_4 to be an operator with polynomial coefficients (instead of being a unitary operator), the series $L_4(Diag(R))$, solution of L_3, is a series with *integer coefficients*:

$$16 - 94464\,x - 127052100\,x^2 - 86146838400\,x^3 - 44244836836200\,x^4 - 19495756524980736\,x^5 - 7791904441995369696\,x^6 + \cdots \qquad (26)$$

The order-three linear differential operator L_3 is MUM (maximal unipotent monodromy [55]) and has a canonical $SO(3, \mathbb{C})$ decomposition $(A_1 B_1 C_1 + A_1 + C_1) \cdot r(x)$, where A_1, B_1, C_1, are order-one self-adjoint operators. It is not the symmetric square of an order-two operator, it is homomorphic to the symmetric square of an order-two operator. In contrast L_4 has the canonical $Sp(4, \mathbb{C})$ decomposition $(A_2 B_2 + 1) \cdot \rho(x)$ where A_2, B_2 are order-two self-adjoint operators.

6. Diagonals of rational functions corresponding to ${}_nF_{n-1}$ hypergeometric functions

Beyond these finite sets of examples of diagonals of rational functions, namely the 210 reflexive 4-polytopes operators [28] with symplectic $Sp(n, \mathbb{C})$ differential Galois groups, and these 879 operators of section (4) associated with diagonals of rational functions of four variables with orthogonal $SO(n, \mathbb{C})$ differential Galois groups, one can find infinite families of diagonals of rational functions for which exact results can be obtained corresponding to ${}_nF_{n-1}$ hypergeometric functions.

For instance, the diagonal of the rational function of three variables $R = 1/(1-x-z-y^n)$, with n being a positive integer, is a ${}_{2n}F_{2n-1}$ hypergeometric function

$$Diag\Big(\frac{1}{1-x-z-y^n}\Big) = {}_{2n}F_{2n-1}\Big([\frac{1}{2n+1}, \frac{2}{2n+1}, \frac{3}{2n+1}, \cdots, \frac{2n}{2n+1}], [\frac{1}{n}, \frac{1}{n}, \frac{2}{n}, \frac{2}{n}, \frac{3}{n}, \frac{3}{n}, \cdots, \frac{n-1}{n}, \frac{n-1}{n}, \frac{n}{n}], \frac{(2n+1)^{2n+1}}{n^{2n}} \cdot x^n\Big). \qquad (27)$$

The corresponding linear differential operator is an order-$2n$ linear differential operator having a *symplectic* $Sp(2\,n, \, \mathbb{C})$ differential Galois group.

The diagonal of the rational functions of three variables $R \, = \, 1/(1-x-z-x^n\, y^n)$, with n being a positive integer, is a $\, _nF_{n\,-1}$ hypergeometric function

$$Diag\Big(\frac{1}{1-x-z\,-x^n\,y^n}\Big) \,\,\, = \,\,\, _nF_{n\,-1}\Big([\frac{1}{n\,+1}, \, \frac{2}{n\,+1}, \, \frac{3}{n\,+1}, \, \cdots \,, \, \frac{n}{n\,+1}], \, [\frac{1}{n}, \, \frac{2}{n}, \, \frac{3}{n}, \, \cdots \,, \, \frac{n\,-1}{n}], \,\, \frac{(n\,+1)^{n\,+1}}{n^n} \cdot x^n\Big). \tag{28}$$

The corresponding linear differential operator is an order-n linear differential operator having an *orthogonal* $O(n, \, \mathbb{C})$ differential Galois group.

The diagonal of the rational functions of three variables $R \, = \, 1/(1-x-z-x\,y^n)$, with n being a positive integer, is a $\, _nF_{n\,-1}$ hypergeometric function. For n even it reads:

$$Diag\Big(\frac{1}{1-x-z\,-x\,y^n}\Big) \,\,\, = \,\,\, _nF_{n\,-1}\Big([\frac{1}{2\,n}, \, \frac{3}{2\,n}, \, \frac{5}{2\,n}, \, \cdots \,, \, \frac{2\,n\,-1}{2\,n}], \, [\frac{1}{n\,-1}, \, \frac{2}{n\,-1}, \, \frac{3}{n\,-1}, \, \cdots \,, \, \frac{n\,-2}{n\,-1}, \, \frac{n\,-1}{n\,-1}], \,\, \frac{4^n \cdot n^n}{(n\,-1)^{n\,-1}} \cdot x^n\Big). \tag{29}$$

The corresponding linear differential operator is an order-n linear differential operator having a *symplectic* $Sp(n, \, \mathbb{C})$ differential Galois group.

For n odd the same formula (29) holds. Note, however, that the argument $1/2$ appears in both the first and second list of arguments of the $\, _nF_{n\,-1}$ hypergeometric function, and, hence, can be avoided, which is thus a $\, _{n-1}F_{n\,-2}$ hypergeometric function. The corresponding linear differential operator is an order-$(n-1)$ operator having a *symplectic* $Sp(n-1, \, C)$ differential Galois group.

7. Diagonals of rational functions associated with operators non homomorphic to their adjoints

After all this accumulation of examples of diagonals of rational functions associated with orthogonal, or symplectic, differential Galois groups, it is tempting to conjecture that counter-examples§ like $\, _3F_2([1/3, \, 1/3, \, 1/3], \, [1, \, 1], \, 3^6\, x)$, are quite "rare exceptional cases" that one can easily detect, and, hopefully, understand, as situations of algebraic varieties "sufficiently singular" *to break the Poincaré duality* [32]. Along this line of seeking for diagonals of rational functions with a "sufficiently singular" denominator, let us try to provide examples of operators that are *not homomorphic to their adjoints*.

Appendix B provides two attempts to find examples of diagonals of rational functions such that their corresponding linear differential operators would not be homomorphic to their adjoints. We first study in Appendix B.1 a set of singular denominators for the rational functions, the polynomials of three variables corresponding to *classifications of singular varieties* performed by V. I. Arnold [56].

§ Given in the introduction corresponding to a $SL(3, \mathbb{C})$ differential Galois group, and thus the corresponding operator cannot be homomorphic to its adjoint *even with an algebraic extension.*

All the corresponding operators yield symplectic $Sp(n, \mathbb{C})$ differential Galois groups: the kind of singular behaviour required to "break the Poincaré duality", considered in V. I. Arnold [56] for three variables, is not "sufficiently singular".

It could be that these "sufficiently singular" situations require much *more than three variables*†. Therefore we have considered, here, rational functions of *six* variables, and since the theory of singularities of algebraic varieties [57, 37] suggests that the situation of algebraic varieties "singular enough" often correspond to denominators that *factor*††, we have considered, in Appendix B.2, rational functions of the form $1/\left((1-P_1)(1-P_2)\right)$ where P_1 and P_2 are two polynomials of two different sets of three variables‡. Again we found that *all*♯ the operators that could be analyzed among the 170 operators in Appendix B.2, are *homomorphic to their adjoints, and have symplectic or orthogonal differential Galois groups*.

We also considered the (minimal order) linear differential operators annihilating large sets¶ of *Hadamard products* [1, 2, 29] *of algebraic functions*, and, *each time*, obtained that these linear differential operators are homomorphic to their adjoints. Surprisingly, finding diagonals of rational functions such that their annihilating linear differential operators are *not* homomorphic to their adjoints, is not so easy, examples like ${}_3F_2([1/3, 1/2, 1/2], [1, 1], 12^2 x)$, being *quite rare*. Let us revisit these simple hypergeometric examples yielding operators that are *not homomorphic* to their adjoints.

7.1. Examples of diagonals of rational functions with no homomorphism to the adjoint

The hypergeometric function ${}_3F_2([1/3, 1/2, 1/2], [1, 1], 12^2 x)$, actually corresponds to a $SL(3, \mathbb{C})$ differential Galois group. It can be seen as the diagonal of a rational function of *six variables*

$$Diag\Big(\frac{1-9xy}{(1-3y-2x+3y^2+9x^2y)\cdot(1-u-z)\cdot(1-v-w)}\Big) \tag{30}$$
$$= \quad 1 + 12x + 648x^2 + 50400x^3 + 4630500x^4 + 468087984x^5 + \cdots$$

The corresponding (order-three) linear differential operator is *not homomorphic to its adjoint*, even with an algebraic extension. This diagonal (30) is of the form $Diag(R_1(x, y)\cdot R_2(u, z)\cdot R_3(v, w))$, (where R_1, R_2 and R_3 are simple rational functions). It is, thus, the Hadamard product [1, 2, 29] of the three diagonals $Diag(R_1(x, y))$, $Diag(R_2(u, z))$ and $Diag(R_2(v, w))$, which are simple *algebraic functions*, respectively:

$$Diag\Big(\frac{1-9xy}{1-2x-3y+3y^2+9x^2y}\Big) \;=\; \frac{1}{(1-9x)^{1/3}}, \tag{31}$$

† Counter-examples, like ${}_3F_2([1/3, 1/3, 1/3], [1, 1], 3^6 x)$, correspond to diagonal of algebraic functions of three variables (which are very simple since they are products of the algebraic functions of one variable), but this means that they are diagonals of rational functions of, at first sight, *six* variables.

†† This is actually the case for the counter-examples, like ${}_3F_2([1/3, 1/3, 1/3], [1, 1], 3^6 x)$ mentioned in the introduction.

‡ This also corresponds to considering the *Hadamard product of the diagonal of* $1/(1-P_1(x, y, z))$ *and of the diagonal of* $1/(1-P_2(u, v, w))$.

♯ Even using Koutschan's creative telescoping program [47], two operators among the 170 were difficult to obtain. Among these 170 operators, 71 are so large that they cannot be analyzed using the DEtools commands, and required to switch to a theta-system approach (see [21]).

¶ Some miscellaneous heuristic examples are given in the supplementary materials [42].

$$Diag\Big(\frac{1}{1\,-\,u\,-\,z}\Big) \,\,\, = \,\,\, Diag\Big(\frac{1}{1\,-\,v\,-\,w}\Big) \,\,\, = \,\,\, \frac{1}{(1\,-\,4\,x)^{1/2}}. \qquad (32)$$

The linear differential operators, annihilating the diagonals (31) and (32), are very simple *order-one* linear differential operators. There is, of course, no relation between their differential Galois groups and the $SL(3, \mathbb{C})$ differential Galois group of the order-three operator annihilating their Hadamard product (30): *the Hadamard product does not preserve algebraic structures*, like the differential Galois group.

Remark: Let us consider the following rational function of six variables:

$$\frac{N(x,\, y,\, z,\, u,\, v,\, w)}{(1-2\,x-3\,y\,+9\,x^2y\,+3\,y^2)\cdot\,(1\,-\,z\,-\,u)\cdot\,(1\,-\,v\,-\,w)} \qquad (33)$$
$$= \,\,\, \frac{1\,-\,9\,xy}{1-2\,x-3\,y\,+9\,x^2y\,+3\,y^2} \,\,+\,\, \frac{1}{1\,-\,z\,-\,u} \,\,+\,\, \frac{1}{1\,-\,v\,-\,w}.$$

It is straightforward to see that the diagonal of this rational function (33) of six variables is nothing but the sum of the three diagonals (31) and (32), namely:

$$\frac{1}{(1\,-\,9\,x)^{1/3}} \,\,+\,\, \frac{2}{(1\,-\,4\,x)^{1/2}}. \qquad (34)$$

This diagonal is solution of an order-two linear differential operator which is (obviously) a direct sum:

$$\Big(D_x \,-\, \frac{3}{1\,-\,9\,x}\Big) \,\oplus\, \Big(D_x \,-\, \frac{2}{1\,-\,4\,x}\Big). \qquad (35)$$

The differential Galois group of the (order-three) operator that annihilates (30), and the one of the (order-two) operator (35) that annihilates (33), are quite different, *even if the denominators of* (30) and (33) *are the same*. In a theory of singularity perspective [37, 57], the statement that the differential Galois group depends essentially on the denominator‡ of the rational function that "encodes the singularities", has to be taken "cum grano salis". The restriction we have imposed on our rational functions by imposing the numerators to be equal to one, is far from innocent.

7.2. More examples of diagonals of rational functions with no homomorphism to the adjoint

A slight modification of the previous example (30) amounts to considering the diagonal of a rational function of *six variables*

$$Diag\Big(\frac{1\,-\,9\,x\,y}{(1-3\,y-2\,x+3\,y^2+8\,x^2\,y)\cdot\,(1\,-\,u\,-\,z)\cdot\,(1\,-\,v\,-\,w)}\Big), \qquad (36)$$

where the coefficient of $x^2\,y$ has been modified ($9\,x^2\,y$ changed into $8\,x^2\,y$). The corresponding (order-five) linear differential operator is *not homomorphic to its adjoint*, even with an algebraic extension, and its differential Galois group is $SL(5, \mathbb{C})$.

Another slight modification of (30) amounts to considering the diagonal of a rational function of *five variables*

$$Diag\Big(\frac{1\,-\,9\,x\,y}{(1-3\,y-2\,x+3\,y^2+9\,x^2\,y)\cdot\,(1\,-\,u\,-\,v\,\,-\,z\,-\,w)}\Big). \qquad (37)$$

‡ Mathematicians would say that changing the numerator may change the n-form one integrates, that one must consider the Gauss-Manin Picard-Vessiot module of these differentials. We just try here, heuristically, to make crystal clear that numerators matter.

The corresponding (order-four) linear differential operator is *not homomorphic to its adjoint*, even with an algebraic extension, and its differential Galois group is $SL(4, \mathbb{C})$.

More examples of diagonals of rational functions with linear differential operators, that are not homomorphic to their adjoints, are given in Appendix C.

To sum-up: The cases such that the diagonals of rational functions do not yield linear differential operators homomorphic to their adjoints, are far from being understood, either from a differential algebra viewpoint, or from a *theory of singularity* perspective. From an experimental mathematics perspective, what we see is that diagonals of rational functions "almost systematically" (but not always !) yield linear differential operators homomorphic to their adjoints, thus giving selected differential Galois groups. Diagonals of rational functions were seen [1, 2] to naturally emerge in physics. However, in a mathematical framework, *not related to physics*, diagonals of rational functions seem to "almost systematically" yield *orthogonal or symplectic* groups, with, at first sight, no obvious "physical interpretation".

8. Conclusion

We have introduced well-defined sets of diagonals of rational functions of three, four and six variables, showing that all these examples yield selected differential Galois groups, namely *orthogonal and symplectic* groups.

It has been seen that, in our set of rational functions of *three* variables with denominators with degree bounded by 1, all diagonals correspond to *modular forms* that can all be written as ${}_2F_1$ hypergeometric functions with *two pull-backs* [35], related by a *modular curve*.

We have seen that a set of 879 diagonal of rational functions of *four* variables correspond to *orthogonal* $SO(n, \mathbb{C})$ differential Galois groups with a remarkable canonical decomposition [21] with a rightmost self-adjoint operator of order three. These results were obtained using the very powerful package "HolonomicFunctions" written by C. Koutschan [47, 48, 49], based on the method of *creative telescoping* [50], which enables to obtain directly, and very efficiently, the linear differential operator annihilating a given diagonal of a rational function, without calculating the series expansion of the corresponding diagonal. In order to find the homomorphisms of these operators to their adjoints, which is the first step towards the analysis of the differential Galois groups of these operators and their "canonical decompositions" [21], we have, for large linear differential operators, also used a new algorithm that requires to work on the linear *theta-system* associated with the operators [21].

We have also seen (Appendix B.2 below) that 170 diagonals of rational functions of six variables of the form $1/\left((1 - P_1(x, y, z))(1 - P_2(u, v, w))\right)$, actually correspond to a quite rich set of linear differential operators with orthogonal or symplectic differential Galois groups. The systematic analysis performed in this paper of these three sets of diagonals of rational functions of respectively three, four and six variables, suggests that diagonals of rational functions "almost systematically" yield orthogonal or symplectic differential Galois groups.

A contrario, we have provided in section (7) a few miscellaneous examples of diagonals of rational functions where the corresponding linear differential operators are *not homomorphic to their adjoints*. These cases, such that the diagonals of rational functions do not yield linear differential operators homomorphic to their adjoints, are far from being fully understood. Is it possible that such cases could also emerge with the diagonals of rational functions appearing in physics? This remains an open and challenging question.

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Appendix A. Diagonals of rational functions of three variables: some modular forms

Let us recall the two *Hauptmoduls* [35] $12^3/j_3$ and $12^3/j_4$ of Table 4 of Maier [46]:

$$H_3(z) \;=\; \frac{12^3 \cdot z}{(z\,+27)\cdot\,(z\,+3)^3}, \qquad H_4(z) \;=\; \frac{12^3 \cdot\, z \cdot (z+16)}{(z^2\,+16\,z\,+16)^3}.$$

• The diagonal of $1/(1\,-\,x\,-\,y\,-\,y\,z\,-\,x\,z)$ *as well as the diagonal of* $1/(1\,-x-y-\,z\,+x\,z)$, correspond to the sequence $[1,4,36,400,4900,63504,\cdots]$ of the complete elliptic integral $K(4\,x^{1/2})$ (oeis number A002894 in Sloane's on-line encyclopedia) can be written as a pullbacked ${}_2F_1$ hypergeometric function:

$$(1\,-16\,x\,+16\,x^2)^{-1/4}\cdot\,{}_2F_1\Big([\frac{1}{12},\,\frac{5}{12}],\,[1],\,H_4(z)\Big) \quad \text{with:} \quad z \;=\; \frac{1-16\,x}{x}. \qquad \text{(A.1)}$$

Recalling Maier's paper [46] one knows that (A.1) can alternatively also be written a pullbacked ${}_2F_1$ hypergeometric function with the $12^3/j_4'$ of Table 5 of Maier [46]. The diagonal of $1/(1\,-x-y-z\,+x\,z)$ is clearly a *modular form.* As a byproduct this suggests that the diagonal of the rational function of three variables

$$\frac{z\,\cdot(1\,-2\,x\,-y)}{(1\,-x-y-z\,+x\,z)\cdot\,(1\,-x-y\,-xz-yz)} \qquad \text{(A.2)}$$

is zero. This can be checked directly.

• The diagonal of $1/(1\,-\,x\,-\,y\,-\,z\,-\,x\,y\,z)$, which corresponds to the sequence $[1,7,115,2371,54091,1307377,\cdots]$ (oeis number A081798 in Sloane's on-line encyclopedia) can be written as a pullbacked ${}_2F_1$ hypergeometric function:

$$(1\,-27\,x\,+3\,x^2\,-\,x^3)^{-1/4}\cdot\,(1\,-\,x)^{-1/4}\cdot\,{}_2F_1\Big([\frac{1}{12},\,\frac{5}{12}],\,[1],\,H_3(z)\Big)$$
$$\text{with:} \qquad z \;=\; \frac{1\,-30\,x\,+3\,x^2\,-x^3}{x}, \qquad \text{(A.3)}$$

or more simply as

$$\frac{1}{1-x}\cdot\,{}_2F_1\Big([\frac{1}{3},\,\frac{2}{3}],\,[1],\,\frac{27\,x}{(1\,-x)^3}\Big). \qquad \text{(A.4)}$$

• The diagonal of $1/(1 - x - y - z - x\,y - x\,z - y\,z)$, which corresponds to the sequence $[1, 12, 366, 13800, 574650, 25335072, \cdots]$ (no oeis number) can be written as a pullbacked ${}_2F_1$ hypergeometric function:

$$(1 \, - 48\,x \, - 24\,x^2)^{-1/4} \cdot \, {}_2F_1\Big([\frac{1}{12}, \, \frac{5}{12}], \, [1], \, H_3(z)\Big)$$
$$\text{with:} \qquad z \; = \; \frac{1 \, - 54\,x \, - 27\,x^2}{x \cdot (x \, + 2)}. \tag{A.5}$$

• The diagonal of $1/(1 - x - y - x\,z - y\,z + x\,y\,z)$, which corresponds to the sequence $[1, 3, 25, 243, 2601, 29403, \cdots]$ (oeis number A245925 in Sloane's on-line encyclopedia) can be written as a pullbacked ${}_2F_1$ hypergeometric function:

$$(1 \, - 12\,x \, - 10\,x^2 \, - 12\,x^3 \, + x^4)^{-1/4} \cdot \, {}_2F_1\Big([\frac{1}{12}, \, \frac{5}{12}], \, [1], \, H_4(z)\Big)$$
$$\text{with:} \qquad z \; = \; - \frac{(1 \, + x)^2}{x}. \tag{A.6}$$

• The diagonal of $1/(1 - x - y - z + x\,y\,z)$, which corresponds to the sequence‡ $[1, 5, 67, 1109, 20251, 391355, \cdots]$ (oeis number A124435 in Sloane's on-line encyclopedia) can be written as a pullbacked ${}_2F_1$ hypergeometric function:

$$(1 \, + x)^{-1/4} \cdot \, (1 \, - 21\,x \, + 3\,x^2 \, + x^3)^{-1/4} \cdot \, {}_2F_1\Big([\frac{1}{12}, \, \frac{5}{12}], \, [1], \, H_3(z)\Big)$$
$$\text{with:} \qquad z \; = \; \frac{1 \, - 24\,x \, + 3\,x^2 \, + x^3}{x}. \tag{A.7}$$

• The diagonal of $1/(1 - x + y + z + x\,y + x\,z - y\,z + x\,y\,z)$, which corresponds to the sequence $[1, 11, 325, 11711, 465601, 19590491, \cdots]$ (no oeis number) can be written as a pullbacked ${}_2F_1$ hypergeometric function:

$$(1 \, - 46\,x \, + x^2)^{-1/4} \cdot \, (1 \, + x)^{-1/2} \cdot \, {}_2F_1\Big([\frac{1}{12}, \, \frac{5}{12}], \, [1], \, H_3(z)\Big)$$
$$\text{with:} \qquad z \; = \; \frac{1 \, - 52\,x \, + x^2}{2\,x}. \tag{A.8}$$

Appendix B. Two attempts to break the Poincaré duality

Along the line, sketched in section (7), which amounts to seeking for diagonals of rational functions with a "sufficiently singular" denominator, let us try to find examples of linear differential operators that are *not homomorphic to their adjoints*. We study here two sets of singular denominators for the rational functions, first polynomials corresponding to classifications of singular varieties performed by V. I. Arnold [56], then denominators that factor into two polynomials.

Appendix B.1. Diagonals of rational functions associated with singular algebraic varieties

If one believes that the situations where the linear differential operator annihilating the diagonal of a rational function should correspond to situations of algebraic

‡ Number of effective multiple alignments of three equal-length sequences.

varieties "singular enough" to break the Poincaré duality [32], it is tempting to study the singular algebraic varieties classified by V. I. Arnold [56]. The classification of the simplest singularities turned out to be related to Lie, Coxeter and Weyl groups, A_n, D_n, E_n, and to the classification of platonic solids in Euclidean three spaces [56]. Arnold's paper gives a set of polynomials of three variables that are obvious candidates to be considered as denominators of rational functions of three variables: $P = x^2 z + y^3 + z^5 + 2 y z^4, \; x^2 z + y z^2 + a y^3 z, \, x^3 + y^3 + z^4 + a x y z^2, \, ...$

With these polynomials, we obtained the linear differential operators annihilating the diagonals of the rational functions of the form $1/P$. These linear differential operators of various orders (order 22 for Q_{10} in [56], order 14 for Q_{12} in [56], order 12 for S_{11}, order 8 for S_{12}, order 10 for U_{12}, order 16 for Z_{13}, order 18 for W_{12}, ...) are *all homomorphic to their adjoints*, their differential Galois groups being $Sp(n, \mathbb{C})$ *symplectic* groups, the canonical decomposition [21] being in terms of only order-two self-adjoint operators.

Appendix B.2. Diagonals of rational functions of six variables

Apparently the kind of singular behaviour required to "break the Poincaré duality", considered in V. I. Arnold [56] for *three variables*, is not "sufficiently singular". It could be that these "sufficiently singular" situations require much more than three variables. Therefore we have considered in this section, rational functions of *six* variables (see section (7.1) and (Appendix C) below). Since the theory of singularities of algebraic varieties suggests that the situation of "singular enough" algebraic varieties often correspond to algebraic varieties that *factor*, we have studied exhaustively the diagonals of rational functions of the form $1/\left((1 - P_1(x, y, z))(1 - P_2(u, v, w))\right)$, where the degree of the two polynomials in each of their three variables is less than one, and their coefficients are 0 or 1. Note that this also corresponds to considering the Hadamard product of the diagonal of $1/(1 - P_1(x, y, z))$ and of the diagonal of $1/(1 - P_2(u, v, w))$. The number of such classes yielding different diagonals is only 170. We have obtained all the corresponding linear differential operators using the Mathematica "HolonomicFunctions" package [47]. The order of the (minimal order) linear differential operators runs from 2 to 12. The number of linear differential operators corresponding to the various orders is given in Table (B1).

Table B1. Number of operators corresponding to the various orders, number of operators with symplectic and orthogonal differential Galois groups.

Order of Oper.	2	3	4	5	6	7	8	9	10	11	12
Number of Oper.	2	3	19	13	39	0	52	0	36	0	6
Number of $Sp(n, \mathbb{C})$	2	0	7	0	35	0	52	0	36	0	6
Number of $SO(n, \mathbb{C})$	0	3	12	13	4	0	0	0	0	0	0

We also give the number of linear differential operators having respectively a symplectic and an orthogonal differential Galois group. Among these 170 linear differential operators, 91 can be analyzed using the DEtools command in order to see that they are homomorphic to their adjoint, find their differential Galois group and their canonical decomposition [21]. The other linear differential operators are too large to be analyzed that way: they *require to switch to a differential theta-*

system [21] in order to find the intertwiners to their adjoint, and, then the canonical decomposition from a simple euclidean division [21]. Note that these calculations (*theta-system calculations*) are still quite massive.

We found that the three operators with a $SO(3, \mathbb{C})$ orthogonal differential Galois group were self-adjoint.

We found that all the seven order-4 linear differential operators with a symplectic differential Galois group (namely $Sp(4, \mathbb{C})$) are *self-adjoint*, or conjugated to their adjoints (by a simple rational function), except one with a $(L_2 \cdot M_2 + 1) \cdot r(x)$ canonical decomposition [21]. We found that the 12 linear differential operators with a $SO(4, \mathbb{C})$ orthogonal differential Galois group had a $(L_1 \cdot M_3 + 1) \cdot r(x)$ canonical decomposition [21].

All the 13 order-5 linear differential operators which have an $SO(5, \mathbb{C})$ orthogonal differential Galois group have a $(L_1 \cdot M_1 \cdot N_3 + L_1 + N_3) \cdot r(x)$ canonical decomposition [21]. The four order-6 linear differential operators with an $SO(6, \mathbb{C})$ orthogonal differential Galois group have a $(L_1 \cdot M_1 \cdot N_1 \cdot P_3 + L_1 \cdot P_3 + N_1 \cdot P_3 + L_1 \cdot M_1 + 1) \cdot r(x)$ canonical decomposition [21].

We found that the other 35 order-6 linear differential operators have a $Sp(6, \mathbb{C})$ symplectic differential Galois group. Among these 35 symplectic operators only two have a $(L_2 \cdot M_2 \cdot N_2 + L_2 + N_4) \cdot r(x)$ canonical decomposition [21], all the other having a $(L_2 \cdot M_4 + 1) \cdot r(x)$ canonical decomposition [21].

We found that all the 52 order-8 linear differential operators have a $Sp(8, \mathbb{C})$ symplectic differential Galois group with a $(L_2 \cdot M_2 \cdot N_4 + L_2 + N_4) \cdot r(x)$ canonical decomposition [21].

All the 36 order-10 and 6 order-12 linear differential operators have respectively $Sp(10, \mathbb{C})$ and $Sp(12, \mathbb{C})$ *symplectic* differential Galois groups, with $(L_2 \cdot M_2 \cdot N_2 \cdot P_4 + \cdots) \cdot r(x)$ and $(L_2 \cdot M_2 \cdot N_2 \cdot P_2 \cdot Q_4 + \cdots) \cdot r(x)$ canonical decomposition [21].

Remark 1: In all the decomposition we have obtained, the rightmost self-adjoint operator [21], is always of *order three* (for orthogonal differential Galois groups) or of *order four* (for symplectic differential Galois groups), except an order-4 operatoir with a $(L_2 \cdot M_2 + 1) \cdot r(x)$ decomposition and the two order-6 linear differential operators corresponding to the diagonals of the two rational functions

$$(1 - xy - xz - yz) \cdot (1 - v - w - uv - uw), \tag{B.1}$$

$$(1 - z - xy - xz - yz - xyz) \cdot (1 - u - v - w - uvw), \tag{B.2}$$

the corresponding operators having a $(L_2 \cdot M_2 \cdot N_2 + L_2 + N_2) \cdot r(x)$ canonical decomposition [21].

Remark 2: Using Koutschan's creative telescoping program [47], three linear differential operators were quite difficult to obtain compared to the others. They correspond to the following denominators of the rational functions:

$$(1 - x - y - z - xz - xyz) \cdot (1 - u - v - w - uw - vw - uvw), \tag{B.3}$$

$$(1 - x - y - z - xz - xyz) \cdot (1 - u - v - w - uv - uw - vw - uvw), \tag{B.4}$$

$$(1 - x - y - z - xz - xyz) \cdot (1 - u - v - w - uv - uw - vw). \tag{B.5}$$

Koutschan's creative telescoping program gives an order-12 linear differential operator for (B.3). Note that one always needs to verify that the operator obtained from this program is actually the *minimal order* operator annihilating the diagonal†. Actually,

† It may not be irreducible: see, for instance, the $L_7 = L_3 \cdot L_4$ operator associated with the diagonal (23) in section (5).

performing *Hadamard products* [7, 29] modulo primes, one can forecast that minimal order for (B.3). We found an order 10 which suggests that the order-12 linear differential operator obtained by Koutschan's creative telescoping program [47], is not the *minimal order* linear differential operator‡: this order-12 *actually factorises.* It is the product of *two order-one* operators and of the minimal order-10 linear differential operator annihilating the diagonal: $L_{12} = L_1 \cdot M_1 \cdot L_{10}$. The factorization of such very large linear differential operators♯ is a *quite difficult task*. In a first step calculations modulo prime have been performed that enable us to find the order of the minimal order operator annihilating all these diagonals. We have, however, been able to perform these factorizations (in characteristic zero, not modulo primes).

Remark 3: When the linear differential operator annihilating the diagonal, obtained from Koutschan's program [47], is *not the minimal order* linear differential operator, one can try to obtain that minimal order operator by factorization (DFactor in Maple). Unfortunately the factorization of (very) large operators like many of these operators, cannot be obtained using straightforwardly DFactor DEtools command in Maple. The factorization of the largest operators has been obtained, in the following way: one first obtains a large set of coefficients of the series of the corresponding diagonals (that is the non-trivial part of the calculation), and then use it as the input of a "guessing procedure¶".

Remark 4: Koutschan's creative telescoping program [47] gave us an order-9 linear differential operator corresponding to the diagonal of $1/((1 - x - y - z - x\,z - x\,y\,z)\,(1 - u - v - w - u\,w - u\,v\,w))$ (i.e. the Hadamard square of the diagonal of $1/(1 - x - y - z - x\,z - x\,y\,z))$. In fact this order-9 linear differential operator is *not minimal*, the minimal order operator being of order 8. This program gave us also four order-11 linear differential operators, like, for instance, the operator corresponding to the diagonal of (B.4). Similarly this order-11 linear differential operator is *not minimal*, the minimal order operator being of order 10 (the degree of the polynomial coefficients is 74). This is also the case for the three other, at first sight, order-11 linear differential operators corresponding to the following denominators of the rational functions:

$$(1 - z - xz - yz - xy) \cdot (1 - u - v - w - uw - vw - uvw), \qquad (B.6)$$

$$(1 - y - z - xz - xy - yz) \cdot (1 - u - v - w - uv - uw - vw - uvw), \qquad (B.7)$$

$$(1 - x - y - z - xz - xyz) \cdot (1 - u - v - w - uw - vw). \qquad (B.8)$$

They are *not minimal order operators*, the minimal order linear differential operators, annihilating the corresponding diagonals being of order 10 (the degree of the polynomial coefficients being respectively 42, 51, 51). Therefore one finds that there is *no order*-9 *or order*-11 *operators* for this set of 170 diagonals.

The linear differential operator annihilating the diagonal of (B.5) was the most difficult to obtain using Koutschan's creative telescoping program♯. This program gives an order-13 linear differential operator. From this exact differential operator one can obtain as long as necessary series of the diagonal, and study, modulo some primes,

‡ One can even forecast the degree of the polynomial coefficients of that order-10 operator: the degree is 74.

♯ Using the DFactor command of DEtools in Maple, or any other method.

¶ Essentially of the same type as in gfun.

♯ The program computes the "telescoper" (the ODE) and the "certificate". It is a general observation that in most examples, the certificate is much larger than the telescoper [58]. We just need the telescoper. An algorithm that computes telescopers without computing the corresponding certificates, has already been built [58, 59].

the linear differential operators annihilating these series. One finds, that way, that the minimal order operator annihilating the diagonal of (B.5) is of order 12 (with degree 85). We actually factorised this order-13 operator, and obtained the minimal order-12 operator that annihilates this diagonal¶.

Remark 5: Seeking for the (large ...) linear differential operators annihilating the diagonals of (B.4), (B.5), it seems natural to take into account the selected form of all these diagonals of rational functions of six variables, which are, actually, the *Hadamard product* [29, 7] of two diagonals of rational functions of three variables. In the case of (B.4), (B.5) (and even (B.3)), the linear differential operators annihilating these various diagonals of rational functions of three variables, are simple order-two operators. One can imagine to obtain these linear differential operators using the gfun command "hadamard product" of two simple order-two operators‡. Unfortunately, the linear differential operators obtained that way are of *order much larger* than the one obtained from Koutschan's creative telescoping of a diagonal of a function of six variables ! The same remarks apply for the analysis of all the 170 diagonals of Table (B1).

Let us give, here, miscellaneous, simple examples of these diagonal of rational functions of six variables (all the exhaustive results being given in a supplementary material [42]).

Appendix B.2.1. Order-two operators: pullbacked $_2F_1$ hypergeometric function

• The diagonal of the rational function of six variables

$$Diag\Big(\frac{1}{(1\,-w\,-u\,v\,-u\,v\,w)\cdot(1\,-\,z\,-\,x\,y)}\Big) \tag{B.9}$$
$$= \;\; 1\,+6\,x\,+78\,x^2\,+1260\,x^3\,+22470\,x^4\,+424116\,x^5\,+8305836\,x^6\,+\,\cdots$$

is annihilated by an order-two linear differential operator. This diagonal is the Hadamard product [7, 29] of the diagonal of $1/(1\,-\,z\,-\,x\,y)$ and of the diagonal of $1/(1\,-w\,-u\,v\,-u\,v\,w)$, namely the *two simple algebraic functions*:

$$(1\,-4\,x)^{-1/2} \tag{B.10}$$
$$= \;\; 1\,+2\,x\,+6\,x^2\,+20\,x^3\,+70\,x^4\,+252\,x^5\,+924\,x^6\,+\,\cdots$$
$$(1\,-6\,x\,+x^2)^{-1/2} \tag{B.11}$$
$$= \;\; 1\,+3\,x\,+13\,x^2\,+63\,x^3\,+321\,x^4+1683\,x^5\,+8989\,x^6\,+\,\cdots$$

This diagonal is actually a pullbacked $_2F_1$ hypergeometric function:

$$(1\,-24\,x\,+48\,x^2)^{-1/4}\cdot\,{}_2F_1\Big([\frac{1}{12},\,\frac{5}{12}],\,[1],\,P_1(x)\Big)$$
$$\text{where:}\qquad P_1(x)\;=\;6912\,\frac{x^4\cdot(1\,-24\,x\,+16\,x^2)}{(1\,-24\,x\,+48\,x^2)^3}. \tag{B.12}$$

¶ After completion of this work we were told by C. Koutschan that giving extra options to "FindCreativeTelescoping" enables the program to find directly the minimal order-12 operator in 84 hours CPU time, instead of the 116 hours CPU time we used to obtain the order-13 operator. With these extra options the telescoper is 232560 bytes when the certificate requires 32743760 bytes, to be compared with 265432 bytes and 38422496 bytes for respectively the telescoper and certificate in the order-13 calculation.

‡ The gfun[hadamardproduct](eq1, eq2, y(z)) command determines the linear differential equation satisfied by the Hadamard product of two holonomic functions, solutions of the linear differential equations eq1 and eq2.

This *Hauptmodul* can be seen to be of the form $12^3/j_2$ (see Table 4 of Maier [46]):

$$P_1(x) \,\, = \,\, 12^3 \cdot \frac{z}{(z\, +16)^3} \qquad \hbox{with:} \qquad z \,\, = \,\, \frac{1\, -24\,x\, +16\,x^2}{2\,x^2}. \tag{B.13}$$

This diagonal (B.9) has another pullbacked ${}_2F_1$ hypergeometric function representation, corresponding to the other pullback $P_2(x) \, = \, 12^3 \cdot \, z^2/(z\, +256)^3$:

$$(1\, -24\,x\, +528\,x^2)^{-1/4} \cdot \, {}_2F_1\Big([\frac{1}{12}, \, \frac{5}{12}], \, [1], \, P_2(x)\Big)$$
$$\hbox{where:} \qquad P_2(x) \,\, = \,\, 3456\,\frac{x^2 \cdot \, (1\, -24\,x\, +16\,x^2)^2}{(1\, -24\,x\, +528\,x^2)^3}. \tag{B.14}$$

The pullbacked ${}_2F_1$ hypergeometric function (B.12) (or equivalently (B.14)) is thus a *modular form*.

Note that this diagonal (B.9) is, in fact, actually identical to the diagonal of the rational function of three variables $1/(1\, -x -y -z\, -x\,y +x\,z)$.

- The diagonal of the rational function of six variables

$$Diag\Big(\frac{1}{(1\, -w\, -u\,v\, -u\,v\,w) \cdot \, (1-\, z\, -x\,y\, -x\,y\,z)}\Big) \tag{B.15}$$
$$= \,\, 1\, +9\,x\, +169\,x^2\, +3969\,x^3\, +103041\,x^4\, +2832489\,x^5\, +80802121\,x^6\, + \, \cdots$$

is annihilated by an order-two linear differential operator. This diagonal is the *Hadamard square* of the diagonal of $1/(1\, -\, z\, -x\,y\, -x\,y\,z)$, namely the Hadamard square of the algebraic function (B.11). This diagonal is actually a pullbacked ${}_2F_1$ hypergeometric function:

$$(1\, -36\,x\, +134\,x^2\, -36\,x^3\, +x^4)^{-1/4} \cdot \, {}_2F_1\Big([\frac{1}{12}, \, \frac{5}{12}], \, [1], \, P_1(x)\Big)$$
$$\hbox{where:} \qquad P_1(x) \,\, = \,\, 27648\,\frac{x^4 \cdot \, (1\, -x)^2 \cdot \, (1\, -34\,x\, +x^2)}{(1\, -36\,x\, +134\,x^2\, -36\,x^3\, +x^4)^3}. \tag{B.16}$$

It is also be written as pullbacked ${}_2F_1$ hypergeometric function:

$$(1\, +444\,x\, +134\,x^2\, +444\,x^3\, +x^4)^{-1/4} \cdot \, {}_2F_1\Big([\frac{1}{12}, \, \frac{5}{12}], \, [1], \, P_2(x)\Big)$$
$$\hbox{where:} \qquad P_2(x) \,\, = \,\, 3456\,\frac{x \cdot \, (1-x)^2 \cdot \, (1\, -34\,x\, +x^2)^4}{(1\, +444\,x\, +134\,x^2\, +444\,x^3\, +x^4)^3}. \tag{B.17}$$

which shows that this diagonal is a *modular form*: these two pullbacks can be seen as the Hauptmoduls $12^3/j_4$, $12^3/j_4'$ in Table 4 and 5 of Maier [46]:

$$P_1(x) \,\, = \,\, 12^3 \cdot \, \frac{z \cdot \, (z\, +16)}{(z^2\, +16\,z\, +16)^3} \qquad \hbox{where:} \qquad z \,\, = \,\, \frac{x^2-34\,x+1}{2\,x}. \tag{B.18}$$

Note that this diagonal (B.15) is, in fact, actually identical to the diagonal of the rational function of three variables $1/(1\, -x -y -z\, -x\,y +x\,z -y\,z\, -x\,y\,z)$.

Appendix B.2.2. Order-four and order-eight operators

- The diagonal of the rational function of six variables

$$Diag\Big(\frac{1}{(1\, -w\, -u\,v) \cdot \, (1-x\,y\, -x\,z\, -y\,z)}\Big) \,\, = \,\, 1\, +36\,x^2\, +6300\,x^4\, + \, \cdots \tag{B.19}$$

is annihilated by a linear differential operator of order four, L_4, which has an orthogonal differential Galois group $SO(4, \mathbb{C})$, with a simple canonical decomposition [21], $L_4 = D_x \cdot L_3 + 12$, where L_3 is an order-three self-adjoint linear differential operator:

$$L_3 = x^2 \cdot (432\,x^2 - 1) \cdot D_x^3 + 3\,x \cdot (864\,x^2 - 1) \cdot D_x^2 + (2868\,x^2 - 1) \cdot D_x + 276\,x \qquad \text{(B.20)}$$

• The diagonal of the rational function

$$Diag\Big(\frac{1}{(1 - u\,v - u\,w - v\,w) \cdot (1 - x\,y - x\,z - y\,z)}\Big) = 1 + 36\,x^2 + 8100\,x^4 + 2822400\,x^6 + \cdots \qquad \text{(B.21)}$$

is annihilated by a linear differential operator of order four, L_4, which is self-adjoint, its differential Galois group being the symplectic group $Sp(4, \mathbb{C})$.

• The diagonal of the rational function

$$Diag\Big(\frac{1}{(1 - u\,v - u\,w - v\,w - u\,v\,w) \cdot (1 - x\,y - x\,z - y\,z)}\Big) = 1 + 42\,x^2 + 13590\,x^4 + 7410480\,x^6 + \cdots \qquad \text{(B.22)}$$

is annihilated by a linear differential operator of order eight, L_8, which has a symplectic differential Galois group $Sp(8, \mathbb{C})$, with a simple canonical decomposition [21], $L_8 = (L_2 \cdot M_2 \cdot N_4 + L_2 + N_4) \cdot r(x)$, where L_2 and M_2 are two order-two self-adjoint operators and N_4 is an *order-four self-adjoint* linear differential operator.

Appendix C. Miscellaneous examples of diagonals of rational functions with operators that are not homomorphic to their adjoints

Another slight modification of (30) amounts to considering the diagonal of a rational function of *six variables*

$$Diag\Big(\frac{1 - 9\,x\,y}{(1 - 3\,y - 2\,x + 3\,y^2 + 9\,x^2\,y) \cdot (1 - z - u - u\,z) \cdot (1 - v - w)}\Big). \qquad \text{(C.1)}$$

The corresponding (order-four) linear differential operator is *not homomorphic to its adjoint*, even with an algebraic extension, and its differential Galois group is $SL(4, \mathbb{C})$.

Similarly, the diagonal of the *five variable* rational function

$$Diag\Big(\frac{1}{(1 - x + 3\,y - 27\,x\,y^3 - 27\,x\,y^2 - 9\,x\,y + 3\,y^2) \cdot (1 - u - v - u\,z - v\,z)}\Big),$$

is annihilated by an order-three linear differential operator which is *not homomorphic to its adjoint* (even with an algebraic extension), and its differential Galois group is $SL(3, \mathbb{C})$.

Similarly, the diagonal of the *five variable* rational function

$$Diag\Big(\frac{1 - 9\,x\,y}{(1 - 3\,y - 2\,x + 3\,y^2 + 9\,x^2\,y) \cdot (1 - u - v - w)}\Big), \qquad \text{(C.2)}$$

is annihilated by an order-three linear differential operator which is *not homomorphic to its adjoint*, and its differential Galois group is $SL(3, \mathbb{C})$.

Similarly, the diagonal of the *five variable* rational function

$$Diag\Big(\frac{1 - 9\,x\,y}{(1 - 3\,y - 2\,x + 3\,y^2 + 8\,x^2\,y) \cdot (1 - u - v - w)}\Big), \qquad \text{(C.3)}$$

is annihilated by an order-five linear differential operator which is *not homomorphic to its adjoint*, and its differential Galois group is $SL(5, \mathbb{C})$.

References

[1] A. Bostan, S. Boukraa, G. Christol, S. Hassani, J-M. Maillard, *Ising n-fold integrals as diagonals of rational functions and integrality of series expansions*, (2013), J. Phys. **A 46**: Math. Theor. 185202 (44 pages), `http://arxiv.org/abs/1211.6645v2`

[2] Bostan A, Boukraa S, Christol G, Hassani S and Maillard J-M Ising n-fold integrals as diagonals of rational functions and integrality of series expansions: integrality versus modularity Preprint, `http://arxiv.org/abs/1211.6031`

[3] Zenine N, Boukraa S, Hassani S and Maillard J-M 2005 Square lattice Ising model susceptibility: connection matrices and singular behaviour of $\chi^{(3)}$ and $\chi^{(4)}$ *J. Phys. A* **38** pp. 9439-9474 `http://arxiv.org/abs/math-ph/0506065`

[4] Bostan A, Boukraa S, Guttmann A J, Hassani S, Jensen I, Maillard J-M and Zenine N 2009 High order Fuchsian equations for the square lattice Ising model: $\tilde{\chi}^{(5)}$ *J. Phys. A* **42** 275209, 32, `http://arxiv.org/abs/0904.1601`

[5] Boukraa S, Hassani S, Jensen I, Maillard J-M and Zenine N 2010 High-order Fuchsian equations for the square lattice Ising model: $\chi^{(6)}$ *J. Phys. A* **43** 115201, 22, `http://arxiv.org/abs/0912.4968`

[6] Boukraa S, Hassani S, Maillard J-M and Zenine N 2007 Singularities of n-fold integrals of the Ising class and the theory of elliptic curves *J. Phys. A* **40** pp. 11713-11748, `http://arxiv.org/abs/0706.3367`

[7] Assis M, Boukraa S, Hassani S, van Hoeij M, Maillard J-M and McCoy B M 2012 Diagonal Ising susceptibility: elliptic integrals, modular forms and Calabi-Yau equations *J. Phys. A* **45** 075205, 32 `http://arxiv.org/abs/1110.1705`

[8] Guttmann A J 2010 Lattice Green's functions in all dimensions *J. Phys. A* **43** 305205, 26, `http://arxiv.org/abs/1004.1435`

[9] Glasser M L and Guttmann A J 1994 Lattice Green function (at 0) for the 4D hypercubic lattice *J. Phys. A* **27** 7011–7014, `http://arxiv.org/abs/cond-mat/9408097`

[10] S. Boukraa, S. Hassani, J-M. Maillard, J-A. Weil, *Differential algebra on lattice Green functions and Calabi-Yau operators*, (2014), J. Phys. **A 48**: Math. Theor. 095203 (37 pages)

[11] N. Zenine, S. Hassani, J-M. Maillard, *Lattice Green Functions: the seven-dimensional face-centred cubic lattice*, (2015), J. Phys. **A 48**: Math. Theor 035205 (18 pages) arXiv:1409.8615v1 [math-ph]

[12] Lipshitz L and van der Poorten A J 1990 Rational functions, diagonals, automata and arithmetic *Number theory (Banff, AB, 1988)* (Berlin: de Gruyter) pp. 339-358

[13] Lipshitz L 1988 The diagonal of a D-finite power series is D-finite *J. Algebra* **113** 373–378 `http://dx.doi.org/10.1016/0021-8693(88)90166-4`

[14] Christol G 1984 Diagonales de fractions rationnelles et équations différentielles *Study group on ultrametric analysis, 10th year: 1982/83, No. 2, Exp. No. 18* (Paris: Inst. Henri Poincaré) pp 1–10 `http://archive.numdam.org/article/GAU_1982-1983__10_2_A4_0.pdf`

[15] Christol G 1985 Diagonales de fractions rationnelles et équations de Picard-Fuchs *Study group on ultrametric analysis, 12th year, 1984/85, No. 1, Exp. No. 13* (Paris: Secrétariat Math.) pp 1–12 `http://archive.numdam.org/article/GAU_1984-1985__12_1_A8_0.pdf`

[16] Christol G 1988 Diagonales de fractions rationnelles *Séminaire de Théorie des Nombres, Paris 1986–87* (*Progr. Math.* vol 75) (Boston, MA: Birkhäuser Boston) pp. 65-90

[17] A. Bostan, S. Boukraa, S. Hassani, J.-M. Maillard, J.-A. Weil, and N. Zenine. *Globally nilpotent differential operators and the square Ising model. J. Phys. A: Math. Theor.* **42** *(2009) 125206 (50pp) and arXiv:0812.4931.*

[18] G. Christol. 1990 *Globally bounded solutions of differential equations*, *Analytic number theory (Tokyo, 1988)* (*Lecture Notes in Math.* **1434** (Berlin: Springer) pp. 45-64 `http://dx.doi.org/10.1007/BFb0097124`

[19] M. van der Put and M. Singer, *Galois Theory of Linear Differential Equations*, Grundlehren der mathematischen Wissenschaften, Volume 328, Springer, 2003.

[20] S. Boukraa, S. Hassani, J-M. Maillard, *The Ising model and Special Geometries*, (2014), J. Phys. **A 47**: Math. Theor. 225204 (32 pages) arXiv:1402.6291v2 [math-ph]

[21] S. Boukraa, S. Hassani, J-M. Maillard, J-A. Weil *Canonical decomposition of linear differential operators with selected differential groups*, (2015), J. Phys. **A 48**: Math. Theor 105202 (40 pages) arXiv:1407.5541v2 [math-ph]

[22] Zenine N, Boukraa S, Hassani S and Maillard J-M 2004 The Fuchsian differential equation of the square lattice Ising model $\chi^{(3)}$ susceptibility *J. Phys. A* **37** 9651–9668 `http://arxiv.org/abs/math-ph/0407060`

[23] Boukraa S, Hassani S, Maillard J-M, McCoy B M and Zenine N 2007 The diagonal Ising

susceptibility *J. Phys. A* **40** 8219–8236 http://arxiv.org/abs/math-ph/0703009

[24] Zenine N, Boukraa S, Hassani S and Maillard J-M 2005 Ising model susceptibility: the Fuchsian differential equation for $\chi^{(4)}$ and its factorization properties *J. Phys. A* **38** pp. 4149-4173 http://arxiv.org/abs/cond-mat/0502155

[25] Bostan A, Boukraa S, Hassani S, van Hoeij M, Maillard J-M, Weil J-A and Zenine N 2011 The Ising model: from elliptic curves to modular forms and Calabi-Yau equations *J. Phys. A* **44** 045204, 44 http://arxiv.org/abs/1007.0535

[26] Y. Namikawa, *Stratified local moduli of Calabi-Yau threefolds*, Topology, **41** (2002), pp. 1219-1237

[27] V. Batyrev and M. Kreuzer, Constructing new Calabi-Yau 3-folds and their mirrors via conifold transitions, In: Adv. Theor. Math. Phys. 14.3 (2010), pp. 879-898.

[28] P. Lairez, 2014, Computing periods of rational integrals, *arXiv:1404.5069v2* and Supplementary material http://pierre.lairez.fr/supp/periods/.

[29] Hadamard J 1892 Essai sur l'étude des fonctions données par leur développement de Taylor *J. Math. Pures Appl.* **4** 101–186 http://portail.mathdoc.fr/JMPA/PDF/JMPA_1892_4_8_A4_0.pdf

[30] Cameron R H and Martin W T 1938 Analytic continuation of diagonals and Hadamard compositions of multiple power series *Trans. Amer. Math. Soc.* **44** 1–7 http://dx.doi.org/10.2307/1990100

[31] C. Sabbah, *Hodge theory, singularities and D-modules*, Lecture Notes (CIR, Luminy, March 2007)

[32] P. Griffiths and J. Harris, *Principles of algebraic geometry*, (1994) Wiley Classics Library, New York: Wiley.

[33] M. Bogner, *Algebraic characterization of differential operators of Calabi-Yau type* http://arxiv.org/pdf/1304.5434v1.pdf

[34] S. C. Coutinho (1995), *A primer of algebraic D-modules*, London Mathematical Society Student Texts 33, Cambridge University Press.

[35] A. Bostan, S. Boukraa, S. Hassani, M. van Hoeij, J-M. Maillard, J-A. Weil, N. J. Zenine, *The Ising model: from elliptic curves to modular forms and Calabi-Yau equations*, J. Phys. **A** **44**: Math. Theor. (2011) (43 pp) 045204 and arXiv: 1007.69804 v1 [math-ph]

[36] P. Lairez, 2014, Périodes d'intégrales rationnelles, Algorithmes et applications, Thèse de doctorat Mathématiques, Ecole Polytechnique, INRIA Saclay.

[37] H. Hauser, *The Hironaka theorem on the resolution of singularities (or; A proof we always wanted to understand)*, Bull. Am. Math. Soc. New Ser. **40**, (2003) pp. 323-403

[38] Furstenberg H 1967 Algebraic functions over finite fields *J. Algebra* **7** pp. 271-277 http://dx.doi.org/10.1016/0021-8693(67)90061-0

[39] Denef J and Lipshitz L 1987 Algebraic power series and diagonals *J. Number Theory* **26** 46–67 http://dx.doi.org/10.1016/0022-314X(87)90095-3

[40] Adamczewski B and Bell J P Diagonalization and Rationalization of algebraic Laurent series Preprint http://arxiv.org/pdf/1205.4090.pdf

[41] Pólya G 1921–1922 Sur les séries entières, dont la somme est une fonction algébrique *Enseignement Math.* **22** 38–47 http://retro.seals.ch/digbib/view?rid=ensmat-001:1921-1922:22::220

[42] Jacques-Arthur Weil, *Diagonals of rational functions and selected differential Galois groups*, http://www.unilim.fr/pages_perso/jacques-arthur.weil/diagonals/

[43] N. J. A. Sloane, The on-line encyclopedia of integer sequences, https://oeis.org/

[44] V. J. Kunwar and M. van Hoeij, *Second order differential equations with hypergeometric solutions of degree three*, Proceeding ISSAC '13 Proceedings of the 38th International Symposium on Symbolic and Algebraic Computation, Pages 235-242, ACM New York, NY, USA (2013)

[45] V. J. Kunwar and M. van Hoeij, http://www.math.fsu.edu/~vkunwar/hypergeomdeg3/

[46] Maier R S 2009 On rationally parametrized modular equations *J. Ramanujan Math. Soc.* **24** pp. 1-73 http://arxiv.org/abs/math/0611041

[47] HolonomicFunctions Package version 1.7.1 (09-Oct-2013) written by Christoph Koutschan, Copyright 2007-2013, Research Institute for Symbolic Computation (RISC), Johannes Kepler University, Linz, Austria

[48] C. Koutschan, HolonomicFunctions (User's Guide) June 17, (2013)

[49] Research Institute for Symbolic Computation (RISC), Johannes Kepler University, Linz, Austria http://www.risc.jku.at/research/combinat/software/ergosum/RISC/HolonomicFunctions.html.

[50] D. Zeilberger, *The Method of Creative Telescoping*, (1991) J. Symbolic Computation **11** pp. 195-204

[51] C. Koutschan, *A Fast Approach to Creative Telescoping*, (2010) Mathematics in Computer Science, **4**, Issue 2-3, pp. 259-266

[52] F. Chyzak, *The ABC of Creative Telescoping Algorithms, Bounds, Complexity* `http://specfun.inria.fr/chyzak/Chyzak-2014-ABC.pdf`

[53] Boukraa S, Guttmann A J, Hassani S, Jensen I, Maillard J-M, Nickel B and Zenine N 2008 Experimental mathematics on the magnetic susceptibility of the square lattice Ising model *J. Phys. A* **41** 455202, 51 `http://arxiv.org/abs/0808.0763`

[54] Nickel B, Jensen I, Boukraa S, Guttmann A J, Hassani S, Maillard J-M and Zenine N 2010 Square lattice Ising model $\tilde{\chi}^{(5)}$ ODE in exact arithmetic *J. Phys. A* **43** 195205, 24 `http://arxiv.org/abs/1002.0161`

[55] B.H. Lian, A. Todorov and S-T. Yau, *Maximal Unipotent Monodromy for Complete Intersection of CY Manifolds*, (2005) American Journal of Mathematics **127**, pp. 1-50

[56] V.I. Arnold, Critical Points of Smooth Functions Proceedings of the International Congress of Mathematicians, Vancouver 1975

[57] G. Christol, *Diagonals of rational fractions*, (2015) Newsletter of the EMS

[58] A. Bostan, P. Lairez and B. Salvy, *Creative telescoping for rational functions using the Griffiths-Dwork method*, Proceedings ISSAC'13, pp. 93-100, ACM Press, 2013. `http://specfun.inria.fr/bostan/publications/BoLaSa13.pdf`

[59] P. Lairez, *Computing periods of rational integrals*, 2015 Mathematics of Computation (to be published)