Universal and complete sets in martingale theory
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To cite this version:

Dominique Lecomte, Miroslav Zeleny. Universal and complete sets in martingale theory. 2015. hal-01242318

HAL Id: hal-01242318
https://hal.science/hal-01242318
Preprint submitted on 11 Dec 2015

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Abstract. The Doob convergence theorem implies that the set of divergence of any martingale has measure zero. We prove that, conversely, any $G_{δσ}$ subset of the Cantor space with Lebesgue-measure zero can be represented as the set of divergence of some martingale. In fact, this is effective and uniform. A consequence of this is that the set of everywhere converging martingales is $Π^1_1$-complete, in a uniform way. We derive from this some universal and complete sets for the whole projective hierarchy, via a general method. We provide some other complete sets for the classes $Π^1_1$ and $Σ^1_2$ in the theory of martingales.
1 Introduction

The reader should see [K2] for the notation used in this paper.

Definition 1.1 We say that a map \( f : 2^{<\omega} \to [0,1] \) is a martingale if \( f(s) = \frac{f(s_0) + f(s_1)}{2} \) for each \( s \in 2^{<\omega} \). The set of martingales is denoted by \( \mathcal{M} \) and is a compact subset of \([0,1]^{2^{<\omega}}\) (equipped with the usual product topology).

This terminology is not the standard one, but the set \( \mathcal{M} \) can be interpreted as the set of all discrete martingales (in the classical sense) taking values in \([0,1]\), as follows. If \( s \in 2^{<\omega} \), then

\[
N_s := \{ \beta \in 2^\omega \mid s \subseteq \beta \}
\]

is the usual basic clopen set. Let \( f \in \mathcal{M} \). If \( n \in \omega \), then let \( S_n \) be the \( \sigma \)-algebra on \( 2^\omega \) generated by \( \{ N_s \mid s \in 2^n \} \), and \( f_n : 2^\omega \to [0,1] \) be defined by \( f_n(\beta) := f(\beta|n) \). Then the sequence \( (f_n)_{n \in \omega} \) is a discrete martingale taking values in \([0,1]\) with respect to the sequence of \( \sigma \)-algebras \( (S_n)_{n \in \omega} \) and the usual Lebesgue product measure \( \lambda \) on \( 2^\omega \). Conversely, if \( (f_n)_{n \in \omega} \) is any such martingale, it can be viewed as an element of \( \mathcal{M} \) by setting \( f(s) := f_{|s}(\alpha) \) if \( \alpha \in N_s \). This definition is correct because \( f_{|s} \), as a function measurable with respect to \( S_{|s} \), has a constant value on \( N_s \).

Definition 1.2 Let \( f \) be a martingale and \( \beta \in 2^\omega \). The oscillation of \( f \) at \( \beta \) is the number

\[
osc(f, \beta) := \inf_{N \in \omega} \sup_{p,q \geq N} |f(\beta|p) - f(\beta|q)|.
\]

The set of divergence of \( f \) is \( D(f) := \{ \beta \in 2^\omega \mid osc(f, \beta) > 0 \} \).

By definition, if \( f \) is a martingale, then

\[
\beta \in D(f) \iff \exists r \in \omega \ \forall N \in \omega \ \exists p,q \geq N \ |f(\beta|p) - f(\beta|q)| > 2^{-r}.
\]

This shows that \( D(f) \in \Sigma^0_3 \). Moreover, \( D(f) \) has \( \lambda \)-measure zero, by Doob’s convergence theorem (see Chapter XI, Section 14 in [D]). So it is natural to ask whether any \( \Sigma^0_3 \) subset of \( 2^\omega \) with \( \lambda \)-measure zero is the set of divergence of some martingale (this question was asked by Louveau). We answer positively:

Theorem 1.3 Let \( B \) be a subset of \( 2^\omega \). Then the following are equivalent:
(a) \( B \) is \( \Sigma^0_3 \) and has \( \lambda \)-measure zero,
(b) there is a martingale \( f \) with \( B = D(f) \).

Definition 1.4 Let \( \Gamma \) be a class of subsets of Polish spaces, \( X,Y \) be Polish spaces, and \( \mathcal{U} \subseteq Y \times X \).
(a) We say that \( \mathcal{U} \) is \( Y \)-universal for the \( \Gamma \) subsets of \( X \) if \( \mathcal{U} \subseteq \Gamma(Y \times X) \) and \( \Gamma(X) = \{ \mathcal{U}_y \mid y \in Y \} \).
(b) We say that \( \mathcal{U} \) is uniformly \( Y \)-universal for the \( \Gamma \) subsets of \( X \) if \( \mathcal{U} \) is \( Y \)-universal for the \( \Gamma \) subsets of \( X \) and, for each \( S \subseteq \Gamma(\omega^\omega \times X) \), there is a Borel map \( b : \omega^\omega \to Y \) such that \( S_\alpha = \mathcal{U}_b(\alpha) \) for each \( \alpha \in \omega^\omega \).

Corollary 1.5 Let \( \mathcal{G} \) be a \( G_\delta \) subset of \( 2^\omega \) with \( \lambda(\mathcal{G}) = 0 \). Then the set \( \{(f, \beta) \in \mathcal{M} \times \mathcal{G} \mid \beta \in D(f)\} \) is \( \mathcal{M} \)-universal for the \( \Sigma^0_3 \) subsets of \( \mathcal{G} \).
In fact, we prove an effective and uniform version of the implication (a) ⇒ (b) in Theorem 1.3. In particular, we can associate, via a Borel map \( F \), a martingale to a code \( \alpha \) of an arbitrary \( G_\delta \) subset \( G \) of \( G \) (as in the previous corollary), in such a way that \( G = D(F(\alpha)) \). A consequence of this is the following:

**Theorem 1.6** The set \( \mathcal{P} \) of everywhere converging martingales is \( \Pi_1 \)-complete.

These statements are in the spirit of some results concerning the differentiability of functions due to Zahorski and Mazurkiewicz (see Section 4 for details). In fact, \( \mathcal{P} \) is \( \Pi_1 \)-complete in a uniform way, which allows to derive some universal and complete sets for the whole projective hierarchy, in spaces of continuous functions, starting from \( \mathcal{P} \). More precisely, let \( P_1 := [0, 1]^{2^{<\omega}} \) and \( C_1 := \mathcal{P} \). We define, for each natural number \( n \geq 1 \),

- the space \( P_{n+1} := \mathcal{C}(2^{<\omega}, P_n) \) of continuous functions from \( 2^{<\omega} \) into \( P_n \), equipped with the topology of uniform convergence (inductively),
- \( C_{n+1} := \{ h \in P_{n+1} \mid \forall \beta \in 2^{<\omega} \ h(\beta) \notin C_n \} \) (inductively),
- \( U_n := \{ (h, \beta) \in P_{n+1} \times 2^{<\omega} \mid h(\beta) \in C_n \} \).

We prove the following:

**Theorem 1.7** Let \( n \geq 1 \) be a natural number. Then
(a) the set \( U_n \) is uniformly \( P_{n+1} \)-universal for the \( \Pi_1 \) subsets of \( 2^{<\omega} \).
(b) the set \( C_n \) is \( \Pi_1 \)-complete.

In fact, our method is more general and works if we start with a \( \Pi_1 \) set which is complete in a uniform way.

Let \( f \) be a martingale. As \( D(f) \) has \( \lambda \)-measure zero, we can associate to \( f \) the partial function \( \psi(f) \) defined \( \lambda \)-almost everywhere by \( \psi(f)(\beta) := \lim_{l \rightarrow \infty} f(\beta|l) \). The partial function \( \psi(f) \) will be called the **associated partial function**. The martingale \( f \) is in \( \mathcal{P} \) if and only if \( \psi(f) \) is total, in which case \( \psi(f) \) is called the **associated function**. Using the work in [B-Ka-L] and [K2] about spaces of continuous functions, we prove the following:

**Theorem 1.8** (a) The set of sequences of everywhere converging martingales whose associated functions converge pointwise is \( \Pi_1 \)-complete.
(b) The set of sequences of everywhere converging martingales whose associated functions converge pointwise to zero is \( \Pi_1 \)-complete.
(c) The set of sequences of everywhere converging martingales having a subsequence whose associated functions converge pointwise to zero is \( \Sigma_2 \)-complete.
2 $\Sigma^0_3$ sets of measure zero

**Notation.** In the sequel, $B$ will be a Borel subset of $2^\omega$, and $M$ will be a $\lambda$-measurable subset of $2^\omega$. If $\beta \in 2^\omega$, then the density of $M$ at $\beta$ is the number $d(M, \beta) := \lim_{n \to \infty} \frac{\lambda(M \cap N_{\beta^n})}{\lambda(N_{\beta^n})}$ when it is defined. Note that $d(B, \beta) = 1$ if $\beta \in B$ and $B$ is open. We first recall the Lebesgue density theorem (see 17.9 in [K2]).

**Theorem 2.1** *(Lebesgue)* The equality $\lambda(M) = \lambda(\{\beta \in M \mid d(M, \beta) = 1\})$ holds for any $\lambda$-measurable subset $M$ of $2^\omega$.

The reader should see [C] for the next lemma. We include a proof to be self-contained and also because we will prove an effective and uniform version of it later.

**Lemma 2.2** *(Lusin-Menchoff)* Let $F$ be a closed subset of $2^\omega$, and $M \supseteq F$ be a $\lambda$-measurable subset of $2^\omega$ such that $d(M, \beta) = 1$ for each $\beta \in F$. Then there is a closed subset $C$ of $2^\omega$ such that

1. $F \subseteq C \subseteq M$,
2. $d(M, \beta) = 1$ for each $\beta \in C$,
3. $d(C, \beta) = 1$ for each $\beta \in F$.

**Proof.** If $F$ is $2^\omega$, then we can take $C := F$. So we may assume that $F$ is not $2^\omega$. We set $s^{-} := s(\lceil s \rceil - 1)$ if $\emptyset \neq s \in 2^{<\omega}$. Note that $\neg F$ is the disjoint union of the elements of a sequence $(N_{s_n})_{n \in \omega}$, where $N_{s_n} \cap F \neq \emptyset$ for each $n \in \omega$. Fix $n \in \omega$. By Theorem 2.1,

$$\lambda(M \cap N_{s_n}) = \lambda(\{\beta \in M \cap N_{s_n} \mid d(M \cap N_{s_n}, \beta) = 1\}).$$

The regularity of $\lambda$ gives a closed subset $F_n$ of $2^\omega$ contained in $\{\beta \in M \cap N_{s_n} \mid d(M \cap N_{s_n}, \beta) = 1\}$ such that $\lambda(F_n) \geq (1 - 2^{-n}) \lambda(M \cap N_{s_n})$. We set $C := F \cup \bigcup_{n \in \omega} F_n$, which is closed since $|s_n| \to \infty$.

As Conditions (1) and (2) are clearly satisfied, pick $\beta \in F$. Note that

$$\lambda(N_{\beta^n} \setminus C) = \sum_{s_n \supseteq \beta^n} \lambda(N_{s_n} \setminus C)$$

$$\leq \sum_{s_n \supseteq \beta^n} \lambda(N_{s_n} \setminus F_n)$$

$$\leq \sum_{s_n \supseteq \beta^n} 2^{-n} \lambda(M \cap N_{s_n}) + \sum_{s_n \supseteq \beta^n} \lambda(N_{s_n} \setminus M)$$

$$\leq \sum_{s_n \supseteq \beta^n} 2^{-n} \lambda(N_{s_n}) + \lambda(N_{\beta^n} \setminus M).$$

This implies that the limit of $\frac{\lambda(N_{\beta^n} \setminus C)}{\lambda(N_{\beta^n})}$ is zero since $d(M, \beta) = 1$. □

The next topology is considered in [Lu-Ma-Z], see Chapter 6.

**Definition 2.3** The $\tau$-topology on $2^\omega$ is generated by

$$\mathcal{F} := \{M \subseteq 2^\omega \mid M \text{ is } \lambda\text{-measurable} \land \forall \beta \in M \ d(M, \beta) = 1\}.$$  

The next result is proved in [Lu-Ma-Z], but in a much more abstract way. This is the reason why we include a much more direct proof here, since it is not too long.
Lemma 2.4  The family $F$ is a topology. In particular, any $\tau$-open set is $\lambda$-measurable.

Proof. Note first that $F$ is closed under finite intersections, so that it is a basis for the $\tau$-topology. Indeed, let $M, M'$ be in $F$, and $\beta \in M \cap M'$. Then we use the facts that $$\lambda(M \cap M' \cap N_{\beta}|) = \lambda(M \cap N_{\beta}|) - \lambda((M \cap N_{\beta}|) \setminus M')$$ and $$\lambda((M \cap N_{\beta}|) \setminus M') \leq \lambda(N_{\beta}| \setminus M').$$

Let $H$ be a subfamily of $F$, and $H := \cup H$. We claim that there is a countable subfamily $C$ of $H$ such that $m := \sup\{\lambda(D) \mid D \subseteq H $ countable $\} = \lambda\cup C$. Indeed, for each $n \in \omega$ there is $D_n \subseteq H$ countable such that $\lambda(\cup D_n) > m - 2^{-n}$, and $C := \cup n \in D_n$ is suitable. Let $C := \cup C$.

Let $\beta \in H$, and $M$ in $H$ with $\beta \in M$. Note that $\lambda(M \cup C) = \lambda(C)$ (consider the family $C \cup \{M\}$). Thus $\lambda(M \setminus C) = 0$. As $d(M, \beta) = 1$, the equality $d(M \cap C, \beta) = 1$ holds, and $d(-C, \beta) = 0$. This implies that $H \setminus C$ is contained in $\{\beta \notin C \mid d(-C, \beta) < 1\}$, which has $\lambda$-measure zero by Theorem 2.1. Therefore $H \setminus C$ has $\lambda$-measure zero and $H = C \cup (H \setminus C)$ is $\lambda$-measurable.

Pick $\beta \in H$, and $M$ in $H$ with $\beta \in M$. Then $d(M, \beta) = 1$, and thus $d(H, \beta) = 1$. Therefore $H \in F$. This finishes the proof. □

The next lemma is in the style of Urysohn’s theorem (see [Lu] for its version on the real line). We include a proof to be self-contained and also because we will prove an effective and uniform version of it later.

Lemma 2.5  Let $C$ be a closed subset of $2^\omega$, and $G$ be a $G_\beta$ subset of $2^\omega$ disjoint from $C$ such that $\lambda(G) = 0$. Then there is a $\tau$-continuous map $h : 2^\omega \to [0, 1]$ such that $h|C \equiv 0$ and $h|G \equiv 1$.

Proof. Let $(F_n)_{n \in \omega}$ be an increasing sequence of closed subsets of $2^\omega$ with union $\neg G$ and $F_0 = C$. We first construct a sequence $(C_{\frac{k}{2^n}})_{n \in \omega}$ of closed subsets of $2^\omega$ with $F_n \subseteq C_{\frac{k}{2^n}} \subseteq \neg G$, $C_{\frac{1}{2^n}} \subseteq C_{\frac{k}{2^n} + \frac{1}{2^n}}$, and $d(C_{\frac{k}{2^n}}, \beta) = 1$ for each $\beta \in C_{\frac{k}{2^n}}$. We first apply Lemma 2.2 to $F := F_0$ and $M := \neg G$, which gives $F_0 \subseteq C_1 \subseteq \neg G$. Then, inductively, we apply Lemma 2.2 to $F := C_{\frac{k}{2^n}} \cup F_{n+1}$ and $M := \neg G$, which gives $C_{\frac{k}{2^n}} \cup F_{n+1} \subseteq C_{\frac{k}{2^n} + \frac{1}{2^n}} \subseteq \neg G$ such that $d(C_{\frac{k}{2^n}}, \beta) = 1$ for each $\beta \in C_{\frac{k}{2^n}}$.

Then we continue $C_{\frac{k+1}{2^n+1}}$ for $0 < k < 2^n - 1$ and $n \geq 2$. This will give us a family $(C_{\frac{k}{2^n}})_{n \in \omega, 0 < k \leq 2^n}$ of closed subsets of $2^\omega$. We want to ensure that $C_\zeta \subseteq C_{\zeta'}$ and $d(C_{\zeta'}, \beta) = 1$ for each $\beta \in C_\zeta$ if $\zeta' < \zeta$. We proceed by induction on $n$. We apply Lemma 2.2 to $F := C_{\frac{k}{2^n+1}}$ and $M := C_{\frac{k}{2^n+1}}$, which gives $C_{\frac{k+1}{2^n+1}} \subseteq C_{\frac{2k+1}{2^n+1}} \subseteq C_{\frac{k}{2^n+1}}$, $d(C_{\frac{k+1}{2^n+1}}, \beta) = 1$ for each $\beta \in C_{\frac{2k+1}{2^n+1}}$, and $d(C_{\frac{2k+1}{2^n+1}}, \beta) = 1$ for each $\beta \in C_{\frac{k+1}{2^n+1}}$. This allows us to define $\tilde{h}$ by

$$\tilde{h}(\beta) := \begin{cases} 0 & \text{if } \beta \in G, \\ \sup\{\zeta \mid \beta \in C_\zeta\} & \text{if } \beta \notin G. \end{cases}$$

It remains to see that $\tilde{h}$ is $\tau$-continuous (and then we will set $h(\beta) := 1 - \tilde{h}(\beta)$). So let $b \in (0, 1]$, and $\beta \in 2^\omega$ with $\tilde{h}(\beta) < b$. Note that there is $\zeta < b$ with $\tilde{h}(\beta) < \zeta$, so that $\beta \notin C_\zeta$. If $\gamma \notin C_\zeta$, then $\tilde{h}(\gamma) \leq \zeta < b$, so that $\neg C_\zeta$ is an open (and thus $\tau$-open since the $\tau$-topology is finer than the usual one) neighborhood of $\beta$ on which $\tilde{h} < b$. In particular, $\tilde{h}$ is Borel.
Now let \( a \in [0, 1] \). It is enough to see that \( B := \{ \gamma \in 2^\omega \mid \hat{h}(\gamma) > a \} \) is \( \tau \)-open. So assume that \( \hat{h}(\gamma) > a \). Note that there are \( \zeta > \zeta' > a \) with \( \hat{h}(\gamma) > \zeta \), so that \( \gamma \in C_{\zeta'} \subseteq C_{\zeta'} \subseteq B \). Thus \( d(C_{\zeta'}, \gamma) = 1 \), by construction of the family. As \( \hat{h} \) is Borel, \( B \) is Borel, \( d(B, \gamma) \) is defined and equal to 1. \( \square \)

**Remark.** We in fact proved that \( h \) is lower semi-continuous.

**Notation.** If \( h: 2^\omega \to [0, 1] \) is a \( \lambda \)-measurable map and \( s \in 2^{<\omega} \), then we set \( \int_{\gamma \in N_s} h \, d\lambda := \int_{\gamma \in N_s} \frac{h(\gamma)}{\lambda(\gamma)} \).

**Lemma 2.6** Let \( h: 2^\omega \to [0, 1] \) be a \( \tau \)-continuous map, and \( \beta \in 2^\omega \). Then

\[
\lim_{l \to \infty} \int_{N_{[\beta,l]}} h \, d\lambda = h(\beta).
\]

**Proof.** Let \( \varepsilon > 0 \), and \( \beta \in M := h^{-1}\left( B(h(\beta), \varepsilon) \right) \). Note that \( d(M, \gamma) = 1 \) for each \( \gamma \in M \) since \( h \) is \( \tau \)-continuous. As \( h \) is \( \lambda \)-measurable, we can write

\[
\int_{N_{[\beta,l]}} h \, d\lambda = \int_{M \cap N_{[\beta,l]}} h \, d\lambda + \int_{N_{[\beta,l]} \setminus M} h \, d\lambda.
\]

Note that \( 0 \leq \int_{N_{[\beta,l]} \setminus M} h \, d\lambda \leq \lambda(N_{\beta,l} \setminus M) \), so that \( 0 \leq \int_{N_{[\beta,l]} \setminus M} h \, d\lambda \leq \frac{\lambda(N_{\beta,l} \setminus M)}{\lambda(N_{[\beta,l]})} \to 0 \). Similarly,

\[
\int_{M \cap N_{[\beta,l]}} h \, d\lambda \leq \left( h(\beta) - \varepsilon \right) \frac{\lambda(M \cap N_{[\beta,l]})}{\lambda(N_{[\beta,l]})}, \quad \left( h(\beta) + \varepsilon \right) \frac{\lambda(M \cap N_{[\beta,l]})}{\lambda(N_{[\beta,l]})}
\]

and we are done since \( \frac{\lambda(M \cap N_{[\beta,l]})}{\lambda(N_{[\beta,l]})} \) tends to 1 as \( l \) tends to \( \infty \). \( \square \)

Now we come to our main lemma, inspired by Zahorski (see [Za]).

**Lemma 2.7** Let \( G \) be a \( G_\delta \) subset of \( 2^\omega \) with \( \lambda \)-measure zero. Then there is a martingale \( f \) with \( G = D(f) \) and \( \{ \text{osc}(f, \beta) \mid \beta \in 2^\omega \} \subseteq \{ 0 \} \cup \left\{ \frac{3}{2}, 1 \right\} \).

**Proof.** Let \( (G_n)_{n \in \omega} \) be a decreasing sequence of open subsets of \( 2^\omega \) with intersection \( G \) and \( G_0 = 2^\omega \).

- We construct \( g_n : 2^\omega \to [0, 1] \), open subsets \( G_n^* \), \( G_n^{**} \) of \( 2^\omega \), and a sequence \((s_j^n)_{j \in I_n}\) of pairwise incompatible finite binary sequences, by induction on \( n \in \omega \), such that, if \( S_n := \sum_{j \leq n} (-1)^j g_j \),

\[
\begin{align*}
(1) & \ G \subseteq G_{n+1}^* \subseteq G_n^* = \bigcup_{j \in I_n} N_{s_j^n}^* \subseteq G_n \ \wedge \ G_0^* = 2^\omega, \\
(2) & \ g_{n|G} = 1 \ \wedge \ g_{n|\neg G} = 0, \\
(3) & \ g_n \text{ is } \tau \text{-continuous}, \\
(4) & \ g_{n+1} \leq g_n, \\
(5) & \ \lambda(G_{n+1}^* \cap N_{s_j^n}) < 2^{-n-3}\lambda(N_{s_j^n}), \\
(6) & \ \left| \int_{N_{[\beta,l]}} S_n \, d\lambda - \lambda(S_n(\beta)) \right| < 2^{-3} \text{ if } \beta \in G \cap N_{s_j^n}.
\end{align*}
\]

We set \( g_0 := 1, G_0^* := 2^\omega, I_0 := \{ 0 \} \) and \( s_0^n := \emptyset \). Assume that our objects are constructed up to \( n \). We first construct an open subset \( G_{n+1}^* \) of \( 2^\omega \) with \( G \subseteq G_{n+1}^* \subseteq G_{n+1}^{**} \cap G_{n+1} \) such that

\[
\lambda(G_{n+1}^* \cap N_{s_j^n}) < 2^{-n-3}\lambda(N_{s_j^n})
\]

if \( j \in I_n \). For each \( j \in I_n \), there is an open set \( O_j \) with \( G \cap N_{s_j^n} \subseteq O_j \subseteq G_{n+1} \cap N_{s_j^n} \) such that \( \lambda(O_j) < 2^{-n-3}\lambda(N_{s_j^n}) \). We then set \( G_{n+1}^* := \bigcup_{j \in I_n} O_j \).
We now apply Lemma 2.5 to $C := -G_{n+1}^*$ and $G$, which gives a $\tau$-continuous map $h : 2^\omega \to [0, 1]$ with $h_{\mid -G_{n+1}^*} \equiv 0$ and $h_G \equiv 1$. We set $g_{n+1} := \min(g_n, h)$, so that $g_{n+1}$ satisfies (2)-(4).

By Lemma 2.6, $\lim_{l \to \infty} \int_{N_{\beta l}} S_n \, d\lambda = S_{n+1}(\beta)$ for each $\beta \in G$. This gives $l(\beta) \in \omega$ minimal with $\left| \int_{N_{\beta l}(\beta)} S_n \, d\lambda - S_{n+1}(\beta) \right| < 2^{-3}$ and $N_{\beta l}(\beta) \subseteq G_{n+1}^*$. The set $G_{n+1}^*$ is the union of the $N_{\beta l}(\beta)$'s, which defines $I_{n+1}$ and $(s_n^{\alpha+1})_j \in I_{n+1}$ ($S_{n+1}(\beta)$ is 0 if $n$ is even and 1 otherwise when $\beta \in G$).

• We then define a partial map $f_\infty : 2^\omega \to [0, 1]$ by $f_\infty := \Sigma_{j \in \omega} (-1)^j g_j$. If $\beta \in G$, then $S_n(\beta)$ takes alternatively the values 1 and 0, depending on the parity of $n$, so that $f_\infty(\beta)$ is not defined. If $\beta \notin G$, then there is $n$ such that $\beta \in -G_{n+1}^* \subseteq -G_{n+2}^* \subseteq \ldots$. This implies that $f_\infty(\beta)$ is defined and equal to $S_n(\beta)$. As $0 \leq \Sigma_{p \leq q} (g_{2p} - g_{2p+1}) = S_{2q+1} \leq S_{2q} = g_0 + \Sigma_{1 \leq p \leq q} (g_{2p} - g_{2p-1}) \leq g_0$, $f_\infty$ takes values in $[0, 1]$. So $f_\infty$ is a partial $\lambda$-measurable map defined $\lambda$-almost everywhere since $\Lambda(\lambda) = 0$ (we use Lemma 2.4).

• This allows us to define $f : 2^{<\omega} \to [0, 1]$ by $f(s) := f_{N_s} f_\infty \, d\lambda$. As $\lambda(N_s) = 2\lambda(N_{s\varepsilon})$ for each $\varepsilon \in 2$, $f(s) = f_{N_s} f_\infty \, d\lambda = \int_{N_{\beta l}} f_\infty \, d\lambda f_{N_{\beta s}} f_\infty = \frac{f(0)}{2} + \frac{f(1)}{2}$ and $f$ is a martingale.

• If $\beta \notin G$, then there is $n$ with $\beta \in G_{n+1}^* \setminus G_{n+1}^*$, so that $f_\infty(\beta) = S_n(\beta)$. By Lemma 2.6, $k \geq n$ implies that $\lim_{l \to \infty} \int_{N_{\beta l}} S_k \, d\lambda = S_{k+1}(\beta) = S_n(\beta)$ since $S_{k+1}$ is $\tau$-continuous. Note that, for each $k \geq n$,

$$\left| \int_{N_{\beta l}} (f_\infty - S_{k+1}) \, d\lambda \right| \leq \lambda(G_{k+2}^* \cap N_{\beta l}) \leq \Sigma_{\beta l \subseteq k+1} \lambda(G_{k+2}^* \cap N_{s_j}) \leq \Sigma_{\beta l \subseteq k+1} 2^{-k-4} \lambda(N_{s_j}) \leq \lambda(N_{\beta l}) 2^{-k-4}.$$  

Moreover,

$$|f(\beta) - f_\infty(\beta)| = |\int_{N_{\beta l}} f_\infty \, d\lambda - f_\infty(\beta)| = |\int_{N_{\beta l}} (f_\infty - S_{k+1}) \, d\lambda + \int_{N_{\beta l}} S_{k+1} \, d\lambda - S_{k+1}(\beta)| \leq 2^{-k-4} + |\int_{N_{\beta l}} S_{k+1} \, d\lambda - S_{k+1}(\beta)|,$$

so that $\lim_{l \to \infty} f(\beta) = f_\infty(\beta)$, $\operatorname{osc}(f, \beta) = 0$ and $\beta \notin D(f)$.

• If $\beta \in G$ and $n \in \omega$, then there is $j \in \omega$ with $\beta \in N_{s_j}$. Note that

$$f(s_j^n) = \int_{N_{s_j}} f_\infty \, d\lambda = \int_{N_{s_j}} S_n \, d\lambda + \int_{N_{s_j}} (f_\infty - S_n) \, d\lambda$$

and $|\int_{N_{s_j}} (f_\infty - S_n) \, d\lambda| \leq \lambda(G_{n+1}^* \cap N_{s_j}) < \frac{1}{8} \lambda(N_{s_j})$, so that $|\int_{N_{s_j}} (f_\infty - S_n) \, d\lambda| < \frac{1}{8}$. By (6), $|f(s_j^n) - S_n(\beta)| < \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$. As $S_n(\beta)$ takes infinitely often the values 1 and 0, $\operatorname{osc}(f, \beta) \geq \frac{1}{2}$ and $\beta \in D(f)$.

The main result will be a consequence of the main lemma and the following.
Lemma 2.8  Let \((f_n)_{n \in \omega}\) be a sequence of martingales such that
\[
\{ \text{osc}(f_n, \beta) \mid (n, \beta) \in \omega \times 2^\omega \} \subseteq \{0\} \cup \left\{ \frac{1}{2}, 1 \right\}.
\]
Then there is a martingale \(f\) with \(D(f) = \bigcup_{n \in \omega} D(f_n)\).

Proof. We first observe the following facts. Let \(g, h : 2^{<\omega} \to \mathbb{R}\) be bounded, \(\beta \in 2^\omega\) and \(a \in \mathbb{R}\).

1. \(\text{osc}(g+h, \beta) \leq \text{osc}(g, \beta) + \text{osc}(h, \beta)\).

   This comes from the triangle inequality.

2. \(\text{osc}(ag, \beta) = |a| \cdot \text{osc}(g, \beta)\).

3. \(\text{osc}(g+h, \beta) = \text{osc}(h, \beta)\) if \(\text{osc}(g, \beta) = 0\).

   By (1), \(\text{osc}(h, \beta) \leq \text{osc}(g+h, \beta) + \text{osc}(-g, \beta) = \text{osc}(g+h, \beta) \leq \text{osc}(g, \beta) + \text{osc}(h, \beta) = \text{osc}(h, \beta)\), so that \(\text{osc}(h, \beta) = \text{osc}(g+h, \beta)\).

4. \(\text{osc}(g, \beta) \leq a\) if \(g(\beta l) \in [0, a]\) for each \(l \in \omega\).

   • We set \(D_n := D(f_n)\) for each \(n \in \omega\), and \(f := \Sigma_{n \in \omega} 4^{-n} f_n\). Note that \(f\) is defined and a martingale.

   • If \(\beta \notin \bigcup_{n \in \omega} D_n\), then \(\text{osc}(f_n, \beta) = 0\) for each \(n \in \omega\). In particular, \(\text{osc}(4^{-n} f_n, \beta) = 0\) for each \(n \in \omega\), by (2). Let \(\varepsilon > 0\), and \(M \in \omega\) with \(\Sigma_{n > M} 4^{-n} \leq \varepsilon\). By (1), \(\text{osc}(\Sigma_{n \leq M} 4^{-n} f_n, \beta) = 0\). By (3) and (4), \(\text{osc}(f, \beta) = \text{osc}(\Sigma_{n > M} 4^{-n} f_n, \beta) \leq \Sigma_{n > M} 4^{-n} \leq \varepsilon\). As \(\varepsilon\) is arbitrary, \(\text{osc}(f, \beta) = 0\), \(\beta \notin D(f)\), which shows that \(D(f) \subseteq \bigcup_{n \in \omega} D_n\).

   • If \(\beta \in \bigcup_{n \in \omega} D_n\), then let \(m\) be minimal such that \(\beta \in D_m\). Note that
\[
f = \Sigma_{n < m} 4^{-n} f_n + 4^{-m} f_m + \Sigma_{n > m} 4^{-n} f_n.
\]
By (2) and (3), \(\text{osc}(f, \beta) = \text{osc}(4^{-m} f_m + \Sigma_{n > m} 4^{-n} f_n, \beta)\). By (1), (2) and (4),
\[
\text{osc}(f, \beta) \geq \text{osc}(4^{-m} f_m, \beta) - \text{osc}(\Sigma_{n > m} 4^{-n} f_n, \beta) \geq 4^{-m} \frac{1}{2} - 4^{-m} \frac{1}{2} > 0.
\]
Thus \(\beta \in D(f)\). \(\square\)

3 Effectivity and uniformity

- We refer to [M] for the basic notions of effective descriptive set theory. We first recall some material present in it.

  • Let \((p_n)_{n \in \omega}\) be the sequence of prime numbers 2, 3, ...

  • If \(l \in \omega\) and \(s \in \omega^l\), then \(\pi := s(0), ..., s(l-1) >:= p_0^{s(0)+1} \cdots p_{l-1}^{s(l-1)+1} \in \omega\) codes \(s\) (if \(l = 0\), then \(< : = 1\).

  • If \(\alpha \in \omega^\omega\) and \(l \in \omega\), then \(\pi(l) := < \alpha(0), ..., \alpha(l-1) > \in \omega\) codes \(\alpha|l \in \omega^l\), and \(\alpha^*\) is defined by removing the first coordinate: \(\alpha^* := (\alpha(1), \alpha(2), ...)\).
• If $\kappa \in \{2, \omega\}$, then $< \ldots >: (\kappa^\omega)^2 \to \kappa^\omega$ is a recursive homeomorphism with inverse map $\alpha \mapsto (\alpha)_0, (\alpha)_1$ defined for example by $(\alpha)_2(n) := \alpha(2n + \varepsilon)$ if $(n, \varepsilon) \in \omega \times 2$ (we will also consider recursive homeomorphisms $< \ldots >: (\kappa^\omega)^3 \to \kappa^\omega$ and $< \ldots >: (\kappa^\omega)^\omega \to \kappa^\omega$).

• If $u \in \omega$, then $\text{Seq}(u)$ means that there are $l \in \omega$ and $s \in \omega^l$ (denoted by $s(u)$) such that $u = < s(0), \ldots, s(l - 1) >$. The natural number $(u)_i$ is $s(i)$ if $i < l$, and 0 otherwise. The number $l$ is the length of $u$ and is denoted by $\text{lh}(u)$. If $k \leq l$, then $u(k) := < s(0), \ldots, s(k - 1) >$, so that $u(l) = u$. The standard basic clopen set is $N^u := \{ \beta \in 2^\omega \mid \forall i < \text{lh}(u) \beta(i) = (u)_i \}$.

Let $X$ be a recursively presented Polish space. Then we will consider the effective basic open set $N(X, u) = B_X(r((u)_1)_0, \ldots, (u)_1)$.

• Let $n \geq 1$ be a natural number. A subset $T$ of $\omega^n$ is a tree if $\text{Seq}(u_i)$ and $\text{lh}(u_i) = \text{lh}(u_0)$ for each $(u_0, \ldots, u_{n - 1}) \in T$ and each $i < n$, and $(u_0(k), \ldots, u_{n - 1}(k)) \in T$ if $(u_0, \ldots, u_{n - 1}) \in T$ and $k \leq \text{lh}(u_0)$.

The next result is a part of 4A.1 in [M].

**Theorem 3.1** Let $m \geq 1$ be a natural number, and $B \in \Sigma_1^0(\omega^\omega \times (\omega^\omega)^m)$. Then there is a recursive subset $T$ of $\omega^\omega \times \omega^m$ such that $(\alpha, \alpha_1, \ldots, \alpha_m) \in B \iff \exists l \in \omega (\alpha, \alpha_1(l), \ldots, \alpha_m(l)) \notin T$, and $T_\alpha := \{ (u_0, \ldots, u_{m - 1}) \in \omega^m \mid (\alpha, u_0, \ldots, u_{m - 1}) \in T \}$ is a tree for each $\alpha \in \omega^\omega$.

The next result is a part of 4A.7 in [M].

**Theorem 3.2** Let $X$ be a recursively presented Polish space and $B \in \Delta_1^1(X)$. Then we can find a recursive function $\pi: \omega^\omega \to X$ and $C \in \Pi_1^0(\omega^\omega)$ such that $\pi$ is injective on $C$ and $\pi[C] = B$.

- We then recall some material from [L].

**Notation.** Let $X$ be a recursively presented Polish space. Recall that there is a pair $(\mathcal{W}^X, C^X)$ such that

- $\mathcal{W}^X \subseteq \omega$ is a $\Pi_1^1$ set of codes for the $\Delta_1^1$ subsets of $X$,
- $C^X \subseteq \omega \times X$ is $\Pi_1^1$ and $\Delta_1^1(X) = \{ C_n^X \mid n \in \mathcal{W}^X \}$, which means that $C^X$ is “universal” for the $\Delta_1^1$ subsets of $X$,
- the relation “$n \in \mathcal{W}^X \land (n, x) \notin C^X$” is $\Pi_1^1$ in $(n, x)$.

If $X = \omega^\omega \times \omega^\omega$, then we simply write $(\mathcal{W}, C) := (\mathcal{W}^X, C^X)$.

The next result will be extremely useful in the sequel.

**The uniformization lemma.** Let $X, Y$ be recursively presented Polish spaces, and $P \in \Pi_1^1(X \times Y)$. Then the set $P^+ := \{ x \in X \mid \exists y \in \Delta_1^1(x) (x, y) \in P \}$ is $\Pi_1^1$, and there is a partial $\Pi_1^1$-recursive map $f: X \to Y$ such that $(x, f(x)) \in P$ for each $x \in P^+$. If moreover $S \subseteq P^+$ is a $\Sigma_1^1$ subset of $X$, then there is a total $\Delta_1^1$-recursive map $g: X \to Y$ such that $(x, g(x)) \in P$ for each $x \in S$.

- The following definition is inspired by 3H.1 in [M].
Definition 3.3 (a) Let \( \Gamma \) be a class of subsets of recursively presented Polish spaces, and \( \Gamma \) be the associated boldface class. A system of sets \( \mathcal{U}^X \subseteq \Gamma(\omega^\omega \times X) \), where is \( X \) is a recursively presented Polish space, is a nice parametrization in \( \Gamma \) for \( \mathcal{U} \) if the following hold:

1. \( \Gamma(X) = \{ U^X_\alpha \mid \alpha \in \omega^\omega \} \).
2. \( \Gamma(X) = \{ U^X_\alpha \mid \alpha \in \omega^\omega \text{ recursive} \} \).

(b) If \( \mathcal{U} \) belongs to a nice parametrization, then we will say that \( \mathcal{U} \) is a good universal set.

(c) If \( \mathcal{U} \) satisfies all these properties except maybe (3), then we will say that \( \mathcal{U} \) is a suitable universal set.

By 3E.2, 3F.6 and 3H.1 in [M], there is a nice parametrization in \( \Pi^1_n \) for \( \Pi^1_n \), for each natural number \( n \geq 1 \).

- We now recall two results that can essentially be found in [K1]. The first one is Theorem 2.2.3.(a) (see also [T1]).

Theorem 3.4 (Tanaka) Let \( U \in \Sigma^1_1(\omega^\omega \times \omega^\omega) \) be \( \omega^\omega \)-universal for the analytic subsets of \( \omega^\omega \). Then \( L(U) := \{ (\alpha, p) \in \omega^\omega \times \omega \mid \lambda(U_\alpha \cap 2^\omega) > (p)_0 \} \) is \( \Sigma^1_1 \).

Corollary 3.5 Let \( B \in \Delta^1_1(\omega^\omega \times 2^\omega) \).
(a) The map \( \lambda_B : \omega^\omega \rightarrow \mathbb{R} \) defined by \( \lambda_B(\alpha) := \lambda(B_\alpha) \) is \( \Delta^1_1 \)-recursive, and the partial function \( (n, \alpha) \mapsto \lambda(C_{n, \alpha}) \) is \( \Pi^1_1 \)-recursive on its domain \( \mathcal{W} \times \omega^\omega \).

(b) Let \( D \subseteq \omega \), \( O_0 \in \Sigma^1_1(\omega \times \omega^\omega \times 2^\omega) \), and \( O_1 \in \Pi^1_1(\omega \times \omega^\omega \times 2^\omega) \) be such that \( \lambda((O_0)_{n, \alpha}) = \lambda((O_1)_{n, \alpha}) \) if \( n \in D \). Then the partial map \( \lambda_O : D \times \omega^\omega \rightarrow \mathbb{R} \) defined by \( \lambda_O(n, \alpha) := \lambda((O_0)_{n, \alpha}) \) is \( \Sigma^1_1 \)-recursive and \( \Pi^1_1 \)-recursive on its domain.

(c) The partial map \( d_B : \omega^\omega \times 2^\omega \rightarrow \mathbb{R} \) defined by \( d_B(\alpha, \beta) := d(B_\alpha, \beta) \) is \( \Delta^1_1 \)-recursive, and the partial map \( (n, \alpha, \beta) \mapsto d(C_{n, \alpha}, \beta) \) is \( \Pi^1_1 \)-recursive on its domain.

\( \{ (n, \alpha, \beta) \in \mathcal{W} \times \omega^\omega \times 2^\omega \mid d(C_{n, \alpha}, \beta) \text{ exists} \} \).

(d) Let \( h : \omega^\omega \times 2^\omega \rightarrow \mathbb{R} \) be \( \Delta^1_1 \)-recursive taking values in \([0, 1] \). Then the partial map \( i_h : \omega^\omega \times \omega \rightarrow \mathbb{R} \) defined by \( i_h(\alpha, u) := \int_{N^u} h(\alpha, \cdot) \) \( d \lambda \) is \( \Delta^1_1 \)-recursive on its \( \Delta^1_1 \) domain \( \omega^\omega \times \{ u \in \omega \mid \text{Seq}(u) \} \).

Proof. (a) It is enough to see that the relations \( P_B(\alpha, p) \iff \lambda(B_\alpha) > r_p := (-1)^{(p)_0} \frac{(p)_1}{(p)_2 + 1} \) and 

\[ Q_B(\alpha, p) \iff \lambda(B_\alpha) < r_p \]

are \( \Delta^1_1 \) to see that \( \lambda_B \) is \( \Delta^1_1 \)-recursive. Note that there is \( \phi : \omega^2 \rightarrow \omega \) recursive with \( r_{\phi(p, l)} = r_p - \frac{1}{l+1} \). Thus

\[ Q_B(\alpha, p) \iff \exists l \in \omega \ \lambda(B_\alpha) \leq r_p - \frac{1}{l+1} \]
\[ \iff \exists l \in \omega \ \neg (\lambda(B_\alpha) > r_p - \frac{1}{l+1}) \]
\[ \iff \exists l \in \omega \ \neg P_B(\alpha, \phi(p, l)) \]

so that it is enough to see that \( P_B \) is \( \Delta^1_1 \).
Now let \( S \in \Sigma_1^1(\omega^\omega \times (\omega^\omega)^2) \) be a good \( \omega^\omega \)-universal for the analytic subsets of \((\omega^\omega)^2\). We set

\[
U(\alpha, \gamma) \Leftrightarrow S((\alpha)_0, (\alpha)_1, \gamma),
\]

so that \( U \in \Sigma_1^1(\omega^\omega \times \omega^\omega) \) is \( \omega^\omega \)-universal for the analytic subsets of \( \omega^\omega \). Let \( A \) be a \( \Sigma_1^1 \) subset of \( \omega^\omega \times 2^\omega \). Then there is \( \alpha_0 \in \omega^\omega \) recursive with \( A = S_{\alpha_0} \), so that

\[
\gamma \in A_\alpha \Leftrightarrow (\alpha_0, \alpha, \gamma) \in S \Leftrightarrow (\alpha_0, \alpha >, \gamma) \in U.
\]

This implies that the relation \( R_A(\alpha, p) \Leftrightarrow \lambda(A_\alpha) > r_p \), equivalent to

\[
((p)_0 \text{ is odd } \land (p)_1 > 0) \lor ((p)_0 \text{ is even } \land (\alpha_0, \alpha >, (p)_1, (p)_2 >) \in L(U),
\]

is \( \Sigma_1^1 \), by Theorem 3.4.

In particular, this applies to \( A := B \), so that \( P_B \) is \( \Sigma_1^1 \). Now note that

\[
P_B(\alpha, p) \Leftrightarrow \lambda(\neg B_{\alpha}) < 1 - r_p \Leftrightarrow Q_{\neg B}(\alpha, \phi'(p)),
\]

for some \( \phi' : \omega \to \omega \) is recursive, so that \( P_B \) is \( \Pi_1^1 \) by the previous computation.

We set \( C' := \{ (\gamma, \beta) \in \omega^\omega \times 2^\omega \mid \gamma(0) \in \mathcal{W} \land (\gamma(0), \gamma^*, \beta) \in C \} \). As \( C' \) is \( \Pi_1^1 \),

\[
A := \{ (\alpha, p) \in \omega^\omega \times \omega \mid \lambda(\neg C'_{\alpha}) > r_p \}
\]

is \( \Sigma_1^1 \), by the previous discussion. So let \( n \in \mathcal{W} \). Note that

\[
\lambda(C_{n, \alpha}) > r_p \Leftrightarrow \lambda(C_{n, \alpha}) < 1 - r_p \Leftrightarrow \lambda(\neg (C'_{n, \alpha}) < 1 - r_p
\]

\[
\Leftrightarrow \exists l \in \omega \left( \lambda((\neg C')_{n, \alpha}) \leq 1 - r_p - \frac{1}{n^l} \right) \Leftrightarrow \exists l \in \omega \left( (n_\alpha, \phi''(p, l)) \notin A, \right.
\]

for some recursive \( \phi'' : \omega^2 \to \omega \). Similarly, the relation \( \lambda(C_{n, \alpha}) < r_p \) is \( \Pi_1^1 \) in \( (n, \alpha, p) \) since the relation \( \lambda(C_{n, \alpha}) < r_p \). Therefore the relation \( R_{O_0}(n, \alpha, p) \Leftrightarrow R_A(n, \alpha, p) \) is \( \Sigma_1^1 \) too. Moreover, \( R_{O_0}(n, \alpha, p) \Leftrightarrow \lambda((O_0)_n, \alpha) > r_p \Leftrightarrow \lambda_O(n, \alpha) > r_p \).

Assume now that \( n \in D \). Then as above there is \( \phi'' : \omega^2 \to \omega \) recursive such that

\[
\lambda_O(n, \alpha) > r_p \Leftrightarrow \lambda((O_1)_n, \alpha) > r_p \Leftrightarrow \lambda(\neg (O_1)_{n, \alpha}) < 1 - r_p
\]

\[
\Leftrightarrow \exists l \in \omega \left( \lambda((\neg O_1)_{n, \alpha}) \leq 1 - r_p - \frac{1}{n^l} \right) \Leftrightarrow \exists l \in \omega \left( (n_\alpha, \phi''(p, l)) \notin A, \right.
\]

which shows the existence of \( R_{O_0} \in \Pi_1^1 \) such that \( \lambda_O(n, \alpha) > r_p \Leftrightarrow R'_{O_0}(n, \alpha, p) \) if \( n \in D \).

Assume that \( n \in D \). Then there is \( \phi' : \omega \to \omega \) recursive such that

\[
\lambda_O(n, \alpha) < r_q \Leftrightarrow \lambda((O_1)_{n, \alpha}) < r_q \Leftrightarrow \lambda((\neg O_1)_{n, \alpha}) > 1 - r_q \Leftrightarrow R_{\neg O_1}(n, \alpha, \phi'(q)),
\]

which shows the existence of \( R''_{O_0} \in \Sigma_1^1 \) such that \( \lambda_O(n, \alpha) < r_q \Leftrightarrow R''_{O_0}(n, \alpha, q) \) if \( n \in D \).
• Assume that \( n \in D \). Then there is \( \phi'' : \omega^2 \rightarrow \omega \) recursive such that
\[
\lambda(O(n, \alpha) < r_q \iff \lambda((O_0)_{n, \alpha}) < r_q \iff \exists l \in \omega \ \lambda((O_0)_{n, \alpha}) \leq 1 - r_q - \frac{1}{l+1} \\
\iff \exists l \in \omega \ (- \lambda((O_0)_{n, \alpha}) > r_{\phi''(q, l)}) \iff \exists l \in \omega \ -R_{O_0}(n, \alpha, \phi''(q, l)),
\]
which shows the existence of \( R''_{O_0} \in \Pi^1_1 \) such that \( \lambda(O(n, \alpha) < r_q \iff R''_{O_0}(n, \alpha, q) \) if \( n \in D \).

• Finally, \( r_p < \lambda(O(n, \alpha) < r_q \iff R_{O_0}(n, \alpha, p) \land R''_{O_0}(n, \alpha, q) \) and
\[
r_p < \lambda(O(n, \alpha) < r_q \iff R''_{O_0}(n, \alpha, p) \land R''_{O_0}(n, \alpha, q)
\]
if \( n \in D \), which shows that \( \lambda(O) \) is \( \Sigma^1_1 \)-recursive and \( \Pi^1_1 \)-recursive on \( D \times \omega \).

(c) We first prove the following. Let \( X, Y \) be a recursively presented Polish spaces and \( g : X \times \omega \rightarrow Y \) be a \( \Delta^1_1 \)-recursive map. Then the partial map \( h : X \rightarrow Y \) defined by
\[
h(x) := \lim_{l \rightarrow \infty} g(x, l)
\]
when this limit exists is \( \Delta^1_1 \)-recursive.

Indeed, the domain \( D \) of \( h \) is \( \{ x \in X \mid \forall r \in \omega \ \exists L \in \omega \ \forall k, l \geq L \ d_Y(g(x, k), g(x, l)) < 2^{-r} \} \), so that \( D \) is \( \Delta^1_1 \). If \( x \in D \), then \( h(x) \in N(Y, u) \) is equivalent to
\[
\exists p, q \in \omega \ \frac{p}{q+1} < \frac{((u)1)}{((u)1)_{2} + 1} \land \exists L \in \omega \ \forall l \geq L \ g(x, l) \in N(Y, \{ 0, < (u)_{0}, p, q > \}),
\]
and we are done.

• We set \( B' := \{ (\alpha, \gamma) \in \omega^2 \times 2^\omega \mid (\alpha)_{0, \gamma} \in B \land \alpha \in N_{\gamma}|(\alpha)_{1}(0) \} \), so that \( B_x \cap N_{\beta|l} = B'_{<\alpha, l\beta>} \) and \( B' \) is \( \Delta^1_1 \). By (a), the map \( g : \omega^2 \times 2^\omega \rightarrow [0, 1] \) defined by \( g(\alpha, \beta, l) := 2^{-l} \lambda(B_x \cap N_{\beta|l}) \) is \( \Delta^1_1 \)-recursive. By the previous point, the partial map \( h : \omega^2 \times 2^\omega \rightarrow [0, 1] \) defined by
\[
h(\alpha, \beta) := \lim_{l \rightarrow \infty} 2^{-l} \lambda(B_x \cap N_{\beta|l})
\]
when it exists is also \( \Delta^1_1 \)-recursive. But \( h = d_B \).

• Fix \( n \in \mathcal{W} \). Then there is \( q(n) \in \mathcal{W} \) such that
\[
\mathcal{C}_{q(n)} = \{ (\gamma, \delta) \in \omega^2 \times 2^\omega \mid (n, (\gamma)_{0}, \delta) \in \mathcal{C} \land (\gamma)_{1}(0) \leq \delta \}.
\]
Moreover, we may assume that \( q \) is \( \Pi^1_1 \)-recursive on \( \mathcal{W} \), by the uniformization lemma. As \( \Pi^1_1 \) has the substitution property, the map \( g' : (n, \alpha, \beta, l) \mapsto 2^{-l} \lambda(\mathcal{C}_{q(n), <\alpha, l\beta>}) = 2^{-l} \lambda(\mathcal{C}_{n, \alpha} \cap N_{\beta|l}) \) is \( \Pi^1_1 \)-recursive on \( \mathcal{W} \times \omega^2 \times 2^\omega \times \omega \). As above, the map
\[
h' : (n, \alpha, \beta) \mapsto \lim_{l \rightarrow \infty} 2^{-l} \lambda(\mathcal{C}_{n, \alpha} \cap N_{\beta|l}) = d(\mathcal{C}_{n, \alpha}, \beta)
\]
is \( \Pi^1_1 \)-recursive on the \( \Pi^1_1 \) set \( \{ (n, \alpha, \beta) \in \mathcal{W} \times \omega^2 \times 2^\omega \mid d(\mathcal{C}_{n, \alpha}, \beta) \) exists\}. 

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(d) The argument here is partly similar to 11.6 and 17.25 in [K2]. We set, for \((k,l) \in \omega^2\),
\[ A_{k,l} := h^{-1}(\left(\frac{k}{2^l}, \frac{k+1}{2^l}\right)) \]
and define \(h_1: \omega^\omega \times 2^\omega \to [0,1]\) by \(h_1 = \Sigma_{k\leq 2^l} 2^k \chi_{A_{k,l}}\). We also define \(R \subseteq \omega^\omega \times 2^\omega \times \omega^3\) by
\[ R(\alpha, \beta, u, k, l) \iff \frac{k}{2^l} \leq h(\alpha, \beta) < \frac{k+1}{2^l} \land \text{Seq}(u) \land \beta \in N^u, \]
so that \(R\) is \(\Delta^1_1\). Then we define \(O \subseteq \omega^\omega \times 2^\omega\) by
\[ O(\alpha, \beta) \iff \text{Seq}(\alpha(0)) \land \text{lh}(\alpha(0)) = 3 \land R\left(\alpha^*, \beta, (\alpha(0))_0, (\alpha(0))_1, (\alpha(0))_2\right), \]
so that \(O\) is \(\Delta^1_1\).

Note that \((h_1)\) is a sequence of Borel functions pointwise converging to \(h\). By Lebesgue’s dominated convergence theorem, \(\int_{N^u} h(\alpha, .) \, d\lambda = \lim_{N^u} \int_{N^u} h_1(\alpha, .) \, d\lambda\) if \(\text{Seq}(u)\). Note that
\[ \int_{N^u} h_1(\alpha, .) \, d\lambda = \int_{N^u} \sum_{k \leq 2^l} \frac{k}{2^l} \chi_{A_{k,l}}(\alpha, .) \, d\lambda = \sum_{k \leq 2^l} \frac{k}{2^l} \lambda((A_{k,l})_\alpha \cap N^u) \]
\[ = \sum_{k \leq 2^l} \frac{k}{2^l} \lambda(R_{k,l}) = \sum_{k \leq 2^l} \frac{k}{2^l} \lambda(O_{<k,l}, \alpha). \]

Using (a), this implies that the map \((\alpha, u, l) \mapsto \int_{N^u} h_1(\alpha, .) \, d\lambda\) is \(\Delta^1_1\)-recursive on its \(\Delta^0_1\) domain \(\omega^\omega \times \{u \in \omega \mid \text{Seq}(u)\} \times \omega\). As in the proof of (c), \(h_1\) is \(\Delta^1_1\)-recursive on its domain. \(\square\)

We now prove a uniform version of Theorem 4.3.2 in [K1] (due to Tanaka, see [T2]).

**Theorem 3.6** Let \(B \in \Delta^1_1(\omega^\omega \times 2^\omega)\), and \(\epsilon: \omega^\omega \to \mathbb{R}\) be \(\Delta^1_1\)-recursive such that \(\epsilon(\alpha) \in (0,1)\) for each \(\alpha \in \omega^\omega\). Then there is \(T \in \Delta^1_1(\omega^\omega \times \omega)\) such that
(a) \(T_\alpha\) is a tree for each \(\alpha \in \omega^\omega\),
(b) if \(K = \{(\alpha, \beta) \in \omega^\omega \times 2^\omega \mid \forall l \in \omega \quad (\alpha, \beta(l)) \in T\}\), then \(K_\alpha \subseteq B_\alpha\) and \(\lambda(K_\alpha) \geq \lambda(B_\alpha) - \epsilon(\alpha)\) for each \(\alpha \in \omega^\omega\).

**Proof.** Theorem 3.2 gives \(\pi: \omega^\omega \to \omega^\omega \times 2^\omega\) recursive and \(C \in \Pi^0_1(\omega^\omega)\) such that \(\pi\) is injective on \(C\) and \(\pi[C] = B\). We set \(Q := \{(\alpha, \beta, \gamma) \in (\omega^\omega)^3 \mid \gamma \in C \land \pi(\gamma) = (\alpha, \beta)\}\). As \(Q \in \Pi^0_1\), Theorem 3.1 gives a recursive subset \(T\) of \(\omega^\omega \times \omega^2\) such that \((\alpha, \beta, \gamma) \in Q \iff \forall l \in \omega \quad (\alpha, \beta(l), \gamma(l)) \in T\) and \(T_\alpha\) is a tree for each \(\alpha \in \omega^\omega\).

- We set, for \(u, v \in \omega\),
\[ u \leq^u v \iff \text{Seq}(u), \text{Seq}(v) \land \text{lh}(u) = \text{lh}(v) \land \forall i < \text{lh}(u) \quad (u)_i \leq (v)_i. \]
- Then we set, for \(u \in \omega\) with \(\text{Seq}(u)\) and \(\alpha \in \omega^\omega\),
\[ B_\alpha^u := \{\beta \in 2^\omega \mid \exists \gamma \in \omega^\omega \quad \gamma(\text{lh}(u)) \leq^u u \land \forall l \in \omega \quad (\alpha, \beta(l), \gamma(l)) \in T\} \]
and \(B' := \{(\alpha, \beta) \in \omega^\omega \times 2^\omega \mid \text{Seq}(\alpha(0)) \land \beta \in B_\alpha(0)\}\). Note that \(B'\) is \(\Sigma^1_1\). In fact, \(B'\) is \(\Delta^1_1\) by uniqueness of the witness \(\gamma\).
• We now define $\delta_\alpha \in \omega^\omega$ as follows. We define $\delta_\alpha(i)$ by induction on $i$. We first set

$$\delta_\alpha(0) := \min \{ k \in \omega \mid \lambda(B_\alpha^{<k^+}) > \lambda(B_\alpha) - \frac{\epsilon(\alpha)}{2} \}.$$ 

This number exists since $B_\alpha$ is the increasing union of the $B_\alpha^{<k^+}$'s. Then

$$\delta_\alpha(i+1) := \min \{ k \in \omega \mid \lambda(B_\alpha^{<\delta_\alpha(0) \ldots \delta_\alpha(i),k^+}) > \lambda(B_\alpha) - \frac{\epsilon(\alpha)}{2} - \ldots - \frac{\epsilon(\alpha)}{2^{i+2}} \}.$$ 

Note that $\delta_\alpha \in \Delta^1_1(\alpha)$, by Corollary 3.5.(a).

• We set $T := \{(\alpha, u) \in \omega^\omega \times \omega \mid \text{Seq}(u) \wedge \exists \omega \leq \delta_\alpha(\text{lh}(u)) \ (\alpha, v, u) \in \bar{T}\}$, so that $T \in \Delta^1_1(\omega^\omega \times \omega)$ and $\bar{T}_\alpha$ is a tree for each $\alpha \in \omega^\omega$.

• We set $K := \{(\alpha, \beta) \in \omega^\omega \times 2^\omega \mid \forall \omega \in \beta \exists \alpha, \bar{\beta}(l) \in T\}$, so that $K_\alpha \subseteq B_\alpha$ and

$$\lambda(K_\alpha) = \lim_{l \to \infty} \lambda(B_\alpha^{\bar{\beta}(l)}) \geq \lambda(B_\alpha) - \epsilon(\alpha)$$ 

for each $\alpha \in \omega^\omega$ since $(B_\alpha^{\bar{\beta}(l)})_{l \in \omega}$ is decreasing. It remains to apply König’s lemma to see that $K = \{(\alpha, \beta) \in \omega^\omega \times 2^\omega \mid \forall \omega \in \beta \exists \alpha, \bar{\beta}(l) \in T\}$ since

$$\{ s \in \omega^\omega \mid \omega \leq \delta_\alpha(\text{lh}(s)) \wedge (\alpha, \bar{\beta}(|s|), s(0), \ldots, s(|s| - 1)) \in \bar{T}\}$$

is a finitely splitting tree. \qed

- We want to prove an effective and uniform version of the Lusin-Menchoff lemma. We first need the following result, which slightly and uniformly refines Theorem A in [L] at the first level of the Borel hierarchy.

**Lemma 3.7** Let $O$ be a $\Delta^1_1$ subset of $\omega^\omega \times 2^\omega$ with open vertical sections. Then there is a $\Delta^1_1$-recursive map $f : \omega^\omega \to \omega^\omega$ such that $O_\alpha$ is the disjoint union $\bigcup \{ N^{f(\alpha)}(u) \mid u \in \omega \wedge \text{Seq}(f(\alpha)(u)) \}$, for each $\alpha \in \omega^\omega$.

**Proof.** Let $P := \{(\alpha, u) \in \omega^\omega \times \omega \mid \text{Seq}(u) \wedge (\text{lh}(u) = 0 \vee (N_u \subseteq O_\alpha \wedge N_u^- \not\subseteq O_\alpha))\}$. Note that $P$ is $\Pi^1_1$, since a nonempty $\Delta^1_1(\alpha)$ closed subset of $2^\omega$ contains a $\Delta^1_1(\alpha)$ point, by 4F.15 in [M]. We then define a relation $R$ on $\omega^\omega \times 2^\omega \times \omega$ by $R(\alpha, \beta, u) \iff P(\alpha, u) \wedge \beta \in N_u$, so that $R$ is $\Pi^1_1$. Note that, for each $(\alpha, \beta) \in O$ there is $u \in R(\alpha, \beta, u)$. By 4B.5 in [M], there is a $\Delta^1_1$-recursive map $g : \omega^\omega \times 2^\omega \to \omega$ such that $R(\alpha, \beta, g(\alpha, \beta))$ for each $(\alpha, \beta) \in O$. Fix $\alpha \in \omega^\omega$. Note that $S^\alpha := \{ g(\alpha, \beta) \mid \beta \in O_\alpha \}$ is a $\Sigma^1_1(\alpha)$ subset of $\omega$ contained in the $\Pi^1_1(\alpha)$ set $P_\alpha$. By 4B.11 and 4C in [M], there is $D^\alpha \in \Delta^1_1(\alpha)$ with $S^\alpha \subseteq D^\alpha \subseteq P_\alpha$. Note that $O_\alpha \subseteq \bigcup_{u \in D^\alpha} N_u \subseteq O_\alpha$, so that $O_\alpha$ is the disjoint union of the sequence $(N_u)_{u \in D^\alpha}$. We define $\delta_\alpha \in \omega^\omega$ by

$$\delta_\alpha(u) := \begin{cases} u & \text{if } u \in D_\alpha, \\ 0 & \text{otherwise}. \end{cases}$$

Note that $\delta_\alpha \in \Delta^1_1(\alpha)$ and $O_\alpha$ is the disjoint union $\bigcup \{ N^{\delta_\alpha(u)}(u) \mid u \in \omega \wedge \text{Seq}(\delta_\alpha(u)) \}$. As the set

$$\{ (\alpha, \delta) \in \omega^\omega \times \omega^\omega \mid \delta \in \Delta^1_1(\alpha) \wedge \text{Seq}(\delta_\alpha(u)) \}$$

is $\Pi^1_1$, it remains to apply the uniformization lemma to get the desired map $f$. \qed

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Notation. We set $\mathcal{W}_1 := \{n \in \mathcal{W} \mid \forall \alpha < \omega^\omega \exists \gamma_n \in \Delta^1_1(\alpha) \ C_{n,\alpha} = \bigcup \{ N^{\gamma_n(u)} | u \in \omega \land \text{Seq}(\gamma_n(u)) \}$, so that, by Lemma 3.7, $\mathcal{W}_1$ is a $\Pi^1_1$ set of codes for the $\Delta^1_1$ subsets of $\omega^\omega \times 2^\omega$ with open vertical sections.

Lemma 3.8 Let $f$ be a $\Delta^1_1$ subset of $\omega^\omega \times 2^\omega$ with closed vertical sections, and $B$ be a $\Delta^1_1$ subset of $\omega^\omega \times 2^\omega$ such that $B \subseteq F$ and $d(B, \beta) = 1$ for each $(\alpha, \beta) \in F$. Then there is a $\Delta^1_1$ subset $C$ of $\omega^\omega \times 2^\omega$ with closed vertical sections such that

1. $F \subseteq C \subseteq B$,
2. $d(B, \beta) = 1$ for each $(\alpha, \beta) \in C$,
3. $d(C, \beta) = 1$ for each $(\alpha, \beta) \in F$.

Proof. Lemma 3.7 gives a $\Delta^1_1$-recursive map $f : \omega^\omega \rightarrow \omega^\omega$, so that $(-F)_\alpha$ is the disjoint union $\bigcup \{ N^{f(\alpha)(u)} | u \in \omega \land \text{Seq}(f(\alpha)(u)) \}$, for each $\alpha < \omega^\omega$. We set

$$B' := \{ (\alpha, \gamma) \in \omega^\omega \times 2^\omega | ((\alpha)_0, \gamma) \in B \land \text{Seq}(f((\alpha)_0)((\alpha)_1(0))) \land \gamma \in N^{f((\alpha)_0)((\alpha)_1(0))} \},$$

so that $B'$ is a $\Delta^1_1$ and $B_{\alpha} \cap N^{f(\alpha)(u)} = B'_{\alpha < \omega^\omega}$ if Seq$(f(\alpha)(u))$. By Corollary 3.5.(c), the partial map $(\alpha, \beta, u) \mapsto d(B_{\alpha} \cap N^{f(\alpha)(u)}, \beta)$ is $\Delta^1_1$-recursive. We then set

$$B'' := \{ (\alpha, \gamma) \in B' \mid d(B_{\alpha} \cap N^{f(\alpha)(u)}((\alpha)_1(0)), \gamma) = 1 \},$$

so that $B''$ is $\Delta^1_1$ and $\{ \beta \in B_{\alpha} \cap N^{f(\alpha)(u)} | d(B_{\alpha} \cap N^{f(\alpha)(u)}, \beta) = 1 \} = B''_{\alpha, \omega^\omega}$ if Seq$(f(\alpha)(u))$. We define $\epsilon : \omega^\omega \rightarrow \mathbb{R}$ by

$$\epsilon(\alpha) := \begin{cases} 2^{-\alpha_1(0)} \lambda(B'_{\alpha}) & \text{if } \lambda(B'_{\alpha}) \neq 0, \\ 1 & \text{otherwise,} \end{cases}$$

so that $\epsilon$ is $\Delta^1_1$-recursive by Corollary 3.5.(a), and $\epsilon(\alpha) \in (0, 1]$ for each $\alpha < \omega^\omega$. Theorem 3.6 gives $T \in \Delta^1_1(\omega^\omega \times \omega)$ such that

(a) $T$ is a tree for each $\alpha < \omega^\omega$,
(b) if $K = \{ (\alpha, \beta) \in \omega^\omega \times 2^\omega | \forall l \in \omega \ (\alpha, \beta(l)) \in T \}$, then $K_{\alpha} \subseteq B''_{\alpha}$ and $\lambda(K_{\alpha}) \geq \lambda(B''_{\alpha}) - \epsilon(\alpha)$ for each $\alpha < \omega^\omega$.

We set, for $u \in \omega$,

$$F_u := \{ (\alpha, \beta) \in \omega^\omega \times 2^\omega | \text{Seq}(f(\alpha)(u)) \land (\alpha, u^\omega \beta) \in K \land \lambda(B'_{\alpha, u^\omega} > 0) \neq 0 \}.$$
Moreover,

$$\lambda(F_0^u) = \lambda(K_{<,u^\omega}) \geq \lambda(B_{<,u^\omega}) - \epsilon(<,u^\omega) = \lambda(B_{<,u^\omega}) - 2^{-u} \lambda(B_{<,u^\omega})$$

$$= (1 - 2^{-u}) \lambda(B_\alpha \cap N f(\alpha(u)))$$

since $$\lambda(B_\alpha \cap N f(\alpha(u))) = \lambda(\{\beta \in B_\alpha \cap N f(\alpha(u)) \mid d(B_\alpha \cap N f(\alpha(u)), \beta) = 1\})$$, by Theorem 2.1. It remains to set $$C := F \cup \bigcup_{u \in \omega} F^u$$. We conclude as in the proof of Lemma 2.2. 

- We now want to prove an effective and uniform version of Lemma 2.5.

**Lemma 3.9** Let $$C$$ be a $$\Delta^1_1$$ subset of $$\omega^\omega \times 2^\omega$$ with closed vertical sections, $$G$$ be a Borel subset of $$2^\omega$$ with $$\lambda(G) = 0$$, and $$F$$ be a $$\Delta^1_1$$ subset of $$\omega^\omega \times 2^\omega$$ with $$G$$ vertical sections, contained in $$\omega^\omega \times G$$ and disjoint from $$C$$. Then there is a $$\Delta^1_1$$-recursive map $$h : \omega^\omega \times 2^\omega \to \mathbb{R}$$ such that $$h(\alpha, \cdot) : 2^\omega \to [0, 1]$$ is $$\tau$$-continuous for each $$\alpha \in \omega^\omega$$, $$h|_G \equiv 0$$ and $$h|_G \equiv 1$$.

**Proof.** By Theorem 3.5 in [L], there is a $$\Delta^1_1$$ subset $$F$$ of $$\omega \times \omega^\omega \times 2^\omega$$ such that $$F_{n,0}$$ is closed for each $$(n, \alpha) \in \omega \times \omega^\omega$$ and $$\neg G = \bigcup_{n \in \omega} F_n$$. Moreover, we may assume that $$(F_n)_{n \in \omega}$$ is increasing and $$F_0 = C$$.

- We will define, by primitive recursion, a partial map $$f : \omega \to \omega$$ which is $$\Pi^1_1$$-recursive on its domain such that $$f(n)$$ essentially codes the set $$C_f$$ constructed in the proof of Lemma 2.5. As this map will in fact be total, it will be $$\Delta^1_1$$-recursive by the uniformization lemma.

We first apply Lemma 3.8 to $$F := F_0$$ and $$B := \neg G$$. This is possible because $$G_\alpha \subseteq G$$, so that $$(\neg G)_\alpha$$ has $$\lambda$$-measure one and therefore density one at any point of $$2^\omega$$, for each $$\alpha \in \omega^\omega$$. Lemma 3.8 gives $$C_1 \in \Delta^1_1$$ with closed vertical sections such that $$\neg G \supseteq C_1 \supseteq F_0$$. Let $$f(0) \in W_1$$ with $$C_f(0) = \neg C_1$$.

More generally, we will have $$C_{f(n)} = \neg C_{f(n)}$$. As mentioned above, $$f$$ will be defined by primitive recursion, which means that will be a partial map $$g : \omega^2 \to \omega$$ such that $$f(n + 1) = g(f(n), n)$$. This partial map $$g$$ will be $$\Pi^1_1$$-recursive on its $$\Pi^1_1$$ domain $$\{m \in W_1 \mid \neg C_m \subseteq \neg G \times \omega\}$$, so that $$f$$ will be $$\Pi^1_1$$-recursive on its domain by 7A.5 in [M]. The map $$g$$ will take values in $$W_1$$, and is constructed in such a way that, if $$A := \neg C_m \subseteq \neg G$$ and $$A' := \neg C_{g(m,n)}$$, then

1. $$A \cup F_{n+1} \subseteq A' \subseteq \neg G$$,
2. $$\forall (\alpha, \beta) \in A' \ d((\neg G)_\alpha, \beta) = 1$$,
3. $$\forall (\alpha, \beta) \in A \cup F_{n+1} \ d(A'_\alpha, \beta) = 1$$.

Lemma 3.8 ensures that such a $$g(m,n) \in \omega$$ exists if $$(m,n) \in \{q \in W_1 \mid \neg C_q \subseteq \neg G \times \omega\}$$. As the properties (1)-(3) are $$\Pi^1_1$$ by Corollary 3.5, the uniformization lemma ensures the existence of $$g$$. So we constructed a $$\Delta^1_1$$-recursive map $$f : \omega \to \omega$$, taking values in $$W_1$$, such that $$C_{f(n)} := \neg C_{f(n)}$$ is a $$\Delta^1_1$$ subset of $$\omega^\omega \times 2^\omega$$ with closed vertical sections, $$F_n \subseteq C_{f(n)} \subseteq \neg G$$, $$C_{f(n)} \subseteq C_{f(n)}$$, and

$$d((C_{f(n)}), \alpha, \beta) = 1$$

if $$(\alpha, \beta) \in C_{f(n)}$$.
• Similarly, we construct a $\Delta^1_1$-recursive map $F: \omega \to \omega$ satisfying the following properties, if

$$D := \{ p \in \omega : \exists \text{Seq}(u) \land \text{lh}(u) = 2 \land 0 < (p)_1 \leq 2^{(p)_0} \}.$$ 

(a) $F(p) \in \mathcal{W}_1$ if $p \in D$, in which case we set $C_p := \neg C_{F(p)}$, 
(b) $C_p \subseteq C_p$ if $p, p' \in D \land \frac{(p)_1}{2^{(p)_0}} \leq \frac{(p')_1}{2^{(p')_0}}$, 
(c) $d((C_p')_\alpha, \beta) = 1$ if $p, p' \in D \land \frac{(p')_1}{2^{(p')_0}} < \frac{(p)_1}{2^{(p)_0}} \land (\alpha, \beta) \in C_p$.

This allows us to define $h$ by

$$1 - h(\alpha, \beta) := \begin{cases} 0 & \text{if } (\alpha, \beta) \in G, \\
\sup \{ \frac{(p)_1}{2^{(p)_0}} \mid p \in D \land (\alpha, \beta) \in C_p \} & \text{if } (\alpha, \beta) \notin G. 
\end{cases}$$

Note that $h$ is $\Delta^1_1$-recursive since $D \in \Delta^0_1$, so that the relation “$p \in D \land (\alpha, \beta) \in C_p$” is $\Delta^1_1$ in $(\alpha, \beta)$. We conclude as in the proof of Lemma 2.5. \hfill \Box

- We are now ready to prove the main lemma in this section. We equip the space $[0, 1]^{2^{<\omega}}$ with the distance defined by $d(f, g) := \sup_{u \in \omega} \frac{|f(s(u)) - g(s(u))|}{2^{\text{lh}(u)}}$. We give a recursive presentation of $([0, 1]^{2^{<\omega}}, d)$. We set

$$f_n(s) := \begin{cases} \frac{(n)_1}{2^{\text{lh}(n)} + 1} & \text{if Seq}(n) \land \forall k < \text{lh}(n) \left( \text{Seq}\left((n)_k \right) \land \text{lh}\left((n)_k\right) = 2 \right) \land \exists \ell < \text{lh}(n), \\
0 & \text{otherwise,}
\end{cases}$$

so that $(f_n)$ is dense in $[0, 1]^{2^{<\omega}}$. It is now routine to check that the relations "$d(f_m, f_n) \leq \frac{p}{q + 1}$" and "$d(f_m, f_n) < \frac{p}{q + 1}$" are recursive in $(m, n, p, q)$. It is also routine to check that $F: \omega^\omega \to [0, 1]^{2^{<\omega}}$ is $\Delta^1_1$-recursive if the map $F': \omega \times \omega^\omega \to \mathbb{R}$ defined by $F'(u, \alpha) := F(\alpha)(s(u))$ if Seq(u), 0 otherwise, is $\Delta^1_1$-recursive $(s(u)$ was defined at the beginning of Section 3).

**Lemma 3.10** Let $\mathcal{V} := \{(f, \beta) \in \mathcal{M} \times 2^\omega \mid \text{osc}(f, \beta) > 0\}$, $G$ be a nonempty $G_\delta \cap \Delta^1_1$ subset of $2^\omega$ with $\lambda(G) = 0$, and $G$ be a $\Delta^1_1$ subset of $\omega^\omega \times 2^\omega$, contained in $\omega^\omega \times G$, and with $G_\delta$ vertical sections. Then there is a $\Delta^1_1$-recursive map $F: \omega^\omega \to [0, 1]^{2^{<\omega}}$, taking values in $\mathcal{V}$, and such that $G_\alpha = \mathcal{V}(F(\alpha))$ for each $\alpha \in \omega^\omega$.

**Proof.** We will define, by primitive recursion, $f : \omega \to \omega^4$ coding $g_n$, $S_n$, $G_n^*$, and $(s_j^n)_{j \in \ell_n}$, defining $G_n^*$ considered in the proof of the Lemma 2.7. We must find $r: \omega^4 \times \omega \to \omega^4$ with $f(n+1) = r(f(n), n)$. In practice,

1. $f_0(n) \in \mathcal{W}_1 \text{ codes } G_n^* \subseteq \omega^\omega \times 2^\omega$,
2. $f_1(n) \in \mathcal{W}[\omega^{\omega} \times 2^\omega \times \mathbb{R}] \text{ codes the graph of } g_n : \omega^\omega \times 2^\omega \to \mathbb{R}$,
3. $f_2(n) \in \mathcal{W}[\omega^{\omega} \times 2^\omega \times \mathbb{R}] \text{ codes the graph of } S_n : \omega^\omega \times 2^\omega \to \mathbb{R}$,
4. $f_3(n) \in \mathcal{W}[\omega^{\omega} \times \omega^\omega] \text{ codes the graph of the function } \alpha \mapsto (s_j^n(\alpha))_{j \in \ell_n}$.

**• By Theorem 3.5 in [L], there is a $\Delta^1_1$ subset $O$ of $\omega \times \omega^\omega \times 2^\omega$ such that $O_n, \alpha$ is open for each $(n, \alpha) \in \omega \times \omega^\omega$ and $G = \bigcap_{n \in \omega} O_n$. Moreover, we may assume that $(O_n)_{n \in \omega}$ is decreasing and $O_0 = \omega^\omega \times 2^\omega$.**
• Let \( n_0 \in \mathcal{W}_1 \) with \( C_{n_0} = \omega^\omega \times 2^\omega \), \( n_1 \in \mathcal{W}_1 \times 2^\omega \times \mathbb{R} \) with \( C_{n_1} = \omega^\omega \times 2^\omega \times \mathbb{R} = \{(\alpha, \beta, r) \in \omega^\omega \times 2^\omega \times \mathbb{R} \mid r = 1\} \), and \( n_3 \in \mathcal{W}_1 \times \omega^\omega \) with \( C_{n_3} = \{(\alpha, \gamma) \in \omega^\omega \times \omega^\omega \mid \gamma = 10^\infty\} \). We set \( f(0) = (n_0, n_1, n_2, n_3) \), so that \( C_{n_0} = G_0^\alpha, C_{n_1} = \omega^\omega \times 2^\omega \times \mathbb{R} = \text{Gr}(g_0) = \text{Gr}(S_0), C_{n_3} \) is \( \text{Gr}(\alpha \rightarrow 10^\infty) \),

\[
\{ u \in \omega \mid \text{Seq}(10^\infty)(u) \} = \{0\} = I_0
\]

and \((10^\infty)(0) = 1 = \ast = s_0^0\). So \( f(0) \) is as desired.

• We now study the induction step. This means that we must define \( r(n_0, n_1, n_2, n_3, n) \in \omega^4 \).

(1) We first define \( r_0(n_0, n_1, n_2, n_3, n) \) coding \( G_{n_0}^* \). Fix \( n_3 \in \mathcal{W}_1 \times \omega^\omega \) coding the graph of a \( \Delta_1^1 \)-recursive function \( \phi : \omega^\omega \rightarrow \omega^\omega \) such that the sequences \( s(\phi(\alpha)(u)) \) coded by the \( u \)'s with \( \text{Seq}(\phi(\alpha)(u)) \) are pairwise incompatible and \( G_\alpha \subseteq \bigcup \{ N(\phi(\alpha)(u)) \mid u \in \omega \wedge \text{Seq}(\phi(\alpha)(u)) \} \) (we call \( P_3 \) the \( \Pi_1^1 \) set of such \( n_3 \)'s). Let \( \alpha \in \omega^\omega \). Assume that \( \text{Seq}(\phi(\alpha)(u)) \) (which intuitively means that \( u \in I_{n,\alpha} \) and \( s_u^{n,\alpha} \) is coded by \( \phi(\alpha)(u) \)). By continuity of \( \lambda \),

\[
0 = \lambda(G_\alpha \cap N^\phi(\alpha)(u)) = \lim_{j \to \infty} \lambda(O_j \cap N^\phi(\alpha)(u)).
\]

This gives \( j(n, \alpha, u) > n \) minimal with \( \lambda(O_j(n,\alpha,u),\alpha \cap N^\phi(\alpha)(u)) < 2^{-n-3-\text{lh}(\phi(\alpha)(u))} \) (note that \( 2^{-\text{lh}(\phi(\alpha)(u))} = \lambda(N^\phi(\alpha)(u)) \)). Moreover, \( G_\alpha \cap N^\phi(\alpha)(u) \subseteq O_j(n,\alpha,u),\alpha \cap N^\phi(\alpha)(u) \subseteq O_{n+1},\alpha \cap N^\phi(\alpha)(u) \), so that \( O_j(n,\alpha,u),\alpha \cap N^\phi(\alpha)(u) \) satisfies the properties of the set \( O_j \) in the proof of Lemma 2.7. We will have \( G_{n+1,\alpha} = \bigcup_{\text{Seq}(\phi(\alpha)(u))} O_{j(n,\alpha,u),\alpha} \cap N^\phi(\alpha)(u) \). By Corollary 3.5 and the uniformization lemma, we may assume that the map \( j \) is \( \Delta_1^1 \)-recursive on its \( \Delta_1^1 \) domain

\[
\{ (n, \alpha, u) \in \omega \times \omega^\omega \times \omega \mid \text{Seq}(\phi(\alpha)(u)) \}.
\]

Note that \( G_{n+1}^* \) is a \( \Delta_1^1 \) subset of \( \omega^\omega \times 2^\omega \) with open vertical sections, which gives \( m \in \mathcal{W}_1 \) such that \( C_m = G_{n+1}^* \). By compatibility, \( G_{n+1,\alpha} \cap N^\phi(\alpha)(u) = O_{j(n,\alpha,u),\alpha} \cap N^\phi(\alpha)(u) \). So we proved that, for each \( (n_3, n) \in P_3 \times \omega \), there is \( m \in \mathcal{W}_1 \) such that, for each \( \alpha \in \omega^\omega \),

\[
\begin{align*}
(1) & \quad G_\alpha \subseteq C_{m,\alpha} \subseteq O_{n+1,\alpha} \cap \bigcup \{ N^\phi(\alpha)(u) \mid u \in \omega \wedge \text{Seq}(\phi(\alpha)(u)) \}, \\
(5) & \quad \lambda(C_{m,\alpha} \cap N^\phi(\alpha)(u)) < 2^{-n-3-\text{lh}(\phi(\alpha)(u))} \text{ if } u \in \omega \wedge \text{Seq}(\phi(\alpha)(u)).
\end{align*}
\]

By Corollary 3.5 and the uniformization lemma, we may assume that the map \( r_0 : (n_3, n) \mapsto m \) is \( \Pi_1^1 \)-recursive on \( P_3 \times \omega \). We set \( r_0(n_0, n_1, n_2, n_3, n) := r_0(n_3, n) \), which defines a partial map \( r_0 \) which is \( \Pi_1^1 \)-recursive on its \( \Pi_1^1 \) domain \( \omega^3 \times P_3 \times \omega \).

(2) We now define \( r_1(n_0, n_1, n_2, n_3, n) \) coding \( g_{n+1} \). We use Lemma 3.9 and its proof. Note that \( r_0(n_0, n_1, n_2, n_3, n) \in D_0 := \{ m \in \mathcal{W}_1 \mid G \subseteq C_m \} \). The proof of Lemma 3.9 shows that for any \( m \in D_0 \) there is \( F_m \in \omega^\omega \cap \Delta_1^1 \) satisfying the conditions (a), (b), (c) and

\[
\begin{align*}
(d) & \quad \forall p \in D \quad -(0 < (p)_1 = 2(p)^n) \lor C_{\tilde{F}_m(p)} \subseteq C_m.
\end{align*}
\]

The uniformization lemma shows that we may assume that the partial map \( \tilde{F} : m \mapsto \tilde{F}_m \) is \( \Pi_1^1 \)-recursive on \( D_0 \).
The definition of $h$ in the proof of Lemma 3.9 and the uniformization lemma show the existence of a partial map $\bar{H}: \omega \to \omega$, which is $\Pi^1_1$-recursive on $D_0$, and such that $\bar{H}(m)$ is in $\mathcal{W}^{\omega^\omega \times 2^\omega \times \mathbb{R}}$ and codes the graph of a $\Delta^1_1$-recursive map $\bar{h}: \omega^\omega \times 2^\omega \to \mathbb{R}$ with

$$1 - h(\alpha, \beta) := \begin{cases} 0 \text{ if } (\alpha, \beta) \in G \\ \sup \left\{ \frac{1}{2^{p_0}} \mid p \in D \land (\alpha, \beta) \notin C_F(m)(p) \right\} \text{ if } (\alpha, \beta) \notin G \end{cases}$$

if $m \in D_0$. We set $P_1 := \{ c \in \mathcal{W}^{\omega^\omega \times 2^\omega \times \mathbb{R}} \mid \mathcal{C}_c \text{ is the graph of a function } \zeta_c \}$. It is routine to check that there is a $\Pi^1_1$-recursive partial map $I: \omega^2 \to \omega$ on its domain $P^2_1$ such that $I(c, c') \in \mathcal{W}^{\omega^\omega \times 2^\omega \times \mathbb{R}}$ is the graph of the function $\min(\zeta_c, \zeta_{c'})$ if $c, c' \in P_1$. We set

$$r_1(n_0, n_1, n_2, n_3, n) := I(n_1, \bar{H}(r_0(n_0, n_1, n_2, n_3, n))),$$

so that $r_1$ is $\Pi^1_1$-recursive on its $\Pi^1_1$ domain $\omega \times P_1 \times \omega \times P_3 \times \omega$.

(3) We now define $r_2(n_0, n_1, n_2, n_3, n)$ coding

$$S_{n+1} = \begin{cases} S_n + g_{n+1} \text{ if } n \text{ is odd}, \\ S_n - g_{n+1} \text{ if } n \text{ is even}. \end{cases}$$

It is routine to check that there is a $\Pi^1_1$-recursive partial map $S: \omega^3 \to \omega$ on its domain $P^2_1 \times \omega$ such that $S(c, c', n) \in \mathcal{W}^{\omega^\omega \times 2^\omega \times \mathbb{R}}$ codes the graph of the function

$$(\alpha, \beta) \mapsto \begin{cases} \zeta_c(\alpha, \beta) + \zeta_{c'}(\alpha, \beta) \text{ if } n \text{ is odd} \\ \zeta_c(\alpha, \beta) - \zeta_{c'}(\alpha, \beta) \text{ if } n \text{ is even} \end{cases}$$

if $(c, c', n) \in P^2_1 \times \omega$. We set $r_2(n_0, n_1, n_2, n_3, n) := S(n_2, r_1(n_0, n_1, n_2, n_3, n), n)$, so that $r_2$ is $\Pi^1_1$-recursive on its $\Pi^1_1$ domain $\omega \times P^2_1 \times P_3 \times \omega$.

(4) We now define $r_3(n_0, n_1, n_2, n_3, n)$ coding the graph of the function $\alpha \mapsto (s_{j+1, \alpha}^n)_{j \in I_{n+1, \alpha}}$. We want to ensure the two following conditions:

1. $G_\alpha \subseteq \bigcup_{j \in I_{n+1, \alpha}} N_{s_{j+1, \alpha}} \subseteq G_{n+1, \alpha}^*$
2. $| \int_{N_{s_{j+1, \alpha}}} S_{n+1}(\alpha, \cdot) \ d\lambda - S_{n+1}(\alpha, \beta) | \leq 2^{-3} \text{ if } j \in I_{n+1, \alpha} \land \beta \in G_\alpha \cap N_{s_{j+1, \alpha}}$

Note first that in practice

$$S_{n+1}(\alpha, \beta) = \begin{cases} 0 \text{ if } n \text{ is even} \\ 1 \text{ if } n \text{ is odd} \end{cases}$$

if $(\alpha, \beta) \in G$ since $g_0(\alpha, \beta) = 1$ for each $p$ in this case. So there is $\psi: \omega \to \mathbb{R}^2$ recursive with

$$| \int_{N_{s_{j+1, \alpha}}} S_{n+1}(\alpha, \cdot) \ d\lambda - S_{n+1}(\alpha, \beta) | < 2^{-3} \iff \psi_0(n) < \int_{N_{s_{j+1, \alpha}}} S_{n+1}(\alpha, \cdot) \ d\lambda < \psi_1(n)$$

if $(\alpha, \beta) \in G$. We use Corollary 3.5 and its proof. Note that $r_3(n_0, n_1, n_2, n_3, n) \in P_1$.
We first consider $n'_0 \in W_1$ and $n'_2 \in P_1$ (coding $G^*_n,1$ and $S_{n+1}$ respectively) as variables. We define $R_0, R_1 \subseteq \omega \times \omega \times 2^\omega \times \omega^3$ by

$$R_0(n'_2, \alpha, \beta, u, k, l) \iff \exists r \in \mathbb{R} \quad \neg (n'_2 \in \mathcal{W} \times \omega^3 \times \mathbb{R} \wedge (n'_2, \alpha, \beta, r) \notin C \times \omega^3 \times \mathbb{R} \wedge \left( \frac{k}{l} \leq r < \frac{k+1}{l} \wedge \text{Seq}(u) \wedge \beta \in \mathbb{N}^u \right)$$

$$R_1(n'_2, \alpha, \beta, u, k, l) \iff \forall r \in \mathbb{R} \quad (n'_2 \in \mathcal{W} \times \omega^3 \times \mathbb{R} \wedge (n'_2, \alpha, \beta, r) \notin C \times \omega^3 \times \mathbb{R} \vee \left( \frac{k}{l} \leq r < \frac{k+1}{l} \wedge \text{Seq}(u) \wedge \beta \in \mathbb{N}^u \right),$$

so that $R_0$ is $\Sigma^1_1$, $R_1$ is $\Pi^1_1$, and $R_0(n'_2, \alpha, \beta, u, k, l) \iff R_1(n'_2, \alpha, \beta, u, k, l)$ if $n'_2 \in P_1$. Then, as in the proof of Corollary 3.5(d), we define $O_0, O_1 \subseteq \omega \times \omega \times 2^\omega$ by

$$O_\varepsilon(n'_2, \alpha, \beta) \iff \text{Seq}(\alpha(0)) \wedge \text{lh}(\alpha(0)) = 3 \wedge R_\varepsilon(n'_2, \alpha^*, \beta, (\alpha(0))_0, (\alpha(0))_1, (\alpha(0))_2)$$

if $\varepsilon < 2$, so that $O_0$ is $\Sigma^1_1$, $O_1$ is $\Pi^1_1$, and $O_0(n'_2, \alpha, \beta) \iff O_1(n'_2, \alpha, \beta)$ if $n'_2 \in P_1$. In particular, $n'_2 \in P_1$ and Seq$(u)$ imply that

$$\int_{\mathbb{N}} S_{n+1}(\alpha, .) \, d\lambda = \lim_{l \to \infty} \sum_{k \leq 2^l} \frac{k}{2^l} \lambda((O_\varepsilon)n'_2, <u, k, l>)$$

for each $\varepsilon < 2$. Thus $a < \int_{\mathbb{N}} S_{n+1}(\alpha, .) \, d\lambda < b$ is in this case equivalent to

$$\exists p_0, p_1, q_0, q_1, N \in \omega \wedge a < \frac{p_0}{p_1 + 1} \wedge \frac{q_0}{q_1 + 1} < b \wedge \forall l \geq N \frac{p_0}{p_1 + 1} \leq \sum_{k \leq 2^l} \frac{k}{2^l} \lambda((O_\varepsilon)n'_2, <u, k, l>) \leq \frac{q_0}{q_1 + 1}.$$  

By Corollary 3.5(b) applied to $D := P_1$, the partial map $\lambda_D : P_1 \times \omega \to \mathbb{R}$ defined by

$$\lambda_D(n'_2, \alpha) := \lambda((O_0)n'_2, \alpha)$$

is $\Sigma^1_1$-recursive and $\Pi^1_1$-recursive on its domain. By 3E.2, 3G.1 and 3G.2 in [M], these two classes of functions are closed under composition. In particular, the partial map

$$(n'_2, \alpha, u, l) \mapsto \sum_{k \leq 2^l} \frac{k}{2^l} \lambda((O_\varepsilon)n'_2, <u, k, l>)$$

is $\Sigma^1_1$-recursive and $\Pi^1_1$-recursive on $P_1 \times \omega \times \omega^2$. This shows the existence of $Q_0 \in \Sigma^1_1(\omega^2 \times \omega \times \omega)$ and $Q_1 \in \Pi^1_1(\omega^2 \times \omega \times \omega)$ such that

$$Q_0(n'_2, n, \alpha, u) \iff Q_1(n'_2, n, \alpha, u) \iff \text{Seq}(u) \wedge \psi_0(n) < \int_{\mathbb{N}} S_{n+1}(\alpha, .) \, d\lambda < \psi_1(n)$$

if $n'_2 \in P_1$. We now consider $n'_0 \in W_1$ and $n'_2 \in P_1$ as parameters. We set

$$P_{n'_0, n'_2}(n, \alpha, u) \iff Q_1(n'_2, n, \alpha, u) \wedge N \subseteq C_{n'_0, \alpha} \wedge \forall k < \text{lh}(u) \left( \neg Q_0(n'_2, n, \alpha, u(k)) \vee N \subseteq C_{n'_0, \alpha} \right).$$

Note that for each $(\alpha, \beta) \in G$ there is $l \in \omega$ minimal with the properties that $N \beta I \subseteq C_{n'_0, \alpha}$ and $Q_1(n'_2, n, \alpha, \beta) \beta (l-1) >$, so that $P_{n'_0, n'_2}(n, \alpha, \beta) \beta (l-1) >$ since $n'_0 \in W_1$ and $n'_2 \in P_1$. As $n'_0 \in W_1, N \subseteq \mathbb{N} \setminus C_{n'_0, \alpha}$ is a $\Delta_1^1(\alpha)$ compact subset of $\omega^3$, so that it contains a $\Delta_1^1(\alpha)$ point if it is not empty (see 4F.15 in [M]). This shows that $P_{n'_0, n'_2} \subseteq \Pi^1_1$.
The uniformization lemma provides a $\Delta^1_1$-recursive map $L: \omega \times \omega^{\omega} \times 2^\omega \to \omega$ such that

$$P_{n'_0, n'_2}(n, \alpha, < \beta(0), \ldots, \beta(L(n, \alpha, \beta) - 1) >)$$

if $(\alpha, \beta) \in G$. Note that the $\Sigma^1_1$ set

$$\sigma := \{(n, \alpha, u) \in \omega \times \omega^{\omega} \times \omega \mid \exists \beta \in G_n \ u = < \beta(0), \ldots, \beta(L(n, \alpha, \beta) - 1) >\}$$

is contained in the $\Pi^1_1$ set $\pi := \{(n, \alpha, u) \in \omega \times \omega^{\omega} \times \omega \mid P_{n'_0, n'_2}(n, \alpha, u)\}$. By 7B.3 in [M], there is a $\Delta^1_1$ subset $\delta$ of $\omega \times \omega^{\omega} \times \omega$ such that $\sigma \subseteq \delta \subseteq \pi$. We now also consider $n$ as a parameter and define $\varphi: \omega^{\omega} \to \omega^{\omega}$ by

$$\varphi(\alpha)(u) := \begin{cases} u \text{ if } (n, \alpha, u) \in \delta, \\ 0 \text{ otherwise.} \end{cases}$$

Note that $\varphi$ is $\Delta^1_1$-recursive, and that $\text{Seq}(\varphi(\alpha)(u))$ is equivalent to $(n, \alpha, u) \in \delta$. In particular,

1. $G_n \subseteq \bigcup \{ N\gamma(\alpha)(u) \mid u \in \omega \land \text{Seq}(\varphi(\alpha)(u)) \} \subseteq C_{n'_0, \alpha}$
2. $f_{N\gamma(\alpha)(u)} S_{n+1}(\alpha, \gamma) d\lambda - S_{n+1}(\alpha, \beta) < 2^{-3}$ if $\text{Seq}(\varphi(\alpha)(u)) \land \beta \in G_n \cap N\gamma(\alpha)(u)$

for each $\alpha \in \omega^{\omega}$. Let $k \in \mathcal{W}_n^{\omega^{\omega} \times \omega^{\omega}}$ such that $C_k^{\omega^{\omega} \times \omega^{\omega}} = \text{Gr}(\varphi)$. We now consider $n'_0$, $n'_2$ and $n$ as variables again. Note that for each $(n'_0, n'_2, n) \in \mathcal{W}_1 \times P_1 \times \omega$ there is $k \in \omega$ such that

$$R(n'_0, n'_2, n, k) \iff \begin{cases} k \in \mathcal{W}_n^{\omega^{\omega} \times \omega^{\omega}} \land \\ \forall \alpha \in \omega^{\omega} \forall \gamma \in \omega^{\omega} \left( k \in \mathcal{W}_n^{\omega^{\omega} \times \omega^{\omega}} \land \neg C_k^{\omega^{\omega} \times \omega^{\omega}}(k, \alpha, \gamma) \right) \lor \\ (1) G_n \subseteq \bigcup \{ N\gamma(u) \mid u \in \omega \land \text{Seq}(\gamma(u)) \} \subseteq C_{n'_0, \alpha} \\ \land (6) \forall u \in \omega \neg \text{Seq}(\gamma(u)) \lor Q_1(n'_2, n, \alpha, u) \} \end{cases}$$

Note that $R \in \Pi^1_1(\omega^4)$. The uniformization lemma provides a partial map $K: \omega^3 \to \omega$ which is $\Pi^1_1$-recursive on its $\Pi^1_1$ domain $\mathcal{W}_1 \times P_1 \times \omega$, and $R(n'_0, n'_2, n, K(n'_0, n'_2, n))$ if

$$(n'_0, n'_2, n) \in \mathcal{W}_1 \times P_1 \times \omega.$$ 

It remains to set $r_3(n_0, n_1, n_2, n_3, n) := K(n'_0, n'_2, n)$ if $n'_0 = r_0(n_0, n_1, n_2, n_3, n)$ and

$$n'_2 = r_2(n_0, n_1, n_2, n_3, n),$$

so that $r_3$ is $\Pi^1_1$-recursive on its $\Pi^1_1$ domain $\mathcal{W}_1 \times P_2 \times P_3 \times \omega$.

Finally, $r$ is $\Pi^1_1$-recursive on $\mathcal{W}_1 \times P_2 \times P_3 \times \omega$, $f$ is $\Pi^1_1$-recursive on $\omega$, and thus $f$ is $\Delta^1_1$-recursive by the uniformization lemma since it is total.

- We are now ready to define the dimension two versions of $G_n^x$, $g_n$, $S_n$, and $(s_j^n)_{j \in I_n}$:

1. $G_n^x := C_{f_0(n),}$
2. $g_n(\alpha, \beta) = \rho \iff (f_1(n), \alpha, \beta, \rho) \in C^{\omega^{\omega} \times 2^\omega \times \mathbb{R}}$
3. $S_n(\alpha, \beta) = \rho \iff (f_2(n), \alpha, \beta, \rho) \in C^{\omega^{\omega} \times 2^\omega \times \mathbb{R}}$
4. $\begin{cases} (i) \ j \in I_{n, \alpha} \iff \exists \delta \in \omega^{\omega} \ (f_3(n), \alpha, \delta) \in C^{\omega^{\omega} \times \omega^{\omega}} \land \text{Seq}(\delta(j)) \\ (ii) \ s_j^n = \delta(j) \text{ if } j \in I_{n, \alpha}. \end{cases}$

By construction of $r$, these objects satisfy the conditions (1)-(6) of the proof of Lemma 2.7.
• Consequently, the martingale $F(\alpha)$ will be defined in such a way that if $u \in \omega$ codes $s \in 2^{<\omega}$, then $F(\alpha)(s) = \int_{N^u} f_\infty(\alpha,.) \, d\lambda$. Note that $G = \bigcap_{n \in \omega} G_n^*$, so that $\neg G$ is the disjoint union of the $G_n^* \setminus G_n^{*+1}$'s. Thus

$$
\int_{N^u} f_\infty(\alpha,.) \, d\lambda = \int_{N^u \cap G_n^*} f_\infty(\alpha,.) \, d\lambda = \sum_{n \in \omega} \int_{N^u \cap (G_n^* \setminus (G_{n+1})_{\omega})} f_\infty(\alpha,.) \, d\lambda
= \lim_{n \to \infty} \sum_{j \leq n} (-1)^j \int_{N^u \cap (G_n^* \setminus (G_{n+1})_{\omega})} g_j(\alpha,.) \, d\lambda
= \lim_{n \to \infty} \sum_{j \leq n} (-1)^j \int_{N^u \cap (G_n^* \setminus (G_{n+1})_{\omega})} g_j(\alpha,.) \, d\lambda.
$$

Consequently, in order to prove that $F$ is $\Delta^1_1$-recursive, it is enough to check that the partial map $(u, \alpha, j, n) \mapsto \int_{N^u \cap (G_n^* \setminus (G_{n+1})_{\omega})} g_j(\alpha,.) \, d\lambda$ is $\Delta^1_1$-recursive from $\{u \in \omega \mid \text{Seq}(u)\} \times \omega \times \omega^2$ into $\mathbb{R}$. By Corollary 3.5, it is enough to check that the map $h: \omega \times 2^{\omega} \to \mathbb{R}$ defined by

$$
h(\alpha, \beta) := \begin{cases} 
g(\alpha(0))_\omega(\alpha^*, \beta) \text{ if } \text{Seq}(\alpha(0)) \land \text{lh}(\alpha(0)) = 2 \land (\alpha^*, \beta) \in G^*_{(\alpha(0))_1} \setminus G^*_{(\alpha(0))_1} + 1 \\
0 \text{ otherwise,}
\end{cases}
$$

is $\Delta^1_1$-recursive. This comes from the facts that

$$(\alpha, \beta) \in G_n^* \iff (f_0(n), \alpha, \beta) \in \mathcal{C} \iff \neg (f_0(n) \in W \land (f_0(n), \alpha, \beta) \notin \mathcal{C})$$

is $\Delta^1_1$ in $(\alpha, \beta, n)$ and

$$g_n(\alpha, \beta) \in N(\mathbb{R}, p) \iff \exists \rho \in \mathbb{R} \neg (f_1(n) \in W_{\omega \times 2^{\omega} \times \mathbb{R}} \land (f_1(n), \alpha, \beta, \rho) \notin \mathcal{C}_{\omega \times 2^{\omega} \times \mathbb{R}}) \land \\
\forall \rho \in \mathbb{R} \left( f_1(n) \in W_{\omega \times 2^{\omega} \times \mathbb{R}} \land (f_1(n), \alpha, \beta, \rho) \notin \mathcal{C}_{\omega \times 2^{\omega} \times \mathbb{R}} \right) \vee \\
\rho \in N(\mathbb{R}, p)$$

is $\Delta^1_1$ in $(\alpha, \beta, n, p)$. 

• Finally, the map $F$ is $\Delta^1_1$-recursive and is as required. 

\[ \square \]

4 First consequences

(A) Universal sets

- We first recall some material from [K2]. The first result can be found in Section 23.F (see also [Za]).

Theorem 4.1 (Zahorski) Let $B$ be a subset of $[0, 1]$. The following are equivalent:

(a) there are $S \in \Sigma^0_2$ and $P \in \Pi^0_3$ with $m(P) = 1$, where $m$ is the Lebesgue measure on $[0, 1]$, such that $B = S \cap P$,

(b) there is $f \in C([0, 1])$ with $B = \{x \in [0, 1] \mid f'(x) \text{ exists} \}$ (we consider only one-sided derivatives at the endpoints).

The second result is 23.23.
Theorem 4.2  Let $\mathcal{G}$ be a $G_\delta$ subset of $(0,1)$ with $m(\mathcal{G}) = 0$. Then
\[
\{(f,x) \in C([0,1]) \times \mathcal{G} \mid f'(x) \text{ exists}\}
\]
is $C([0,1])$-universal for $\Pi^0_3(\mathcal{G})$.

We prove results in that spirit here.

Theorem 4.3  Let $B$ be a subset of $2^\omega$. Then the following are equivalent:
(a) $B$ is $\Sigma^0_3$ and has $\lambda$-measure zero,
(b) there is $f \in M$ with $B = \{\beta \in 2^\omega \mid \operatorname{osc}(f,\beta) > 0\}$.

Proof. (a) $\Rightarrow$ (b) Write $B = \bigcup_{n \in \omega} G_n$, where the $G_n$’s are $G_\delta$. Lemma 2.7 gives, for each $n$, a martingale $f_n$ with $G_n = D(f_n)$ and $\{\operatorname{osc}(f_n,\beta) \mid \beta \in 2^\omega\} \subseteq \{0\} \cup [\frac{1}{2},1]$. Lemma 2.8 gives $f \in M$ with $D(f) = B$.

(b) $\Rightarrow$ (a) We already noticed in the introduction that $B$ is $\Sigma^0_3$. By Doob’s theorem, $B$ has $\lambda$-measure zero (see [D]). \qed

Corollary 4.4  Let $\mathcal{G}$ be a $G_\delta$ subset of $2^\omega$ with $\lambda(\mathcal{G}) = 0$. Then $\{(f,\beta) \in M \times \mathcal{G} \mid \operatorname{osc}(f,\beta) > 0\}$ is $M$-universal for $\Sigma^0_3(\mathcal{G})$.

For example, $\{\beta \in 2^\omega \mid \forall n \in \omega \ \beta(2n) = 0\}$ is a $\Pi^0_1$ copy of $2^\omega$ and has $\lambda$-measure zero.

(B) Complete sets

- By 33.G in [K2], there is a uniform version of Zahorski’s theorem, which allows to prove the following result

Theorem 4.5  (Mazurkiewicz) The set of differentiable functions in $C([0,1])$ is $\Pi^1_1$-complete.

- Here again, there is a result in that spirit.

Theorem 4.6  The set $\mathcal{P} := \{f \in M \mid \forall \beta \in 2^\omega \ \operatorname{osc}(f,\beta) = 0\}$ is $\Pi^1_1$-complete.

Notation. Let $\mathcal{K} := \{\beta \in 2^\omega \mid \forall n \in \omega \ \beta(2n) = 0\}$, which is a $\Pi^0_1$ copy of the Cantor space $2^\omega$ with $\lambda(\mathcal{K}) = 0$. In particular, $\mathcal{K}$ is a nonempty $G_\delta \cap \Delta^1_1$ subset of $2^\omega$.

Proof. Let $U \in \Pi^1_1(\omega^\omega \times 2^\omega)$ be $\omega^\omega$-universal for the co-analytic subsets of $2^\omega$, and
\[
\Pi := \{\alpha \in \omega^\omega \mid ((\alpha)_0, (\alpha)_1) \in U\}.
\]
Note that $\Pi \in \Pi^1_1$. If $P \in \Pi^1_1(2^\omega)$, then $P = U_\alpha$ for some $\alpha \in \omega^\omega$, so that the map $\beta \mapsto \langle \alpha, \beta \rangle$ is a continuous reduction of $P$ to $\Pi$ and $\Pi$ is $\Pi^1_1$-complete. Let $H \in \Pi^0_1(\omega^\omega \times 2^\omega)$ with $\neg \Pi = \Pi_0[H]$. We set $G := \{\langle \alpha, \beta \rangle \in \omega^\omega \times 2^\omega \mid ((\alpha)_0, (\alpha)_1) \in H \ \land \ \beta \in \mathcal{K}\} \subseteq \omega^\omega \times \mathcal{K}$. Lemma 3.10 gives $F : \omega^\omega \to M$ Borel such that $G_\alpha = \mathcal{V}_{F(\alpha)}$ for each $\alpha \in \omega^\omega$.
Thus
\[ \alpha \notin \Pi \Leftrightarrow \exists \beta \in 2^\omega \quad (\alpha, \beta) \in H \Leftrightarrow \exists \beta \in 2^\omega \quad (\alpha, \beta) \in G \Leftrightarrow \exists \beta \in 2^\omega \quad (F(\alpha), \beta) \in V \Leftrightarrow F(\alpha) \notin \mathcal{P}. \]

Thus \( \Pi = F^{-1}(\mathcal{P}) \) and \( \mathcal{P} \) is Borel \( \Pi \)\(^1\)-complete. By 26.C in [K2], \( \mathcal{P} \) is \( \Pi \)\(^1\)-complete. \( \square \)

- We now prove Theorem 1.8. Let \( X \) be a metrizable compact space and \( Y \) be a Polish space. We equip \( \mathcal{C}(X, Y) \) with the topology of uniform convergence, so that it is a Polish space (see 4.19 in [K2]). We use the map \( \psi \) defined before Theorem 1.8.

**Theorem 4.7**

(a) The set \( \mathcal{P}_1 := \{(f_k)_{k \in \omega} \in \mathcal{P}^\omega \mid (\psi(f_k))_{k \in \omega} \text{ pointwise converges} \} \) is \( \Pi \)\(^1\)-complete.

(b) The set \( \mathcal{P}_2 := \{(f_k)_{k \in \omega} \in \mathcal{P}^\omega \mid (\psi(f_k))_{k \in \omega} \text{ pointwise converges to zero} \} \) is \( \Pi \)\(^1\)-complete.

(c) The set \( \mathcal{S} := \{(f_k)_{k \in \omega} \in \mathcal{P}^\omega \mid \exists \gamma \in \omega^\omega \quad (\psi(f_{\gamma(i)}))_{i \in \omega} \text{ pointwise converges to zero} \} \) is \( \Sigma \)\(^2\)-complete.

**Proof.** We define \( \varphi : \mathcal{C}(2^\omega, [0, 1]) \to \mathcal{M} \) by \( \varphi(h)(s) := \int_{N_s} h \, d\lambda \). As in the proof of Lemma 2.7, \( \varphi \) is well-defined. It is also continuous, and injective: if \( h \neq h' \), then we can find \( q \in \omega \) and \( s \in 2^\omega \) such that \( h(\beta) - h'(\beta) > 2^{-q} \) for each \( \beta \in N_s \) or \( h'(\beta) - h(\beta) > 2^{-q} \) for each \( \beta \in N_s \), so that
\[
|\varphi(h)(s) - \varphi(h')(s)| = \frac{1}{\lambda(N_s)} |\int_{N_s} h \, d\lambda - \int_{N_s} h' \, d\lambda| \geq 2^{-q}.
\]
This implies that the range \( \mathcal{R} \) of \( \varphi \) is Borel and \( \psi := \varphi^{-1} : \mathcal{R} \to \mathcal{C}(2^\omega, [0, 1]) \) is Borel. As every continuous map \( h : 2^\omega \to [0, 1] \) is \( \tau \)-continuous,
\[
\lim_{l \to \infty} \varphi(h)(\beta)(l) = \lim_{l \to \infty} \int_{N_s \cap f^{-1}(l)} h \, d\lambda = h(\beta)
\]
for each \( \beta \in 2^\omega \), by Lemma 2.6. This implies that \( f \in \mathcal{P} \) and \( \psi(f)(\beta) = \lim_{l \to \infty} f(\beta)(l) \) for each \( \beta \in 2^\omega \) if \( f \in \mathcal{R} \).

(a) Note that the proof of 33.11 in [K2] shows that the set
\[
P_1 := \{(h_k)_{k \in \omega} \in (\mathcal{C}(2^\omega, [0, 1]))^\omega \mid (h_k)_{k \in \omega} \text{ pointwise converges} \}
\]
is \( \Pi \)\(^1\)-complete. As \( \mathcal{E} := \{(f_k)_{k \in \omega} \in \mathcal{R}^\omega \mid (\psi(f_k))_{k \in \omega} \text{ pointwise converges} \} = (\psi^{-1})^{-1}(P_1) \), the equalities \( P_1 = (\psi^{-1})(E) = (\psi^{-1})(P_1) \) hold and \( P_1 \) is \( \Pi \)\(^1\)-complete.

(b) We argue as in (a).

(c) As in [B-Ka-L], the set
\[
S := \{(h_k)_{k \in \omega} \in (\mathcal{C}(2^\omega, [0, 1]))^\omega \mid \exists \gamma \in \omega^\omega \quad (h_{\gamma(i)})_{i \in \omega} \text{ pointwise converges to zero} \},
\]
is \( \Sigma \)\(^2\)-complete. Indeed, fix \( Q \in \Sigma \)\(^2\)(2^\omega).
Lemma 2.2 in [B-Ka-L] gives \((g_k)_{k \in \omega} \in \left(\mathcal{C}(2^\omega \times 2^\omega, 2)\right)^\omega\) such that, for each \(\delta \in 2^\omega\), the following are equivalent:

(i) \(\delta \in Q\),

(ii) \(\exists \gamma \in \omega^\omega \ \forall \beta \in 2^\omega \ \lim_{i \to \infty} g_{\gamma(i)}(\delta, \beta) = 0\).

We define, \(g: 2^\omega \to (\mathcal{C}(2^\omega, [0, 1]))^\omega\) by \(g(\delta)(k)(\beta) := g_k(\delta, \beta)\). Then \(g\) is continuous and reduces \(Q\) to \(S\). As

\[
E' := \{ (f_k)_{k \in \omega} \in R^\omega \mid \exists \gamma \in \omega^\omega \ \left( \psi(f_{\gamma(i)}) \right)_{i \in \omega} \text{ pointwise converges to zero} \} = (\varphi^\omega)^{-1}(S),
\]

\(S = (\varphi^\omega)^{-1}(E') = (\varphi^\omega)^{-1}(S)\) and \(S\) is \(\Sigma^0_3\)-complete. \(\square\)

5 Universal and complete sets in the spaces \(C(2^\omega, X)\)

- It is known that if \(\Gamma\) is a self-dual Wadge class and \(X\) is a Polish space, then there is no set which is \(X\)-universal for the subsets of \(X\) in \(\Gamma\) (see 22.7 in [K2]). This is no longer the case if the space of codes is different from the space of coded sets.

**Proposition 5.1** Let \(X\) be a Polish space, \(\Gamma\) be a Wadge class with complete set \(C \in \Gamma(X)\), and \(U^\Gamma := \{(h, \beta) \in \mathcal{C}(2^\omega, X) \times 2^\omega \mid h(\beta) \in C\}\). Then \(U^\Gamma\) is \(C(2^\omega, X)\)-universal for the \(\Gamma\) subsets of \(2^\omega\).

**Proof.** As the evaluation map \((h, \beta) \mapsto h(\beta)\) is continuous, \(U^\Gamma \in \Gamma\). If \(A \in \Gamma(2^\omega)\), then \(A = h^{-1}(C)\) for some \(h \in C(2^\omega, X)\), so that \(A = U^\Gamma_h\). \(\square\)

We will partially strengthen this result to get our uniform universal sets.

- Recall that it is proved in [K3] that a Borel \(\Pi^1_1\)-complete set is actually \(\Pi^1_1\)-complete. In fact, Kechris’s proof shows the result for the classes \(\Pi^1_n\). Our main tool is a uniform version of this. Kechris’s result has recently been strengthened in [P] as follows.

**Theorem 5.2** (Pawlikowski) Let \(n \geq 1\) be a natural number, and \(C \subseteq X \subseteq 2^\omega\). If Borel functions from \(2^\omega\) into \(X\) give as preimages of \(C\) all \(\Pi^1_n\) subsets of \(2^\omega\), then so do continuous injections.

The main tool mentioned above is the following:

**Theorem 5.3** Let \(n \geq 1\) be a natural number; \(U^\Pi_n: 2^\omega \to \Pi^1_n\) be a suitable \(\omega^\omega\)-universal set for the \(\Pi^1_n\) subsets of \(2^\omega\), \(X\) be a recursively presented Polish space, \(C \in \Pi^1_n(X)\), \(\mathcal{R} : \omega^\omega \times \omega^\omega \to \omega^\omega\) be a recursive map, and \(b : \omega^\omega \to X\) be a \(\Delta^1_1\)-recursive map such that

\[
(\alpha, \beta) \in U^\Pi_n \iff b(\mathcal{R}(\alpha, \beta)) \in C
\]

for each \((\alpha, \beta) \in \omega^\omega \times \omega^\omega\). Then there is a \(\Delta^1_1\)-recursive map \(f : \omega^\omega \to C(2^\omega, X)\) such that

\[
(\alpha, \beta) \in U^\Pi_n \iff f(\alpha)(\beta) \in C
\]

for each \((\alpha, \beta) \in \omega^\omega \times \omega^\omega\).
- We first recall some material from [K3].

**Definition 5.4** (a) A coding system for nonempty perfect binary trees is a pair \((D, O)\), where \(D \subseteq 2^\omega\) and \(O : D \rightarrow \{T \in 2^{2^\omega} \mid T \text{ is a nonempty perfect binary tree}\}\) is onto.

(b) A coding system \((D, O)\) is nice if

(i) for any \(\alpha \in \omega^\omega\) and any \(\Delta_1^1(\alpha)\)-recursive map \(H : 2^\omega \times 2^\omega \rightarrow \omega\), we can find \(\beta \in D \cap \Delta_1^1(\alpha)\) and \(k \in \omega\) such that \(H(\beta, \delta) = k\) for each \(\delta\) in the body \([O(\beta)]\) of \(O(\beta)\).

(ii) \(D\) is \(\Pi_1^1\) and, for \(\beta \in D\), the relation

\[
R(m, \beta) \iff \text{Seq}(m) \land \left( (m)_0, \ldots, (m)_{lh(m)-1} \right) \in O(\beta)
\]

is \(\Delta_1^1\), i.e., there are \(\Pi_1\) relations \(\Pi_0, \Pi_1\) such that \(R(m, \beta) \iff \Pi_0(m, \beta) \iff -\Pi_1(m, \beta)\) if \(\beta \in D\).

Nice coding systems exist. If \(\beta \in D\), then there is a canonical homeomorphism \(\beta^*\) from \([O(\beta)]\) onto \(2^\omega\). We now check that the construction of \(\beta^*\) is effective.

**Lemma 5.5** (a) The partial function \(\epsilon : (\beta, \delta) \mapsto \beta^*(\delta)\) is \(\Pi_1^1\)-recursive on its \(\Pi_1\) domain

\[
\text{Domain}(\epsilon) := \{(\beta, \delta) \in D \times 2^\omega \mid \delta \in [O(\beta)]\}.
\]

(b) The partial function \(\nu : (\beta, \gamma) \mapsto \text{the unique} \ \delta \in [O(\beta)] \text{ with} \ \beta^*(\delta) = \gamma \text{ is} \ \Pi_1^1\)-recursive on its \(\Pi_1\) domain \(D \times 2^\omega\).

**Proof.** (a) We define a \(\Pi_1\) relation \(Q\) on \(\omega^2 \times (2^\omega)^2\) by

\[
Q(p, p', \beta, \delta) \iff \left( (\forall \varepsilon \in 2 \ \Pi_0((\delta)p\varepsilon, \beta)) \land (\forall p \leq p'' < p' \ \exists \varepsilon \in 2 \ \Pi_1((\delta)p''\varepsilon, \beta)) \right).
\]

Note that

\[
\beta^*(\delta)(n) = \varepsilon \iff \exists l \in \omega \ \text{Seq}(l) \land lh(l) = n + 1 \land \delta(l)_n = \varepsilon \land Q(0, (l)_0, \beta, \delta) \land (\forall m < n \ (l)_m < (l)_{m+1} \land Q((l)_m+1, (l)_{m+1}, \beta, \delta))
\]

if \(\beta \in D\). The proof of (b) is similar. \(\square\)

- Let \(X\) be a recursively presented Polish space, and \(d_X\) and \((r^n_X)_{n \in \omega}\) be respectively a distance function and a recursive presentation of \(X\). We now give a recursive presentation of \(C(2^\omega, X)\), equipped with the usual distance defined by

\[
d(h, h') := \sup_{\beta \in 2^\omega} d_X(h(\beta), h'(\beta)),
\]

since this is not present in [M]. We define, by primitive recursion, a recursive map \(\nu : \omega \rightarrow \omega\) such that \(\nu(i)\) enumerates \(\{s \in 2^{<\omega} \mid |s| = i\}\). We first set \(\nu(0) := 1 = \langle \rangle\). Then

\[
\nu(i+1) = k \iff \text{Seq}(k) \land lh(k) = 2^{i+1} \land (\forall l < 2^i \ \forall \varepsilon \in 2 \ (k)_l+1 = s(\nu(i))_l \varepsilon).
\]

If \(\text{Seq}(n)\) and \(lh(n) = 2^i\) for some \(i < n\), then we define \(h_n : 2^\omega \rightarrow X\) by \(h_n(\beta) := r^n_X(\beta)\) if

\[
|\beta|_i = s_i = s(\nu(i))_i.
\]

If \(\neg\text{Seq}(n)\) or \(lh(n) \neq 2^i\) for each \(i\), then we define \(h_n : 2^\omega \rightarrow X\) by \(h_n(\beta) := r^n_0\) if \(\beta \in 2^\omega\). In any case, \(h_n \in C(2^\omega, X)\) and takes finitely many values.
Lemma 5.6 Let $X$ be a recursively presented Polish space. Then the sequence $(h_n)_{n \in \omega}$ is a recursive presentation of $C(2^\omega, X)$, equipped with $d$.

Proof. We have to see that $(h_n)$ is dense in $C(2^\omega, X)$. So let $h \in C(2^\omega, X)$, $\epsilon > 0$ and $m \in \omega$ with $2^{-m} < \frac{\epsilon}{2}$. As $h$ is uniformly continuous, there is $i \in \omega$ such that $d_X(h(\beta), h(\delta)) < 2^{-m}$ if $\beta|_i = \delta|_i$. We choose, for each $l < 2^i$, $n_l \in \omega$ such that $d_X(r_l^X, h(s_l^00^\omega)) < 2^{-m}$. We set $n := < n_0, ..., n_{2^i-1}>$. If $\beta \in 2^\omega$ and $\beta|_i = s_l^i$, then $d_X(h(\beta), h_n(\beta)) \leq d_X(h(\beta), h(s_l^i0^\omega)) + d_X(h(s_l^i0^\omega), r_l^X) \leq 2^{-m+2^-m}$, so that $d(h, h_n) < \epsilon$. It is routine to check that the relations “$d(h_m, h_n) \leq \frac{\epsilon}{2^m}$” and “$d(h_m, h_n) < \frac{\epsilon}{2^m}$” are recursive in $(m, n, p, q)$.

We saw in the proof of Proposition 5.1 that the evaluation map $(h, \beta) \mapsto h(\beta)$ is continuous from $C(2^\omega, X) \times 2^\omega$ into $X$. We can say more if $X$ is recursively presented.

Lemma 5.7 Let $X$ be a recursively presented Polish space. Then the evaluation map is recursive.

Proof. Note that

$$h(\beta) \in N(X, n) \iff d_X(h(\beta), r_l^X) < \frac{\epsilon}{(n_1)_{12+1}}$$

$$\iff \exists m, i, l \in \omega \text{ Seq}(m) \land lh(m) = 2^i \land \beta|_i = s_l^i \land (m)i = (n_1)i \land d(h, h_m) < \frac{\epsilon}{(n_1)_{12+1}},$$

which gives the result.

- We then strengthen 7A.3 in [M] about primitive recursion as follows. If $Z, Y$ are recursively presented Polish spaces, $g : Z \to Y$ and $h : Y \times \omega \times Z \to Y$ are $\Pi^1_1$-recursive and $f : \omega \times Z \to Y$ is defined by

$$\begin{cases} f(0, z) := g(z), \\ f(n+1, z) := h(f(n, z), n, z), \end{cases}$$

then $f$ is also $\Pi^1_1$-recursive. If $m : Z \to Z$ is $\Pi^1_1$-recursive, then the proof of 7A.3 in [M] shows that the map $f' : \omega \times Z \to Y$ defined by

$$\begin{cases} f'(0, z) := g(z), \\ f'(n+1, z) := h'(f(n, m(z)), n, z), \end{cases}$$

is also $\Pi^1_1$-recursive. As in 7A.5 in [M], this can be extended to partial functions which are $\Pi^1_1$-recursive on their domain.

- We are ready for the proof of our main tool.

Proof of Theorem 5.3. 3E.6 in [M] provides $\pi : \omega^\omega \to X$ recursive, $F \in \Pi^1_0(\omega^\omega)$ and a $\Delta^1_1$-recursive injection $\rho : X \to \omega^\omega$ such that $\pi F$ is injective, $\rho F = X$ and $\rho$ is the inverse of $\pi F$. Let us show that the map $\mu : h \mapsto \pi \circ h$ is $\Delta^1_1$-recursive from $C(2^\omega, \omega^\omega)$ into $C(2^\omega, X)$. More generally, let $Y$ be a recursively presented Polish space, and $\psi : Y \to C(2^\omega, X)$. Note that

$$\psi(y) \in N(C(2^\omega, X), n) \iff d_X(\psi(y), h((n_1)_0)) < \frac{(n_1)_1}{(n_1)_{12+1}}$$

$$\iff \exists m \in \omega \sup_{\beta \in \omega} d_X(\psi(y)(\beta), h((n_1)_0)(\beta)) < \frac{(n_1)_1}{(n_1)_{12+1}}$$

$$\iff \exists m \in \omega \forall \beta \in 2^\omega d_X(\psi(y)(\beta), h((n_1)_0)(\beta)) < \frac{(n_1)_1}{(n_1)_{12+1}}$$

and $h((n_1)_0)(\beta) = r_l^X g_{l, \beta}$ for some recursive map $g : \omega \times 2^\omega \to \omega$. 27
In the present case, \( Y = C(2^{\omega}, \omega^\omega) \) and \( \psi(y)(\beta) = \pi(y(\beta)) \). Thus

\[
d_X \left( \psi(y)(\beta), h_{(n_1)_0}(\beta) \right) < \frac{(m_1)_1}{(m_1)_2 + 1} \iff d_X \left( \pi(y(\beta)), r^n_X(Y_{\gamma(n,\beta)}) \right) < \frac{(m_1)_1}{(m_1)_2 + 1}
\]

\[
\iff \pi(y(\beta)) \in N_X(\{0, < g(n, \beta), ((m_1)_1, (m_1)_2) >\})
\]

\[
\iff \left( y(\beta), \{0, < g(n, \beta), ((m_1)_1, (m_1)_2) >\} \right) \in G^\pi
\]

where \( G^\pi \) is the \( \Sigma^0_1 \) neighborhood diagram of \( \pi \). As the evaluation map is recursive, \( h = \pi \circ h \) is \( \Pi^1_1 \)-recursive and total, and thus \( \Delta^1_1 \)-recursive.

- Let us show that there is a \( \Delta^1_1 \)-recursive map \( f: \omega^\omega \to C(2^{\omega^\omega}, X) \) such that \( U^X_{\alpha} = (f(\alpha))^{-1}(C) \) for each \( \alpha \in \omega^\omega \). We adapt the proof of the main result in [K3]. We set \( A := \pi^{-1}(C) \). As \( C \in \Pi^1_1(X) \), \( A \in \Pi^1_1(\omega^\omega) \). If \( \beta^0, \delta^0 > \epsilon \in 2^{\omega^\omega} \), then we inductively define, for \( i \in \omega, m_i, \beta^i+1, \delta^i+1 \) as follows. If \((\beta^i, \delta^i)\) is given and in Domain(\(e\)), then \((\beta^i)^*(\delta^i) = \in x_i, \beta^i+1, \delta^i+1 \) and

\[
m_i := \begin{cases} \text{ the location of the first } 0 \text{ in } x_i \text{ if it exists,} \\
2 \text{ otherwise.} 
\end{cases}
\]

We then set \( Q := \{ (\alpha, < \beta^0, \delta^0 >) \in \omega^\omega \times 2^{\omega^\omega} \ | \ \forall i \in \omega, (\beta^i, \delta^i) \in \text{Domain}(e) \land (\alpha, (m_i)) \in U^X_{\alpha} \} \) and \( B^* := Q \alpha \), so that \( Q \in \Pi^1_1(\omega^\omega \times 2^{\omega^\omega}) \) and \( B^* = \{ (\alpha, \beta) \in Q \} \) for each \( (\alpha, \beta) \in \omega^\omega \times 2^{\omega^\omega} \) (note that \( B^* \) depends on \( \alpha \), but we denote it like this to keep the notation of [K3]). We define \( I: \omega^\omega \to 2^{\omega^\omega} \) by \( I(\alpha) := 0^{\alpha(0)}1^{\alpha(1)} \ldots \). Note that \( I \) is a \( \Delta^1_1 \)-recursive injection onto the \( \Pi^1_2 \) set

\[
P^\infty := \{ \beta \in 2^{\omega^\omega} \ | \ \forall p \in \omega, \exists q \geq p, \beta(q) = 1 \},
\]

so that there is a \( \Delta^1_1 \)-recursive map \( \phi: 2^{\omega^\omega} \to \omega^\omega \) which is the inverse of \( I \) on \( P^\infty \). We set

\[
Q' := \{ \delta \in 2^{\omega^\omega} \ | \ (\delta)_0 \in P^\infty \land (\phi((\delta)_0), (\delta)_1) \in Q \},
\]

so that \( Q' \in \Pi^1_1(2^{\omega^\omega}) \). As \( U^X_{\alpha} \) is suitable, there is \( \alpha_Q \in \omega^\omega \) recursive with \( Q' = U^X_{\alpha_Q} \). Note that

\[
\beta \in B^* \iff (\alpha, \beta) \in Q \iff I(\alpha), \beta > Q' \iff (\alpha_Q, (I(\alpha), \beta >) \in U^X_{\alpha_Q} \)
\]

\[
\iff b(R(\alpha_Q, (I(\alpha), \beta >))) \in C \iff \rho(b(R(\alpha_Q, (I(\alpha), \beta >))) \in A.
\]

We set \( G := \rho(b(R(\alpha_Q, (I(\alpha), \beta >))) \), so that \( G: 2^{\omega^\omega} \to \omega^\omega \) is \( \Delta^1_1(\alpha) \)-recursive and \( < \beta^0, \delta^0 > \) is in \( B^* \) if and only if \( G(< \beta^0, \delta^0 >) \in A \).

- As in [K3], we can find \( F: 2^{<\omega^\omega} \to (2^{\omega^\omega} \times \omega)^{<\omega} \) satisfying the following properties:

1. \( t \subseteq t' \Rightarrow F(t) \subseteq F(t') \)
2. \( |F(t)| = |t| + 1 \)
3. (i) if \( F(\emptyset) = (\beta^0, k_0) \), then \( \beta^0 \in D \land \forall \delta^0 \in [\mathcal{O}(\beta^0)] \ G(< \beta^0, \delta^0 >)(0) = k_0 
\)
   (ii) if \( F(\varepsilon_0, \ldots, \varepsilon_n) = (\beta^0, k_0, \beta^1, k_1, \ldots, \beta^{n+1}, k_{n+1}) \), then
   \( a \) \( \forall i \leq n + 1 \ \beta^i \in D \)
   \( b \) for all \( \delta^{n+1} \in [\mathcal{O}(\beta^{n+1})] \), if \( \delta^n, \ldots, \delta^0 \) are the uniquely determined members of \( [\mathcal{O}(\beta^n)], \ldots, [\mathcal{O}(\beta^0)] \) such that \( \forall i \leq n \ (\beta^i)^*(\delta^i) = \in \varepsilon_i, \beta^{i+1}, \delta^{i+1} \), where \( \varepsilon_i = 1^{\varepsilon_i}\alpha_i \), then \( \forall i \leq n + 1 \ G(< \beta^0, \delta^0 >)(i) = k_i \).

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We will need an effective version of this, so that we give the details of the construction of $F$. In fact, the $\beta^i$'s involved in the definition of $F$ can be $\Delta^1_1(\alpha)$. In order to see this, we first define

$$H_0: 2^\omega \times 2^\omega \to \omega$$

by $H_0(\beta, \delta) := G(\beta, \delta)(0)$. As $G$ is $\Delta^1_1(\alpha)$-recursive, $H_0$ too, and the niceness of the coding system gives $\beta^0 \in D \cap \Delta^1_1(\alpha)$ and $k_0 \in \omega$ such that $G(\beta^0, \delta^0)(0) = k_0$ for each $\delta^0 \in [\mathcal{O}(\beta^0)]$. Now suppose that $n \in \omega$, $(\varepsilon_0, \ldots, \varepsilon_n)$ and $F(\varepsilon_0, \ldots, \varepsilon_{n-1}) = (\beta^0, k_0, \ldots, \beta^n, k_n)$ are given. We define

$$H_{n+1}: 2^\omega \times 2^\omega \to \omega$$

as follows. Given $(\beta, \delta) \in 2^\omega \times 2^\omega$, let $\delta^n, \ldots, \delta^0$ be the uniquely determined members of $[\mathcal{O}(\beta^n)], \ldots, [\mathcal{O}(\beta^0)]$ resp., such that $(\beta^n)^*(\delta^n) = \langle \varepsilon_n, \beta, \delta \rangle$, and $(\beta^i)^*(\delta^i) = \langle \varepsilon_i, \beta^{i+1}, \delta^{i+1} \rangle$ if $i < n$. Put $H_{n+1}(\beta, \delta) := G(\beta^0, \delta^0)(n+1)$. As $H_{n+1}$ is $\Delta^1_1(\alpha)$ (it is total and $\Pi^1_1(\alpha)$-recursive since $\iota$ is $\Pi^1_1$-recursive), the niceness of the coding system gives $\beta^{n+1} \in D \cap \Delta^1_1(\alpha)$ and $k_{n+1} \in \omega$ such that $G(\beta^0, \delta^0)(n+1) = k_{n+1}$ for each $\delta^{n+1} \in [\mathcal{O}(\beta^{n+1})]$. Then

$$F(\varepsilon_0, \ldots, \varepsilon_n) := (\beta^0, k_0, \ldots, \beta^{n+1}, k_{n+1}),$$

so that $F$ is as desired. So we can assume that the $\beta^i$'s are $\Delta^1_1(\alpha)$ in the conditions required for $F$.

- By [K3] again, the map $h_\alpha : (\varepsilon_i) \mapsto (k_i)$ is continuous and $U^{\Pi^1_1, 2^\omega}_\alpha = h^{-1}_\alpha(A)$. As this is not too long to prove, we give the details for completeness. The map $h_\alpha$ is in fact more than continuous: it is Lipschitz, by definition. Fix $(\varepsilon_i)$. We apply $F$ to the initial segments of $(\varepsilon_i)$, which gives $(\beta^i)$. For each $n$, we define perfect sets $C^n_0, C^1_n, \ldots, C^n_n \subseteq 2^\omega$ with $C^n_i \subseteq [\mathcal{O}(\beta^i)]$ if $i \leq n$, as follows:

$$C^n_i := \{ \delta^n \in [\mathcal{O}(\beta^n)] \mid \exists \delta^{n+1} \in 2^\omega : (\beta^n)^*(\delta^n) = \langle \varepsilon_n, \beta^{n+1}, \delta^{n+1} \rangle \},$$

$$C^n_{i-1} := \{ \delta^{n-1} \in [\mathcal{O}(\beta^{n-1})] \mid \exists \delta^n \in C^n_i : (\beta^{n-1})^*(\delta^{n-1}) = \langle \varepsilon_{n-1}, \beta^n, \delta^n \rangle \},$$

$$\ldots$$

$$C^n_0 := \{ \delta^0 \in [\mathcal{O}(\beta^0)] \mid \exists \delta^1 \in C^1_1 : (\beta^0)^*(\delta^0) = \langle \varepsilon_0, \beta^1, \delta^1 \rangle \}.$$ 

Note that

(4) $\delta^0 \in C^n_0 \Rightarrow \beta^i, \delta^i \in \text{Domain}(\varepsilon)$ for each $i \leq n$, where $\delta^1, \ldots, \delta^n$ are computed according to the formula in (3).(ii).(b),

(5) $n' \geq n \Rightarrow \forall i \leq n : C^n_i \subseteq C^{n'}_i$.

This implies that $[\mathcal{O}(\beta^0)] \supseteq C^0_0 \supseteq C^1_0 \supseteq \ldots$ and $\cap_{n \in \omega} C^n_0$ contains some $\delta^0$. Note that $\beta^i, \delta^i$ is in $\text{Domain}(\varepsilon)$, and $(\beta^i)^*(\delta^i) = \langle \varepsilon_i, \beta^{i+1}, \delta^{i+1} \rangle$ for each $i \in \omega$. By (3).(ii).(b),

$$G(\beta^0, \delta^0) = k_i$$

for each $i \in \omega$. As $\beta^0, \delta^0 \in B^* \Leftrightarrow G(\beta^0, \delta^0) \in A$,

$$\forall i \in \omega : \beta^i, \delta^i \in \text{Domain}(\varepsilon) \land (\varepsilon_i) \in U^{\Pi^1_1, 2^\omega}_\alpha \Leftrightarrow (k_i) \in A.$$ 

As $\beta^i, \delta^i$ is in $\text{Domain}(\varepsilon)$ for each $i \in \omega$, $(\varepsilon_i) \in U^{\Pi^1_1, 2^\omega}_\alpha \Leftrightarrow h_\alpha((\varepsilon_i)) = (k_i) \in A$. 

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• So we found, for each $\alpha \in \omega^\omega$, $h_\alpha \in C(2^\omega, \omega^\omega)$ such that $U_{\Pi^0_1} \circ h_\alpha = (\pi \circ h_\alpha)^{-1} \circ (C) = (\mu(h_\alpha))^{-1} \circ (C)$. It remains to see that the map $\psi : \alpha \mapsto h_\alpha$, from $\omega^\omega$ into $\omega(2^\omega, \omega^\omega)$, can be $\Delta^1_1$-recursive (then $f$ will be $\mu \circ \psi$). By the previous discussion, it is enough to see that the relation “$k_i = k$” is $\Delta^1_1$ in $\alpha, (\varepsilon_i), i, k) \in \omega^\omega \times 2^\omega \times \omega^2$.

• We will define, by primitive recursion, a $\Delta^1_1$-recursive map $\tilde{f} : \omega \times \omega^\omega \times 2^\omega \to \omega^\omega \times \omega$ such that $\tilde{f}(n, \alpha, (\varepsilon_i))$ will be of the form $(< \beta^0, ..., \beta^n, \beta^n, ..., < k^0, ..., k^n >)$ and can play the role of $F(\varepsilon_0, ..., \varepsilon_{n-1})$. We first set

$$P := \left\{ (\alpha, (\varepsilon_i), \beta, k) \in \omega^\omega \times (2^\omega)^2 \times \omega \mid \nu i \in \omega \ (\beta)_i = (\beta)_0 \in D \cap \Delta^1_1(\alpha) \land \forall \delta \in [O((\beta)_0)] \ G(< (\beta)_0, \delta >)(0) = k \right\}.$$  

Note that $P$ is $\Pi^1_1$ and for any $(\alpha, (\varepsilon_i)) \in \omega^\omega \times 2^\omega$ there is $(\beta, k) \in \omega^\omega \times \omega$ such that $(\alpha, (\varepsilon_i), \beta, k) \in P$. The uniformization lemma gives a $\Delta^1_1$-recursive map $\tilde{g} : \omega^\omega \times 2^\omega \to \omega^\omega \times \omega$ such that

$$\left(\alpha, (\varepsilon_i), \tilde{g}(\alpha, (\varepsilon_i))\right) \in P$$

for each $(\alpha, (\varepsilon_i)) \in \omega^\omega \times 2^\omega$. Then we set

$$D := \left\{ (\beta, p, n, \alpha, (\varepsilon_i)) \in 2^\omega \times 2^\omega \times \omega^\omega \times 2^\omega \mid \text{Seq}(p) \land \text{lh}(p) = n + 1 \land \forall q \in \omega \ (\beta)_q \in D \cap \Delta^1_1(\alpha) \right\}.$$  

Note that $D$ is $\Pi^1_1$, as well as

\[ R := \left\{ (\beta, p, n, \alpha, (\varepsilon_i), \beta', k') \in D \times 2^\omega \times \omega \mid \forall i > n \ (\beta')_i = (\beta')_{n+1} \in D \cap \Delta^1_1(\alpha) \land \text{Seq}(k') \land \text{lh}(k') = n + 2 \land \forall i \leq n \ (\beta')_i = (\beta)_i \land (k')_i = (p)_i \land \forall \delta \in 2^\omega \ (\exists i \leq n + 1 \ (\delta)_i \notin [O((\beta')_i)] \lor \exists i \leq n \ (\beta')_i \land ((\delta)_i) \land (\delta)_{i+1} > \forall i \leq n + 1 \ G(\nu (\beta')_0, (\delta)_0 > (i) = (k')_i) \right\}. \]

Moreover, for each $(\beta, p, n, \alpha, (\varepsilon_i)) \in D = \Pi^1_1 \times 2^\omega \times \omega^\omega \times 2^\omega \times [R]$ there is $(\beta', k') \in (2^\omega \times \Delta^1_1(\alpha)) \times \omega$ such that $(\beta, p, n, \alpha, (\varepsilon_i), \beta', k') \in R$. The uniformization lemma gives a partial map

$$\tilde{h} : 2^\omega \times 2^\omega \times 2^\omega \times 2^\omega \to 2^\omega \times \omega$$

which is $\Pi^1_1$-recursive on its domain $D$, and such that $\left(\beta, p, n, \alpha, (\varepsilon_i), \tilde{h}(\beta, p, n, \alpha, (\varepsilon_i))\right) \in R$ if $(\beta, p, n, \alpha, (\varepsilon_i)) \in D$. This implies that the partial map $\tilde{f}$ defined by

\[
\begin{align*}
\tilde{f}(0, \alpha, (\varepsilon_i)) := & \tilde{g}(\alpha, (\varepsilon_i)), \\
\tilde{f}(n+1, \alpha, (\varepsilon_i)) := & \tilde{h}(\tilde{f}(n, \alpha, (\varepsilon_i)), n, \alpha, (\varepsilon_i)),
\end{align*}
\]

is $\Pi^1_1$-recursive.
Moreover, an induction shows that \( \left( \tilde{f}(n, \alpha, (\varepsilon_i)), n, \alpha, (\varepsilon_i) \right) \in \mathcal{D} \) for each \( n, \alpha, (\varepsilon_i) \), so that \( \tilde{f} \) is in fact total, and thus \( \Delta_1^1 \)-recursive. More precisely, \( \tilde{f}(n, \alpha, (\varepsilon_i)) \) is of the form
\[
< \beta^0, \ldots, \beta^n, \beta^n, \ldots >/ < k_0, \ldots, k_n >,
\]
where \((\varepsilon_0, \ldots, \varepsilon_{n-1}) \mapsto (\beta^0, k_0, \ldots, \beta^n, k_n) \) satisfies the properties (1)-(3) of \( F \). It remains to note that \( k_i = \tilde{f}(i, \alpha, (\varepsilon_i))(1) \).

- We now prove the consequences of our main tool.

**Definition 5.8** Let \( \Gamma \) be a class of subsets of recursively presented Polish spaces, \( \Gamma \) be the corresponding boldface class, \( X, Y \) be recursively presented Polish spaces, and \( \mathcal{U} \in \Gamma(Y \times X) \). We say that \( \mathcal{U} \) is **effectively uniformly \( Y \)-universal for the \( \Gamma \) subsets of \( X \)** if the following hold:

1. \( \Gamma(X) = \{ \mathcal{U}_y \mid y \in Y \} \),
2. \( \Gamma(X) = \{ \mathcal{U}_y \mid y \in Y \}\) \( \Delta_1^1 \)-recursive,
3. for each \( S \in \Gamma(\omega^\omega \times X) \), there is a Borel map \( b : \omega^\omega \rightarrow Y \) such that \( S_\alpha = \mathcal{U}_0(\alpha) \) for each \( \alpha \in \omega^\omega \),
4. for each \( S \in \Gamma(\omega^\omega \times X) \), there is a \( \Delta_1^1 \)-recursive map \( b : \omega^\omega \rightarrow Y \) such that \( S_\alpha = \mathcal{U}_0(\alpha) \) for each \( \alpha \in \omega^\omega \).

**Notation.** Let \( \mathcal{U}_{\Pi_1}^{\Pi_1} \in \Pi_1^1 \) be a good \( \omega^\omega \)-universal for the \( \Pi_1^1 \) subsets of \( 2^\omega \), \( X_1 \) be a recursively presented Polish space, and \( C_1 \) be a \( \Pi_1^1 \) subset of \( X_1 \) for which there is a \( \Delta_1^1 \)-recursive map \( b : \omega^\omega \rightarrow X_1 \) such that
\[
(\alpha, \beta) \in \mathcal{U}_{\Pi_1}^{\Pi_1} \iff b(< \alpha, \beta >) \in C_1
\]
if \( (\alpha, \beta) \in \omega^\omega \times 2^\omega \). We define, for each natural number \( n \geq 1 \),

- \( X_{n+1} := C(2^\omega, X_n) \) (inductively),
- \( C_{n+1} := \{ h \in X_{n+1} \mid \forall \beta \in 2^\omega \ h(\beta) \notin C_n \} \) (inductively),
- \( \mathcal{U}_n := \{ (h, \beta) \in X_{n+1} \times 2^\omega \mid h(\beta) \in C_n \} \).

**Theorem 5.9** Let \( n \geq 1 \) be a natural number. Then
(a) the set \( \mathcal{U}_n \) is effectively uniformly \( X_{n+1} \)-universal for the \( \Pi_{n+1}^1 \) subsets of \( 2^{\omega^\omega} \),
(b) the set \( C_n \) is \( \Pi_{n+1}^1 \)-complete.

**Proof.** We argue by induction on \( n \).

(a) Assume first that \( n = 1 \), and fix \( S \in \Pi_1^1(\omega^\omega \times 2^\omega) \). Our assumption gives \( b_1 : \omega^\omega \rightarrow X_1 \). As \( \mathcal{U}_{\Pi_1}^{\Pi_1} \in \Pi_1^1 \) is a good \( \omega^\omega \)-universal for the \( \Pi_1^1 \) subsets of \( 2^\omega \), there is by Theorem 5.3 a \( \Delta_1^1 \)-recursive map \( f_1 : \omega^\omega \rightarrow C(2^\omega, X_1) \) such that \( (\alpha, \beta) \in \mathcal{U}_{\Pi_1}^{\Pi_1} \iff f_1(\alpha)(\beta) \in C_1 \) if \( (\alpha, \beta) \in \omega^\omega \times 2^\omega \). Let \( \alpha_S \in \omega^\omega \) with \( S = \mathcal{U}_{\alpha_S}^{\Pi_1^{\omega^\omega \times 2^\omega}} \). Note that
\[
(\alpha, \beta) \in S \iff (\mathcal{R}(\alpha_S, \alpha), \beta) \in \mathcal{U}_{\Pi_1}^{\Pi_1} \iff f_1(\mathcal{R}(\alpha_S, \alpha))(\beta) \in C_1 \iff f_1(\mathcal{R}(\alpha_S, \alpha), \beta) \in \mathcal{U}_1.
\]
As \( C_1 \) is \( \Pi_1^1 \), \( \mathcal{U}_1 \) too. If \( A \in \Pi_1^1(2^\omega) \), then \( A = \mathcal{U}_{\alpha}^{\Pi_1^{\omega^\omega \times 2^\omega}} \) for some \( \alpha \in \omega^\omega \). Applying the previous discussion to \( S := \mathcal{U}_{\Pi_1^{\omega^\omega \times 2^\omega}} \), we get \( A = (\mathcal{U}_1)_{f_1(\mathcal{R}(\alpha_S, \alpha))} \), so that \( \mathcal{U}_1 \) is \( X_2 \)-universal for the \( \Pi_1^1 \) subsets of \( 2^{\omega^\omega} \), effectively and uniformly.
We now study \( U_{n+1} \). Fix \( S \in \Pi^1_{n+1}(\omega^\omega \times 2^\omega) \). Let \( U^{\Pi^1_n,2^\omega} \) be a good \( \omega^\omega \)-universal for the \( \Pi^1_n \) subsets of \( 2^\omega \). We set \( V^{\Pi^1_{n+1},2^\omega} := \{ (\alpha,\beta) \in \omega^\omega \times 2^\omega \mid \forall \delta \in 2^\omega \left( (\mathcal{R}(\alpha,\beta),\delta) \notin U^{\Pi^1_n,2^\omega} \right) \} \), so that \( V^{\Pi^1_{n+1},2^\omega} \) is a suitable \( \omega^\omega \)-universal for the \( \Pi^1_{n+1} \) subsets of \( 2^\omega \). Moreover, the induction assumption gives a \( \Delta^0_1 \)-recursive map \( b_{n+1} : \omega^\omega \rightarrow X_{n+1} \) such that

\[
(\alpha,\beta) \in V^{\Pi^1_{n+1},2^\omega} \iff \forall \delta \in 2^\omega \left( (\mathcal{R}(\alpha,\beta),\delta) \notin U^{\Pi^1_n,2^\omega} \Rightarrow \forall \delta \in 2^\omega \left( b_{n+1}(\mathcal{R}(\alpha,\beta)),\delta \right) \notin U_n \Rightarrow \forall \delta \in 2^\omega \left( b_{n+1}(\mathcal{R}(\alpha,\beta)),\delta \right) \notin \varnothing \right) \in C_{n+1} \Rightarrow (\alpha,\beta) \in C_{n+1},
\]

Theorem 5.3 gives a \( \Delta^0_1 \)-recursive map \( f_{n+1} \) such that \( (\alpha,\beta) \in V^{\Pi^1_{n+1},2^\omega} \iff f_{n+1}(\alpha)(\beta) \in C_{n+1} \) if \( (\alpha,\beta) \in \omega^\omega \times 2^\omega \). Let

\[
Q \in \Pi^1_n(\omega^\omega \times 2^\omega \times 2^\omega) \subseteq \Pi^1_n(\omega^\omega \times 2^\omega \times 2^\omega)
\]
such that \( (\alpha,\beta) \in S \iff \forall \delta \in 2^\omega \left( (\alpha,\beta,\delta) \notin Q \right), \), and \( \alpha Q \in \omega^\omega \) such that \( Q = U^{\Pi^1_n,\omega^\omega \times \omega^\omega \times 2^\omega} \). Note that

\[
(\alpha,\beta) \in S \iff \forall \delta \in 2^\omega \left( (\mathcal{R}(\alpha Q,\alpha),\beta,\delta) \notin U^{\Pi^1_n,2^\omega} \iff (\mathcal{R}(\alpha Q,\alpha),\beta) \in V^{\Pi^1_{n+1},2^\omega} \iff f_{n+1}(\mathcal{R}(\alpha Q,\alpha),\beta) \in C_{n+1} \iff \left( f_{n+1}(\mathcal{R}(\alpha Q,\alpha),\beta) \right) \in U_{n+1}.
\]

As \( C_n \in \Pi^1_n \), \( C_n + 1 \in \Pi^1_{n+1} \) and \( U_{n+1} \in \Pi^1_{n+1} \). If \( A \in \Pi^1_1(2^\omega) \), then \( A = U^{\Pi^1_n,\omega^\omega \times \omega^\omega \times 2^\omega} \) for some \( \alpha \in \omega^\omega \). Applying the previous discussion to \( S := U^{\Pi^1_{n+1},2^\omega} \), we get \( A = (U_{n+1})_{f_{n+1}(\mathcal{R}(\alpha Q,\alpha))} \), so that \( U_{n+1} \) is \( X_{n+1} \)-universal for the analytic subsets of \( 2^\omega \), effectively and uniformly.

(b) By definition, \( C_1 \in \Pi^1_1 \), and \( C_{n+1} \in \Pi^1_1 \) if \( C_n \in \Pi^1_1 \). Assume first that \( E \in \Pi^1_1(2^\omega) \). Then \( E = (U_n)_h \) for some \( h \in C(2^\omega, X_n) \), by (a). Thus \( E = h^{-1}(C_n) \). If \( Z \) is a zero-dimensional Polish space and \( D \in \Pi^1_1(Z) \), then we may assume that \( Z \) is a \( G_\delta \) subset of \( 2^\omega \) by 7.8 in [K2], so that \( D \in \Pi^1_1(2^\omega) \). The previous discussion gives \( g \in C(2^\omega, X_n) \) with \( D = g^{-1}(C_n) \). Thus \( D = (g|Z)^{-1}(C_n) \) and \( C_n \) is \( \Pi^0_1 \)-complete.

**Proof of Theorem 1.7.** By Theorem 5.9, it is enough to show that if \( U^{\Pi^1_1,2^\omega} \in \Pi^1_1 \) is a good \( \omega^\omega \)-universal set for the \( \Pi^1_1 \) subsets of \( 2^\omega \), then there is a \( \Delta^0_1 \)-recursive map \( b : \omega^\omega \rightarrow [0,1]^{[2^\omega]} \) such that \( (\alpha,\beta) \in U^{\Pi^1_1,2^\omega} \iff b(\alpha,\beta) \in \mathcal{P} \) if \( (\alpha,\beta) \in \omega^\omega \times 2^\omega \). Let \( H \in \Pi^1_0(\omega^\omega \times 2^\omega \times 2^\omega) \) such that \( \neg U^{\Pi^1_1,2^\omega} = \Pi^1_0(\omega^\omega \times 2^\omega \times 2^\omega) \). We set \( G := \{ (\alpha,\beta) \in \omega^\omega \times 2^\omega \mid ((\alpha)_0, (\beta)_1) \in H \} \), so that \( G \in \Delta^0_1(\omega^\omega \times 2^\omega) \), has \( G_\delta \) vertical sections and \( G \subseteq \omega^\omega \times \mathcal{K} \). Lemma 3.10 gives a \( \Delta^0_1 \)-recursive map \( F : \omega^\omega \rightarrow [0,1]^{[2^\omega]} \), taking values in \( \mathcal{M} \), and such that \( G_\alpha = V_{b(\alpha)} \) for each \( \alpha \in \omega^\omega \). If \( (\alpha,\beta) \in \omega^\omega \times 2^\omega \), then

\[
(\alpha,\beta) \notin U^{\Pi^1_1,2^\omega} \iff \exists \delta \in 2^\omega \ (\alpha,\beta,\delta) \in H \iff \exists \delta \in 2^\omega \ (\alpha,\beta,\delta) \in G \iff \exists \delta \in 2^\omega \ (b(\alpha,\beta) > \delta) \in \mathcal{V} \iff b(\alpha,\beta) > \delta \not\in \mathcal{P}.
\]

This finishes the proof.

**Questions.** Let \( U \) be a \( \Pi^0_2 \) subset of \( \omega^\omega \times 2^\omega \) which is universal for \( \Pi^0_2(2^\omega) \). We set

\[
G := \{ (\alpha,\beta) \in \omega^\omega \times \mathcal{K} \mid (\alpha, (\beta)_1) \in U \}.
\]

Note that \( G \) is a \( \Pi^0_2 \) subset of \( \omega^\omega \times 2^\omega \) contained in \( \omega^\omega \times \mathcal{K} \) which is universal for \( \Pi^0_2(\mathcal{K}) \). Indeed, fix \( H \in \Pi^0_2(\mathcal{K}) \). Then \( H' := \{ \gamma \in 2^\omega \mid < 0^\infty, \gamma > \in H \} \) is \( \Pi^0_2 \), which gives \( \alpha_0 \in \omega^\omega \) with \( H' = U_{\alpha_0} \). Then \( H = G_{\alpha_0} \).
Let $\alpha \mapsto (\alpha_k)_{k \in \omega}$ be a homeomorphism between $\omega^\omega$ and $((\omega^\omega)^\omega)^\omega$, with inverse map

$$(\alpha_k)_{k \in \omega} \mapsto <\alpha_0, \alpha_1, \ldots>.$$  

We set $S' := \{\alpha \in \omega^\omega \mid \exists \gamma \in \omega^\omega \ \forall i \in \omega \ \forall \beta \in 2^\omega \ \beta \notin G_{(\alpha)_{\gamma(i)}}\}$. Note that $S'$ is $\Sigma^1_2$.

1) Is $S'$ a Borel $\Sigma^1_2$-complete set?

Assume that this is the case. Then the set $S_2 := \{(f_k)_{k \in \omega} \in M^\omega \mid \exists \gamma \in \omega^\omega \ \forall i \in \omega \ \forall \beta \in 2^\omega \ \beta \notin \mathcal{D}(f_{\gamma(i)})\}$ of sequences of martingales having a subsequence made of everywhere converging martingales is Borel $\Sigma^1_2$-complete. Indeed, Lemma 3.10 gives a Borel map $F : \omega^\omega \to M$ such that $G_{(\alpha)_{\gamma(i)}} = V_{F(\alpha)}$ for each $\alpha \in \omega^\omega$. The map $\tilde{F} : \omega^\omega \to M^\omega$ defined by $\tilde{F}(\alpha)(k) := F((\alpha_k)_k)$ is Borel. Moreover,

$$\tilde{F}(\alpha) \in S_2 \iff \exists \gamma \in \omega^\omega \ \forall i \in \omega \ \forall \beta \in 2^\omega \ \beta \notin D(F((\alpha)_{\gamma(i)}))$$

$$\iff \exists \gamma \in \omega^\omega \ \forall i \in \omega \ \forall \beta \in 2^\omega \ \beta \notin V_{F(F((\alpha)_{\gamma(i)}))}$$

$$\iff \exists \gamma \in \omega^\omega \ \forall i \in \omega \ \forall \beta \in 2^\omega \ \beta \notin G_{(\alpha)_{\gamma(i)}}$$

$$\iff \alpha \in S',$$

so that $S' = \tilde{F}^{-1}(S_2)$.

2) Is there a Borel map $f : C(2^\omega, [0, 1]) \to \omega^\omega$ such that, for each $(h_k)_{k \in \omega} \in \left(C(2^\omega, [0, 1])\right)^\omega$ and each $\beta \in 2^\omega$, the following are equivalent:

(a) $\lim_{k \to \infty} h_k(\beta) = 0,$

(b) $\forall k \in \omega \ \beta \notin G_f(h_k)$?

Assume that this is the case. Then $S'$ (and therefore $S_2$) is Borel $\Sigma^1_2$-complete, and thus $\Sigma^1_2$-complete (see [P]). We define $F : \left(C(2^\omega, [0, 1])\right)^\omega \to \omega^\omega$ by $F((h_k)_{k \in \omega}) := <f(h_0), f(h_1), \ldots>$, so that $F$ is Borel. Note that

$$F((h_k)_{k \in \omega}) \in S' \iff \exists \gamma \in \omega^\omega \ \forall i \in \omega \ \forall \beta \in 2^\omega \ \beta \notin G_{f(h_{\gamma(i)})}$$

$$\iff \exists \gamma \in \omega^\omega \ \forall \beta \in 2^\omega \ \lim_{i \to \infty} h_{\gamma(i)}(\beta) = 0$$

$$\iff (h_k)_{k \in \omega} \in S',$$

so that $S = F^{-1}(S')$. 

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6 References

[T2] H. Tanaka, A basis result for the $\Pi_1^1$ sets of positive measure, *Comment. Math. Univ. St. Paul.* 16 (1968), 115-127