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Multisymplectic formulation of vielbein gravity

I. De Donder-Weyl formulation, Hamiltonian $(n-1)$-forms

Dimitri VEY[1]

Abstract. We consider the De Donder-Weyl (DW) Hamiltonian formulation of the Palatini action of vielbein gravity formulated in terms of the solder form and spin connection, which are treated as independent variables. The basic geometrical constructions necessary for the DW Hamiltonian theory of vielbein gravity are presented. We reproduce the DW Hamilton equations in the multisymplectic and pre-multisymplectic formulations. We also give basic examples of Hamiltonian $(n-1)$-forms and related Poisson brackets.

1 Introduction

The canonical Hamiltonian theory of the Palatini action of vierbein (tetrad) gravity has been studied by Deser and Isham [15] and Heanneaux *et al.* [52]. In the canonical formulation, *space* and *time* are treated asymmetrically and the canonical variables are defined on spacelike hypersurfaces. Therefore, the dynamics implies a global spacelike foliation of the space-time manifold. The canonical commutation relations are defined on the equal time hypersurfaces. Accordingly, the Dirac canonical quantization is related to the instantaneous Hamiltonian formalism, which adds an additional structure of global hyperbolicity on the relativistic space-time. In this paper, we consider the De Donder-Weyl (DW) Hamiltonian formulation of vielbein gravity in the broader context of Multisymplectic Geometry (MG). The finite dimensional DW theory is a covariant Hamiltonian-like formulation for field theory, where the *space* and *time* coordinates are treated symmetrically. Hence, MG may give a profound geometrical road to field quantization (see *e.g.* [47, 60]). The DW Hamiltonian formulation of vielbein gravity based on the first order Palatini action is already found in some papers. A constraints analysis of the Ashtekar theory based on the multisymplectic formalism is found in the paper by Esposito *et al.* [23]. For a glimpse of the DW formulation of vierbein gravity, see also Rovelli [94, 95]. The work of Bruno, Cianci, and Vignolo [5, 6] gives a more detailed development at the crossroad of the natural bundles theory and the jet bundle formalism. Finally, the papers of Kanatchikov [57, 58] focus on the problem of constraints and precanonical quantization [61] of vielbein gravity in the DW formulation.

In this paper, we first outline in section 1 the basic ingredients needed for the subsequent study such as the MG, Palatini formulation and the configuration space of vielbein gravity. Then, in section 2, we present the DW Hamiltonian formulation of the first order Palatini action of vielbein gravity. More precisely, in section 2.1 we describe the Legendre correspondence in the DW setting. We define the constraint hypersurface $\mathcal{C} \subset \mathcal{M}_{\mathsf{DW}}$ in section 2.3. In section 2.4 we give the expression of the DW Hamiltonian density related to the Palatini action *i.e.* $\mathcal{H}^{\mathsf{Palatini}} := \iota^\star \mathcal{H}^{\mathsf{DW}}$, where ι is the canonical inclusion $\iota : \mathcal{C} \hookrightarrow \mathcal{M}_{\mathsf{DW}}$. In section 2.5 we calculate its exterior derivative $\mathrm{d}\mathcal{H}^{\mathsf{Palatini}} := \iota^\star \mathrm{d}\mathcal{H}^{\mathsf{DW}}$. Then, in section 2.6, we present a brief comment on the primary constraints set and the extended DW Hamiltonian. Finally,

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in section 2.7 we derive the DW Hamilton equations in three and four dimensional cases. In section 3 we discuss the pre-multisymplectic formulation of vielbein gravity, *i.e.* we work on the level set $\mathcal{C}_\circ := (\mathcal{H}^{\mathsf{DW}})^{-1}(0) \subset \mathcal{M}_{\mathsf{DW}}$. Thus, the pre-multisymplectic formulation of dreibein and vierbein gravity is presented in sections 3.1 and 3.2, respectively. In section 4 we focus on the notion of Hamiltonian $(n-1)$-forms. In particular, we explore its relation to homotopy Lie algebra and to the graded Poisson bracket in sections 4.1 and 4.2, respectively. We also present some simple examples of Hamiltonian $(n-1)$-forms in sections 4.3 and 4.4. Finally, in section 4.5 we give succinct comments on canonically conjugate forms for vielbein gravity.

1.1 Multisymplectic geometry

Let us recall that MG is a generalization of symplectic geometry to field theory. It allows us to construct a general framework for the calculus of variations with several independent variables. The origins of MG are connected with the names of Carathéodory [10], Weyl [108] on one hand and De Donder [19, 20] on the other. We make this distinction since the motivations involved were different. Carathéodory and Weyl were interested in the generalization of the Hamilton-Jacobi equation to the case of several independent variables and the line of development stemming from their work is concerned with the solutions of variational problems given by an action functional. On the other hand, Cartan [12] recognized the crucial importance of developing an invariant language not dependent on local coordinates. De Donder carried this development further by exploring, in the context of field theory, the relation between Hamilton equations and the theory of integral invariants. The DW system of Hamiltonian equations, as noted in [19, 47], has been discovered already by Volterra [105, 106] at the end of the ninetieth century. Hence, the Hamilton-Volterra system of equations is today termed the DW Hamilton equations with the reference to the work by De Donder [19, 20] and Weyl [108]. As was first noted by Lepage [78, 79, 80], the DW theory is a special case of a more general theory. The geometrical constructions permitting a fully general treatment were provided by Dedecker [16, 17, 18]. Note also that the line of research focusing on the related Lepagean equivalents was developed in particular by Krupka [70, 71, 72], Krupková and Smetanová [73, 74, 75]. Finally, we refer to the review paper by Kastrup [62], the book by Rund [96], Gotay [40, 42], and Olver [85, 86] for more details about the Lepagean equivalents. The Legendre correspondence, *i.e.* the generalization of the Legendre transform in the context of the Lepage-Dedecker theory, the description of observables and the construction of the Poisson brackets are the cornerstones of the covariant Hamiltonian formalism for field theories. For example, in the context of the Lepage-Dedecker theory, the papers by Hélein and Kouneiher [50, 51] develop an insightful classification of observable forms in terms of *algebraic observable forms* and *observable forms*.

A fruitful step in the development of MG and its relation to classical field theories was taken in the seventies of the past century. In particular, the Polish school formulated important ideas and developed the «*multisymplectic*», or «*multiphase-space*», formalism in the work of Tulczyjew [100, 101], Kijowski [63, 64], Kijowski and Tulczyjew [67], Kijowski and Szczyrba [65, 66], and Gawedski [34]. We find the notion of an observable form already in their work. A formulation of the notion of a dynamical observable used in [50, 51] already emerges in the work of Kijowski [63]. Parallel to this development, the paper by Goldschmidt and Sternberg

[43], gave a formulation of the DW Hamilton equations in terms of the Poincaré-Cartan form and the underlying jet bundles geometry, and a related approach was also developed by the Spanish school: García [31, 32] and García and Pérez-Rendún [33].

In this paper, we use the multisymplectic formulation based on the DW «*multimomentum phase space*». Let us consider a theory with a covariant configuration space given by a fiber bundle $(\mathcal{Y}, \mathcal{X}, \pi)$, where $\pi : \mathcal{Y} \longrightarrow \mathcal{X}$ is the bundle projection. Let us denote by $\{x^\mu\}_{1\leq\mu\leq n}$ local coordinates on $\mathcal{X}$ the base space. The dimension of the space-time manifold is $\dim(\mathcal{X}) = n$. We denote also by $\{y^i\}_{1\leq i\leq k}$ local coordinates on $\mathcal{Y}_x$, where $\mathcal{Y}_x := \pi^{-1}(x)$ is the fiber over a point of the space-time manifold. The dimension of the fiber is $\dim(\mathcal{Y}_x) = k$. Local coordinates on the total space $\mathcal{Y}$ are denoted by (x^μ, y^i). We denote $\Lambda^n_1 T^\star\mathcal{Y}$ the vector subbundle of $\Lambda^n T^\star\mathcal{Y}$ whose fiber at $y \in \mathcal{Y}$ consists of all $\boldsymbol{\varphi} \in \Lambda^n_y T^\star\mathcal{Y}$ such that for any vertical vector fields $\zeta^{\mathsf{V}}, \chi^{\mathsf{V}} \in \mathsf{V}\mathcal{Y}$ *i.e.* $\Lambda^n_1 T^\star\mathcal{Y} = \big\{\boldsymbol{\varphi} \in \Lambda^n_y T^\star\mathcal{Y} \ / \ \zeta^{\mathsf{V}} \lrcorner\, \chi^{\mathsf{V}} \lrcorner\, \boldsymbol{\varphi} = 0\big\}$. We also denote $\Lambda^n_0 T^\star\mathcal{Y}$ the space of horizontal n-forms on $\mathcal{Y}$. Thus, we denote by $\mathcal{M}_{\mathsf{DW}} := \mathcal{M}_{\mathsf{DW}}(\mathcal{Y}) := \Lambda^n_1 T^\star\mathcal{Y}$ the DW multimomentum phase space. The bundle $\Lambda^n_1 T^\star\mathcal{Y} \to \mathcal{Y}$ carries a canonical structure $\theta^{\mathsf{DW}} = \varkappa\beta + p^\mu_i \mathrm{d}y^i \wedge \beta_\mu$ and leads to the multisymplectic structure: $\boldsymbol{\omega}^{\mathsf{DW}} = \mathrm{d}\varkappa \wedge \beta + \mathrm{d}p^\mu_i \wedge \mathrm{d}y^i \wedge \beta_\mu$, with $\beta = \mathrm{d}x^1 \wedge ... \wedge \mathrm{d}x^n$ a volume n-form on $\mathcal{X}$ and $\beta_\mu := \partial_\mu \lrcorner\, \beta$ is a $(n-1)$-form.

To conclude this overview we mention examples of more recent papers in the field. We refer to Binz, Sniatycki and Fischer [4], Günther [44], De León, Cariñena, Crampin, Ibort [9, 11], Forger, Paufler and Römer [28, 29, 30], Gotay *et al.* [39, 40, 41, 42], Hélein [46, 47], Hélein and Kouneiher [49, 50, 51], Kanatchikov [53, 54, 55, 56], and Sardanashvily *et al.* [36, 37, 38, 97]. Most of the literature on the subject focuses on the contact structure and jet bundles formalism. For a general presentation of multisymplectic, k-symplectic and k-cosymplectic geometries, we refer to the review paper by Román-Roy [92] and the book by De León, Salgado and Vilariño [21]. The multiplicity of formalisms is illustrated by the polysemy of the term «*polysymplectic*», first introduced by Günther [44]. Thus, Günther's *polysymplectic* (or k-symplectic, see [21]) formalism is different from the polysymplectic approaches developed later by Kanatchikov [53] and Sardanashvily *et al.* [37], respectively. In the former, the polysymplectic formulation is based on the *polymomentum phase space i.e.* the quotient bundle $\mathcal{M}^{\mathsf{Poly}}_{\mathsf{DW}}(\mathcal{Y}) = \Lambda^n_1 T^\star\mathcal{Y}/\Lambda^n_0 T^\star\mathcal{Y}$. The polysymplectic structure on $\mathcal{M}^{\mathsf{Poly}}_{\mathsf{DW}}(\mathcal{Y})$ is described as an equivalence class of canonical forms while the main object is $\boldsymbol{\omega}^{\mathsf{V}} := \mathrm{d}p^\mu_i \wedge \mathrm{d}y^i \wedge \beta_\mu$, the vertical part of the multisymplectic form $\boldsymbol{\omega}^{\mathsf{DW}}$. In the latter approach, the *polymomentum phase space* is defined as $\mathcal{M}^{\mathsf{Poly}}(\mathcal{Y}) = \pi^\star T\mathcal{X} \otimes \mathsf{V}^\star(\mathcal{Y}) \otimes \pi^\star\Lambda^n T^\star\mathcal{X}$ and the canonical polysymplectic form is given by $\boldsymbol{\omega}^{\mathsf{Poly}} = \mathrm{d}p^\mu_i \wedge \mathrm{d}y^i \wedge \beta \otimes \partial_\mu$.

1.1.1 Poincaré-Cartan n-form, multisymplectic $(n+1)$-form

In this section, we introduce the multimomentum phase space in MG, *i.e.* the bundle $\mathcal{M} := \Lambda^n T^\star\mathcal{Y}$ of n-forms over the configuration space $\mathcal{Y}$. This is a generalization of the phase space, *i.e.* of the cotangent bundle introduced in symplectic geometry. We will follow the terminology found in [47, 49, 50, 51].

Definition 1.1.1. *A multisymplectic manifold $\big(\mathcal{M}, \boldsymbol{\omega}\big)$ is a manifold $\mathcal{M}$ together with $\boldsymbol{\omega}$, a closed and non degenerate differential $(n+1)$-form on $\mathcal{M}$.*

In field theory we are led to think of solutions of variational problems as n-dimensional

submanifolds Γ embedded in the multimomentum phase space. We introduce the notion of a Hamiltonian n-curve, see [50, 51]:

Definition 1.1.2. *Let $\mathcal{H} : \mathcal{M} \to \mathbb{R}$ be a smooth Hamiltonian function (such that $\mathrm{d}\mathcal{H} \neq 0$). A Hamiltonian n-curve is a n-dimensional oriented submanifold $\Gamma \subset \mathcal{M}$ such that*

$$\forall m \in \Gamma, \qquad \exists X \in \Lambda^n T_m\Gamma, \qquad X \lrcorner\, \boldsymbol{\omega}_m = (-1)^n \mathrm{d}\mathcal{H}_m. \tag{1}$$

A Hamiltonian n-curve is parametrized by a map $x \mapsto (q(x), p(x))$ from the space-time manifold $\mathcal{X}$ to the multimomentum phase space $\mathcal{M}$. Actually, in definition 1.1.2, the generalized Hamilton equations are written in geometric form as $X \lrcorner\, \boldsymbol{\omega}_m = (-1)^n \mathrm{d}\mathcal{H}_m$.

The Poincaré-Cartan n-form θ on $\Lambda^n T^\star\mathcal{Y}$ is defined as

$$\forall q \in \mathcal{Y}, \quad \forall p \in \Lambda^n T^\star_q\mathcal{Y}, \qquad \theta_{(q,p)}(X_1, \cdots, X_n) = p(\Pi_\star(X_1), \cdots, \Pi_\star(X_n)), \tag{2}$$

where $\Pi : \mathcal{M} := \Lambda^n T^\star\mathcal{Y} \xrightarrow{\Pi} \mathcal{Y}$ is the bundle projection on the configuration bundle and $\Pi_\star := \mathrm{d}\Pi : T\Lambda^n T^\star\mathcal{Y} \xrightarrow{\Pi_\star} T\mathcal{Y}$. Note that the dimension of a fiber at $q \in \mathcal{Y}$ is $\dim\left(\Lambda^n T^\star_q\mathcal{Y}\right) = (n+k)!/(n!k!)$, whereas the dimension of the total space of the fiber bundle is $\dim\left(\Lambda^n T^\star\mathcal{Y}\right) = n + k + (n+k)!/(n!k!)$.

Strictly speaking, the object defined by (2) is the most general Lepagean equivalent of the Poincaré-Cartan form. Nevertheless, we term it the «*Poincaré-Cartan*» form, according to the terminology found in [50, 51]. Let $(q^{\boldsymbol{\mu}})_{1\leq\boldsymbol{\mu}\leq n+k}$ be the local coordinates on $\mathcal{Y}$, *i.e.* $q^{\boldsymbol{\mu}} := (x^\mu, y^i)$. Let the family $(\mathrm{d}q^{\boldsymbol{\mu}_1} \wedge ... \wedge \mathrm{d}q^{\boldsymbol{\mu}_n})_{1\leq\boldsymbol{\mu}_1<...\boldsymbol{\mu}_n<n+k}$ be a basis of $\Lambda^n T^\star_q\mathcal{Y}$. We denote by $p_{\boldsymbol{\mu}_1\cdots\boldsymbol{\mu}_n}$ the local coordinates of the Poincaré-Cartan form on $\Lambda^n T^\star_q\mathcal{Y}$ in the basis $\mathrm{d}q^{\boldsymbol{\mu}_1} \wedge ... \wedge \mathrm{d}q^{\boldsymbol{\mu}_n}$. In particular, we denote $\varkappa := p_{1\cdots n}$, $p^\mu_i := p_{1\cdots(\mu-1)i(\mu+1)\cdots n}$, $p^{\mu_1\mu_2}_{i_1i_2} := p_{1\cdots(\mu_1-1)i_1(\mu_1+1)\cdots(\mu_2-1)i_2(\mu_2+1)\cdots n}$, ... Finally, we use also the notations $\beta_{\mu_1\cdots\mu_p} := \partial_{\mu_1} \wedge \cdots \wedge \partial_{\mu_p} \lrcorner\, \beta$, and $\beta^{i_1\cdots i_p}_{\mu_1\cdots\mu_p} := \mathrm{d}y^{i_1} \wedge \cdots \wedge \mathrm{d}y^{i_p} \wedge \left(\partial_{\mu_1} \wedge \cdots \wedge \partial_{\mu_p} \lrcorner\, \beta\right)$. In local coordinates, the Poincaré-Cartan n-form θ is written as

$$\begin{aligned} \theta &= \sum_{1\leq\boldsymbol{\mu}_1<\cdots<\boldsymbol{\mu}_n<n+k} p_{\boldsymbol{\mu}_1\cdots\boldsymbol{\mu}_n} \mathrm{d}q^{\boldsymbol{\mu}_1} \wedge \cdots \wedge \mathrm{d}q^{\boldsymbol{\mu}_n}, \\ &= \varkappa\beta + \sum_{j=1}^{n} \sum_{\mu_1<\cdots<\mu_j} \sum_{i_1<\cdots<i_j} p^{\mu_1\cdots\mu_j}_{i_1\cdots i_j} \beta^{i_1\cdots i_j}_{\mu_1\cdots\mu_j}. \end{aligned} \tag{3}$$

The multisymplectic $(n+1)$-form $\boldsymbol{\omega} := \mathrm{d}\theta$ (called also the «*pataplectic form*» in [49]) is the exterior derivative of the Poincaré-Cartan form. Traditionaly the term «*multisymplectic form*» refers to Kijowski's multisymplectic form [63, 64] *i.e.* in the DW formulation *only*. Nonetheless, we will follow the terminology introduced in [50, 51]. In local coordinates, the multisymplectic $(n+1)$-form $\boldsymbol{\omega} := \mathrm{d}\theta$ is written as

$$\begin{aligned} \boldsymbol{\omega} &= \sum_{1\leq\boldsymbol{\mu}_1<\cdots<\boldsymbol{\mu}_n<n+k} \mathrm{d}p_{\boldsymbol{\mu}_1\cdots\boldsymbol{\mu}_n} \wedge \mathrm{d}q^{\boldsymbol{\mu}_1} \wedge \cdots \wedge \mathrm{d}q^{\boldsymbol{\mu}_n}, \\ &= \mathrm{d}\varkappa \wedge \beta + \sum_{j=1}^{n} \sum_{\mu_1<\cdots<\mu_j} \sum_{i_1<\cdots<i_j} \mathrm{d}p^{\mu_1\cdots\mu_j}_{i_1\cdots i_j} \wedge \beta^{i_1\cdots i_j}_{\mu_1\cdots\mu_j}. \end{aligned} \tag{4}$$

1.1.2 Bundle of field derivatives

We now describe the Lagrangian side of the formulation of a variational problem on fields $\varphi : \mathcal{X} \to \mathcal{Y}$. The Lagrangian density $L(q,v) = L(x^\mu, y^i, v^i_\mu) = L(x^\mu, y^i, \partial_\mu y^i)$ is defined on the bundle $\mathcal{P}$ of field derivatives. We associate to φ the bundle $\varphi^\star T\mathcal{Y} \otimes T^\star\mathcal{X}$ over $\mathcal{X}$. A point $(x,v) \in \varphi^\star T\mathcal{Y} \otimes T^\star\mathcal{X}$ is given by $v = \sum_{1\leq\mu\leq n}\sum_{1\leq i\leq k} v^i_\mu \frac{\partial}{\partial y^i} \otimes \mathrm{d}x^\mu$. On the bundle $\mathcal{P}^\varphi := \varphi^\star T\mathcal{Y} \otimes T^\star\mathcal{X}$, which is included in the bundle $\mathcal{P} = \{(x,y,v)/(x,y) \in \mathcal{Y},\ v \in T_y\mathcal{Y} \otimes T^\star_x\mathcal{X}\}$, the local coordinates are (x^μ, y^i, v^i_μ). Note that the dimension of the fiber is $\dim(\mathcal{P}_{(x,y)}) = nk$, whereas the dimension the bundle is $\dim(\mathcal{P}) = n+k+nk$. They can be equivalently thought of as the local coordinates on the first order jet bundle $J^1(\mathcal{Y})$. We refer to Saunders [98] for an introduction to the jet bundle formalism, and to Cariñena *et al.* [11], and Gotay *et al.* [39] for the use of it in the multisymplectic context.

Using the variational principle we obtain for the action $\mathcal{S}[\varphi] = \int_\mathcal{X} L(x, \varphi(x), \mathrm{d}\varphi(x))\beta$ the related Euler-Lagrange

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial L}{\partial v^i_\mu}(x, \varphi(x), \mathrm{d}\varphi(x))\right) = \frac{\partial L}{\partial y^i}(x, \varphi(x), \mathrm{d}\varphi(x)). \tag{5}$$

We denote by $\Lambda^n_{\mathrm{N}}T\mathcal{Y}$ the normalized space of decomposable n-vector fields on $\mathcal{Y}$: for any $z_1, \cdots, z_n \in T_q\mathcal{Y}$, $\Lambda^n_{\mathrm{N}}T\mathcal{Y} := \{(q,z) \in \Lambda^n T\mathcal{Y}/z = z_1 \wedge ... \wedge z_n \text{ and } \beta(z_1, ..., z_n) = 1\}$. We construct a diffeomorphism between $\Lambda^n_{\mathrm{N}}T\mathcal{Y}$ and $\mathcal{P}$. More precisely, for any $(x^\mu, y^i) \in \mathcal{Y}$ the fiber $\Lambda^n_{\mathrm{N}}T_{(x,y)}\mathcal{Y}$ is identified with $\mathcal{P}_{(x,y)}$ using the diffeomorphism $\sum_\mu\sum_i v^i_\mu \frac{\partial}{\partial y^i} \otimes \mathrm{d}x^\mu \mapsto z = z_1 \wedge ... \wedge z_n$, where for any $1 \leq \alpha \leq n$, $z_\alpha = \frac{\partial}{\partial x^\alpha} + \sum_{1\leq i\leq k} v^i_\alpha \frac{\partial}{\partial y^i}$, see [49].

1.1.3 DW Multimomentum manifold

The DW multimomentum manifold $\mathcal{M}_{\mathsf{DW}}$ is a submanifold of $\mathcal{M} := \Lambda^n T^\star\mathcal{Y}$. For any $(q,p) \in \Lambda^n T^\star\mathcal{Y}$ we restrict ourselves to the case where the interior product $\zeta^{\mathsf{V}} \lrcorner\, \chi^{\mathsf{V}} \lrcorner\, p = 0$ is identically vanishing, where $\zeta^{\mathsf{V}}, \chi^{\mathsf{V}} \in \mathsf{V}\mathcal{Y}$ are any two vertical vector fields. Let us recall that a vector field is vertical if any $\xi \in T_y\mathcal{Y}$ such that $\pi_\star(\xi) := \mathrm{d}\pi(\xi) = 0$, where π is the bundle projection on the space-time manifold $\mathcal{Y} \xrightarrow{\pi} \mathcal{X}$. Then, by definition

$$\mathcal{M}_{\mathsf{DW}} := \Lambda^n_1 T^\star\mathcal{Y} = \left\{(q,p) \in \Lambda^n T^\star\mathcal{Y} \ / \ \forall \zeta^{\mathsf{V}}, \chi^{\mathsf{V}} \in \mathsf{V}\mathcal{Y}, \quad \zeta^{\mathsf{V}} \wedge \chi^{\mathsf{V}} \lrcorner\, p = 0\right\}. \tag{6}$$

Let $\iota_1 : \mathcal{M}_{\mathsf{DW}} \hookrightarrow \mathcal{M}$ be the canonical inclusion. Note that $\theta^{\mathsf{DW}} := \iota_1^\star\theta \in \Gamma(\mathcal{M}_{\mathsf{DW}}, \Lambda^n T^\star(\mathcal{M}_{\mathsf{DW}}))$ where $\theta \in \Gamma(\mathcal{M}, \Lambda^n T^\star(\mathcal{M}))$. Since $\mathrm{d}\big(\iota_1^\star\theta\big) = \iota_1^\star\mathrm{d}\theta = \iota_1^\star\boldsymbol{\omega}$, we obtain $\boldsymbol{\omega}^{\mathsf{DW}} = \mathrm{d}\theta^{\mathsf{DW}} = \iota_1^\star\boldsymbol{\omega}$. We denote by $\theta^{\mathsf{DW}} := \theta|_{\mathcal{M}_{\mathsf{DW}}}$ the restriction of θ to $\mathcal{M}_{\mathsf{DW}}$. Working on $\mathcal{M}_{\mathsf{DW}}$ is equivalent to setting $p^{\mu_1\cdots\mu_j}_{i_1\cdots i_j} = 0$ for all $j > 1$ in the expression of θ given in (3). In local coordinates, the Poincaré-Cartan n-form is written as $\theta^{\mathsf{DW}} = \varkappa\beta + p^\mu_i \mathrm{d}y^i \wedge \beta_\mu$. Then, following the terminology used by *e.g.* Kijowski [63, 64], Cantrijn, Ibort and León [9], and Hélein [47], we introduce also the multisymplectic $(n+1)$-form

$$\boldsymbol{\omega}^{\mathsf{DW}} = \mathrm{d}\varkappa \wedge \beta + \sum_\mu\sum_i \mathrm{d}p^\mu_i \wedge \mathrm{d}y^i \wedge \beta_\mu. \tag{7}$$

1.1.4 Hamilton equation in DW formulation

The DW Hamiltonian function $H(x^\mu, y^i, p_i^\mu) = p_i^\mu v_\mu^i - L(x^\mu, y^i, v_\mu^i)$ is defined by introducing the Legendre transform $(x^\mu, y^i, v_\mu^i) \mapsto (x^\mu, y^i, p_i^\mu)$, with the multimomenta $p_i^\mu := \partial L/\partial v_\mu^i(x^\mu, y^i, v_\mu^i)$. If the Legendre transform is non singular, *i.e.* $\det(\partial^2 L/\partial v_\mu^i \partial v_\nu^j) \neq 0$, the Euler-Lagrange equations (5) are equivalent to the DW Hamilton equations:

$$\frac{\partial \varphi^i}{\partial x^\mu}(x) = \frac{\partial H}{\partial p_i^\mu}(x^\mu, \varphi^i(x), p_i^\mu(x)), \qquad \sum_\mu \frac{\partial p_i^\mu}{\partial x^\mu}(x) = -\frac{\partial H}{\partial y^i}(x^\mu, \varphi^i(x), p_i^\mu(x)), \tag{8}$$

Following [49], we introduce the Legendre correspondence in the context of the most general Lepagean theory by the function $W : \Lambda^n T\mathcal{Y} \times \Lambda^n T^\star\mathcal{Y} \to \mathbb{R}, (q, v, p) \mapsto \langle p, v\rangle - L(q, v)$, where

$$\langle p, v\rangle \cong \langle p, z\rangle = \langle p, z_1 \wedge, \cdots, \wedge z_n\rangle = \sum_{\boldsymbol{\mu}_1 < \cdots < \boldsymbol{\mu}_n} p_{\boldsymbol{\mu}_1, \cdots, \boldsymbol{\mu}_n} z_1^{\boldsymbol{\mu}_1} \cdots z_n^{\boldsymbol{\mu}_n}. \tag{9}$$

We have denoted by $z_\alpha^{\boldsymbol{\mu}}$ the coordinates of the vector fields $z_\alpha = \sum_{1 \leq \boldsymbol{\mu} \leq n+k} z_\alpha^{\boldsymbol{\mu}} \partial/\partial q^{\boldsymbol{\mu}} \in T_q\mathcal{Y}$, which are used to construct the decomposable n-vector field $z = z_1 \wedge \cdots \wedge z_n \in \Lambda_{\mathrm{N}}^n T\mathcal{Y} \in \Lambda_{\mathrm{N}}^n T\mathcal{Y} \cong \mathcal{P}$, see section 1.1.2. The Legendre correspondence is satisfied if and only if, for any $(q, v, w) \in \Lambda_{\mathrm{N}}^n T\mathcal{Y} \times \mathbb{R}$, and for any $(q, p) \in \Lambda^n T^\star\mathcal{Y}$, we have

$$\langle p, v\rangle - L(q, v) = w \qquad \text{and} \qquad \frac{\partial W}{\partial v}(q, v, p) = 0. \tag{10}$$

When the Legendre hypothesis is satisfied, *c.f.* [49, 50, 51], we denote $(q, v, w) \leftrightarrow (q, p)$. To obtain the DW Hamilton equations, we restrict ourselves to the manifold $\mathcal{M}_{\mathsf{DW}}$ with a Hamiltonian function $\mathcal{H} : \mathcal{M}_{\mathsf{DW}} = \Lambda_1^n T^\star\mathcal{Y} \subset \Lambda^n T^\star\mathcal{Y} \to \mathbb{R}$. Only when the Legendre correspondence is non degenerate we have a unique correspondence $(q, v) \leftrightarrow (q, p)$, *i.e.* for any $(q, p) \in \mathcal{M}_{\mathsf{DW}}$ there exists a unique element $(q, v) \in T\mathcal{Y} \otimes T^\star\mathcal{X}$ such that $(p, v) \leftrightarrow (q, p)$. The DW Hamilton equations (or the generalized Hamilton equations, as termed in [49, 50]) are to be thought of as necessary and sufficient conditions on the map $x \mapsto (q(x), p(x)) := (x^\mu, \varphi^i(x), \varkappa(x), p_i^\mu(x))$ such that there exist fields $x \mapsto \varphi(x)$ for which:

- The Legendre condition is satisfied for any $x \in \mathcal{X}$, $(x, \varphi(x), \mathrm{d}\varphi(x)) \leftrightarrow (q(x), p(x))$.
- The fields $x \mapsto \varphi(x)$ are solutions of the Euler-Lagrange equations (5), which are related to the Lagrangian density $L(x, \varphi(x), \mathrm{d}\varphi(x))$.

Note that we can always write $\mathcal{H}(q, p) = \mathcal{H}(x^\mu, y^i, \varkappa, p_i^\mu) = \varkappa + H(x^\mu, y^i, p_i^\mu) = \varkappa + H(q, p)$ and then work on the level set $\mathcal{H}^{-1}(0)$. The variable $\varkappa = p_{1\cdots n}$ is seen as the canonical variable conjugate to the volume form β, see [50, 51]. If we fix $\mathcal{H}(q, p) = 0$, then $\varkappa = -H(q, p)$. In this case, the pre-multisymplectic $(n+1)$-form $\boldsymbol{\omega}^\circ := \boldsymbol{\omega}^{\mathsf{DW}}|_{\mathcal{H}=0}$ is

$$\boldsymbol{\omega}^\circ = \mathrm{d}p_i^\mu \wedge \mathrm{d}z^i \wedge \beta_\mu - \mathrm{d}H \wedge \beta, \tag{11}$$

the exterior derivative of the Poincaré-Cartan n-form $\theta^{\mathsf{PC}} := p_i^\mu \mathrm{d}z^i \wedge \beta_\mu - H\beta$, see Gotay [40, 41, 42], the analogue of the Poincaré-Cartan form of mechanics in the multisymplectic context.

We denote by $\mathcal{C}_\circ$ the level set $\mathcal{H}^{-1}(0) = \big\{(q,p) \in \mathcal{M} = \Lambda^n_1 T^\star \mathbf{3} / \mathcal{H}(q,p) = 0\big\}$. The triple $(\mathcal{C}_\circ := \mathcal{H}^{-1}(0), \boldsymbol{\omega}|_{\mathcal{C}_\circ}, \beta|_{\mathcal{C}_\circ})$ is a n-phase space, where $\beta|_{\mathcal{C}_\circ}$ is a nowhere vanishing volume n-form, and $\boldsymbol{\omega}|_{\mathcal{C}_\circ}$ is a closed $(n+1)$-form, see Kijowski and Szczyrba [63, 64, 65, 66] and Hélein [47]. We consider the n-dimensional submanifold $\Gamma \subset \mathcal{M}_{\mathsf{DW}}$, *i.e.* the Hamiltonian n-curve defined by $\Gamma = \{(x^\mu, y^i, p^\mu_i) \ / \ y^i = \varphi^i(x) \ , \ p^\mu_i = \partial L / \partial v^i_\mu(x^\mu, \varphi^i(x), \partial_\mu \varphi^i(x))\}$. Then, on the level set $\mathcal{C}_\circ \subset \mathcal{M}_{\mathsf{DW}}$, the DW system is written in geometric form as

$$\forall m \in \Gamma, \quad \forall X \in \Lambda^n T_m \Gamma, \quad X \lrcorner \boldsymbol{\omega}^\circ = 0 \quad \text{and} \quad \exists X \in \Lambda^n T_m \Gamma, \quad X \lrcorner \beta_m \neq 0. \tag{12}$$

We refer to section 3 for more details on the pre-multisymplectic scenario, where we reproduce the DW Hamilton system of equations, which in turn is equivalent to the Einstein system.

1.2 First order Palatini formulation of vielbein gravity

Dynamics of General Relativity (GR) is described by the Einstein's equations. They are obtained from the Einstein-Hilbert action functional

$$\mathcal{S}_{\mathsf{EH}}[\mathsf{g}_{\mu\nu}] = \kappa \int_{\mathcal{X}} L_{\mathsf{EH}}[\mathsf{g}_{\mu\nu}]\beta = \kappa \int_{\mathcal{X}} R\sqrt{-\mathsf{g}}\beta, \tag{13}$$

where $\kappa := (16\pi G)^{-1}$. The Einstein-Hilbert Lagrangian density is $L_{\mathsf{EH}}[\mathsf{g}_{\mu\nu}]$. The functional (13) depends on the metric $\mathsf{g}_{\mu\nu}$ and its first and second derivatives. In this approach the metric is the dynamical variable and it satisfies the Euler-Lagrange equations. The fundamental objects: the Levi-Civita connection $\Gamma^\rho_{\mu\nu}$ and the curvature tensor $R^\rho{}_{\mu\nu\sigma}$, are expressed via the metric ant its derivatives. In such a framework, GR is described as a *metric* theory. The variational principle is applied to the functional $\mathcal{S}_{\mathsf{EH}}[\mathsf{g}_{\mu\nu}]$. Variations with respect to the metric $\mathsf{g}_{\mu\nu}$ lead to the vacuum Einstein field equations

$$G_{\mu\nu} = R_{\mu\nu} - (1/2)\mathsf{g}_{\mu\nu} R = 0. \tag{14}$$

Classical GR can be also formulated in terms of the *vierbein* e^I_μ, or *vielbein* in the n-dimensional case, and the spin connection ω^{IJ}_μ, see section 1.3 for details. The passage from GR seen as a metric theory to the first order Palatini action of vielbein gravity is built, as emphasized in [93], in two steps. The first step is the Palatini first order theory. We consider the metric g and the connection Γ as independent variables. We write

$$\mathcal{S}_{\mathsf{Palatini}}[\mathsf{g}, \Gamma] = \kappa \int_{\mathcal{X}} \sqrt{-\mathsf{g}}\mathsf{g}^{\mu\nu} R_{\mu\nu}[\Gamma]\beta, \tag{15}$$

and we perform the variations of Γ and g independently. The variations with respect to the connection coefficients set the connection Γ to be the Levi-Civita affine connection, while variations with respect to the metric yield the Einstein vacuum equations (14). The second step concerns the use of the vierbein (tetrad) field. The Einstein-Palatini first order theory is given by the action

$$\mathcal{S}_{\mathsf{Palatini}}[e, \omega] = \frac{\kappa}{2} \int \epsilon_{IJKL} e^I \wedge e^J \wedge F^{KL}, \tag{16}$$

which uses of two independent dynamical fields: the *co-frame field* e^I, or the *solder form*, and the *spin connection* ω^{IJ}. We refer to appendix A for details on the action functional (16).

Using this formulation the Einstein's equations (14) are equivalent to the Euler-Lagrange system of equations

$$\begin{aligned} \mathrm{d}_\omega e^I = \mathrm{d}e^I + \omega^I{}_J \wedge e^J &= 0, \\ \epsilon_{IJKL} e^J \wedge F^{KL} &= 0, \end{aligned} \tag{17}$$

see [3, 45]. We call the action functional $\mathcal{S}_{\mathsf{Palatini}}[e,\omega]$ given in (16) the first order Palatini action of vielbein gravity.

1.3 Vielbein gravity: dynamical fields

As emphasized in many papers, *e.g.* [25, 45, 68, 69], the concept of *orthonormal moving frame*, or vielbein, is distinct from the concept of the *solder form*. A *moving frame* $\mathbf{e}_\mu(x)$, or *repère mobile* of Cartan [13, 14], is thought of as a section $\mathbf{e}_\mu(x) : \mathcal{X} \to L(\mathcal{X})$ of the linear frame bundle $L(\mathcal{X})$. In the same way, an orthonormal moving frame $\mathbf{e}_I(x)$ is a section of the Lorentz frame bundle $L_{SO(1,3)}(\mathcal{X})$. We denote a local frame as $\{\mathbf{e}_\mu^{(\alpha)}\}$ defined on an open subset $\mathcal{U}_{(\alpha)} \subset \mathcal{X}$, where the index (α) is related to a choice of trivialization. If the space-time manifold is parallelizable, the local nature of the moving frame extends to a well-defined global object. The vielbein field is written as $\mathbf{e}_I = e_I^\mu(x)\partial_\mu$ and is related to the metric by the formula $\mathsf{g}_{\mu\nu} = e_\mu^I e_\nu^J \mathsf{h}_{IJ}$. Note that the dual object is $\mathbf{e}^I = e_\mu^I(x)\mathrm{d}x^\mu$. In the next section, the solder form is given as a global section of the bundle $\mathcal{V} \otimes T^\star\mathcal{X}$ over $\mathcal{X}$, see the right side of figure 1. The solder form is canonically represented by a family of local frames $\{\mathbf{e}_\mu^{(\alpha)}\}$ on the space-time manifold and is termed alternatively the vielbein field or *co-frame field*. In the subsequent section, we offer some basic remarks about the interplay between the concept of vielbein, *i.e* a section of the orthonormal frame bundle, as opposed to the one of *solder form*, or «*forme de soudure*» [22], and the related description of the co-frame field as a bundle isomorphism.

1.3.1 Co-frame field: the solder form

In the first order Palatini formulation of vielbein gravity, space-time is represented by an n-dimensional oriented manifold $\mathcal{X}$ which is not equipped with a metric *a priori*. The metric is obtained via the pullback along the co-frame field, or solder form $e : T\mathcal{X} \to \mathcal{V}$. Then, we work in terms of the bundle isomorphism $e : T\mathcal{X} \to \mathcal{V}$ between the not necessarily trivial tangent bundle $T\mathcal{X} \to \mathcal{X}$ and the vector bundle $\mathcal{V} \to \mathcal{X}$, see figure 1-[1]. The isomorphism e is equivalently seen as a section of the vector bundle $\mathcal{V} \otimes T^\star\mathcal{X} \to \mathcal{X}$ such that for any $x \in \mathcal{X}$, e_x is an isomorphism, see figure 1-[2]. Note that $\mathcal{V}_x$ is the *internal* space. The notion of solder form was introduced by Ehresmann in [22], see also [68, 69]. As emphasized in [3, 109], the name *co-frame* is related to the case the manifold is parallelizable, the tangent bundle is trivial, and the bundle isomorphism $e : T\mathcal{X} \to \mathcal{V} = \mathcal{X} \times \mathbb{R}^{1,3}$ is equivalent to a choice of trivialization. In this context, the solder form is identified locally, on any tangent space $T_x\mathcal{X}$, with the co-frame $e_x : T_x\mathcal{X} \to \mathbb{R}^{1,3}$.

1.3.2 The co-frame field: covariant exterior derivative

In this section, we consider the solder form $e \in \Omega^1(\mathcal{X},\mathcal{V}) = \Gamma(\mathcal{V}) \otimes \Omega^1(\mathcal{X})$ previously introduced in section 1.3.1. Let $\mathfrak{e}_I$ be a frame on the vector space $\mathcal{V}_x := \mathbb{R}^{1,3}$, the Minkowski

$$\begin{tikzcd} T\mathcal{X} \arrow[rr, "e"] \arrow[dr] & & \mathcal{V} \arrow[dl] \\ & \mathcal{X} & \end{tikzcd} \quad [1] \qquad \begin{tikzcd} T^\star\mathcal{X}\otimes\mathcal{V} \arrow[d] \\ \mathcal{X} \arrow[u, bend left, "e"] \end{tikzcd} \quad [2]$$

Figure 1. [1] The solder form the co-frame field as a bundle isomorphism. [2] Equivalently the co-frame field is pictured as a global section of $T^\star\mathcal{X}\otimes\mathcal{V}\longrightarrow\mathcal{X}$. Note that $\mathcal{V}$ does not need to be trivial.

space. Let $\mathbf{e}^\mu$ be a moving co-frame, locally defined on $\Omega^1(\mathcal{X})$ (on an open subset $\mathcal{U}_{(\alpha)}\subset\mathcal{X}$). Locally, for $x\in\mathcal{U}_{(\alpha)}$, we write $e=e^I_\mu\mathbf{e}^\mu\otimes\mathfrak{e}_I=\mathfrak{e}_Ie^I_\mu\mathbf{e}^\mu=\mathfrak{e}_Ie^I$, *i.e.* e is decomposed with respect to the basis $\mathfrak{e}_I\otimes\mathbf{e}^\mu$ without any reference to space-time indices. We use the covariant derivative $D:\Gamma(\mathcal{V})\longrightarrow\Omega^1(\mathcal{X},\mathcal{V})=\Gamma(\mathcal{V})\otimes\Omega^1(\mathcal{X})$. Let σ be a section of the vector bundle $\mathcal{V}\to\mathcal{X}$ so that $D\sigma$ is a section 1-form, $D\sigma\in\Gamma(\mathcal{V})\otimes\Omega^n(\mathcal{X})=\Omega^n(\mathcal{X},\mathcal{V})$. By means of the covariant exterior derivative defined for any $\lambda=(1/n!)\lambda^I_{\mu_1\cdots\mu_n}\mathbf{e}^{\mu_1}\wedge\cdots\wedge\mathbf{e}^{\mu_n}\otimes\mathfrak{e}_I\in\Omega^n(\mathcal{X},\mathcal{V})$ by

$$\begin{array}{rcl} \mathrm{d}_\omega:\Omega^n(\mathcal{X},\mathcal{V}) & \longrightarrow & \Omega^{n+1}(\mathcal{X},\mathcal{V}), \\ \lambda & \longmapsto & \mathrm{d}_\omega\lambda. \end{array} \tag{18}$$

We obtain the expression of $\mathrm{d}_\omega e\in\Gamma(\mathcal{V})\otimes\Omega^2(\mathcal{X})$, *i.e.*

$$\begin{array}{rcl} \mathrm{d}_\omega e & = & \mathrm{d}_\omega\big(\mathfrak{e}_Ie^I_\mu\big)\wedge\mathbf{e}^\mu+\mathfrak{e}_Ie^I_\mu\mathrm{d}\mathbf{e}^\mu=(D\mathfrak{e}_I)e^I_\mu\wedge\mathbf{e}^\mu+\mathfrak{e}_I\mathrm{d}e^I_\mu\wedge\mathbf{e}^\mu+\mathfrak{e}_Ie^I_\mu\mathrm{d}\mathbf{e}^\mu, \\ & = & \omega^J_{\nu I}\mathfrak{e}_J\mathbf{e}^\nu e^I_\mu\wedge\mathbf{e}^\mu+\mathfrak{e}_I\partial_\nu e^I_\mu\mathbf{e}^\mu\wedge\mathbf{e}^\nu+\mathfrak{e}_Ie^I_\mu\mathrm{d}\mathbf{e}^\mu, \end{array} \tag{19}$$

where we have used in (19) the formula $D\mathfrak{e}_I=\omega^J_{\nu I}\mathfrak{e}_Je^\nu$ as well as $\mathrm{d}e^I_\mu=\partial_\nu e^I_\mu e^\mu$. We refer to the section 1.3.4 for details on the connection ω^{IJ}_μ. For a non integrable moving co-frame we obtain $\mathrm{d}\mathbf{e}^\mu=-1/2\mathfrak{c}^\mu_{\rho\nu}\mathbf{e}^\rho\wedge\mathbf{e}^\nu$. Hence, in this case $\mathrm{d}_\omega e=\mathfrak{e}_J\big(\partial_\nu e^J_\mu+\omega^J_{\nu I}e^I_\mu-1/2e^J_\rho\mathfrak{c}^\rho_{\nu\mu}\big)\mathbf{e}^\nu\wedge\mathbf{e}^\mu$. For an integrable moving co-frame $\mathbf{e}^\mu=\mathrm{d}x^\mu$, we have $\mathrm{d}\mathbf{e}^\mu=0$, and we obtain

$$\mathrm{d}_\omega e=\mathfrak{e}_J\big(\partial_\nu e^J_\mu+\omega^J_{\nu I}e^I_\mu\big)\mathrm{d}x^\nu\wedge\mathrm{d}x^\mu\in\Gamma(\mathcal{V})\otimes\Omega^1(\mathcal{X}). \tag{20}$$

Now we write the object $\mathrm{d}_\omega e$ decomposed with respect to a basis of $\Gamma(\mathcal{V})\otimes\Omega^2(\mathcal{X})$, *i.e.* $\mathfrak{e}_I\otimes\mathbf{e}^\mu\wedge\mathbf{e}^\nu$. Hence $\mathrm{d}_\omega e$ is written as $\mathrm{d}_\omega e=(1/2)\,(\mathrm{d}_\omega e)^I_{\mu\nu}\,\mathfrak{e}_I\otimes\mathbf{e}^\mu\wedge\mathbf{e}^\nu$. The covariant exterior derivative d_ω and the gauge covariant derivative $\mathcal{D}$ are related by $\mathrm{d}_\omega e=\mathfrak{e}_I\mathcal{D}e^I$, where $\mathcal{D}e^I=\mathrm{d}e^I+\omega^I_J\wedge e^J$. Since $\omega^I{}_J=\omega^I_{\mu J}\mathrm{d}x^\mu$ and $\mathrm{d}e^I=\mathrm{d}(e^I_\mu\mathrm{d}x^\mu)=\mathrm{d}e^I_\mu\wedge\mathrm{d}x^\mu$ we obtain $\mathcal{D}e^I=\big(\partial_\mu e^I_\nu+\omega^I_{\mu J}e^J_\nu\big)\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu$.

1.3.3 The Lorentz spin connection

Let $(\mathcal{P},\mathcal{X},\pi,SO(1,3))$ be a principal fiber bundle with a gauge group $SO(1,3)$. We denote by $\mathfrak{g}$ the $\mathfrak{so}(1,3)$-Lie algebra. Equivalently, $\mathcal{P}$ is thought to be the total space of the h-orthonormal frame bundle over the space-time manifold. Here, h is the Minkowski metric. We consider an Ehresmann connection on $\mathcal{P}$ *i.e.* a smooth distribution of horizontal subspaces, see [22], along with an equivariance property. In a given trivialization we obtain from the connection 1-form $\boldsymbol{\omega}\in\Omega^1(\mathcal{P},\mathfrak{g})$ on $\mathcal{P}$ the local connection form a 1-form $\omega\in\Omega^1(\mathcal{X},\mathfrak{g})$ on $\mathcal{X}$. Note that the local connection form or *gauge potential* is the pull back of the connection

form $\boldsymbol{\omega}$ by a section $\sigma^{(\alpha)} : \mathcal{U}_\alpha \subset \mathcal{X} \to \mathcal{P}$ - and denoted as $\omega = (\sigma^{(\alpha)})^\star(\boldsymbol{\omega}) \in T^\star\mathcal{X} \otimes \mathfrak{g}$. The local connection form is only described *in* the local trivialization $\sigma^{(\alpha)}$ and therefore is a notion that depends on the choice of trivialization. In the context of vierbein gravity, the Lorentz spin connection is written as $\omega = \omega_\mu \mathrm{d}x^\mu = \mathfrak{b}_i \omega^i_\mu \mathrm{d}x^\mu = \omega^i_\mu \mathrm{d}x^\mu \otimes \mathfrak{b}_i$, where $(\mathfrak{b}_1, \cdots, \mathfrak{b}_6)$ is a basis of $\mathfrak{g}$. Note that in the formulation of dreibein gravity, the basis of the $\mathfrak{so}(1,2)$-Lie algebra is denoted $(\mathfrak{b}^{(1,2)}_1, \cdots, \mathfrak{b}^{(1,2)}_3)$. We induce a connection on associated bundles $\mathcal{P} \times_\rho \mathcal{V}$ via a representation ρ of the $SO(1,3)$ group, see [68, 69]. The image $\rho(\omega)$ of the gauge potential ω via the representation ρ gives the matrix connection $\rho(\omega) = \rho(\mathfrak{b}_i \omega^i_\mu \mathrm{d}x^\mu) = \omega^i_\mu \rho(\mathfrak{b}_i)\mathrm{d}x^\mu = \omega^i_\mu (\rho(\mathfrak{b}_i))^I_J \mathrm{d}x^\mu$. We denote $\rho(\mathfrak{b}_i) := \Delta_i = (\Delta_i)^I_J$, where $0 \leq I, J \leq 3$ are Lorentz Lie algebra indices. Working in a given representation, we simply denote the matrix elements by $\omega^I_J = \omega^I_{\mu J}\mathrm{d}x^\mu$ with $\omega^I_{\mu J} = \omega^i_\mu (\Delta_i)^I_J$. Alternatively, in section 1.3.5 the Lorentz spin connection is constructed on the vector bundle $\mathcal{V}$.

1.3.4 Lorentz spin connection: curvature and covariant exterior derivative

The curvature $F^{\boldsymbol{\omega}} \in \Omega^2(\mathcal{P}, \mathfrak{g})$ of the connection $\boldsymbol{\omega} \in \Omega^1(\mathcal{P}, \mathfrak{g})$ is written as $F^{\boldsymbol{\omega}} := \mathrm{d}_{\boldsymbol{\omega}}\boldsymbol{\omega} = \mathrm{d}\boldsymbol{\omega} + (1/2)[\boldsymbol{\omega}, \boldsymbol{\omega}]$, where for any $\boldsymbol{\lambda} = (1/n!)\boldsymbol{\lambda}^I_{\mu_1 \cdots \mu_n} \mathbf{e}^{\mu_1} \wedge \cdots \wedge \mathbf{e}^{\mu_n} \otimes \mathfrak{b}_I \in \Omega^n(\mathcal{P}, \mathfrak{g})$,

$$\begin{array}{rcl} \mathrm{d}_{\boldsymbol{\omega}} : \Omega^n(\mathcal{P}, \mathfrak{g}) & \longrightarrow & \Omega^{n+1}(\mathcal{P}, \mathfrak{g}), \\ \boldsymbol{\lambda} & \longmapsto & \mathrm{d}_{\boldsymbol{\omega}}\boldsymbol{\lambda}, \end{array} \tag{21}$$

is the covariant exterior derivative relative to $\boldsymbol{\omega}$. The pullback by a section $\sigma^{(\alpha)}$ gives the local expression of the connection form $\omega = (\sigma^{(\alpha)})^\star(\boldsymbol{\omega}) \in T^\star\mathcal{X} \otimes \mathfrak{g}$ and the curvature 2-form $F^{\boldsymbol{\omega}} \in \Omega^2(\mathcal{P}, \mathfrak{g})$, *i.e* $F^\omega = (\sigma^{(\alpha)})^\star F^{\boldsymbol{\omega}} \in \Omega^2(\mathcal{X}, \mathfrak{g})$. The Lie algebra-valued 2-form on space-time $F^\omega_{\mu\nu} = F_{\mu\nu}$ is written as $F = (1/2)F^i_{\mu\nu}\mathfrak{b}_i \otimes \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$. The curvature 2-form on the associated bundle is $\rho(F) \in \Omega^2(\mathcal{X}, \mathsf{End}(\mathcal{V}))$. In that case $F = (1/2)F_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$, where $F_{\mu\nu} = F^i_{\mu\nu}\Delta_i$. We denote $F^I_{\mu\nu J} := F^i_{\mu\nu}(\Delta_i)^I_J$. The curvature of the spin connection ω^{IJ}_μ is written as [3, 93]

$$F^{IJ}_{\mu\nu}[\omega] = 2\partial_{[\mu}\omega^{IJ}_{\nu]} + [\omega_\mu, \omega_\nu]^{IJ} = \partial_\mu \omega^{IJ}_\nu - \partial_\nu \omega^{IJ}_\mu + \omega^I_{\mu K}\omega^{KJ}_\nu - \omega^I_{\nu K}\omega^{KJ}_\mu. \tag{22}$$

Note that the covariant exterior derivative $\mathrm{d}_\omega \omega = \mathfrak{b}_i \mathcal{D}\omega^i$, or equivalently $\mathrm{d}_\omega \omega = \Delta_{IJ}\mathcal{D}\omega^{IJ}$, is given by means of the object

$$\mathcal{D}\omega^{IJ} \quad = \mathrm{d}\omega^{IJ} + \omega^I_K \wedge \omega^{KJ} + \omega^J_K \wedge \omega^{IK} = \mathrm{d}\omega^{IJ} + \omega^I_K \wedge \omega^{KJ} - \omega^J_K \wedge \omega^{KI}, \tag{23}$$

written in components $(\mathcal{D}\omega)^{IJ}_{\mu\nu} = 2\partial_{[\mu}\omega^{IJ}_{\nu]} + 2\omega_{[\mu}{}^I{}_K\omega^{KJ}_{\nu]} - 2\omega_{[\mu}{}^J{}_K\omega^{KI}_{\nu]}$. The variation $\delta F^{IJ}_{\mu\nu}$ of the curvature of the Lorentz spin connection is expressed via the covariant exterior derivative $\delta F^{IJ}_{\mu\nu} = 2\mathcal{D}_{[\mu}\delta\omega^{IJ}_{\nu]}$, see *e.g.* [3, 93].

1.3.5 The pullbacks $\mathsf{g} = e^\star\mathsf{h}$ and $\nabla = e^\star D$

If we have a metric h on $\mathcal{V}$, then we obtain a metric on $\mathcal{X}$ by pullback $\mathsf{g} = e^\star\mathsf{h}$, where $\forall x \in \mathcal{X}, \forall \xi, \zeta \in T_x\mathcal{X}$: $(e^\star\mathsf{h})_x(\xi, \zeta) = \mathsf{g}_x(e_x(\xi), e_x(\zeta))$. In this case, the vector space $\mathcal{V}$ is equipped with a connection D, so that we obtain the connection $\nabla = e^\star D$ on $T\mathcal{X}$ described as follows: $\forall \xi, \sigma \in \Gamma(\mathcal{X}) = T\mathcal{X}$, $\nabla_\xi \sigma = e^\star(D_\xi e(\sigma))$, where $D : \Gamma(\mathcal{X}) \times \Gamma(\mathcal{X}, \mathcal{V}) \longrightarrow \Gamma(\mathcal{X}, \mathcal{V}) : (X, \sigma) \mapsto D_X\sigma$. The set of 1-forms ω^J_I defined on an open subset $\mathcal{U}_{(\alpha)} \subset \mathcal{X}$ by $\omega^J_I = \omega^J_{\mu I}\mathrm{d}x^\mu$

gives, for any $\xi \in \mathfrak{X}(\mathcal{U}_{(\alpha)})$, $\xi^\mu \omega^J_{\mu I} = \omega^J_I(\xi)$. Then $D_\xi s = D_\xi(\sigma^I e_I) = \mathrm{d}s^I(\xi)e_I + \omega^J_I(\xi)\sigma^I \mathfrak{e}_J$. We have $(D_\mu \sigma)^I = \partial_\mu \sigma^I + \omega^I_{\mu J}\sigma^J$. Now, using the solder form we obtain a connection on $T\mathcal{X}$. Pulling back the connection on $\mathcal{V}$ via $\nabla_\xi \sigma = e^\star(D_\xi e(\sigma))$, we get the covariant derivative's components:

$$(\nabla_\mu \xi)^\nu = \partial_\mu \xi^\nu + (e^\nu_I \partial_\mu e^I_\rho + e^\nu_I \omega^I_{\mu J} e^J_\rho)\xi^\rho. \tag{24}$$

However, we have also $(\nabla_\mu \xi)^\nu = \partial_\mu \xi^\nu + \Gamma^\nu_{\mu\rho}\xi^\rho$. Therefore, $\Gamma^\nu_{\mu\rho} = e^\nu_I \partial_\mu e^I_\rho + e^\nu_I \omega^I_{\mu J} e^J_\rho$ and we reproduce the well known relation between the spin connection coefficients and the Christoffel symbol $\Gamma^\rho_{\mu\nu}$: $\partial_\mu e^I_\nu + e^K_\nu \omega^I_{\mu K} - \Gamma^\rho_{\mu\nu} e^I_\rho = 0$. We summarize the two pull-backs of interest:

$$\begin{tikzcd} \mathsf{g} = e^\star \mathsf{h} \text{ on } T\mathcal{X} & \mathsf{h} \text{ on } \mathcal{V}, \arrow[l, "\mathsf{g}_m(e_m(\xi), e_m(\sigma)) = (e^\star \mathsf{h})_m(\xi,\sigma)"'] \\ \nabla = e^\star D \text{ on } T\mathcal{X} & D \text{ on } \mathcal{V}, \arrow[l, "\nabla_\xi \sigma = e^\star(D_\xi e(\sigma))"'] \end{tikzcd}$$

which are related to the metric and to the spin connection, respectively. The bundle isomorphism gives a correspondence between objects on the tangent bundle $T\mathcal{X}$ and the internal bundle $\mathcal{V}$. The curvature of the connection D is the 2-form $F^{IJ} = \mathrm{d}\omega^{IJ} + \omega^I_K \wedge \omega^{KJ}$, written in components as $F^{IJ}_{\mu\nu} = \omega^{IJ}_{[\mu,\nu]} + \omega^I_{[\mu K}\omega^{KJ}_{\nu]}$. The bundle isomorphism e maps the curvature of D to that of ∇ with the relation $R^{\rho\sigma}_{\mu\nu} = F^{IJ}_{\mu\nu} e^\rho_I e^\sigma_J$. Finally, we recall the expressions of the Ricci tensor $R_\mu{}^\nu = F^{IJ}_{\mu\sigma} e^\sigma_I e^\nu_J$ and the scalar curvature $R = R^\mu_\mu = e^\sigma_I e^\rho_J F^{IJ}_{\sigma\rho}$.

1.4 Configuration space

In section 1.4.1, we first briefly present two *fully* covariant formulations *i.e.* that does not rely on any choice of trivialization of some principal bundle. Then, in section 1.4.2, we present the less sophisticated configuration space that we will use in sections 2 - 4. The latter being dependent of a given trivialization of the principal bundle $(\mathcal{P}, \mathcal{X}, \pi, SO(1,3))$.

1.4.1 Fully covariant configuration space

We mention two formalisms to take into account the viewpoint of the geometry of the principal bundle $(\mathcal{P}, \mathcal{X}, \pi, SO(1,3))$. The first is related to the Gauge Natural Bundle approach, see Nijenhuis [83], Eck [24], Kolář, Michor and Slovák [69], Fatibene and Francaviglia [25]. We construct the gauge natural bundle $\mathcal{P}_\rho := \big(\mathcal{P} \times_{\mathcal{X}} L(\mathcal{X})\big) \times GL(n)$ associated to the $SO(1,3)$-principal bundle $\mathcal{P}$, see [25, 81]. We denote by $\mathcal{Y}^{\mathsf{GNB}}_{\mathsf{purely\text{-}frame}} := \mathcal{P}_\rho$ the covariant configuration space of the purely-frame gravitational theory. In the frame-affine framework, *i.e.* based on the Palatini action of vielbein gravity, the covariant configuration space is $\mathcal{Y}^{\mathsf{GNB}}_{\mathsf{frame\text{-}affine}} := \mathcal{P}_\rho \times \mathcal{Y}_{\mathcal{P}}$, where $\mathcal{Y}_{\mathcal{P}}$ is the space of connection of the principal $SO(1,3)$-bundle. This fruitful approach has been used in the context of gravity and Einstein-Cartan gravity by Fatibene and Francaviglia [25, 26], and Matteucci [81]. Afterward, the gauge natural approach blends with the multisymplectic viewpoint in the papers by Bruno, Cianci and Vignolo [5, 6]. We refer also to [7, 8] for the similar treatment of the Yang-Mills fields. In this framework, the gauge symmetry is obtained via some reduction of the geometry of connections on the principal bundle.

Another fully covariant multisymplectic formulation for the Yang-Mills fields is given by Hélein [48]. We give a brief idea of the corresponding multimomentum phase space for vielbein gravity, following this line of thought. Let $\mathfrak{p} := \mathfrak{iso}(1,3) = \mathfrak{so}(1,3) \ltimes \mathbb{R}^{1,3}$ be the Poincaré Lie algebra. We consider a $\mathfrak{p}$-valued connection 1-form $\boldsymbol{\eta} \in \Omega^1(\mathcal{P}, \mathfrak{p})$ defined on the principal fiber bundle $(\mathcal{P}, \mathcal{X}, \pi, SO(1,3))$ which satisfies some normalization and equivariance conditions. The covariant configuration space is $\mathcal{Y}^{\mathsf{cov}} := \mathfrak{p} \otimes T^\star\mathcal{P} \to \mathcal{P}$. The multimomentum phase space is $\mathcal{M}^{\mathsf{cov}}_{\mathsf{DW}} = \Lambda^m_1 T^\star(\mathfrak{p} \otimes T^\star\mathcal{P})$, the DW multisymplectic manifold fibered over $\mathfrak{p} \otimes T^\star\mathcal{P}$. We refer to [48] for more details on the dimension $m = n + r$, where $n = \dim(\mathcal{X})$, and $r = \dim(\mathfrak{p})$.

1.4.2 Trivialization dependent covariant configuration space

Any connection D on the internal bundle $\mathcal{V}$ can be written as $D = D^\circ + \omega$, where $\omega \in \Omega^1(\mathcal{X}, \mathsf{End}(\mathcal{V}))$ is the matrix connection and $D^\circ : \Gamma(\mathcal{X}) \otimes \Gamma(\mathcal{V}) \to \Gamma(\mathcal{V})$ is the standard flat connection. Note that $D^\circ_\zeta \sigma^{(\alpha)} = \zeta(\sigma^{(\alpha)}) = \zeta((\sigma^\alpha)^I)\mathfrak{e}_I$ is trivialization dependent. We restrict ourselves, as suggested in [50], to this local approach which depends on a particular choice of trivialization of the principal bundle $(\mathcal{P}, \mathcal{X}, \pi, SO(1,3))$. The *covariant configuration space* is the bundle $\mathcal{Y} := \mathfrak{iso}(1,3) \otimes T^\star\mathcal{X}$ over $\mathcal{X}$. Albeit non fully covariant from the viewpoint of the geometry of gauge fields, we nevertheless use this approach in sections 2 - 4.

2 DW formulation of vielbein gravity

In this section we describe the DW Hamiltonian formulation of the first order Palatini action of vielbein gravity. First, let us begin with the notations and the geometrical background related to the covariant configuration space used in the paper.

2.1 Geometrical setting and notations

Two independent dynamical fields are $e \in \mathcal{V} \otimes T^\star\mathcal{X}$ and $\omega \in \mathfrak{so}(1,3) \otimes T^\star\mathcal{X} := \mathfrak{g} \otimes T^\star\mathcal{X}$. The former is the solder form (or co-frame field), locally seen as a $\mathbb{R}^{(1,3)}$-valued 1-form, whereas the latter is the Lorentzian spin connection, a $\mathfrak{g}$-valued 1-form. Let $\mathcal{Y} = \mathfrak{p} \otimes T^\star\mathcal{X}$ be the bundle of $\mathfrak{p} := \mathfrak{iso}(1,3)$-valued 1-forms over the space-time manifold $\mathcal{X}$, *i.e.* the covariant configuration space. A point in $\mathcal{Y} = \mathfrak{p} \otimes T^\star\mathcal{X}$ is denoted as (x, e_x, ω_x), where $x \in \mathcal{X}$, $e_x \in \mathcal{Y}^e_x := \mathbb{R}^{1,3} \otimes T^\star_x\mathcal{X}$ and $\omega_x \in \mathcal{Y}^\omega_x := \mathfrak{g} \otimes T^\star_x\mathcal{X}$. Let us consider the maps $e : \mathcal{X} \to \mathcal{Y}^e = \mathbb{R}^{(1,3)} \otimes T^\star\mathcal{X}$ and

$$\begin{tikzcd} \mathcal{Y}^e = \mathbb{R}^{1,3} \otimes T^\star\mathcal{X} \arrow[d, "\pi^e_{\mathcal{X}}"] \\ \mathcal{X} \arrow[u, bend left, "e"] \end{tikzcd} \quad [1] \qquad \begin{tikzcd} \mathcal{Y}^\omega = \mathfrak{so}(1,3) \otimes T^\star\mathcal{X} \arrow[d, "\pi^\omega_{\mathcal{X}}"] \\ \mathcal{X} \arrow[u, bend left, "\omega"] \end{tikzcd} \quad [2] \qquad \begin{tikzcd} \mathcal{Y} = \mathfrak{iso}(1,3) \otimes T^\star\mathcal{X} \arrow[d, "\pi_{\mathcal{X}}"] \\ \mathcal{X} \arrow[u, bend left, "(e,\omega)"] \end{tikzcd} \quad [3]$$

Figure 2. [1] The fiber bundle $\mathcal{Y}^e := \mathbb{R}^{1,3} \otimes T^\star\mathcal{X}$ over $\mathcal{X}$. [2] The fiber bundle $\mathcal{Y}^\omega := \mathfrak{so}(1,3) \otimes T^\star\mathcal{X}$ over $\mathcal{X}$. [3] The covariant configuration space is the fiber bundle $\mathcal{Y} := \mathfrak{p} \otimes T^\star\mathcal{X} = \mathfrak{iso}(1,3) \otimes T^\star\mathcal{X}$ over the space-time manifold $\mathcal{X}$.

$\omega : \mathcal{X} \mapsto \mathcal{Y}^\omega = \mathfrak{g} \otimes T^\star\mathcal{X}$ written as

$$\begin{array}{rcl} \mathcal{X} & \to & \mathcal{Y}^e = \mathbb{R}^{(1,3)} \otimes T^\star\mathcal{X}, \\ x & \mapsto & (x, e(x)) = (x, e^I_\mu(x)\mathrm{d}x^\mu \otimes \mathfrak{e}_I), \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathcal{X} & \to & \mathcal{Y}^\omega = \mathfrak{g} \otimes T^\star\mathcal{X}, \\ x & \mapsto & (x, \omega(x)) = (x, \omega^{IJ}_\mu(x)\mathrm{d}x^\mu \otimes \Delta_{IJ}). \end{array}$$

These maps are equivalently thought of as sections of $\mathcal{Y}^e$ and $\mathcal{Y}^\omega$ (see figure 2-[1] and 2-[2], respectively). We introduce also the map $(e, \omega) : \mathcal{X} \to \mathcal{Y}$, that is written as

$$\begin{array}{rcl} \mathcal{X} & \to & \mathcal{Y} = \mathfrak{p} \otimes T^\star\mathcal{X}, \\ x & \mapsto & (x, e(x), \omega(x)) = (x, e^I_\mu(x)\mathrm{d}x^\mu \otimes \mathfrak{e}_I, \omega^{IJ}_\mu(x)\mathrm{d}x^\mu \otimes \Delta_{IJ}). \end{array} \tag{25}$$

Any choice as $(e(x), \omega(x))$ is equivalent to the data of an n-dimensional submanifold of the fiber bundle $\mathcal{Y}$ and is equivalently thought of as a section $\sigma^{(\alpha)} : \mathcal{U}^{(\alpha)} \subset \mathcal{X} \to \mathcal{Y}$, where $\sigma : x \mapsto \sigma(x) = (x, e(x), \omega(x))$, see figure 2-[3]. Finally, the set of local coordinates in the covariant configuration bundle $\mathcal{Y}$ is equivalently denoted as $(x^\mu, e^I_\mu, \omega^{IJ}_\mu)$.

2.2 The bundle $\mathcal{P} = T\mathcal{Y} \otimes_{\mathcal{Y}} T^\star\mathcal{X}$

For any point $x \in \mathcal{X}$ the differential $(\mathrm{d}e)_x : T_x\mathcal{X} \mapsto T_{(x,e_x)}\mathcal{Y}^e$ is seen as an element of $T^\star_x\mathcal{X} \otimes T_{(x,e_x)}(T^\star\mathcal{X} \otimes \mathcal{V})$ canonically identified with $T^\star_x\mathcal{X} \otimes T^\star_x\mathcal{X} \otimes \mathcal{V}$. Analogously, $(\mathrm{d}\omega)_x : T_x\mathcal{X} \mapsto T_{(x,\omega_x)}\mathcal{Y}^\omega$ is seen as an element of $T^\star_x\mathcal{X} \otimes T_{(x,\omega_x)}(T^\star\mathcal{X} \otimes \mathfrak{g})$ canonically identified with $T^\star_x\mathcal{X} \otimes T^\star_x\mathcal{X} \otimes \mathfrak{g}$. Let us consider the bundles $\mathcal{P}^e := e^\star T\mathcal{Y}^e \otimes T^\star\mathcal{X}$ and $\mathcal{P}^\omega := \omega^\star T\mathcal{Y}^\omega \otimes T^\star\mathcal{X}$ over the space-time manifold $\mathcal{X}$. These bundles enable us to describe the differentials $\mathrm{d}e$ and $\mathrm{d}\omega$ of the map e and ω as sections of the bundles $\mathcal{P}^e$ and $\mathcal{P}^\omega$, respectively. In particular, the points $(x, v^e) \in \mathcal{P}^e$ and $(x, v^\omega) \in \mathcal{P}^\omega$ are described by

$$v^e = \sum_{\mu,\nu}\sum_I v^I_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu \otimes \mathfrak{e}_I, \qquad v^\omega = \sum_{\mu,\nu}\sum_{I<J} v^{IJ}_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu \otimes \Delta_{IJ}, \tag{26}$$

where $v^I_{\mu\nu} := \partial_\mu e^I_\nu$ and $v^{IJ}_{\mu\nu} := \partial_\mu \omega^{IJ}_\nu$, respectively. Local coordinates on $\mathcal{P}^e$ and $\mathcal{P}^\omega$ are denoted by $(x^\mu, v^I_{\mu\nu})$ and $(x^\mu, v^{IJ}_{\mu\nu})$, respectively. Using the map (25), we introduce the bundle $\mathcal{P}^{(e,\omega)} := (e, \omega)^\star T\mathcal{Y} \otimes T^\star\mathcal{X}$ over $\mathcal{X}$. Note that $\mathcal{P}^{(e,\omega)} \subset \mathcal{P} := T\mathcal{Y} \otimes_{\mathcal{Y}} T^\star\mathcal{X}$. This bundle is the bundle over $\mathcal{Y} := \mathfrak{p} \otimes T^\star\mathcal{X}$, such that the fiber over (x, e_x, ω_x) is $T_{(x,e_x,\omega_x)}(\mathfrak{p} \otimes T^\star\mathcal{X}) \otimes T^\star_x\mathcal{X}$. In terms of local coordinates:

$$\mathcal{P} = \left\{(x^\mu, e^I_\mu, \omega^I_\mu, v^I_{\mu\nu}, v^{IJ}_{\mu\nu}) \ / \ (x^\mu, e^I_\mu, \omega^I_\mu) \in \mathcal{Y},\ (v^I_{\mu\nu}, v^{IJ}_{\mu\nu}) \in T_{(x,e_x,\omega_x)}\mathcal{Y} \otimes T^\star_x\mathcal{X}\right\}. \tag{27}$$

Subsequently, the covariant exterior derivatives $\mathrm{d}_\omega e$ and $\mathrm{d}_\omega\omega$ are described as sections of the bundle $\mathcal{P}$. Recall that

$$\begin{array}{rcl} \mathrm{d}_\omega e & = & (1/2)\,(\mathrm{d}_\omega e)^I_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \otimes \mathfrak{e}_I = (1/2)\left(\partial_\mu e^I_\nu + \omega_\mu{}^I_J e^J_\nu\right)\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \otimes \mathfrak{e}_I, \\ \mathrm{d}_\omega\omega & = & (1/2)\,(\mathrm{d}_\omega\omega)^{IJ}_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \otimes \Delta_{IJ} = (1/2)\left(\partial_\mu\omega^{IJ}_\nu + \omega_\mu{}^I_K\omega^{KJ}_\nu - \omega_\mu{}^J_K\omega^{KI}_\nu\right) \otimes \Delta_{IJ}. \end{array}$$

We now consider the fiber bundle of n-vector fields $\Lambda^n_{\mathrm{N}}T\mathcal{Y}$ over $\mathcal{Y}$. For any $(x, e_x, \omega_x) \in \mathcal{Y}$ the fiber $\Lambda^n_{\mathrm{N}}T_{(x,e_x,\omega_x)}(\mathfrak{p} \otimes T^\star\mathcal{X}) = \Lambda^n_{\mathrm{N}}T_{(x,e_x,\omega_x)}\mathcal{Y}$ can be identified with $\mathcal{P}_{(x,e_x,\omega_x)}$ via the diffeomorphism:

$$\begin{array}{rcl} T_{(x,e_x,\omega_x)}(\mathfrak{p} \otimes T^\star\mathcal{X}) \otimes T^\star_x\mathcal{X} & \to & \Lambda^n_{\mathrm{N}}T_{(x,e_x,\omega_x)}(\mathfrak{p} \otimes T^\star\mathcal{X}), \\ (\sum_{\mu,\nu}\sum_I (\mathrm{d}_\omega e)^I_{\mu\nu}\,\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu \otimes \mathfrak{e}_I, \sum_{\mu,\nu}\sum_{I<J} (\mathrm{d}_\omega\omega)^{IJ}_{\mu\nu}\,\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu \otimes \Delta_{IJ}) & \mapsto & z = z_1 \wedge ... \wedge z_n, \end{array}$$

where for any $1 \leq \alpha \leq n$, $z_\alpha = \dfrac{\partial}{\partial x^\alpha} + \sum_{1\leq\beta\leq n} z^I_{\alpha\beta}\dfrac{\partial}{\partial e^I_\beta} + \sum_{1\leq\beta\leq n} z^{IJ}_{\alpha\beta}\dfrac{\partial}{\partial \omega^{IJ}_\beta}$,

$$\begin{aligned} z^I_{\alpha\beta} &:= \partial_\alpha e^I_\beta + \omega_\alpha{}^I_J e^J_\beta, \\ z^{IJ}_{\alpha\beta} &:= \partial_\alpha \omega^{IJ}_\beta + \omega_\alpha{}^I_K \omega_\beta{}^{KJ} - \omega_\alpha{}^J_K \omega_\beta{}^{KI}. \end{aligned} \tag{28}$$

Now we consider the first order Palatini density $L_{\mathsf{Palatini}}[e,\omega] = \kappa \mathsf{vol}(e) e^\mu_I e^\nu_J F^{IJ}_{\mu\nu}[\omega]$, equivalently written as $L_{\mathsf{Palatini}}[e,\omega]\beta = (\kappa/4)\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma} e^I_\mu e^J_\nu F^{KL}_{\rho\sigma}[\omega]\beta$ (see appendix A for details). We now set the constant $\kappa := 1/2$, so that

$$\begin{aligned} L_{\mathsf{Palatini}}[e,\omega] &= (1/8)\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma} e^I_\mu e^J_\nu \big(\partial_\rho \omega^{KL}_\sigma - \partial_\sigma \omega^{KL}_\rho + \omega^K_{\rho M}\omega^{ML}_\sigma - \omega^K_{\sigma M}\omega^{ML}_\rho\big), \\ &= (1/2) e \left(e^\mu_I e^\nu_J - e^\nu_I e^\mu_J\right)\left(\partial_\mu \omega^{IJ}_\nu + \omega^I_{\mu K}\omega^{KJ}_\nu\right), \\ &= E^{[\mu}_I e^{\nu]}_J (\partial_\mu \omega^{IJ}_\nu + \omega^I_{\mu K}\omega^{KJ}_\nu), \end{aligned} \tag{29}$$

where we used the identity $E^{[\mu}_I e^{\nu]}_J := e e^{[\mu}_I e^{\nu]}_J = (1/4)\epsilon_{IJKL} e^K_\rho e^L_\sigma \epsilon^{\mu\nu\rho\sigma}$, see appendix B.5. The Lagrangian density $L[e,\omega] : \mathcal{P} \to \mathbb{R}$ is thought of as a function defined on the bundle $\mathcal{P}$, *i.e.* the bundle over $\mathcal{Y}$ with the fiber over a point $(x, e_x, \omega_x) \in \mathcal{Y}$ given by $T_{(x,e_x,\omega_x)}\mathcal{Y} \otimes_{\mathcal{Y}} T^\star_x\mathcal{X}$. Then, the set of local coordinates $(x^\mu, e^I_\mu, \omega^{IJ}_\mu, v^I_{\mu\nu}, v^{IJ}_{\mu\nu})$ on $\mathcal{P}$ is equivalently described, using the definitions (28), by the set $(x^\mu, e^I_\mu, \omega^{IJ}_\mu, z^I_{\mu\nu}, z^{IJ}_{\mu\nu})$ on $\Lambda^n_{\mathrm{N}} T\mathcal{Y}$. Alternatively, we can use the set of coordinates on the first jet bundle $J^1(\mathcal{Y})$, see for example [5, 6, 23]. We summarize these constructions in figure 3-[1].

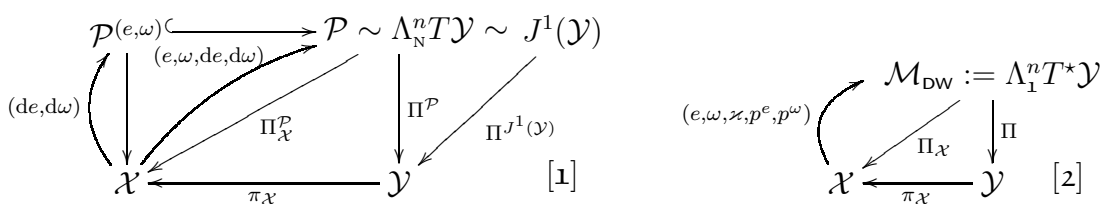

Figure 3. [1] The fiber bundles $\mathcal{P}^{(e,\omega)} = (e,\omega)^\star T\mathcal{Y} \otimes_{\mathcal{X}} T^\star\mathcal{X}$ and $\mathcal{P} := T\mathcal{Y} \otimes_{\mathcal{Y}} T^\star\mathcal{X}$, on which the Lagrangian density is defined. The latter is identified with the bundle of decomposable n-vector fields $\Lambda^n_{\mathrm{N}} T(\mathcal{Y})$ on the covariant configuration space $\mathcal{Y} = \mathfrak{p} \otimes T^\star\mathcal{X}$. [2] The DW multimomentum bundle $\mathcal{M}_{\mathsf{DW}} := \Lambda^n_1 T^\star\mathcal{Y}$ as a fiber bundle over $\mathcal{Y}$.

2.3 DW multisymplectic manifold and Legendre correspondence

Now we describe the DW multisymplectic manifold for the Palatini action of vielbein gravity. The multimomentum phase space is constructed on the covariant configuration space $\mathcal{Y} := \mathfrak{p} \otimes T^\star\mathcal{X}$, see the construction in figure 3-[2]. We present the notations used for the DW submanifold $\mathcal{M}_{\mathsf{DW}} \subset \mathcal{M} := \Lambda^n T^\star(\mathfrak{p} \otimes T^\star\mathcal{X})$, as introduced in section 1.1.3. The DW manifold is

$$\mathcal{M}_{\mathsf{DW}} = \{(x,e,\omega,p)/x \in \mathcal{X}, e \in \mathbb{R}(1,3) \otimes T^\star_x\mathcal{X}; \omega \in \mathfrak{g} \otimes T^\star_x\mathcal{X}, p \in \Lambda^n_1 T^\star(\mathfrak{p} \otimes T^\star\mathcal{X})\}, \tag{30}$$

In the DW formulation we consider all the components of the Poincaré-Cartan form, see (3), equal to zero except $p_{1\dots n} := \varkappa$, $p_{1\dots(\nu-1)(e^{IJ}_\mu)(\nu+1)\dots n} := p^{e_\mu\nu}_{IJ}$, and $p_{1\dots(\nu-1)(\omega^{IJ}_\mu)(\nu+1),\dots n} := p^{\omega_\mu\nu}_{IJ}$. Thus, we restrict ourselves to n-forms $p \in \Lambda^n T^\star_{(x,e_x,\omega_x)}\mathcal{Y}$ such that $\partial_{e^I_\mu} \wedge \partial_{e^I_\nu} \lrcorner\, p$, $\partial_{\omega^{IJ}_\mu} \wedge \partial_{\omega^{IJ}_\nu} \lrcorner\, p$, and $\partial_{e^I_\mu} \wedge \partial_{\omega^{IJ}_\nu} \lrcorner\, p$ are identically vanishing. Equivalently, the DW multisymplectic manifold is specified as

$$\mathcal{M}_{\mathsf{DW}} = \left\{(x,e,\omega,\varkappa\beta + p^{e_\mu\nu}_I \mathrm{d}e^I_\mu \wedge \beta_\nu + p^{\omega_\mu\nu}_{IJ}\mathrm{d}\omega^{IJ}_\mu \wedge \beta_\nu)/(x,e,\omega) \in \mathcal{Y}, \varkappa, p^{e_\mu\nu}_I, p^{\omega_\mu\nu}_{IJ} \in \mathbb{R}\right\}.$$

We consider the following Poincaré-Cartan $\theta^{\mathsf{DW}}_{(q,p)}$ n-form, for any $(q,p) \in \mathcal{M}_{\mathsf{DW}} := \Lambda^n_1 T^\star\mathcal{Y}$

$$\theta^{\mathsf{DW}}_{(q,p)} := \varkappa\beta + p^{e_\mu\nu}_I \mathrm{d}e^I_\mu \wedge \beta_\nu + p^{\omega_\mu\nu}_{IJ}\mathrm{d}\omega^{IJ}_\mu \wedge \beta_\nu. \tag{31}$$

Now, we describe the Legendre correspondence for the DW formulation of the first order Palatini Lagrangian $L_{\mathsf{Palatini}}[e,\omega] = E^{[\rho\sigma]}_{IJ}\big(\partial_\rho\omega^{IJ}_\sigma + \omega^I_{\rho M}\omega^{MJ}_\sigma\big)$, where we denote $E^{[\rho\sigma]}_{IJ} := E^{[\rho}_I e^{\sigma]}_J$, see appendix B.5. The Legendre correspondence $(q,v) \leftrightarrow (q,p)$ for the formulation of vielbein gravity is given by

$$\begin{aligned} \mathcal{P} \sim \Lambda^n_{\mathrm{N}} T(\mathfrak{p} \otimes T^\star\mathcal{X}) = T\mathcal{Y} \otimes_{\mathcal{Y}} T^\star\mathcal{X} \;\leftrightarrow\; \mathcal{M}_{\mathsf{DW}} := \Lambda^n_1 T^\star(\mathfrak{p} \otimes T^\star\mathcal{X}) = \Lambda^n_1 T^\star\mathcal{Y}, \\ (q,v) \sim (x^\mu, e^I_\mu, \omega^{IJ}_\mu, z^I_{\mu\nu}, z^{IJ}_{\mu\nu}) \;\leftrightarrow\; (q,p) = (x^\mu, e^I_\mu, \omega^{IJ}_\mu, \varkappa, p^{e_\mu\nu}_I, p^{\omega_\mu\nu}_{IJ}). \end{aligned} \tag{32}$$

In particular, the construction of the Legendre correspondence involves the relation

$$(q,v) \leftrightarrow (q,p) \quad \Longleftrightarrow \quad \frac{\partial\langle p,v\rangle}{\partial v} = \frac{\partial L(q,v)}{\partial v}, \tag{33}$$

between (q,v) and (q,p), where we denote $\langle p,v\rangle := \theta^{\mathsf{DW}}_{(q,p)}(\mathcal{Z})$ and $\mathcal{Z} \in \Lambda^n_{\mathrm{N}} T\mathcal{Y}$. We consider a decomposable multivector field $\mathcal{Z} = \mathcal{Z}_1 \wedge \mathcal{Z}_2 \wedge \mathcal{Z}_3 \wedge \mathcal{Z}_4 \in \Lambda^4_{\mathrm{N}} T\mathcal{Y}$, where for any $1 \le \mu \le 4$:

$$\begin{aligned} \mathcal{Z}_\mu &= \frac{\partial}{\partial x^\mu} + \mathcal{Z}^I_{\mu\nu}\frac{\partial}{\partial e^I_\nu} + \mathcal{Z}^{IJ}_{\mu\nu}\frac{\partial}{\partial \omega^{IJ}_\nu}, \\ &= \frac{\partial}{\partial x^\mu} + \big(\partial_\mu e^I_\nu + \omega_\mu{}^I{}_J e^J_\nu\big)\frac{\partial}{\partial e^I_\nu} + \big(\partial_\mu\omega^{IJ}_\nu + \omega_\mu{}^I{}_K\omega_\nu{}^{KJ} - \omega_\mu{}^J{}_K\omega_\nu{}^{KI}\big)\frac{\partial}{\partial\omega^{IJ}_\nu}. \end{aligned} \tag{34}$$

Let us note that the multivector field $\mathcal{Z}$ is written as

$$\mathcal{Z} = \sum_{\boldsymbol{\mu}_1<\dots<\boldsymbol{\mu}_4} \mathcal{Z}^{\boldsymbol{\mu}_1\dots\boldsymbol{\mu}_4}_{1\dots 4}\frac{\partial}{\partial q^{\boldsymbol{\mu}_1}} \wedge \dots \wedge \frac{\partial}{\partial q^{\boldsymbol{\mu}_4}} := \sum_{\boldsymbol{\mu}_1<\dots<\boldsymbol{\mu}_4} \begin{vmatrix} \mathcal{Z}^{\boldsymbol{\mu}_1}_1 & \dots & \mathcal{Z}^{\boldsymbol{\mu}_1}_4 \\ \vdots & & \vdots \\ \mathcal{Z}^{\boldsymbol{\mu}_4}_1 & \dots & \mathcal{Z}^{\boldsymbol{\mu}_4}_4 \end{vmatrix} \frac{\partial}{\partial q^{\boldsymbol{\mu}_1}} \wedge \dots \wedge \frac{\partial}{\partial q^{\boldsymbol{\mu}_4}}.$$

Now, for any $(q,p) \in \Lambda^n_1 T^\star\mathcal{Y}$ and $\mathcal{Z} \in \Lambda^n_{\mathrm{N}} T\mathcal{Y}$, we make the straightforward calculation

$$\begin{aligned} \langle p,v\rangle &= \theta^{\mathsf{DW}}_p(\mathcal{Z}) = \varkappa\beta(\mathcal{Z}) + p^{e_\mu\nu}_I \mathrm{d}e^I_\mu \wedge \beta_\nu(\mathcal{Z}) + p^{\omega_\mu\nu}_{IJ}\mathrm{d}\omega^{IJ}_\mu \wedge \beta_\nu(\mathcal{Z}), \\ &= \varkappa + p^{e_\mu\nu}_I \mathcal{Z}^I_{\nu\mu} + p^{\omega_\mu\nu}_{IJ}\mathcal{Z}^{IJ}_{\nu\mu}. \end{aligned} \tag{35}$$

Let us compute the two parts involved in the Legendre correspondence. We calculate the partial derivatives with respect to the field derivatives $\partial_\nu e^I_\mu$ and $\partial_\nu\omega^{IJ}_\mu$

$$\begin{aligned} \frac{\partial\langle p,v\rangle}{\partial(\partial_\nu\omega^{IJ}_\mu)} &= p^{\omega_\mu\nu}_{IJ}, & \frac{\partial L_{\mathsf{Palatini}}[e,\omega]}{\partial(\partial_\mu\omega^{IJ}_\nu)} &= \frac{\partial}{\partial(\partial_\mu\omega^{IJ}_\nu)}\Big(ee^\mu_I e^\nu_J(\partial_{[\mu}\omega^{IJ}_{\nu]} + \omega^I_{[\mu K}\omega^{KJ}_{\nu]})\Big) = E^{[\mu}_I e^{\nu]}_J, \\ \frac{\partial\langle p,v\rangle}{\partial(\partial_\nu e^I_\mu)} &= p^{e_\mu\nu}_I, & \frac{\partial L_{\mathsf{Palatini}}[e,\omega]}{\partial(\partial_\mu e^I_\nu)} &= \frac{\partial}{\partial(\partial_\mu e^I_\nu)}\Big(ee^\mu_I e^\nu_J(\partial_{[\mu}\omega^{IJ}_{\nu]} + \omega^I_{[\mu K}\omega^{KJ}_{\nu]})\Big) = 0. \end{aligned}$$

Therefore, the Legendre correspondence yields

$$p_{IJ}^{\omega_\mu{}^\nu} = -E_I^{[\mu} e_J^{\nu]} = -(1/4)\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma} e_\rho^K e_\sigma^L, \qquad p_I^{e_\mu{}^\nu} = 0, \tag{36}$$

for the multimomenta related to ω_μ^{IJ} and e_μ^I, respectively. Then, the Legendre correspondence yields $p_{IJ}^{\omega_\nu{}^\mu} + p_{IJ}^{\omega_\mu{}^\nu} = 0$ and $p_I^{e_\mu{}^\nu} = 0$. It is an example of the set of Dirac primary constraints in the DW multisymplectic formalism. Therefore, we shall be restricted to the submanifold $\mathcal{C} \subset \mathcal{M}_{\mathsf{DW}}$ for taking into account the primary constraints:

$$\mathcal{C} \subset \mathcal{M}_{\mathsf{DW}} = \{(x, e, \omega, p) \in \mathcal{M}_{\mathsf{DW}} \ / \ p_{IJ}^{\omega_\mu{}^\nu} = -E_I^{[\mu} e_J^{\nu]}, \quad p_I^{e_\mu{}^\nu} = 0\}. \tag{37}$$

The Legendre transformation is degenerate since we cannot express arbitrary field derivative via multimomenta. Let us note that the multimomenta $p_{IJ}^{\omega_\mu{}^\nu} := p_{IJ}^{\omega_\mu{}^\nu}(x, e)$ are functions of the vierbein.

2.4 DW Hamiltonian of the Palatini action

Now we present the DW Hamiltonian function of the Palatini action of vielbein gravity. The Legendre correspondence is generated by the function $W^{\mathsf{DW}}(q, v, p) := \langle p, v\rangle - L(q, v)$, i.e.

$$\begin{aligned} W^{\mathsf{DW}}(q, v, p) &= = \varkappa + p_I^{e_\mu{}^\nu} \mathcal{Z}_{\nu\mu}^I + p_{IJ}^{\omega_\mu{}^\nu} \mathcal{Z}_{\nu\mu}^{IJ} - E_I^{[\mu} e_J^{\nu]} \big(\partial_\mu \omega_\nu^{IJ} + \omega_{\mu K}^I \omega_\nu^{KJ}\big), \\ &= \varkappa + p_I^{e_\mu{}^\nu} \mathcal{Z}_{\nu\mu}^I + p_{IJ}^{\omega_\mu{}^\nu} \big(\partial_\nu \omega_\mu^{IJ} + \omega_{\nu K}{}^I \omega_\mu{}^{KJ} - \omega_{\nu K}{}^J \omega_\mu{}^{KI}\big), \\ &\quad - E_I^{[\nu} e_J^{\mu]} \big(\partial_\nu \omega_\mu^{IJ} + \omega_{\nu K}^I \omega_\mu^{KJ}\big). \end{aligned}$$

Let us work on $\mathcal{C} \subset \mathcal{M}_{\mathsf{DW}}$. We introduce the Hamiltonian function $\mathcal{H} : \mathcal{M}_{\mathsf{DW}} \to \mathbb{R}$ defined by $\mathcal{H} = \langle p, \nu(q, p)\rangle - L(q, p, \nu(q, p))$, where $\nu(q, p)$ is such that $(q, \nu(q, p)) \leftrightarrow (q, p)$. For any $v \in T_{(x, e_x, \omega_x)}\mathcal{Y} \otimes T_x^\star \mathcal{X}$ the equation (33) has a solution $p \in \mathcal{M}_{\mathsf{DW}}$ if and only if $p \in \mathcal{C}$ with

$$\mathcal{C} = \{(x, e, \omega, \varkappa\beta - ee_I^{[\mu} e_J^{\nu]} \mathrm{d}\omega_\mu^{IJ} \wedge \beta_\nu) \ / \ (x, e, \omega) \in \mathcal{Y} = \mathfrak{p} \otimes T^\star \mathcal{X}, \ \varkappa \in \mathbb{R}\}.$$

The use of the constraint (36), *i.e.* $p_I^{e_\mu{}^\nu} = 0$ and $p_{IJ}^{\omega_\mu{}^\nu} = -ee_I^{[\mu} e_J^{\nu]}$, leads to the expression of the Hamiltonian function restricted to the hypersurface of constraints $\mathcal{C}$. Thus, $\mathcal{H}^{\mathsf{Palatini}}(q, p) := \iota^\star \mathcal{H}^{\mathsf{DW}}(q, p)$ is written as

$$\begin{aligned} \mathcal{H}^{\mathsf{Palatini}}(q, p) &= \varkappa + p_{IJ}^{\omega_\mu{}^\nu} \big(\partial_\nu \omega_\mu^{IJ} + \omega_{\nu K}{}^I \omega_\mu{}^{KJ} - \omega_{\nu K}{}^J \omega_\mu{}^{KI}\big) - p_{IJ}^{\omega_\mu{}^\nu} \big(\partial_\nu \omega_\mu^{IJ} + \omega_{\nu K}^I \omega_\mu^{KJ}\big), \\ &= \varkappa - p_{IJ}^{\omega_\mu{}^\nu} \omega_{\nu K}{}^J \omega_\mu{}^{KI} = \varkappa - ee_I^{[\mu} e_J^{\nu]} \omega_{\mu K}{}^J \omega_\nu{}^{KI}. \end{aligned}$$

The Hamiltonian function $\mathcal{H}^{\mathsf{Palatini}}(q, p) : \mathcal{C} \to \mathbb{R}$ is equivalently written as $\mathcal{H}^{\mathsf{Palatini}}(q, p) = \varkappa + H^{\mathsf{Palatini}}(q, p)$, where $H^{\mathsf{Palatini}}(e, \omega) := \iota^\star H^{\mathsf{DW}}(q, p)$ is the DW Hamiltonian [57, 58] evaluated on the constraint hypersurface $\mathcal{C}$. In section 3, we explore the n-phase space approach, we fix $\varkappa = ee_I^\mu e_J^\nu \omega_{[\mu K}^J \omega_{\nu]}^{KI}$. Note that we can always choose $\varkappa(x)$ such that $\mathcal{H}(x, e(x), \omega(x), \varkappa(x), p(x))$ is constant, see [50, 51].

2.5 Exterior derivative of the DW Hamiltonian

In this section we derive the exterior derivative of the DW Hamiltonian function for the formulation of dreibein and vierbein gravity. First, let us consider the case of dreibein gravity. We denote the exterior derivative by $\mathrm{d}\mathcal{H}_{\mathsf{3D}}^{\mathsf{Palatini}}$. We have

$$\mathrm{d}\mathcal{H}_{\mathsf{3D}}^{\mathsf{Palatini}}(q, p) = \mathrm{d}\varkappa - E_I^{[\mu} e_J^{\nu]} \mathrm{d}\left(\omega_{\mu K}^J \omega_\nu^{KI}\right) - \mathrm{d}\left(E_I^{[\mu} e_J^{\nu]}\right) \omega_{\mu K}^J \omega_\nu^{KI}. \tag{38}$$

When the dimension of the space-time manifold is $n=3$, we have the algebraic relation $-E_I^{[\mu}e_J^{\nu]}=-ee_I^{[\mu}e_J^{\nu]}=-(1/2)\epsilon^{\mu\nu\rho}\epsilon_{IJK}e_\rho^K$. Then the second term in (38) takes the form

$$\begin{aligned}-E_I^{[\mu}e_J^{\nu]}\mathrm{d}\left(\omega_{\mu K}^J\omega_\nu^{KI}\right) &= -(1/2)\epsilon^{\mu\nu\rho}\epsilon_{IJK}e_\rho^K\left(\omega_{\mu K}^J\mathrm{d}\omega_\nu^{KI}+\mathrm{d}(\omega_\mu^{JM}\mathsf{h}_{MK})\omega_\nu^{KI}\right),\\ &= -\epsilon_{IJM}\epsilon^{\mu\nu\lambda}e_\lambda^M\omega_{\nu K}^J\mathrm{d}\omega_\mu^{KI}=-\epsilon_{IJK}\epsilon^{\mu\nu\rho}e_\rho^K\omega_{\nu M}^J\mathrm{d}\omega_\mu^{MI},\end{aligned}$$

where we have used the algebraic relation

$$\epsilon^{\mu\rho\sigma}\epsilon_{IJK}e_\mu^I\omega_\sigma{}^J{}_M\mathrm{d}\omega_\rho^{MK}=-1/2\epsilon^{\mu\rho\sigma}\epsilon_{LJK}e_\mu^I\omega_\sigma{}^L{}_I\mathrm{d}\omega_\rho^{JK}=1/2\epsilon^{\mu\nu\rho}\epsilon_{LJI}e_\rho^K\omega_\nu{}^L{}_K\mathrm{d}\omega_\mu^{IJ}. \tag{39}$$

This relation is used also to decompose the interior product $X^{\mathcal{C}}\lrcorner\,\boldsymbol{\omega}^{\mathsf{Palatini}}$ in the basis 1-forms $\mathrm{d}\omega_\mu^{IJ}$. Also, since $\mathrm{d}(ee_I^\mu e_J^\nu)=\mathrm{d}(\epsilon_{IJM}\epsilon^{\mu\nu\lambda}e_\lambda^M)=\epsilon_{IJM}\epsilon^{\mu\nu\lambda}\mathrm{d}e_\lambda^M$, the third term in (38) is written as $(1/2)\epsilon_{IJM}\epsilon^{\mu\nu\lambda}\omega_{\mu K}^J\omega_\nu^{KI}\mathrm{d}e_\lambda^M$. Now, we obtain the expression for the 1-form $\mathrm{d}\mathcal{H}_{\mathsf{3D}}^{\mathsf{Palatini}}(q,p)$, namely

$$\mathrm{d}\mathcal{H}_{\mathsf{3D}}^{\mathsf{Palatini}}(q,p)=\mathrm{d}\varkappa+(1/2)\epsilon_{IJM}\epsilon^{\mu\nu\lambda}\omega_{\mu K}^J\omega_\nu^{KI}\mathrm{d}e_\lambda^M+(1/2)\epsilon^{\mu\nu\rho}\epsilon_{LJI}e_\rho^K\omega_\nu{}^L{}_K\mathrm{d}\omega_\mu^{IJ}. \tag{40}$$

When $n=4$, $E_{IJ}^{[\mu\nu]}=(1/4)\epsilon_{IJKL}e_\rho^Ke_\sigma^L\epsilon^{\mu\nu\rho\sigma}$, therefore $\mathrm{d}\big(E_{IJ}^{[\mu\nu]}\big)=(1/2)\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e_\rho^K\mathrm{d}e_\sigma^L$. Thus, we have

$$\begin{aligned}\mathrm{d}\mathcal{H}_{\mathsf{4D}}^{\mathsf{Palatini}} &= \mathrm{d}\varkappa-\mathrm{d}\big(E_I^{[\mu}e_J^{\nu]}\big)(\omega_{\mu K}^J\omega_\nu^{KI})-E_I^{[\mu}e_J^{\nu]}\mathrm{d}\big(\omega_{\mu K}^J\omega_\nu^{KI}\big),\\ &= \mathrm{d}\varkappa-(1/2)\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e_\rho^K\omega_{\mu M}^J\omega_\nu^{MI}\mathrm{d}e_\sigma^L,\\ &\quad+(1/2)\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e_\rho^Ke_\sigma^L\omega_{\nu M}^J\mathrm{d}\omega_\mu^{MI}.\end{aligned} \tag{41}$$

Using the algebraic relation

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{IJKL}e_\mu^Ie_\nu^J\omega_\sigma{}^K{}_M\mathrm{d}\omega_\rho^{ML}=-\epsilon^{\mu\nu\rho\sigma}\epsilon_{INKL}e_\mu^Ie_\nu^J\omega_\sigma{}^N{}_J\mathrm{d}\omega_\rho^{KL}, \tag{42}$$

see [5, 6], the last term in (41) is equivalently written as

$$\begin{aligned}\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e_\rho^Ke_\sigma^L\omega_{\nu M}^J\mathrm{d}\omega_\mu^{MI} &= -\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e_\mu^Ie_\nu^J\omega_{\sigma M}^K\mathrm{d}\omega_\rho^{ML}=\epsilon^{\mu\nu\rho\sigma}\epsilon_{INKL}e_\mu^Ie_\nu^J\omega_\sigma{}^N{}_J\mathrm{d}\omega_\rho^{KL},\\ &= -\epsilon^{\mu\nu\rho\sigma}\epsilon_{IJKN}e_\rho^Ke_\nu^L\omega_\sigma{}^N{}_L\mathrm{d}\omega_\mu^{IJ}=-\epsilon^{\mu\nu\rho\sigma}\epsilon_{IJKL}e_\rho^Ke_\nu^N\omega_\sigma{}^L{}_N\mathrm{d}\omega_\mu^{IJ}.\end{aligned}$$

Therefore, the exterior derivative of the DW Hamiltonian function related to the Palatini action of vierbein gravity is given by

$$\mathrm{d}\mathcal{H}_{\mathsf{4D}}^{\mathsf{Palatini}}(q,p)=\mathrm{d}\varkappa-(1/2)\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e_\rho^K\omega_{\mu M}^J\omega_\nu^{MI}\mathrm{d}e_\sigma^L+(1/2)\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e_\rho^Ke_\nu^N\omega_\sigma{}^L{}_N\mathrm{d}\omega_\mu^{IJ}. \tag{43}$$

2.6 Primary constraints and the extended Hamiltonian

The set of primary constraints that weakly vanish on the constraint hypersurface, following the terminology of Dirac, are $p_I^{e_\mu\nu}\approx 0$ and $p_{IJ}^{\omega_\mu\nu}\approx -ee_I^{[\mu}e_J^{\nu]}$. An extension of the traditional method developed by Dirac in the DW formulation involves the construction of an extended Hamiltonian,

$$\mathcal{H}^{\mathsf{Ext}}=\varkappa-ee_I^{[\mu}e_J^{\nu]}\omega_\mu{}^J{}_K\omega_\nu{}^{KI}+\lambda_{\nu\mu}^Ip_I^{e_\mu\nu}+\lambda_{\nu\mu}^{IJ}\big(p_{IJ}^{\omega_\mu\nu}+ee_I^{[\mu}e_J^{\nu]}\big).$$

The extended DW Hamiltonian is $\mathcal{H}^{\mathsf{Ext}} = \mathcal{H}^{\mathsf{Palatini}} + \lambda^I_{\nu\mu} p_I^{e_\mu\nu} + \lambda^{IJ}_{\nu\mu}\big(p_{IJ}^{\omega_\mu\nu} + e e_I^{[\mu} e_J^{\nu]}\big)$. Here, $\lambda^I_{\nu\mu}$ and $\lambda^{IJ}_{\nu\mu}$ are Lagrange multipliers. We postulate, since there is no reason to assume they are valid *a priori*, the DW Hamilton equations

$$\begin{aligned} \frac{\partial \omega_\mu^{IJ}}{\partial x^\nu}(x) &= \frac{\partial \mathcal{H}^{\mathsf{Ext}}}{\partial p_{IJ}^{\omega_\mu\nu}}(x,e,\omega,\varkappa,p), & \sum_\nu \frac{\partial p_{IJ}^{\omega_\mu\nu}}{\partial x^\nu}(x) &= -\frac{\partial \mathcal{H}^{\mathsf{Ext}}}{\partial \omega_\mu^{IJ}}(x,e,\omega,\varkappa,p), \\ \frac{\partial e_\mu^I}{\partial x^\nu}(x) &= \frac{\partial \mathcal{H}^{\mathsf{Ext}}}{\partial p_{IJ}^{\omega_\mu\nu}}(x,e,\omega,\varkappa,p), & \sum_\nu \frac{\partial p_I^{e_\mu\nu}}{\partial x^\nu}(x) &= -\frac{\partial \mathcal{H}^{\mathsf{Ext}}}{\partial e_\mu^I}(x,e,\omega,\varkappa,p), \end{aligned} \tag{44}$$

In the context of the polysymplectic formalism [53], the extended DW Hamiltonian function is written as $H^{\mathsf{Ext}} = H^{\mathsf{Palatini}} + \lambda^I_{\nu\mu} p_I^{e_\mu\nu} + \lambda^{IJ}_{\nu\mu}\big(p_{IJ}^{\omega_\mu\nu} + e e_I^{[\mu} e_J^{\nu]}\big)$. Then, the system of DW Hamilton equation is given as

$$\begin{aligned} \partial_\nu \omega_\mu^{IJ}(x) &= \lambda^{IJ}_{\nu\mu}, & \partial_\nu p_{IJ}^{\omega_\mu\nu}(x) &= -(\partial H^{\mathsf{Ext}}/\partial \omega_\mu^{IJ})(x,e,\omega,p), \\ \partial_\nu e_\mu^I(x) &= \lambda^I_{\nu\mu}, & \partial_\nu p_I^{e_\mu\nu}(x) &= -(\partial H^{\mathsf{Ext}}/\partial e_\mu^I)(x,e,\omega,p). \end{aligned} \tag{45}$$

For a detailed analysis of constraints within the polysymplectic approach to the DW Hamiltonian formalism, we refer to [57, 58]. Note that our conventions here differ from those of Kanatchikov: the polymomenta have opposite sign.

2.7 DW Hamilton equations on $(\mathcal{C}, \boldsymbol{\omega}^{\mathsf{Palatini}})$

The canonical DW multisymplectic $(n+1)$-form $\boldsymbol{\omega}^{\mathsf{DW}} = \mathrm{d}\theta^{\mathsf{DW}}$ previously introduced in (7) is written as $\boldsymbol{\omega}^{\mathsf{DW}} = \mathrm{d}\varkappa \wedge \beta + \mathrm{d}p_{IJ}^{\omega_\mu\nu} \wedge \mathrm{d}\omega_\mu^{IJ} \wedge \beta_\nu$. Let us introduce the $(n+1)$-form $\boldsymbol{\omega}^{\mathsf{Palatini}} := \iota^\star \boldsymbol{\omega}^{\mathsf{DW}}$, where $\iota : \mathcal{C} \hookrightarrow \mathcal{M}_{\mathsf{DW}}$ is the canonical inclusion. In local coordinates,

$$\boldsymbol{\omega}^{\mathsf{Palatini}} = \mathrm{d}\varkappa \wedge \beta - \mathrm{d}\left(e e_I^{[\mu} e_J^{\nu]}\right) \wedge \mathrm{d}\omega_\mu^{IJ} \wedge \beta_\nu. \tag{46}$$

Using (46) we can now describe the Einstein equations in the DW Hamilton formulation, where the DW Hamilton equations in geometric form are written as

$$X \lrcorner\, \boldsymbol{\omega}^{\mathsf{Palatini}} = (-1)^n \mathrm{d}\mathcal{H}^{\mathsf{Palatini}}. \tag{47}$$

Let $\Xi^{\mathsf{DW}} \in \Gamma(\mathcal{M}_{\mathsf{DW}}, T\mathcal{M}_{\mathsf{DW}})$ be a vector field on $\mathcal{M}_{\mathsf{DW}}$ and $X^{\mathsf{DW}} \in \Gamma(\mathcal{M}_{\mathsf{DW}}, \Lambda^n T\mathcal{M}_{\mathsf{DW}})$ be a n-vector field on $\mathcal{M}_{\mathsf{DW}}$. Then, we construct on the constraint hypersurface $\mathcal{C}$ the vector field $\Xi^{\mathcal{C}} := \boldsymbol{\pi}_\star \Xi^{\mathsf{DW}} \in \Gamma(\mathcal{C}, T\mathcal{C})$ and the n-vector field $X^{\mathcal{C}} := \boldsymbol{\pi}_\star \Xi^{\mathsf{DW}} \in \Gamma(\mathcal{C}, \Lambda^n T\mathcal{C})$, respectively. We have denoted by $\boldsymbol{\pi}$ the canonical projection $\boldsymbol{\pi} : \mathcal{M}_{\mathsf{DW}} \to \mathcal{C}$ such that $\boldsymbol{\pi} \circ \iota = \mathsf{Id}_{\mathcal{C}}$.

Note that, because of the primary constraints, there is no reason *a priori* that the set of DW Hamilton equations is in a one-to-one correspondence with the Euler-Lagrange system of equations. Nevertheless, working on $\mathcal{C} \hookrightarrow \mathcal{M}_{\mathsf{DW}}$, the DW Hamilton equations in geometric form (47) reproduces the Einstein system. The DW Hamilton equations $X^{\mathcal{C}} \lrcorner\, (\iota^\star \boldsymbol{\omega}^{\mathsf{DW}}) = (-1)^n \mathrm{d}(\iota^\star \mathcal{H}^{\mathsf{DW}})$ are presented for dreibein and vierbein gravity in section 2.7.1 and 2.7.2, respectively.

2.7.1 DW Hamilton equations of dreibein gravity

First, we consider the DW Hamilton equations for the Palatini action of dreibein gravity. Let $X^{\mathcal{C}} = X_1^{\mathcal{C}} \wedge X_2^{\mathcal{C}} \wedge X_3^{\mathcal{C}} \in \Lambda^3 T\mathcal{C}$ be a decomposable 3-vector field, where for any $1 \leq \nu \leq 3$,

$$X_\nu^{\mathcal{C}} = \frac{\partial}{\partial x^\nu} + \Theta^I_{\nu\mu}\frac{\partial}{\partial e^I_\mu} + \Theta^{IJ}_{\nu\mu}\frac{\partial}{\partial \omega^I_\mu} + \Upsilon_\nu \frac{\partial}{\partial \varkappa}. \tag{48}$$

First, we re-express $\boldsymbol{\omega}^{\mathsf{Palatini}}$ as follows:

$$\boldsymbol{\omega}^{\mathsf{Palatini}} = \mathrm{d}\varkappa \wedge \beta - \mathrm{d}\big(E^{[\mu}_I e^{\nu]}_J\big) \wedge \mathrm{d}\omega^{IJ}_\mu \wedge \beta_\nu = \mathrm{d}\varkappa \wedge \beta - (1/2)\epsilon_{IJM}\epsilon^{\mu\nu\lambda}\mathrm{d}e^M_\lambda \wedge \mathrm{d}\omega^{IJ}_\mu \wedge \beta_\nu.$$

The left hand side of (47) is given by the interior product $X^{\mathcal{C}} \lrcorner\, \boldsymbol{\omega}^{\mathsf{Palatini}}$. Then,

$$\begin{aligned} X^{\mathcal{C}} \lrcorner\, \boldsymbol{\omega}^{\mathsf{Palatini}} &= -(1/2)\epsilon_{IJL}\epsilon^{\mu\nu\alpha}\big((\mathrm{d}\omega^{IJ}_\mu \wedge \beta_\nu)(X)\mathrm{d}e^L_\alpha - (\mathrm{d}e^L_\alpha \wedge \beta_\nu)(X)\mathrm{d}\omega^{IJ}_\mu\big), \\ &\quad -(1/2)\epsilon_{IJL}\epsilon^{\mu\nu\alpha}\big((\mathrm{d}e^L_\alpha \wedge \mathrm{d}\omega^{IJ}_\mu \wedge \beta_{\rho\nu})(X)\mathrm{d}x^\rho\big), \\ &= \mathrm{d}\varkappa - (\mathrm{d}\varkappa \wedge \beta_\rho)(X)\mathrm{d}x^\rho - (1/2)\epsilon_{IJL}\epsilon^{\mu\nu\alpha}(\mathrm{d}e^L_\alpha \wedge \mathrm{d}\omega^{IJ}_\mu \wedge \beta_{\rho\nu})(X)\mathrm{d}x^\rho, \\ &\quad -(1/2)\epsilon_{IJL}\epsilon^{\mu\nu\alpha}(\mathrm{d}\omega^{IJ}_\mu \wedge \beta_\nu)(X)\mathrm{d}e^L_\alpha + (1/2)\epsilon_{IJL}\epsilon^{\mu\nu\alpha}(\mathrm{d}e^L_\alpha \wedge \beta_\nu)(X)\mathrm{d}\omega^{IJ}_\mu. \end{aligned}$$

Finally, the expression becomes

$$\begin{aligned} X^{\mathcal{C}} \lrcorner\, \boldsymbol{\omega}^{\mathsf{Palatini}} &= \mathrm{d}\varkappa - \Upsilon_\rho \mathrm{d}x^\rho - (1/2)\epsilon_{IJL}\epsilon^{\mu\nu\alpha}\big(\Theta^{IJ}_{\nu\mu}\Theta^L_{\rho\alpha} - \Theta^{IJ}_{\rho\mu}\Theta^L_{\nu\alpha}\big)\mathrm{d}x^\rho, \\ &\quad -(1/2)\epsilon_{IJL}\epsilon^{\mu\nu\alpha}\Theta^{IJ}_{\nu\mu}\mathrm{d}e^L_\alpha + (1/2)\epsilon_{IJL}\epsilon^{\mu\nu\alpha}\Theta^L_{\nu\alpha}\mathrm{d}\omega^{IJ}_\mu. \end{aligned} \tag{49}$$

which is equal to the right hand side of (47)

$$\mathrm{d}\mathcal{H}^{\mathsf{Palatini}}_{\mathsf{3D}}(q,p) = \mathrm{d}\varkappa + (1/2)\epsilon_{IJM}\epsilon^{\mu\nu\lambda}\omega^J_{\mu K}\omega^{KI}_\nu \mathrm{d}e^M_\lambda + (1/2)\epsilon^{\mu\nu\rho}\epsilon_{LJI}e^K_\rho \omega_\nu{}^L{}_K \mathrm{d}\omega^{IJ}_\mu. \tag{50}$$

The equality between (49) and (50) leads to the DW Hamilton system of equations

$$\begin{aligned} \epsilon_{IJL}\epsilon^{\mu\nu\alpha}\big(\Theta^{IJ}_{\nu\mu} + \omega^J_{\mu K}\omega^{KI}_\nu\big) &= 0, \\ \epsilon_{IJL}\epsilon^{\mu\nu\alpha}\big(\Theta^L_{\nu\alpha} + e^K_\alpha \omega_\nu{}^L{}_K\big) &= 0, \\ -\epsilon_{IJL}\epsilon^{\mu\nu\alpha}\big(\Theta^{IJ}_{\nu\mu}\Theta^L_{\rho\alpha} - \Theta^{IJ}_{\rho\mu}\Theta^L_{\nu\alpha}\big) &= \Upsilon_\rho. \end{aligned} \tag{51}$$

The system (51) is the DW Hamilton equations associated to the first order Palatini action of dreibein gravity and is written as

$$\epsilon_{IJK}F^{JK} = 0, \qquad \epsilon_{IJK}\mathrm{d}_\omega e^K = 0, \tag{52}$$

with the additional equation $\Upsilon_\rho = \partial_\rho \varkappa = -\epsilon_{IJL}\epsilon^{\mu\nu\alpha}\big(\Theta^{IJ}_{\nu\mu}\Theta^L_{\rho\alpha} - \Theta^{IJ}_{\rho\mu}\Theta^L_{\nu\alpha}\big)$.

2.7.2 DW Hamilton equations of vierbein gravity

Now, we are interested in the DW Hamilton equations for the Palatini action of vierbein gravity. We consider the 5-form

$$\boldsymbol{\omega}^{\mathsf{Palatini}} = \mathrm{d}\varkappa \wedge \beta - \mathrm{d}(e e^{[\mu}_I e^{\nu]}_J) \wedge \mathrm{d}\omega^{IJ}_\mu \wedge \beta_\nu = \mathrm{d}\varkappa \wedge \beta - (1/2)\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^K_\rho \mathrm{d}e^L_\sigma \wedge \mathrm{d}\omega^{IJ}_\mu \wedge \beta_\nu. \tag{53}$$

Let us consider a multivector field $X^{\mathcal{C}} = X_1^{\mathcal{C}} \wedge X_2^{\mathcal{C}} \wedge X_3^{\mathcal{C}} \wedge X_4^{\mathcal{C}} \in \Lambda^4 T\mathcal{C}$, where for any $1 \leq \nu \leq 4$,

$$X_\nu^{\mathcal{C}} = \frac{\partial}{\partial x^\nu} + \Theta^I_{\nu\mu}\frac{\partial}{\partial e^I_\mu} + \Theta^{IJ}_{\nu\mu}\frac{\partial}{\partial \omega^I_\mu} + \Upsilon_\nu \frac{\partial}{\partial \varkappa}. \tag{54}$$

Then

$$\begin{aligned} X^{\mathcal{C}} \lrcorner\, \boldsymbol{\omega}^{\mathsf{Palatini}} &= X \lrcorner\, (\mathrm{d}\varkappa \wedge \beta) - (1/2)\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma} e^K_\rho X \lrcorner\, (\mathrm{d}e^L_\sigma \wedge \mathrm{d}\omega^{IJ}_\mu \wedge \beta_\nu), \\ &= \beta(X)\mathrm{d}\varkappa - (\mathrm{d}\varkappa \wedge \beta_\rho)(X)\mathrm{d}x^\rho, \\ &\quad -(1/2)\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma} e^K_\rho \big((\mathrm{d}\omega^{IJ}_\mu \wedge \beta_\nu)(X)\mathrm{d}e^L_\sigma - (\mathrm{d}e^L_\sigma \wedge \beta_\nu)(X)\mathrm{d}\omega^{IJ}_\mu\big), \\ &\quad -(1/2)\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma} e^K_\rho \big((\mathrm{d}e^L_\sigma \wedge \mathrm{d}\omega^{IJ}_\mu \wedge \beta_{\lambda\nu})(X)\mathrm{d}x^\lambda\big). \end{aligned}$$

Since, $(\mathrm{d}\omega^{IJ}_\mu \wedge \beta_\nu)(X) = \Theta^{IJ}_{\nu\mu}$, $(\mathrm{d}e^L_\sigma \wedge \beta_\nu)(X) = \Theta^{I}_{\nu\mu}$ and $(\mathrm{d}e^L_\sigma \wedge \mathrm{d}\omega^{IJ}_\mu \wedge \beta_{\lambda\nu})(X) = (\Theta^{IJ}_{\nu\mu}\Theta^L_{\lambda\sigma} - \Theta^{IJ}_{\lambda\mu}\Theta^L_{\nu\sigma})$, the left hand side of (47) is written as

$$X^{\mathcal{C}} \lrcorner\, \boldsymbol{\omega}^{\mathsf{Palatini}} = \mathrm{d}\varkappa - \Upsilon_\rho \mathrm{d}x^\rho - (1/2)\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma} e^K_\rho \big(\Theta^{IJ}_{\nu\mu}\mathrm{d}e^L_\sigma - \Theta^L_{\nu\sigma}\mathrm{d}\omega^{IJ}_\mu + (\Theta^{IJ}_{\nu\mu}\Theta^L_{\lambda\sigma} - \Theta^{IJ}_{\lambda\mu}\Theta^L_{\nu\sigma})\mathrm{d}x^\lambda\big).$$

The DW Hamilton equations (47) are obtained by equalizing the interior product $X^{\mathcal{C}} \lrcorner\, \boldsymbol{\omega}^{\mathsf{Palatini}}$ with the expression of $\mathrm{d}\mathcal{H}^{\mathsf{Palatini}}_{4\mathrm{D}}(q,p)$ found in (43). We obtain

$$\begin{aligned} -\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma} e^K_\rho \Theta^{IJ}_{\nu\mu} &= -\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma} e^K_\rho \omega^J_{\mu M}\omega^{MI}_\nu, \\ \epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma} e^K_\rho \Theta^L_{\nu\sigma} &= \epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma} e^K_\rho e^N_\nu \omega_\sigma{}^L{}_N, \\ -\Upsilon_\lambda - (1/2)\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma} e^K_\rho \big(\Theta^{IJ}_{\nu\mu}\Theta^L_{\lambda\sigma} - \Theta^{IJ}_{\lambda\mu}\Theta^L_{\nu\sigma}\big) &= 0. \end{aligned} \tag{55}$$

Therefore, we obtain the DW Hamilton system of equations

$$\begin{aligned} \epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma} e^K_\rho \big(\Theta^{IJ}_{\mu\nu} + \omega^I_{\mu M}\omega^{MJ}_\nu\big) &= 0, \\ \epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma} e^K_\rho \big(\Theta^L_{\nu\sigma} + \omega^L_{\nu N} e^N_\sigma\big) &= 0, \\ -\Upsilon_\lambda - (1/2)\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma} e^K_\rho \big(\Theta^{IJ}_{\nu\mu}\Theta^L_{\lambda\sigma} - \Theta^{IJ}_{\lambda\mu}\Theta^L_{\nu\sigma}\big) &= 0. \end{aligned} \tag{56}$$

We reproduce the results obtained by Bruno, Cianci and Vignolo [5, 6]. The equations of motion (56) are equivalent to the Einstein's equations (17) written as

$$\epsilon_{IJKL} e^J \wedge F^{KL} = 0, \qquad \epsilon_{IJKL} e^I \wedge \mathrm{d}_\omega e^J = 0, \tag{57}$$

with the additional equation $\partial_\rho \varkappa := -(1/2)\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma} e^K_\rho \big(\Theta^{IJ}_{\nu\mu}\Theta^L_{\lambda\sigma} - \Theta^{IJ}_{\lambda\mu}\Theta^L_{\nu\sigma}\big)$.

3 n-phase space formulation of vielbein gravity

In this section we concentrate on the study of the pre-multisymplectic space defined by the constraint $\mathcal{H}^{\mathsf{Palatini}}(q,p) = \varkappa - e e^{[\mu}_I e^{\nu]}_J (\omega^J_{\mu K}\omega^{KI}_\nu) = 0$. This formulation is related to the n-phase space framework introduced by Kijowski and Szczyrba [63, 64, 65, 66] and further developed by Hélein [47]. Let us begin with some definitions, see [47]:

Definition 3.0.1. *A n-multimomentum phase space (or simply an n-phase space) is a triple $(\mathcal{M}, \boldsymbol{\omega}, \beta)$, where $\mathcal{M}$ is a smooth manifold, $\boldsymbol{\omega}$ is a closed $(n+1)$-form and β is an everywhere non-vanishing n-form.*

Definition 3.0.2. *A pre-multisymplectic manifold is a pair $\big(\mathcal{M}, \boldsymbol{\omega}\big)$, where $\mathcal{M}$ is a smooth manifold $\mathcal{M}$ and $\boldsymbol{\omega}$ is a closed $(n+1)$-form on $\mathcal{M}$.*

In the DW n-phase space formulation we express the dynamical structure on the level set of $\mathcal{H}$ *i.e.* by means of the constraint $\mathcal{H}=0$. We can canonically construct a n-phase space $(\mathcal{C}_\circ, \boldsymbol{\omega}|_{\mathcal{C}_\circ}, \beta = \Xi \lrcorner\, \boldsymbol{\omega}|_{\mathcal{C}_\circ})$, where $\mathcal{C}_\circ := \mathcal{H}^{-1}(0) := \{(q,p)\in\mathcal{M}/\ \mathcal{H}(q,p)=0\}$ and Ξ is a vector field such that $\mathrm{d}\mathcal{H}(\Xi)=1$. The dynamical equations in the pre-multisymplectic formulation, already presented in geometrical form (12), are equivalently written as

$$\forall \Xi \in C^\infty(\boldsymbol{\mathcal{M}}, T_m\mathcal{M}), \qquad (\Xi \lrcorner\, \boldsymbol{\omega})\big|_\Gamma = 0 \qquad \text{and} \qquad \beta|_\Gamma \neq 0, \tag{58}$$

see [46, 47]. We denote by $\mathcal{C}_\circ$ the hypersuface of constraints contained in the level set $\mathcal{C}_\circ$, *i.e.* we have the inclusion of spaces $\mathcal{C}_\circ \subset \mathcal{C}_\circ \hookrightarrow \mathcal{M}_{\mathsf{DW}}$. Using the primary constraints, the hypersurface of constraints is now

$$\mathcal{C}_\circ := \{(x,e,\omega,p)\in\mathcal{M}_{\mathsf{DW}}\ /\ \varkappa = e e_I^{[\mu} e_J^{\nu]}(\omega^J_{\mu K}\omega^{KI}_\nu), \quad p^{\omega_\mu\nu}_{IJ} = -\mathsf{E}_I^{[\mu} e_J^{\nu]}, \quad p_I^{e_\mu\nu} = 0\}. \tag{59}$$

Now we give the pre-multisymplectic formulation of dreibein and vierbein gravity. Note that we introduce the canonical inclusion $\iota_\circ : \mathcal{C}_\circ \hookrightarrow \mathcal{M}_{\mathsf{DW}}$ and the projection $\boldsymbol{\pi}_\circ : \mathcal{M}_{\mathsf{DW}} \to \mathcal{C}_\circ$. Then, we consider n-vector fields $X^{\mathcal{C}_\circ} \in \Gamma(\mathcal{C}_\circ, \Lambda^n\mathcal{C}_\circ)$ obtained by the push-forward $X^{\mathcal{C}_\circ} = (\boldsymbol{\pi}_\circ)_\star X^{\mathsf{DW}}$.

3.1 Pre-Multisymplectic formulation of dreibein gravity

In this section, we consider the first order Palatini functional of dreibein gravity $\mathcal{S}_{\mathsf{Palatini}}[e,\omega] = \int \epsilon_{IJK} e^I \wedge F^{JK}$, where $F^{JK} = \mathrm{d}\omega^{JK} + \omega^J{}_L \wedge \omega^{LK}$ is the curvature 2-form.

3.1.1 Canonical forms

Since $e^I = e^I_\mu \mathrm{d}x^\mu$ and $\omega^{JK} = \omega^{JK}_\mu \mathrm{d}x^\mu$, we obtain the following expression for the Poincaré-Cartan 3-form, identified with the Palatini action 3-form itself *i.e.* $\epsilon_{IJK} e^I \wedge F^{JK}$:

$$\theta^\circ = \ \epsilon_{IJK}\epsilon^{\mu\rho\sigma}\big(e^I_\mu \mathrm{d}\omega^{JK}_\sigma \wedge \beta_\rho + e^I_\mu \omega_\rho{}^J{}_L \omega^{LK}_\sigma \beta\big). \tag{60}$$

We demonstrate (60) by direct calculation $\theta^\circ = \epsilon_{IJK} e^I_\mu \mathrm{d}x^\mu \wedge \mathrm{d}\omega^{JK}_\sigma \wedge \mathrm{d}x^\sigma + \epsilon_{IJK} e^I_\mu \omega_\rho{}^J{}_L \omega^{LK}_\sigma \mathrm{d}x^\mu \wedge \mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma$. The Poincaré-Cartan 3-form is written as $\theta^\circ = \theta^\circ_1 + \theta^\circ_2$, where

$$\theta^\circ_1 = \epsilon_{IJK} e^I_\mu \omega_\rho{}^J{}_L \omega^{LK}_\sigma \mathrm{d}x^\mu \wedge \mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma, \quad \theta^\circ_2 = \epsilon_{IJK} e^I_\mu \mathrm{d}x^\mu \wedge \mathrm{d}\omega^{JK}_\sigma \wedge \mathrm{d}x^\sigma. \tag{61}$$

We re-express the terms θ°_1 and θ°_2 using the following lemma.

Lemma 3.1. *The terms θ°_1 and θ°_2 are given by*

$$\begin{aligned} \theta^\circ_1 &= \epsilon_{IJK}\epsilon^{\mu\rho\sigma} e^I_\mu \omega_\rho{}^J{}_L \omega^{LK}_\sigma \beta, \\ \theta^\circ_2 &= -\epsilon_{IJK}\epsilon^{\mu\rho\sigma} e^I_\mu \mathrm{d}\omega^{JK}_\rho \wedge \beta_\sigma. \end{aligned} \tag{62}$$

Proof. The formula for θ°_2 is straightforward. Since $\beta_1 = \mathrm{d}x^2 \wedge \mathrm{d}x^3$, $\beta_2 = -\mathrm{d}x^1 \wedge \mathrm{d}x^3$, and $\beta_3 = \mathrm{d}x^1 \wedge \mathrm{d}x^2$ we find $\epsilon_{IJK}\epsilon^{\mu\rho\sigma} e^I_\mu \mathrm{d}\omega^{JK}_\rho \wedge \beta_\sigma = -\epsilon_{IJK} e^I_\mu \mathrm{d}x^\mu \wedge \mathrm{d}\omega^{JK}_\sigma \wedge \mathrm{d}x^\sigma = -\theta^\circ_2$. Now we focus on the first term θ°_1. Using $\beta = \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 = (1/3!)\epsilon_{\alpha\beta\gamma}\mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta \wedge \mathrm{d}x^\gamma$, we have

$$\begin{aligned} \epsilon_{IJK}\epsilon^{\mu\rho\sigma} e^I_\mu \omega_\rho{}^J{}_L \omega^{LK}_\sigma \beta &= \epsilon_{IJK}\epsilon^{\mu\rho\sigma} e^I_\mu \omega_\rho{}^J{}_L \omega^{LK}_\sigma \beta, \\ &= (1/3!)\epsilon_{IJK}\epsilon^{\mu\rho\sigma}\epsilon_{\alpha\beta\gamma} e^I_\mu \omega_\rho{}^J{}_L \omega^{LK}_\sigma \mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta \wedge \mathrm{d}x^\gamma. \end{aligned}$$

Using the formula $\epsilon^{\mu\rho\sigma}\epsilon_{\alpha\beta\gamma} = 3!\delta^{[\mu}_{\alpha}\delta^{\rho}_{\beta}\delta^{\sigma]}_{\gamma} = (3!/3)\left(\delta^{\mu}_{\alpha}\delta^{[\rho}_{\beta}\delta^{\sigma]}_{\gamma} - \delta^{\rho}_{\alpha}\delta^{[\mu}_{\beta}\delta^{\sigma]}_{\gamma} + \delta^{\sigma}_{\alpha}\delta^{[\mu}_{\beta}\delta^{\rho]}_{\gamma}\right)$, see appendix B, we obtain

$$\epsilon_{IJK}\epsilon^{\mu\rho\sigma}e^{I}_{\mu}\omega_{\rho}{}^{J}{}_{L}\omega^{LK}_{\sigma}\beta \;=\; \epsilon_{IJK}e^{I}_{\mu}\omega_{\rho}{}^{J}{}_{L}\omega^{LK}_{\sigma}\mathrm{d}x^{\mu}\wedge\mathrm{d}x^{\rho}\wedge\mathrm{d}x^{\sigma}.$$

Note that $\epsilon^{\mu\rho\sigma}\beta = (1/3!)\epsilon^{\mu\rho\sigma}\epsilon_{\alpha\beta\gamma}\mathrm{d}x^{\alpha}\wedge\mathrm{d}x^{\beta}\wedge\mathrm{d}x^{\gamma} = \mathrm{d}x^{\mu}\wedge\mathrm{d}x^{\rho}\wedge\mathrm{d}x^{\sigma}$. Using lemma 3.1, the Poincaré-Cartan 3-form (60) is written as $\theta^{\circ} = \epsilon_{IJK}\epsilon^{\mu\rho\sigma}\big(e^{I}_{\mu}\mathrm{d}\omega^{JK}_{\sigma}\wedge\beta_{\rho} + e^{I}_{\mu}\omega_{\rho}{}^{J}{}_{L}\omega^{LK}_{\sigma}\beta\big)$. We are now interested in the exterior derivative $\mathrm{d}\theta^{\circ}$. The exterior derivative is decomposed in two terms $\mathrm{d}\theta^{\circ} = \mathrm{d}\theta^{\circ}_{1} + \mathrm{d}\theta^{\circ}_{2}$, where

$$\begin{aligned}\mathrm{d}\theta^{\circ}_{1} &= \epsilon_{IJK}\epsilon^{\mu\rho\sigma}\omega_{\rho}{}^{J}{}_{L}\omega^{LK}_{\sigma}\mathrm{d}e^{I}_{\mu}\wedge\beta + \epsilon_{IJK}\epsilon^{\mu\rho\sigma}e^{I}_{\mu}\big(\mathrm{d}\omega_{\rho}{}^{J}{}_{L}\omega^{LK}_{\sigma} + \omega_{\rho}{}^{J}{}_{L}\mathrm{d}\omega^{LK}_{\sigma}\big)\wedge\beta,\\ \mathrm{d}\theta^{\circ}_{2} &= \epsilon_{IJK}\epsilon^{\mu\rho\sigma}\mathrm{d}e^{I}_{\mu}\wedge\mathrm{d}\omega^{JK}_{\sigma}\wedge\beta_{\rho}.\end{aligned} \tag{63}$$

Note that the exterior derivative $\mathrm{d}\theta^{\circ}_{1}$ is given as

$$\begin{aligned}\mathrm{d}\theta^{\circ}_{1} &= \mathrm{d}(\epsilon_{IJK}\epsilon^{\mu\rho\sigma}e^{I}_{\mu}\omega_{\rho}{}^{J}{}_{L}\omega^{LK}_{\sigma}\beta) = \epsilon_{IJK}\epsilon^{\mu\rho\sigma}\mathrm{d}(e^{I}_{\mu}\omega_{\rho}{}^{J}{}_{L}\omega^{LK}_{\sigma})\wedge\beta,\\ &= \epsilon_{IJK}\epsilon^{\mu\rho\sigma}\omega_{\rho}{}^{J}{}_{L}\omega^{LK}_{\sigma}\mathrm{d}e^{I}_{\mu}\wedge\beta + \epsilon_{IJK}\epsilon^{\mu\rho\sigma}e^{I}_{\mu}(\mathrm{d}\omega_{\rho}{}^{J}{}_{L}\omega^{LK}_{\sigma} + \omega_{\rho}{}^{J}{}_{L}\mathrm{d}\omega^{LK}_{\sigma})\wedge\beta.\end{aligned} \tag{64}$$

where we have used $\mathrm{d}(e^{I}_{\mu}\omega_{\rho}{}^{J}{}_{L}\omega^{LK}_{\sigma}) = \mathrm{d}(e^{I}_{\mu})\omega_{\rho}{}^{J}{}_{L}\omega^{LK}_{\sigma} + e^{I}_{\mu}\mathrm{d}(\omega_{\rho}{}^{J}{}_{L})\omega^{LK}_{\sigma} + e^{I}_{\mu}\omega_{\rho}{}^{J}{}_{L}\mathrm{d}(\omega^{LK}_{\sigma})$.

Using (63), the multisymplectic 4-form $\boldsymbol{\omega}^{\circ} = \mathrm{d}\theta^{\circ} = \mathrm{d}\theta^{\circ}_{1} + \mathrm{d}\theta^{\circ}_{2}$ is now written as

$$\begin{aligned}\boldsymbol{\omega}^{\circ} &= \epsilon_{IJK}\epsilon^{\mu\rho\sigma}\mathrm{d}e^{I}_{\mu}\wedge\mathrm{d}\omega^{JK}_{\sigma}\wedge\beta_{\rho} + \epsilon_{IJK}\epsilon^{\mu\rho\sigma}(\omega_{\rho}{}^{J}{}_{L}\omega^{LK}_{\sigma})\mathrm{d}e^{I}_{\mu}\wedge\beta,\\ &\quad -\epsilon_{LJK}\epsilon^{\mu\rho\sigma}(e^{I}_{\mu}\omega_{\sigma}{}^{L}{}_{I})\mathrm{d}\omega^{JK}_{\rho}\wedge\beta.\end{aligned} \tag{65}$$

3.1.2 DW Hamilton equations

In the pre-multisymplectic formulation, we work on the level set $\mathcal{C}_{\circ} := \mathcal{H}^{-1}(0)$. The submanifold of interest is the constraint hypersurface $\mathcal{C}_{\circ}\subset\mathcal{C}_{\circ}$. The DW Hamilton equations are written in geometric form as $X^{\mathcal{C}_{\circ}}\lrcorner\,\boldsymbol{\omega}^{\circ}|_{\Gamma} = 0$. We evaluate the interior product of the vector field $X^{\mathcal{C}_{\circ}}$ with the terms $\mathrm{d}\theta^{\circ}_{1}$ and $\mathrm{d}\theta^{\circ}_{2}$, respectively. First, we find the term

$$\begin{aligned}X^{\mathcal{C}_{\circ}}\lrcorner\,\mathrm{d}\theta^{\circ}_{1} &= X^{\mathcal{C}_{\circ}}\lrcorner\,\big(\epsilon_{IJK}\epsilon^{\mu\rho\sigma}\omega_{\rho}{}^{J}{}_{L}\omega^{LK}_{\sigma}\mathrm{d}e^{I}_{\mu}\wedge\beta - \epsilon_{LJK}\epsilon^{\mu\rho\sigma}e^{I}_{\mu}\omega_{\rho}{}^{L}{}_{I}\mathrm{d}\omega^{JK}_{\sigma}\wedge\beta\big),\\ &= \epsilon_{IJK}\epsilon^{\mu\rho\sigma}\omega_{\rho}{}^{J}{}_{L}\omega^{LK}_{\sigma}\big(\beta(X)\mathrm{d}e^{I}_{\mu} - (\mathrm{d}e^{I}_{\mu}\wedge\beta_{\lambda})(X)\mathrm{d}x^{\lambda}\big),\\ &\quad -\epsilon_{IJK}\epsilon^{\mu\rho\sigma}\big(e^{L}_{\mu}\omega_{\rho}{}^{I}{}_{L}\beta(X)\mathrm{d}\omega^{JK}_{\sigma} - (\mathrm{d}\omega^{JK}_{\sigma}\wedge\beta_{\lambda})(X)\mathrm{d}x^{\lambda}\big),\end{aligned} \tag{66}$$

where we have used $\beta(X) = 1$. Then, we find the other term

$$\begin{aligned}X^{\mathcal{C}_{\circ}}\lrcorner\,\mathrm{d}\theta^{\circ}_{2} &= X^{\mathcal{C}_{\circ}}\lrcorner\,\big(\epsilon_{IJK}\epsilon^{\mu\rho\sigma}\mathrm{d}e^{I}_{\mu}\wedge\mathrm{d}\omega^{JK}_{\sigma}\wedge\beta_{\rho}\big),\\ &= \epsilon_{IJK}\epsilon^{\mu\rho\sigma}\big((\mathrm{d}\omega^{JK}_{\sigma}\wedge\beta_{\rho})(X)\mathrm{d}e^{I}_{\mu} - (\mathrm{d}e^{I}_{\mu}\wedge\beta_{\rho})(X)\mathrm{d}\omega^{JK}_{\sigma}\big),\\ &\quad +\epsilon_{IJK}\epsilon^{\mu\rho\sigma}\big(\mathrm{d}e^{I}_{\mu}\wedge\mathrm{d}\omega^{JK}_{\sigma}\wedge\beta_{\lambda\rho}\big)(X)\mathrm{d}x^{\lambda}\big).\end{aligned} \tag{67}$$

Now, using the equations (67) and (66),

$$X^{\mathcal{C}_{\circ}}\lrcorner\,\boldsymbol{\omega}^{\circ} = \epsilon_{IJK}\epsilon^{\mu\rho\sigma}\Big(\big(\Theta^{JK}_{\rho\sigma} + \omega_{\rho}{}^{J}{}_{L}\omega^{LK}_{\sigma}\big)\mathrm{d}e^{I}_{\mu} - \big(\Theta^{I}_{\rho\mu} + e^{L}_{\mu}\omega_{\rho}{}^{I}{}_{L}\big)\mathrm{d}\omega^{JK}_{\sigma}\big) + \Upsilon_{\lambda}\mathrm{d}x^{\lambda}\Big),$$

with $\Upsilon_{\lambda} = e^{L}_{\mu}\omega_{\rho}{}^{I}{}_{L}\Theta^{JK}_{\lambda\sigma} - \omega_{\rho}{}^{J}{}_{L}\omega^{LK}_{\sigma}\Theta^{I}_{\lambda\mu} + \big(\Theta^{JK}_{\lambda\sigma}\Theta^{I}_{\rho\mu} - \Theta^{JK}_{\rho\sigma}\Theta^{I}_{\lambda\mu}\big)$. Then, the DW Hamilton equations in the pre-multisymplectic formulation (*i.e.* $X\lrcorner\,\boldsymbol{\omega}^{\circ}|_{\Gamma} = 0$) are given by

$$\begin{aligned}\epsilon_{IJK}\epsilon^{\mu\rho\sigma}\big(\Theta^{JK}_{\rho\sigma} + \omega_{\rho}{}^{J}{}_{L}\omega^{LK}_{\sigma}\big) &= 0,\\ \epsilon_{IJK}\epsilon^{\mu\rho\sigma}\big(\Theta^{I}_{\rho\mu} + e^{L}_{\mu}\omega_{\rho}{}^{I}{}_{L}\big) &= 0,\\ \epsilon_{IJK}\epsilon^{\mu\rho\sigma}\Upsilon_{\lambda} &= 0,\end{aligned} \tag{68}$$

Remarks: (1) Note that if the first two conditions in (68) are satisfied, then the last one is automatically verified.

$$\begin{aligned}\epsilon_{IJK}\epsilon^{\mu\rho\sigma}\Upsilon_\lambda &= \epsilon_{IJK}\epsilon^{\mu\rho\sigma}e^L_\mu\omega_\rho{}^I{}_L\Theta^{JK}_{\lambda\sigma} - \epsilon_{IJK}\epsilon^{\mu\rho\sigma}\omega_\rho{}^J{}_L\omega^{LK}_\sigma\Theta^I_{\lambda\mu} + \big(\Theta^{JK}_{\lambda\sigma}\Theta^I_{\rho\mu} - \Theta^{JK}_{\rho\sigma}\Theta^I_{\lambda\mu}\big),\\ &= -\Theta^I_{\rho\mu}\Theta^{JK}_{\lambda\sigma} + \Theta^{JK}_{\rho\sigma}\Theta^I_{\lambda\mu} + \big(\Theta^{JK}_{\lambda\sigma}\Theta^I_{\rho\mu} - \Theta^{JK}_{\rho\sigma}\Theta^I_{\lambda\mu}\big) = 0.\end{aligned}$$

(2) The system (68) reproduces the Einstein's equations and is equivalently written as the following two equations: $\epsilon_{IJK}F^{JK}$ and $\epsilon_{IJK}\mathrm{d}_\omega e^I = 0$.

Proof Note that $\epsilon^{\rho\sigma\mu}\beta_\mu = \mathrm{d}x^\rho\wedge\mathrm{d}x^\sigma$, where $\beta_\mu\neq 0$. We straightforwardly obtain

$$\begin{aligned}\epsilon_{IJK}F^{JK} &= \epsilon_{IJK}\big(\partial_{[\rho}\omega^{JK}_{\sigma]} + \omega^J_{[\rho}{}_L\omega^{LK}_{\sigma]}\big)\mathrm{d}x^\rho\wedge\mathrm{d}x^\sigma = \epsilon_{IJK}\epsilon^{\mu\rho\sigma}\big(\Theta^{JK}_{\sigma\rho} + \omega^J_{\rho}{}_L\omega^{LK}_\sigma\big)\beta_\mu,\\ \epsilon_{IJK}\mathrm{d}_\omega e^K &= \epsilon_{IJK}\big(\partial_{[\rho}e^K_{\sigma]} + \omega^K_{[\rho}{}_L e^L_{\sigma]}\big)\mathrm{d}x^\rho\wedge\mathrm{d}x^\sigma = \epsilon_{IJK}\epsilon^{\rho\mu\sigma}\big(\Theta^K_{\mu\rho} + \omega^K_\rho{}_L e^L_\mu\big)\beta_\sigma.\end{aligned}$$

3.2 Pre-multisymplectic formulation of vierbein gravity

In this section we are interested in the pre-multisymplectic formulation of vierbein gravity. Here we will reproduce some results found in Bruno *et al.* [5, 6] and Rovelli [94, 95].

Let us consider the action functional $\mathcal{S}_{\mathsf{Palatini}}[e,\omega] = (1/2)\int\epsilon_{IJKL}e^I\wedge e^J\wedge F^{KL}$, where $F^{KL} = \mathrm{d}\omega^{KL} + \omega^K{}_M\wedge\omega^{ML}$.

3.2.1 Canonical forms

Since $e^I := e^I_\mu\mathrm{d}x^\mu$ and $\omega^{KL} := \omega^{KL}_\mu\mathrm{d}x^\mu$, we obtain the following expression for the Poincaré-Cartan 4-form $\theta^\circ = (1/2)\left(\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^I_\mu e^J_\nu\mathrm{d}\omega^{KL}_\rho\wedge\beta_\sigma + \epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^I_\mu e^J_\nu\omega_\sigma{}^K{}_M\omega^{ML}_\rho\beta\right)$. By direct calculation

$$\begin{aligned}\theta^\circ &= (1/2)\epsilon_{IJKL}e^I_\mu\mathrm{d}x^\mu\wedge e^J_\nu\mathrm{d}x^\nu\wedge(\mathrm{d}(\omega^{KL}_\rho\mathrm{d}x^\rho) + \omega_\rho{}^K{}_M\mathrm{d}x^\rho\wedge\omega^{ML}_\sigma\mathrm{d}x^\sigma),\\ &= (1/2)\epsilon_{IJKL}e^I_\mu e^J_\nu\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\wedge\mathrm{d}\omega^{KL}_\sigma\wedge\mathrm{d}x^\sigma,\\ &\quad +(1/2)\epsilon_{IJKL}e^I_\mu e^J_\nu\omega_\rho{}^K{}_M\omega^{ML}_\sigma\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\wedge\mathrm{d}x^\rho\wedge\mathrm{d}x^\sigma.\end{aligned}\tag{69}$$

The Poincaré-Cartan form is written as $\theta^\circ = \theta^\circ_1 + \theta^\circ_2$, where

$$\begin{aligned}\theta^\circ_1 &= (1/2)\epsilon_{IJKL}e^I_\mu e^J_\nu\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\wedge\mathrm{d}\omega^{JK}_\sigma\wedge\mathrm{d}x^\sigma,\\ \theta^\circ_2 &= (1/2)\epsilon_{IJKL}e^I_\mu e^J_\nu\omega_\rho{}^K{}_M\omega^{ML}_\sigma\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\wedge\mathrm{d}x^\rho\wedge\mathrm{d}x^\sigma.\end{aligned}\tag{70}$$

Since $\epsilon^{\mu\nu\rho\sigma}\beta_\sigma = (1!(4-1)!/3!)\,\delta^{[\mu}_\alpha\delta^\nu_\beta\delta^{\rho]}_\gamma\mathrm{d}x^\alpha\wedge\mathrm{d}x^\beta\wedge\mathrm{d}x^\gamma$, we obtain $\epsilon^{\mu\nu\rho\sigma}\beta_\sigma = \mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\wedge\mathrm{d}x^\rho$. Then, $\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\wedge\mathrm{d}\omega^{KL}_\sigma\wedge\mathrm{d}x^\sigma = \epsilon^{\mu\nu\rho\sigma}\mathrm{d}\omega^{KL}_\rho\wedge\beta_\sigma$. Hence,

$$\epsilon_{IJKL}e^I_\mu e^J_\nu\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\wedge\mathrm{d}\omega^{JK}_\sigma\wedge\mathrm{d}x^\sigma = (1/2)\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^I_\mu e^J_\nu\mathrm{d}\omega^{KL}_\rho\wedge\beta_\sigma = \theta^\circ_1.\tag{71}$$

Note that the volume form $\beta = \mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3\wedge\mathrm{d}x^4$ is equivalently written $\beta = (1/4!)\epsilon_{\alpha\beta\gamma\delta}\mathrm{d}x^\alpha\wedge\mathrm{d}x^\beta\wedge\mathrm{d}x^\gamma\wedge\mathrm{d}x^\delta$, then the second term in (69) is written as

$$\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^I_\mu e^J_\nu\omega_\sigma{}^K{}_M\omega^{ML}_\rho\beta = (1/2)\epsilon_{IJKL}e^I_\mu e^J_\nu\omega_\rho{}^K{}_M\omega^{ML}_\sigma\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\wedge\mathrm{d}x^\rho\wedge\mathrm{d}x^\sigma = \theta^\circ_2$$

where we have used the formula (113) for the expression $\epsilon^{\mu\nu\rho\sigma}\epsilon_{\alpha\beta\gamma\delta}$.

Let us compute the pre-multisymplectic 5-form $\boldsymbol{\omega}^\circ = \mathrm{d}\theta^\circ$:

$$\begin{aligned}\boldsymbol{\omega}^\circ &= \epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^I_\mu\big(\mathrm{d}e^J_\nu\wedge\mathrm{d}\omega^{KL}_\rho\wedge\beta_\sigma+\omega^K_{\sigma M}\omega^{ML}_\rho\mathrm{d}e^J_\nu\wedge\beta\big),\\ &\quad+(1/2)\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^I_\mu e^J_\nu\omega^{ML}_\rho\mathrm{d}\big(\omega^K_{\sigma M}\big)\wedge\beta+(1/2)\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^I_\mu e^J_\nu\omega_\sigma{}^K{}_M\mathrm{d}\big(\omega^{ML}_\rho\big)\wedge\beta,\\ &= \epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^I_\mu\big(\mathrm{d}e^J_\nu\wedge\mathrm{d}\omega^{KL}_\rho\wedge\beta_\sigma+\omega^K_{\sigma M}\omega^{ML}_\rho\mathrm{d}e^J_\nu\wedge\beta\big),\\ &\quad+\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^I_\mu e^J_\nu\omega_\sigma{}^K{}_M\mathrm{d}\omega^{ML}_\rho\wedge\beta.\end{aligned}$$

Using the algebraic relation $\epsilon^{\mu\nu\rho\sigma}\epsilon_{IJKL}e^I_\mu e^J_\nu\omega_\sigma{}^K{}_M\mathrm{d}\omega^{ML}_\rho = -\epsilon^{\mu\nu\rho\sigma}\epsilon_{INKL}e^I_\mu e^J_\nu\omega_\sigma{}^N{}_J\mathrm{d}\omega^{KL}_\rho$, the pre-multisymplectic 5-form is written as

$$\begin{aligned}\boldsymbol{\omega}^\circ &= \epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^I_\mu\mathrm{d}e^J_\nu\wedge\mathrm{d}\omega^{KL}_\rho\wedge\beta_\sigma+\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^I_\mu\omega_\sigma{}^K{}_M\omega^{ML}_\rho\mathrm{d}e^J_\nu\wedge\beta,\\ &\quad-\epsilon^{\mu\nu\rho\sigma}\epsilon_{INKL}e^I_\mu e^J_\nu\omega_\sigma{}^N{}_J\mathrm{d}\omega^{KL}_\rho\wedge\beta.\end{aligned}\tag{72}$$

3.2.2 DW Hamilton equations

In the pre-multisymplectic setting we work with the constraint $\mathcal{H}=0$. The dynamics is expressed on the level set $\mathcal{C}_\circ := \mathcal{H}^{-1}(0)$ and the DW Hamilton equations are written as

$$X^{\mathcal{C}_\circ}\lrcorner\,\boldsymbol{\omega}^\circ\big|_\Gamma = 0.\tag{73}$$

We now evaluate, for vierbein gravity, the interior product of the multivector field $X^{\mathcal{C}_\circ}\in\Lambda^4T\mathcal{C}_\circ$, with the three terms in (72). We choose a 4-vector $X^{\mathcal{C}_\circ} = X_1^{\mathcal{C}_\circ}\wedge X_2^{\mathcal{C}_\circ}\wedge X_3^{\mathcal{C}_\circ}\wedge X_4^{\mathcal{C}_\circ}$, where for any $1\leq\alpha\leq 4$, the vector field $X_\alpha\in\mathfrak{X}^1(\mathcal{C}_\circ)$ is

$$X^{\mathcal{C}_\circ}_\alpha = \frac{\partial}{\partial x^\alpha}+\Theta^I_{\alpha\mu}\frac{\partial}{\partial e^I_\mu}+\Theta^{IJ}_{\alpha\mu}\frac{\partial}{\partial\omega^I_\mu}.$$

The left side of (73) is written as

$$\begin{aligned}X^{\mathcal{C}_\circ}\lrcorner\,\boldsymbol{\omega}^\circ &= -\epsilon^{\mu\nu\rho\sigma}\epsilon_{INKL}e^I_\mu e^J_\nu\omega_\sigma{}^N{}_J\mathrm{d}\omega^{KL}_\rho\wedge\beta-\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^I_\mu(\mathrm{d}e^J_\nu\wedge\beta_\sigma)(X),\\ &\quad-\epsilon_{INKL}\epsilon^{\mu\nu\rho\sigma}e^I_\mu e^J_\nu\omega_\sigma{}^N{}_J\big((\beta)(X)\mathrm{d}\omega^{KL}_\rho+(\mathrm{d}\omega^{KL}_\rho\wedge\beta_\lambda)(X)\mathrm{d}x^\lambda\big),\\ &\quad+\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^I_\mu\left((\mathrm{d}e^J_\nu\wedge\mathrm{d}\omega^{KL}_\rho\wedge\beta_{\lambda\sigma})(X)\mathrm{d}x^\lambda\right),\\ &= \epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^I_\mu\big((\Theta^{KL}_{\sigma\rho}+\omega_\sigma{}^K{}_M\omega^{ML}_\rho)\mathrm{d}e^J_\nu-(e^I_\mu e^N_\nu\omega_\sigma{}^J{}_N+\Theta^I_{\sigma\nu})\mathrm{d}\omega^{KL}_\rho+\Upsilon_\lambda\mathrm{d}x^\lambda\big),\end{aligned}$$

where $\Upsilon_\lambda = e^N_\nu\omega_\rho{}^J{}_N\Theta^{KL}_{\lambda\sigma}-\omega_\sigma{}^K{}_M\omega^{ML}_\rho\Theta^J_{\lambda\nu}+\big(\Theta^{KL}_{\lambda\sigma}\Theta^J_{\rho\nu}-\Theta^{KL}_{\rho\sigma}\Theta^J_{\lambda\nu}\big)$. In the pre-multisymplectic setting we find the DW Hamilton equations for the Palatini action

$$\begin{aligned}\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^I_\mu\big(\Theta^{KL}_{\sigma\rho}+\omega_\sigma{}^K{}_M\omega^{ML}_\rho\big) &= 0,\\ \epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^I_\mu\big(\Theta^J_{\sigma\nu}+e^N_\nu\omega_\sigma{}^J{}_N\big) &= 0,\\ \epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}\Upsilon_\lambda &= 0.\end{aligned}\tag{74}$$

Analogously to the dreibein case, see the end of the section 3.1.2, we obtain the Einstein's system of equations in term of differential forms. We have, see also (57), $\epsilon_{IJKL}e^I\wedge\mathrm{d}_\omega e^J=0$ and $\epsilon_{IJKL}e^I\wedge e^J\wedge F^{KL}=0$, together with the equation $\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}\Upsilon_\lambda=0$.

4 Hamiltonian $(n-1)$-forms and brackets

4.1 Hamiltonian $(n-1)$-forms, homotopy Lie algebra

We begin this section with the definition of Hamiltonian $(n-1)$-forms and their related Hamiltonian vector fields, *c.f.* Cariñena, Crampin and Ibort [11], Kanatchikov [53, 54, 55], Forger *et al.* [28, 29, 30], Hélein and Kouneiher [49, 50, 51].

Definition 4.1.1. *Let $(\mathcal{M},\boldsymbol{\omega})$ be a multisymplectic manifold. An $(n-1)$-form $\boldsymbol{\varphi}$ is called a Hamiltonian $(n-1)$-form if and only if there exists $\Xi_{\boldsymbol{\varphi}} \in \mathfrak{X}(\mathcal{M})$ such that $\Xi_{\boldsymbol{\varphi}} \lrcorner\, \boldsymbol{\omega} + \mathrm{d}\boldsymbol{\varphi} = 0$.*

We denote by $\Omega^{n-1}_{\mathsf{Ham}}(\mathcal{M})$ the set of all Hamiltonian $(n-1)$-forms. For any $\boldsymbol{\varphi},\boldsymbol{\rho} \in \Omega^{n-1}_{\mathsf{Ham}}(\mathcal{M})$, let us define the bracket

$$\big\{\boldsymbol{\varphi},\boldsymbol{\rho}\big\} := \Xi_{\boldsymbol{\varphi}} \wedge \Xi_{\boldsymbol{\rho}} \lrcorner\, \boldsymbol{\omega} = \Xi_{\boldsymbol{\varphi}} \lrcorner\, \mathrm{d}\boldsymbol{\rho} = -\Xi_{\boldsymbol{\rho}} \lrcorner\, \mathrm{d}\boldsymbol{\varphi}, \tag{75}$$

where $\{\boldsymbol{\varphi},\boldsymbol{\rho}\} \in \Omega^{n-1}_{\mathsf{Ham}}(\mathcal{M})$. For any form $\boldsymbol{\eta} \in \Omega^*(\mathcal{M})$ and any decomposable multivector field $\Xi := \Xi_1 \wedge \cdots \wedge \Xi_n \in \mathfrak{X}^n(\mathcal{M})$, we have $\Xi \lrcorner\, \boldsymbol{\eta} = (\Xi_1 \wedge \cdots \wedge \Xi_n) \lrcorner\, \boldsymbol{\eta} := \Xi_n \lrcorner\, \cdots \lrcorner\, \Xi_1 \lrcorner\, \boldsymbol{\eta}$. This definition is the natural analogue of the Poisson bracket in classical mechanics. The bracket defined in 4.1.1 satisfies the antisymmetry property: $\{\boldsymbol{\varphi},\boldsymbol{\rho}\} + \{\boldsymbol{\rho},\boldsymbol{\varphi}\} = 0$, but the Jacobi condition is only satisfied modulo an exact term, see [49, 90]. For any $\boldsymbol{\varphi},\boldsymbol{\rho},\boldsymbol{\eta} \in \mathfrak{P}^{n-1}_{\circ}(\mathcal{M})$

$$\{\{\boldsymbol{\rho},\boldsymbol{\eta}\},\boldsymbol{\varphi}\} + \{\{\boldsymbol{\eta},\boldsymbol{\varphi}\},\boldsymbol{\rho}\} + \{\{\boldsymbol{\varphi},\boldsymbol{\rho}\},\boldsymbol{\eta}\} = \mathrm{d}(\Xi_{\boldsymbol{\varphi}} \wedge \Xi_{\boldsymbol{\rho}} \wedge \Xi_{\boldsymbol{\eta}} \lrcorner\, \boldsymbol{\omega}). \tag{76}$$

Using the Cartan formula, *i.e.* $\mathcal{L}_{\Xi}\boldsymbol{\omega} = \mathrm{d}(\Xi \lrcorner\, \boldsymbol{\omega}) + \Xi \lrcorner\, \mathrm{d}\boldsymbol{\omega} = 0$, we define a locally Hamiltonian vector field of $(\mathcal{M},\boldsymbol{\omega})$ to be a vector field $\Xi \in \Gamma(\mathcal{M},T\mathcal{M})$, such that $\mathcal{L}_{\Xi}\boldsymbol{\omega} = 0$ (since $\mathrm{d}\boldsymbol{\omega} = 0$). We are looking for vector fields $\Xi \in \Gamma(\mathcal{M},T\mathcal{M})$, such that $\mathrm{d}(\Xi \lrcorner\, \boldsymbol{\omega}) = 0$. We denote by $\mathfrak{X}^1_{\mathsf{Ham}}(\mathcal{M})$ the set of locally Hamiltonian vector fields of the multisymplectic manifold $(\mathcal{M},\boldsymbol{\omega})$, *i.e.*

$$\mathfrak{X}^1_{\mathsf{Ham}}(\mathcal{M}) = \{\Xi \in \Gamma(\mathcal{M},T\mathcal{M}) \ / \ \mathrm{d}(\Xi \lrcorner\, \boldsymbol{\omega}) = 0\} = \{\Xi \in \Gamma(\mathcal{M},T\mathcal{M}) \ / \ \mathcal{L}_{\Xi}\boldsymbol{\omega} = 0\}\,. \tag{77}$$

Although antisymmetric, the bracket (75) nevertheless fails to respect the Jacobi property which is necessary to obtain a strict Lie algebraic structure. Thus, $\big(\Omega^{n-1}_{\mathsf{Ham}}(\mathcal{M}), \{\cdot,\cdot\}\big)$ is not a Lie algebra. The fact that this bracket satisfies the Jacobi identity only up to an exact form was already noted by Goldschmidt and Sternberg in [43]. This co-cycle obstruction reveals the connection with homotopy Lie algebra, see [76, 77]. We refer to the paper by Baez and *al.* [1, 2], where the Lie 2-algebra is used to describe the dynamics of the classical bosonic string. More generally, the relation between MG and L_∞-algebra is found in Rogers [90, 91], Richter [88, 89], and Vitagliano [104], where a L_∞-algebra is a chain complex equipped with an antisymmetric bracket operation that satisfies the Jacobi identity up to coherent homotopy [1, 91].

4.2 Hamiltonian forms, graded Poisson bracket

In Kanatchikov's approach [53, 54, 55, 56] the polysymplectic form $\boldsymbol{\omega}^{\vee} = \mathrm{d}p^{\mu}_i \wedge \mathrm{d}y^i \wedge \beta_{\mu}$ is used to construct the graded Poisson bracket on forms of arbitrary degrees. Let $\boldsymbol{\varphi} \in \Omega^p_{\mathsf{Ham}}(\mathcal{M})$,

$\boldsymbol{\rho} \in \Omega^q_{\mathsf{Ham}}(\mathcal{M})$ and $\boldsymbol{\eta} \in \Omega^r_{\mathsf{Ham}}(\mathcal{M})$ (where $0 \leq p, q, r \leq n-1$) be Hamiltonian forms, as defined in [53], of degrees $\mathsf{deg}(\boldsymbol{\varphi}) := p$, $\mathsf{deg}(\boldsymbol{\rho}) := q$, and $\mathsf{deg}(\boldsymbol{\eta}) := r$, respectively. The graded Poisson bracket on Hamiltonian $(p-1)$-forms of arbitrary degrees is

$$\{\overset{p}{\boldsymbol{\varphi}}, \overset{q}{\boldsymbol{\rho}}\} = (-1)^{n-p} \Xi_{\boldsymbol{\varphi}} \lrcorner\, \Xi_{\boldsymbol{\rho}} \lrcorner\, \boldsymbol{\omega}^{\mathsf{v}} = (-1)^{n-p} \Xi_{\boldsymbol{\varphi}} \lrcorner\, \mathrm{d}^{\mathsf{v}} \overset{q}{\boldsymbol{\rho}}, \tag{78}$$

where d^{v} is the vertical exterior derivative and the respective Hamiltonian multivector fields related to $\boldsymbol{\varphi}$ and $\boldsymbol{\rho}$ are $\Xi_{\boldsymbol{\varphi}} \in \mathfrak{X}^{n-p}_{\mathsf{Ham}}(\mathcal{M}), \Xi_{\boldsymbol{\rho}} \in \mathfrak{X}^{n-q}_{\mathsf{Ham}}(\mathcal{M})$. The graded Poisson bracket (78) is graded antisymmetric, *i.e.*

$$\{\overset{p}{\boldsymbol{\varphi}}, \overset{q}{\boldsymbol{\rho}}\} \;=\; -(-1)^{(n-p-1)(n-q-1)} \{\overset{q}{\boldsymbol{\rho}}, \overset{p}{\boldsymbol{\varphi}}\}, \tag{79}$$

and satisfies the graded Jacobi identity

$$(-1)^{d_{\boldsymbol{\varphi}} d_{\boldsymbol{\eta}}} \{\overset{p}{\boldsymbol{\varphi}} \{\overset{q}{\boldsymbol{\rho}}, \overset{r}{\boldsymbol{\eta}}\}\} + (-1)^{d_{\boldsymbol{\rho}} d_{\boldsymbol{\varphi}}} \{\overset{p}{\boldsymbol{\rho}} \{\overset{q}{\boldsymbol{\eta}}, \overset{q}{\boldsymbol{\varphi}}\}\} + (-1)^{d_{\boldsymbol{\eta}} d_{\boldsymbol{\rho}}} \{\overset{p}{\boldsymbol{\eta}} \{\overset{q}{\boldsymbol{\varphi}}, \overset{r}{\boldsymbol{\rho}}\}\} = 0, \tag{80}$$

where we have denoted by $d_{\boldsymbol{\varphi}} := n - \mathsf{deg}(\boldsymbol{\varphi}) - 1$, $d_{\boldsymbol{\eta}} := n - \mathsf{deg}(\boldsymbol{\eta}) - 1$, and $d_{\boldsymbol{\rho}} := n - \mathsf{deg}(\boldsymbol{\rho}) - 1$. Note that $\mathsf{deg}(\boldsymbol{\eta})$ denote the degree of the Hamiltonian form $\boldsymbol{\eta}$. The Poisson bracket of Hamiltonian forms is obtained using the Schouten-Nijenhuis bracket $[,]$ of the related Hamiltonian multivector fields $-\mathrm{d}\{\boldsymbol{\varphi}, \boldsymbol{\rho}\} = [\Xi_{\boldsymbol{\varphi}}, \Xi_{\boldsymbol{\rho}}] \lrcorner\, \boldsymbol{\omega}^{\mathsf{v}}$. The Schouten-Nijenhuis bracket, see [83, 84, 99], *i.e.* a bilinear map $[,] : \mathfrak{X}^*_{\mathsf{Ham}}(\mathcal{M}) \times \mathfrak{X}^*_{\mathsf{Ham}}(\mathcal{M}) \to \mathfrak{X}^*_{\mathsf{Ham}}(\mathcal{M})$, that obeys the graded antisymmetric property and the graded Leibniz rule

$$\begin{aligned} [\Xi_1, \Xi_2] &= -(-1)^{(\mathsf{deg}(\Xi_1)-1)(\mathsf{deg}(\Xi_2)-1)} [\Xi_2, \Xi_2], \\ [\Xi_1, \Xi_2 \wedge \Xi_3] &= [\Xi_1, \Xi_2] \wedge \Xi_3 + (-1)^{(\mathsf{deg}(\Xi_1)-1)\mathsf{deg}(\Xi_2)} \Xi_2 \wedge [\Xi_1, \Xi_3], \end{aligned} \tag{81}$$

as well as the graded Jacobi identity

$$0 \;=\; (-1)^{d_1 d_3} [\Xi_1, [\Xi_2, \Xi_3]] + (-1)^{d_2 d_3} [\Xi_3, [\Xi_1, \Xi_2]] + (-1)^{d_1 d_2} [\Xi_2, [\Xi_3, \Xi_1]], \tag{82}$$

where $d_i := \mathsf{deg}(\Xi_i) - 1$ and $\mathsf{deg}(\Xi_i)$ denote the degrees of the respective multivector fields. On vector fields, the Schouten-Nijenhuis bracket reduces to the standard Lie bracket. However, the exterior product of two Hamiltonian forms $\overset{p}{\boldsymbol{\varphi}} \wedge \overset{q}{\boldsymbol{\rho}}$ is not Hamiltonian in general. Kanatchikov introduces the co-exterior product $\bullet$ of horizontal forms $\boldsymbol{\varphi} \bullet \boldsymbol{\rho} = \star^{-1}(\star\boldsymbol{\varphi} \wedge \star\boldsymbol{\rho})$, see [55]. The space of Hamiltonian forms is closed with respect to the co-exterior product. Thus, $\mathfrak{A}^{\mathsf{Poly}}_{\mathsf{DW}} = \{\Omega^*_{\mathsf{Ham}}(\mathcal{M}^{\mathsf{Poly}}_{\mathsf{DW}}), \{,\}, \bullet\}$ is a Gerstenhaber algebra [35]. As an illustration of the use of the higher dimensional algebraic structures in field theory we refer to the example of the classical string. The DW Hamiltonian formulation of Nambu-Goto string, using the polysymplectic formalism and the Poisson-Gerstenhaber algebra [56], is given by Kanatchikov in [53, 54].

In section 4.3 and 4.4 we will consider Hamiltonian $(n-1)$-forms $\boldsymbol{\varphi} = \boldsymbol{\varphi}^\mu \beta_\mu \in \Omega^{n-1}_{\mathsf{Ham}}(\mathcal{M})$. In that case, the graded Poisson structure reduces to a Poisson structure. For any $\boldsymbol{\varphi}, \boldsymbol{\rho} \in \Omega^{n-1}(\mathcal{M}^{\mathsf{Poly}}_{\mathsf{DW}})$, the bracket is defined as $\{\boldsymbol{\varphi}, \boldsymbol{\rho}\} := -\Xi_{\boldsymbol{\varphi}} \lrcorner\, \Xi_{\boldsymbol{\rho}} \lrcorner\, \boldsymbol{\omega}^{\mathsf{v}} = (-1)^{n-r} \Xi_{\boldsymbol{\varphi}} \lrcorner\, \mathrm{d}^{\mathsf{v}} \boldsymbol{\rho}$, where $\Xi_{\boldsymbol{\varphi}}, \Xi_{\boldsymbol{\rho}} \in \mathfrak{X}^1_{\mathsf{Ham}}(\mathcal{M})$. The Poisson bracket has the antisymmetry property $\{\boldsymbol{\varphi}, \boldsymbol{\rho}\} + \{\boldsymbol{\rho}, \boldsymbol{\varphi}\} = 0$ and it satisfies the Jacobi identity $\{\boldsymbol{\varphi}\{\boldsymbol{\rho}, \boldsymbol{\eta}\}\} + \{\boldsymbol{\rho}\{\boldsymbol{\eta}, \boldsymbol{\varphi}\}\} + \{\boldsymbol{\eta}\{\boldsymbol{\varphi}, \boldsymbol{\rho}\}\} = 0$.

4.3 Hamiltonian $(n-1)$-forms

In this section we consider Hamiltonian $(n-1)$-forms and their related Hamiltonian vector fields on the DW manifold $\mathcal{M}_{\mathsf{DW}}$. We will work with the multisymplectic manifold $(\mathcal{M}_{\mathsf{DW}}, \boldsymbol{\omega}^{\mathsf{DW}}) = (\mathcal{M}_{\mathsf{DW}}, \iota_1^\star \boldsymbol{\omega})$ and with the pair $(\mathcal{C}, \iota^\star \Omega^{\mathsf{DW}})$, respectively.

First, we use the results of Hélein and Kouneiher [51], see, in particular, section 5.2, page 771. We consider the general formula which describes the Hamiltonian vector fields and their related Hamiltonian $(n-1)$-forms. In the terminology by Hélein and Kouneiher those objects are termed «*algebraic observable* $(n-1)$*-forms*» and «*infinitesimal symplectomorphisms*», respectively (see [51]). This formulation corresponds to the algebraic structure described in section 4.1.

Let $\Xi \in \Gamma(\mathcal{M}_{\mathsf{DW}}, T\mathcal{M}_{\mathsf{DW}})$ be an arbitrary vector field on $\mathcal{M}_{\mathsf{DW}}$ written as

$$\Xi := \boldsymbol{X}^\nu \frac{\partial}{\partial x^\nu} + \boldsymbol{\Theta}^M_\lambda \frac{\partial}{\partial e^M_\lambda} + \boldsymbol{\Theta}^{IJ}_\mu \frac{\partial}{\partial \omega^{IJ}_\mu} + \boldsymbol{\Upsilon} \frac{\partial}{\partial \varkappa} + \boldsymbol{\Upsilon}^{e_\mu \nu}_I \frac{\partial}{\partial p_I^{e_\mu \nu}} + \boldsymbol{\Upsilon}^{\omega_\mu \nu}_{IJ} \frac{\partial}{\partial p_{IJ}^{\omega_\mu \nu}}, \tag{83}$$

such that $\mathrm{d}(\Xi \lrcorner \Omega^{\mathsf{DW}}) = 0$. Note that $\boldsymbol{X}^\nu, \boldsymbol{\Theta}^M_\lambda, \boldsymbol{\Theta}^{IJ}_\mu, \boldsymbol{\Upsilon}, \boldsymbol{\Upsilon}^{e_\mu \nu}_I$ and $\boldsymbol{\Upsilon}^{\omega_\mu \nu}_{IJ}$ are smooth functions on $\mathcal{M}_{\mathsf{DW}}$. The set of all infinitesimal symplectomorphisms, *i.e.* locally Hamiltonian vector fields, of $(\mathcal{M}_{\mathsf{DW}}, \Omega^{\mathsf{DW}})$ is described by vector fields $\Xi = \Xi(\mathbf{Q}) + \Xi(\mathbf{P})$, where

$$\begin{aligned}
\Xi(\mathbf{Q}) &= \Upsilon \frac{\partial}{\partial \varkappa} + \Upsilon^{e_\mu \alpha}_I \frac{\partial}{\partial p_I^{e_\mu \alpha}} + \Upsilon^{\omega_\mu \alpha}_{IJ} \frac{\partial}{\partial p_{IJ}^{\omega_\mu \alpha}}, \quad \text{with} \quad \frac{\partial \Upsilon}{\partial \omega^{IJ}_\mu} - \frac{\partial \Upsilon^{e_\mu \nu}_I}{\partial x^\nu} - \frac{\partial \Upsilon^{\omega_\mu \nu}_{IJ}}{\partial x^\nu} = 0, \\
\Xi(\mathbf{P}) &= X^\nu \frac{\partial}{\partial x^\nu} + \Theta^M_\lambda \frac{\partial}{\partial e^M_\lambda} + \Theta^{IJ}_\mu \frac{\partial}{\partial \omega^{IJ}_\mu} - \left(\varkappa(\frac{\partial X^\nu}{\partial x^\nu}) + \frac{\partial \Theta^{IJ}_\mu}{\partial x^\nu} p_{IJ}^{\omega_\mu \nu} + \frac{\partial \Theta^I_\mu}{\partial x^\nu} p_I^{e_\mu \nu} \right) \frac{\partial}{\partial \varkappa}, \\
&\quad + \left(p_K^{e_\rho \sigma} \delta^\mu_\rho \left(\delta^K_I \left[(\frac{\partial X^\nu}{\partial x^\sigma}) - \delta^\nu_\sigma (\frac{\partial X^\lambda}{\partial x^\lambda}) \right] - (\frac{\partial \Theta^K_\sigma}{\partial \omega^I_\nu}) \right) - \varkappa(\frac{\partial X^\nu}{\partial e^I_\mu}) \right) \frac{\partial}{\partial p_I^{e_\mu \nu}}, \\
&\quad + \left(p_{KL}^{\omega_\rho \sigma} \delta^\mu_\rho \left(\delta^K_I \delta^L_J \left[(\frac{\partial X^\nu}{\partial x^\sigma}) - \delta^\nu_\sigma (\frac{\partial X^\lambda}{\partial x^\lambda}) \right] - (\frac{\partial \Theta^{KL}_\sigma}{\partial \omega^{IJ}_\nu}) \right) - \varkappa(\frac{\partial X^\nu}{\partial \omega^{IJ}_\mu}) \right) \frac{\partial}{\partial p_{IJ}^{\omega_\mu \nu}},
\end{aligned}$$

and $X^\nu, \Theta^M_\lambda, \Theta^{IJ}_\mu, \Upsilon, \Upsilon^{e_\mu \nu}_I$ and $\Upsilon^{\omega_\mu \nu}_{IJ}$ are smooth functions on $\mathcal{Y}$. We hope to present elsewhere [102] a detailed analysis of all algebraic observable $(n-1)$-forms, *i.e.* of all Hamiltonian $(n-1)$-forms as defined in section 4.1, in the DW formulation of vielbein gravity.

We now restrict ourselves to simple examples of Hamiltonian $(n-1)$-forms in $\Omega^{n-1}_{\mathsf{Ham}}(\mathcal{M}_{\mathsf{DW}})$. Let us consider the $(n-1)$-forms $\mathbf{Q}_{e,\chi} = \mathbf{Q}^I_{e,\chi} \otimes \mathfrak{e}_I$, $\mathbf{Q}_{\omega,\psi} = \mathbf{Q}^{IJ}_{\omega,\psi} \otimes \Delta_{IJ}$, $\mathbf{P}^{\omega,\varphi}_{IJ} = \mathbf{P}^{\omega,\varphi}_{IJ} \otimes \Delta^{IJ}$, and $\mathbf{P}^{e,\zeta}_I = \mathbf{P}^{e,\zeta}_I \otimes \mathfrak{e}^I$, where

$$\begin{aligned}
\mathbf{Q}_{e,\chi} &= \chi^{\mu\nu}_I(x) e^I_\mu \beta_\nu, & \mathbf{P}_{e,\zeta} &= \zeta^I_\mu(x) p_I^{e_\mu \nu} \beta_\mu, \\
\mathbf{Q}_{\omega,\psi} &= \psi^{\mu\nu}_{IJ}(x) \omega^{IJ}_\mu \beta_\nu, & \mathbf{P}_{\omega,\varphi} &= \varphi^{IJ}_\mu(x) p_{IJ}^{\omega_\mu \nu} \beta_\nu.
\end{aligned} \tag{84}$$

If we evaluate those different $(n-1)$-forms on the hypersurface of constraints $\mathcal{C}$ defined in section 2.3, we obtain

$$\begin{aligned}
\mathbf{Q}_{e,\chi}\Big|_{\mathcal{C}} &= \iota^\star \mathbf{Q}_{\omega,\psi} = \chi^{\mu\nu}_I(x) e^I_\mu \beta_\nu = \mathbf{Q}_{e,\chi} \\
\mathbf{Q}_{\omega,\psi}\Big|_{\mathcal{C}} &= \iota^\star \mathbf{Q}_{\omega,\psi} = \psi^{\mu\nu}_{IJ}(x) \omega^{IJ}_\mu \beta_\nu = \mathbf{Q}_{\omega,\psi}, \\
\mathbf{P}_{e,\zeta}\Big|_{\mathcal{C}} &= \iota^\star \mathbf{P}_{e,\zeta} = 0, \\
\mathbf{P}_{\omega,\varphi}\Big|_{\mathcal{C}} &= \iota^\star \mathbf{P}_{\omega,\varphi} = -(1/4) \varphi_\mu(x) \epsilon_{IJKL} \epsilon^{\mu\nu\rho\sigma} e^K_\rho e^L_\sigma \beta_\nu.
\end{aligned} \tag{85}$$

The exterior derivative of $(n-1)$-forms $\mathbf{Q}_{e,\chi}$, $\mathbf{Q}_{\omega,\psi}$, $\mathbf{P}_{e,\zeta}$, and $\mathbf{P}_{\omega,\varphi}$, are given by

$$\begin{aligned} \mathrm{d}\mathbf{Q}_{e,\chi} &= e^I_\mu \partial_\nu \chi^{\mu\nu}_I(x)\beta + \chi^{\mu\nu}_I(x)\mathrm{d}e^I_\mu \wedge \beta_\nu, \\ \mathrm{d}\mathbf{Q}_{\omega,\psi} &= \omega^{IJ}_\mu \partial_\nu \psi^{\mu\nu}_{IJ}(x)\beta + \psi^{\mu\nu}_{IJ}(x)\mathrm{d}\omega^{IJ}_\mu \wedge \beta_\nu, \\ \mathrm{d}\mathbf{P}_{e,\zeta} &= \zeta^I_\mu(x)\mathrm{d}p^{e_\mu\nu}_I \wedge \beta_\nu + p^{e_\mu\nu}_I \partial_\nu \zeta^I_\mu(x)\beta, \\ \mathrm{d}\mathbf{P}_{\omega,\varphi} &= \varphi^{IJ}_\mu(x)\mathrm{d}p^{\omega_\mu\nu}_{IJ} \wedge \beta_\nu + p^{\omega_\mu\nu}_{IJ} \partial_\nu \varphi^{IJ}_\mu(x)\beta. \end{aligned} \tag{86}$$

The Hamiltonian $(n-1)$-form $\mathbf{Q}_{\omega,\psi}$ is equivalently written as $\mathbf{Q}_{\omega,\psi} = (1/2)\psi^{\mu\nu}(x)\omega^{IJ} \wedge \beta_{\mu\nu}$, where $\psi^{\mu\nu}(x)$ is a real function such that $\psi^{\mu\nu} = -\psi^{\nu\mu}$. Then, $\mathbf{Q}_{\omega,\psi} = (1/2)\psi^{\mu\nu}(x)\omega^{IJ}_\rho \mathrm{d}x^\rho \wedge \beta_{\mu\nu} = (1/2)\psi^{\mu\nu}(x)\big(\omega^{IJ}_\mu \beta_\nu - \omega^{IJ}_\nu \beta_\mu\big) = \psi^{\mu\nu}(x)\omega^{IJ}_\mu \beta_\nu$. The exterior derivative of the $(n-1)$-forms is

$$\begin{aligned} \mathrm{d}\mathbf{Q}_{\omega,\psi} &= \mathrm{d}(\psi^{\mu\nu}(x)\omega^{IJ}_\mu \beta_\nu) = \omega^{IJ}_\mu \partial_\sigma \psi^{\mu\nu}_{IJ}(x)\mathrm{d}x^\sigma \wedge \beta_\nu + \psi^{\mu\nu}_{IJ}(x)\mathrm{d}\omega^{IJ}_\mu \wedge \beta_\nu, \\ &= \omega^{IJ}_\mu \partial_\nu \psi^{\mu\nu}_{IJ}(x)\beta + \psi^{\mu\nu}_{IJ}(x)\mathrm{d}\omega^{IJ}_\mu \wedge \beta_\nu, \end{aligned} \tag{87}$$

whereas the exterior derivative of the $(n-1)$-form $\mathbf{P}_{\omega,\varphi}$ is written as

$$\begin{aligned} \mathrm{d}\mathbf{P}_{\omega,\varphi} &= \mathrm{d}\big(\varphi^{IJ}_\mu(x)p^{\omega_\mu\nu}_{IJ}\beta_\nu\big) = \varphi^{IJ}_\mu(x)\mathrm{d}p^{\omega_\mu\nu}_{IJ} \wedge \beta_\nu + p^{\omega_\mu\nu}_{IJ}\mathrm{d}\varphi^{IJ}_\mu(x) \wedge \beta_\nu, \\ &= \varphi^{IJ}_\mu(x)\mathrm{d}p^{\omega_\mu\nu}_{IJ} \wedge \beta_\nu + p^{\omega_\mu\nu}_{IJ}\partial_\nu \varphi^{IJ}_\mu(x)\beta. \end{aligned} \tag{88}$$

Using the constraints (37), the exterior derivatives of the Hamiltonian $(n-1)$-forms of type $\mathbf{Q}_{\omega,\psi}\big|_{\mathcal{C}}$ and $\mathbf{P}_{\omega,\varphi}\big|_{\mathcal{C}}$ are now written as

$$\begin{aligned} \mathrm{d}\mathbf{Q}_{\omega,\psi}\big|_{\mathcal{C}} &= \omega^{IJ}_\mu \partial_\nu \psi^{\mu\nu}_{IJ}(x)\beta + \psi^{\mu\nu}_{IJ}(x)\mathrm{d}\omega^{IJ}_\mu \wedge \beta_\nu = \mathrm{d}\mathbf{Q}_{\omega,\psi}, \\ \mathrm{d}\mathbf{P}_{\omega,\varphi}\big|_{\mathcal{C}} &= -(1/4)\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^K_\rho e^L_\sigma \mathrm{d}\varphi^{IJ}_\mu(x) \wedge \beta_\nu - (1/2)\varphi^{IJ}_\mu(x)\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^K_\rho \mathrm{d}e^L_\sigma \wedge \beta_\nu. \end{aligned}$$

Lemma 4.1. *The Hamiltonian vector fields related to the Hamiltonian $(n-1)$-forms $\mathbf{Q}_{e,\chi}$, $\mathbf{Q}_{\omega,\psi}$, $\mathbf{P}_{e,\zeta}$, and $\mathbf{P}_{\omega,\varphi}$, which are denoted as $\Xi(\mathbf{Q}_{e,\chi})$, $\Xi(\mathbf{Q}_{\omega,\psi})$, $\Xi(\mathbf{P}_{e,\zeta})$, and $\Xi(\mathbf{P}_{\omega,\varphi})$, are given by*

$$\begin{aligned} \Xi(\mathbf{Q}_{e,\chi}) &= -e^I_\mu\left(\partial_\nu \chi^{\mu\nu}_I\right)\frac{\partial}{\partial \varkappa} - \chi^{\mu\nu}_I \frac{\partial}{\partial p^{e_\mu\nu}_I}, & \Xi(\mathbf{P}_{e,\zeta}) &= \zeta^I_\mu \frac{\partial}{\partial e^I_\mu} - p^{e_\mu\nu}_I\left(\partial_\nu \zeta^I_\mu\right)\frac{\partial}{\partial \varkappa}, \\ \Xi(\mathbf{Q}_{\omega,\psi}) &= -\omega^{IJ}_\mu\left(\partial_\nu \psi^{\mu\nu}_{IJ}\right)\frac{\partial}{\partial \varkappa} - \psi^{\mu\nu}_{IJ}\frac{\partial}{\partial p^{\omega_\mu\nu}_{IJ}}, & \Xi(\mathbf{P}_{\omega,\varphi}) &= \varphi^{IJ}_\mu \frac{\partial}{\partial \omega^{IJ}_\mu} - p^{\omega_\mu\nu}_{IJ}\left(\partial_\nu \varphi^{IJ}_\mu\right)\frac{\partial}{\partial \varkappa}. \end{aligned}$$

Proof. Let us compute the contractions on the multisymplectic manifold $(\mathcal{M}_{\mathsf{DW}}, \boldsymbol{\omega}^{\mathsf{DW}})$, where the vector field $\Xi(\mathbf{P}_{\omega,\varphi})$ on $\mathcal{M}_{\mathsf{DW}}$ is given as in lemma 4.1. By the straightforward calculation,

$$\begin{aligned} \Xi(\mathbf{P}_{\omega,\varphi}) \lrcorner\, \boldsymbol{\omega}^{\mathsf{DW}} &= \big(\varphi_\mu(x)\partial/\partial\omega^{IJ}_\mu - \big(\partial_\nu \varphi_\mu(x)p^{\omega_\mu\nu}_{IJ}\big)\partial/\partial\varkappa\big) \lrcorner \big(\mathrm{d}\varkappa \wedge \beta + \mathrm{d}p^{\omega_\mu\nu}_{IJ} \wedge \mathrm{d}\omega^{IJ}_\mu \wedge \beta_\nu\big), \\ &= -\big(\partial_\nu \varphi_\mu(x)p^{\omega_\mu\nu}_{IJ}\big)\beta - \varphi_\mu(x)\mathrm{d}p^{\omega_\mu\nu}_{IJ} \wedge \beta_\nu = -\mathrm{d}\mathbf{P}_{\omega,\varphi}, \end{aligned}$$

$$\begin{aligned} \Xi(\mathbf{Q}_{\omega,\psi}) \lrcorner\, \boldsymbol{\omega}^{\mathsf{DW}} &= -\left((\omega^{IJ}_\mu \partial_\nu \psi^{\mu\nu})\partial/\partial\varkappa + \psi^{\mu\nu}(x)\partial/\partial p^{\omega_\mu\nu}_{IJ}\right) \lrcorner \left(\mathrm{d}\varkappa \wedge \beta + \mathrm{d}p^{\omega_\mu\nu}_{IJ} \wedge \mathrm{d}\omega^{IJ}_\mu \wedge \beta_\nu\right), \\ &= -\omega^{IJ}_\mu\left(\partial_\nu \psi^{\mu\nu}(x)\right)\beta - \psi^{\mu\nu}(x)\mathrm{d}\omega^{IJ}_\mu \wedge \beta_\nu = -\mathrm{d}\mathbf{Q}_{\omega,\psi}. \end{aligned}$$

Analogously, a straightforward calculation yields the Hamiltonian vector fields on the constraints hypersurface $\mathcal{C}$ defined in section 2.3. More precisely, working on $(\mathcal{C}, \iota^\star\Omega^{\mathsf{DW}})$ we obtain:

Lemma 4.2. *The Hamiltonian vector fields related to the Hamiltonian $(n-1)$-forms $\mathbf{Q}_{e,\chi}\big|_{\mathcal{C}}$, $\mathbf{P}_{e,\zeta}\big|_{\mathcal{C}}$, $\mathbf{Q}_{\omega,\psi}\big|_{\mathcal{C}}$, and $\mathbf{P}_{\omega,\varphi}\big|_{\mathcal{C}}$ are given by*

$$\begin{aligned}
\Xi(\mathbf{Q}_{e,\chi})\big|_{\mathcal{C}} &= -e^I_\mu\left(\partial_\nu\chi^{\mu\nu}_I\right)\frac{\partial}{\partial\varkappa}+\chi^{\mu\nu}_I\frac{\partial}{\partial p^{e_\mu\nu}_I},\\
\Xi(\mathbf{P}_{e,\zeta})\big|_{\mathcal{C}} &= \zeta^I_\mu(x)\frac{\partial}{\partial e^I_\mu},\\
\Xi(\mathbf{Q}_{\omega,\psi})\big|_{\mathcal{C}} &= -\big(\omega^{IJ}_\mu\frac{\partial\psi^{\mu\nu}}{\partial x^\nu}\big)\frac{\partial}{\partial\varkappa}-\left(\frac{1}{6}\psi^{\mu\nu}(x)\epsilon^{IJKL}\epsilon_{\mu\nu\rho\sigma}e^\rho_K\right)\frac{\partial}{\partial e^L_\sigma},\\
\Xi(\mathbf{P}_{\omega,\varphi})\big|_{\mathcal{C}} &= \varphi^{IJ}_\mu\frac{\partial}{\partial\omega^{IJ}_\mu}+\left(\frac{1}{4}\epsilon_{IJOP}\epsilon^{\mu\nu\alpha\beta}\frac{\partial\varphi_\mu}{\partial x^\nu}e^O_\alpha e^P_\beta\right)\frac{\partial}{\partial\varkappa}.
\end{aligned}\tag{89}$$

We present the explicit calculation for the $(n-1)$-forms $\mathbf{Q}_{\omega,\psi}$ and $\mathbf{P}_{\omega,\varphi}$. The interior product $\Xi(\mathbf{P}_{\omega,\varphi}\big|_{\mathcal{C}})\lrcorner\,\boldsymbol{\omega}^{\mathsf{Palatini}}$ yields

$$\begin{aligned}
\Xi(\mathbf{P}_{\omega,\varphi}\big|_{\mathcal{C}})\lrcorner\,\boldsymbol{\omega}^{\mathsf{Palatini}} &= \big(\varphi_\mu(x)\partial/\partial\omega^{IJ}_\mu+\big((1/4)\epsilon_{IJOP}\epsilon^{\mu\nu\alpha\beta}\partial_\nu\varphi_\mu e^O_\alpha e^P_\beta\big)\partial/\partial\varkappa\big),\\
&\quad\lrcorner\,\big(\mathrm{d}\varkappa\wedge\beta-(1/2)\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^K_\rho\mathrm{d}e^L_\sigma\wedge\mathrm{d}\omega^{IJ}_\mu\wedge\beta_\nu\big),\\
&= (1/2)\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}\varphi_\mu(x)e^K_\rho\mathrm{d}e^L_\sigma\wedge\beta_\nu+(1/4)\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^K_\rho e^L_\sigma\partial_\nu\varphi_\mu(x)\beta.
\end{aligned}$$

Therefore, $\Xi(\mathbf{P}_{\omega,\varphi})\big|_{\mathcal{C}}\lrcorner\,\boldsymbol{\omega}^{\mathsf{Palatini}}=-\mathrm{d}\mathbf{P}_{\omega,\varphi}\big|_{\mathcal{C}}$. Now, we calculate $\Xi(\mathbf{Q}_{\omega,\psi})\big|_{\mathcal{C}}\lrcorner\,\boldsymbol{\omega}^{\mathsf{Palatini}}$. Let us contract both sides of $p^{\omega_\mu\nu}_{IJ}=-(1/4)\epsilon_{IJMN}\epsilon^{\mu\nu\alpha\beta}e^M_\alpha e^N_\beta$ with $\epsilon^{IJKL}\epsilon_{\mu\nu\rho\sigma}e^\rho_K$. We obtain

$$\underbrace{\epsilon^{IJKL}\epsilon_{\mu\nu\rho\sigma}p^{\omega_\mu\nu}_{IJ}e^\rho_K}_{(1)}=\underbrace{-(1/4)\epsilon^{IJKL}\epsilon_{\mu\nu\rho\sigma}\epsilon_{IJMN}\epsilon^{\mu\nu\alpha\beta}e^M_\alpha e^N_\beta e^\rho_K}_{(2)},$$

where

$$\begin{aligned}
(2) &= -(1/4)\epsilon_{IJMN}\epsilon^{IJKL}\epsilon_{\mu\nu\rho\sigma}\epsilon^{\mu\nu\alpha\beta}e^M_\alpha e^N_\beta e^\rho_K=-(1/4)(2!)(2!)\delta^{[K}_M\delta^{L]}_N(2!)(2!)\delta^{[\alpha}_\rho\delta^{\beta]}_\sigma e^M_\alpha e^N_\beta e^\rho_K,\\
&= -2\delta^{[K}_M\delta^{L]}_N(e^M_\rho e^N_\sigma e^\rho_K-e^M_\sigma e^N_\rho e^\rho_K)=-2\delta^{[K}_M\delta^{L]}_N(\delta^M_K e^N_\sigma-e^M_\sigma\delta^N_K),\\
&= -\big((\delta^K_K e^L_\sigma-e^K_\sigma\delta^L_K)-(\delta^L_K e^K_\sigma-e^L_\sigma\delta^K_K)\big)=-\big((4e^L_\sigma-e^L_\sigma)-(e^L_\sigma-4e^L_\sigma)\big)=-6e^L_\sigma.
\end{aligned}$$

Then, we obtain

$$e^L_\sigma=-(1/3!)\cdot(1)=-(1/6)\epsilon^{IJKL}\epsilon_{\mu\nu\rho\sigma}p^{\omega_\mu\nu}_{IJ}e^\rho_K.\tag{90}$$

We directly verify this result by the straightforward calculation:

$$\begin{aligned}
-(1/6)\epsilon^{IJKL}\epsilon_{\mu\nu\rho\sigma}p^{\omega_\mu\nu}_{IJ}e^\rho_K &= (1/4)(1/6)\epsilon^{IJKL}\epsilon_{IJOP}\epsilon_{\mu\nu\rho\sigma}\epsilon^{\mu\nu\alpha\beta}(e^O_\alpha e^P_\beta)e^\rho_K,\\
&= (1/24)\big[(2!)(2!)\delta^{[K}_O\delta^{L]}_P\big]\big[(2!)(2!)\delta^{[\alpha}_\rho\delta^{\beta]}_\sigma\big]e^O_\alpha e^P_\beta e^\rho_K\\
&= (1/6)\big[(e^K_\rho e^L_\sigma e^\rho_K-e^K_\sigma e^L_\rho e^\rho_K)-(e^L_\rho e^K_\sigma e^\rho_K-e^L_\sigma e^K_\rho e^\rho_K)\big]\\
&= 2(1/6)(e^K_\rho e^L_\sigma e^\rho_K-e^K_\sigma e^L_\rho e^\rho_K)=2(1/6)(\delta^\rho_\rho e^L_\sigma-\delta^\rho_\sigma e^L_\rho),\\
&= 2(1/6)(4e^L_\sigma-e^L_\sigma)=6\cdot(1/6)e^L_\sigma=e^L_\sigma.
\end{aligned}$$

Using (90), we obtain $\dfrac{\partial}{\partial p^{\omega_\mu\nu}_{IJ}}=(\dfrac{\partial e^L_\sigma}{\partial p^{\omega_\mu\nu}_{IJ}})\dfrac{\partial}{\partial e^L_\sigma}=\big(-(1/6)\epsilon^{IJKL}\epsilon_{\mu\nu\rho\sigma}e^\rho_K\big)\dfrac{\partial}{\partial e^L_\sigma}$, so that

$$\Xi(\mathbf{Q}_{\omega,\psi})=-\big(\omega^{IJ}_\mu\frac{\partial\psi^{\mu\nu}}{\partial x^\nu}\big)\frac{\partial}{\partial\varkappa}-(1/6)\psi^{\mu\nu}(x)\epsilon^{IJKL}\epsilon_{\mu\nu\rho\sigma}e^\rho_K\frac{\partial}{\partial e^L_\sigma}.\tag{91}$$

Finally, we obtain the expression

$$\begin{aligned}\Xi(\mathbf{Q}_{\omega,\psi})\big|_{\mathcal{C}} \lrcorner\, \boldsymbol{\omega}^{\mathsf{Palatini}} &= \Xi(\mathbf{Q}_{\omega,\psi}) \lrcorner \left(\mathrm{d}\varkappa \wedge \beta - (1/2)\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma} e^K_\rho \mathrm{d}e^L_\sigma \wedge \mathrm{d}\omega^{IJ}_\mu \wedge \beta_\nu\right),\\ &= -\omega^{IJ}_\mu \left(\partial_\nu \psi^{\mu\nu}_{IJ}\right)\beta \underbrace{-(1/2)\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma} e^K_\rho \mathrm{d}e^L_\sigma (\Xi(\mathbf{Q}_{\omega,\psi}))\mathrm{d}\omega^{IJ}_\mu \wedge \beta_\nu}_{(3)}\end{aligned}$$

In the appendix C, we explicitly prove that $(3) = \psi^{\mu\nu}_{IJ}\mathrm{d}\omega^{IJ}_\mu \wedge \beta_\nu$.

Finally, we also consider the Hamiltonian $(n-1)$-form $\mathbf{Q}_{\varkappa,\tau} = \tau^\nu_\mu(x)X^\mu\beta_\nu$ and $\mathbf{P}_\varkappa = \varkappa X^\alpha\beta_\alpha - p^{\omega_\mu{}^\nu}_{IJ}X^\alpha \mathrm{d}\omega^{IJ}_\mu \wedge \beta_{\alpha\nu}$. We will use them in section 4.4.2 to give an example of an homotopy Lie structure. Working on the constraint hypersurface $\mathcal{C}$ defined in section 2.3,

$$\begin{aligned}\mathbf{Q}_\varkappa\big|_{\mathcal{C}} &= = \iota^\star \mathbf{Q}_\varkappa = \tau^\nu_\mu(x)X^\mu\beta_\nu = \mathbf{Q}_e,\\ \mathbf{P}_\varkappa\big|_{\mathcal{C}} &= \iota^\star \mathbf{P}_\varkappa = \varkappa X^\alpha\beta_\alpha - e e^{[\mu}_I e^{\nu]}_J X^\alpha \mathrm{d}\omega^{IJ}_\mu \wedge \beta_{\alpha\nu}.\end{aligned} \tag{92}$$

Lemma 4.3. *The Hamiltonian vector field related to the Hamiltonian $(n-1)$-form $\mathbf{P}_\varkappa$ is* $\Xi(\mathbf{P}_\varkappa) = X^\mu(x)\partial_\mu - (\varkappa\,(\partial_\mu X^\mu))\,\partial/\partial\varkappa + p^{\omega_\mu{}^\sigma}_{IJ}\left((\partial_\sigma X^\nu) - \delta^\nu_\sigma(\partial X^\lambda/\partial x^\lambda)\right)\partial/\partial p^{\omega_\mu{}^\nu}_{IJ}$.

Proof. The interior product $\Xi(\mathbf{P}_\varkappa) \lrcorner\, \boldsymbol{\omega}^{\mathsf{DW}}$ yields

$$\begin{aligned}\Xi(\mathbf{P}_\varkappa) \lrcorner\, \boldsymbol{\omega}^{\mathsf{DW}} &= (X^\rho(x)\partial/\partial x^\rho - \varkappa\,(\partial_\rho X^\rho)\,\partial/\partial\varkappa) \lrcorner \left(\mathrm{d}\varkappa \wedge \beta + \mathrm{d}p^{\omega_\mu{}^\nu}_{IJ} \wedge \mathrm{d}\omega^{IJ}_\mu \wedge \beta_\nu\right),\\ &\quad + p^{\omega_\lambda{}^\sigma}_{KL}\left((\partial_\sigma X^\rho) - \delta^\rho_\sigma(\partial_\kappa X^\kappa)\right)\partial/\partial p^{\omega_\lambda{}^\rho}_{KL} \lrcorner \left(\mathrm{d}p^{\omega_\mu{}^\nu}_{IJ} \wedge \mathrm{d}\omega^{IJ}_\mu \wedge \beta_\nu\right),\\ &= -\varkappa\,(\partial_\mu X^\mu)\,\beta - X^\mu \mathrm{d}\varkappa \wedge \beta_\mu + X^\rho \mathrm{d}p^{\omega_\mu{}^\nu}_{IJ} \wedge \mathrm{d}\omega^{IJ}_\mu \wedge \beta_{\rho\nu},\\ &\quad + p^{\omega_\mu{}^\sigma}_{IJ}\left((\partial_\sigma X^\nu) - \delta^\nu_\sigma(\partial_\lambda X^\lambda)\right)\mathrm{d}\omega^{IJ}_\mu \wedge \beta_\nu.\end{aligned}$$

Note that $\mathrm{d}\mathbf{P}_\varkappa = X^\alpha \mathrm{d}\varkappa \wedge \beta_\alpha + \varkappa(\partial_\mu X^\mu)\beta - X^\alpha \mathrm{d}p^{\omega_\mu{}^\nu}_{IJ} \wedge \mathrm{d}\omega^{IJ}_\mu \wedge \beta_{\alpha\nu} - p^{\omega_\mu{}^\nu}_{IJ}\mathrm{d}X^\alpha \wedge \mathrm{d}\omega^{IJ}_\mu \wedge \beta_{\alpha\nu}$. The last term is equivalently written as $-\left(p^{\omega_\mu{}^\rho}_{IJ}(\partial_\rho X^\nu) - p^{\omega_\mu{}^\nu}_{IJ}(\partial_\rho X^\rho)\right)\mathrm{d}\omega^{IJ}_\mu \wedge \beta_\nu$, where we have used $\mathrm{d}x^\alpha \wedge \beta_{\rho\nu} = \delta^\alpha_\rho\beta_\nu - \delta^\alpha_\nu\beta_\rho$.

4.4 Brackets of Hamiltonian $(n-1)$-forms, Lie and homotopy Lie structures

In this section, we study bracket operations between Hamiltonian $(n-1)$-forms. In particular, the exactness or the failure of the Jacobi property is clarified along with simple examples. First, in section 4.4.1 we give an example of an exact Lie algebra $\mathfrak{A}_1 := \{\mathfrak{a}_1, \{,\}\}$, where $\mathfrak{a}_1$ is the set of Hamiltonian $(n-1)$-forms $\{\mathbf{Q}_{e,\chi}, \mathbf{Q}_{\omega,\psi}, \mathbf{P}_{e,\zeta}, \mathbf{P}_{\omega,\varphi}\}$. Then, in section 4.4.2 we present some aspects of an homotopy Lie algebra $\mathfrak{A}_2 := \{\mathfrak{a}_2, \{,\}\}$, where $\mathfrak{a}_2$ is the set of Hamiltonian $(n-1)$-forms $\{\mathbf{Q}_{e,\chi}, \mathbf{Q}_{\omega,\psi}, \mathbf{P}_\varkappa, \mathbf{P}_{e,\zeta}, \mathbf{P}_{\omega,\varphi}\}$. Finally, in section 4.4.3, we present a third algebraic structure on the set of Hamiltonian $(n-1)$-forms $\mathfrak{a}_3 := \{\boldsymbol{\mathcal{C}}_{e^I_\mu}, \boldsymbol{\mathcal{C}}_{\omega^{IJ}_\mu}\}$. This one reproduces some aspects of the formulation of vielbein gravity in polymomentum variables [57, 58].

4.4.1 Lie algebraic structure

We construct some bracket relations with the Hamiltonian $(n-1)$-forms introduced in section 4.3. Let us consider the Hamiltonian $(n-1)$-forms $\mathbf{Q}_{\omega,\psi} = \psi^{\mu\nu}_{IJ}(x)\omega^{IJ}_\mu\beta_\nu$, $\mathbf{Q}_{\omega,\overline{\psi}} = \overline{\psi}^{\mu\nu}_{IJ}(x)\omega^{IJ}_\mu\beta_\nu$, $\mathbf{P}_{\omega,\varphi} = \varphi^{IJ}_\mu(x)p^{\omega_\mu{}^\nu}_{IJ}\beta_\nu$, and $\mathbf{P}_{\omega,\overline{\varphi}} = \overline{\varphi}^{IJ}_\mu(x)p^{\omega_\mu{}^\nu}_{IJ}\beta_\nu$. Note that $\phi^{IJ}_\mu(x), \overline{\phi}^{IJ}_\mu(x), \psi^{\mu\nu}_{IJ}(x)$, and $\overline{\psi}^{\mu\nu}_{IJ}(x)$ are smooth functions on the space-time manifold $\mathcal{X}$, where $\psi^{\mu\nu}(x) = -\psi^{\nu\mu}(x)$ and $\overline{\psi}^{\mu\nu}(x) = -\overline{\psi}^{\nu\mu}(x)$.

Proposition 4.1. *On the multisymplectic manifold, $(\mathcal{M}_{\mathsf{DW}}, \boldsymbol{\omega}^{\mathsf{DW}})$, the brackets on the set of Hamiltonian $(n-1)$-forms $\mathbf{Q}_{\omega,\psi}, \mathbf{Q}_{\omega,\overline{\psi}}, \mathbf{P}_{\omega,\varphi}$, and $\mathbf{P}_{\omega,\overline{\varphi}} \in \Omega^{n-1}_{\mathsf{Ham}}(\mathcal{M}_{\mathsf{DW}})$ are given by*

$$\{\mathbf{Q}_{\omega,\psi}, \mathbf{Q}_{\omega,\overline{\psi}}\} = \{\mathbf{P}_{\omega,\varphi}, \mathbf{P}_{\omega,\overline{\varphi}}\} = 0, \qquad \{\mathbf{Q}_{\omega,\psi}, \mathbf{P}_{\omega,\varphi}\} = -\psi^{\mu\nu}_{IJ}(x)\varphi^{IJ}_{\mu}(x)\mathrm{d}\mathfrak{y}_{\nu}. \tag{93}$$

Proof. The brackets are easily computed using lemma 4.1

$$\begin{aligned}
\{\mathbf{Q}_{\omega,\psi}, \mathbf{Q}_{\omega,\overline{\psi}}\} &= -\Xi(\mathbf{Q}_{\omega,\psi}) \lrcorner\, \Xi(\mathbf{Q}_{\omega,\overline{\psi}}) \lrcorner\, \boldsymbol{\omega}^{\mathsf{DW}} = \Xi(\mathbf{Q}_{\omega,\psi}) \lrcorner\, \mathrm{d}\mathbf{Q}_{\omega,\overline{\psi}}, \\
&= \Xi(\mathbf{Q}_{\omega,\psi}) \lrcorner\, \left(\omega^{IJ}_{\mu}\left(\partial_{\nu}\overline{\psi}^{\mu\nu}_{IJ}\right)\beta + \overline{\psi}^{\mu\nu}_{IJ}\mathrm{d}\omega^{IJ}_{\mu} \wedge \beta_{\nu}\right) = 0, \\
\\
\{\mathbf{P}_{\omega,\varphi}, \mathbf{P}_{\omega,\overline{\varphi}}\} &= -\Xi(\mathbf{P}_{\omega,\varphi}) \lrcorner\, \Xi(\mathbf{P}_{\omega,\overline{\varphi}}) \lrcorner\, \boldsymbol{\omega}^{\mathsf{DW}} = \Xi(\mathbf{P}_{\omega,\varphi}) \lrcorner\, \mathrm{d}\mathbf{P}_{\omega,\overline{\varphi}}, \\
&= \Xi(\mathbf{P}_{\omega,\varphi}) \lrcorner\, \left(\varphi^{IJ}_{\mu}(x)\mathrm{d}p^{\omega_{\mu}\nu}_{IJ} \wedge \beta_{\nu} + p^{\omega_{\mu}\nu}_{IJ}\partial_{\nu}\varphi^{IJ}_{\mu}(x)\beta\right) = 0, \\
\\
\{\mathbf{Q}_{\omega,\psi}, \mathbf{P}_{\omega,\varphi}\} &= \Xi(\mathbf{Q}_{\omega,\psi}) \lrcorner\, \left(\varphi^{IJ}_{\mu}(x)\mathrm{d}p^{\omega_{\mu}\nu}_{IJ} \wedge \beta_{\nu} + p^{\omega_{\mu}\nu}_{IJ}\partial_{\nu}\varphi^{IJ}_{\mu}(x)\beta\right), \\
&= -[\delta^{I}_{K}\delta^{J}_{L}\delta^{\rho}_{\mu}\delta^{\sigma}_{\nu}]\psi^{\rho\sigma}_{KL}(x)\varphi^{IJ}_{\mu}(x)\beta_{\nu} = -\psi^{\mu\nu}_{IJ}(x)\phi^{IJ}_{\mu}(x)\beta_{\nu}.
\end{aligned}$$

Proposition 4.2. $\mathfrak{A}_1$ *is a Lie algebra.*

Proof. We consider the Hamiltonian $(n-1)$-forms $\mathbf{Q}_{\omega,\psi_1}, \mathbf{Q}_{\omega,\psi_2}, \mathbf{Q}_{\omega,\psi_3} \in \Omega^{n-1}_{\mathsf{Ham}}(\mathcal{M}_{\mathsf{DW}})$ and $\mathbf{P}_{\omega,\varphi_1}, \mathbf{P}_{\omega,\varphi_2}, \mathbf{P}_{\omega,\varphi_3} \in \Omega^{n-1}_{\mathsf{Ham}}(\mathcal{M}_{\mathsf{DW}})$. The brackets $\{\{\mathbf{Q}_{\omega,\psi_1}, \mathbf{Q}_{\omega,\psi_2}\}, \mathbf{Q}_{\omega,\psi_3}\}$, $\{\{\mathbf{Q}_{\omega,\psi_1}, \mathbf{Q}_{\omega,\psi_3}\}, \mathbf{Q}_{\omega,\psi_2}\}$, $\{\{\mathbf{Q}_{\omega,\psi_2}, \mathbf{Q}_{\omega,\psi_3}\}, \mathbf{Q}_{\omega,\psi_1}\}$, $\{\{\mathbf{P}_{\omega,\varphi_1}, \mathbf{P}_{\omega,\varphi_2}\}, \mathbf{P}_{\omega,\varphi_3}\}$, $\{\{\mathbf{P}_{\omega,\varphi_1}, \mathbf{P}_{\omega,\varphi_3}\}, \mathbf{P}_{\omega,\varphi_2}\}$, as well as the bracket $\{\{\mathbf{P}_{\omega,\varphi_2}, \mathbf{P}_{\omega,\varphi_3}\}, \mathbf{P}_{\omega,\varphi_1}\}$ are identically vanishing. We also have

$$\begin{aligned}
\{\{\mathbf{Q}_{\omega,\psi_1}, \mathbf{P}_{\omega,\varphi}\}, \mathbf{Q}_{\omega,\psi_2}\} &= -\Xi(\mathbf{Q}_{\omega,\psi_2}) \lrcorner\, \mathrm{d}\{\mathbf{Q}_{\omega,\psi_1}, \mathbf{P}_{\omega,\varphi}\}, \\
&= \Xi(\mathbf{Q}_{\omega,\psi_2}) \lrcorner\, \left(\partial_{\nu}\psi^{\mu\nu}_{IJ}(x)\varphi^{IJ}_{\mu}(x) + \psi^{\mu\nu}_{IJ}(x)\partial_{\nu}\varphi^{IJ}_{\mu}(x)\right)\mathrm{d}\mathfrak{y} = 0.
\end{aligned}$$

Analogously, $\{\{\mathbf{P}_{\omega,\varphi}, \mathbf{Q}_{\omega,\psi_2}\}, \mathbf{Q}_{\omega,\psi_1}\} = \{\{\mathbf{Q}_{\omega,\psi_2}, \mathbf{Q}_{\omega,\psi_1}\}, \mathbf{P}_{\omega,\varphi}\} = 0$. Finally, the last brackets $\{\{\mathbf{P}_{\omega,\varphi_1}, \mathbf{Q}_{\omega,\psi}\}, \mathbf{P}_{\omega,\varphi_2}\}$, $\{\{\mathbf{Q}_{\omega,\psi}, \mathbf{P}_{\omega,\varphi_2}\}, \mathbf{P}_{\omega,\varphi_1}\}$, and $\{\{\mathbf{P}_{\omega,\varphi_2}, \mathbf{P}_{\omega,\varphi_1}\}, \mathbf{Q}_{\omega,\psi}\}$ are also identically vanishing. The Jacobi property is satisfied exactly, *i.e.*

$$\begin{aligned}
0 &= \{\{\mathbf{Q}_{\omega,\psi_1}, \mathbf{P}_{\omega,\varphi}\}, \mathbf{Q}_{\omega,\psi_2}\} + \{\{\mathbf{P}_{\omega,\varphi}, \mathbf{Q}_{\omega,\psi_2}\}, \mathbf{Q}_{\omega,\psi_1}\} + \{\{\mathbf{Q}_{\omega,\psi_2}, \mathbf{Q}_{\omega,\psi_1}\}, \mathbf{P}_{\omega,\varphi}\}, \\
0 &= \{\{\mathbf{P}_{\omega,\varphi_1}, \mathbf{Q}_{\omega,\psi}\}, \mathbf{P}_{\omega,\varphi_2}\} + \{\{\mathbf{Q}_{\omega,\psi}, \mathbf{P}_{\omega,\varphi_2}\}, \mathbf{P}_{\omega,\varphi_1}\} + \{\{\mathbf{P}_{\omega,\varphi_2}, \mathbf{P}_{\omega,\varphi_1}\}, \mathbf{Q}_{\omega,\psi}\}.
\end{aligned} \tag{94}$$

Note that $[\Xi(\mathbf{Q}_{\omega,\psi}), \Xi(\mathbf{P}_{\omega,\varphi})] \lrcorner\, \boldsymbol{\omega}^{\mathsf{DW}} = \Xi(\{\mathbf{Q}_{\omega,\psi}, \mathbf{P}_{\omega,\varphi}\}) \lrcorner\, \boldsymbol{\omega}^{\mathsf{DW}} = -\mathrm{d}(\{\mathbf{Q}_{\omega,\psi_1}, \mathbf{P}_{\omega,\varphi}\})$. In this case the Hamiltonian vector field is $\Xi(\{\mathbf{Q}_{\omega,\psi_1}, \mathbf{P}_{\omega,\varphi}\}) = \left(\partial_{\nu}\psi^{\mu\nu}_{IJ}(x)\varphi^{IJ}_{\mu}(x) + \psi^{\mu\nu}_{IJ}(x)\partial_{\nu}\varphi^{IJ}_{\mu}(x)\right)\partial/\partial\varkappa$.

4.4.2 Homotopy Lie Algebraic structure

In this section, we work with the set of Hamiltonian $(n-1)$-forms $\mathfrak{a}_2 := \{\mathfrak{a}_1, \mathbf{P}_{\varkappa}\}$. We present the failure of the Jacobi identity, *i.e.* the homotopy type of the Lie algebraic structure. Here we only focus on the brackets between the $(n-1)$-forms $\mathbf{P}_{\varkappa}, \mathbf{Q}_{\omega,\psi}, \mathbf{P}_{\omega,\varphi} \in \Omega^{n-1}_{\mathsf{Ham}}(\mathcal{M}_{\mathsf{DW}})$.

Proposition 4.3. *On the multisymplectic manifold $(\mathcal{M}_{\mathsf{DW}}, \boldsymbol{\omega}^{\mathsf{DW}})$, the bracket operations between the Hamiltonian $(n-1)$-forms $\mathbf{P}_{\varkappa} \in \Omega^{n-1}_{\mathsf{Ham}}(\mathcal{M}_{\mathsf{DW}})$ and $\mathbf{Q}_{\omega,\psi}, \mathbf{P}_{\omega,\varphi} \in \Omega^{n-1}_{\mathsf{Ham}}(\mathcal{M}_{\mathsf{DW}})$ are given by*

$$\begin{aligned}
\{\mathbf{P}_{\varkappa}, \mathbf{Q}_{\omega,\psi}\} &= X^{\rho}(x)\left(\omega^{IJ}_{\mu}\left(\partial_{\nu}\psi^{\mu\nu}_{IJ}(x)\right)\beta_{\rho} - \psi^{\mu\nu}_{IJ}(x)\mathrm{d}\omega^{IJ}_{\mu} \wedge \beta_{\rho\nu}\right), \\
\{\mathbf{P}_{\varkappa}, \mathbf{P}_{\omega,\varphi}\} &= X^{\rho}(x)\left(p^{\omega_{\mu}\nu}_{IJ}\partial_{\nu}\varphi^{IJ}_{\mu}(x)\beta_{\rho} - \varphi^{IJ}_{\mu}(x)\mathrm{d}p^{\omega_{\mu}\nu}_{IJ} \wedge \beta_{\rho\nu}\right), \\
&\quad + p^{\omega_{\mu}\sigma}_{IJ}(\partial_{\sigma}X^{\nu})\varphi^{IJ}_{\mu}(x)\beta_{\nu} - p^{\omega_{\mu}\nu}_{IJ}(\partial_{\lambda}X^{\lambda})\varphi^{IJ}_{\mu}(x)\beta_{\nu}.
\end{aligned}$$

Proof. By a straightforward calculation, using lemma 4.1 and lemma 4.2, we obtain

$$\begin{aligned}\left\{\mathbf{P}_{\varkappa},\mathbf{Q}_{\omega,\psi}\right\} &= \Xi(\mathbf{P}_{\varkappa})\lrcorner\,\mathrm{d}\mathbf{Q}_{\omega,\psi}=X^{\rho}(x)\partial_{\rho}\lrcorner\left(\omega_{\mu}^{IJ}\left(\partial_{\nu}\psi_{IJ}^{\mu\nu}(x)\right)\beta+\psi_{IJ}^{\mu\nu}(x)\mathrm{d}\omega_{\mu}^{IJ}\wedge\beta_{\nu}\right),\\ &= -X^{\rho}(x)\omega_{\mu}^{IJ}\mathrm{d}\psi_{IJ}^{\mu\nu}\wedge\beta_{\rho\nu}-X^{\rho}(x)\psi_{IJ}^{\mu\nu}(x)\mathrm{d}\omega_{\mu}^{IJ}\wedge\beta_{\rho\nu},\end{aligned}$$

$$\begin{aligned}\left\{\mathbf{P}_{\varkappa},\mathbf{P}_{\omega,\varphi}\right\} &= \Xi(\mathbf{P}_{\varkappa})\lrcorner\,\mathrm{d}\mathbf{P}_{\omega,\psi}=X^{\rho}(x)\partial_{\rho}\lrcorner\left(\varphi_{\mu}^{IJ}(x)\mathrm{d}p_{IJ}^{\omega_{\mu}\nu}\wedge\beta_{\nu}+p_{IJ}^{\omega_{\mu}\nu}\partial_{\nu}\varphi_{\mu}^{IJ}(x)\beta\right),\\ &\quad +p_{KL}^{\omega_{\alpha}\sigma}\left(\left(\partial_{\sigma}X^{\beta}\right)-\delta_{\sigma}^{\beta}(\partial_{\lambda}X^{\lambda})\right)\partial/\partial p_{KL}^{\omega_{\alpha}\beta}\lrcorner\left(\varphi_{\mu}^{IJ}(x)\mathrm{d}p_{IJ}^{\omega_{\mu}\nu}\wedge\beta_{\nu}\right),\\ &= -X^{\rho}(x)p_{IJ}^{\omega_{\mu}\nu}\mathrm{d}\varphi_{\mu}^{IJ}\wedge\beta_{\rho\nu}-X^{\rho}(x)\varphi_{\mu}^{IJ}(x)\mathrm{d}p_{IJ}^{\omega_{\mu}\nu}\wedge\beta_{\rho\nu},\\ &\quad +\varphi_{\mu}^{IJ}(x)\left(p_{IJ}^{\omega_{\mu}\rho}(\partial_{\rho}X^{\nu})-p_{IJ}^{\omega_{\mu}\nu}(\partial_{\rho}X^{\rho})\right)\beta_{\nu}.\end{aligned}$$

Then,

$$\begin{aligned}\mathrm{d}\left(\left\{\mathbf{P}_{\varkappa},\mathbf{Q}_{\omega,\psi}\right\}\right) &= -\left(X^{\rho}(x)\partial_{\rho}\psi_{IJ}^{\mu\nu}(x)+\psi_{IJ}^{\mu\nu}(x)\partial_{\rho}X^{\rho}(x)\right)\mathrm{d}\omega_{\mu}^{IJ}\wedge\beta_{\nu},\\ &\quad +\left(X^{\nu}(x)\partial_{\rho}\psi_{IJ}^{\mu\rho}(x)+\psi_{IJ}^{\mu\rho}(x)\partial_{\rho}X^{\nu}(x)\right)\mathrm{d}\omega_{\mu}^{IJ}\wedge\beta_{\nu},\\ &\quad +(\partial_{\nu}\psi_{IJ}^{\mu\nu})X^{\rho}(x)\mathrm{d}\omega_{\mu}^{IJ}\wedge\beta_{\rho}-(\partial_{\rho}\psi_{IJ}^{\mu\nu})X^{\rho}(x)\mathrm{d}\omega_{\mu}^{IJ}\wedge\beta_{\nu},\\ &\quad +\omega_{\mu}^{IJ}\partial_{\rho}X^{\rho}\beta-\omega_{\mu}^{IJ}\partial_{\nu}X^{\rho}\beta,\end{aligned}\tag{95}$$

$$\begin{aligned}\mathrm{d}\left\{\mathbf{P}_{\varkappa},\mathbf{P}_{\omega,\varphi}\right\} &= -X^{\rho}(x)\partial_{\rho}\varphi_{\mu}^{IJ}\mathrm{d}p_{IJ}^{\omega_{\mu}\nu}\wedge\beta_{\nu}+X^{\rho}(x)\partial_{\nu}\varphi_{\mu}^{IJ}\mathrm{d}p_{IJ}^{\omega_{\mu}\nu}\wedge\beta_{\rho},\\ &\quad +p_{IJ}^{\omega_{\mu}\nu}(\partial_{\rho}X^{\rho}(x))\partial_{\nu}\varphi_{\mu}^{IJ}\beta,-\left(p_{IJ}^{\omega_{\mu}\nu}\partial_{\nu}X^{\rho}(x)\right)\partial_{\rho}\varphi_{\mu}^{IJ}\beta,\\ &\quad +\left(X^{\rho}(x)\partial_{\rho}\varphi_{\mu}^{IJ}(x)+\varphi_{\mu}^{IJ}(x)\partial_{\rho}X^{\rho}(x)\right)\mathrm{d}p_{IJ}^{\omega_{\mu}\nu}\wedge\beta_{\nu},\\ &\quad -\left(X^{\rho}(x)\partial_{\nu}\varphi_{\mu}^{IJ}(x)+\varphi_{\mu}^{IJ}(x)\partial_{\nu}X^{\rho}(x)\right)\mathrm{d}p_{IJ}^{\omega_{\mu}\nu}\wedge\beta_{\rho},\\ &\quad +(\partial_{\rho}X^{\nu})\varphi_{\mu}^{IJ}(x)\mathrm{d}p_{IJ}^{\omega_{\mu}\rho}\wedge\beta_{\nu}+(\partial_{\rho}X^{\nu})p_{IJ}^{\omega_{\mu}\rho}\partial_{\nu}\varphi_{\mu}^{IJ}(x)\beta,\\ &\quad -(\partial_{\rho}X^{\rho})\varphi_{\mu}^{IJ}(x)\mathrm{d}p_{IJ}^{\omega_{\mu}\nu}\wedge\beta_{\nu}-(\partial_{\rho}X^{\rho})p_{IJ}^{\omega_{\mu}\nu}\partial_{\nu}\varphi_{\mu}^{IJ}(x)\beta.\end{aligned}\tag{96}$$

We have used in (95) and (96) the definition $\beta_{\mu\nu}:=\partial_{\mu}\lrcorner\,\partial_{\nu}\lrcorner\,\beta:=\partial_{\nu}\wedge\partial_{\mu}\lrcorner\,\beta$ and the algebraic identity $\mathrm{d}x^{\alpha}\wedge\beta_{\rho\nu}=\delta_{\rho}^{\alpha}\beta_{\nu}-\delta_{\nu}^{\alpha}\beta_{\rho}$. The brackets obtained by cyclic permutations are given by $\{\{\mathbf{P}_{\varkappa},\mathbf{Q}_{\omega,\psi}\},\mathbf{P}_{\omega,\varphi}\}=-\Xi(\mathbf{P}_{\omega,\varphi})\lrcorner\,\mathrm{d}\left(\{\mathbf{P}_{\varkappa},\mathbf{Q}_{\omega,\psi}\}\right)$, $\{\{\mathbf{Q}_{\omega,\psi},\mathbf{P}_{\omega,\varphi}\},\mathbf{P}_{\varkappa}\}=-\Xi(\mathbf{P}_{\varkappa})\lrcorner\,\mathrm{d}\left(\{\mathbf{Q}_{\omega,\psi},\mathbf{P}_{\omega,\varphi}\}\right)$, and $\{\{\mathbf{P}_{\omega,\varphi},\mathbf{P}_{\varkappa}\},\mathbf{Q}_{\omega,\psi}\}=\Xi(\mathbf{Q}_{\omega,\psi})\lrcorner\,\mathrm{d}\left(\{\mathbf{P}_{\varkappa},\mathbf{P}_{\omega,\varphi}\}\right)$. Thus, we obtain

$$\begin{aligned}\{\{\mathbf{P}_{\varkappa},\mathbf{Q}_{\omega,\psi}\},\mathbf{P}_{\omega,\varphi}\} &= -\varphi_{\mu}^{IJ}\left(X^{\rho}(x)\partial_{\rho}\psi_{IJ}^{\mu\nu}(x)+\psi_{IJ}^{\mu\nu}(x)\partial_{\rho}X^{\rho}(x)\right)\beta_{\nu},\\ &\quad +\varphi_{\mu}^{IJ}\left(X^{\nu}(x)\partial_{\rho}\psi_{IJ}^{\mu\rho}(x)+\psi_{IJ}^{\mu\rho}(x)\partial_{\rho}X^{\nu}(x)\right)\beta_{\nu},\\ &\quad +\varphi_{\mu}^{IJ}(\partial_{\nu}\psi_{IJ}^{\mu\nu})X^{\rho}(x)\beta_{\rho}-\varphi_{\mu}^{IJ}(\partial_{\rho}\psi_{IJ}^{\mu\nu})X^{\rho}(x)\beta_{\nu},\end{aligned}\tag{97}$$

$$\begin{aligned}\{\{\mathbf{Q}_{\omega,\psi},\mathbf{P}_{\omega,\varphi}\},\mathbf{P}_{\varkappa}\} &= -X^{\rho}(x)\psi_{IJ}^{\mu\nu}(x)\mathrm{d}\varphi_{\mu}^{IJ}\wedge\mathrm{d}\mathfrak{y}_{\rho\nu}-X^{\rho}(x)\varphi_{\mu}^{IJ}(x)\mathrm{d}\psi_{IJ}^{\mu\nu}\wedge\mathrm{d}\mathfrak{y}_{\rho\nu},\\ &= -X^{\rho}(x)\psi_{IJ}^{\mu\nu}(x)\partial_{\rho}\varphi_{\mu}^{IJ}\beta_{\nu}+X^{\rho}(x)\psi_{IJ}^{\mu\nu}(x)\partial_{\nu}\varphi_{\mu}^{IJ}\beta_{\rho},\\ &\quad -X^{\rho}(x)\varphi_{\mu}^{IJ}(x)\partial_{\rho}\psi_{IJ}^{\mu\nu}\beta_{\nu}+X^{\rho}(x)\varphi_{\mu}^{IJ}(x)\partial_{\nu}\psi_{IJ}^{\mu\nu}\beta_{\rho},\end{aligned}\tag{98}$$

$$\begin{aligned}\{\{\mathbf{P}_{\omega,\varphi},\mathbf{P}_{\varkappa}\},\mathbf{Q}_{\omega,\psi}\} &= -\psi_{IJ}^{\mu\nu}\left(-X^{\rho}(x)\partial_{\rho}\varphi_{\mu}^{IJ}\beta_{\nu}+X^{\rho}(x)\partial_{\nu}\varphi_{\mu}^{IJ}\beta_{\rho}\right),\\ &\quad -\psi_{IJ}^{\mu\nu}\left(\left(X^{\rho}(x)\partial_{\rho}\varphi_{\mu}^{IJ}(x)+\varphi_{\mu}^{IJ}(x)\partial_{\rho}X^{\rho}(x)\right)\beta_{\nu}\right),\\ &\quad +\psi_{IJ}^{\mu\nu}\left(\left(X^{\rho}(x)\partial_{\nu}\varphi_{\mu}^{IJ}(x)+\varphi_{\mu}^{IJ}(x)\partial_{\nu}X^{\rho}(x)\right)\beta_{\rho}\right),\\ &\quad -\psi_{IJ}^{\mu\nu}\left((\partial_{\rho}X^{\nu})\varphi_{\mu}^{IJ}(x)\mathrm{d}p_{IJ}^{\omega_{\mu}\rho}\wedge\beta_{\nu}-(\partial_{\rho}X^{\rho})\varphi_{\mu}^{IJ}(x)\beta_{\nu}\right)=0.\end{aligned}\tag{99}$$

Let us denote $(\mathsf{cyc}):=\{\{\mathbf{P}_{\varkappa},\mathbf{Q}_{\omega,\psi}\},\mathbf{P}_{\omega,\varphi}\}+\{\{\mathbf{Q}_{\omega,\psi},\mathbf{P}_{\omega,\varphi}\},\mathbf{P}_{\varkappa}\}+\{\{\mathbf{P}_{\omega,\varphi},\mathbf{P}_{\varkappa}\},\mathbf{Q}_{\omega,\psi}\}$, the sum of cyclic permutations. Using (97) - (99), we obtain

$$\begin{aligned}(\mathsf{cyc}) &= -\varphi_{\mu}^{IJ}\psi_{IJ}^{\mu\nu}(x)\partial_{\rho}X^{\rho}(x)\beta_{\nu}+\varphi_{\mu}^{IJ}\psi_{IJ}^{\mu\rho}(x)\partial_{\rho}X^{\nu}(x)\beta_{\nu}-X^{\rho}(x)\psi_{IJ}^{\mu\nu}(x)\partial_{\rho}\varphi_{\mu}^{IJ}\beta_{\nu},\\ &\quad -X^{\nu}(x)\psi_{IJ}^{\mu\rho}(x)\partial_{\rho}\varphi_{\mu}^{IJ}\beta_{\nu},-\varphi_{\mu}^{IJ}X^{\rho}(x)\partial_{\rho}\psi_{IJ}^{\mu\nu}(x)\beta_{\nu}+\varphi_{\mu}^{IJ}X^{\nu}(x)\partial_{\rho}\psi_{IJ}^{\mu\rho}(x)\beta_{\nu}.\end{aligned}$$

We denote by $\mathfrak{S}$ the $(n-2)$-form $\Xi(\mathbf{P}_\varkappa)\wedge\Xi(\mathbf{Q}_{\omega,\psi})\wedge\Xi(\mathbf{P}_{\omega,\varphi})\lrcorner\,\boldsymbol{\omega}^{\mathsf{DW}}\in\Omega^{n-2}(\mathcal{M}_{\mathsf{DW}})$. Then, $\mathfrak{S}=\Xi(\mathbf{P}_{\omega,\varphi})\lrcorner\,\Xi(\mathbf{Q}_{\omega,\psi})\lrcorner\,\Xi(\mathbf{P}_\varkappa)\lrcorner\,\boldsymbol{\omega}^{\mathsf{DW}}$. Then,

$$\begin{aligned}
\mathfrak{S} &= \Xi(\mathbf{P}_{\omega,\varphi})\lrcorner\,\Xi(\mathbf{Q}_{\omega,\psi})\lrcorner\,\left(-\varkappa\,(\partial_\mu X^\mu)\,\beta-X^\mu\mathrm{d}\varkappa\wedge\beta_\mu+X^\rho\mathrm{d}p^{\omega_\mu\nu}_{IJ}\wedge\mathrm{d}\omega^{IJ}_\mu\wedge\beta_{\rho\nu}\right),\\
&\quad+\Xi(\mathbf{P}_{\omega,\varphi})\lrcorner\,\Xi(\mathbf{Q}_{\omega,\psi})\lrcorner\,\left(\left(p^{\omega_\mu\rho}_{IJ}\partial_\rho X^\nu(x)-p^{\omega_\mu\nu}_{IJ}\partial_\rho X^\rho(x)\right)\mathrm{d}\omega^{IJ}_\mu\wedge\beta_\nu\right),\\
&= \Xi(\mathbf{P}_{\omega,\varphi})\lrcorner\,\left(-\omega^{IJ}_\mu\,(\partial_\nu\psi^{\mu\nu}_{IJ})\,\partial/\partial\varkappa\right)\lrcorner\,(-X^\mu\mathrm{d}\varkappa\wedge\beta_\mu),\\
&\quad+\Xi(\mathbf{P}_{\omega,\varphi})\lrcorner\,\left(-\psi^{\alpha\beta}_{KL}\partial/\partial p^{\omega_\alpha\beta}_{KL}\right)\lrcorner\,\left(X^\rho\mathrm{d}p^{\omega_\mu\nu}_{IJ}\wedge\mathrm{d}\omega^{IJ}_\mu\wedge\beta_{\rho\nu}\right),\\
&= \Xi(\mathbf{P}_{\omega,\varphi})\lrcorner\,\left(X^\mu\omega^{IJ}_\mu\,(\partial_\nu\psi^{\mu\nu}_{IJ})\,\beta_\mu-X^\rho\psi^{\mu\nu}_{IJ}\mathrm{d}\omega^{IJ}_\mu\wedge\beta_{\rho\nu}\right),\\
&= -\left(\varphi^{KL}_\alpha\partial/\partial\omega^{KL}_\alpha\right)\lrcorner\,\left(X^\rho\psi^{\mu\nu}_{IJ}\mathrm{d}\omega^{IJ}_\mu\wedge\beta_{\rho\nu}\right)=-X^\rho\psi^{\mu\nu}_{IJ}\varphi^{IJ}_\mu\beta_{\rho\nu}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathrm{d}\mathfrak{S} &= -\left(\psi^{\mu\nu}_{IJ}(x)\varphi^{IJ}_\mu(x)\mathrm{d}X^\rho(x)+X^\rho(x)\psi^{\mu\nu}_{IJ}(x)\mathrm{d}\varphi^{IJ}_\mu(x)+\varphi^{IJ}_\mu(x)X^\rho(x)\mathrm{d}\psi^{\mu\nu}_{IJ}(x)\right)\wedge\beta_{\rho\nu},\\
&= -\psi^{\mu\nu}_{IJ}(x)\varphi^{IJ}_\mu(x)\partial_\rho X^\rho(x)\beta_\nu+\psi^{\mu\rho}_{IJ}(x)\varphi^{IJ}_\mu(x)\partial_\rho X^\nu(x)\beta_\nu-X^\rho(x)\psi^{\mu\nu}_{IJ}(x)\partial_\rho\varphi^{IJ}_\mu(x)\beta_\nu,\\
&\quad+X^\nu(x)\psi^{\mu\rho}_{IJ}(x)\partial_\rho\varphi^{IJ}_\mu(x)\beta_\nu-\varphi^{IJ}_\mu(x)X^\rho(x)\partial_\rho\psi^{\mu\nu}_{IJ}(x)\beta_\nu+\varphi^{IJ}_\mu(x)X^\nu(x)\partial_\rho\psi^{\mu\rho}_{IJ}(x)\beta_\nu,
\end{aligned}$$

is identically equal to the sum of cyclic permutations: $\mathrm{d}\mathfrak{S}=(\mathsf{cyc})$. Hence, we have proven the Jacobi property up to coherent homotopy, *i.e.*

$$\mathrm{d}\mathfrak{S} = \{\{\mathbf{P}_\varkappa,\mathbf{Q}_{\omega,\psi}\},\mathbf{P}_{\omega,\varphi}\}+\{\{\mathbf{Q}_{\omega,\psi},\mathbf{P}_{\omega,\varphi}\},\mathbf{P}_\varkappa\}+\{\{\mathbf{P}_{\omega,\varphi},\mathbf{P}_\varkappa\},\mathbf{Q}_{\omega,\psi}\}.$$

Using the notation $\mathfrak{S}_{[n]}:=(\Xi_1\wedge\cdots\wedge\Xi_n)\lrcorner\,\boldsymbol{\omega}^{\mathsf{DW}}$ (where $\Xi_1,\cdots,\Xi_n\in\Omega^{n-1}_{\mathsf{Ham}}(\mathcal{M}_{\mathsf{DW}})$ are Hamiltonian vectors fields), the Jacobi identity, up to a coherent homotopy, is equivalently contained in the formula

$$\mathrm{d}\mathfrak{S}_{[n]}=(-1)^n\sum_{1\leq i<j\leq n}\left([\Xi_i,\Xi_j]\wedge\Xi_1\wedge\cdots\Xi_{i-1}\wedge\Xi_{i+1}\wedge\cdots\wedge\Xi_{j-1}\wedge\Xi_{j+1}\wedge\cdots\Xi_n\right)\lrcorner\,\boldsymbol{\omega}^{\mathsf{DW}}.$$

For a detailed proof, we refer to [90], page 25. Applying it to our example with $\mathfrak{S}_{[n]}:=\mathfrak{S}=\Xi(\mathbf{P}_{\omega,\varphi})\lrcorner\,\Xi(\mathbf{Q}_{\omega,\psi})\lrcorner\,\Xi(\mathbf{P}_\varkappa)\lrcorner\,\boldsymbol{\omega}^{\mathsf{DW}})$, we obtain

$$\begin{aligned}
\mathrm{d}\mathfrak{S} &= -\left([\Xi(\mathbf{P}_\varkappa),\Xi(\mathbf{Q}_{\omega,\psi})]\wedge\Xi(\mathbf{P}_{\omega,\varphi})\lrcorner\,\boldsymbol{\omega}^{\mathsf{DW}}\right)-\left([\Xi(\mathbf{P}_\varkappa),\Xi(\mathbf{P}_{\omega,\varphi})]\wedge\Xi(\mathbf{Q}_{\omega,\psi})\lrcorner\,\boldsymbol{\omega}^{\mathsf{DW}}\right),\\
&\quad-\left([\Xi(\mathbf{Q}_{\omega,\psi}),\Xi(\mathbf{P}_{\omega,\varphi})]\wedge\Xi(\mathbf{P}_\varkappa)\lrcorner\,\boldsymbol{\omega}^{\mathsf{DW}}\right),\\
&= -\left(\Xi(\mathbf{P}_{\omega,\varphi})\lrcorner\,[\Xi(\mathbf{P}_\varkappa),\Xi(\mathbf{Q}_{\omega,\psi})]\lrcorner\,\boldsymbol{\omega}^{\mathsf{DW}}\right)-\left(\Xi(\mathbf{Q}_{\omega,\psi})\lrcorner\,[\Xi(\mathbf{P}_\varkappa),\Xi(\mathbf{P}_{\omega,\varphi})]\lrcorner\,\boldsymbol{\omega}^{\mathsf{DW}}\right),\\
&\quad-\left(\Xi(\mathbf{P}_\varkappa)\lrcorner\,[\Xi(\mathbf{Q}_{\omega,\psi}),\Xi(\mathbf{P}_{\omega,\varphi})]\lrcorner\,\boldsymbol{\omega}^{\mathsf{DW}}\right),
\end{aligned}$$

which is easily verified.

4.4.3 Algebraic structure on $\boldsymbol{\mathcal{C}}_{e^I_\mu},\boldsymbol{\mathcal{C}}_{\omega^{IJ}_\mu}$

Let us denote by $\mathfrak{a}_3$ the set of two $(n-1)$-forms $\boldsymbol{\mathcal{C}}_{e^I_\mu},\boldsymbol{\mathcal{C}}_{\omega^{IJ}_\mu}$, where $\boldsymbol{\mathcal{C}}_{e^I_\mu}:=p^{e_\mu\nu}_I\beta_\nu$ and $\boldsymbol{\mathcal{C}}_{\omega^{IJ}_\mu}:=p^{\omega_\mu\nu}_{IJ}\beta_\nu+E^{[\mu}_I e^{\nu]}_J\beta_\nu$. Note that $\mathrm{d}\boldsymbol{\mathcal{C}}_{e^I_\mu}=\mathrm{d}p^{e_\mu\nu}_I\wedge\beta_\nu$ and $\mathrm{d}\boldsymbol{\mathcal{C}}_{\omega^{IJ}_\mu}=\mathrm{d}p^{\omega_\mu\nu}_{IJ}\wedge\beta_\nu+(1/2)\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^K_\rho\mathrm{d}e^L_\sigma\wedge\beta_\nu$. The related Hamiltonian vector fields $\Xi(\boldsymbol{\mathcal{C}}_{e^I_\mu})$ and $\Xi(\boldsymbol{\mathcal{C}}_{\omega^{IJ}_\mu})$ are given by

$$\Xi(\boldsymbol{\mathcal{C}}_{e^I_\mu})=\partial/\partial e^I_\mu,\qquad\qquad \Xi(\boldsymbol{\mathcal{C}}_{\omega^{IJ}_\mu})=\partial/\partial\omega^{IJ}_\mu-(1/2)\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^K_\rho\partial/\partial p^{e_\sigma\nu}_L. \tag{100}$$

The interior products of the Hamiltonian vector fields $\Xi(\boldsymbol{\mathcal{C}}_{e^I_\mu})$ and $\Xi(\boldsymbol{\mathcal{C}}_{\omega^{IJ}_\mu})$ with the multisymplectic form give $\Xi(\boldsymbol{\mathcal{C}}_{e^I_\mu})\lrcorner\,\boldsymbol{\omega}^{\mathsf{DW}}=-\mathrm{d}p^{e_\mu\nu}_I\wedge\beta_\nu=-\mathrm{d}\boldsymbol{\mathcal{C}}_{e^I_\mu}$ and $\Xi(\boldsymbol{\mathcal{C}}_{\omega^{IJ}_\mu})\lrcorner\,\boldsymbol{\omega}^{\mathsf{DW}}=-\mathrm{d}p^{\omega_\mu\nu}_{IJ}\wedge$

$\beta_\nu - (1/2)\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^K_\rho \mathrm{d}e^L_\sigma \wedge \beta_\nu = -\mathrm{d}\boldsymbol{\mathcal{C}}_{\omega^{IJ}_\mu}$, respectively. Note that, by definition, $\boldsymbol{\mathcal{C}}_{e^I_\mu}|_{\mathcal{C}} = \boldsymbol{\mathcal{C}}_{\omega^{IJ}_\mu}|_{\mathcal{C}} = 0$. We now calculate the bracket operations between the Hamiltonian $(n-1)$-forms $\boldsymbol{\mathcal{C}}_{e^I_\mu} \in \Omega^{n-1}_{\mathsf{Ham}}(\mathcal{M}_{\mathsf{DW}})$ and $\boldsymbol{\mathcal{C}}_{\omega^{IJ}_\mu} \in \Omega^{n-1}_{\mathsf{Ham}}(\mathcal{M}_{\mathsf{DW}})$:

$$\begin{aligned}
\left\{\boldsymbol{\mathcal{C}}_{e^I_\mu}, \boldsymbol{\mathcal{C}}_{\overline{e}^I_\mu}\right\} &= -\Xi(\boldsymbol{\mathcal{C}}_{e^I_\mu}) \lrcorner\, \Xi(\boldsymbol{\mathcal{C}}_{\overline{e}^I_\mu}) \lrcorner \left(\mathrm{d}p^{e_\mu\nu}_I \wedge \mathrm{d}e^I_\mu \wedge \beta_\nu + \mathrm{d}p^{\omega_\mu\nu}_{IJ} \wedge \mathrm{d}\omega^{IJ}_\mu \wedge \beta_\nu\right), \\
&= -\Xi(\boldsymbol{\mathcal{C}}_{e^I_\mu}) \lrcorner \left(\mathrm{d}p^{e_\mu\nu}_I \wedge \beta_\nu\right) = 0,
\end{aligned}$$

$$\begin{aligned}
\left\{\boldsymbol{\mathcal{C}}_{\omega^{IJ}_\mu}, \boldsymbol{\mathcal{C}}_{\overline{\omega}^{IJ}_\mu}\right\} &= -\Xi(\boldsymbol{\mathcal{C}}_{\omega^{IJ}_\mu}) \lrcorner\, \Xi(\boldsymbol{\mathcal{C}}_{\overline{\omega}^{IJ}_\mu}) \lrcorner \left(\mathrm{d}p^{e_\mu\nu}_I \wedge \mathrm{d}e^I_\mu \wedge \beta_\nu + \mathrm{d}p^{\omega_\mu\nu}_{IJ} \wedge \mathrm{d}\omega^{IJ}_\mu \wedge \beta_\nu\right), \\
&= -\Xi(\boldsymbol{\mathcal{C}}_{\omega^{IJ}_\mu}) \lrcorner \left(\mathrm{d}p^{\omega_\mu\nu}_{IJ} \wedge \beta_\nu + (1/2)\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^K_\rho \mathrm{d}e^L_\sigma \wedge \beta_\nu\right) = 0,
\end{aligned}$$

$$\begin{aligned}
\left\{\boldsymbol{\mathcal{C}}_{e^L_\sigma}, \boldsymbol{\mathcal{C}}_{\omega^{IJ}_\mu}\right\} &= -\Xi(\boldsymbol{\mathcal{C}}_{e^L_\sigma}) \lrcorner\, \Xi(\boldsymbol{\mathcal{C}}_{\omega^{IJ}_\mu}) \lrcorner \left(\mathrm{d}p^{e_\mu\nu}_I \wedge \mathrm{d}e^I_\mu \wedge \beta_\nu + \mathrm{d}p^{\omega_\mu\nu}_{IJ} \wedge \mathrm{d}\omega^{IJ}_\mu \wedge \beta_\nu\right), \\
&= -\Xi(\boldsymbol{\mathcal{C}}_{e^L_\sigma}) \lrcorner \left(\mathrm{d}p^{\omega_\mu\nu}_{IJ} \wedge \beta_\nu + (1/2)\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^K_\rho \mathrm{d}e^L_\sigma \wedge \beta_\nu\right), \\
&= -(1/2)\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^K_\rho \beta_\nu.
\end{aligned}$$

Note that $\partial/\partial e^L_\sigma\big(E^{[\mu}_I e^{\nu]}_J\big) = -\partial/\partial e^L_\sigma\big((1/4)\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^K_\rho e^L_\sigma\big) = -(1/2)\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^K_\rho$. We reproduce the result of Kanatchikov [57, 58], which underlines his constraints analysis of DW formulation of vielbein gravity and its precanonical quantization. In particular, we refer to equations (19) page 6 in [58]. The brackets are written as

$$\left\{\boldsymbol{\mathcal{C}}_{e^I_\mu}, \boldsymbol{\mathcal{C}}_{\overline{e}^J_\nu}\right\} = \left\{\boldsymbol{\mathcal{C}}_{\omega^{IJ}_\mu}, \boldsymbol{\mathcal{C}}_{\overline{\omega}^{KL}_\nu}\right\} = 0, \qquad \left\{\boldsymbol{\mathcal{C}}_{e^L_\sigma}, \boldsymbol{\mathcal{C}}_{\omega^{IJ}_\mu}\right\} = -\frac{\partial}{\partial e^L_\sigma}\big(E^{[\mu}_I e^{\nu]}_J\big)\beta_\nu. \tag{101}$$

Proposition 4.4. *$\mathfrak{A}_3 := \{\mathfrak{a}_3\ ;\ \{,\}\}$ is a Lie algebra, where $\mathfrak{a}_3$ is the set of forms $\boldsymbol{\mathcal{C}}_{e^L_\sigma}, \boldsymbol{\mathcal{C}}_{\omega^{IJ}_\mu}$ and where the bracket operation is $\{,\}$.*

Proof. We consider the Hamiltonian $(n-1)$-forms $\boldsymbol{\mathcal{C}}_{e^M_\lambda}, \boldsymbol{\mathcal{C}}_{e^L_\sigma}, \boldsymbol{\mathcal{C}}_{\omega^{IJ}_\mu}$. The following bracket operations based on the cyclic permutations are found:

$$\begin{aligned}
\{\{\boldsymbol{\mathcal{C}}_{e^M_\lambda}, \boldsymbol{\mathcal{C}}_{e^L_\sigma}\}, \boldsymbol{\mathcal{C}}_{\omega^{IJ}_\mu}\} &= -\Xi(\boldsymbol{\mathcal{C}}_{\omega^{IJ}_\mu}) \lrcorner\, \mathrm{d}\{\boldsymbol{\mathcal{C}}_{e^M_\lambda}, \boldsymbol{\mathcal{C}}_{e^L_\sigma}\} = 0, \\
\{\{\boldsymbol{\mathcal{C}}_{e^L_\sigma}, \boldsymbol{\mathcal{C}}_{\omega^{IJ}_\mu}\}, \boldsymbol{\mathcal{C}}_{e^M_\lambda}\} &= -\Xi(\boldsymbol{\mathcal{C}}_{e^M_\lambda}) \lrcorner\, \mathrm{d}\{\boldsymbol{\mathcal{C}}_{e^L_\sigma}, \boldsymbol{\mathcal{C}}_{\omega^{IJ}_\mu}\} = (1/2)\epsilon_{IJML}\epsilon^{\mu\nu\lambda\sigma}\beta_\nu, \\
\{\{\boldsymbol{\mathcal{C}}_{\omega^{IJ}_\mu}, \boldsymbol{\mathcal{C}}_{e^M_\lambda}\}, \boldsymbol{\mathcal{C}}_{e^L_\sigma}\} &= -\Xi(\boldsymbol{\mathcal{C}}_{e^L_\sigma}) \lrcorner\, \mathrm{d}\{\boldsymbol{\mathcal{C}}_{\omega^{IJ}_\mu}, \boldsymbol{\mathcal{C}}_{e^M_\lambda}\} = -(1/2)\epsilon_{IJML}\epsilon^{\mu\nu\lambda\sigma}\beta_\nu.
\end{aligned} \tag{102}$$

Also, let us consider the Hamiltonian $(n-1)$-forms $\boldsymbol{\mathcal{C}}_{e^M_\lambda}, \boldsymbol{\mathcal{C}}_{\omega^{IJ}_\mu}, \boldsymbol{\mathcal{C}}_{\omega^{KL}_\nu} \in \Omega^{n-1}_{\mathsf{Ham}}(\mathcal{M}_{\mathsf{DW}})$. The brackets based on the cyclic permutations of the Jacobi identity are

$$\begin{aligned}
\{\{\boldsymbol{\mathcal{C}}_{e^M_\lambda}, \boldsymbol{\mathcal{C}}_{\omega^{KL}_\nu}\}, \boldsymbol{\mathcal{C}}_{\omega^{IJ}_\mu}\} &= -\Xi(\boldsymbol{\mathcal{C}}_{\omega^{IJ}_\mu}) \lrcorner\, \mathrm{d}\{\boldsymbol{\mathcal{C}}_{e^M_\lambda}, \boldsymbol{\mathcal{C}}_{\omega^{KL}_\nu}\} = 0, \\
\{\{\boldsymbol{\mathcal{C}}_{\omega^{KL}_\nu}, \boldsymbol{\mathcal{C}}_{\omega^{IJ}_\mu}\}, \boldsymbol{\mathcal{C}}_{e^M_\lambda}\} &= -\Xi(\boldsymbol{\mathcal{C}}_{e^M_\lambda}) \lrcorner\, \mathrm{d}\{\boldsymbol{\mathcal{C}}_{\omega^{KL}_\nu}, \boldsymbol{\mathcal{C}}_{\omega^{IJ}_\mu}\} = 0, \\
\{\{\boldsymbol{\mathcal{C}}_{\omega^{IJ}_\mu}, \boldsymbol{\mathcal{C}}_{e^M_\lambda}\}, \boldsymbol{\mathcal{C}}_{\omega^{KL}_\nu}\} &= -\Xi(\boldsymbol{\mathcal{C}}_{\omega^{KL}_\nu}) \lrcorner\, \mathrm{d}\{\boldsymbol{\mathcal{C}}_{\omega^{IJ}_\mu}, \boldsymbol{\mathcal{C}}_{e^M_\lambda}\} = 0.
\end{aligned} \tag{103}$$

Then, using (102) and (103), we obtain the Jacobi identity

$$\begin{aligned}
0 &= \{\{\boldsymbol{\mathcal{C}}_{e^M_\lambda}, \boldsymbol{\mathcal{C}}_{e^L_\sigma}\}, \boldsymbol{\mathcal{C}}_{\omega^{IJ}_\mu}\} + \{\{\boldsymbol{\mathcal{C}}_{e^L_\sigma}, \boldsymbol{\mathcal{C}}_{\omega^{IJ}_\mu}\}, \boldsymbol{\mathcal{C}}_{e^M_\lambda}\} + \{\{\boldsymbol{\mathcal{C}}_{\omega^{IJ}_\mu}, \boldsymbol{\mathcal{C}}_{e^M_\lambda}\}, \boldsymbol{\mathcal{C}}_{e^L_\sigma}\}, \\
0 &= \{\{\boldsymbol{\mathcal{C}}_{e^M_\lambda}, \boldsymbol{\mathcal{C}}_{\omega^{KL}_\nu}\}, \boldsymbol{\mathcal{C}}_{\omega^{IJ}_\mu}\} + \{\{\boldsymbol{\mathcal{C}}_{\omega^{KL}_\nu}, \boldsymbol{\mathcal{C}}_{\omega^{IJ}_\mu}\}, \boldsymbol{\mathcal{C}}_{e^M_\lambda}\} + \{\{\boldsymbol{\mathcal{C}}_{\omega^{IJ}_\mu}, \boldsymbol{\mathcal{C}}_{e^M_\lambda}\}, \boldsymbol{\mathcal{C}}_{\omega^{KL}_\nu}\}.
\end{aligned} \tag{104}$$

4.5 Towards the canonical forms for vielbein gravity

The quantization of gravity within the MG formulation is still in its infancy. However, some progress have been made by Kanatchikov within his precanonical quantization based on his polysymplectic approach. The description of fundamental brackets, using the graded structure presented in 4.2, between Hamiltonian $(n-1)$-forms and Hamiltonian 0-forms is found in [57, 58]. In particular, the constraints analysis involves a generalization of the Dirac bracket to the polysymplectic context, see [59].

Another example of canonical Poisson bracket, *i.e.* a bracket between canonically conjugate forms, is obtained by using the copolarization of algebraic observable forms developed in the work of Hélein and Kouneiher [51]. We present briefly the formulation of a Poisson bracket on observable functionals for vierbein gravity. The functionals are built on the pair $(\omega, \boldsymbol{\varpi})$ of canonically conjugate forms, *i.e* $\{\boldsymbol{\varpi}, \omega\} = 1$, where $\omega := \omega^{IJ} \otimes \Delta_{IJ} \in \Omega^1_{\mathsf{Ham}}(\mathcal{M}_{\mathsf{DW}}) \otimes \mathfrak{g}$ and $\boldsymbol{\varpi} := \boldsymbol{\varpi}_{IJ} \otimes \Delta^{IJ} \in \Omega^{n-2}_{\mathsf{Ham}}(\mathcal{M}_{\mathsf{DW}}) \otimes \mathfrak{g}^\star$. We denote $\boldsymbol{\varpi}_I := (1/2) \sum_{\mu,\nu} p_I^{e_\mu \nu} \beta_{\mu\nu} \in \Omega^{n-2}_{\mathsf{Ham}}(\mathcal{M}_{\mathsf{DW}})$ and $\boldsymbol{\varpi}_{IJ} := (1/2) \sum_{\mu,\nu} p_{IJ}^{\omega_\mu \nu} \beta_{\mu\nu} \in \Omega^{n-2}_{\mathsf{Ham}}(\mathcal{M}_{\mathsf{DW}})$. When restricted to the constraint hypersurface $\boldsymbol{\mathcal{C}}$, the $(n-2)$-forms are denoted $\boldsymbol{\varpi}_I|_{\boldsymbol{\mathcal{C}}} := \iota^\star \boldsymbol{\varpi}_I = 0$ and $\boldsymbol{\varpi}_{IJ}|_{\boldsymbol{\mathcal{C}}} := \iota^\star \boldsymbol{\varpi}_{IJ} = -(1/2) \sum_{\mu,\nu} e e_I^{[\mu} e_J^{\nu]} \beta_{\mu\nu} = -(1/8) \epsilon^{\mu\nu\sigma\rho} \epsilon_{IJKL} e^K_\sigma e^L_\rho \beta_{\mu\nu}$. Since

$$\begin{aligned} \mathrm{d}e^I \wedge \mathrm{d}\boldsymbol{\varpi}_I &= (1/2) \mathrm{d}p_I^{e_\mu \nu} \wedge \mathrm{d}e^I_\rho \wedge \left(\delta^\rho_\mu \beta_\nu - \delta^\rho_\nu \beta_\mu\right) = \mathrm{d}p_I^{e_\mu \nu} \wedge \mathrm{d}e^I_\mu \wedge \beta_\nu, \\ \mathrm{d}\omega^{IJ} \wedge \mathrm{d}\boldsymbol{\varpi}_{IJ} &= (1/2) \mathrm{d}p_{IJ}^{\omega_\mu \nu} \wedge \mathrm{d}\omega^{IJ}_\rho \wedge \left(\delta^\rho_\mu \beta_\nu - \delta^\rho_\nu \beta_\mu\right) = \mathrm{d}p_{IJ}^{\omega_\mu \nu} \wedge \mathrm{d}\omega^{IJ}_\mu \wedge \beta_\nu, \end{aligned}$$

the multisymplectic form is written as $\boldsymbol{\omega}^{\mathsf{DW}} = \mathrm{d}\varkappa \wedge \beta + \mathrm{d}e^I \wedge \mathrm{d}\boldsymbol{\varpi}_I + \mathrm{d}\omega^{IJ} \wedge \mathrm{d}\boldsymbol{\varpi}_{IJ}$. Following the method found in [50, 51], we construct a bracket between the observable functionals $F[\omega, \Sigma \cap \boldsymbol{\gamma}_{\boldsymbol{\omega}}] := \int_{\Sigma \cap \boldsymbol{\gamma}_{\boldsymbol{\omega}}} \omega$ and $F[\boldsymbol{\varpi}, \Sigma \cap \boldsymbol{\gamma}_{\boldsymbol{\varpi}}] := \int_{\Sigma \cap \boldsymbol{\gamma}_{\boldsymbol{\varpi}}} \boldsymbol{\varpi}$, where Σ is a 1-codimensional slice [50], and $\Sigma \cap \boldsymbol{\gamma}_{\boldsymbol{\omega}}$ and $\Sigma \cap \boldsymbol{\gamma}_{\boldsymbol{\varpi}}$ are submanifolds of codimension $n-2$ and $n-3$, respectively. We construct the Poisson bracket $\left\{ \int_{\Sigma \cap \boldsymbol{\gamma}_{\boldsymbol{\varpi}}} \boldsymbol{\varpi}, \int_{\Sigma \cap \boldsymbol{\gamma}_{\boldsymbol{\omega}}} \omega \right\}(\boldsymbol{\Gamma}) = \sum_{m \in \Sigma \cap \boldsymbol{\gamma}_{\boldsymbol{\varpi}} \cap \boldsymbol{\gamma}_{\boldsymbol{\omega}} \cap \boldsymbol{\Gamma}} \boldsymbol{c}(m)$, where $\boldsymbol{c}(m)$ is a counting function and Γ is a Hamiltonian n-curve. We refer to a forthcoming paper [103] for an analysis of canonically conjugate forms and Poisson brackets in the DW Hamiltonian formulation of vielbein gravity.

5 Conclusion

In this paper, we have presented several geometrical frameworks for the DW Hamiltonian formulation of vielbein gravity. We have chosen to work in a local trivialization of the principal fiber bundle $(\mathcal{P}, \mathcal{X}, \pi, SO(1,3))$. The covariant configuration space is the fiber bundle $\mathcal{Y} := \mathfrak{iso}(1,3) \otimes T^\star \mathcal{X}$ over $\mathcal{X}$, see section 2. We have described the DW Hamilton equations in geometrical form in sections 2 and 3. In section 2 we studied the Hamilton equations in the multimomentum phase space $\mathcal{M}_{\mathsf{DW}} := \Lambda^n_1 T^\star \mathcal{Y}$, which is described by the set of local coordinates $(x^\mu, e^I_\mu, \omega^{IJ}_\mu, \varkappa, p_I^{e_\mu \nu}, p_{IJ}^{\omega_\mu \nu})$. Working with $(\boldsymbol{\mathcal{C}}, \iota^\star \Omega^{\mathsf{DW}})$, the DW Hamilton equations $X^{\boldsymbol{\mathcal{C}}} \lrcorner (\iota^\star \boldsymbol{\omega}^{\mathsf{DW}}) = (-1)^n \mathrm{d}(\iota^\star \mathcal{H}^{\mathsf{DW}})$, reproduce the Einstein system of equations. In section 3 we consider the n-phase space formulation of dreibein and vierbein gravity, following the formalism developed by Kijowski and Szczyrba [63, 64, 65, 66], and Hélein [47]. We present the DW Hamilton equations on the pre-multisymplectic phase space $(\boldsymbol{\mathcal{C}}_\circ, \boldsymbol{\omega}^\circ)$. Then, in the multisymplectic case, when working on the constraint hypersurface $\boldsymbol{\mathcal{C}}$, the DW Hamilton

equations are given by (51) and (56) for dreibein and vierbein gravity, respectively. In the pre-multisymplectic case, and working on ($\boldsymbol{\mathcal{C}}_\circ$), the equations are given by (68) and (74). This fact is related to the first order nature of the Einstein-Palatini gravity. We have reproduced in the context of the DW Hamiltonian formulation developed in [46, 49, 50, 51] some of the results found in [5, 6, 23, 94, 95]. In section 4 we give some examples of Hamiltonian $(n-1)$-forms, their related Hamiltonian vectors fields, and some Poisson brackets, which lead to the Lie or homotopy Lie algebra.

One of the interesting questions beyond the scope of the DW formulation is to find a multisymplectic manifold $(\mathcal{M}_{\mathsf{Lepage}}, \iota_2^\star\Omega)$ contained in the following inclusion of spaces: $\mathcal{M}_{\mathsf{DW}} \hookrightarrow \mathcal{M}_{\mathsf{Lepage}} \hookrightarrow \mathcal{M}$, such that a more general Lepagean Legendre correspondence [49, 50, 51] is non singular. Note that $\iota_2 : \mathcal{M}_{\mathsf{Lepage}} \hookrightarrow \mathcal{M}$ is the canonical inclusion. The idea is to use a formulation based on a higher Lepagean equivalent of the Poincaré-Cartan n-form, denoted by $\theta^{\mathsf{Lepage}} := \iota_2^\star\theta$. In such a context we use the multimomentum phase space $\mathcal{M}_{\mathsf{Lepage}} := \Lambda_2^n T^\star(\mathfrak{p} \otimes T^\star\mathcal{X})$. Then, for any point (q,p) in $\mathcal{M}_{\mathsf{Lepage}}$,

$$\begin{aligned} \theta^{\mathsf{Lepage}}_{(q,p)} \quad := \quad & \theta^{\mathsf{DW}}_{(q,p)} + p^{e^I_\rho \omega^{JK}_\sigma \mu\nu} \mathrm{d}e^I_\rho \wedge \mathrm{d}\omega^{JK}_\sigma \wedge \beta_{\mu\nu} + p^{e^i_\rho e^J_\sigma \mu\nu} \mathrm{d}e^I_\rho \wedge \mathrm{d}e^J_\sigma \wedge \beta_{\mu\nu}, \\ & + p^{\omega^{IJ}_\rho \omega^{KL}_\sigma \mu\nu} \mathrm{d}\omega^{IJ}_\rho \wedge \mathrm{d}\omega^{KL}_\sigma \wedge \beta_{\mu\nu}, \end{aligned} \tag{105}$$

where we have introduced additional multimomenta $p^{e^K_\alpha \omega^{IJ}_\beta \mu\nu}, p^{e^I_\alpha e^J_\beta \mu\nu}$, and $p^{\omega^{IJ}_\alpha \omega^{KL}_\beta \mu\nu}$. Within this geometrical formulation we could be able to construct an isomorphism between a subset of the multimomenta and the field derivatives $\partial_\mu e^i_\nu$ and $\partial_\mu \omega^{IJ}_\nu$. This viewpoint might allows us to avoid the primary constraints at all, and eventually shed new light on the problem of quantization. Another problem for further research, already mentioned in section 1.4, is to describe a fully covariant setting for vielbein gravity and to establish connections with the work of Bruno *et al.* [5, 6, 7, 8] and Hélein [48].

The most interesting problem related on the quantization of vielbein gravity would include the classification of the full set of algebraic and dynamical observable forms and the search of good conjugate forms. We hope to present elsewhere [103] results on the construction of canonical forms $(\boldsymbol{\varpi}_{IJ}, \omega^{IJ})$, canonical brackets and a pre-quantum theory, in the sense of geometric quantization, for vielbein gravity. The canonically conjugate forms are the connection 1-form $\omega^{IJ} = \omega^{IJ}_\mu \mathrm{d}x^\mu$ and the 2-form $\boldsymbol{\varpi}_{IJ} = (1/2)\sum_{\mu,\nu} e e^{[\nu}_I e^{\mu]}_J \beta_{\mu\nu}$. Note that interesting results have been obtained by Kanatchikov within his precanonical quantization scheme for vielbein gravity [57, 58].

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A First order Palatini action of vielbein gravity

First, we consider the first order Palatini action functional of vierbein gravity

$$\mathcal{S}_{\mathsf{Palatini}}[e,\omega] = \kappa \int_{\mathcal{X}} \mathsf{vol}(e) e^\mu_I e^\nu_J F^{IJ}_{\mu\nu}[\omega], \tag{106}$$

also called the «*Hilbert-Palatini*» action functional in Peldan's review [87], and which corresponds to the «*frame-affine*» framework in [25]. The functionals $\mathcal{S}_{\mathsf{EH}}[e] := \kappa \int_{\mathcal{X}} \mathrm{vol}(e) e^{\mu}_{I} e^{\nu}_{J} F^{IJ}_{\mu\nu}[\omega(e)]$ and $\mathcal{S}_{\mathsf{EP}}[\mathsf{g},\Gamma] := \kappa \int_{\mathcal{X}} \sqrt{-\mathsf{g}} R[\Gamma]\beta := \kappa \int_{\mathcal{X}} R\mathrm{vol}(\mathsf{g})$ are termed the «*Einstein-Hilbert*» and the «*Einstein-Palatini*» action functional in Peldan's review [87]. They correspond, in the framework developed by Fatibene and Francaviglia [25], to the «*purely-frame*» and the «*metric-affine*» formulations, respectively. Let us sketch the passage from $\mathcal{S}_{\mathsf{EP}}[\mathsf{g},\Gamma]$ to $\mathcal{S}_{\mathsf{Palatini}}[e,\omega]$, using some vielbein algebraic relations.

Lemma A.1. *The Palatini action functional $\mathcal{S}_{\mathsf{Palatini}}[e,\omega]$ is written as*

$$\mathcal{S}_{\mathsf{Palatini}}[e,\omega] = \frac{\kappa}{4}\int_{\mathcal{X}} \epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma} e^{I}_{\mu} e^{J}_{\nu} F^{KL}_{\rho\sigma}[\omega]\beta = \frac{1}{64\pi G}\int_{\mathcal{X}} \epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma} e^{I}_{\mu} e^{J}_{\nu} F^{KL}_{\rho\sigma}[\omega]\beta. \tag{107}$$

Proof. Note that $\mathsf{vol}(e) e^{\mu}_{I} e^{\nu}_{J} F^{IJ}_{\mu\nu} = \beta e e^{\mu}_{I} e^{\nu}_{J} F^{IJ}_{\mu\nu} = \beta\sqrt{-\mathsf{g}} R^{\mu\nu}_{\mu\nu} = \beta\sqrt{-\mathsf{g}} R = \mathsf{vol}(\mathsf{g}) R$. Alternatively, we have the straightforward calculation:

$$\begin{aligned}
\sqrt{-\mathsf{g}} R\beta &= \sqrt{-\mathsf{g}}\delta^{\rho}_{[\alpha}\delta^{\sigma}_{\beta]} R^{\alpha\beta}{}_{\rho\sigma}\beta = (1/4)\sqrt{-\mathsf{g}}\epsilon_{\mu\nu\alpha\beta}\epsilon^{\mu\nu\rho\sigma} R^{\alpha\beta}{}_{\rho\sigma}\beta, \\
&= (1/4)\boldsymbol{\epsilon}_{\mu\nu\alpha\beta}\epsilon^{\mu\nu\rho\sigma} R^{\alpha\beta}{}_{\rho\sigma}\beta = (1/4)\epsilon_{IJKL} e^{I}_{\mu} e^{J}_{\nu} e^{K}_{\alpha} e^{L}_{\beta}\epsilon^{\mu\nu\rho\sigma} R^{\alpha\beta}{}_{\rho\sigma}\beta, \\
&= (1/4)\epsilon_{IJKL} e^{I}_{\mu} e^{J}_{\nu}\epsilon^{\mu\nu\rho\sigma}[e^{K}_{\alpha} e^{L}_{\beta} R^{\alpha\beta}{}_{\rho\sigma}]\beta = (1/4)\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma} e^{I}_{\mu} e^{J}_{\nu} F^{KL}_{\rho\sigma}[\omega]\beta,
\end{aligned} \tag{108}$$

where we have used $\delta^{\rho}_{[\alpha}\delta^{\sigma}_{\beta]} = (1/2)[\delta^{\rho}_{\alpha}\delta^{\sigma}_{\beta} - \delta^{\rho}_{\beta}\delta^{\sigma}_{\alpha}] = (1/4)\epsilon_{\mu\nu\alpha\beta}\epsilon^{\mu\nu\rho\sigma}$. In the first line of (108) the Levi-Civita tensor is written as $\boldsymbol{\epsilon}_{\mu\nu\alpha\beta} = \sqrt{-\mathsf{g}}\epsilon_{\mu\nu\alpha\beta}$. We have used $\boldsymbol{\epsilon}_{\mu\nu\alpha\beta} = \epsilon_{IJKL} e^{I}_{\mu} e^{J}_{\nu} e^{K}_{\alpha} e^{L}_{\beta}$ and $e^{K}_{\alpha} e^{L}_{\beta} R^{\alpha\beta}{}_{\rho\sigma} = F^{KL}_{\rho\sigma}$ in the second and the last line of (108), respectively. We pass from the Einstein-Palatini action functional $\mathcal{S}_{\mathsf{EP}}[\mathsf{g},\Gamma] = \kappa\int_{\mathcal{X}} L_{\mathsf{EP}}[\mathsf{g},\Gamma]\beta$ to the functional

$$\mathcal{S}_{\mathsf{Palatini}}[e,\omega] = \frac{\kappa}{2}\int \epsilon_{IJKL} e^{I}\wedge e^{J}\wedge F^{KL} = \frac{1}{32\pi G}\epsilon_{IJKL} e^{I}\wedge e^{J}\wedge F^{KL}, \tag{109}$$

written in terms of differential forms.

Proof. Let us evaluate $\mathrm{vol}(\mathsf{g}) R = \beta\sqrt{-\mathsf{g}} R$, the integrand of the Einstein-Hilbert action. Contracting the Riemannn curvature tensor we have the following equality $R = R^{\alpha\beta}{}_{\rho\sigma}\delta^{\rho}_{[\alpha}\delta^{\sigma}_{\beta]}$. Therefore,

$$L_{\mathsf{EH}}[\mathsf{g}]\mathrm{vol}(\mathsf{g}) = \kappa\mathrm{vol}(\mathsf{g}) R = \kappa\mathrm{vol}(\mathsf{g})\delta^{\rho}_{[\alpha}\delta^{\sigma}_{\beta]} R^{\alpha\beta}{}_{\rho\sigma} = \frac{\kappa}{4}\mathrm{vol}(\mathsf{g})(-1)^{s}\boldsymbol{\epsilon}_{\mu\nu\alpha\beta}\boldsymbol{\epsilon}^{\mu\nu\rho\sigma} R^{\alpha\beta}{}_{\rho\sigma},$$

where we use the relation $\delta^{\rho}_{[\alpha}\delta^{\sigma}_{\beta]} p!(n-p)!(-1)^{s} = \boldsymbol{\epsilon}_{\mu\nu\alpha\beta}\boldsymbol{\epsilon}^{\mu\nu\rho\sigma}$ (see the algebraic identity (113) in appendix B.2, with $n = 4$ and $\mathrm{p} = 2$). Then, in a integrable moving co-frame $\mathbf{e}^{\mu} := \mathrm{d}x^{\mu}$, the volume form $\mathrm{vol}(\mathsf{g}) = \sqrt{-\mathsf{g}}\mathrm{d}x^{0}\wedge\mathrm{d}x^{1}\wedge\mathrm{d}x^{2}\wedge\mathrm{d}x^{3}$ is written as

$$\mathrm{vol}(\mathsf{g}) = \frac{\sqrt{-\mathsf{g}}}{4!}\epsilon_{\lambda\kappa\tau\gamma}\mathrm{d}x^{\lambda}\wedge\mathrm{d}x^{\kappa}\wedge\mathrm{d}x^{\tau}\wedge\mathrm{d}x^{\gamma} = \frac{1}{4!}\boldsymbol{\epsilon}_{\lambda\kappa\tau\gamma}\mathrm{d}x^{\lambda}\wedge\mathrm{d}x^{\kappa}\wedge\mathrm{d}x^{\tau}\wedge\mathrm{d}x^{\gamma}.$$

We refer to appendix B.3 for details on the relation between the volume form and the Levi-Civita symbols. Since, see the formula (113), $\boldsymbol{\epsilon}^{\mu\nu\rho\sigma}\boldsymbol{\epsilon}_{\lambda\kappa\tau\gamma} = (-1)^{s} 4!\delta^{[\mu}_{\lambda}\delta^{\nu}_{\kappa}\delta^{\rho}_{\tau}\delta^{\sigma]}_{\gamma}$ the Einstein-Palatini functional is written as

$$\begin{aligned}
\mathcal{S}_{\mathsf{EP}}[\mathsf{g},\Gamma] &= \frac{\kappa}{4}\int_{\mathcal{X}}\frac{(-1)^{s}}{4!}\boldsymbol{\epsilon}_{\mu\nu\alpha\beta}\boldsymbol{\epsilon}^{\mu\nu\rho\sigma}\boldsymbol{\epsilon}_{\lambda\kappa\tau\gamma} R^{\alpha\beta}{}_{\rho\sigma}\mathrm{d}x^{\lambda}\wedge\mathrm{d}x^{\kappa}\wedge\mathrm{d}x^{\tau}\wedge\mathrm{d}x^{\gamma}, \\
&= \frac{\kappa}{4}\int_{\mathcal{X}}\frac{(-1)^{s}(-1)^{s}4!}{4!}\boldsymbol{\epsilon}_{\mu\nu\alpha\beta}\delta^{[\mu}_{\lambda}\delta^{\nu}_{\kappa}\delta^{\rho}_{\tau}\delta^{\sigma]}_{\gamma} R^{\alpha\beta}{}_{\rho\sigma}\mathrm{d}x^{\lambda}\wedge\mathrm{d}x^{\kappa}\wedge\mathrm{d}x^{\tau}\wedge\mathrm{d}x^{\gamma}, \\
&= (\kappa/4)\int_{\mathcal{X}}\boldsymbol{\epsilon}_{\mu\nu\alpha\beta} R^{\alpha\beta}{}_{\rho\sigma}\mathrm{d}x^{\mu}\wedge\mathrm{d}x^{\nu}\wedge\mathrm{d}x^{\rho}\wedge\mathrm{d}x^{\sigma} = (\kappa/2)\int_{\mathcal{X}}\boldsymbol{\epsilon}_{\mu\nu\alpha\beta}\mathrm{d}x^{\mu}\wedge\mathrm{d}x^{\nu}\wedge R^{\alpha\beta},
\end{aligned}$$

where in the last equality we use the curvature 2-form $R^{\alpha\beta} = (1/2)R^{\alpha\beta}{}_{\rho\sigma}\mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma$. Finally, using the relation $\boldsymbol{\epsilon}_{\mu\nu\alpha\beta} = e^I_\mu e^J_\nu e^K_\alpha e^L_\beta \epsilon_{IJKL}$ (between the volume element $\boldsymbol{\epsilon}_{\mu\nu\alpha\beta}$ of $\mathsf{g}_{\mu\nu} = e^I_\mu e^J_\nu \mathsf{h}_{IJ}$ and the volume element ϵ_{IJKL} of the Minkowski metric h_{IJ}), the Palatini functional action is written as

$$\begin{aligned}\mathcal{S}_{\mathsf{Palatini}}[e,\omega] &= \frac{\kappa}{2}\int_{\mathcal{X}} e^I_\mu e^J_\nu e^K_\alpha e^L_\beta \epsilon_{IJKL}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge R^{\alpha\beta},\\ &= \frac{\kappa}{2}\int_{\mathcal{X}} \epsilon_{IJKL} e^I_\mu \mathrm{d}x^\mu \wedge e^J_\nu \mathrm{d}x^\nu \wedge e^K_\alpha e^L_\beta R^{\alpha\beta} = \frac{\kappa}{2}\int \epsilon_{IJKL} e^I \wedge e^J \wedge F^{KL}.\end{aligned}$$

Analogously, in the formulation of dreibein gravity, the Einstein-Hilbert action functional $\mathcal{S}_{\mathsf{EH}}[\mathsf{g}_{\mu\nu}] = \int_{\mathcal{X}} \sqrt{-\mathsf{g}}R\beta$ is equivalent to the action functional $\mathcal{S}_{\mathsf{Palatini}} = \int \epsilon_{IJK} e^I \wedge R^{JK}$.

Proof. Let us evaluate $\mathrm{vol}(\mathsf{g})R = \beta\sqrt{-\mathsf{g}}R$, the integrand of the Einstein-Hilbert action. Contracting the Riemann curvature tensor, we have $R = R^{\alpha\beta}{}_{\rho\sigma}\delta^\rho_{[\alpha}\delta^\sigma_{\beta]}$. Then,

$$\mathcal{S}_{\mathsf{EP}}[\mathsf{g},\Gamma] = \int_{\mathcal{X}} \mathrm{vol}(\mathsf{g})R = \int_{\mathcal{X}} \mathrm{vol}(\mathsf{g})\delta^\rho_{[\alpha}\delta^\sigma_{\beta]}R^{\alpha\beta}{}_{\rho\sigma}.$$

We also have the relation $\delta^\rho_{[\alpha}\delta^\sigma_{\beta]}1!2!(-1)^s = \boldsymbol{\epsilon}_{\mu\alpha\beta}\boldsymbol{\epsilon}^{\mu\rho\sigma}$, see the algebraic identity (113) in appendix B.2, with $n = 3$ and $p = 1$. Thus,

$$\mathcal{S}_{\mathsf{EP}}[\mathsf{g},\Gamma] = \frac{1}{2}\int_{\mathcal{X}} \mathrm{vol}(\mathsf{g})(-1)^s \boldsymbol{\epsilon}_{\mu\alpha\beta}\boldsymbol{\epsilon}^{\mu\rho\sigma}R^{\alpha\beta}{}_{\rho\sigma}.$$

The volume form is written: $\mathrm{vol}(\mathsf{g}) = \sqrt{-\mathsf{g}}\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 = (\sqrt{-\mathsf{g}}/3!)\epsilon_{\lambda\kappa\tau}\mathrm{d}x^\lambda \wedge \mathrm{d}x^\kappa \wedge \mathrm{d}x^\tau = (1/3!)\boldsymbol{\epsilon}_{\lambda\kappa\tau}\mathrm{d}x^\lambda \wedge \mathrm{d}x^\kappa \wedge \mathrm{d}x^\tau$. Then, we have

$$\begin{aligned}\mathcal{S}_{\mathsf{EP}}[\mathsf{g},\Gamma] &= \frac{1}{2}\int_{\mathcal{X}} \frac{(-1)^s}{3!}\boldsymbol{\epsilon}_{\mu\alpha\beta}\boldsymbol{\epsilon}^{\mu\rho\sigma}\boldsymbol{\epsilon}_{\lambda\kappa\tau}R^{\alpha\beta}{}_{\rho\sigma}\mathrm{d}x^\lambda \wedge \mathrm{d}x^\kappa \wedge \mathrm{d}x^\tau,\\ &= \frac{1}{2}\int_{\mathcal{X}} \boldsymbol{\epsilon}_{\mu\alpha\beta}R^{\alpha\beta}{}_{\rho\sigma}\mathrm{d}x^\mu \wedge \mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma = \int_{\mathcal{X}} \boldsymbol{\epsilon}_{\mu\alpha\beta}\mathrm{d}x^\mu \wedge R^{\alpha\beta},\end{aligned}$$

where we used $\boldsymbol{\epsilon}^{\mu\rho\sigma}\boldsymbol{\epsilon}_{\lambda\kappa\tau} = (-1)^s 3!\delta^{[\mu}_\lambda\delta^\rho_\kappa\delta^{\sigma]}_\tau$ and since the curvature 2-form is written as $R^{\alpha\beta} = (1/2)R^{\alpha\beta}{}_{\rho\sigma}\mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma$. Using the identity $\boldsymbol{\epsilon}_{\mu\alpha\beta} = e^I_\mu e^J_\alpha e^K_\beta \epsilon_{IJK}$, we finally obtain

$$\mathcal{S}_{\mathsf{Palatini}}[e,\omega] = \int_{\mathcal{X}} e^I_\mu e^J_\alpha e^K_\beta \epsilon_{IJK}\mathrm{d}x^\mu \wedge R^{\alpha\beta} = \int_{\mathcal{X}} \epsilon_{IJK} e^I_\mu \mathrm{d}x^\mu e^J_\alpha e^K_\beta R^{\alpha\beta} = \int \epsilon_{IJK} e^I \wedge F^{JK}.$$

B Algebraic relations, volume form and vielbein

In this section we present the basic algebraic properties of the Levi-Civita symbols, generalized Kronecker symbols, Levi-Civita tensors, and densities constructed on the vielbein field.

B.1 Levi-Civita symbols

We denote by $\epsilon_{\mu_1,\cdots,\mu_n}$ the Levi-Civita symbol and by $\boldsymbol{\epsilon}_{\mu_1,\cdots,\mu_n}$ the Levi-Civita tensor. Let $\mathcal{S}_n$ be the set of all permutations of n elements. The signature of the permutation $\sigma \in \mathcal{S}_n$ is

denoted by $\mathsf{sgn}(\sigma)$ with value 1 and -1, when the permutation is even or odd, respectively. By definition, $\epsilon_{\mu_1,\cdots,\mu_n}=+1$ if $(\mu_1,\cdots,\mu_n)$ is an even permutation of $(1,\cdots,n)$, $\epsilon_{\mu_1,\cdots,\mu_n}=-1$ if $(\mu_1,\cdots,\mu_n)$ is an odd permutation of $(1,\cdots,n)$, and $\epsilon_{\mu_1,\cdots,\mu_n}=0$ otherwise.

The determinant $\det(\mathrm{M})$ of a matrix $\mathrm{M}=\{\mathrm{M}^\mu{}_\nu\}_{1\leq\mu,\nu\leq n}$ is given by the Leibniz formula

$$\det(\mathrm{M}) = \sum_{\sigma\in\mathcal{S}_n}\mathsf{sgn}(\sigma)\mathrm{M}^{\sigma(1)}{}_1\cdots\mathrm{M}^{\sigma(n)}{}_n=\sum_{\sigma\in\mathcal{S}_n}\mathsf{sgn}(\sigma)\mathrm{M}^1{}_{\sigma(1)}\cdots\mathrm{M}^n{}_{\sigma(n)}, \tag{110}$$

and is equivalently written as $\det(\mathrm{M})=\sum_{1\leq\mu_1\cdots\mu_n\leq n}\epsilon_{\mu_1,\cdots,\mu_n}\mathrm{M}^{\mu_1}{}_1\cdots\mathrm{M}^{\mu_n}{}_n$.

B.2 Generalized Kronecker symbols

We introduce the generalized Kronecker symbols $\delta^{\mu_1,\cdots,\mu_n}_{\nu_1,\cdots,\nu_n}$. By definition $\delta^{\mu_1,\cdots,\mu_n}_{\nu_1,\cdots,\nu_n}=+1$ if $(\mu_1,\cdots,\mu_n)$ is an even permutation of $(\nu_1,\cdots,\nu_n)$, $\delta^{\mu_1,\cdots,\mu_n}_{\nu_1,\cdots,\nu_n}=-1$ if $(\mu_1,\cdots,\mu_n)$ is an an odd permutation of $(\nu_1,\cdots,\nu_n)$, and $\epsilon_{\mu_1,\cdots,\mu_n}=0$ otherwise. The generalized Kronecker symbol provides a way to write the anti-symmetric Levi-Civita symbols $\epsilon_{\mu_1,\cdots,\mu_n}=\delta^{1\cdots n}_{\mu_1,\cdots,\mu_n}$ and $\epsilon^{\mu_1,\cdots,\mu_n}=\delta^{\mu_1,\cdots,\mu_n}_{1\cdots n}$. We adopt the anti-symmetry conventions of Wald [107] *i.e.*

$$\delta^{\mu_1}_{[\nu_1}\cdots\delta^{\mu_n}_{\nu_n]}=\frac{1}{n!}\sum_{\sigma\in\mathcal{S}_n}\mathsf{sgn}(\sigma)\delta^{\mu_1}_{\sigma(\nu_1)}\cdots\delta^{\mu_n}_{\sigma(\nu_n)}=\frac{1}{n!}\epsilon^{\nu_1,\cdots,\nu_n}\delta^{\mu_1}_{\nu_1}\cdots\delta^{\mu_n}_{\nu_n}, \tag{111}$$

then, $\delta^{\mu_1,\cdots,\mu_n}_{\nu_1,\cdots,\nu_n}=n!\delta^{\mu_1}_{[\nu_1}\cdots\delta^{\mu_n}_{\nu_n]}=\sum_{\sigma\in\mathcal{S}_n}\mathsf{sgn}(\sigma)\delta^{\mu_1}_{\sigma(\nu_1)}\cdots\delta^{\mu_n}_{\sigma(\nu_n)}=\epsilon^{\nu_1,\cdots,\nu_n}\delta^{\mu_1}_{\nu_1}\cdots\delta^{\mu_n}_{\nu_n}$. For any $1\leq p\leq n$, we also have the identity

$$(1/p!)\epsilon^{\mu_1\cdots\mu_{n-p}\rho_1\cdots\rho_p}\epsilon_{\nu_1\cdots\nu_{n-p}\rho_1\cdots\rho_p}=\delta^{\mu_1\cdots\mu_{n-p}}_{\nu_1\cdots\nu_{n-p}}. \tag{112}$$

The identity (112) is very useful and give

$$\begin{aligned}\epsilon^{\mu_1\ldots\mu_p\alpha_1\ldots\alpha_{n-p}}\epsilon_{\mu_1\ldots\mu_p\beta_1\ldots\beta_{n-p}} &= p!(n-p)!\delta^{[\alpha_1}_{\beta_1}\ldots\delta^{\alpha_{n-p}]}_{\beta_{n-p}},\\ \epsilon^{\mu_1\cdots\mu_n}\epsilon_{\nu_1\ldots\nu_n} &= n!\delta^{[\mu_1}_{\nu_1}\ldots\delta^{\mu_n]}_{\nu_n},\\ \epsilon^{\mu_1\cdots\mu_n}\epsilon_{\mu_1\ldots\mu_n} &= n!.\end{aligned} \tag{113}$$

Finally, using the generalized Kronecker symbol, the general formula for the determinant of a matrix $\mathrm{M}\in\mathsf{Mat}_n(\mathbb{R})$ is written as $\det(\mathrm{M})=(1/n!)\sum_{\mu_1\cdots\mu_n\nu_1\cdots\nu_n}\delta^{\nu_1,\cdots,\nu_n}_{\mu_1,\cdots,\mu_n}\mathrm{M}^{\mu_1}{}_{\nu_1}\cdots\mathrm{M}^{\mu_n}{}_{\nu_n}$.

B.3 Volume form, Levi-Civita tensor, Levi-Civita tensor density

Let $(\mathcal{X},\mathsf{g})$ be a Riemannian manifold. The canonical volume form, a nowhere vanishing n-form on $\mathcal{X}$ is denoted by $\mathrm{vol}(\mathsf{g})\in\Lambda^nT^\star\mathcal{X}$ is related to the metric $\mathsf{g}_{\mu\nu}$ by $\mathrm{vol}(\mathsf{g})=\sqrt{\mathsf{g}}\mathrm{d}x^1\wedge\ldots\wedge\mathrm{d}x^n=\sqrt{\mathsf{g}}\beta$, where $\mathsf{g}:=|\mathsf{g}|:=|\det(\mathsf{g}_{\mu\nu})|$. The Levi-Civita tensor is connected to the volume form $\mathrm{vol}(\mathsf{g})$ by the following formulae:

$$\boldsymbol{\epsilon}_{\mu_1\ldots\mu_n}=\sqrt{|\mathsf{g}|}\epsilon_{\mu_1\ldots\mu_n},\qquad \boldsymbol{\epsilon}^{\mu_1\cdots\mu_n}=(-1)^{\boldsymbol{\sigma}}(1/\sqrt{|\mathsf{g}|})\epsilon^{\mu_1\cdots\mu_n}, \tag{114}$$

where $\boldsymbol{\sigma}$ is the number of negative values in the signature of the metric *i.e.* $(-1)^{\boldsymbol{\sigma}}=1$ and $(-1)^{\boldsymbol{\sigma}}=-1$ in the Riemannian and Lorentzian cases, respectively. We construct the tensorial invariant volume n-form $\mathrm{vol}(\mathsf{g})=\sqrt{|\mathsf{g}|}\beta$, where $\beta=\mathrm{d}x^1\wedge\ldots\wedge\mathrm{d}x^n=(1/n!)\epsilon_{\mu_1\ldots\mu_n}\mathrm{d}x^{\mu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_n}$. We have $\mathrm{vol}(\mathsf{g})=(1/n!)\boldsymbol{\epsilon}_{\mu_1\ldots\mu_n}\mathrm{d}x^{\mu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_n}=(1/n!)\sqrt{|\mathsf{g}|}\epsilon_{\mu_1\ldots\mu_n}\mathrm{d}x^{\mu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_n}$.

Finally, the important formula $\boldsymbol{\epsilon}^{\mu_1\cdots\mu_p\alpha_1\ldots\alpha_{n-p}}\boldsymbol{\epsilon}_{\mu_1\ldots\mu_p\beta_1\ldots\beta_{n-p}}=(-1)^{\boldsymbol{\sigma}}p!(n-p)!\delta^{[\alpha_1}_{\beta_1}\ldots\delta^{\alpha_{n-p}]}_{\beta_{n-p}}$ specializes to $\boldsymbol{\epsilon}^{\mu_1\cdots\mu_n}\boldsymbol{\epsilon}_{\nu_1\ldots\nu_n}=(-1)^{\boldsymbol{\sigma}}n!\delta^{[\mu_1}_{\nu_1}\ldots\delta^{\mu_n]}_{\nu_n}$ and $\boldsymbol{\epsilon}^{\mu_1\cdots\mu_n}\boldsymbol{\epsilon}_{\mu_1\ldots\mu_n}=(-1)^{\boldsymbol{\sigma}}n!$.

B.4 Volume form and vielbein

We introduce the covariant volume form vol(g), from the vielbein viewpoint. We denote $\mathrm{vol}(e) = e\beta = e^{I_1} \wedge \cdots \wedge e^{I_n}$, where $\beta = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n = (1/n!)\epsilon_{\mu_1,\cdots,\mu_n}\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_n}$. The space-time Levi-Civita symbols $\epsilon^{\mu_1,\cdots,\mu_n}$ and $\epsilon_{\mu_1,\cdots,\mu_n}$ have a counterpart in the vielbein setting. They correspond to the alternating symbols with tangent space indices $\epsilon^{I_1\cdots I_n}$ and $\epsilon_{I_1\cdots I_n}$, respectively. For any $1 \leq j \leq n$ we have $e^{I_j} := e^{I_j}_{\mu_j}\mathrm{d}x^{\mu_j}$, thus $\mathrm{vol}(e)$ is written as

$$\mathrm{vol}(e) = (1/n!)\epsilon_{I_1\cdots I_n}e^{I_1} \wedge \cdots \wedge e^{I_n} = (1/n!)\epsilon_{I_1\cdots I_n}e^{I_1}_{\mu_1}\cdots e^{I_n}_{\mu_n}\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_n}. \tag{115}$$

Using $e = \det(e^I_\mu) = (1/n!)\epsilon_{I_1\cdots I_n}\epsilon^{\mu_1,\cdots,\mu_n}e^{I_1}_{\mu_1}\cdots e^{I_n}_{\mu_n}$ (the formula for the determinant of the vielbein matrix), (115) is now written as $\mathrm{vol}(e) = (1/n!)\epsilon_{I_1\cdots I_n}e^{I_1}_{\mu_1}\cdots e^{I_n}_{\mu_n}(n!)\epsilon^{\mu_1,\cdots,\mu_n}\beta = e\beta$. Then, the determinant $e_{[3]} := \det(e^I_\mu)$ of the dreibein and the determinant $e_{[4]} := \det(e^I_\mu)$ of the vierbein are given by

$$e_{[3]} = (1/3!)\epsilon_{IJK}\epsilon^{\mu\nu\rho}e^I_\mu e^J_\nu e^K_\rho, \qquad e_{[4]} = (1/4!)\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^I_\mu e^J_\nu e^K_\rho e^L_\sigma, \tag{116}$$

respectively. The determinant of the inverse vielbein matrix $\det\big((e^I_\mu)^{-1}\big)$ is given by

$$e^{-1} = \det\big((e^I_\mu)^{-1}\big) = \big(\det(e^\mu_I)\big)^{-1} = \det(e^\mu_I) = (1/n!)\epsilon^{I_1\cdots I_n}\epsilon_{\mu_1,\cdots,\mu_n}e^{\mu_1}_{I_1}\cdots e^{\mu_n}_{I_n}. \tag{117}$$

Let us note that the formula in (114) are equivalently, in the vielbein formalism, written as

$$\boldsymbol{\epsilon}_{\mu_1\ldots\mu_n} = e\epsilon_{\mu_1\ldots\mu_n}, \qquad \boldsymbol{\epsilon}^{\mu_1\ldots\mu_n} = (-1)^{\boldsymbol{\sigma}}e^{-1}\epsilon^{\mu_1\ldots\mu_n}. \tag{118}$$

We have $\boldsymbol{\epsilon}_{\mu\nu\alpha} = \epsilon_{IJK}e^I_\mu e^J_\nu e^K_\alpha$, and $\boldsymbol{\epsilon}_{\mu\nu\alpha\beta} = \epsilon_{IJKL}e^I_\mu e^J_\nu e^K_\alpha e^L_\beta$, where $\boldsymbol{\epsilon}_{\mu\nu\alpha} = e\epsilon_{\mu\nu\alpha}$ and $\boldsymbol{\epsilon}_{\mu\nu\alpha\beta} = e\epsilon_{\mu\nu\alpha\beta}$, for the dreibein and vierbein formulation, respectively. Note that we have also the relations $\epsilon_{IJK} = \boldsymbol{\epsilon}_{\mu\nu\alpha}e^\mu_I e^\nu_J e^\alpha_K$ and $\epsilon_{IJKL} = \boldsymbol{\epsilon}_{\mu\nu\alpha\beta}e^\mu_I e^\nu_J e^\alpha_K e^\beta_L$.

B.5 Vielbein densities

We introduce the vielbein densities, denoted by $E^{\mu_1\cdots\mu_p}_{I_1\cdots I_p}$, with $1 \leq p \leq n$. They are constructed on the determinant of the vielbein $\det(e)$ and p vielbeins $e^{\mu_1}_{I_1}\cdots e^{\mu_p}_{I_p}$ such that $E^{\mu_1\cdots\mu_p}_{I_1\cdots I_p} = \det(e)\prod_j e^{\mu_j}_{I_j} = ee^{\mu_1}_{I_1}\cdots e^{\mu_p}_{I_p}$. We consider the anti-symmetrized object, *i.e.*

$$\begin{aligned} E^{[\mu_1\cdots\mu_p]}_{I_1\cdots I_p} &= \frac{1}{p!}\sum_{\sigma\in\mathcal{S}_n} E^{\mu_{\sigma(1)}\cdots\mu_{\sigma(p)}}_{I_1\cdots I_p} = \frac{1}{p!}\delta^{\mu_1\cdots\mu_p}_{\nu_1\cdots\nu_p}E^{\nu_1\cdots\nu_p}_{I_1\cdots I_p} = \frac{1}{p!}\delta^{\mu_1\cdots\mu_p}_{\nu_1\cdots\nu_p}E^{[\nu_1\cdots\nu_p]}_{I_1\cdots I_p} \\ &= \frac{1}{(n-p)!}\epsilon_{\mu_1\cdots\mu_p\rho_1\cdots\rho_{n-p}}\frac{1}{p!}\epsilon^{\nu_1\cdots\nu_p\rho_1\cdots\rho_{n-p}}E^{\mu_1\cdots\mu_p}_{I_1\cdots I_p} \end{aligned} \tag{119}$$

First, we are interested by the density $E^\mu_I = ee^\mu_I = \det(e^\mu_I)e^\mu_I$. We have, for $p := 1$ (n is the dimension of the space-time manifold), $E^\mu_I = \big((1/n!)\epsilon_{I_1\cdots I_n}\epsilon^{\mu_1,\cdots,\mu_n}e^{I_1}_{\mu_1}\cdots e^{I_n}_{\mu_n}\big)e^\mu_I$ or equivalently $E^\mu_I = (1/(n-1)!)\epsilon^{\mu\mu_1\cdots\mu_{n-1}}\epsilon_{II_1\ldots I_{n-1}}e^{I_1}_{\mu_1}\cdots e^{I_{n-1}}_{\mu_{n-1}}$. This relation is straightforwardly derived. Let us denote $(\mathbf{1}) := \epsilon^{\mu\mu_1\cdots\mu_{n-1}}\epsilon_{II_1\ldots I_{n-1}}e^{I_1}_{\mu_1}\cdots e^{I_{n-1}}_{\mu_{n-1}}$. Using the algebraic relation $\epsilon_{II_1\ldots I_{n-1}} = \boldsymbol{\epsilon}_{\nu\nu_1\ldots\nu_{n-1}}e^\nu_I e^{\nu_1}_{I_1}\cdots e^{\nu_{n-1}}_{I_{n-1}}$, we obtain

$$\begin{aligned} (\mathbf{1}) &= \epsilon^{\mu\mu_1\cdots\mu_{n-1}}\big(\boldsymbol{\epsilon}_{\nu\nu_1\ldots\nu_{n-1}}e^\nu_I e^{\nu_1}_{I_1}\cdots e^{\nu_{n-1}}_{I_{n-1}}\big)e^{I_1}_{\mu_1}\cdots e^{I_{n-1}}_{\mu_{n-1}} = e\delta^{\mu\mu_1\cdots\mu_{n-1}}_{\nu\nu_1\ldots\nu_{n-1}}e^\nu_I e^{\nu_1}_{I_1}\cdots e^{\nu_{n-1}}_{I_{n-1}}e^{I_1}_{\mu_1}\cdots e^{I_{n-1}}_{\mu_{n-1}}, \\ &= e\delta^{\mu\mu_1\cdots\mu_{n-1}}_{\nu\nu_1\ldots\nu_{n-1}}e^\nu_I\big(\delta^{\nu_1}_{\mu_1}\cdots\delta^{\nu_{n-1}}_{\mu_{n-1}}\big) = e\delta^{\mu\mu_1\cdots\mu_{n-1}}_{\nu\mu_1\ldots\mu_{n-1}}e^\nu_I = e(n-1)!\delta^\mu_\nu e^\nu_I = e(n-1)!e^\mu_I. \end{aligned}$$

Now, we are interested in the density $E^{[\mu\nu]}_{IJ} = ee^{[\mu}_I e^{\nu]}_J = (1/2)e(e^\mu_I e^\nu_J - e^\nu_I e^\mu_J)$, which is written as $E^{[\mu\nu]}_{IJ} = \left(\frac{1}{n!}\epsilon_{I_1\cdots I_n}\epsilon^{\mu_1,\cdots,\mu_n}e^{I_1}_{\mu_1}\cdots e^{I_n}_{\mu_n}\right)e^{[\mu}_I e^{\nu]}_J = \frac{1}{2!(n-2)!}\epsilon^{\mu\nu\mu_1\cdots\mu_{n-2}}\epsilon_{IJI_1\cdots I_{n-2}}e^{I_1}_{\mu_1}\cdots e^{I_{n-2}}_{\mu_{n-2}}$. This relation is obtained as follows. Let us denote $(2) := \epsilon^{\mu\nu\mu_1\cdots\mu_{n-2}}\epsilon_{IJI_1\cdots I_{n-2}}e^{I_1}_{\mu_1}\cdots e^{I_{n-2}}_{\mu_{n-2}}$. By the straightforward calculation

$$\begin{aligned}
(2) &= \epsilon^{\mu\nu\mu_1\cdots\mu_{n-2}}\left(\boldsymbol{\epsilon}_{\rho\sigma\nu_1\cdots\nu_{n-2}}e^\rho_I e^\sigma_J e^{\nu_1}_{I_1}\cdots e^{\nu_{n-2}}_{I_{n-2}}\right)e^{I_1}_{\mu_1}\cdots e^{I_{n-2}}_{\mu_{n-2}},\\
&= e\delta^{\mu\nu\mu_1\cdots\mu_{n-2}}_{\rho\sigma\nu_1\cdots\nu_{n-2}}e^\rho_I e^\sigma_J e^{\nu_1}_{I_1}\cdots e^{\nu_{n-2}}_{I_{n-2}}e^{I_1}_{\mu_1}\cdots e^{I_{n-2}}_{\mu_{n-2}} = e\delta^{\mu\nu\mu_1\cdots\mu_{n-2}}_{\rho\sigma\nu_1\cdots\nu_{n-2}}e^\rho_I e^\sigma_J\left(\delta^{\nu_1}_{\mu_1}\cdots\delta^{\nu_{n-2}}_{\mu_{n-2}}\right),\\
&= e\delta^{\mu\nu\mu_1\cdots\mu_{n-2}}_{\rho\sigma\mu_1\cdots\mu_{n-2}}e^\rho_I e^\sigma_J = ee^\rho_I e^\sigma_J(n-2)!\delta^{\mu\nu}_{\rho\sigma} = ee^\rho_I e^\sigma_J(n-2)!2!\delta^{[\mu}_\rho\delta^{\nu]}_\sigma = (n-2)!2!E^{[\mu\nu]}_{IJ},
\end{aligned}$$

where we use the formula $\delta^{\mu\nu\mu_1\cdots\mu_{n-2}}_{\rho\sigma\mu_1\cdots\mu_{n-2}} = \frac{(n-2)!}{(n-2-(n-2))!}\delta^{\mu\nu}_{\rho\sigma} = (n-2)!\delta^{\mu\nu}_{\rho\sigma}$ to pass from the second to the third line.

Lemma B.1. *Let us consider the vielbein density* (119)*, with* $p = 2$, i.e. $E^{[\mu\nu]}_{IJ} = ee^{[\mu}_I e^{\nu]}_J$. *Then,* $E^{[\mu\nu]}_{IJ} = \frac{1}{2!(n-2)!}\epsilon_{IJI_1\cdots I_{n-2}}e^{I_1}_{\rho_1}\cdots e^{I_{n-2}}_{\rho_{n-2}}\epsilon^{\mu\nu\rho_1\cdots\rho_{n-2}}$.

Proof. By the straightforward calculation

$$\begin{aligned}
E^{[\mu\nu]}_{IJ} &= \left((1/n!)\epsilon_{I_1\cdots I_n}\epsilon^{\mu_1,\cdots,\mu_n}e^{I_1}_{\mu_1}\cdots e^{I_n}_{\mu_n}\right)\left(\delta^{[\mu}_\rho\delta^{\nu]}_\sigma\right)e^\rho_I e^\sigma_J,\\
&= \left((1/n!)\epsilon_{I_1\cdots I_n}\epsilon^{\mu_1,\cdots,\mu_n}e^{I_1}_{\mu_1}\cdots e^{I_n}_{\mu_n}\right)(1/(2!(n-2)!))^{-1}\left(\epsilon^{\nu_1\cdots\nu_{n-2}\mu\nu}\epsilon_{\nu_1\cdots\nu_{n-2}\rho\sigma}\right)e^\rho_I e^\sigma_J,\\
&= (1/(2!(n-2)!))^{-1}\epsilon_{I_1\cdots I_n}\epsilon^{\nu_1\cdots\nu_{n-2}\mu\nu}\delta^{[\mu_1}_{\nu_1}\cdots\delta^{\mu_{n-2}}_{\nu_{n-2}}\delta^{\mu_{n-1}}_\rho\delta^{\mu_n]}_\sigma\left(e^{I_1}_{\mu_1}\cdots e^{I_n}_{\mu_n}e^\rho_I e^\sigma_J\right),\\
&= (1/(2!(n-2)!))^{-1}\epsilon_{I_1\cdots I_n}\epsilon^{\mu_1\cdots\mu_{n-2}\mu\nu}\left(e^{I_1}_{\mu_1}\cdots e^{I_{n-2}}_{\mu_{n-2}}e^{I_{n-1}}_\rho e^{I_n}_\sigma e^\rho_I e^\sigma_J\right),\\
&= (1/(2!(n-2)!))^{-1}\epsilon_{IJI_1\cdots I_{n-2}}e^{I_1}_{\rho_1}\cdots e^{I_{n-2}}_{\rho_{n-2}}\epsilon^{\mu\nu\rho_1\cdots\rho_{n-2}}.
\end{aligned}$$

In particular when $n = 3$ and $n = 4$ we have:

Lemma B.2. *The densities* $E^{[\mu\nu]}_{IJ} = ee^{[\mu}_I e^{\nu]}_J$*, which are constructed with two dreibeins and vierbeins are given by* $E^{[\mu\nu]}_{IJ} = (1/2)\epsilon_{IJK}e^K_\rho\epsilon^{\mu\nu\rho}$, and $E^{[\mu\nu]}_{IJ} = (1/4)\epsilon_{IJKL}e^K_\rho e^L_\sigma\epsilon^{\mu\nu\rho\sigma}$*, in the case where the dimension of the space-time manifold is* $n = 3$ *and* $n = 4$*, respectively.*[2]

C Calculation of $\Xi(\mathbf{Q}_{\omega,\psi}) \lrcorner\, \boldsymbol{\omega}^{\text{Palatini}}$

The interior product $\Xi(\mathbf{Q}_{\omega,\psi}) \lrcorner\, \boldsymbol{\omega}^{\text{Palatini}}$ is given by the straightforward computation:

$$\begin{aligned}
(3) &= -(1/2)\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^K_\rho \mathrm{d}e^L_\sigma\left(-(1/6)\psi^{\alpha\beta}(x)\epsilon^{OPQL}\epsilon_{\alpha\beta\eta\sigma}e^\eta_Q\partial/\partial e^L_\sigma\right)\mathrm{d}\omega^{IJ}_\mu\wedge\beta_\nu,\\
&= (1/12)\epsilon^{OPQL}\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}\epsilon_{\alpha\beta\eta\sigma}e^K_\rho e^\eta_Q\psi^{\alpha\beta}(x)\mathrm{d}\omega^{IJ}_\mu\wedge\beta_\nu,\\
&= (1/12)(3!)\delta^{[O}_I\delta^P_J\delta^{Q]}_K(3!)\delta^{[\mu}_\alpha\delta^\nu_\beta\delta^{\rho]}_\eta e^K_\rho e^\eta_Q\psi^{\alpha\beta}(x)\mathrm{d}\omega^{IJ}_\mu\wedge\beta_\nu,
\end{aligned}\tag{120}$$

[2]In Peldan's review [87] we found the relation $ee^{[\mu}_I e^{\nu]}_J = (1/2)\epsilon_{IJKL}e^K_\rho e^L_\sigma\epsilon^{\mu\nu\rho\sigma}$ since there the terms in antisymmetric sums are weighted with 1, *e.g.* $e^{[\mu}_I e^{\nu]}_J = e^\mu_I e^\nu_J - e^\nu_I e^\mu_J$. In our conventions, $e^{[\mu}_I e^{\nu]}_J = (1/2)(e^\mu_I e^\nu_J - e^\nu_I e^\mu_J)$, thus $ee^{[\mu}_I e^{\nu]}_J = (1/4)\epsilon_{IJKL}e^K_\rho e^L_\sigma\epsilon^{\mu\nu\rho\sigma}$.

Since $\epsilon^{\mu\nu\rho\sigma}\epsilon_{\alpha\beta\eta\sigma} = (3!)\delta^{[\mu}_{\alpha}\delta^{\nu}_{\beta}\delta^{\rho]}_{\eta} = ((3!)/3)\big(\delta^{[\mu}_{\alpha}\delta^{\nu]}_{\beta}\delta^{\rho}_{\eta} + \delta^{[\rho}_{\alpha}\delta^{\mu]}_{\beta}\delta^{\nu}_{\eta} + \delta^{[\nu}_{\alpha}\delta^{\rho]}_{\beta}\delta^{\mu}_{\eta}\big)$, we obtain

$$
\begin{aligned}
(3) &= (1/12)\delta^{[O}_{I}\delta^{P}_{J}\delta^{Q]}_{K}\Big(\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta}\delta^{\rho}_{\eta} - \delta^{\mu}_{\alpha}\delta^{\rho}_{\beta}\delta^{\nu}_{\eta} - (\delta^{\nu}_{\alpha}\delta^{\mu}_{\beta}\delta^{\rho}_{\eta} - \delta^{\nu}_{\alpha}\delta^{\rho}_{\beta}\delta^{\mu}_{\eta}) + \delta^{\rho}_{\alpha}\delta^{\mu}_{\beta}\delta^{\nu}_{\eta} - \delta^{\rho}_{\alpha}\delta^{\nu}_{\beta}\delta^{\mu}_{\eta}\Big)e^{K}_{\rho}e^{\eta}_{Q}\psi^{\alpha\beta}\mathrm{d}\omega^{IJ}_{\mu}\wedge\beta_{\nu},\\
&= (1/12)\delta^{[O}_{I}\delta^{P}_{J}\delta^{Q]}_{K}\Big(\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta}\delta^{\rho}_{\eta}e^{K}_{\rho}e^{\eta}_{Q}\psi^{\alpha\beta}\mathrm{d}\omega^{IJ}_{\mu}\wedge\beta_{\nu} - \delta^{\mu}_{\alpha}\delta^{\rho}_{\beta}\delta^{\nu}_{\eta}e^{K}_{\rho}e^{\eta}_{Q}\psi^{\alpha\beta}\mathrm{d}\omega^{IJ}_{\mu}\wedge\beta_{\nu} - \delta^{\nu}_{\alpha}\delta^{\mu}_{\beta}\delta^{\rho}_{\eta}e^{K}_{\rho}e^{\eta}_{Q}\psi^{\alpha\beta}\mathrm{d}\omega^{IJ}_{\mu}\wedge\beta_{\nu},\\
&\quad +\delta^{\nu}_{\alpha}\delta^{\rho}_{\beta}\delta^{\mu}_{\eta}e^{K}_{\rho}e^{\eta}_{Q}\psi^{\alpha\beta}\mathrm{d}\omega^{IJ}_{\mu}\wedge\beta_{\nu} + \delta^{\rho}_{\alpha}\delta^{\mu}_{\beta}\delta^{\nu}_{\beta}e^{K}_{\rho}e^{\eta}_{Q}\psi^{\alpha\beta}\mathrm{d}\omega^{IJ}_{\mu}\wedge\beta_{\nu} - \delta^{\rho}_{\alpha}\delta^{\nu}_{\beta}\delta^{\mu}_{\beta}e^{K}_{\rho}e^{\eta}_{Q}\psi^{\alpha\beta}\mathrm{d}\omega^{IJ}_{\mu}\wedge\beta_{\nu}\Big),\\
&= (1/12)\delta^{[O}_{I}\delta^{P}_{J}\delta^{Q]}_{K}\Big(e^{K}_{\rho}e^{\rho}_{Q}\psi^{\mu\nu} - e^{K}_{\rho}e^{\nu}_{Q}\psi^{\mu\rho} - e^{K}_{\rho}e^{\rho}_{Q}\psi^{\nu\mu} + e^{K}_{\rho}e^{\mu}_{Q}\psi^{\nu\rho} + e^{K}_{\rho}e^{\nu}_{Q}\psi^{\rho\mu} - e^{K}_{\rho}e^{\mu}_{Q}\psi^{\rho\nu}\Big)\mathrm{d}\omega^{IJ}_{\mu}\wedge\beta_{\nu},\\
&= (1/12)\delta^{[O}_{I}\delta^{P}_{J}\delta^{Q]}_{K}\Big(\delta^{K}_{Q}\psi^{\mu\nu} - e^{K}_{\rho}e^{\nu}_{Q}\psi^{\mu\rho} - \delta^{K}_{Q}\psi^{\nu\mu} + e^{K}_{\rho}e^{\mu}_{Q}\psi^{\nu\rho} + e^{K}_{\rho}e^{\nu}_{Q}\psi^{\rho\mu} - e^{K}_{\rho}e^{\mu}_{Q}\psi^{\rho\nu}\Big)\mathrm{d}\omega^{IJ}_{\mu}\wedge\beta_{\nu},\\
&= (1/6)\delta^{[O}_{I}\delta^{P}_{J}\delta^{Q]}_{K}\Big(\delta^{K}_{Q}\psi^{\mu\nu} - e^{K}_{\rho}e^{\nu}_{Q}\psi^{\mu\rho} + e^{K}_{\rho}e^{\mu}_{Q}\psi^{\nu\rho}\Big)\mathrm{d}\omega^{IJ}_{\mu}\wedge\beta_{\nu},
\end{aligned}
$$

and since $\epsilon^{OPQ}\epsilon_{IJK} = \delta^{O}_{I}\delta^{P}_{J}\delta^{Q}_{K} - \delta^{O}_{I}\delta^{Q}_{J}\delta^{P}_{K} - (\delta^{P}_{I}\delta^{O}_{J}\delta^{Q}_{K} - \delta^{P}_{I}\delta^{Q}_{J}\delta^{O}_{K}) + \delta^{Q}_{I}\delta^{O}_{J}\delta^{P}_{K} - \delta^{Q}_{I}\delta^{P}_{J}\delta^{O}_{K}$, then (3) is written as

$$
\begin{aligned}
(3) &= (1/6)\big(\delta^{O}_{I}\delta^{P}_{J}\delta^{Q}_{K} - \delta^{O}_{I}\delta^{Q}_{J}\delta^{P}_{K} - (\delta^{P}_{I}\delta^{O}_{J}\delta^{Q}_{K} - \delta^{P}_{I}\delta^{Q}_{J}\delta^{O}_{K}) + \delta^{Q}_{I}\delta^{O}_{J}\delta^{P}_{K} - \delta^{Q}_{I}\delta^{P}_{J}\delta^{O}_{K}\big)\\
&\quad \big(\delta^{K}_{Q}\psi^{\mu\nu} - e^{K}_{\rho}e^{\nu}_{Q}\psi^{\mu\rho} + e^{K}_{\rho}e^{\mu}_{Q}\psi^{\nu\rho}\big)\mathrm{d}\omega^{IJ}_{\mu}\wedge\beta_{\nu},\\
&= (1/6)\Big[\delta^{O}_{I}\delta^{P}_{J}\delta^{Q}_{K}\Big(\delta^{K}_{Q}\psi^{\mu\nu}\mathrm{d}\omega^{IJ}_{\mu}\wedge\beta_{\nu}\Big) - \delta^{O}_{I}\delta^{Q}_{J}\delta^{P}_{K}\Big(\delta^{K}_{Q}\psi^{\mu\nu}\mathrm{d}\omega^{IJ}_{\mu}\wedge\beta_{\nu}\Big) - \delta^{P}_{I}\delta^{O}_{J}\delta^{Q}_{K}\Big(\delta^{K}_{Q}\psi^{\mu\nu}\mathrm{d}\omega^{IJ}_{\mu}\wedge\beta_{\nu}\Big),\\
&\quad +\delta^{P}_{I}\delta^{Q}_{J}\delta^{O}_{K}\Big(\delta^{K}_{Q}\psi^{\mu\nu}\mathrm{d}\omega^{IJ}_{\mu}\wedge\beta_{\nu}\Big) + \delta^{Q}_{I}\delta^{O}_{J}\delta^{P}_{K}\Big(\delta^{K}_{Q}\psi^{\mu\nu}\mathrm{d}\omega^{IJ}_{\mu}\wedge\beta_{\nu}\Big) - \delta^{Q}_{I}\delta^{P}_{J}\delta^{O}_{K}\Big(\delta^{K}_{Q}\psi^{\mu\nu}\mathrm{d}\omega^{IJ}_{\mu}\wedge\beta_{\nu}\Big),\\
&\quad +\delta^{O}_{I}\delta^{P}_{J}\delta^{Q}_{K}\Big(-e^{K}_{\rho}e^{\nu}_{Q}\psi^{\mu\rho}\mathrm{d}\omega^{IJ}_{\mu}\wedge\beta_{\nu}\Big) - \delta^{O}_{I}\delta^{Q}_{J}\delta^{P}_{K}\Big(-e^{K}_{\rho}e^{\nu}_{Q}\psi^{\mu\rho}\mathrm{d}\omega^{IJ}_{\mu}\wedge\beta_{\nu}\Big) - \delta^{P}_{I}\delta^{O}_{J}\delta^{Q}_{K}\Big(-e^{K}_{\rho}e^{\nu}_{Q}\psi^{\mu\rho}\mathrm{d}\omega^{IJ}_{\mu}\wedge\beta_{\nu}\Big),\\
&\quad +\delta^{P}_{I}\delta^{Q}_{J}\delta^{O}_{K}\Big(-e^{K}_{\rho}e^{\nu}_{Q}\psi^{\mu\rho}\mathrm{d}\omega^{IJ}_{\mu}\wedge\beta_{\nu}\Big) + \delta^{Q}_{I}\delta^{O}_{J}\delta^{P}_{K}\Big(-e^{K}_{\rho}e^{\nu}_{Q}\psi^{\mu\rho}\mathrm{d}\omega^{IJ}_{\mu}\wedge\beta_{\nu}\Big) - \delta^{Q}_{I}\delta^{P}_{J}\delta^{O}_{K}\Big(-e^{K}_{\rho}e^{\nu}_{Q}\psi^{\mu\rho}\mathrm{d}\omega^{IJ}_{\mu}\wedge\beta_{\nu}\Big),\\
&\quad +\delta^{O}_{I}\delta^{P}_{J}\delta^{Q}_{K}\Big(+e^{K}_{\rho}e^{\mu}_{Q}\psi^{\nu\rho}\mathrm{d}\omega^{IJ}_{\mu}\wedge\beta_{\nu}\Big) - \delta^{O}_{I}\delta^{Q}_{J}\delta^{P}_{K}\Big(+e^{K}_{\rho}e^{\mu}_{Q}\psi^{\nu\rho}\mathrm{d}\omega^{IJ}_{\mu}\wedge\beta_{\nu}\Big) - \delta^{P}_{I}\delta^{O}_{J}\delta^{Q}_{K}\Big(+e^{K}_{\rho}e^{\mu}_{Q}\psi^{\nu\rho}\mathrm{d}\omega^{IJ}_{\mu}\wedge\beta_{\nu}\Big),\\
&\quad +\delta^{P}_{I}\delta^{Q}_{J}\delta^{O}_{K}\Big(+e^{K}_{\rho}e^{\mu}_{Q}\psi^{\nu\rho}\mathrm{d}\omega^{IJ}_{\mu}\wedge\beta_{\nu}\Big) + \delta^{Q}_{I}\delta^{O}_{J}\delta^{P}_{K}\Big(+e^{K}_{\rho}e^{\mu}_{Q}\psi^{\nu\rho}\mathrm{d}\omega^{IJ}_{\mu}\wedge\beta_{\nu}\Big) - \delta^{Q}_{I}\delta^{P}_{J}\delta^{O}_{K}\Big(+e^{K}_{\rho}e^{\mu}_{Q}\psi^{\nu\rho}\mathrm{d}\omega^{IJ}_{\mu}\wedge\beta_{\nu}\Big)\Big],\\
&= (1/6)\Big[\Big(+4\psi^{\mu\nu}\mathrm{d}\omega^{PO}_{\mu}\wedge\beta_{\nu}\Big) + \Big(-\delta^{\nu}_{\rho}\psi^{\mu\rho}\mathrm{d}\omega^{OP}_{\mu}\wedge\beta_{\nu}\Big) + \Big(e^{P}_{\rho}e^{\nu}_{Q}\psi^{\mu\rho}\mathrm{d}\omega^{OQ}_{\mu}\wedge\beta_{\nu}\Big) + \Big(\delta^{\nu}_{\rho}\psi^{\mu\rho}\mathrm{d}\omega^{PO}_{\mu}\wedge\beta_{\nu}\Big),\\
&\quad +\Big(-e^{O}_{\rho}e^{\nu}_{Q}\psi^{\mu\rho}\mathrm{d}\omega^{PQ}_{\mu}\wedge\beta_{\nu}\Big) + \Big(-e^{P}_{\rho}e^{\nu}_{Q}\psi^{\mu\rho}\mathrm{d}\omega^{QO}_{\mu}\wedge\beta_{\nu}\Big) - \Big(-e^{O}_{\rho}e^{\nu}_{Q}\psi^{\mu\rho}\mathrm{d}\omega^{QP}_{\mu}\wedge\beta_{\nu}\Big),\\
&\quad +\Big(+\delta^{\mu}_{\rho}\psi^{\nu\rho}\mathrm{d}\omega^{OP}_{\mu}\wedge\beta_{\nu}\Big) - \Big(+e^{P}_{\rho}e^{\mu}_{Q}\psi^{\nu\rho}\mathrm{d}\omega^{OQ}_{\mu}\wedge\beta_{\nu}\Big) - \Big(+\delta^{\mu}_{\rho}\psi^{\nu\rho}\mathrm{d}\omega^{PO}_{\mu}\wedge\beta_{\nu}\Big),\\
&\quad +\Big(+e^{O}_{\rho}e^{\mu}_{Q}\psi^{\nu\rho}\mathrm{d}\omega^{PQ}_{\mu}\wedge\beta_{\nu}\Big) + \Big(+e^{P}_{\rho}e^{\mu}_{Q}\psi^{\nu\rho}\mathrm{d}\omega^{QO}_{\mu}\wedge\beta_{\nu}\Big) - \Big(+e^{O}_{\rho}e^{\mu}_{Q}\psi^{\nu\rho}\mathrm{d}\omega^{QP}_{\mu}\wedge\beta_{\nu}\Big)\Big],\\
&= (1/6)\Big[\Big(+6\psi^{\mu\nu}\mathrm{d}\omega^{PO}_{\mu}\wedge\beta_{\nu}\Big) + \Big(\Big(e^{P}_{\rho}e^{\nu}_{Q}\psi^{\mu\rho}\mathrm{d}\omega^{OQ}_{\mu}\wedge\beta_{\nu}\Big) - \Big(e^{O}_{\rho}e^{\nu}_{Q}\psi^{\mu\rho}\mathrm{d}\omega^{PQ}_{\mu}\wedge\beta_{\nu}\Big),\\
&\quad +\Big(-e^{P}_{\rho}e^{\nu}_{Q}\psi^{\mu\rho}\mathrm{d}\omega^{QO}_{\mu}\wedge\beta_{\nu}\Big) - \Big(-e^{O}_{\rho}e^{\nu}_{Q}\psi^{\mu\rho}\mathrm{d}\omega^{QP}_{\mu}\wedge\beta_{\nu}\Big) + \Big(e^{O}_{\rho}e^{\mu}_{Q}\psi^{\nu\rho}\mathrm{d}\omega^{PQ}_{\mu}\wedge\beta_{\nu}\Big),\\
&\quad -\Big(e^{P}_{\rho}e^{\mu}_{Q}\psi^{\nu\rho}\mathrm{d}\omega^{OQ}_{\mu}\wedge\beta_{\nu}\Big) + \Big(e^{P}_{\rho}e^{\mu}_{Q}\psi^{\nu\rho}\mathrm{d}\omega^{QO}_{\mu}\wedge\beta_{\nu}\Big) - \Big(e^{O}_{\rho}e^{\mu}_{Q}\psi^{\nu\rho}\mathrm{d}\omega^{QP}_{\mu}\wedge\beta_{\nu}\Big)\Big],\\
&= \Big[\Big(+6(1/6)\psi^{\mu\nu}\mathrm{d}\omega^{PO}_{\mu}\wedge\beta_{\nu}\Big) + (1/6)\Big(2\Big(e^{P}_{\rho}e^{\nu}_{Q}\psi^{\mu\rho}\mathrm{d}\omega^{OQ}_{\mu}\wedge\beta_{\nu}\Big) - (1/6)\Big(2e^{O}_{\rho}e^{\nu}_{Q}\psi^{\mu\rho}\mathrm{d}\omega^{PQ}_{\mu}\wedge\beta_{\nu}\Big),\\
&\quad +(1/6)\Big(2e^{O}_{\rho}e^{\mu}_{Q}\psi^{\nu\rho}\mathrm{d}\omega^{PQ}_{\mu}\wedge\beta_{\nu}\Big) - (1/6)\Big(2e^{P}_{\rho}e^{\mu}_{Q}\psi^{\nu\rho}\mathrm{d}\omega^{OQ}_{\mu}\wedge\beta_{\nu}\Big)\Big].
\end{aligned}
$$

Therefore, we conclude that $(3) = 6(1/6)\psi^{\mu\nu}\mathrm{d}\omega^{IJ}_{\mu}\wedge\beta_{\nu} = (1/6)((1/6))^{-1}\psi^{\mu\nu}\mathrm{d}\omega^{IJ}_{\mu}\wedge\beta_{\nu} = \psi^{\mu\nu}\mathrm{d}\omega^{IJ}_{\mu}\wedge\beta_{\nu}$.

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