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Joseph Vidal-Rosset

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Why Intuitionistic Relevant Logic Cannot Be a Core Logic

Joseph Vidal-Rosset

Abstract At the end of the eighties Tennant invented a logical system that he called "intuitionistic relevant logic" (for short **IR**). Now he calls this same system "Core logic". In section 1, by reference to the rules of natural deduction for **IR**, I explain why **IR** is a relevant logic in a subtle way. Sections 2, 3 and 4 give three reasons to assert that **IR** cannot be a Core logic.

1 IR in a nutshell

1.1 Motivation of IR . Intuitionistic Relevant logic (for short **IR**) is a logical system invented by Tennant at the end of the eighties in order to prove all *theorems* of Heyting logic without accepting the intuitionistic absurdity rule $\bot E$ i.e. $\frac{\bot}{B}\ {\scriptstyle \bot E}$.

Tennant claims that, in his logical system, neither the "First Lewis Paradox" i.e. the sequent provable in intuitionistic logic by application of the intuitionistic absurdity rule $\bot E$, i.e. *Ex Contradictione Quodlibet*

$$\neg A, A \vdash_i B \tag{ECQ}$$

nor the negate-conclusion version of the First Lewis Paradox, i.e. the following sequent provable in minimal logic:

$$\neg A, A \vdash_m \neg B \tag{ECQ$\neg$}$$

are provable in **IR**. The subtle goal of **IR** can be understood thanks to Tennant's slogan [8, p. 711]:

> If one needs a slogan to help with orientation, ours is a method of relevantizing "at the level of the turnstile." As a result, there will be some tweaking of the rules for the logical operators (in the natural deduction or sequent setting); but these tweakings are not as to change their established meanings.

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1.2 Tweaking the rules with IR. Before seeing how **IR** tweaks the rules of intuitionistic logic, note the unchanged rules of minimal logic belonging also to **IR**:

$$\dfrac{\neg A \qquad A}{\bot}\,\bot E \qquad\qquad \dfrac{A \qquad B}{A \wedge B}\,\wedge I \qquad\qquad \dfrac{A}{A \vee B}\,\vee I \quad \dfrac{B}{A \vee B}\,\vee I$$

Now let see the "tweaking" that Tennant [8, 7, 5] has made. In the group of introduction rules there is an additional rule $\to I_{ir}$.(Labels $\to I_{ir}$ and $\vee E_{ir}$ are mines.) The other changes consist in notification of obligatory or permissible discharge:

$$\begin{array}{c} \dfrac{}{A}\,\Box(i) \\ \vdots \\ \dfrac{\bot}{\neg A}\,\neg I,\ (i) \end{array} \qquad\qquad \begin{array}{c} \dfrac{}{A}\,\Box(i) \\ \vdots \\ \dfrac{\bot}{A \to B}\,\to I_{ir},\ (i) \end{array} \qquad\qquad \begin{array}{c} \dfrac{}{A}\,\Diamond(i) \\ \vdots \\ \dfrac{B}{A \to B}\,\to I,\ (i) \end{array}$$

The symbol $\Box$ means that the discharge of A is obligatory. If you can deduce a contradiction from an open assumption A, then either you can conclude $\neg A$ by application of rule $\neg I$, or you can introduce a conditional with A as antecedent and any well formed formula as consequent, like B, but in both cases you have to discharge A. In the other case, when you can deduce a formula B from A, if B is not a contradiction, the discharge of A is $\Diamond$ i.e. just permissible. I must specify now a crucial meta-rule of **IR**:

Definition 1.1 (Proof in IR) A deduction $\mathscr{D}$ is a proof in **IR** if and only if only **IR**-rules of inference are used in $\mathscr{D}$ and if and only if $\mathscr{D}$ is in *normal form*. According to Prawitz [3, p. 34], a proof is in normal form "if no formula occurrence is both the consequence of an application of an I-rule and major premise of an application of an E-rule". A proof is said to be "in normal form" if it is "redex free" i.e. if no introduction rule for a connective is followed immediately by an elimination rule of the same connective. If a deduction is not in normal form, it is not a proof in **IR**.

1.2.1 Why (ECQ) is claimed to be unprovable in **IR**

At this stage, we do not need to know more about the rules of **IR** to understand why Tennant claims that (ECQ¬) and (ECQ) are not among the deducibilities of **IR**, while for example the formula expressing the *ex falso quodlibet* in the axiomatic of Heyting logic i.e.

$$\vdash_i \neg A \to (A \to B) \tag{1}$$

is nevertheless also *theorem* of **IR**:

Proof

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\neg A}\,1 \qquad \dfrac{}{A}\,2}{\bot}}{A \to B}\,\to I_{ir},\ 2}{\neg A \to (A \to B)}\,\to I,\ 1$$

$\Box$

Note that this proof of (1) does not prove (ECQ) and that it is impossible to use a conditional elimination rule in order to deduce (ECQ): such a proof would be an abnormal proof i.e. a non-proof in **IR** (see definition 1.1).

1.2.2 Why (ECQ¬) is claimed to be unprovable in **IR**

To prevent a proof (ECQ¬) in **IR**, vacuous discharge (or Weakening) is banned from **IR** when the assumptions base is reducible to contradiction. Indeed, suppose that there is no restriction on the use of vacuous discharge, as in minimal logic and in intuitionistic logic, then the following deduction schema

$$\dfrac{\bot}{B \to \bot}\ {\scriptstyle \to I}$$

is a sufficient proof of (ECQ¬) because by the intuitionistic definition of negation the formula $B \to \bot$ is $\neg B$. But one must add that if vacuous discharge is replaced by the following rule of Weakening.[1] defined by David *et al.* [1, p. 38]:

$$\dfrac{\begin{array}{c}\vdots\\ A\end{array} \qquad \dfrac{}{B}\ {\scriptstyle (i)}}{A}\ {\scriptstyle Wk}$$

then this deduction schema shows that (ECQ¬) is always provable:

$$\dfrac{\dfrac{\dfrac{\neg A \qquad A}{\bot} \qquad \dfrac{}{B}\ {\scriptstyle 1}}{\bot}\ {\scriptstyle Wk}}{\neg B}\ {\scriptstyle \neg I,\ 1}$$

To avoid the provability in **IR**, one must therefore ban this instance of Weakening [2]:

$$\dfrac{\begin{array}{c}\vdots\\ \bot\end{array} \qquad \dfrac{}{B}\ {\scriptstyle (i)}}{\bot}\ {\scriptstyle Wk}$$

1.2.3 Tweaking of elimination rules and acceptance of Disjunctive Syllogism For computational reasons explained in *Autologic* [5], Tennant replaces the usual rules $\wedge E$ and rule $\to E$ by these "parallelized" rules:

$$\dfrac{A \wedge B \qquad \begin{array}{c}\underbrace{\dfrac{}{A\ ,\ B}\ {\scriptstyle \Box(i)}}_{\text{Two } \Box\text{: at least one of } A, B \text{ must have been used}}\ {\scriptstyle \Box(i)}\\ \vdots\\ C\end{array}}{C}\ {\scriptstyle \wedge E(i)} \qquad \dfrac{A \to B \qquad \begin{array}{c}\vdots\\ A\end{array} \qquad \begin{array}{c}\dfrac{}{B}\ {\scriptstyle \Box(i)}\\ \vdots\\ C\end{array}}{C}\ {\scriptstyle \to E\ (i)}$$

Remark 1.2 The usual (i.e. non-parallelized) rules $\wedge E$ and $\to E$ of natural deduction are trivially derivable in **IR**, by replacing C by either A or B in the former and by B in the latter.

Last, the following liberalized rule $\vee E_{ir}$

$$\dfrac{A \vee B \qquad \begin{array}{c}\dfrac{}{A}\ {\scriptstyle \Box\ (i)}\\ \vdots\\ \bot/C\end{array} \qquad \begin{array}{c}\dfrac{}{B}\ {\scriptstyle \Box\ (i)}\\ \vdots\\ \bot/C\end{array}}{\bot/C}\ {\scriptstyle \vee E_{ir}\ (i)}$$

has been invented by Tennant in order to allow in **IR** the proof of Disjunctive Syllogism i.e.

$$(A \vee B), \neg A \vdash B \tag{DS}$$

without making use of the intuitionistic rule $\bot E$.

Tennant [8, p. 714, in [4]] explains this rule as follows:

> By stating graphically [rule $\vee E_{ir}$] as we have, we are providing for the possibility that one of the case assumptions might lead to absurdity ($\bot$). We are then permitted to bring down as the main conclusion whatever is concluded from the *other* case assumption. [...] Liberalizing proof by cases in this way in entirely natural, given how we reason informally. Suppose one is told that $A \vee B$ hold, along with certain other assumptions X, and one is required to prove that C follows from the combined assumption $X, A \vee B$. If one assumes A and discovers that it is inconsistent with X, one simply stop one's investigation of that case, and turns to the case B. If C follows in the latter case, one concludes C as required. One does *not* go back to the conclusion of absurdity in the first case, and artificially dress it up with an application of the absurdity rule so as to make it also " yield" the conclusion C.

Our paper focuses only on natural deduction for **IR** and does not deal with the corresponding sequent calculus for **IR**. But, to conclude this section, I stress on the treatment of the Deduction Theorem and its converse. About this basic theorem, Tennant [6, p. 344] says:

> The orthodox Deduction Theorem for logical calculi states that:
>
> $$X \vdash B \text{ only if } X \backslash \{A\} \vdash A \to B; \text{ and } X \vdash A \to B \text{ only if } X, A \vdash B$$
>
> In **IR**, the first implication holds. But the converse fails, for in **IR** we have $\neg A \vdash A \to B$ but not: $\neg A, A \vdash B$. The lesson here is that we have to be careful to distinguish between asserting a conditional and making an inference. We have determined a conditional that is exactly what is needed in order to have the best possible system of relevant inference. The price of 'unpacking' a conditional is eternal vigilance with regard to the joint consistency of the new set of assumptions in play, and with regard to the possible logical truth of the new conclusion.

2 IR changes the meaning of disjunction elimination

Statement 2.1 It is provable that rule $\vee E_{ir}$ changes the established meaning of $\vee$ when $\vee$ is on the left of the turnstile.

Proof

$$A \vee B \vdash C \tag{2}$$

means in minimal logic that C is a syntactical consequence of *both* A and B, i.e. (2) is provable if and only if it is provable that $A \vdash_m C$ *and* if it is provable that $B \vdash_m C$, *whatever* A, B, C *are.* In **IR** the meaning of (2) changes if either A or B is equivalent to $\bot$, because *via* rule $\vee E_{ir}$

$$\bot \vee B \vdash B \tag{3}$$

but only because of the axiom $B \vdash B$ and in spite of the claim that $\bot \nvdash B$ in **IR**. Therefore **IR** changes the meaning of $\vee$ when $\vee$ is on the left of the turnstile. $\square$

3 The law of substitution of logical equivalents fails in IR

Theorem 3.1 *The law of substitution of logical equivalents fails in* **IR**.

Proof

The logical equivalence in **IR** between $(\bot \vee B)$ and B is provable:

$$\dfrac{\dfrac{\dfrac{\overline{B}^{\,1}}{B \vee \bot}\,\vee I}{B \to (B \vee \bot)}\,\to I,\,1 \qquad \dfrac{\dfrac{\overline{B \vee \bot}^{\,1} \quad \overline{B}^{\,2} \quad \overline{\bot}^{\,3}}{B}\,\vee E_{ir},\,2,3}{(B \vee \bot) \to B}\,\to I,\,1}{(B \vee \bot) \leftrightarrow B}\,\wedge I$$

Nevertheless, $(\bot \vee B)$ and B are not always substitutable *salva veritate* in **IR**. Indeed

$$\bot \vdash \bot \vee B \tag{4}$$

is provable in **IR** (*via* rule $\vee I$), but if one replaces $\bot \vee B$ by B one gets

$$\bot \vdash B \tag{5}$$

in contradiction with the claim of "relevance at the level of the turnstile" in **IR**, i.e. $\bot \nvdash B$. Therefore the law of substitution of logical equivalents fails in **IR** because it is not true in **IR** that, in *any* context Γ

$$\Gamma \vdash \bot \vee B \tag{6}$$

if and only if

$$\Gamma \vdash B \tag{7}$$

□

4 A heavy rule of absurdity in IR

IR does not differ from minimal logic only by rejecting the Weakening rule. Contrary to minimal logic and in agreement with intuitionistic logic, $\bot$ is in **IR** an inference marker occurring with contradiction and being equivalent to any contradiction; but this difference with minimal logic entails some problematic theorems, such as the following.

Theorem 4.1 *Unlike minimal logic, any contradiction is logically equivalent to any other one in* **IR**, *i.e.*

$$\vdash (\neg A \wedge A) \leftrightarrow (\neg B \wedge B) \tag{8}$$

Proof

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg A \wedge A}\,1}{\neg A}\,\wedge E \quad \dfrac{\dfrac{}{\neg A \wedge A}\,1}{A}\,\wedge E}{\bot}\,\neg E}{(\neg A \wedge A) \to (\neg B \wedge B)}\,\to I_{ir}\,1 \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\neg B \wedge B}\,2}{\neg B}\,\wedge E \quad \dfrac{\dfrac{}{\neg B \wedge B}\,2}{B}\,\wedge E}{\bot}\,\neg E}{(\neg B \wedge B) \to (\neg A \wedge A)}\,\to I_{ir}\,2}{(\neg A \wedge A) \leftrightarrow (\neg B \wedge B)}\,\wedge I$$

□

Theorem 4.1 leads to a way of avoiding in **IR** the relevance at the level of the turnstile:

Theorem 4.2 *In* **IR**, *B is deducible "at the level of the turnstile" from the assumption of the conjunction of contradiction $\neg A \wedge A$ and theorem $(\neg A \wedge A) \leftrightarrow (\neg B \wedge B)$, i.e.*

$$\neg A \wedge A \wedge ((\neg A \wedge A) \leftrightarrow (\neg B \wedge B)) \vdash B \tag{9}$$

is provable in **IR** via *a proof in perfect normal form, using only elimination rules.*

Proof

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg A \wedge A \wedge ((\neg A \wedge A) \leftrightarrow (\neg B \wedge B))}{\neg A \wedge A \wedge ((\neg A \wedge A) \to (\neg B \wedge B))}\,\wedge E}{(\neg A \wedge A) \to (\neg B \wedge B)}\,\wedge E \quad \dfrac{\dfrac{\neg A \wedge A \wedge ((\neg A \wedge A) \leftrightarrow (\neg B \wedge B))}{\neg A \wedge A \wedge ((\neg B \wedge B) \to (\neg A \wedge A))}\,\wedge E}{\neg A \wedge A}\,\wedge E}{\neg B \wedge B}\,\to E}{B}\,\wedge E$$

□

Remark 4.3 This proof is a proof in **IR**: it is in normal form, and usual rules $\wedge E$ and $\rightarrow E$ are rules of **IR** because they are derivable in **IR** (remark 1.2).

Remark 4.4 Sequent (9) is semantically reducible to

$$\bot \wedge (\bot \leftrightarrow \bot) \vdash B \tag{10}$$

and therefore to

$$\bot \wedge \top \vdash B \tag{11}$$

and last to

$$\bot \vdash B \tag{12}$$

5 Conclusion

The arguments of section 2, 3 and 4 provide three sufficient reasons to deny that **IR** is a Core logic.

1. We expect that a Core logic respects the univocal meaning of logical rules. It is not the case with **IR**: rule $\vee E_{ir}$ has not the same meaning when it is applied to a sequent like

$$A \vee \neg\neg\neg\neg A \vdash \neg\neg A \tag{13}$$

and when it is applied to

$$(\neg A \wedge A) \vee B \vdash B \tag{14}$$

Sequent (13) is valid in **IR** because two sequents are provable, i.e. $A \vdash \neg\neg A$ and $\neg\neg\neg\neg A \vdash \neg\neg A$, but sequent (14) is valid in **IR** *only* because $B \vdash B$ is an axiom (of course $\neg A \wedge A \vdash B$ is claimed to be unprovable in **IR**).

2. At least since Leibniz, this law of substitution of logical equivalents is regarded as a basic logical law. To quote Leibniz' example [2, pp. 85-87], if A and B have the same *meaning*, for example "trilateral plane figure" and "plane figure triangular", A can be replaced with B *salva veritate* in all statements (provided a context as referentially transparent as Euclidean geometry). Conversely, it is generally accepted in mathematical logic that, if A cannot be replaced by B in system **S**, it is because A and B have different meanings i.e. are not logically equivalent in **S**. For example $\neg\neg A$ cannot be replaced by A *salva veritate* in intuitionistic logic, and $\bot \vee B$ cannot be replaced by B in minimal logic. This law is essential to our understanding of the rules of logical systems in general. By proving that Leibniz' law fails in **IR**, section 3 helps explain why this so-called "Core logic" is so difficult to understand. Of course, Tennant could reply that the virtue of **IR** is to prevent the transitivity of deductions from sequents $\Gamma \vdash \bot \vee B$ to $\Gamma \vdash B$ when Γ is itself $\bot$. But the trouble is that the logical equivalence between $\bot \vee B$ and B is *syntactically* provable in **IR** regardless of context, and therefore should be *semantically* valid in **IR** also regardless of context; proof has been given in this paper that it is not the case in **IR**, and it is a serious problem if harmony between syntax and semantics is expected in Core logic.

3. To my knowledge, nowhere has Tennant clearly explained the difference between a provable inference "at the level of the turnstile" and a provable conditional, i.e. an implication. After all, why (ECQ) should be unprovable, while the conditional $(\neg A \wedge A) \rightarrow B$ is a theorem of **IR**? This lack of harmony between provable sequents and provable conditionals is shocking. The proof given in section 4 shows that Tennant's slogan is not even in harmony with the theorem of **IR** according to which any contradiction $(\neg A \wedge A)$ is equivalent to another one, say $(\neg B \wedge B)$. From this point of view, minimal logic is a better candidate to be a Core logic. With minimal logic, by contrast with **IR**, you are not embarrassed to recognize that if you assume that any contradiction is equivalent to any other one, it is easy to show, at the level of the turnstile, that from this assumption and the assumption of $\neg A \wedge A$, you can infer B (i.e. (9) is provable in minimal logic). In my opinion, it is contrary to the meaning of "Core logic" to contradict minimal logic, and that is probably the main reason why Tennant's logic cannot be a Core logic.

Notes

1. Tennant calls this rule "Dilution". I use the most usual name for this rule.

2. In [5, p. 189] Tennant repeats that **IR** "requires non-vacuous discharge of assumptions", but he adds in footnote 6 at the same page "except, crucially, in one half of the rule of implication introduction!". It means that it is *only* for the other half of implication introduction, i.e. for rule $\rightarrow I_{ir}$, that vacuous discharge i.e. Weakening is forbidden.

References

[1] David, R. & Nour, K. & Raffalli, C., *Introduction à la logique (Théorie de la démonstration, Cours et exercices corrigés)*, Dunod, Paris, 2001, 2003. 3

[2] Leibniz, G. W., *Recherches générales sur l'analyse des notions et des vérités: 24 thèses métaphysiques et autres textes logiques et métaphysiques,* edited by Rauzy, Épiméthée. Presses universitaires de France, Paris, France, 1998. 6

[3] Prawitz, D., *Natural Deduction - A Proof Theoretical Study*, Almqvist & Wilksell, Uppsala, 1965. 2nd ed. 2006 by Dover Publications, Inc. Mineola, New-York. 2

[4] Shapiro, S., editor, *The Oxford Handbook of Philosophy of Mathematics*, Oxford, 2005. Oxford University Press. 3, 7

[5] Tennant, N., *Autologic*, Edinburgh Information and Technology Series. Edinburgh University Press, Edinburgh, 1992. 2, 3, 7

[6] Tennant, N., *The Taming of the True*, Oxford University Press, Oxford, New York, 1997. 4

[7] Tennant, N., "Ultimate Normal Forms for Parallelized Natural Deductions," *Logic Journal of the IGPL*, vol. 10 (2002), pp. 299–337. 2

[8] Tennant, N., "Relevance in Reasoning," in Shapiro, S. [4], pp. 696–726. 1, 2, 3

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Joseph Vidal-Rosset
Université de Lorraine - Département de philosophie
Archives Poincaré - CNRS - UMR 7117
91 Avenue de la Libération
BP 454 - 54001 NANCY CEDEX
FRANCE
Email: joseph.vidal-rosset@univ-lorraine.fr