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# Universal and complete sets in martingale theory

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**Abstract.** The Doob convergence theorem implies that the set of divergence of any martingale has measure zero. We prove that, conversely, any  $G_{\delta\sigma}$  subset of the Cantor space with Lebesgue-measure zero can be represented as the set of divergence of some martingale. In fact, this is effective and uniform. A consequence of this is that the set of everywhere converging martingales is  $\Pi_1^1$ -complete, in a uniform way. We derive from this some universal and complete sets for the whole projective hierarchy, via a general method. We provide some other complete sets for the classes  $\Pi_1^1$  and  $\Sigma_2^1$  in the theory of martingales.

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# 1 Introduction

The reader should see [K2] for the notation used in this paper.

**Definition 1.1** We say that a map  $f : 2^{<\omega} \rightarrow [0, 1]$  is a **martingale** if  $f(s) = \frac{f(s0)+f(s1)}{2}$  for each  $s \in 2^{<\omega}$ . The set of martingales is denoted by  $\mathcal{M}$  and is a compact subset of  $[0, 1]^{2^{<\omega}}$  (equipped with the usual product topology).

This terminology is not the standard one, but the set  $\mathcal{M}$  can be interpreted as the set of all discrete martingales (in the classical sense) taking values in  $[0, 1]$ , as follows. If  $s \in 2^{<\omega}$ , then

$$N_s := \{\beta \in 2^\omega \mid s \subseteq \beta\}$$

is the usual basic clopen set. Let  $f \in \mathcal{M}$ . If  $n \in \omega$ , then let  $\mathcal{S}_n$  be the  $\sigma$ -algebra on  $2^\omega$  generated by  $\{N_s \mid s \in 2^n\}$ , and  $f_n : 2^\omega \rightarrow [0, 1]$  be defined by  $f_n(\beta) := f(\beta|n)$ . Then the sequence  $(f_n)_{n \in \omega}$  is a discrete martingale taking values in  $[0, 1]$  with respect to the sequence of  $\sigma$ -algebras  $(\mathcal{S}_n)_{n \in \omega}$  and the usual Lebesgue product measure  $\lambda$  on  $2^\omega$ . Conversely, if  $(f_n)_{n \in \omega}$  is any such martingale, it can be viewed as an element of  $\mathcal{M}$  by setting  $f(s) := f_{|s|}(\alpha)$  if  $\alpha \in N_s$ . This definition is correct because  $f_{|s|}$ , as a function measurable with respect to  $\mathcal{S}_{|s|}$ , has a constant value on  $N_s$ .

**Definition 1.2** Let  $f$  be a martingale and  $\beta \in 2^\omega$ . The **oscillation** of  $f$  at  $\beta$  is the number

$$\text{osc}(f, \beta) := \inf_{N \in \omega} \sup_{p, q \geq N} |f(\beta|p) - f(\beta|q)|.$$

The **set of divergence** of  $f$  is  $D(f) := \{\beta \in 2^\omega \mid \text{osc}(f, \beta) > 0\}$ .

By definition, if  $f$  is a martingale, then

$$\beta \in D(f) \Leftrightarrow \exists r \in \omega \ \forall N \in \omega \ \exists p, q \geq N \ |f(\beta|p) - f(\beta|q)| > 2^{-r}.$$

This shows that  $D(f) \in \Sigma_3^0$ . Moreover,  $D(f)$  has  $\lambda$ -measure zero, by Doob's convergence theorem (see Chapter XI, Section 14 in [D]). So it is natural to ask whether any  $\Sigma_3^0$  subset of  $2^\omega$  with  $\lambda$ -measure zero is the set of divergence of some martingale (this question was asked by Louveau). We answer positively:

**Theorem 1.3** Let  $B$  be a subset of  $2^\omega$ . Then the following are equivalent:

- (a)  $B$  is  $\Sigma_3^0$  and has  $\lambda$ -measure zero,
- (b) there is a martingale  $f$  with  $B = D(f)$ .

**Definition 1.4** Let  $\Gamma$  be a class of subsets of Polish spaces,  $X, Y$  be Polish spaces, and  $\mathcal{U} \subseteq Y \times X$ .

- (a) We say that  $\mathcal{U}$  is  **$Y$ -universal for the  $\Gamma$  subsets of  $X$**  if  $\mathcal{U} \in \Gamma(Y \times X)$  and  $\Gamma(X) = \{\mathcal{U}_y \mid y \in Y\}$ .
- (b) We say that  $\mathcal{U}$  is **uniformly  $Y$ -universal for the  $\Gamma$  subsets of  $X$**  if  $\mathcal{U}$  is  $Y$ -universal for the  $\Gamma$  subsets of  $X$  and, for each  $S \in \Gamma(\omega^\omega \times X)$ , there is a Borel map  $b : \omega^\omega \rightarrow Y$  such that  $S_\alpha = \mathcal{U}_{b(\alpha)}$  for each  $\alpha \in \omega^\omega$ .

**Corollary 1.5** Let  $\mathcal{G}$  be a  $G_\delta$  subset of  $2^\omega$  with  $\lambda(\mathcal{G}) = 0$ . Then the set  $\{(f, \beta) \in \mathcal{M} \times \mathcal{G} \mid \beta \in D(f)\}$  is  $\mathcal{M}$ -universal for the  $\Sigma_3^0$  subsets of  $\mathcal{G}$ .

In fact, we prove an effective and uniform version of the implication (a)  $\Rightarrow$  (b) in Theorem 1.3. In particular, we can associate, via a Borel map  $F$ , a martingale to a code  $\alpha$  of an arbitrary  $G_\delta$  subset  $G$  of  $\mathcal{G}$  (as in the previous corollary), in such a way that  $G = D(F(\alpha))$ . A consequence of this is the following:

**Theorem 1.6** *The set  $\mathcal{P}$  of everywhere converging martingales is  $\mathbf{\Pi}_1^1$ -complete.*

These statements are in the spirit of some results concerning the differentiability of functions due to Zahorski and Mazurkiewicz (see Section 4 for details). In fact,  $\mathcal{P}$  is  $\mathbf{\Pi}_1^1$ -complete in a uniform way, which allows to derive some universal and complete sets for the whole projective hierarchy, in spaces of continuous functions, starting from  $\mathcal{P}$ . More precisely, let  $P_1 := [0, 1]^{2^{<\omega}}$  and  $C_1 := \mathcal{P}$ . We define, for each natural number  $n \geq 1$ ,

- the space  $P_{n+1} := \mathcal{C}(2^\omega, P_n)$  of continuous functions from  $2^\omega$  into  $P_n$ , equipped with the topology of uniform convergence (inductively),
- $C_{n+1} := \{h \in P_{n+1} \mid \forall \beta \in 2^\omega \ h(\beta) \notin C_n\}$  (inductively),
- $U_n := \{(h, \beta) \in P_{n+1} \times 2^\omega \mid h(\beta) \in C_n\}$ .

We prove the following:

**Theorem 1.7** *Let  $n \geq 1$  be a natural number. Then*

- (a) *the set  $U_n$  is uniformly  $P_{n+1}$ -universal for the  $\mathbf{\Pi}_n^1$  subsets of  $2^\omega$ ,*
- (b) *the set  $C_n$  is  $\mathbf{\Pi}_n^1$ -complete.*

In fact, our method is more general and works if we start with a  $\mathbf{\Pi}_1^1$  set which is complete in a uniform way.

Let  $f$  be a martingale. As  $D(f)$  has  $\lambda$ -measure zero, we can associate to  $f$  the partial function  $\psi(f)$  defined  $\lambda$ -almost everywhere by  $\psi(f)(\beta) := \lim_{l \rightarrow \infty} f(\beta|l)$ . The partial function  $\psi(f)$  will be called the **associated partial function**. The martingale  $f$  is in  $\mathcal{P}$  if and only if  $\psi(f)$  is total, in which case  $\psi(f)$  is called the **associated function**. Using the work in [B-Ka-L] and [K2] about spaces of continuous functions, we prove the following:

**Theorem 1.8** (a) *The set of sequences of everywhere converging martingales whose associated functions converge pointwise is  $\mathbf{\Pi}_1^1$ -complete.*

(b) *The set of sequences of everywhere converging martingales whose associated functions converge pointwise to zero is  $\mathbf{\Pi}_1^1$ -complete.*

(c) *The set of sequences of everywhere converging martingales having a subsequence whose associated functions converge pointwise to zero is  $\mathbf{\Sigma}_2^1$ -complete.*

## 2 $\Sigma_3^0$ sets of measure zero

**Notation.** In the sequel,  $B$  will be a Borel subset of  $2^\omega$ , and  $M$  will be a  $\lambda$ -measurable subset of  $2^\omega$ . If  $\beta \in 2^\omega$ , then the **density of  $M$  at  $\beta$**  is the number  $d(M, \beta) := \lim_{l \rightarrow \infty} \frac{\lambda(M \cap N_{\beta|l})}{\lambda(N_{\beta|l})}$  when it is defined. Note that  $d(B, \beta) = 1$  if  $\beta \in B$  and  $B$  is open. We first recall the Lebesgue density theorem (see 17.9 in [K2]).

**Theorem 2.1 (Lebesgue)** *The equality  $\lambda(M) = \lambda(\{\beta \in M \mid d(M, \beta) = 1\})$  holds for any  $\lambda$ -measurable subset  $M$  of  $2^\omega$ .*

The reader should see [C] for the next lemma. We include a proof to be self-contained and also because we will prove an effective and uniform version of it later.

**Lemma 2.2 (Lusin-Menchoff)** *Let  $F$  be a closed subset of  $2^\omega$ , and  $M \supseteq F$  be a  $\lambda$ -measurable subset of  $2^\omega$  such that  $d(M, \beta) = 1$  for each  $\beta \in F$ . Then there is a closed subset  $C$  of  $2^\omega$  such that*

- (1)  $F \subseteq C \subseteq M$ ,
- (2)  $d(M, \beta) = 1$  for each  $\beta \in C$ ,
- (3)  $d(C, \beta) = 1$  for each  $\beta \in F$ .

**Proof.** If  $F$  is  $2^\omega$ , then we can take  $C := F$ . So we may assume that  $F$  is not  $2^\omega$ . We set  $s^- := s(|s| - 1)$  if  $\emptyset \neq s \in 2^{<\omega}$ . Note that  $\neg F$  is the disjoint union of the elements of a sequence  $(N_{s_n})_{n \in \omega}$ , where  $N_{s_n}^- \cap F \neq \emptyset$  for each  $n \in \omega$ . Fix  $n \in \omega$ . By Theorem 2.1,

$$\lambda(M \cap N_{s_n}) = \lambda(\{\beta \in M \cap N_{s_n} \mid d(M \cap N_{s_n}, \beta) = 1\}).$$

The regularity of  $\lambda$  gives a closed subset  $F_n$  of  $2^\omega$  contained in  $\{\beta \in M \cap N_{s_n} \mid d(M \cap N_{s_n}, \beta) = 1\}$  such that  $\lambda(F_n) \geq (1 - 2^{-n})\lambda(M \cap N_{s_n})$ . We set  $C := F \cup \bigcup_{n \in \omega} F_n$ , which is closed since  $|s_n| \rightarrow \infty$ .

As Conditions (1) and (2) are clearly satisfied, pick  $\beta \in F$ . Note that

$$\begin{aligned} \lambda(N_{\beta|l} \setminus C) &= \sum_{s_n \supseteq \beta|l} \lambda(N_{s_n} \setminus C) \\ &\leq \sum_{s_n \supseteq \beta|l} \lambda(N_{s_n} \setminus F_n) \\ &\leq \sum_{s_n \supseteq \beta|l} 2^{-n} \lambda(M \cap N_{s_n}) + \sum_{s_n \supseteq \beta|l} \lambda(N_{s_n} \setminus M) \\ &\leq \sum_{s_n \supseteq \beta|l} 2^{-n} \lambda(N_{s_n}) + \lambda(N_{\beta|l} \setminus M). \end{aligned}$$

This implies that the limit of  $\frac{\lambda(N_{\beta|l} \setminus C)}{\lambda(N_{\beta|l})}$  is zero since  $d(M, \beta) = 1$ . □

The next topology is considered in [Lu-Ma-Z], see Chapter 6.

**Definition 2.3** *The  $\tau$ -topology on  $2^\omega$  is generated by*

$$\mathcal{F} := \{M \subseteq 2^\omega \mid M \text{ is } \lambda\text{-measurable} \wedge \forall \beta \in M \ d(M, \beta) = 1\}.$$

The next result is proved in [Lu-Ma-Z], but in a much more abstract way. This is the reason why we include a much more direct proof here, since it is not too long.

**Lemma 2.4** *The family  $\mathcal{F}$  is a topology. In particular, any  $\tau$ -open set is  $\lambda$ -measurable.*

**Proof.** Note first that  $\mathcal{F}$  is closed under finite intersections, so that it is a basis for the  $\tau$ -topology. Indeed, let  $M, M'$  be in  $\mathcal{F}$ , and  $\beta \in M \cap M'$ . Then we use the facts that

$$\lambda(M \cap M' \cap N_{\beta|l}) = \lambda(M \cap N_{\beta|l}) - \lambda((M \cap N_{\beta|l}) \setminus M')$$

and  $\lambda((M \cap N_{\beta|l}) \setminus M') \leq \lambda(N_{\beta|l} \setminus M')$ .

Let  $\mathcal{H}$  be a subfamily of  $\mathcal{F}$ , and  $H := \cup \mathcal{H}$ . We claim that there is a countable subfamily  $\mathcal{C}$  of  $\mathcal{H}$  such that  $m := \sup\{\lambda(\cup \mathcal{D}) \mid \mathcal{D} \subseteq \mathcal{H} \text{ countable}\} = \lambda(\cup \mathcal{C})$ . Indeed, for each  $n \in \omega$  there is  $\mathcal{D}_n \subseteq \mathcal{H}$  countable such that  $\lambda(\cup \mathcal{D}_n) > m - 2^{-n}$ , and  $\mathcal{C} := \cup_{n \in \omega} \mathcal{D}_n$  is suitable. Let  $C := \cup \mathcal{C}$ .

Let  $\beta \in H$ , and  $M$  in  $\mathcal{H}$  with  $\beta \in M$ . Note that  $\lambda(M \cup C) = \lambda(C)$  (consider the family  $\mathcal{C} \cup \{M\}$ ). Thus  $\lambda(M \setminus C) = 0$ . As  $d(M, \beta) = 1$ , the equality  $d(M \cap C, \beta) = 1$  holds, and  $d(\neg C, \beta) = 0$ . This implies that  $H \setminus C$  is contained in  $\{\beta \notin C \mid d(\neg C, \beta) < 1\}$ , which has  $\lambda$ -measure zero by Theorem 2.1. Therefore  $H \setminus C$  has  $\lambda$ -measure zero and  $H = C \cup (H \setminus C)$  is  $\lambda$ -measurable.

Pick  $\beta \in H$ , and  $M \in \mathcal{H}$  with  $\beta \in M$ . Then  $d(M, \beta) = 1$ , and thus  $d(H, \beta) = 1$ . Therefore  $H \in \mathcal{F}$ . This finishes the proof.  $\square$

The next lemma is in the style of Urysohn's theorem (see [Lu] for its version on the real line). We include a proof to be self-contained and also because we will prove an effective and uniform version of it later.

**Lemma 2.5** *Let  $C$  be a closed subset of  $2^\omega$ , and  $G$  be a  $G_\delta$  subset of  $2^\omega$  disjoint from  $C$  such that  $\lambda(G) = 0$ . Then there is a  $\tau$ -continuous map  $h: 2^\omega \rightarrow [0, 1]$  such that  $h|_C \equiv 0$  and  $h|_G \equiv 1$ .*

**Proof.** Let  $(F_n)_{n \in \omega}$  be an increasing sequence of closed subsets of  $2^\omega$  with union  $\neg G$  and  $F_0 = C$ . We first construct a sequence  $(C_{\frac{1}{2^n}})_{n \in \omega}$  of closed subsets of  $2^\omega$  with  $F_n \subseteq C_{\frac{1}{2^n}} \subseteq \neg G$ ,  $C_{\frac{1}{2^n}} \subseteq C_{\frac{1}{2^{n+1}}}$ , and  $d(C_{\frac{1}{2^{n+1}}}, \beta) = 1$  for each  $\beta \in C_{\frac{1}{2^n}}$ . We first apply Lemma 2.2 to  $F := F_0$  and  $M := \neg G$ , which gives  $F_0 \subseteq C_1 \subseteq \neg G$ . Then, inductively, we apply Lemma 2.2 to  $F := C_{\frac{1}{2^n}} \cup F_{n+1}$  and  $M := \neg G$ , which gives  $C_{\frac{1}{2^n}} \cup F_{n+1} \subseteq C_{\frac{1}{2^{n+1}}} \subseteq \neg G$  such that  $d(C_{\frac{1}{2^{n+1}}}, \beta) = 1$  for each  $\beta \in C_{\frac{1}{2^n}}$ .

Then we construct  $C_{\frac{2k+1}{2^n}}$ , for  $0 < k < 2^{n-1}$  and  $n \geq 2$ . This will give us a family  $(C_{\frac{k}{2^n}})_{n \in \omega, 0 < k \leq 2^n}$  of closed subsets of  $2^\omega$ . We want to ensure that  $C_\zeta \subseteq C_{\zeta'}$  and  $d(C_{\zeta'}, \beta) = 1$  for each  $\beta \in C_\zeta$  if  $\zeta' < \zeta$ . We proceed by induction on  $n$ . We apply Lemma 2.2 to  $F := C_{\frac{k+1}{2^{n-1}}}$  and  $M := C_{\frac{k}{2^{n-1}}}$ , which gives  $C_{\frac{2k+1}{2^n}}$  such that  $C_{\frac{k+1}{2^{n-1}}} \subseteq C_{\frac{2k+1}{2^n}} \subseteq C_{\frac{k}{2^{n-1}}}$ ,  $d(C_{\frac{k}{2^{n-1}}}, \beta) = 1$  for each  $\beta \in C_{\frac{2k+1}{2^n}}$ , and  $d(C_{\frac{2k+1}{2^n}}, \beta) = 1$  for each  $\beta \in C_{\frac{k+1}{2^{n-1}}}$ . This allows us to define  $\tilde{h}$  by

$$\tilde{h}(\beta) := \begin{cases} 0 & \text{if } \beta \in G, \\ \sup\{\zeta \mid \beta \in C_\zeta\} & \text{if } \beta \notin G. \end{cases}$$

It remains to see that  $\tilde{h}$  is  $\tau$ -continuous (and then we will set  $h(\beta) := 1 - \tilde{h}(\beta)$ ). So let  $b \in (0, 1]$ , and  $\beta \in 2^\omega$  with  $\tilde{h}(\beta) < b$ . Note that there is  $\zeta < b$  with  $\tilde{h}(\beta) < \zeta$ , so that  $\beta \notin C_\zeta$ . If  $\gamma \notin C_\zeta$ , then  $\tilde{h}(\gamma) \leq \zeta < b$ , so that  $\neg C_\zeta$  is an open (and thus  $\tau$ -open since the  $\tau$ -topology is finer than the usual one) neighborhood of  $\beta$  on which  $\tilde{h} < b$ . In particular,  $\tilde{h}$  is Borel.

Now let  $a \in [0, 1)$ . It is enough to see that  $B := \{\gamma \in 2^\omega \mid \tilde{h}(\gamma) > a\}$  is  $\tau$ -open. So assume that  $\tilde{h}(\gamma) > a$ . Note that there are  $\zeta > \zeta' > a$  with  $\tilde{h}(\gamma) > \zeta$ , so that  $\gamma \in C_\zeta \subseteq C_{\zeta'} \subseteq B$ . Thus  $d(C_{\zeta'}, \gamma) = 1$ , by construction of the family. As  $h$  is Borel,  $B$  is Borel,  $d(B, \gamma)$  is defined and equal to 1.  $\square$

**Remark.** We in fact proved that  $h$  is lower semi-continuous.

**Notation.** If  $h: 2^\omega \rightarrow [0, 1]$  is a  $\lambda$ -measurable map and  $s \in 2^{<\omega}$ , then we set  $\int_{N_s} h d\lambda := \frac{\int_{N_s} h d\lambda}{\lambda(N_s)}$ .

**Lemma 2.6** *Let  $h: 2^\omega \rightarrow [0, 1]$  be a  $\tau$ -continuous map, and  $\beta \in 2^\omega$ . Then*

$$\lim_{l \rightarrow \infty} \int_{N_{\beta|l}} h d\lambda = h(\beta).$$

**Proof.** Let  $\varepsilon > 0$ , and  $\beta \in M := h^{-1}(B(h(\beta), \varepsilon))$ . Note that  $d(M, \gamma) = 1$  for each  $\gamma \in M$  since  $h$  is  $\tau$ -continuous. As  $h$  is  $\lambda$ -measurable, we can write

$$\int_{N_{\beta|l}} h d\lambda = \int_{M \cap N_{\beta|l}} h d\lambda + \int_{N_{\beta|l} \setminus M} h d\lambda.$$

Note that  $0 \leq \int_{N_{\beta|l} \setminus M} h d\lambda \leq \lambda(N_{\beta|l} \setminus M)$ , so that  $0 \leq \int_{N_{\beta|l} \setminus M} h d\lambda \leq \frac{\lambda(N_{\beta|l} \setminus M)}{\lambda(N_{\beta|l})} \rightarrow 0$ . Similarly,

$$\int_{M \cap N_{\beta|l}} h d\lambda \in \left[ (h(\beta) - \varepsilon) \frac{\lambda(M \cap N_{\beta|l})}{\lambda(N_{\beta|l})}, (h(\beta) + \varepsilon) \frac{\lambda(M \cap N_{\beta|l})}{\lambda(N_{\beta|l})} \right],$$

and we are done since  $\frac{\lambda(M \cap N_{\beta|l})}{\lambda(N_{\beta|l})}$  tends to 1 as  $l$  tends to  $\infty$ .  $\square$

Now we come to our main lemma, inspired by Zahorski (see [Za]).

**Lemma 2.7** *Let  $G$  be a  $G_\delta$  subset of  $2^\omega$  with  $\lambda$ -measure zero. Then there is a martingale  $f$  with  $G = D(f)$  and  $\{osc(f, \beta) \mid \beta \in 2^\omega\} \subseteq \{0\} \cup [\frac{1}{2}, 1]$ .*

**Proof.** Let  $(G_n)_{n \in \omega}$  be a decreasing sequence of open subsets of  $2^\omega$  with intersection  $G$  and  $G_0 = 2^\omega$ .

• We construct  $g_n: 2^\omega \rightarrow [0, 1]$ , open subsets  $G_n^*, G_n^{**}$  of  $2^\omega$ , and a sequence  $(s_j^n)_{j \in I_n}$  of pairwise incompatible finite binary sequences, by induction on  $n \in \omega$ , such that, if  $S_n := \sum_{j \leq n} (-1)^j g_j$ ,

- (1)  $G \subseteq G_{n+1}^* \subseteq G_n^{**} = \bigcup_{j \in I_n} N_{s_j^n} \subseteq G_n^* \subseteq G_n \wedge G_0^* = 2^\omega$ ,
- (2)  $g_n|_G \equiv 1 \wedge g_n|_{-G_n^*} \equiv 0$ ,
- (3)  $g_n$  is  $\tau$ -continuous,
- (4)  $g_{n+1} \leq g_n$ ,
- (5)  $\lambda(G_{n+1}^* \cap N_{s_j^n}) < 2^{-n-3} \lambda(N_{s_j^n})$ ,
- (6)  $|\int_{N_{s_j^n}} S_n d\lambda - S_n(\beta)| < 2^{-3}$  if  $\beta \in G \cap N_{s_j^n}$ .

We set  $g_0 \equiv 1$ ,  $G_0^*, G_0^{**} := 2^\omega$ ,  $I_0 := \{0\}$  and  $s_0^0 := \emptyset$ . Assume that our objects are constructed up to  $n$ . We first construct an open subset  $G_{n+1}^*$  of  $2^\omega$  with  $G \subseteq G_{n+1}^* \subseteq G_n^{**} \cap G_{n+1}$  such that

$$\lambda(G_{n+1}^* \cap N_{s_j^n}) < 2^{-n-3} \lambda(N_{s_j^n})$$

if  $j \in I_n$ . For each  $j \in I_n$ , there is an open set  $O_j$  with  $G \cap N_{s_j^n} \subseteq O_j \subseteq G_{n+1} \cap N_{s_j^n}$  such that  $\lambda(O_j) < 2^{-n-3} \lambda(N_{s_j^n})$ . We then set  $G_{n+1}^* := \bigcup_{j \in I_n} O_j$ .

We now apply Lemma 2.5 to  $C := \neg G_{n+1}^*$  and  $G$ , which gives a  $\tau$ -continuous map  $h: 2^\omega \rightarrow [0, 1]$  with  $h|_{\neg G_{n+1}^*} \equiv 0$  and  $h|_G \equiv 1$ . We set  $g_{n+1} := \min(g_n, h)$ , so that  $g_{n+1}$  satisfies (2)-(4).

By Lemma 2.6,  $\lim_{l \rightarrow \infty} \int_{N_{\beta|l}} S_{n+1} d\lambda = S_{n+1}(\beta)$  for each  $\beta \in G$ . This gives  $l(\beta) \in \omega$  minimal with  $|\int_{N_{\beta|l(\beta)}} S_{n+1} d\lambda - S_{n+1}(\beta)| < 2^{-3}$  and  $N_{\beta|l(\beta)} \subseteq G_{n+1}^*$ . The set  $G_{n+1}^{**}$  is the union of the  $N_{\beta|l(\beta)}$ 's, which defines  $I_{n+1}$  and  $(s_j^{n+1})_{j \in I_{n+1}}$  ( $S_{n+1}(\beta)$  is 0 if  $n$  is even and 1 otherwise when  $\beta \in G$ ).

- We then define a partial map  $f_\infty: 2^\omega \rightarrow [0, 1]$  by  $f_\infty := \sum_{j \in \omega} (-1)^j g_j$ . If  $\beta \in G$ , then  $S_n(\beta)$  takes alternatively the values 1 and 0, depending on the parity of  $n$ , so that  $f_\infty(\beta)$  is not defined. If  $\beta \notin G$ , then there is  $n$  such that  $\beta \in \neg G_{n+1}^* \subseteq \neg G_{n+2}^* \subseteq \dots$ . This implies that  $f_\infty(\beta)$  is defined and equal to  $S_n(\beta)$ . As  $0 \leq \sum_{p \leq q} (g_{2p} - g_{2p+1}) = S_{2q+1} \leq S_{2q} = g_0 + \sum_{1 \leq p \leq q} (g_{2p} - g_{2p-1}) \leq g_0$ ,  $f_\infty$  takes values in  $[0, 1]$ . So  $f_\infty$  is a partial  $\lambda$ -measurable map defined  $\lambda$ -almost everywhere since  $\lambda(G) = 0$  (we use Lemma 2.4).

- This allows us to define  $f: 2^{<\omega} \rightarrow [0, 1]$  by  $f(s) := \int_{N_s} f_\infty d\lambda$ . As  $\lambda(N_s) = 2\lambda(N_{s\varepsilon})$  for each  $\varepsilon \in 2$ ,  $f(s) = \int_{N_s} f_\infty d\lambda = \frac{\int_{N_{s0}} f_\infty d\lambda + \int_{N_{s1}} f_\infty d\lambda}{\lambda(N_s)} = \frac{f(s0)}{2} + \frac{f(s1)}{2}$  and  $f$  is a martingale.

- If  $\beta \notin G$ , then there is  $n$  with  $\beta \in G_n^* \setminus G_{n+1}^*$ , so that  $f_\infty(\beta) = S_n(\beta)$ . By Lemma 2.6,  $k \geq n$  implies that  $\lim_{l \rightarrow \infty} \int_{N_{\beta|l}} S_{k+1} d\lambda = S_{k+1}(\beta) = S_n(\beta)$  since  $S_{k+1}$  is  $\tau$ -continuous. Note that, for each  $k \geq n$ ,

$$\begin{aligned} \left| \int_{N_{\beta|l}} (f_\infty - S_{k+1}) d\lambda \right| &\leq \lambda(G_{k+2}^* \cap N_{\beta|l}) \\ &\leq \sum_{\beta|l \subseteq s_j^{k+1}} \lambda(G_{k+2}^* \cap N_{s_j^{k+1}}) \\ &\leq \sum_{\beta|l \subseteq s_j^{k+1}} 2^{-k-4} \lambda(N_{s_j^{k+1}}) \\ &\leq \lambda(N_{\beta|l}) 2^{-k-4}. \end{aligned}$$

Moreover,

$$\begin{aligned} |f(\beta|l) - f_\infty(\beta)| &= \left| \int_{N_{\beta|l}} f_\infty d\lambda - f_\infty(\beta) \right| = \left| \int_{N_{\beta|l}} (f_\infty - S_{k+1}) d\lambda + \int_{N_{\beta|l}} S_{k+1} d\lambda - S_{k+1}(\beta) \right| \\ &\leq 2^{-k-4} + \left| \int_{N_{\beta|l}} S_{k+1} d\lambda - S_{k+1}(\beta) \right|, \end{aligned}$$

so that  $\lim_{l \rightarrow \infty} f(\beta|l) = f_\infty(\beta)$ ,  $\text{osc}(f, \beta) = 0$  and  $\beta \notin D(f)$ .

- If  $\beta \in G$  and  $n \in \omega$ , then there is  $j \in \omega$  with  $\beta \in N_{s_j^n}$ . Note that

$$f(s_j^n) = \int_{N_{s_j^n}} f_\infty d\lambda = \int_{N_{s_j^n}} S_n d\lambda + \int_{N_{s_j^n}} (f_\infty - S_n) d\lambda$$

and  $|\int_{N_{s_j^n}} (f_\infty - S_n) d\lambda| \leq \lambda(G_{n+1}^* \cap N_{s_j^n}) < \frac{1}{8} \lambda(N_{s_j^n})$ , so that  $|\int_{N_{s_j^n}} (f_\infty - S_n) d\lambda| < \frac{1}{8}$ . By (6),  $|f(s_j^n) - S_n(\beta)| < \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$ . As  $S_n(\beta)$  takes infinitely often the values 1 and 0,  $\text{osc}(f, \beta) \geq \frac{1}{2}$  and  $\beta \in D(f)$ .  $\square$

The main result will be a consequence of the main lemma and the following.



**Lemma 2.8** Let  $(f_n)_{n \in \omega}$  be a sequence of martingales such that

$$\{\text{osc}(f_n, \beta) \mid (n, \beta) \in \omega \times 2^\omega\} \subseteq \{0\} \cup [\frac{1}{2}, 1].$$

Then there is a martingale  $f$  with  $D(f) = \bigcup_{n \in \omega} D(f_n)$ .

**Proof.** We first observe the following facts. Let  $g, h: 2^{<\omega} \rightarrow \mathbb{R}$  be bounded,  $\beta \in 2^\omega$  and  $a \in \mathbb{R}$ .

$$(1) \text{osc}(g+h, \beta) \leq \text{osc}(g, \beta) + \text{osc}(h, \beta).$$

This comes from the triangle inequality.

$$(2) \text{osc}(ag, \beta) = |a| \cdot \text{osc}(g, \beta).$$

$$(3) \text{osc}(g+h, \beta) = \text{osc}(h, \beta) \text{ if } \text{osc}(g, \beta) = 0.$$

By (1),  $\text{osc}(h, \beta) \leq \text{osc}(g+h, \beta) + \text{osc}(-g, \beta) = \text{osc}(g+h, \beta) \leq \text{osc}(g, \beta) + \text{osc}(h, \beta) = \text{osc}(h, \beta)$ , so that  $\text{osc}(h, \beta) = \text{osc}(g+h, \beta)$ .

$$(4) \text{osc}(g, \beta) \leq a \text{ if } g(\beta|l) \in [0, a] \text{ for each } l \in \omega.$$

• We set  $D_n := D(f_n)$  for each  $n \in \omega$ , and  $f := \sum_{n \in \omega} 4^{-n} f_n$ . Note that  $f$  is defined and a martingale.

• If  $\beta \notin \bigcup_{n \in \omega} D_n$ , then  $\text{osc}(f_n, \beta) = 0$  for each  $n \in \omega$ . In particular,  $\text{osc}(4^{-n} f_n, \beta) = 0$  for each  $n \in \omega$ , by (2). Let  $\varepsilon > 0$ , and  $M \in \omega$  with  $\sum_{n > M} 4^{-n} \leq \varepsilon$ . By (1),  $\text{osc}(\sum_{n \leq M} 4^{-n} f_n, \beta) = 0$ . By (3) and (4),  $\text{osc}(f, \beta) = \text{osc}(\sum_{n > M} 4^{-n} f_n, \beta) \leq \sum_{n > M} 4^{-n} \leq \varepsilon$ . As  $\varepsilon$  is arbitrary,  $\text{osc}(f, \beta) = 0$ ,  $\beta \notin D(f)$ , which shows that  $D(f) \subseteq \bigcup_{n \in \omega} D_n$ .

• If  $\beta \in \bigcup_{n \in \omega} D_n$ , then let  $m$  be minimal such that  $\beta \in D_m$ . Note that

$$f = \sum_{n < m} 4^{-n} f_n + 4^{-m} f_m + \sum_{n > m} 4^{-n} f_n.$$

By (2) and (3),  $\text{osc}(f, \beta) = \text{osc}(4^{-m} f_m + \sum_{n > m} 4^{-n} f_n, \beta)$ . By (1), (2) and (4),

$$\text{osc}(f, \beta) \geq \text{osc}(4^{-m} f_m, \beta) - \text{osc}(\sum_{n > m} 4^{-n} f_n, \beta) \geq 4^{-m} \frac{1}{2} - 4^{-m} \frac{1}{3} > 0.$$

Thus  $\beta \in D(f)$ . □

### 3 Effectivity and uniformity

- We refer to [M] for the basic notions of effective descriptive set theory. We first recall some material present in it.

• Let  $(p_n)_{n \in \omega}$  be the sequence of prime numbers 2, 3, ...

• If  $l \in \omega$  and  $s \in \omega^l$ , then  $\bar{s} := \langle s(0), \dots, s(l-1) \rangle := p_0^{s(0)+1} \dots p_{l-1}^{s(l-1)+1} \in \omega$  codes  $s$  (if  $l = 0$ , then  $\langle \rangle := 1$ ).

• If  $\alpha \in \omega^\omega$  and  $l \in \omega$ , then  $\bar{\alpha}(l) := \langle \alpha(0), \dots, \alpha(l-1) \rangle \in \omega$  codes  $\alpha|l \in \omega^l$ , and  $\alpha^*$  is defined by removing the first coordinate:  $\alpha^* := (\alpha(1), \alpha(2), \dots)$ .

- If  $\kappa \in \{2, \omega\}$ , then  $\langle \cdot, \cdot \rangle: (\kappa^\omega)^2 \rightarrow \kappa^\omega$  is a recursive homeomorphism with inverse map  $\alpha \mapsto ((\alpha)_0, (\alpha)_1)$  defined for example by  $(\alpha)_\varepsilon(n) := \alpha(2n + \varepsilon)$  if  $(n, \varepsilon) \in \omega \times 2$  (we will also consider recursive homeomorphisms  $\langle \cdot, \cdot, \cdot \rangle: (\kappa^\omega)^3 \rightarrow \kappa^\omega$  and  $\langle \cdot, \cdot, \dots \rangle: (\kappa^\omega)^\omega \rightarrow \kappa^\omega$ ).

- If  $u \in \omega$ , then  $\text{Seq}(u)$  means that there are  $l \in \omega$  and  $s \in \omega^l$  (denoted by  $s(u)$ ) such that  $u = \langle s(0), \dots, s(l-1) \rangle$ . The natural number  $(u)_i$  is  $s(i)$  if  $i < l$ , and 0 otherwise. The number  $l$  is the **length** of  $u$  and is denoted by  $\text{lh}(u)$ . If  $k \leq l$ , then  $\underline{u}(k) := \langle s(0), \dots, s(k-1) \rangle$ , so that  $\underline{u}(l) = u$ . The standard basic clopen set is  $N^u := \{\beta \in 2^\omega \mid \forall i < \text{lh}(u) \beta(i) = (u)_i\}$ . We set  $u^- := \langle (u)_0, \dots, (u)_{\text{lh}(u)-2} \rangle$  ( $u^- := \langle \rangle$  if  $\text{lh}(u) \leq 1$ ).

- Let  $X$  be a recursively presented Polish space. Then we will consider the effective basic open set  $N(X, u) = B_X(r_{((u)_1)_0}, \frac{((u)_1)_1}{((u)_1)_2 + 1})$ .

- Let  $n \geq 1$  be a natural number. A subset  $T$  of  $\omega^n$  is a **tree** if  $\text{Seq}(u_i)$  and  $\text{lh}(u_i) = \text{lh}(u_0)$  for each  $(u_0, \dots, u_{n-1}) \in T$  and each  $i < n$ , and  $(\underline{u}_0(k), \dots, \underline{u}_{n-1}(k)) \in T$  if  $(u_0, \dots, u_{n-1}) \in T$  and  $k \leq \text{lh}(u_0)$ .

- The next result is a part of 4A.1 in [M].

**Theorem 3.1** *Let  $m \geq 1$  be a natural number, and  $B \in \Sigma_1^0(\omega^\omega \times (\omega^\omega)^m)$ . Then there is a recursive subset  $T$  of  $\omega^\omega \times \omega^m$  such that  $(\alpha, \alpha_1, \dots, \alpha_m) \in B \Leftrightarrow \exists l \in \omega \ (\alpha, \underline{\alpha}_1(l), \dots, \underline{\alpha}_m(l)) \notin T$ , and  $T_\alpha := \{(u_0, \dots, u_{m-1}) \in \omega^m \mid (\alpha, u_0, \dots, u_{m-1}) \in T\}$  is a tree for each  $\alpha \in \omega^\omega$ .*

- The next result is a part of 4A.7 in [M].

**Theorem 3.2** *Let  $X$  be a recursively presented Polish space and  $B \in \Delta_1^1(X)$ . Then we can find a recursive function  $\pi: \omega^\omega \rightarrow X$  and  $C \in \Pi_1^0(\omega^\omega)$  such that  $\pi$  is injective on  $C$  and  $\pi[C] = B$ .*

- We then recall some material from [L].

**Notation.** Let  $X$  be a recursively presented Polish space. Recall that there is a pair  $(\mathcal{W}^X, \mathcal{C}^X)$  such that

- $\mathcal{W}^X \subseteq \omega$  is a  $\Pi_1^1$  set of codes for the  $\Delta_1^1$  subsets of  $X$ ,
- $\mathcal{C}^X \subseteq \omega \times X$  is  $\Pi_1^1$  and  $\Delta_1^1(X) = \{\mathcal{C}_n^X \mid n \in \mathcal{W}^X\}$ , which means that  $\mathcal{C}^X$  is “universal” for the  $\Delta_1^1$  subsets of  $X$ ,
- the relation “ $n \in \mathcal{W}^X \wedge (n, x) \notin \mathcal{C}^X$ ” is  $\Pi_1^1$  in  $(n, x)$ .

If  $X = \omega^\omega \times 2^\omega$ , then we simply write  $(\mathcal{W}, \mathcal{C}) := (\mathcal{W}^X, \mathcal{C}^X)$ .

The next result will be extremely useful in the sequel.

**The uniformization lemma.** *Let  $X, Y$  be recursively presented Polish spaces, and  $P \in \Pi_1^1(X \times Y)$ . Then the set  $P^+ := \{x \in X \mid \exists y \in \Delta_1^1(x) \ (x, y) \in P\}$  is  $\Pi_1^1$ , and there is a partial  $\Pi_1^1$ -recursive map  $f: X \rightarrow Y$  such that  $(x, f(x)) \in P$  for each  $x \in P^+$ . If moreover  $S \subseteq P^+$  is a  $\Sigma_1^1$  subset of  $X$ , then there is a total  $\Delta_1^1$ -recursive map  $g: X \rightarrow Y$  such that  $(x, g(x)) \in P$  for each  $x \in S$ .*

- The following definition is inspired by 3H.1 in [M].

**Definition 3.3** (a) Let  $\Gamma$  be a class of subsets of recursively presented Polish spaces, and  $\mathbf{\Gamma}$  be the associated boldface class. A system of sets  $\mathcal{U}^X \in \Gamma(\omega^\omega \times X)$ , where  $X$  is a recursively presented Polish space, is a **nice parametrization** in  $\Gamma$  for  $\mathbf{\Gamma}$  if the following hold:

$$(1) \mathbf{\Gamma}(X) = \{\mathcal{U}_\alpha^X \mid \alpha \in \omega^\omega\},$$

$$(2) \Gamma(X) = \{\mathcal{U}_\alpha^X \mid \alpha \in \omega^\omega \text{ recursive}\},$$

(3) if  $X$  is a recursively presented Polish space, then there is  $\mathcal{R} : \omega^\omega \times \omega^\omega \rightarrow \omega^\omega$  recursive such that  $(\alpha, \gamma, x) \in \mathcal{U}^{\omega^\omega \times X} \Leftrightarrow (\mathcal{R}(\alpha, \gamma), x) \in \mathcal{U}^X$  if  $(\alpha, \gamma, x) \in \omega^\omega \times \omega^\omega \times X$ .

(b) If  $\mathcal{U}$  belongs to a nice parametrization, then we will say that  $\mathcal{U}$  is a **good universal set**.

(c) If  $\mathcal{U}$  satisfies all these properties except maybe (3), then we will say that  $\mathcal{U}$  is a **suitable universal set**.

By 3E.2, 3F.6 and 3H.1 in [M], there is a nice parametrization in  $\Pi_n^1$  for  $\mathbf{\Pi}_n^1$ , for each natural number  $n \geq 1$ .

- We now recall two results that can essentially be found in [K1]. The first one is Theorem 2.2.3.(a) (see also [T1]).

**Theorem 3.4** (Tanaka) Let  $U \in \Sigma_1^1(\omega^\omega \times \omega^\omega)$  be  $\omega^\omega$ -universal for the analytic subsets of  $\omega^\omega$ . Then  $L(U) := \{(\alpha, p) \in \omega^\omega \times \omega \mid \lambda(U_\alpha \cap 2^\omega) > \frac{\binom{p}{0}}{\binom{p}{1+1}}\}$  is  $\Sigma_1^1$ .

**Corollary 3.5** Let  $B \in \Delta_1^1(\omega^\omega \times 2^\omega)$ .

(a) The map  $\lambda_B : \omega^\omega \rightarrow \mathbb{R}$  defined by  $\lambda_B(\alpha) := \lambda(B_\alpha)$  is  $\Delta_1^1$ -recursive, and the partial function  $(n, \alpha) \mapsto \lambda(C_{n,\alpha})$  is  $\Pi_1^1$ -recursive on its domain  $\mathcal{W} \times \omega^\omega$ .

(b) Let  $D \subseteq \omega$ ,  $O_0 \in \Sigma_1^1(\omega \times \omega^\omega \times 2^\omega)$ , and  $O_1 \in \Pi_1^1(\omega \times \omega^\omega \times 2^\omega)$  be such that  $\lambda((O_0)_{n,\alpha}) = \lambda((O_1)_{n,\alpha})$  if  $n \in D$ . Then the partial map  $\lambda_O : D \times \omega^\omega \rightarrow \mathbb{R}$  defined by  $\lambda_O(n, \alpha) := \lambda((O_0)_{n,\alpha})$  is  $\Sigma_1^1$ -recursive and  $\Pi_1^1$ -recursive on its domain.

(c) The partial map  $d_B : \omega^\omega \times 2^\omega \rightarrow \mathbb{R}$  defined by  $d_B(\alpha, \beta) := d(B_\alpha, \beta)$  is  $\Delta_1^1$ -recursive, and the partial map  $(n, \alpha, \beta) \mapsto d(C_{n,\alpha}, \beta)$  is  $\Pi_1^1$ -recursive on its  $\Pi_1^1$  domain

$$\{(n, \alpha, \beta) \in \mathcal{W} \times \omega^\omega \times 2^\omega \mid d(C_{n,\alpha}, \beta) \text{ exists}\}.$$

(d) Let  $h : \omega^\omega \times 2^\omega \rightarrow \mathbb{R}$  be  $\Delta_1^1$ -recursive taking values in  $[0, 1]$ . Then the partial map  $i_h : \omega^\omega \times \omega \rightarrow \mathbb{R}$  defined by  $i_h(\alpha, u) := \int_{N^u} h(\alpha, \cdot) d\lambda$  is  $\Delta_1^1$ -recursive on its  $\Delta_1^0$  domain  $\omega^\omega \times \{u \in \omega \mid \text{Seq}(u)\}$ .

**Proof.** (a) It is enough to see that the relations  $P_B(\alpha, p) \Leftrightarrow \lambda(B_\alpha) > r_p := (-1)^{(p)_0} \cdot \frac{\binom{p}{1}}{\binom{p}{2+1}}$  and

$$Q_B(\alpha, p) \Leftrightarrow \lambda(B_\alpha) < r_p$$

are  $\Delta_1^1$  to see that  $\lambda_B$  is  $\Delta_1^1$ -recursive. Note that there is  $\phi : \omega^2 \rightarrow \omega$  recursive with  $r_{\phi(p,l)} = r_p - \frac{1}{l+1}$ . Thus

$$\begin{aligned} Q_B(\alpha, p) &\Leftrightarrow \exists l \in \omega \lambda(B_\alpha) \leq r_p - \frac{1}{l+1} \\ &\Leftrightarrow \exists l \in \omega \neg(\lambda(B_\alpha) > r_p - \frac{1}{l+1}) \\ &\Leftrightarrow \exists l \in \omega \neg P_B(\alpha, \phi(p, l)), \end{aligned}$$

so that it is enough to see that  $P_B$  is  $\Delta_1^1$ .

- Now let  $S \in \Sigma_1^1(\omega^\omega \times (\omega^\omega)^2)$  be a good  $\omega^\omega$ -universal for the analytic subsets of  $(\omega^\omega)^2$ . We set

$$U(\alpha, \gamma) \Leftrightarrow S((\alpha)_0, (\alpha)_1, \gamma),$$

so that  $U \in \Sigma_1^1(\omega^\omega \times \omega^\omega)$  is  $\omega^\omega$ -universal for the analytic subsets of  $\omega^\omega$ . Let  $A$  be a  $\Sigma_1^1$  subset of  $\omega^\omega \times 2^\omega$ . Then there is  $\alpha_0 \in \omega^\omega$  recursive with  $A = S_{\alpha_0}$ , so that

$$\gamma \in A_\alpha \Leftrightarrow (\alpha_0, \alpha, \gamma) \in S \Leftrightarrow \langle \alpha_0, \alpha \rangle, \gamma \in U.$$

This implies that the relation  $R_A(\alpha, p) \Leftrightarrow \lambda(A_\alpha) > r_p$ , equivalent to

$$((p)_0 \text{ is odd} \wedge (p)_1 > 0) \vee ((p)_0 \text{ is even} \wedge \langle \alpha_0, \alpha \rangle, \langle (p)_1, (p)_2 \rangle \in L(U)),$$

is  $\Sigma_1^1$ , by Theorem 3.4.

- In particular, this applies to  $A := B$ , so that  $P_B$  is  $\Sigma_1^1$ . Now note that

$$P_B(\alpha, p) \Leftrightarrow \lambda((-B)_\alpha) < 1 - r_p \Leftrightarrow Q_{-B}(\alpha, \phi'(p)),$$

for some  $\phi' : \omega \rightarrow \omega$  is recursive, so that  $P_B$  is  $\Pi_1^1$  by the previous computation.

- We set  $\mathcal{C}' := \{(\gamma, \beta) \in \omega^\omega \times 2^\omega \mid \gamma(0) \in \mathcal{W} \wedge (\gamma(0), \gamma^*, \beta) \in \mathcal{C}\}$ . As  $\mathcal{C}'$  is  $\Pi_1^1$ ,

$$\mathcal{A} := \{(\alpha, p) \in \omega^\omega \times \omega \mid \lambda((- \mathcal{C}')_\alpha) > r_p\}$$

is  $\Sigma_1^1$ , by the previous discussion. So let  $n \in \mathcal{W}$ . Note that

$$\begin{aligned} \lambda(\mathcal{C}_{n,\alpha}) > r_p &\Leftrightarrow \lambda(-\mathcal{C}_{n,\alpha}) < 1 - r_p \Leftrightarrow \lambda((- \mathcal{C}')_{n\alpha}) < 1 - r_p \\ &\Leftrightarrow \exists l \in \omega \lambda((- \mathcal{C}')_{n\alpha}) \leq 1 - r_p - \frac{1}{l+1} \Leftrightarrow \exists l \in \omega (n\alpha, \phi''(p, l)) \notin \mathcal{A}, \end{aligned}$$

for some recursive  $\phi'' : \omega^2 \rightarrow \omega$ . Similarly, the relation “ $\lambda(\mathcal{C}_{n,\alpha}) < r_p$ ” is  $\Pi_1^1$  in  $(n, \alpha, p)$  since the relation “ $n \in \mathcal{W} \wedge (n, \alpha, \beta) \notin \mathcal{C}$ ” is  $\Pi_1^1$ , so that  $(n, \alpha) \mapsto \lambda(\mathcal{C}_{n,\alpha})$  is  $\Pi_1^1$ -recursive on  $\mathcal{W} \times \omega^\omega$ .

(b) Let  $A := \{(\alpha, \beta) \in \omega^\omega \times 2^\omega \mid (\alpha(0), \alpha^*, \beta) \in O_0\}$ . Note that  $A$  is  $\Sigma_1^1$ . By (a), the relation  $R_A(\alpha, p) \Leftrightarrow \lambda(A_\alpha) > r_p$  is  $\Sigma_1^1$ . Therefore the relation  $R_{O_0}(n, \alpha, p) \Leftrightarrow R_A(n\alpha, p)$  is  $\Sigma_1^1$  too. Moreover,  $R_{O_0}(n, \alpha, p) \Leftrightarrow \lambda((O_0)_{n,\alpha}) > r_p \Leftrightarrow \lambda_O(n, \alpha) > r_p$ .

- Assume now that  $n \in D$ . Then as above there is  $\phi'' : \omega^2 \rightarrow \omega$  recursive such that

$$\begin{aligned} \lambda_O(n, \alpha) > r_p &\Leftrightarrow \lambda((O_1)_{n,\alpha}) > r_p \Leftrightarrow \lambda((-O_1)_{n,\alpha}) < 1 - r_p \\ &\Leftrightarrow \exists l \in \omega \lambda((-O_1)_{n,\alpha}) \leq 1 - r_p - \frac{1}{l+1} \Leftrightarrow \exists l \in \omega \neg(\lambda((-O_1)_{n,\alpha}) > r_{\phi''(p,l)}) \\ &\Leftrightarrow \exists l \in \omega \neg R_{-O_1}(n, \alpha, \phi''(p, l)), \end{aligned}$$

which shows the existence of  $R'_{O_0} \in \Pi_1^1$  such that  $\lambda_O(n, \alpha) > r_p \Leftrightarrow R'_{O_0}(n, \alpha, p)$  if  $n \in D$ .

- Assume that  $n \in D$ . Then there is  $\phi' : \omega \rightarrow \omega$  recursive such that

$$\lambda_O(n, \alpha) < r_q \Leftrightarrow \lambda((O_1)_{n,\alpha}) < r_q \Leftrightarrow \lambda((-O_1)_{n,\alpha}) > 1 - r_q \Leftrightarrow R_{-O_1}(n, \alpha, \phi'(q)),$$

which shows the existence of  $R''_{O_0} \in \Sigma_1^1$  such that  $\lambda_O(n, \alpha) < r_q \Leftrightarrow R''_{O_0}(n, \alpha, q)$  if  $n \in D$ .

- Assume that  $n \in D$ . Then there is  $\phi'' : \omega^2 \rightarrow \omega$  recursive such that

$$\begin{aligned} \lambda_O(n, \alpha) < r_q &\Leftrightarrow \lambda((O_0)_{n, \alpha}) < r_q \Leftrightarrow \exists l \in \omega \lambda((O_0)_{n, \alpha}) \leq 1 - r_q - \frac{1}{l+1} \\ &\Leftrightarrow \exists l \in \omega \neg \left( \lambda((O_0)_{n, \alpha}) > r_{\phi''(q, l)} \right) \Leftrightarrow \exists l \in \omega \neg R_{O_0}(n, \alpha, \phi''(q, l)), \end{aligned}$$

which shows the existence of  $R''_{O_0} \in \Pi_1^1$  such that  $\lambda_O(n, \alpha) < r_q \Leftrightarrow R''_{O_0}(n, \alpha, q)$  if  $n \in D$ .

- Finally,  $r_p < \lambda_O(n, \alpha) < r_q \Leftrightarrow R_{O_0}(n, \alpha, p) \wedge R''_{O_0}(n, \alpha, q)$  and

$$r_p < \lambda_O(n, \alpha) < r_q \Leftrightarrow R'_{O_0}(n, \alpha, p) \wedge R'''_{O_0}(n, \alpha, q)$$

if  $n \in D$ , which shows that  $\lambda_O$  is  $\Sigma_1^1$ -recursive and  $\Pi_1^1$ -recursive on  $D \times \omega$ .

(c) We first prove the following. Let  $X, Y$  be a recursively presented Polish spaces and  $g : X \times \omega \rightarrow Y$  be a  $\Delta_1^1$ -recursive map. Then the partial map  $h : X \rightarrow Y$  defined by

$$h(x) := \lim_{l \rightarrow \infty} g(x, l)$$

when this limit exists is  $\Delta_1^1$ -recursive.

Indeed, the domain  $D$  of  $h$  is  $\{x \in X \mid \forall r \in \omega \exists L \in \omega \forall k, l \geq L d_Y(g(x, k), g(x, l)) < 2^{-r}\}$ , so that  $D$  is  $\Delta_1^1$ . If  $x \in D$ , then  $h(x) \in N(Y, u)$  is equivalent to

$$\exists p, q \in \omega \frac{p}{q+1} < \frac{((u)_1)_1}{((u)_1)_2 + 1} \wedge \exists L \in \omega \forall l \geq L g(x, l) \in N(Y, \langle 0, \langle (u)_1 \rangle_0, p, q \rangle),$$

and we are done.

- We set  $B' := \{(\alpha, \gamma) \in \omega^\omega \times 2^\omega \mid ((\alpha)_0, \gamma) \in B \wedge \gamma \in N_{(\alpha)_1^* | (\alpha)_1(0)}\}$ , so that  $B_\alpha \cap N_{\beta|l} = B'_{\langle \alpha, l \beta \rangle}$  and  $B'$  is  $\Delta_1^1$ . By (a), the map  $g : \omega^\omega \times 2^\omega \times \omega \rightarrow [0, 1]$  defined by  $g(\alpha, \beta, l) := 2^{-l} \lambda(B_\alpha \cap N_{\beta|l})$  is  $\Delta_1^1$ -recursive. By the previous point, the partial map  $h : \omega^\omega \times 2^\omega \rightarrow [0, 1]$  defined by

$$h(\alpha, \beta) := \lim_{l \rightarrow \infty} 2^{-l} \lambda(B_\alpha \cap N_{\beta|l})$$

when it exists is also  $\Delta_1^1$ -recursive. But  $h = d_B$ .

- Fix  $n \in \mathcal{W}$ . Then there is  $q(n) \in \mathcal{W}$  such that

$$\mathcal{C}_{q(n)} = \{(\gamma, \delta) \in \omega^\omega \times 2^\omega \mid (n, (\gamma)_0, \delta) \in \mathcal{C} \wedge (\gamma)_1^* | (\gamma)_1(0) \subseteq \delta\}.$$

Moreover, we may assume that  $q$  is  $\Pi_1^1$ -recursive on  $\mathcal{W}$ , by the uniformization lemma. As  $\Pi_1^1$  has the substitution property, the map  $g' : (n, \alpha, \beta, l) \mapsto 2^{-l} \lambda(\mathcal{C}_{q(n), \langle \alpha, l \beta \rangle}) = 2^{-l} \lambda(\mathcal{C}_{n, \alpha} \cap N_{\beta|l})$  is  $\Pi_1^1$ -recursive on  $\mathcal{W} \times \omega^\omega \times 2^\omega \times \omega$ . As above, the map

$$h' : (n, \alpha, \beta) \mapsto \lim_{l \rightarrow \infty} 2^{-l} \lambda(\mathcal{C}_{n, \alpha} \cap N_{\beta|l}) = d(\mathcal{C}_{n, \alpha}, \beta)$$

is  $\Pi_1^1$ -recursive on the  $\Pi_1^1$  set  $\{(n, \alpha, \beta) \in \mathcal{W} \times \omega^\omega \times 2^\omega \mid d(\mathcal{C}_{n, \alpha}, \beta) \text{ exists}\}$ .

(d) The argument here is partly similar to 11.6 and 17.25 in [K2]. We set, for  $(k, l) \in \omega^2$ ,

$$A_{k,l} := h^{-1}\left(\left[\frac{k}{2^l}, \frac{k+1}{2^l}\right)\right)$$

and define  $h_l : \omega^\omega \times 2^\omega \rightarrow [0, 1]$  by  $h_l = \sum_{k \leq 2^l} \frac{k}{2^l} \chi_{A_{k,l}}$ . We also define  $R \subseteq \omega^\omega \times 2^\omega \times \omega^3$  by

$$R(\alpha, \beta, u, k, l) \Leftrightarrow \frac{k}{2^l} \leq h(\alpha, \beta) < \frac{k+1}{2^l} \wedge \text{Seq}(u) \wedge \beta \in N^u,$$

so that  $R$  is  $\Delta_1^1$ . Then we define  $O \subseteq \omega^\omega \times 2^\omega$  by

$$O(\alpha, \beta) \Leftrightarrow \text{Seq}(\alpha(0)) \wedge \text{lh}(\alpha(0)) = 3 \wedge R(\alpha^*, \beta, (\alpha(0))_0, (\alpha(0))_1, (\alpha(0))_2),$$

so that  $O$  is  $\Delta_1^1$ .

Note that  $(h_l)$  is a sequence of Borel functions pointwise converging to  $h$ . By Lebesgue's dominated convergence theorem,  $\int_{N^u} h(\alpha, \cdot) d\lambda = \lim_{l \rightarrow \infty} \int_{N^u} h_l(\alpha, \cdot) d\lambda$  if  $\text{Seq}(u)$ . Note that

$$\begin{aligned} \int_{N^u} h_l(\alpha, \cdot) d\lambda &= \int_{N^u} \sum_{k \leq 2^l} \frac{k}{2^l} \chi_{A_{k,l}}(\alpha, \cdot) d\lambda = \sum_{k \leq 2^l} \frac{k}{2^l} \lambda((A_{k,l})_\alpha \cap N^u) \\ &= \sum_{k \leq 2^l} \frac{k}{2^l} \lambda(R_{\alpha, u, k, l}) = \sum_{k \leq 2^l} \frac{k}{2^l} \lambda(O_{\langle u, k, l \rangle \alpha}). \end{aligned}$$

Using (a), this implies that the map  $(\alpha, u, l) \mapsto \int_{N^u} h_l(\alpha, \cdot) d\lambda$  is  $\Delta_1^1$ -recursive on its  $\Delta_1^0$  domain  $\omega^\omega \times \{u \in \omega \mid \text{Seq}(u)\} \times \omega$ . As in the proof of (c),  $i_h$  is  $\Delta_1^1$ -recursive on its domain.  $\square$

We now prove a uniform version of Theorem 4.3.2 in [K1] (due to Tanaka, see [T2]).

**Theorem 3.6** *Let  $B \in \Delta_1^1(\omega^\omega \times 2^\omega)$ , and  $\epsilon : \omega^\omega \rightarrow \mathbb{R}$  be  $\Delta_1^1$ -recursive such that  $\epsilon(\alpha) \in (0, 1]$  for each  $\alpha \in \omega^\omega$ . Then there is  $T \in \Delta_1^1(\omega^\omega \times \omega)$  such that*

(a)  $T_\alpha$  is a tree for each  $\alpha \in \omega^\omega$ ,

(b) if  $K = \{(\alpha, \beta) \in \omega^\omega \times 2^\omega \mid \forall l \in \omega \ (\alpha, \bar{\beta}(l)) \in T\}$ , then  $K_\alpha \subseteq B_\alpha$  and  $\lambda(K_\alpha) \geq \lambda(B_\alpha) - \epsilon(\alpha)$  for each  $\alpha \in \omega^\omega$ .

**Proof.** Theorem 3.2 gives  $\pi : \omega^\omega \rightarrow \omega^\omega \times 2^\omega$  recursive and  $C \in \Pi_1^0(\omega^\omega)$  such that  $\pi$  is injective on  $C$  and  $\pi[C] = B$ . We set  $Q := \{(\alpha, \beta, \gamma) \in (\omega^\omega)^3 \mid \gamma \in C \wedge \pi(\gamma) = (\alpha, \beta)\}$ . As  $Q \in \Pi_1^0$ , Theorem 3.1 gives a recursive subset  $\bar{T}$  of  $\omega^\omega \times \omega^2$  such that  $(\alpha, \beta, \gamma) \in Q \Leftrightarrow \forall l \in \omega \ (\alpha, \bar{\beta}(l), \bar{\gamma}(l)) \in \bar{T}$  and  $\bar{T}_\alpha$  is a tree for each  $\alpha \in \omega^\omega$ .

• We set, for  $u, v \in \omega$ ,

$$u \leq^a v \Leftrightarrow \text{Seq}(u), \text{Seq}(v) \wedge \text{lh}(u) = \text{lh}(v) \wedge \forall i < \text{lh}(u) \ (u)_i \leq (v)_i.$$

• Then we set, for  $u \in \omega$  with  $\text{Seq}(u)$  and  $\alpha \in \omega^\omega$ ,

$$B_\alpha^u := \{\beta \in 2^\omega \mid \exists \gamma \in \omega^\omega \ \bar{\gamma}(\text{lh}(u)) \leq^a u \wedge \forall l \in \omega \ (\alpha, \bar{\beta}(l), \bar{\gamma}(l)) \in \bar{T}\}$$

and  $B' := \{(\alpha, \beta) \in \omega^\omega \times 2^\omega \mid \text{Seq}(\alpha(0)) \wedge \beta \in B_{\alpha^*}^{\alpha(0)}\}$ . Note that  $B'$  is  $\Sigma_1^1$ . In fact,  $B'$  is  $\Delta_1^1$  by uniqueness of the witness  $\gamma$ .

- We now define  $\delta_\alpha \in \omega^\omega$  as follows. We define  $\delta_\alpha(i)$  by induction on  $i$ . We first set

$$\delta_\alpha(0) := \min\{k \in \omega \mid \lambda(B_\alpha^{<k>}) > \lambda(B_\alpha) - \frac{\epsilon(\alpha)}{2}\}.$$

This number exists since  $B_\alpha$  is the increasing union of the  $B_\alpha^{<k>}$ 's. Then

$$\delta_\alpha(i+1) := \min\{k \in \omega \mid \lambda(B_\alpha^{<\delta_\alpha(0), \dots, \delta_\alpha(i), k>}) > \lambda(B_\alpha) - \frac{\epsilon(\alpha)}{2} - \dots - \frac{\epsilon(\alpha)}{2^{i+2}}\}.$$

Note that  $\delta_\alpha \in \Delta_1^1(\alpha)$ , by Corollary 3.5.(a).

- We set  $T := \{(\alpha, v) \in \omega^\omega \times \omega \mid \text{Seq}(v) \wedge \exists u \leq^a \bar{\delta}_\alpha(\text{lh}(v)) \ (\alpha, v, u) \in \bar{T}\}$ , so that  $T \in \Delta_1^1(\omega^\omega \times \omega)$  and  $T_\alpha$  is a tree for each  $\alpha \in \omega^\omega$ .

- We set  $K := \{(\alpha, \beta) \in \omega^\omega \times 2^\omega \mid \forall l \in \omega \ \beta \in B_\alpha^{\bar{\delta}_\alpha(l)}\}$ , so that  $K_\alpha \subseteq B_\alpha$  and

$$\lambda(K_\alpha) = \lim_{l \rightarrow \infty} \lambda(B_\alpha^{\bar{\delta}_\alpha(l)}) \geq \lambda(B_\alpha) - \epsilon(\alpha)$$

for each  $\alpha \in \omega^\omega$  since  $(B_\alpha^{\bar{\delta}_\alpha(l)})_{l \in \omega}$  is decreasing. It remains to apply König's lemma to see that  $K = \{(\alpha, \beta) \in \omega^\omega \times 2^\omega \mid \forall l \in \omega \ (\alpha, \bar{\beta}(l)) \in T\}$  since

$$\{s \in \omega^{<\omega} \mid \langle s(0), \dots, s(|s|-1) \rangle \leq^a \bar{\delta}_\alpha(|s|) \wedge (\alpha, \bar{\beta}(|s|), \langle s(0), \dots, s(|s|-1) \rangle) \in \bar{T}\}$$

is a finitely splitting tree. □

- We want to prove an effective and uniform version of the Lusin-Menchoff lemma. We first need the following result, which slightly and uniformly refines Theorem A in [L] at the first level of the Borel hierarchy.

**Lemma 3.7** *Let  $O$  be a  $\Delta_1^1$  subset of  $\omega^\omega \times 2^\omega$  with open vertical sections. Then there is a  $\Delta_1^1$ -recursive map  $f: \omega^\omega \rightarrow \omega^\omega$  such that  $O_\alpha$  is the disjoint union  $\bigcup \{N^{f(\alpha)(u)} \mid u \in \omega \wedge \text{Seq}(f(\alpha)(u))\}$ , for each  $\alpha \in \omega^\omega$ .*

**Proof.** Let  $P := \{(\alpha, u) \in \omega^\omega \times \omega \mid \text{Seq}(u) \wedge (\text{lh}(u) = 0 \vee (N^u \subseteq O_\alpha \wedge N^{u^-} \not\subseteq O_\alpha))\}$ . Note that  $P$  is  $\Pi_1^1$ , since a nonempty  $\Delta_1^1(\alpha)$  closed subset of  $2^\omega$  contains a  $\Delta_1^1(\alpha)$  point, by 4F.15 in [M]. We then define a relation  $R$  on  $\omega^\omega \times 2^\omega \times \omega$  by  $R(\alpha, \beta, u) \Leftrightarrow P(\alpha, u) \wedge \beta \in N^u$ , so that  $R$  is  $\Pi_1^1$ . Note that, for each  $(\alpha, \beta) \in O$  there is  $u$  with  $R(\alpha, \beta, u)$ . By 4B.5 in [M], there is a  $\Delta_1^1$ -recursive map  $g: \omega^\omega \times 2^\omega \rightarrow \omega$  such that  $R(\alpha, \beta, g(\alpha, \beta))$  for each  $(\alpha, \beta) \in O$ . Fix  $\alpha \in \omega^\omega$ . Note that  $S^\alpha := \{g(\alpha, \beta) \mid \beta \in O_\alpha\}$  is a  $\Sigma_1^1(\alpha)$  subset of  $\omega$  contained in the  $\Pi_1^1(\alpha)$  set  $P_\alpha$ . By 4B.11 and 4C in [M], there is  $D^\alpha \in \Delta_1^1(\alpha)$  with  $S^\alpha \subseteq D^\alpha \subseteq P_\alpha$ . Note that  $O_\alpha \subseteq \bigcup_{u \in D^\alpha} N^u \subseteq O_\alpha$ , so that  $O_\alpha$  is the disjoint union of the sequence  $(N^u)_{u \in D^\alpha}$ . We define  $\delta_\alpha \in \omega^\omega$  by

$$\delta_\alpha(u) := \begin{cases} u & \text{if } u \in D_\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Note that  $\delta_\alpha \in \Delta_1^1(\alpha)$  and  $O_\alpha$  is the disjoint union  $\bigcup \{N^{\delta_\alpha(u)} \mid u \in \omega \wedge \text{Seq}(\delta_\alpha(u))\}$ . As the set  $\{(\alpha, \delta) \in \omega^\omega \times \omega^\omega \mid \delta \in \Delta_1^1(\alpha) \wedge O_\alpha \text{ is the disjoint union } \bigcup \{N^{\delta(u)} \mid u \in \omega \wedge \text{Seq}(\delta(u))\}\}$  is  $\Pi_1^1$ , it remains to apply the uniformization lemma to get the desired map  $f$ . □

**Notation.** We set  $\mathcal{W}_1 := \{n \in \mathcal{W} \mid \forall \alpha \in \omega^\omega \exists \gamma_n \in \Delta_1^1(\alpha) \mathcal{C}_{n,\alpha} = \bigcup \{N^{\gamma_n(u)} \mid u \in \omega \wedge \text{Seq}(\gamma_n(u))\}$ , so that, by Lemma 3.7,  $\mathcal{W}_1$  is a  $\Pi_1^1$  set of codes for the  $\Delta_1^1$  subsets of  $\omega^\omega \times 2^\omega$  with open vertical sections.

**Lemma 3.8** *Let  $F$  be a  $\Delta_1^1$  subset of  $\omega^\omega \times 2^\omega$  with closed vertical sections, and  $B$  be a  $\Delta_1^1$  subset of  $\omega^\omega \times 2^\omega$  such that  $B \supseteq F$  and  $d(B_\alpha, \beta) = 1$  for each  $(\alpha, \beta) \in F$ . Then there is a  $\Delta_1^1$  subset  $C$  of  $\omega^\omega \times 2^\omega$  with closed vertical sections such that*

- (1)  $F \subseteq C \subseteq B$ ,
- (2)  $d(B_\alpha, \beta) = 1$  for each  $(\alpha, \beta) \in C$ ,
- (3)  $d(C_\alpha, \beta) = 1$  for each  $(\alpha, \beta) \in F$ .

**Proof.** Lemma 3.7 gives a  $\Delta_1^1$ -recursive map  $f : \omega^\omega \rightarrow \omega^\omega$  such that  $(\neg F)_\alpha$  is the disjoint union  $\bigcup \{N^{f(\alpha)(u)} \mid u \in \omega \wedge \text{Seq}(f(\alpha)(u))\}$ , for each  $\alpha \in \omega^\omega$ . We set

$$B' := \left\{ (\alpha, \gamma) \in \omega^\omega \times 2^\omega \mid ((\alpha)_0, \gamma) \in B \wedge \text{Seq}\left(f((\alpha)_0)((\alpha)_1(0))\right) \wedge \gamma \in N^{f((\alpha)_0)((\alpha)_1(0))} \right\},$$

so that  $B'$  is  $\Delta_1^1$  and  $B_\alpha \cap N^{f(\alpha)(u)} = B'_{\langle \alpha, u^\infty \rangle}$  if  $\text{Seq}(f(\alpha)(u))$ . By Corollary 3.5.(c), the partial map  $(\alpha, \beta, u) \mapsto d(B_\alpha \cap N^{f(\alpha)(u)}, \beta)$  is  $\Delta_1^1$ -recursive. We then set

$$B'' := \{(\alpha, \gamma) \in B' \mid d(B_{(\alpha)_0} \cap N^{f((\alpha)_0)((\alpha)_1(0))}, \gamma) = 1\},$$

so that  $B''$  is  $\Delta_1^1$  and  $\{\beta \in B_\alpha \cap N^{f(\alpha)(u)} \mid d(B_\alpha \cap N^{f(\alpha)(u)}, \beta) = 1\} = B''_{\langle \alpha, u^\infty \rangle}$  if  $\text{Seq}(f(\alpha)(u))$ . We define  $\epsilon : \omega^\omega \rightarrow \mathbb{R}$  by

$$\epsilon(\alpha) := \begin{cases} 2^{-(\alpha)_1(0)} \lambda(B'_\alpha) & \text{if } \lambda(B'_\alpha) \neq 0, \\ 1 & \text{otherwise,} \end{cases}$$

so that  $\epsilon$  is  $\Delta_1^1$ -recursive by Corollary 3.5.(a), and  $\epsilon(\alpha) \in (0, 1]$  for each  $\alpha \in \omega^\omega$ . Theorem 3.6 gives  $T \in \Delta_1^1(\omega^\omega \times \omega)$  such that

- (a)  $T_\alpha$  is a tree for each  $\alpha \in \omega^\omega$ ,
- (b) if  $K = \{(\alpha, \beta) \in \omega^\omega \times 2^\omega \mid \forall l \in \omega \ (\alpha, \bar{\beta}(l)) \in T\}$ , then  $K_\alpha \subseteq B''_\alpha$  and  $\lambda(K_\alpha) \geq \lambda(B''_\alpha) - \epsilon(\alpha)$  for each  $\alpha \in \omega^\omega$ .

We set, for  $u \in \omega$ ,

$$F^u := \{(\alpha, \beta) \in \omega^\omega \times 2^\omega \mid \text{Seq}(f(\alpha)(u)) \wedge (\langle \alpha, u^\infty \rangle, \beta) \in K \wedge \lambda(B'_{\langle \alpha, u^\infty \rangle}) \neq 0\}.$$

As  $K$  is  $\Delta_1^1$  with closed vertical sections, so is  $F^u$ . If  $\text{Seq}(f(\alpha)(u))$  and  $\lambda(B'_{\langle \alpha, u^\infty \rangle}) = 0$ , then  $\lambda(B_\alpha \cap N^{f(\alpha)(u)}) = 0$  and  $F_\alpha^u = \emptyset$ , so that  $F_\alpha^u \subseteq \{\beta \in B_\alpha \cap N^{f(\alpha)(u)} \mid d(B_\alpha \cap N^{f(\alpha)(u)}, \beta) = 1\}$  and  $\lambda(F_\alpha^u) \geq (1 - 2^{-u}) \lambda(B_\alpha \cap N^{f(\alpha)(u)})$ . If  $\text{Seq}(f(\alpha)(u))$  and  $\lambda(B'_{\langle \alpha, u^\infty \rangle}) \neq 0$ , then

$$F_\alpha^u = K_{\langle \alpha, u^\infty \rangle} \subseteq B''_{\langle \alpha, u^\infty \rangle} = \{\beta \in B_\alpha \cap N^{f(\alpha)(u)} \mid d(B_\alpha \cap N^{f(\alpha)(u)}, \beta) = 1\}.$$



Moreover,

$$\begin{aligned}\lambda(F_\alpha^u) &= \lambda(K_{\langle \alpha, u^\infty \rangle}) \geq \lambda(B''_{\langle \alpha, u^\infty \rangle}) - \epsilon(\langle \alpha, u^\infty \rangle) = \lambda(B''_{\langle \alpha, u^\infty \rangle}) - 2^{-u} \lambda(B'_{\langle \alpha, u^\infty \rangle}) \\ &= (1 - 2^{-u}) \lambda(B_\alpha \cap N^{f(\alpha)(u)})\end{aligned}$$

since  $\lambda(B_\alpha \cap N^{f(\alpha)(u)}) = \lambda(\{\beta \in B_\alpha \cap N^{f(\alpha)(u)} \mid d(B_\alpha \cap N^{f(\alpha)(u)}, \beta) = 1\})$ , by Theorem 2.1. It remains to set  $C := F \cup \bigcup_{u \in \omega} F^u$ . We conclude as in the proof of Lemma 2.2.  $\square$

- We now want to prove an effective and uniform version of Lemma 2.5.

**Lemma 3.9** *Let  $C$  be a  $\Delta_1^1$  subset of  $\omega^\omega \times 2^\omega$  with closed vertical sections,  $\mathcal{G}$  be a Borel subset of  $2^\omega$  with  $\lambda(\mathcal{G}) = 0$ , and  $G$  be a  $\Delta_1^1$  subset of  $\omega^\omega \times 2^\omega$  with  $G_\delta$  vertical sections, contained in  $\omega^\omega \times \mathcal{G}$  and disjoint from  $C$ . Then there is a  $\Delta_1^1$ -recursive map  $h : \omega^\omega \times 2^\omega \rightarrow \mathbb{R}$  such that  $h(\alpha, \cdot) : 2^\omega \rightarrow [0, 1]$  is  $\tau$ -continuous for each  $\alpha \in \omega^\omega$ ,  $h|_C \equiv 0$  and  $h|_G \equiv 1$ .*

**Proof.** By Theorem 3.5 in [L], there is a  $\Delta_1^1$  subset  $F$  of  $\omega \times \omega^\omega \times 2^\omega$  such that  $F_{n,\alpha}$  is closed for each  $(n, \alpha) \in \omega \times \omega^\omega$  and  $\neg G = \bigcup_{n \in \omega} F_n$ . Moreover, we may assume that  $(F_n)_{n \in \omega}$  is increasing and  $F_0 = C$ .

• We will define, by primitive recursion, a partial map  $f : \omega \rightarrow \omega$  which is  $\Pi_1^1$ -recursive on its domain such that  $f(n)$  essentially codes the set  $C_{\frac{1}{2^n}}$  constructed in the proof of Lemma 2.5. As this map will in fact be total, it will be  $\Delta_1^1$ -recursive by the uniformization lemma.

We first apply Lemma 3.8 to  $F := F_0$  and  $B := \neg G$ . This is possible because  $G_\alpha \subseteq \mathcal{G}$ , so that  $(\neg G)_\alpha$  has  $\lambda$ -measure one and therefore density one at any point of  $2^\omega$ , for each  $\alpha \in \omega^\omega$ . Lemma 3.8 gives  $C_1 \in \Delta_1^1$  with closed vertical sections such that  $\neg G \supseteq C_1 \supseteq F_0$ . Let  $f(0) \in \mathcal{W}_1$  with  $\mathcal{C}_{f(0)} = \neg C_1$ .

More generally, we will have  $\mathcal{C}_{f(n)} = \neg C_{\frac{1}{2^n}}$ . As mentioned above,  $f$  will be defined by primitive recursion, which means that there will be a partial map  $g : \omega^2 \rightarrow \omega$  such that  $f(n+1) = g(f(n), n)$ . This partial map  $g$  will be  $\Pi_1^1$ -recursive on its  $\Pi_1^1$  domain  $\{m \in \mathcal{W}_1 \mid \neg C_m \subseteq \neg G\} \times \omega$ , so that  $f$  will be  $\Pi_1^1$ -recursive on its domain by 7A.5 in [M]. The map  $g$  will take values in  $\mathcal{W}_1$ , and is constructed in such a way that, if  $A := \neg C_m \subseteq \neg G$  and  $A' := \neg C_{g(m,n)}$ , then

- (1)  $A \cup F_{n+1} \subseteq A' \subseteq \neg G$ ,
- (2)  $\forall (\alpha, \beta) \in A' \quad d((\neg G)_\alpha, \beta) = 1$ ,
- (3)  $\forall (\alpha, \beta) \in A \cup F_{n+1} \quad d(A'_\alpha, \beta) = 1$ .

Lemma 3.8 ensures that such a  $g(m, n) \in \omega$  exists if  $(m, n) \in \{q \in \mathcal{W}_1 \mid \neg C_q \subseteq \neg G\} \times \omega$ . As the properties (1)-(3) are  $\Pi_1^1$  by Corollary 3.5, the uniformization lemma ensures the existence of  $g$ . So we constructed a  $\Delta_1^1$ -recursive map  $f : \omega \rightarrow \omega$ , taking values in  $\mathcal{W}_1$ , such that  $C_{\frac{1}{2^n}} := \neg \mathcal{C}_{f(n)}$  is a  $\Delta_1^1$  subset of  $\omega^\omega \times 2^\omega$  with closed vertical sections,  $F_n \subseteq C_{\frac{1}{2^n}} \subseteq \neg G$ ,  $C_{\frac{1}{2^n}} \subseteq C_{\frac{1}{2^{n+1}}}$ , and

$$d((C_{\frac{1}{2^{n+1}}})_\alpha, \beta) = 1$$

if  $(\alpha, \beta) \in C_{\frac{1}{2^n}}$ .

- Similarly, we construct a  $\Delta_1^1$ -recursive map  $\tilde{F} : \omega \rightarrow \omega$  satisfying the following properties, if

$$D := \{p \in \omega \mid \text{Seq}(p) \wedge \text{lh}(p) = 2 \wedge 0 < (p)_1 \leq 2^{(p)_0}\}.$$

- (a)  $\tilde{F}(p) \in \mathcal{W}_1$  if  $p \in D$ , in which case we set  $C_p := \neg \mathcal{C}_{\tilde{F}(p)}$ ,
- (b)  $C_p \subseteq C_{p'}$  if  $p, p' \in D \wedge \frac{(p)_1}{2^{(p)_0}} \leq \frac{(p')_1}{2^{(p')_0}}$ ,
- (c)  $d((C_{p'})_\alpha, \beta) = 1$  if  $p, p' \in D \wedge \frac{(p)_1}{2^{(p)_0}} < \frac{(p')_1}{2^{(p')_0}} \wedge (\alpha, \beta) \in C_p$ .

- This allows us to define  $h$  by

$$1 - h(\alpha, \beta) := \begin{cases} 0 & \text{if } (\alpha, \beta) \in G, \\ \sup\{\frac{(p)_1}{2^{(p)_0}} \mid p \in D \wedge (\alpha, \beta) \in C_p\} & \text{if } (\alpha, \beta) \notin G. \end{cases}$$

Note that  $h$  is  $\Delta_1^1$ -recursive since  $D \in \Delta_1^0$ , so that the relation “ $p \in D \wedge (\alpha, \beta) \in C_p$ ” is  $\Delta_1^1$  in  $(p, \alpha, \beta)$ . We conclude as in the proof of Lemma 2.5.  $\square$

- We are now ready to prove the main lemma in this section. We equip the space  $[0, 1]^{2^{<\omega}}$  with the distance defined by  $d(f, g) := \sum_{u \in \omega, \text{Seq}(u)} \frac{|f(s(u)) - g(s(u))|}{2^{u+1}}$ . We give a recursive presentation of  $([0, 1]^{2^{<\omega}}, d)$ . We set

$$f_n(s) := \begin{cases} \frac{\binom{(n)_\bar{s}}{0}}{\binom{(n)_\bar{s}}{0} + \binom{(n)_\bar{s}}{1} + 1} & \text{if } \text{Seq}(n) \wedge \forall k < \text{lh}(n) \left( \text{Seq}((n)_k) \wedge \text{lh}((n)_k) = 2 \right) \wedge \bar{s} < \text{lh}(n), \\ 0 & \text{otherwise,} \end{cases}$$

so that  $(f_n)$  is dense in  $[0, 1]^{2^{<\omega}}$ . It is now routine to check that the relations “ $d(f_m, f_n) \leq \frac{p}{q+1}$ ” and “ $d(f_m, f_n) < \frac{p}{q+1}$ ” are recursive in  $(m, n, p, q)$ . It is also routine to check that  $F : \omega^\omega \rightarrow [0, 1]^{2^{<\omega}}$  is  $\Delta_1^1$ -recursive if the map  $F' : \omega \times \omega^\omega \rightarrow \mathbb{R}$  defined by  $F'(u, \alpha) := F(\alpha)(s(u))$  if  $\text{Seq}(u)$ , 0 otherwise, is  $\Delta_1^1$ -recursive ( $s(u)$  was defined at the beginning of Section 3).

**Lemma 3.10** *Let  $\mathcal{V} := \{(f, \beta) \in \mathcal{M} \times 2^\omega \mid \text{osc}(f, \beta) > 0\}$ ,  $\mathcal{G}$  be a nonempty  $G_\delta \cap \Delta_1^1$  subset of  $2^\omega$  with  $\lambda(\mathcal{G}) = 0$ , and  $G$  be a  $\Delta_1^1$  subset of  $\omega^\omega \times 2^\omega$ , contained in  $\omega^\omega \times \mathcal{G}$ , and with  $G_\delta$  vertical sections. Then there is a  $\Delta_1^1$ -recursive map  $F : \omega^\omega \rightarrow [0, 1]^{2^{<\omega}}$ , taking values in  $\mathcal{M}$ , and such that  $G_\alpha = \mathcal{V}_{F(\alpha)}$  for each  $\alpha \in \omega^\omega$ .*

**Proof.** We will define, by primitive recursion,  $f : \omega \rightarrow \omega^4$  coding  $g_n, S_n, G_n^*$ , and  $(s_j^n)_{j \in I_n}$  defining  $G_n^{**}$  considered in the proof of the Lemma 2.7. We must find  $r : \omega^4 \times \omega \rightarrow \omega^4$  with  $f(n+1) = r(f(n), n)$ . In practice,

- (1)  $f_0(n) \in \mathcal{W}_1$  codes  $G_n^* \subseteq \omega^\omega \times 2^\omega$ ,
- (2)  $f_1(n) \in \mathcal{W}^{\omega^\omega \times 2^\omega \times \mathbb{R}}$  codes the graph of  $g_n : \omega^\omega \times 2^\omega \rightarrow \mathbb{R}$ ,
- (3)  $f_2(n) \in \mathcal{W}^{\omega^\omega \times 2^\omega \times \mathbb{R}}$  codes the graph of  $S_n : \omega^\omega \times 2^\omega \rightarrow \mathbb{R}$ ,
- (4)  $f_3(n) \in \mathcal{W}^{\omega^\omega \times \omega^\omega}$  codes the graph of the function  $\alpha \mapsto (s_j^{n, \alpha})_{j \in I_{n, \alpha}}$ .

- By Theorem 3.5 in [L], there is a  $\Delta_1^1$  subset  $O$  of  $\omega \times \omega^\omega \times 2^\omega$  such that  $O_{n, \alpha}$  is open for each  $(n, \alpha) \in \omega \times \omega^\omega$  and  $G = \bigcap_{n \in \omega} O_n$ . Moreover, we may assume that  $(O_n)_{n \in \omega}$  is decreasing and  $O_0 = \omega^\omega \times 2^\omega$ .

• Let  $n_0 \in \mathcal{W}_1$  with  $\mathcal{C}_{n_0} = \omega^\omega \times 2^\omega$ ,  $n_1 \in \mathcal{W}^{\omega^\omega \times 2^\omega \times \mathbb{R}}$  with  $\mathcal{C}_{n_1}^{\omega^\omega \times 2^\omega \times \mathbb{R}} = \{(\alpha, \beta, r) \in \omega^\omega \times 2^\omega \times \mathbb{R} \mid r = 1\}$ , and  $n_3 \in \mathcal{W}^{\omega^\omega \times \omega^\omega}$  with  $\mathcal{C}_{n_3}^{\omega^\omega \times \omega^\omega} = \{(\alpha, \gamma) \in \omega^\omega \times \omega^\omega \mid \gamma = 10^\infty\}$ . We set  $f(0) := (n_0, n_1, n_1, n_3)$ , so that  $\mathcal{C}_{n_0} = G_0^*$ ,  $\mathcal{C}_{n_1}^{\omega^\omega \times 2^\omega \times \mathbb{R}} = \text{Gr}(g_0) = \text{Gr}(S_0)$ ,  $\mathcal{C}_{n_3}^{\omega^\omega \times \omega^\omega} = \text{Gr}(\alpha \mapsto 10^\infty)$ ,

$$\{u \in \omega \mid \text{Seq}((10^\infty)(u))\} = \{0\} = I_0$$

and  $(10^\infty)(0) = 1 = \langle \rangle = s_0^0$ . So  $f(0)$  is as desired.

• We now study the induction step. This means that we must define  $r(n_0, n_1, n_2, n_3, n) \in \omega^4$ .

(1) We first define  $r_0(n_0, n_1, n_2, n_3, n)$  coding  $G_{n+1}^*$ . Fix  $n_3 \in \mathcal{W}^{\omega^\omega \times \omega^\omega}$  coding the graph of a  $\Delta_1^1$ -recursive function  $\phi : \omega^\omega \rightarrow \omega^\omega$  such that the sequences  $s(\phi(\alpha)(u))$  coded by the  $u$ 's with  $\text{Seq}(\phi(\alpha)(u))$  are pairwise incompatible and  $G_\alpha \subseteq \bigcup \{N^{\phi(\alpha)(u)} \mid u \in \omega \wedge \text{Seq}(\phi(\alpha)(u))\}$  (we call  $P_3$  the  $\Pi_1^1$  set of such  $n_3$ 's). Let  $\alpha \in \omega^\omega$ . Assume that  $\text{Seq}(\phi(\alpha)(u))$  (which intuitively means that  $u \in I_{n,\alpha}$  and  $s_u^{n,\alpha}$  is coded by  $\phi(\alpha)(u)$ ). By continuity of  $\lambda$ ,

$$0 = \lambda(G_\alpha \cap N^{\phi(\alpha)(u)}) = \lim_{j \rightarrow \infty} \lambda(O_{j,\alpha} \cap N^{\phi(\alpha)(u)}).$$

This gives  $j(n, \alpha, u) > n$  minimal with  $\lambda(O_{j(n,\alpha,u),\alpha} \cap N^{\phi(\alpha)(u)}) < 2^{-n-3-\text{lh}(\phi(\alpha)(u))}$  (note that  $2^{-\text{lh}(\phi(\alpha)(u))} = \lambda(N^{\phi(\alpha)(u)})$ ). Moreover,  $G_\alpha \cap N^{\phi(\alpha)(u)} \subseteq O_{j(n,\alpha,u),\alpha} \cap N^{\phi(\alpha)(u)} \subseteq O_{n+1,\alpha} \cap N^{\phi(\alpha)(u)}$ , so that  $O_{j(n,\alpha,u),\alpha} \cap N^{\phi(\alpha)(u)}$  satisfies the properties of the set  $O_j$  in the proof of Lemma 2.7. We will have  $G_{n+1,\alpha}^* = \bigcup_{\text{Seq}(\phi(\alpha)(u))} O_{j(n,\alpha,u),\alpha} \cap N^{\phi(\alpha)(u)}$ . By Corollary 3.5 and the uniformization lemma, we may assume that the map  $j$  is  $\Delta_1^1$ -recursive on its  $\Delta_1^1$  domain

$$\{(n, \alpha, u) \in \omega \times \omega^\omega \times \omega \mid \text{Seq}(\phi(\alpha)(u))\}.$$

Note that  $G_{n+1}^*$  is a  $\Delta_1^1$  subset of  $\omega^\omega \times 2^\omega$  with open vertical sections, which gives  $m \in \mathcal{W}_1$  such that  $\mathcal{C}_m = G_{n+1}^*$ . By incompatibility,  $G_{n+1,\alpha}^* \cap N^{\phi(\alpha)(u)} = O_{j(n,\alpha,u),\alpha} \cap N^{\phi(\alpha)(u)}$ . So we proved that, for each  $(n_3, n) \in P_3 \times \omega$ , there is  $m \in \mathcal{W}_1$  such that, for each  $\alpha \in \omega^\omega$ ,

- (1)  $G_\alpha \subseteq \mathcal{C}_{m,\alpha} \subseteq O_{n+1,\alpha} \cap \bigcup \{N^{\phi(\alpha)(u)} \mid u \in \omega \wedge \text{Seq}(\phi(\alpha)(u))\}$ ,
- (5)  $\lambda(\mathcal{C}_{m,\alpha} \cap N^{\phi(\alpha)(u)}) < 2^{-n-3-\text{lh}(\phi(\alpha)(u))}$  if  $u \in \omega \wedge \text{Seq}(\phi(\alpha)(u))$ .

By Corollary 3.5 and the uniformization lemma, we may assume that the map  $\tilde{r}_0 : (n_3, n) \mapsto m$  is  $\Pi_1^1$ -recursive on  $P_3 \times \omega$ . We set  $r_0(n_0, n_1, n_2, n_3, n) := \tilde{r}_0(n_3, n)$ , which defines a partial map  $r_0$  which is  $\Pi_1^1$ -recursive on its  $\Pi_1^1$  domain  $\omega^3 \times P_3 \times \omega$ .

(2) We now define  $r_1(n_0, n_1, n_2, n_3, n)$  coding  $g_{n+1}$ . We use Lemma 3.9 and its proof. Note that  $r_0(n_0, n_1, n_2, n_3, n) \in D_0 := \{m \in \mathcal{W}_1 \mid G \subseteq \mathcal{C}_m\}$ . The proof of Lemma 3.9 shows that for any  $m \in D_0$  there is  $\tilde{F}_m \in \omega^\omega \cap \Delta_1^1$  satisfying the conditions (a), (b), (c) and

$$(d) \forall p \in D \neg(0 < (p)_1 = 2^{(p)_0}) \vee \mathcal{C}_{\tilde{F}_m(p)} \subseteq \mathcal{C}_m.$$

The uniformization lemma shows that we may assume that the partial map  $\tilde{F} : m \mapsto \tilde{F}_m$  is  $\Pi_1^1$ -recursive on  $D_0$ .

The definition of  $h$  in the proof of Lemma 3.9 and the uniformization lemma show the existence of a partial map  $\tilde{H} : \omega \rightarrow \omega$ , which is  $\Pi_1^1$ -recursive on  $D_0$ , and such that  $\tilde{H}(m)$  is in  $\mathcal{W}^{\omega^\omega \times 2^\omega \times \mathbb{R}}$  and codes the graph of a  $\Delta_1^1$ -recursive map  $h : \omega^\omega \times 2^\omega \rightarrow \mathbb{R}$  with

$$1-h(\alpha, \beta) := \begin{cases} 0 & \text{if } (\alpha, \beta) \in G \\ \sup\left\{\frac{(p)_1}{2^{(p)_0}} \mid p \in D \wedge (\alpha, \beta) \notin \mathcal{C}_{\tilde{H}(m)(p)}\right\} & \text{if } (\alpha, \beta) \notin G \end{cases}$$

if  $m \in D_0$ . We set  $P_1 := \{c \in \mathcal{W}^{\omega^\omega \times 2^\omega \times \mathbb{R}} \mid \mathcal{C}_c \text{ is the graph of a function } \zeta_c\}$ . It is routine to check that there is a  $\Pi_1^1$ -recursive partial map  $I : \omega^2 \rightarrow \omega$  on its domain  $P_1^2$  such that  $I(c, c') \in \mathcal{W}^{\omega^\omega \times 2^\omega \times \mathbb{R}}$  is the graph of the function  $\min(\zeta_c, \zeta_{c'})$  if  $c, c' \in P_1$ . We set

$$r_1(n_0, n_1, n_2, n_3, n) := I\left(n_1, \tilde{H}(r_0(n_0, n_1, n_2, n_3, n))\right),$$

so that  $r_1$  is  $\Pi_1^1$ -recursive on its  $\Pi_1^1$  domain  $\omega \times P_1 \times \omega \times P_3 \times \omega$ .

(3) We now define  $r_2(n_0, n_1, n_2, n_3, n)$  coding

$$S_{n+1} = \begin{cases} S_n + g_{n+1} & \text{if } n \text{ is odd,} \\ S_n - g_{n+1} & \text{if } n \text{ is even.} \end{cases}$$

It is routine to check that there is a  $\Pi_1^1$ -recursive partial map  $S : \omega^3 \rightarrow \omega$  on its domain  $P_1^2 \times \omega$  such that  $S(c, c', n) \in \mathcal{W}^{\omega^\omega \times 2^\omega \times \mathbb{R}}$  codes the graph of the function

$$(\alpha, \beta) \mapsto \begin{cases} \zeta_c(\alpha, \beta) + \zeta_{c'}(\alpha, \beta) & \text{if } n \text{ is odd} \\ \zeta_c(\alpha, \beta) - \zeta_{c'}(\alpha, \beta) & \text{if } n \text{ is even} \end{cases}$$

if  $(c, c', n) \in P_1^2 \times \omega$ . We set  $r_2(n_0, n_1, n_2, n_3, n) := S(n_2, r_1(n_0, n_1, n_2, n_3, n), n)$ , so that  $r_2$  is  $\Pi_1^1$ -recursive on its  $\Pi_1^1$  domain  $\omega \times P_1^2 \times P_3 \times \omega$ .

(4) We now define  $r_3(n_0, n_1, n_2, n_3, n)$  coding the graph of the function  $\alpha \mapsto (s_j^{n+1, \alpha})_{j \in I_{n+1, \alpha}}$ . We want to ensure the two following conditions:

- (1)  $G_\alpha \subseteq \bigcup_{j \in I_{n+1, \alpha}} N_{s_j^{n+1, \alpha}} \subseteq G_{n+1, \alpha}^*$
- (6)  $\left| \int_{N_{s_j^{n+1, \alpha}}} S_{n+1}(\alpha, \cdot) d\lambda - S_{n+1}(\alpha, \beta) \right| < 2^{-3}$  if  $j \in I_{n+1, \alpha} \wedge \beta \in G_\alpha \cap N_{s_j^{n+1, \alpha}}$

Note first that in practice

$$S_{n+1}(\alpha, \beta) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$$

if  $(\alpha, \beta) \in G$  since  $g_p(\alpha, \beta) = 1$  for each  $p$  in this case. So there is  $\psi : \omega \rightarrow \mathbb{R}^2$  recursive with

$$\left| \int_{N_{s_j^{n+1, \alpha}}} S_{n+1}(\alpha, \cdot) d\lambda - S_{n+1}(\alpha, \beta) \right| < 2^{-3} \Leftrightarrow \psi_0(n) < \int_{N_{s_j^{n+1, \alpha}}} S_{n+1}(\alpha, \cdot) d\lambda < \psi_1(n)$$

if  $(\alpha, \beta) \in G$ . We use Corollary 3.5 and its proof. Note that  $r_2(n_0, n_1, n_2, n_3, n) \in P_1$ .

We first consider  $n'_0 \in \mathcal{W}_1$  and  $n'_2 \in P_1$  (coding  $G_{n+1}^*$  and  $S_{n+1}$  respectively) as variables. We define  $R_0, R_1 \subseteq \omega \times \omega^\omega \times 2^\omega \times \omega^3$  by

$$R_0(n'_2, \alpha, \beta, u, k, l) \Leftrightarrow \exists r \in \mathbb{R} \neg (n'_2 \in \mathcal{W}^{\omega^\omega \times 2^\omega \times \mathbb{R}} \wedge (n'_2, \alpha, \beta, r) \notin \mathcal{C}^{\omega^\omega \times 2^\omega \times \mathbb{R}}) \wedge \left( \frac{k}{2^l} \leq r < \frac{k+1}{2^l} \wedge \text{Seq}(u) \wedge \beta \in N^u \right)$$

$$R_1(n'_2, \alpha, \beta, u, k, l) \Leftrightarrow \forall r \in \mathbb{R} (n'_2 \in \mathcal{W}^{\omega^\omega \times 2^\omega \times \mathbb{R}} \wedge (n'_2, \alpha, \beta, r) \notin \mathcal{C}^{\omega^\omega \times 2^\omega \times \mathbb{R}}) \vee \left( \frac{k}{2^l} \leq r < \frac{k+1}{2^l} \wedge \text{Seq}(u) \wedge \beta \in N^u \right),$$

so that  $R_0$  is  $\Sigma_1^1$ ,  $R_1$  is  $\Pi_1^1$ , and  $R_0(n'_2, \alpha, \beta, u, k, l) \Leftrightarrow R_1(n'_2, \alpha, \beta, u, k, l)$  if  $n'_2 \in P_1$ . Then, as in the proof of Corollary 3.5.(d), we define  $O_0, O_1 \subseteq \omega \times \omega^\omega \times 2^\omega$  by

$$O_\varepsilon(n'_2, \alpha, \beta) \Leftrightarrow \text{Seq}(\alpha(0)) \wedge \text{lh}(\alpha(0)) = 3 \wedge R_\varepsilon(n'_2, \alpha^*, \beta, (\alpha(0))_0, (\alpha(0))_1, (\alpha(0))_2)$$

if  $\varepsilon \in 2$ , so that  $O_0$  is  $\Sigma_1^1$ ,  $O_1$  is  $\Pi_1^1$ , and  $O_0(n'_2, \alpha, \beta) \Leftrightarrow O_1(n'_2, \alpha, \beta)$  if  $n'_2 \in P_1$ . In particular,  $n'_2 \in P_1$  and  $\text{Seq}(u)$  imply that

$$\int_{N^u} S_{n+1}(\alpha, \cdot) d\lambda = \lim_{l \rightarrow \infty} \sum_{k \leq 2^l} \frac{k}{2^l} \lambda((O_\varepsilon)_{n'_2, \langle u, k, l \rangle} > \alpha)$$

for each  $\varepsilon \in 2$ . Thus  $a < \int_{N^u} S_{n+1}(\alpha, \cdot) d\lambda < b$  is in this case equivalent to

$$\exists p_0, p_1, q_0, q_1, N \in \omega \ a < \frac{p_0}{p_1+1} \wedge \frac{q_0}{q_1+1} < b \wedge \forall l \geq N \ \frac{p_0}{p_1+1} \leq \sum_{k \leq 2^l} \frac{k}{2^l} \lambda((O_\varepsilon)_{n'_2, \langle u, k, l \rangle} > \alpha) \leq \frac{q_0}{q_1+1}.$$

By Corollary 3.5.(b) applied to  $D := P_1$ , the partial map  $\lambda_O : P_1 \times \omega^\omega \rightarrow \mathbb{R}$  defined by

$$\lambda_O(n'_2, \alpha) := \lambda((O_0)_{n'_2, \alpha})$$

is  $\Sigma_1^1$ -recursive and  $\Pi_1^1$ -recursive on its domain. By 3E.2, 3G.1 and 3G.2 in [M], these two classes of functions are closed under composition. In particular, the partial map

$$(n'_2, \alpha, u, l) \mapsto \sum_{k \leq 2^l} \frac{k}{2^l} \lambda((O_\varepsilon)_{n'_2, \langle u, k, l \rangle} > \alpha)$$

is  $\Sigma_1^1$ -recursive and  $\Pi_1^1$ -recursive on  $P_1 \times \omega^\omega \times \omega^2$ . This shows the existence of  $Q_0 \in \Sigma_1^1(\omega^2 \times \omega^\omega \times \omega)$  and  $Q_1 \in \Pi_1^1(\omega^2 \times \omega^\omega \times \omega)$  such that

$$Q_0(n'_2, n, \alpha, u) \Leftrightarrow Q_1(n'_2, n, \alpha, u) \Leftrightarrow \text{Seq}(u) \wedge \psi_0(n) < \int_{N^u} S_{n+1}(\alpha, \cdot) d\lambda < \psi_1(n)$$

if  $n'_2 \in P_1$ . We now consider  $n'_0 \in \mathcal{W}_1$  and  $n'_2 \in P_1$  as parameters. We set

$$P_{n'_0, n'_2}(n, \alpha, u) \Leftrightarrow Q_1(n'_2, n, \alpha, u) \wedge N^u \subseteq \mathcal{C}_{n'_0, \alpha} \wedge \forall k < \text{lh}(u) \ (-Q_0(n'_2, n, \alpha, \underline{u}(k)) \vee N^{\underline{u}(k)} \not\subseteq \mathcal{C}_{n'_0, \alpha}).$$

Note that for each  $(\alpha, \beta) \in G$  there is  $l \in \omega$  minimal with the properties that  $N_{\beta|l} \subseteq \mathcal{C}_{n'_0, \alpha}$  and  $Q_1(n'_2, n, \alpha, \langle \beta(0), \dots, \beta(l-1) \rangle)$ , so that  $P_{n'_0, n'_2}(n, \alpha, \langle \beta(0), \dots, \beta(l-1) \rangle)$  since  $n'_0 \in \mathcal{W}_1$  and  $n'_2 \in P_1$ . As  $n'_0 \in \mathcal{W}_1$ ,  $N^{\underline{u}(k)} \setminus \mathcal{C}_{n'_0, \alpha}$  is a  $\Delta_1^1(\alpha)$  compact subset of  $2^\omega$ , so that it contains a  $\Delta_1^1(\alpha)$  point if it is not empty (see 4F.15 in [M]). This shows that  $P_{n'_0, n'_2}$  is  $\Pi_1^1$ .

The uniformization lemma provides a  $\Delta_1^1$ -recursive map  $L : \omega \times \omega^\omega \times 2^\omega \rightarrow \omega$  such that

$$P_{n'_0, n'_2} \left( n, \alpha, \langle \beta(0), \dots, \beta(L(n, \alpha, \beta) - 1) \rangle \right)$$

if  $(\alpha, \beta) \in G$ . Note that the  $\Sigma_1^1$  set

$$\sigma := \{ (n, \alpha, u) \in \omega \times \omega^\omega \times \omega \mid \exists \beta \in G_\alpha \ u = \langle \beta(0), \dots, \beta(L(n, \alpha, \beta) - 1) \rangle \}$$

is contained in the  $\Pi_1^1$  set  $\pi := \{ (n, \alpha, u) \in \omega \times \omega^\omega \times \omega \mid P_{n'_0, n'_2}(n, \alpha, u) \}$ . By 7B.3 in [M], there is a  $\Delta_1^1$  subset  $\delta$  of  $\omega \times \omega^\omega \times \omega$  such that  $\sigma \subseteq \delta \subseteq \pi$ . We now also consider  $n$  as a parameter and define  $\varphi : \omega^\omega \rightarrow \omega^\omega$  by

$$\varphi(\alpha)(u) := \begin{cases} u & \text{if } (n, \alpha, u) \in \delta, \\ 0 & \text{otherwise.} \end{cases}$$

Note that  $\varphi$  is  $\Delta_1^1$ -recursive, and that  $\text{Seq}(\varphi(\alpha)(u))$  is equivalent to  $(n, \alpha, u) \in \delta$ . In particular,

- (1)  $G_\alpha \subseteq \bigcup \{ N^{\varphi(\alpha)(u)} \mid u \in \omega \wedge \text{Seq}(\varphi(\alpha)(u)) \} \subseteq \mathcal{C}_{n'_0, \alpha}$
- (6)  $\mid f_{N^{\varphi(\alpha)(u)}} S_{n+1}(\alpha, \cdot) \ d\lambda - S_{n+1}(\alpha, \beta) \mid < 2^{-3}$  if  $\text{Seq}(\varphi(\alpha)(u)) \wedge \beta \in G_\alpha \cap N^{\varphi(\alpha)(u)}$

for each  $\alpha \in \omega^\omega$ . Let  $k \in \mathcal{W}^{\omega^\omega \times \omega^\omega}$  such that  $\mathcal{C}_k^{\omega^\omega \times \omega^\omega} = \text{Gr}(\varphi)$ . We now consider  $n'_0, n'_2$  and  $n$  as variables again. Note that for each  $(n'_0, n'_2, n) \in \mathcal{W}_1 \times P_1 \times \omega$  there is  $k \in \omega$  such that

$$R(n'_0, n'_2, n, k) \Leftrightarrow \begin{cases} k \in \mathcal{W}^{\omega^\omega \times \omega^\omega} \wedge \\ \left( \forall \alpha \in \omega^\omega \ \forall \gamma \in \omega^\omega \ (k \in \mathcal{W}^{\omega^\omega \times \omega^\omega} \wedge \neg \mathcal{C}^{\omega^\omega \times \omega^\omega}(k, \alpha, \gamma)) \vee \right. \\ \left. ((1) \ G_\alpha \subseteq \bigcup \{ N^{\gamma(u)} \mid u \in \omega \wedge \text{Seq}(\gamma(u)) \} \subseteq \mathcal{C}_{n'_0, \alpha} \right. \\ \left. \wedge (6) \ \forall u \in \omega \ \neg \text{Seq}(\gamma(u)) \vee Q_1(n'_2, n, \alpha, u) \right) \end{cases}$$

Note that  $R \in \Pi_1^1(\omega^4)$ . The uniformization lemma provides a partial map  $K : \omega^3 \mapsto \omega$  which is  $\Pi_1^1$ -recursive on its  $\Pi_1^1$  domain  $\mathcal{W}_1 \times P_1 \times \omega$ , and  $R(n'_0, n'_2, n, K(n'_0, n'_2, n))$  if

$$(n'_0, n'_2, n) \in \mathcal{W}_1 \times P_1 \times \omega.$$

It remains to set  $r_3(n_0, n_1, n_2, n_3, n) := K(n'_0, n'_2, n)$  if  $n'_0 = r_0(n_0, n_1, n_2, n_3, n)$  and

$$n'_2 = r_2(n_0, n_1, n_2, n_3, n),$$

so that  $r_3$  is  $\Pi_1^1$ -recursive on its  $\Pi_1^1$  domain  $\mathcal{W}_1 \times P_1^2 \times P_3 \times \omega$ .

Finally,  $r$  is  $\Pi_1^1$ -recursive on  $\mathcal{W}_1 \times P_1^2 \times P_3 \times \omega$ ,  $f$  is  $\Pi_1^1$ -recursive on  $\omega$ , and thus  $f$  is  $\Delta_1^1$ -recursive by the uniformization lemma since it is total.

• We are now ready to define the dimension two versions of  $G_n^*$ ,  $g_n$ ,  $S_n$ , and  $(s_j^n)_{j \in I_n}$ :

- (1)  $G_n^* := \mathcal{C}_{f_0(n)}$ ,
- (2)  $g_n(\alpha, \beta) = \rho \Leftrightarrow (f_1(n), \alpha, \beta, \rho) \in \mathcal{C}^{\omega^\omega \times 2^\omega \times \mathbb{R}}$ ,
- (3)  $S_n(\alpha, \beta) = \rho \Leftrightarrow (f_2(n), \alpha, \beta, \rho) \in \mathcal{C}^{\omega^\omega \times 2^\omega \times \mathbb{R}}$ ,
- (4)  $\begin{cases} (i) \ j \in I_{n, \alpha} \Leftrightarrow \exists \delta \in \omega^\omega \ (f_3(n), \alpha, \delta) \in \mathcal{C}^{\omega^\omega \times \omega^\omega} \wedge \text{Seq}(\delta(j)), \\ (ii) \ s_j^{n, \alpha} = \delta(j) \text{ if } j \in I_{n, \alpha}. \end{cases}$

By construction of  $r$ , these objects satisfy the conditions (1)-(6) of the proof of Lemma 2.7.

• Consequently, the martingale  $F(\alpha)$  will be defined in such a way that if  $u \in \omega$  codes  $s \in 2^{<\omega}$ , then  $F(\alpha)(s) = \int_{N^u} f_\infty(\alpha, \cdot) d\lambda$ . Note that  $G = \bigcap_{n \in \omega} G_n^*$ , so that  $\neg G$  is the disjoint union of the  $G_n^* \setminus G_{n+1}^*$ 's. Thus

$$\begin{aligned} \int_{N^u} f_\infty(\alpha, \cdot) d\lambda &= \int_{N^u \setminus G_\alpha} f_\infty(\alpha, \cdot) d\lambda = \sum_{n \in \omega} \int_{N^u \cap (G_n^* \setminus G_{n+1}^*)_\alpha} f_\infty(\alpha, \cdot) d\lambda \\ &= \sum_{n \in \omega} \sum_{j \leq n} (-1)^j \int_{N^u \cap (G_n^* \setminus G_{n+1}^*)_\alpha} g_j(\alpha, \cdot) d\lambda \\ &= \lim_{l \rightarrow \infty} \sum_{n \leq l} \sum_{j \leq n} (-1)^j \int_{N^u \cap (G_n^* \setminus G_{n+1}^*)_\alpha} g_j(\alpha, \cdot) d\lambda. \end{aligned}$$

Consequently, in order to prove that  $F$  is  $\Delta_1^1$ -recursive, it is enough to check that the partial map  $(u, \alpha, j, n) \mapsto \int_{N^u \cap (G_n^* \setminus G_{n+1}^*)_\alpha} g_j(\alpha, \cdot) d\lambda$  is  $\Delta_1^1$ -recursive from  $\{u \in \omega \mid \text{Seq}(u)\} \times \omega^\omega \times \omega^2$  into  $\mathbb{R}$ . By Corollary 3.5, it is enough to check that the map  $h: \omega^\omega \times 2^\omega \rightarrow \mathbb{R}$  defined by

$$h(\alpha, \beta) := \begin{cases} g_{(\alpha(0))_0}(\alpha^*, \beta) & \text{if } \text{Seq}(\alpha(0)) \wedge \text{lh}(\alpha(0)) = 2 \wedge (\alpha^*, \beta) \in G_{(\alpha(0))_1}^* \setminus G_{(\alpha(0))_{1+1}}^*, \\ 0 & \text{otherwise,} \end{cases}$$

is  $\Delta_1^1$ -recursive. This comes from the facts that

$$(\alpha, \beta) \in G_n^* \Leftrightarrow (f_0(n), \alpha, \beta) \in \mathcal{C} \Leftrightarrow \neg(f_0(n) \in \mathcal{W} \wedge (f_0(n), \alpha, \beta) \notin \mathcal{C})$$

is  $\Delta_1^1$  in  $(\alpha, \beta, n)$  and

$$\begin{aligned} g_n(\alpha, \beta) \in N(\mathbb{R}, p) &\Leftrightarrow \exists \rho \in \mathbb{R} \neg \left( f_1(n) \in \mathcal{W}^{\omega \times 2^\omega \times \mathbb{R}} \wedge (f_1(n), \alpha, \beta, \rho) \notin \mathcal{C}^{\omega \times 2^\omega \times \mathbb{R}} \right) \wedge \\ &\quad \rho \in N(\mathbb{R}, p) \\ &\Leftrightarrow \forall \rho \in \mathbb{R} \left( f_1(n) \in \mathcal{W}^{\omega \times 2^\omega \times \mathbb{R}} \wedge (f_1(n), \alpha, \beta, \rho) \notin \mathcal{C}^{\omega \times 2^\omega \times \mathbb{R}} \right) \vee \\ &\quad \rho \in N(\mathbb{R}, p) \end{aligned}$$

is  $\Delta_1^1$  in  $(\alpha, \beta, n, p)$ .

• Finally, the map  $F$  is  $\Delta_1^1$ -recursive and is as required. □

## 4 First consequences

### (A) Universal sets

- We first recall some material from [K2]. The first result can be found in Section 23.F (see also [Za]).

**Theorem 4.1** (Zahorski) *Let  $B$  be a subset of  $[0, 1]$ . The following are equivalent:*

- there are  $S \in \Sigma_2^0$  and  $P \in \Pi_3^0$  with  $m(P) = 1$ , where  $m$  is the Lebesgue measure on  $[0, 1]$ , such that  $B = S \cap P$ ,
- there is  $f \in C([0, 1])$  with  $B = \{x \in [0, 1] \mid f'(x) \text{ exists}\}$  (we consider only one-sided derivatives at the endpoints).

The second result is 23.23.

**Theorem 4.2** Let  $\mathcal{G}$  be a  $G_\delta$  subset of  $(0, 1)$  with  $m(\mathcal{G})=0$ . Then

$$\{(f, x) \in C([0, 1]) \times \mathcal{G} \mid f'(x) \text{ exists}\}$$

is  $C([0, 1])$ -universal for  $\Pi_3^0(\mathcal{G})$ .

- We prove results in that spirit here.

**Theorem 4.3** Let  $B$  be a subset of  $2^\omega$ . Then the following are equivalent:

- (a)  $B$  is  $\Sigma_3^0$  and has  $\lambda$ -measure zero,
- (b) there is  $f \in \mathcal{M}$  with  $B = \{\beta \in 2^\omega \mid \text{osc}(f, \beta) > 0\}$ .

**Proof.** (a)  $\Rightarrow$  (b) Write  $B = \bigcup_{n \in \omega} G_n$ , where the  $G_n$ 's are  $G_\delta$ . Lemma 2.7 gives, for each  $n$ , a martingale  $f_n$  with  $G_n = D(f_n)$  and  $\{\text{osc}(f_n, \beta) \mid \beta \in 2^\omega\} \subseteq \{0\} \cup [\frac{1}{2}, 1]$ . Lemma 2.8 gives  $f \in \mathcal{M}$  with  $D(f) = B$ .

(b)  $\Rightarrow$  (a) We already noticed in the introduction that  $B$  is  $\Sigma_3^0$ . By Doob's theorem,  $B$  has  $\lambda$ -measure zero (see [D]). □

**Corollary 4.4** Let  $\mathcal{G}$  be a  $G_\delta$  subset of  $2^\omega$  with  $\lambda(\mathcal{G}) = 0$ . Then  $\{(f, \beta) \in \mathcal{M} \times \mathcal{G} \mid \text{osc}(f, \beta) > 0\}$  is  $\mathcal{M}$ -universal for  $\Sigma_3^0(\mathcal{G})$ .

For example,  $\{\beta \in 2^\omega \mid \forall n \in \omega \beta(2n) = 0\}$  is a  $\Pi_1^0$  copy of  $2^\omega$  and has  $\lambda$ -measure zero.

## (B) Complete sets

- By 33.G in [K2], there is a uniform version of Zahorski's theorem, which allows to prove the following result

**Theorem 4.5** (Mazurkiewicz) The set of differentiable functions in  $C([0, 1])$  is  $\Pi_1^1$ -complete.

- Here again, there is a result in that spirit.

**Theorem 4.6** The set  $\mathcal{P} := \{f \in \mathcal{M} \mid \forall \beta \in 2^\omega \text{ osc}(f, \beta) = 0\}$  is  $\Pi_1^1$ -complete.

**Notation.** Let  $\mathcal{K} := \{\beta \in 2^\omega \mid \forall n \in \omega \beta(2n) = 0\}$ , which is a  $\Pi_1^0$  copy of the Cantor space  $2^\omega$  with  $\lambda(\mathcal{K}) = 0$ . In particular,  $\mathcal{K}$  is a nonempty  $G_\delta \cap \Delta_1^1$  subset of  $2^\omega$ .

**Proof.** Let  $U \in \Pi_1^1(\omega^\omega \times 2^\omega)$  be  $\omega^\omega$ -universal for the co-analytic subsets of  $2^\omega$ , and

$$\Pi := \{\alpha \in \omega^\omega \mid ((\alpha)_0, (\alpha)_1) \in U\}.$$

Note that  $\Pi \in \Pi_1^1$ . If  $P \in \Pi_1^1(2^\omega)$ , then  $P = U_\alpha$  for some  $\alpha \in \omega^\omega$ , so that the map  $\beta \mapsto \langle \alpha, \beta \rangle$  is a continuous reduction of  $P$  to  $\Pi$  and  $\Pi$  is  $\Pi_1^1$ -complete. Let  $H \in \Pi_2^0(\omega^\omega \times 2^\omega)$  with  $\neg \Pi = \Pi_0[H]$ . We set  $G := \{(\alpha, \beta) \in \omega^\omega \times 2^\omega \mid (\alpha, (\beta)_1) \in H \wedge \beta \in \mathcal{K}\}$ , so that  $G \in \Delta_1^1(\omega^\omega \times 2^\omega)$ , has  $G_\delta$  vertical sections and  $G \subseteq \omega^\omega \times \mathcal{K}$ . Lemma 3.10 gives  $F : \omega^\omega \rightarrow \mathcal{M}$  Borel such that  $G_\alpha = \mathcal{V}_{F(\alpha)}$  for each  $\alpha \in \omega^\omega$ .



Thus

$$\alpha \notin \Pi \Leftrightarrow \exists \beta \in 2^\omega \quad (\alpha, \beta) \in H \Leftrightarrow \exists \beta \in 2^\omega \quad (\alpha, \beta) \in G \Leftrightarrow \exists \beta \in 2^\omega \quad (F(\alpha), \beta) \in \mathcal{V} \Leftrightarrow F(\alpha) \notin \mathcal{P}.$$

Thus  $\Pi = F^{-1}(\mathcal{P})$  and  $\mathcal{P}$  is Borel  $\Pi_1^1$ -complete. By 26.C in [K2],  $\mathcal{P}$  is  $\Pi_1^1$ -complete.  $\square$

- We now prove Theorem 1.8. Let  $X$  be a metrizable compact space and  $Y$  be a Polish space. We equip  $\mathcal{C}(X, Y)$  with the topology of uniform convergence, so that it is a Polish space (see 4.19 in [K2]). We use the map  $\psi$  defined before Theorem 1.8.

**Theorem 4.7** (a) *The set  $\mathcal{P}_1 := \{(f_k)_{k \in \omega} \in \mathcal{P}^\omega \mid (\psi(f_k))_{k \in \omega} \text{ pointwise converges}\}$  is  $\Pi_1^1$ -complete.*

(b) *The set  $\mathcal{P}_2 := \{(f_k)_{k \in \omega} \in \mathcal{P}^\omega \mid (\psi(f_k))_{k \in \omega} \text{ pointwise converges to zero}\}$  is  $\Pi_1^1$ -complete.*

(c) *The set  $\mathcal{S} := \{(f_k)_{k \in \omega} \in \mathcal{P}^\omega \mid \exists \gamma \in \omega^\omega \quad (\psi(f_{\gamma(i)}))_{i \in \omega} \text{ pointwise converges to zero}\}$  is  $\Sigma_2^1$ -complete.*

**Proof.** We define  $\varphi: \mathcal{C}(2^\omega, [0, 1]) \rightarrow \mathcal{M}$  by  $\varphi(h)(s) := \int_{N_s} h \, d\lambda$ . As in the proof of Lemma 2.7,  $\varphi$  is well-defined. It is also continuous, and injective: if  $h \neq h'$ , then we can find  $q \in \omega$  and  $s \in 2^{< \omega}$  such that  $h(\beta) - h'(\beta) > 2^{-q}$  for each  $\beta \in N_s$  or  $h'(\beta) - h(\beta) > 2^{-q}$  for each  $\beta \in N_s$ , so that

$$|\varphi(h)(s) - \varphi(h')(s)| = \frac{1}{\lambda(N_s)} \left| \int_{N_s} h \, d\lambda - \int_{N_s} h' \, d\lambda \right| \geq 2^{-q}.$$

This implies that the range  $\mathcal{R}$  of  $\varphi$  is Borel and  $\psi := \varphi^{-1}: \mathcal{R} \rightarrow \mathcal{C}(2^\omega, [0, 1])$  is Borel. As every continuous map  $h: 2^\omega \rightarrow [0, 1]$  is  $\tau$ -continuous,

$$\lim_{l \rightarrow \infty} \varphi(h)(\beta|l) = \lim_{l \rightarrow \infty} \int_{N_{\beta|l}} h \, d\lambda = h(\beta)$$

for each  $\beta \in 2^\omega$ , by Lemma 2.6. This implies that  $f \in \mathcal{P}$  and  $\psi(f)(\beta) = \lim_{l \rightarrow \infty} f(\beta|l)$  for each  $\beta \in 2^\omega$  if  $f \in \mathcal{R}$ .

(a) Note that the proof of 33.11 in [K2] shows that the set

$$P_1 := \{(h_k)_{k \in \omega} \in (\mathcal{C}(2^\omega, [0, 1]))^\omega \mid (h_k)_{k \in \omega} \text{ pointwise converges}\}$$

is  $\Pi_1^1$ -complete. As  $\mathcal{E} := \{(f_k)_{k \in \omega} \in \mathcal{R}^\omega \mid (\psi(f_k))_{k \in \omega} \text{ pointwise converges}\} = (\psi^\omega)^{-1}(P_1)$ , the equalities  $P_1 = (\varphi^\omega)^{-1}(\mathcal{E}) = (\varphi^\omega)^{-1}(P_1)$  hold and  $\mathcal{P}_1$  is  $\Pi_1^1$ -complete.

(b) We argue as in (a).

(c) As in [B-Ka-L], the set

$$S := \{(h_k)_{k \in \omega} \in (\mathcal{C}(2^\omega, [0, 1]))^\omega \mid \exists \gamma \in \omega^\omega \quad (h_{\gamma(i)})_{i \in \omega} \text{ pointwise converges to zero}\},$$

is  $\Sigma_2^1$ -complete. Indeed, fix  $Q \in \Sigma_2^1(2^\omega)$ .

Lemma 2.2 in [B-Ka-L] gives  $(g_k)_{k \in \omega} \in (\mathcal{C}(2^\omega \times 2^\omega, 2))^\omega$  such that, for each  $\delta \in 2^\omega$ , the following are equivalent:

- (i)  $\delta \in Q$ ,
- (ii)  $\exists \gamma \in \omega^\omega \forall \beta \in 2^\omega \lim_{i \rightarrow \infty} g_{\gamma(i)}(\delta, \beta) = 0$ .

We define,  $g: 2^\omega \rightarrow (\mathcal{C}(2^\omega, [0, 1]))^\omega$  by  $g(\delta)(k)(\beta) := g_k(\delta, \beta)$ . Then  $g$  is continuous and reduces  $Q$  to  $S$ . As

$$\mathcal{E}' := \left\{ (f_k)_{k \in \omega} \in \mathcal{R}^\omega \mid \exists \gamma \in \omega^\omega \left( \psi(f_{\gamma(i)}) \right)_{i \in \omega} \text{ pointwise converges to zero} \right\} = (\psi^\omega)^{-1}(S),$$

$S = (\varphi^\omega)^{-1}(\mathcal{E}') = (\varphi^\omega)^{-1}(S)$  and  $\mathcal{S}$  is  $\Sigma_2^1$ -complete.  $\square$

## 5 Universal and complete sets in the spaces $\mathcal{C}(2^\omega, X)$

- It is known that if  $\Gamma$  is a self-dual Wadge class and  $X$  is a Polish space, then there is no set which is  $X$ -universal for the subsets of  $X$  in  $\Gamma$  (see 22.7 in [K2]). This is no longer the case if the space of codes is different from the space of coded sets.

**Proposition 5.1** *Let  $X$  be a Polish space,  $\Gamma$  be a Wadge class with complete set  $C \in \Gamma(X)$ , and  $\mathcal{U}^\Gamma := \{(h, \beta) \in \mathcal{C}(2^\omega, X) \times 2^\omega \mid h(\beta) \in C\}$ . Then  $\mathcal{U}^\Gamma$  is  $\mathcal{C}(2^\omega, X)$ -universal for the  $\Gamma$  subsets of  $2^\omega$ .*

**Proof.** As the evaluation map  $(h, \beta) \mapsto h(\beta)$  is continuous,  $\mathcal{U}^\Gamma \in \Gamma$ . If  $A \in \Gamma(2^\omega)$ , then  $A = h^{-1}(C)$  for some  $h \in \mathcal{C}(2^\omega, X)$ , so that  $A = \mathcal{U}_h^\Gamma$ .  $\square$

We will partially strengthen this result to get our uniform universal sets.

- Recall that it is proved in [K3] that a Borel  $\Pi_1^1$ -complete set is actually  $\Pi_1^1$ -complete. In fact, Kechris's proof shows the result for the classes  $\Pi_n^1$ . Our main tool is a uniform version of this. Kechris's result has recently been strengthened in [P] as follows.

**Theorem 5.2** (Pawlikowski) *Let  $n \geq 1$  be a natural number, and  $C \subseteq X \subseteq 2^\omega$ . If Borel functions from  $2^\omega$  into  $X$  give as preimages of  $C$  all  $\Pi_n^1$  subsets of  $2^\omega$ , then so do continuous injections.*

The main tool mentioned above is the following:

**Theorem 5.3** *Let  $n \geq 1$  be a natural number,  $\mathcal{U}^{\Pi_n^1, 2^\omega}$  be a suitable  $\omega^\omega$ -universal set for the  $\Pi_n^1$  subsets of  $2^\omega$ ,  $X$  be a recursively presented Polish space,  $C \in \Pi_n^1(X)$ ,  $\mathcal{R}: \omega^\omega \times \omega^\omega \rightarrow \omega^\omega$  be a recursive map, and  $b: \omega^\omega \rightarrow X$  be a  $\Delta_1^1$ -recursive map such that*

$$(\alpha, \beta) \in \mathcal{U}^{\Pi_n^1, 2^\omega} \Leftrightarrow b(\mathcal{R}(\alpha, \beta)) \in C$$

for each  $(\alpha, \beta) \in \omega^\omega \times 2^\omega$ . Then there is a  $\Delta_1^1$ -recursive map  $f: \omega^\omega \rightarrow \mathcal{C}(2^\omega, X)$  such that

$$(\alpha, \beta) \in \mathcal{U}^{\Pi_n^1, 2^\omega} \Leftrightarrow f(\alpha)(\beta) \in C$$

for each  $(\alpha, \beta) \in \omega^\omega \times 2^\omega$ .

- We first recall some material from [K3].

**Definition 5.4** (a) A **coding system** for nonempty perfect binary trees is a pair  $(\mathcal{D}, \mathcal{O})$ , where  $\mathcal{D} \subseteq 2^\omega$  and  $\mathcal{O} : \mathcal{D} \rightarrow \{T \in 2^{2^{<\omega}} \mid T \text{ is a nonempty perfect binary tree}\}$  is onto.

(b) A coding system  $(\mathcal{D}, \mathcal{O})$  is **nice** if

(i) for any  $\alpha \in \omega^\omega$  and any  $\Delta_1^1(\alpha)$ -recursive map  $H : 2^\omega \times 2^\omega \rightarrow \omega$ , we can find  $\beta \in \mathcal{D} \cap \Delta_1^1(\alpha)$  and  $k \in \omega$  such that  $H(\beta, \delta) = k$  for each  $\delta$  in the body  $[\mathcal{O}(\beta)]$  of  $\mathcal{O}(\beta)$ ,

(ii)  $\mathcal{D}$  is  $\Pi_1^1$  and, for  $\beta \in \mathcal{D}$ , the relation

$$R(m, \beta) \Leftrightarrow \text{Seq}(m) \wedge ((m)_0, \dots, (m)_{\text{lh}(m)-1}) \in \mathcal{O}(\beta)$$

is  $\Delta_1^1$ , i.e., there are  $\Pi_1^1$  relations  $\Pi_0, \Pi_1$  such that  $R(m, \beta) \Leftrightarrow \Pi_0(m, \beta) \Leftrightarrow \neg \Pi_1(m, \beta)$  if  $\beta \in \mathcal{D}$ .

Nice coding systems exist. If  $\beta \in \mathcal{D}$ , then there is a canonical homeomorphism  $\beta^*$  from  $[\mathcal{O}(\beta)]$  onto  $2^\omega$ . We now check that the construction of  $\beta^*$  is effective.

**Lemma 5.5** (a) The partial function  $e : (\beta, \delta) \mapsto \beta^*(\delta)$  is  $\Pi_1^1$ -recursive on its  $\Pi_1^1$  domain

$$\text{Domain}(e) := \{(\beta, \delta) \in \mathcal{D} \times 2^\omega \mid \delta \in [\mathcal{O}(\beta)]\}.$$

(b) The partial function  $\iota : (\beta, \gamma) \mapsto$  the unique  $\delta \in [\mathcal{O}(\beta)]$  with  $\beta^*(\delta) = \gamma$  is  $\Pi_1^1$ -recursive on its  $\Pi_1^1$  domain  $\mathcal{D} \times 2^\omega$ .

**Proof.** (a) We define a  $\Pi_1^1$  relation  $\mathcal{Q}$  on  $\omega^2 \times (2^\omega)^2$  by

$$\mathcal{Q}(p, p', \beta, \delta) \Leftrightarrow \left( (\forall \varepsilon \in 2 \ \Pi_0(\overline{(\delta|p')\varepsilon}, \beta)) \wedge (\forall p \leq p' < p' \ \exists \varepsilon \in 2 \ \Pi_1(\overline{(\delta|p'')\varepsilon}, \beta)) \right).$$

Note that

$$\beta^*(\delta)(n) = \varepsilon \Leftrightarrow \begin{cases} \exists l \in \omega \ \text{Seq}(l) \wedge \text{lh}(l) = n+1 \wedge \delta((l)_n) = \varepsilon \wedge \mathcal{Q}(0, (l)_0, \beta, \delta) \wedge \\ \forall m < n \ (l)_m < (l)_{m+1} \wedge \mathcal{Q}((l)_{m+1}, (l)_{m+1}, \beta, \delta) \end{cases}$$

if  $\beta \in \mathcal{D}$ . The proof of (b) is similar.  $\square$

- Let  $X$  be a recursively presented Polish space, and  $d_X$  and  $(r_n^X)_{n \in \omega}$  be respectively a distance function and a recursive presentation of  $X$ . We now give a **recursive presentation of  $\mathcal{C}(2^\omega, X)$** , equipped with the usual distance defined by

$$d(h, h') := \sup_{\beta \in 2^\omega} d_X(h(\beta), h'(\beta)),$$

since this is not present in [M]. We define, by primitive recursion, a recursive map  $\nu : \omega \rightarrow \omega$  such that  $\nu(i)$  enumerates  $\{s \in 2^{<\omega} \mid |s| = i\}$ . We first set  $\nu(0) := 1 = \langle \rangle$ . Then

$$\nu(i+1) = k \Leftrightarrow \text{Seq}(k) \wedge \text{lh}(k) = 2^{i+1} \wedge \forall l < 2^i \ \forall \varepsilon \in 2 \ (k)_{\varepsilon 2^i + l} = s\left(\overline{(\nu(i))_l}\right)_\varepsilon.$$

If  $\text{Seq}(n)$  and  $\text{lh}(n) = 2^i$  for some  $i (< n)$ , then we define  $h_n : 2^\omega \rightarrow X$  by  $h_n(\beta) := r_{(n)_l}^X$  if

$$\beta|_i = s_l^i := s\left(\overline{(\nu(i))_l}\right).$$

If  $\neg \text{Seq}(n)$  or  $\text{lh}(n) \neq 2^i$  for each  $i$ , then we define  $h_n : 2^\omega \rightarrow X$  by  $h_n(\beta) := r_0^X$  if  $\beta \in 2^\omega$ . In any case,  $h_n \in \mathcal{C}(2^\omega, X)$  and takes finitely many values.

**Lemma 5.6** *Let  $X$  be a recursively presented Polish space. Then the sequence  $(h_n)_{n \in \omega}$  is a recursive presentation of  $\mathcal{C}(2^\omega, X)$ , equipped with  $d$ .*

**Proof.** We have to see that  $(h_n)$  is dense in  $\mathcal{C}(2^\omega, X)$ . So let  $h \in \mathcal{C}(2^\omega, X)$ ,  $\epsilon > 0$  and  $m \in \omega$  with  $2^{-m} < \frac{\epsilon}{2}$ . As  $h$  is uniformly continuous, there is  $i \in \omega$  such that  $d_X(h(\beta), h(\delta)) < 2^{-m}$  if  $\beta|_i = \delta|_i$ . We choose, for each  $l < 2^i$ ,  $n_l \in \omega$  such that  $d_X(r_{n_l}^X, h(s_l^i 0^\infty)) < 2^{-m}$ . We set  $n := \langle n_0, \dots, n_{2^i-1} \rangle$ . If  $\beta \in 2^\omega$  and  $\beta|_i = s_l^i$ , then  $d_X(h(\beta), h_n(\beta)) \leq d_X(h(\beta), h(s_l^i 0^\infty)) + d_X(h(s_l^i 0^\infty), r_{n_l}^X) \leq 2^{-m} + 2^{-m}$ , so that  $d(h, h_n) < \epsilon$ . It is routine to check that the relations “ $d(h_m, h_n) \leq \frac{p}{q+1}$ ” and “ $d(h_m, h_n) < \frac{p}{q+1}$ ” are recursive in  $(m, n, p, q)$ .  $\square$

We saw in the proof of Proposition 5.1 that the evaluation map  $(h, \beta) \mapsto h(\beta)$  is continuous from  $\mathcal{C}(2^\omega, X) \times 2^\omega$  into  $X$ . We can say more if  $X$  is recursively presented.

**Lemma 5.7** *Let  $X$  be a recursively presented Polish space. Then the evaluation map is recursive.*

**Proof.** Note that

$$\begin{aligned} h(\beta) \in N(X, n) &\Leftrightarrow d_X(h(\beta), r_{((n)_1)_0}^X) < \frac{((n)_1)_1}{((n)_1)_{2+1}} \\ &\Leftrightarrow \exists m, i, l \in \omega \text{ Seq}(m) \wedge \text{lh}(m) = 2^i \wedge \beta|_i = s_l^i \wedge (m)_l = ((n)_1)_0 \wedge \\ &\quad d(h, h_m) < \frac{((n)_1)_1}{((n)_1)_{2+1}}, \end{aligned}$$

which gives the result.  $\square$

- We then strengthen 7A.3 in [M] about **primitive recursion** as follows. If  $Z, Y$  are recursively presented Polish spaces,  $g : Z \rightarrow Y$  and  $h : Y \times \omega \times Z \rightarrow Y$  are  $\Pi_1^1$ -recursive and  $f : \omega \times Z \rightarrow Y$  is defined by

$$\begin{cases} f(0, z) := g(z), \\ f(n+1, z) := h(f(n, z), n, z), \end{cases}$$

then  $f$  is also  $\Pi_1^1$ -recursive. If  $m : Z \rightarrow Z$  is  $\Pi_1^1$ -recursive, then the proof of 7A.3 in [M] shows that the map  $f' : \omega \times Z \rightarrow Y$  defined by

$$\begin{cases} f'(0, z) := g(z), \\ f'(n+1, z) := h(f'(n, m(z)), n, z), \end{cases}$$

is also  $\Pi_1^1$ -recursive. As in 7A.5 in [M], this can be extended to partial functions which are  $\Pi_1^1$ -recursive on their domain.

- We are ready for the proof of our main tool.

**Proof of Theorem 5.3.** 3E.6 in [M] provides  $\pi : \omega^\omega \rightarrow X$  recursive,  $\mathcal{F} \in \Pi_1^0(\omega^\omega)$  and a  $\Delta_1^1$ -recursive injection  $\rho : X \rightarrow \omega^\omega$  such that  $\pi|_{\mathcal{F}}$  is injective,  $\pi[\mathcal{F}] = X$  and  $\rho$  is the inverse of  $\pi|_{\mathcal{F}}$ . Let us show that the map  $\mu : h \mapsto \pi \circ h$  is  $\Delta_1^1$ -recursive from  $\mathcal{C}(2^\omega, \omega^\omega)$  into  $\mathcal{C}(2^\omega, X)$ . More generally, let  $Y$  be a recursively presented Polish space, and  $\psi : Y \rightarrow \mathcal{C}(2^\omega, X)$ . Note that

$$\begin{aligned} \psi(y) \in N(\mathcal{C}(2^\omega, X), n) &\Leftrightarrow d(\psi(y), h_{((n)_1)_0}) < \frac{((n)_1)_1}{((n)_1)_{2+1}} \\ &\Leftrightarrow \exists m \in \omega \sup_{\beta \in 2^\omega} d_X(\psi(y)(\beta), h_{((n)_1)_0}(\beta)) < \frac{((m)_1)_1}{((m)_1)_{2+1}} < \frac{((n)_1)_1}{((n)_1)_{2+1}} \\ &\Leftrightarrow \exists m \in \omega \forall \beta \in 2^\omega d_X(\psi(y)(\beta), h_{((n)_1)_0}(\beta)) < \frac{((m)_1)_1}{((m)_1)_{2+1}} < \frac{((n)_1)_1}{((n)_1)_{2+1}} \end{aligned}$$

and  $h_{((n)_1)_0}(\beta) = r_{g(n, \beta)}^X$  for some recursive map  $g : \omega \times 2^\omega \rightarrow \omega$ .

In the present case,  $Y = \mathcal{C}(2^\omega, \omega^\omega)$  and  $\psi(y)(\beta) = \pi(y(\beta))$ . Thus

$$\begin{aligned} d_X(\psi(y)(\beta), h_{((n)_1)_0}(\beta)) &< \frac{\binom{(m)_1_1}{(m)_1_2+1}}{\binom{(m)_1_1}{(m)_1_2+1}} \Leftrightarrow d_X\left(\pi(y(\beta)), r_{g(n,\beta)}^X\right) < \frac{\binom{(m)_1_1}{(m)_1_2+1}}{\binom{(m)_1_1}{(m)_1_2+1}} \\ &\Leftrightarrow \pi(y(\beta)) \in N\left(X, \langle 0, \langle g(n, \beta), ((m)_1)_1, ((m)_1)_2 \rangle \rangle\right) \\ &\Leftrightarrow (y(\beta), \langle 0, \langle g(n, \beta), ((m)_1)_1, ((m)_1)_2 \rangle \rangle) \in G^\pi, \end{aligned}$$

where  $G^\pi$  is the  $\Sigma_1^0$  neighborhood diagram of  $\pi$ . As the evaluation map is recursive,  $h \mapsto \pi \circ h$  is  $\Pi_1^1$ -recursive and total, and thus  $\Delta_1^1$ -recursive.

• Let us show that there is a  $\Delta_1^1$ -recursive map  $f: \omega^\omega \rightarrow \mathcal{C}(2^\omega, X)$  such that  $\mathcal{U}_\alpha^{\Pi_n^1, 2^\omega} = (f(\alpha))^{-1}(C)$  for each  $\alpha \in \omega^\omega$ . We adapt the proof of the main result in [K3]. We set  $A := \pi^{-1}(C)$ . As  $C \in \Pi_n^1(X)$ ,  $A \in \Pi_n^1(\omega^\omega)$ . If  $\langle \beta^0, \delta^0 \rangle \in 2^\omega$ , then we inductively define, for  $i \in \omega$ ,  $m_i, \beta^{i+1}, \delta^{i+1}$  as follows. If  $(\beta^i, \delta^i)$  is given and in  $\text{Domain}(e)$ , then  $(\beta^i)^*(\delta^i) = \langle x_i, \beta^{i+1}, \delta^{i+1} \rangle$  and

$$m_i := \begin{cases} \text{the location of the first 0 in } x_i \text{ if it exists,} \\ 2 \text{ otherwise.} \end{cases}$$

We then set  $Q := \{(\alpha, \langle \beta^0, \delta^0 \rangle) \in \omega^\omega \times 2^\omega \mid \forall i \in \omega (\beta^i, \delta^i) \in \text{Domain}(e) \wedge (\alpha, (m_i)) \in \mathcal{U}^{\Pi_n^1, 2^\omega}\}$  and  $B^* := Q_\alpha$ , so that  $Q \in \Pi_n^1(\omega^\omega \times 2^\omega)$  and  $\beta \in B^* \Leftrightarrow (\alpha, \beta) \in Q$  for each  $(\alpha, \beta) \in \omega^\omega \times 2^\omega$  (note that  $B^*$  depends on  $\alpha$ , but we denote it like this to keep the notation of [K3]). We define  $I: \omega^\omega \rightarrow 2^\omega$  by  $I(\alpha) := 0^{\alpha(0)}10^{\alpha(1)}1\dots$ . Note that  $I$  a  $\Delta_1^1$ -recursive injection onto the  $\Pi_2^0$  set

$$\mathbb{P}_\infty := \{\beta \in 2^\omega \mid \forall p \in \omega \exists q \geq p \beta(q) = 1\},$$

so that there is a  $\Delta_1^1$ -recursive map  $\phi: 2^\omega \rightarrow \omega^\omega$  which is the inverse of  $I$  on  $\mathbb{P}_\infty$ . We set

$$Q' := \left\{ \delta \in 2^\omega \mid (\delta)_0 \in \mathbb{P}_\infty \wedge \left( \phi((\delta)_0), (\delta)_1 \right) \in Q \right\},$$

so that  $Q' \in \Pi_n^1(2^\omega)$ . As  $\mathcal{U}^{\Pi_n^1, 2^\omega}$  is suitable, there is  $\alpha_Q \in \omega^\omega$  recursive with  $Q' = \mathcal{U}_{\alpha_Q}^{\Pi_n^1, 2^\omega}$ . Note that

$$\begin{aligned} \beta \in B^* &\Leftrightarrow (\alpha, \beta) \in Q \Leftrightarrow \langle I(\alpha), \beta \rangle \in Q' \Leftrightarrow (\alpha_Q, \langle I(\alpha), \beta \rangle) \in \mathcal{U}^{\Pi_n^1, 2^\omega} \\ &\Leftrightarrow b(\mathcal{R}(\alpha_Q, \langle I(\alpha), \beta \rangle)) \in C \Leftrightarrow \rho\left(b(\mathcal{R}(\alpha_Q, \langle I(\alpha), \beta \rangle))\right) \in A. \end{aligned}$$

We set  $G := \rho\left(b(\mathcal{R}(\alpha_Q, \langle I(\alpha), \cdot \rangle))\right)$ , so that  $G: 2^\omega \rightarrow \omega^\omega$  is  $\Delta_1^1(\alpha)$ -recursive and  $\langle \beta^0, \delta^0 \rangle$  is in  $B^*$  if and only if  $G(\langle \beta^0, \delta^0 \rangle) \in A$ .

• As in [K3], we can find  $F: 2^{<\omega} \rightarrow (2^\omega \times \omega)^{<\omega}$  satisfying the following properties:

- (1)  $t \subseteq t' \Rightarrow F(t) \subseteq F(t')$
- (2)  $|F(t)| = |t| + 1$
- (3) (i) if  $F(\emptyset) = (\beta^0, k_0)$ , then  $\beta^0 \in \mathcal{D} \wedge \forall \delta^0 \in [\mathcal{O}(\beta^0)] G(\langle \beta^0, \delta^0 \rangle)(0) = k_0$   
(ii) if  $F(\varepsilon_0, \dots, \varepsilon_n) = (\beta^0, k_0, \beta^1, k_1, \dots, \beta^{n+1}, k_{n+1})$ , then
  - (a)  $\forall i \leq n+1 \beta^i \in \mathcal{D}$
  - (b) for all  $\delta^{n+1} \in [\mathcal{O}(\beta^{n+1})]$ , if  $\delta^n, \dots, \delta^0$  are the uniquely determined members of  $[\mathcal{O}(\beta^n)], \dots, [\mathcal{O}(\beta^0)]$  such that  $\forall i \leq n (\beta^i)^*(\delta^i) = \langle \bar{\varepsilon}_i, \beta^{i+1}, \delta^{i+1} \rangle$ , where  $\bar{\varepsilon}_i = 1^{\varepsilon_i} 0 1^\infty$ , then  $\forall i \leq n+1 G(\langle \beta^0, \delta^0 \rangle)(i) = k_i$ .

We will need an effective version of this, so that we give the details of the construction of  $F$ . In fact, the  $\beta^i$ 's involved in the definition of  $F$  can be  $\Delta_1^1(\alpha)$ . In order to see this, we first define

$$H_0: 2^\omega \times 2^\omega \rightarrow \omega$$

by  $H_0(\beta, \delta) := G(\langle \beta, \delta \rangle)(0)$ . As  $G$  is  $\Delta_1^1(\alpha)$ -recursive,  $H_0$  too, and the niceness of the coding system gives  $\beta^0 \in \mathcal{D} \cap \Delta_1^1(\alpha)$  and  $k_0 \in \omega$  such that  $G(\langle \beta^0, \delta^0 \rangle)(0) = k_0$  for each  $\delta^0 \in [\mathcal{O}(\beta^0)]$ . Now suppose that  $n \in \omega$ ,  $(\varepsilon_0, \dots, \varepsilon_n)$  and  $F(\varepsilon_0, \dots, \varepsilon_{n-1}) = (\beta^0, k_0, \dots, \beta^n, k_n)$  are given. We define

$$H_{n+1}: 2^\omega \times 2^\omega \rightarrow \omega$$

as follows. Given  $(\beta, \delta) \in 2^\omega \times 2^\omega$ , let  $\delta^n, \dots, \delta^0$  be the uniquely determined members of  $[\mathcal{O}(\beta^n)], \dots, [\mathcal{O}(\beta^0)]$  resp., such that  $(\beta^n)^*(\delta^n) = \langle \overline{\varepsilon}_n, \beta, \delta \rangle$ , and  $(\beta^i)^*(\delta^i) = \langle \overline{\varepsilon}_i, \beta^{i+1}, \delta^{i+1} \rangle$  if  $i < n$ . Put  $H_{n+1}(\beta, \delta) := G(\langle \beta^0, \delta^0 \rangle)(n+1)$ . As  $H_{n+1}$  is  $\Delta_1^1(\alpha)$  (it is total and  $\Pi_1^1(\alpha)$ -recursive since  $\iota$  is  $\Pi_1^1$ -recursive), the niceness of the coding system gives  $\beta^{n+1} \in \mathcal{D} \cap \Delta_1^1(\alpha)$  and  $k_{n+1} \in \omega$  such that  $G(\langle \beta^0, \delta^0 \rangle)(n+1) = k_{n+1}$  for each  $\delta^{n+1} \in [\mathcal{O}(\beta^{n+1})]$ . Then

$$F(\varepsilon_0, \dots, \varepsilon_n) := (\beta^0, k_0, \dots, \beta^{n+1}, k_{n+1}),$$

so that  $F$  is as desired. So we can assume that the  $\beta^i$ 's are  $\Delta_1^1(\alpha)$  in the conditions required for  $F$ .

• By [K3] again, the map  $h_\alpha : (\varepsilon_i) \mapsto (k_i)$  is continuous and  $\mathcal{U}_\alpha^{\Pi_n^1, 2^\omega} = h_\alpha^{-1}(A)$ . As this is not too long to prove, we give the details for completeness. The map  $h_\alpha$  is in fact more than continuous: it is Lipschitz, by definition. Fix  $(\varepsilon_i)$ . We apply  $F$  to the initial segments of  $(\varepsilon_i)$ , which gives  $(\beta^i)$ . For each  $n$ , we define perfect sets  $C_0^n, C_1^n, \dots, C_n^n \subseteq 2^\omega$  with  $C_i^n \subseteq [\mathcal{O}(\beta^i)]$  if  $i \leq n$ , as follows:

$$\begin{aligned} C_n^n &:= \{\delta^n \in [\mathcal{O}(\beta^n)] \mid \exists \delta^{n+1} \in 2^\omega \ (\beta^n)^*(\delta^n) = \langle \overline{\varepsilon}_n, \beta^{n+1}, \delta^{n+1} \rangle\}, \\ C_{n-1}^n &:= \{\delta^{n-1} \in [\mathcal{O}(\beta^{n-1})] \mid \exists \delta^n \in C_n^n \ (\beta^{n-1})^*(\delta^{n-1}) = \langle \overline{\varepsilon}_{n-1}, \beta^n, \delta^n \rangle\}, \\ &\dots \\ C_0^n &:= \{\delta^0 \in [\mathcal{O}(\beta^0)] \mid \exists \delta^1 \in C_1^n \ (\beta^0)^*(\delta^0) = \langle \overline{\varepsilon}_0, \beta^1, \delta^1 \rangle\}. \end{aligned}$$

Note that

(4)  $\delta^0 \in C_0^n \Rightarrow \langle \beta^i, \delta^i \rangle \in \text{Domain}(e)$  for each  $i \leq n$ , where  $\delta^1, \dots, \delta^n$  are computed according to the formula in (3).(ii).(b),

(5)  $n' \geq n \Rightarrow \forall i \leq n \ C_i^{n'} \subseteq C_i^n$ .

This implies that  $[\mathcal{O}(\beta^0)] \supseteq C_0^0 \supseteq C_0^1 \supseteq C_0^2 \supseteq \dots$  and  $\bigcap_{n \in \omega} C_0^n$  contains some  $\delta^0$ . Note that  $\langle \beta^i, \delta^i \rangle$  is in  $\text{Domain}(e)$ , and  $(\beta^i)^*(\delta^i) = \langle \overline{\varepsilon}_i, \beta^{i+1}, \delta^{i+1} \rangle$  for each  $i \in \omega$ . By (3).(ii).(b),

$$G(\langle \beta^0, \delta^0 \rangle) = k_i$$

for each  $i \in \omega$ . As  $\langle \beta^0, \delta^0 \rangle \in B^* \Leftrightarrow G(\langle \beta^0, \delta^0 \rangle) \in A$ ,

$$(\forall i \in \omega \ \langle \beta^i, \delta^i \rangle \in \text{Domain}(e) \wedge (\varepsilon_i) \in \mathcal{U}_\alpha^{\Pi_n^1, 2^\omega}) \Leftrightarrow (k_i) \in A.$$

As  $\langle \beta^i, \delta^i \rangle$  is in  $\text{Domain}(e)$  for each  $i \in \omega$ ,  $(\varepsilon_i) \in \mathcal{U}_\alpha^{\Pi_n^1, 2^\omega} \Leftrightarrow h_\alpha((\varepsilon_i)) = (k_i) \in A$ .

• So we found, for each  $\alpha \in \omega^\omega$ ,  $h_\alpha \in \mathcal{C}(2^\omega, \omega^\omega)$  such that  $\mathcal{U}_\alpha^{\Pi_1^1, 2^\omega} = (\pi \circ h_\alpha)^{-1}(C) = (\mu(h_\alpha))^{-1}(C)$ . It remains to see that the map  $\psi : \alpha \mapsto h_\alpha$ , from  $\omega^\omega$  into  $\mathcal{C}(2^\omega, \omega^\omega)$ , can be  $\Delta_1^1$ -recursive (then  $f$  will be  $\mu \circ \psi$ ). By the previous discussion, it is enough to see that the relation “ $k_i = k$ ” is  $\Delta_1^1$  in  $(\alpha, (\varepsilon_i), i, k) \in \omega^\omega \times 2^\omega \times \omega^2$ .

• We will define, by primitive recursion, a  $\Delta_1^1$ -recursive map  $\tilde{f} : \omega \times \omega^\omega \times 2^\omega \rightarrow 2^\omega \times \omega$  such that  $\tilde{f}(n, \alpha, (\varepsilon_i))$  will be of the form  $\langle \tilde{\beta}^0, \dots, \tilde{\beta}^n, \tilde{\beta}^n, \dots \rangle, \langle \tilde{k}^0, \dots, \tilde{k}^n \rangle$  and can play the role of  $F(\varepsilon_0, \dots, \varepsilon_{n-1})$ . We first set

$$P := \left\{ (\alpha, (\varepsilon_i), \beta, k) \in \omega^\omega \times (2^\omega)^2 \times \omega \mid \forall i \in \omega \ (\beta)_i = (\beta)_0 \in \mathcal{D} \cap \Delta_1^1(\alpha) \wedge \forall \delta \in [\mathcal{O}((\beta)_0)] \ G(\langle (\beta)_0, \delta \rangle)(0) = k \right\}.$$

Note that  $P$  is  $\Pi_1^1$  and for any  $(\alpha, (\varepsilon_i)) \in \omega^\omega \times 2^\omega$  there is  $(\beta, k) \in 2^\omega \times \omega$  such that  $(\alpha, (\varepsilon_i), \beta, k) \in P$ . The uniformization lemma gives a  $\Delta_1^1$ -recursive map  $\tilde{g} : \omega^\omega \times 2^\omega \rightarrow 2^\omega \times \omega$  such that

$$(\alpha, (\varepsilon_i), \tilde{g}(\alpha, (\varepsilon_i))) \in P$$

for each  $(\alpha, (\varepsilon_i)) \in \omega^\omega \times 2^\omega$ . Then we set

$$D := \left\{ (\beta, p, n, \alpha, (\varepsilon_i)) \in 2^\omega \times \omega^2 \times \omega^\omega \times 2^\omega \mid \text{Seq}(p) \wedge \text{lh}(p) = n+1 \wedge \forall q \in \omega \ (\beta)_q \in \mathcal{D} \cap \Delta_1^1(\alpha) \right\}.$$

Note that  $D$  is  $\Pi_1^1$ , as well as

$$R := \left\{ (\beta, p, n, \alpha, (\varepsilon_i), \beta', k') \in D \times 2^\omega \times \omega \mid \forall i > n \ (\beta')_i = (\beta')_{n+1} \in \mathcal{D} \cap \Delta_1^1(\alpha) \wedge \text{Seq}(k') \wedge \text{lh}(k') = n+2 \wedge \forall i \leq n \ (\beta')_i = (\beta)_i \wedge (k')_i = (p)_i \wedge \forall \delta \in 2^\omega \left( \exists i \leq n+1 \ (\delta)_i \notin [\mathcal{O}((\beta')_i)] \vee \exists i \leq n \ (\beta')_i^*((\delta)_i) \neq \overline{\varepsilon}_i, (\beta')_{i+1}, (\delta)_{i+1} > \vee \forall i \leq n+1 \ G(\langle (\beta')_0, (\delta)_0 \rangle)(i) = (k')_i \right) \right\}.$$

Moreover, for each  $(\beta, p, n, \alpha, (\varepsilon_i)) \in D = \Pi_{2^\omega \times \omega^2 \times \omega^\omega \times 2^\omega} [R]$  there is  $(\beta', k') \in (2^\omega \cap \Delta_1^1(\alpha)) \times \omega$  such that  $(\beta, p, n, \alpha, (\varepsilon_i), \beta', k') \in R$ . The uniformization lemma gives a partial map

$$\tilde{h} : 2^\omega \times \omega^2 \times \omega^\omega \times 2^\omega \rightarrow 2^\omega \times \omega$$

which is  $\Pi_1^1$ -recursive on its domain  $D$ , and such that  $(\beta, p, n, \alpha, (\varepsilon_i), \tilde{h}(\beta, p, n, \alpha, (\varepsilon_i))) \in R$  if  $(\beta, p, n, \alpha, (\varepsilon_i)) \in D$ . This implies that the partial map  $\tilde{f}$  defined by

$$\begin{cases} \tilde{f}(0, \alpha, (\varepsilon_i)) := \tilde{g}(\alpha, (\varepsilon_i)), \\ \tilde{f}(n+1, \alpha, (\varepsilon_i)) := \tilde{h}(\tilde{f}(n, \alpha, (\varepsilon_i)), n, \alpha, (\varepsilon_i)), \end{cases}$$

is  $\Pi_1^1$ -recursive.

Moreover, an induction shows that  $(\tilde{f}(n, \alpha, (\varepsilon_i)), n, \alpha, (\varepsilon_i)) \in D$  for each  $(n, \alpha, (\varepsilon_i))$ , so that  $\tilde{f}$  is in fact total, and thus  $\Delta_1^1$ -recursive. More precisely,  $\tilde{f}(n, \alpha, (\varepsilon_i))$  is of the form

$$\langle \beta^0, \dots, \beta^n, \beta^n, \dots \rangle, \langle k_0, \dots, k_n \rangle,$$

where  $(\varepsilon_0, \dots, \varepsilon_{n-1}) \mapsto (\beta^0, k_0, \dots, \beta^n, k_n)$  satisfies the properties (1)-(3) of  $F$ . It remains to note that  $k_i = \tilde{f}(i, \alpha, (\varepsilon_i))(1)(i)$ .  $\square$

- We now prove the consequences of our main tool.

**Definition 5.8** Let  $\Gamma$  be a class of subsets of recursively presented Polish spaces,  $\mathbf{\Gamma}$  be the corresponding boldface class,  $X, Y$  be recursively presented Polish spaces, and  $\mathcal{U} \in \Gamma(Y \times X)$ . We say that  $\mathcal{U}$  is **effectively uniformly  $Y$ -universal for the  $\mathbf{\Gamma}$  subsets of  $X$**  if the following hold:

- (1)  $\Gamma(X) = \{\mathcal{U}_y \mid y \in Y\}$ ,
- (2)  $\Gamma(X) = \{\mathcal{U}_y \mid y \in Y \text{ } \Delta_1^1\text{-recursive}\}$ ,
- (3) for each  $S \in \Gamma(\omega^\omega \times X)$ , there is a Borel map  $b: \omega^\omega \rightarrow Y$  such that  $S_\alpha = \mathcal{U}_{b(\alpha)}$  for each  $\alpha \in \omega^\omega$ ,
- (4) for each  $S \in \Gamma(\omega^\omega \times X)$ , there is a  $\Delta_1^1$ -recursive map  $b: \omega^\omega \rightarrow Y$  such that  $S_\alpha = \mathcal{U}_{b(\alpha)}$  for each  $\alpha \in \omega^\omega$ .

**Notation.** Let  $\mathcal{U}^{\Pi_1^1, 2^\omega} \in \Pi_1^1$  be a good  $\omega^\omega$ -universal for the  $\Pi_1^1$  subsets of  $2^\omega$ ,  $X_1$  be a recursively presented Polish space, and  $\mathcal{C}_1$  be a  $\Pi_1^1$  subset of  $X_1$  for which there is a  $\Delta_1^1$ -recursive map  $b: \omega^\omega \rightarrow X_1$  such that

$$(\alpha, \beta) \in \mathcal{U}^{\Pi_1^1, 2^\omega} \Leftrightarrow b(\langle \alpha, \beta \rangle) \in \mathcal{C}_1$$

if  $(\alpha, \beta) \in \omega^\omega \times 2^\omega$ . We define, for each natural number  $n \geq 1$ ,

- $X_{n+1} := \mathcal{C}(2^\omega, X_n)$  (inductively),
- $\mathcal{C}_{n+1} := \{h \in X_{n+1} \mid \forall \beta \in 2^\omega \ h(\beta) \notin \mathcal{C}_n\}$  (inductively),
- $\mathcal{U}_n := \{(h, \beta) \in X_{n+1} \times 2^\omega \mid h(\beta) \in \mathcal{C}_n\}$ .

**Theorem 5.9** Let  $n \geq 1$  be a natural number. Then

- (a) the set  $\mathcal{U}_n$  is effectively uniformly  $X_{n+1}$ -universal for the  $\Pi_n^1$  subsets of  $2^\omega$ ,
- (b) the set  $\mathcal{C}_n$  is  $\Pi_n^1$ -complete.

**Proof.** We argue by induction on  $n$ .

(a) Assume first that  $n = 1$ , and fix  $S \in \Pi_1^1(\omega^\omega \times 2^\omega)$ . Our assumption gives  $b_1: \omega^\omega \rightarrow X_1$ . As  $\mathcal{U}^{\Pi_1^1, 2^\omega} \in \Pi_1^1$  is a good  $\omega^\omega$ -universal for the  $\Pi_1^1$  subsets of  $2^\omega$ , there is by Theorem 5.3 a  $\Delta_1^1$ -recursive map  $f_1: \omega^\omega \rightarrow \mathcal{C}(2^\omega, X_1)$  such that  $(\alpha, \beta) \in \mathcal{U}^{\Pi_1^1, 2^\omega} \Leftrightarrow f_1(\alpha)(\beta) \in \mathcal{C}_1$  if  $(\alpha, \beta) \in \omega^\omega \times 2^\omega$ . Let  $\alpha_S \in \omega^\omega$  with  $S = \mathcal{U}_{\alpha_S}^{\Pi_1^1, \omega^\omega \times 2^\omega}$ . Note that

$$(\alpha, \beta) \in S \Leftrightarrow (\mathcal{R}(\alpha_S, \alpha), \beta) \in \mathcal{U}^{\Pi_1^1, 2^\omega} \Leftrightarrow f_1(\mathcal{R}(\alpha_S, \alpha))(\beta) \in \mathcal{C}_1 \Leftrightarrow (f_1(\mathcal{R}(\alpha_S, \alpha)), \beta) \in \mathcal{U}_1.$$

As  $\mathcal{C}_1$  is  $\Pi_1^1$ ,  $\mathcal{U}_1$  too. If  $A \in \Pi_1^1(2^\omega)$ , then  $A = \mathcal{U}_\alpha^{\Pi_1^1, 2^\omega}$  for some  $\alpha \in \omega^\omega$ . Applying the previous discussion to  $S := \mathcal{U}^{\Pi_1^1, 2^\omega}$ , we get  $A = (\mathcal{U}_1)_{f_1(\mathcal{R}(\alpha_S, \alpha))}$ , so that  $\mathcal{U}_1$  is  $X_2$ -universal for the  $\Pi_1^1$  subsets of  $2^\omega$ , effectively and uniformly.



We now study  $\mathcal{U}_{n+1}$ . Fix  $S \in \Pi_{n+1}^1(\omega^\omega \times 2^\omega)$ . Let  $\mathcal{U}^{\Pi_n^1, 2^\omega}$  be a good  $\omega^\omega$ -universal for the  $\Pi_n^1$  subsets of  $2^\omega$ . We set  $\mathcal{V}^{\Pi_{n+1}^1, 2^\omega} := \{(\alpha, \beta) \in \omega^\omega \times 2^\omega \mid \forall \delta \in 2^\omega \ (\mathcal{R}(\alpha, \beta), \delta) \notin \mathcal{U}^{\Pi_n^1, 2^\omega}\}$ , so that  $\mathcal{V}^{\Pi_{n+1}^1, 2^\omega}$  is a suitable  $\omega^\omega$ -universal for the  $\Pi_{n+1}^1$  subsets of  $2^\omega$ . Moreover, the induction assumption gives a  $\Delta_1^1$ -recursive map  $b_{n+1}: \omega^\omega \rightarrow X_{n+1}$  such that

$$\begin{aligned} (\alpha, \beta) \in \mathcal{V}^{\Pi_{n+1}^1, 2^\omega} &\Leftrightarrow \forall \delta \in 2^\omega \ (\mathcal{R}(\alpha, \beta), \delta) \notin \mathcal{U}^{\Pi_n^1, 2^\omega} \Leftrightarrow \forall \delta \in 2^\omega \ (b_{n+1}(\mathcal{R}(\alpha, \beta)), \delta) \notin \mathcal{U}_n \\ &\Leftrightarrow \forall \delta \in 2^\omega \ b_{n+1}(\mathcal{R}(\alpha, \beta))(\delta) \notin \mathcal{C}_n \Leftrightarrow b_{n+1}(\mathcal{R}(\alpha, \beta)) \in \mathcal{C}_{n+1} \end{aligned}$$

Theorem 5.3 gives a  $\Delta_1^1$ -recursive map  $f_{n+1}$  such that  $(\alpha, \beta) \in \mathcal{V}^{\Pi_{n+1}^1, 2^\omega} \Leftrightarrow f_{n+1}(\alpha)(\beta) \in \mathcal{C}_{n+1}$  if  $(\alpha, \beta) \in \omega^\omega \times 2^\omega$ . Let

$$Q \in \Pi_n^1(\omega^\omega \times 2^\omega \times 2^\omega) \subseteq \Pi_n^1(\omega^\omega \times \omega^\omega \times 2^\omega)$$

such that  $(\alpha, \beta) \in S \Leftrightarrow \forall \delta \in 2^\omega \ (\alpha, \beta, \delta) \notin Q$ , and  $\alpha_Q \in \omega^\omega$  such that  $Q = \mathcal{U}_{\alpha_Q}^{\Pi_n^1, \omega^\omega \times \omega^\omega \times 2^\omega}$ . Note that

$$\begin{aligned} (\alpha, \beta) \in S &\Leftrightarrow \forall \delta \in 2^\omega \ (\mathcal{R}(\mathcal{R}'(\alpha_Q, \alpha), \beta), \delta) \notin \mathcal{U}^{\Pi_n^1, 2^\omega} \Leftrightarrow (\mathcal{R}'(\alpha_Q, \alpha), \beta) \in \mathcal{V}^{\Pi_{n+1}^1, 2^\omega} \\ &\Leftrightarrow f_{n+1}(\mathcal{R}'(\alpha_Q, \alpha))(\beta) \in \mathcal{C}_{n+1} \Leftrightarrow (f_{n+1}(\mathcal{R}'(\alpha_Q, \alpha)), \beta) \in \mathcal{U}_{n+1}. \end{aligned}$$

As  $\mathcal{C}_n \in \Pi_n^1$ ,  $\mathcal{C}_{n+1} \in \Pi_{n+1}^1$  and  $\mathcal{U}_{n+1} \in \Pi_{n+1}^1$ . If  $A \in \Pi_{n+1}^1(2^\omega)$ , then  $A = \mathcal{U}_\alpha^{\Pi_{n+1}^1, 2^\omega}$  for some  $\alpha \in \omega^\omega$ . Applying the previous discussion to  $S := \mathcal{U}^{\Pi_{n+1}^1, 2^\omega}$ , we get  $A = (\mathcal{U}_{n+1})_{f_{n+1}(\mathcal{R}'(\alpha_Q, \alpha))}$ , so that  $\mathcal{U}_{n+1}$  is  $X_{n+2}$ -universal for the analytic subsets of  $2^\omega$ , effectively and uniformly.

(b) By definition,  $\mathcal{C}_1 \in \Pi_1^1$ , and  $\mathcal{C}_{n+1} \in \Pi_{n+1}^1$  if  $\mathcal{C}_n \in \Pi_n^1$ . Assume first that  $E \in \Pi_n^1(2^\omega)$ . Then  $E = (\mathcal{U}_n)_h$  for some  $h \in \mathcal{C}(2^\omega, X_n)$ , by (a). Thus  $E = h^{-1}(\mathcal{C}_n)$ . If  $Z$  is a zero-dimensional Polish space and  $D \in \Pi_n^1(Z)$ , then we may assume that  $Z$  is a  $G_\delta$  subset of  $2^\omega$  by 7.8 in [K2], so that  $D \in \Pi_n^1(2^\omega)$ . The previous discussion gives  $g \in \mathcal{C}(2^\omega, X_n)$  with  $D = g^{-1}(\mathcal{C}_n)$ . Thus  $D = (g|_Z)^{-1}(\mathcal{C}_n)$  and  $\mathcal{C}_n$  is  $\Pi_n^1$ -complete.  $\square$

**Proof of Theorem 1.7.** By Theorem 5.9, it is enough to show that if  $\mathcal{U}^{\Pi_1^1, 2^\omega} \in \Pi_1^1$  is a good  $\omega^\omega$ -universal set for the  $\Pi_1^1$  subsets of  $2^\omega$ , then there is a  $\Delta_1^1$ -recursive map  $b: \omega^\omega \rightarrow [0, 1]^{2^{<\omega}}$  such that  $(\alpha, \beta) \in \mathcal{U}^{\Pi_1^1, 2^\omega} \Leftrightarrow b(\langle \alpha, \beta \rangle) \in \mathcal{P}$  if  $(\alpha, \beta) \in \omega^\omega \times 2^\omega$ . Let  $H \in \Pi_2^0(\omega^\omega \times 2^\omega \times 2^\omega)$  such that  $\neg \mathcal{U}^{\Pi_1^1, 2^\omega} = \Pi_{\omega^\omega \times 2^\omega}[H]$ . We set  $G := \{(\alpha, \beta) \in \omega^\omega \times 2^\omega \mid ((\alpha)_0, (\alpha)_1, (\beta)_1) \in H \wedge \beta \in \mathcal{K}\}$ , so that  $G \in \Delta_1^1(\omega^\omega \times 2^\omega)$ , has  $G_\delta$  vertical sections and  $G \subseteq \omega^\omega \times \mathcal{K}$ . Lemma 3.10 gives a  $\Delta_1^1$ -recursive map  $F: \omega^\omega \rightarrow [0, 1]^{2^{<\omega}}$ , taking values in  $\mathcal{M}$ , and such that  $G_\alpha = \mathcal{V}_{b(\alpha)}$  for each  $\alpha \in \omega^\omega$ . If  $(\alpha, \beta) \in \omega^\omega \times 2^\omega$ , then

$$\begin{aligned} (\alpha, \beta) \notin \mathcal{U}^{\Pi_1^1, 2^\omega} &\Leftrightarrow \exists \delta \in 2^\omega \ (\alpha, \beta, \delta) \in H \Leftrightarrow \exists \delta \in 2^\omega \ (\langle \alpha, \beta \rangle, \delta) \in G \\ &\Leftrightarrow \exists \delta \in 2^\omega \ (b(\langle \alpha, \beta \rangle), \delta) \in \mathcal{V} \Leftrightarrow b(\langle \alpha, \beta \rangle) \notin \mathcal{P}. \end{aligned}$$

This finishes the proof.  $\square$

**Questions.** Let  $U$  be a  $\Pi_2^0$  subset of  $\omega^\omega \times 2^\omega$  which is universal for  $\Pi_2^0(2^\omega)$ . We set

$$G := \{(\alpha, \beta) \in \omega^\omega \times \mathcal{K} \mid (\alpha, (\beta)_1) \in U\}.$$

Note that  $G$  is a  $\Pi_2^0$  subset of  $\omega^\omega \times 2^\omega$  contained in  $\omega^\omega \times \mathcal{K}$  which is universal for  $\Pi_2^0(\mathcal{K})$ . Indeed, fix  $H \in \Pi_2^0(\mathcal{K})$ . Then  $H' := \{\gamma \in 2^\omega \mid \langle 0^\infty, \gamma \rangle \in H\}$  is  $\Pi_2^0$ , which gives  $\alpha_0 \in \omega^\omega$  with  $H' = U_{\alpha_0}$ . Then  $H = G_{\alpha_0}$ .

Let  $\alpha \mapsto ((\alpha)_k)_{k \in \omega}$  be a homeomorphism between  $\omega^\omega$  and  $(\omega^\omega)^\omega$ , with inverse map

$$(\alpha_k)_{k \in \omega} \mapsto \langle \alpha_0, \alpha_1, \dots \rangle.$$

We set  $S' := \{\alpha \in \omega^\omega \mid \exists \gamma \in \omega^\omega \forall i \in \omega \forall \beta \in 2^\omega \beta \notin G_{(\alpha)_{\gamma(i)}}\}$ . Note that  $S'$  is  $\Sigma_2^1$ .

(1) Is  $S'$  a Borel  $\Sigma_2^1$ -complete set?

Assume that this is the case. Then the set  $S_2 := \{(f_k)_{k \in \omega} \in \mathcal{M}^\omega \mid \exists \gamma \in \omega^\omega \forall i \in \omega f_{\gamma(i)} \in \mathcal{P}\}$  of sequences of martingales having a subsequence made of everywhere converging martingales is Borel  $\Sigma_2^1$ -complete. Indeed, Lemma 3.10 gives a Borel map  $F : \omega^\omega \rightarrow \mathcal{M}$  such that  $G_\alpha = \mathcal{V}_{F(\alpha)}$  for each  $\alpha \in \omega^\omega$ . The map  $\tilde{F} : \omega^\omega \rightarrow \mathcal{M}^\omega$  defined by  $\tilde{F}(\alpha)(k) := F((\alpha)_k)$  is Borel. Moreover,

$$\begin{aligned} \tilde{F}(\alpha) \in S_2 &\Leftrightarrow \exists \gamma \in \omega^\omega \forall i \in \omega \forall \beta \in 2^\omega \beta \notin D(F((\alpha)_{\gamma(i)})) \\ &\Leftrightarrow \exists \gamma \in \omega^\omega \forall i \in \omega \forall \beta \in 2^\omega \beta \notin \mathcal{V}_{F((\alpha)_{\gamma(i)})} \\ &\Leftrightarrow \exists \gamma \in \omega^\omega \forall i \in \omega \forall \beta \in 2^\omega \beta \notin G_{(\alpha)_{\gamma(i)}} \\ &\Leftrightarrow \alpha \in S', \end{aligned}$$

so that  $S' = \tilde{F}^{-1}(S_2)$ .

(2) Is there a Borel map  $f : \mathcal{C}(2^\omega, [0, 1]) \rightarrow \omega^\omega$  such that, for each  $(h_k)_{k \in \omega} \in (\mathcal{C}(2^\omega, [0, 1]))^\omega$  and each  $\beta \in 2^\omega$ , the following are equivalent:

- (a)  $\lim_{k \rightarrow \infty} h_k(\beta) = 0$ ,
- (b)  $\forall k \in \omega \beta \notin G_{f(h_k)}$ ?

Assume that this is the case. Then  $S'$  (and therefore  $S_2$ ) is Borel  $\Sigma_2^1$ -complete, and thus  $\Sigma_2^1$ -complete (see [P]). We define  $F : (\mathcal{C}(2^\omega, [0, 1]))^\omega \rightarrow \omega^\omega$  by  $F((h_k)_{k \in \omega}) := \langle f(h_0), f(h_1), \dots \rangle$ , so that  $F$  is Borel. Note that

$$\begin{aligned} F((h_k)_{k \in \omega}) \in S' &\Leftrightarrow \exists \gamma \in \omega^\omega \forall i \in \omega \forall \beta \in 2^\omega \beta \notin G_{f(h_{\gamma(i)})} \\ &\Leftrightarrow \exists \gamma \in \omega^\omega \forall \beta \in 2^\omega \lim_{i \rightarrow \infty} h_{\gamma(i)}(\beta) = 0 \\ &\Leftrightarrow (h_k)_{k \in \omega} \in S, \end{aligned}$$

so that  $S = F^{-1}(S')$ .

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