Spectral action in noncommutative geometry: An example

B Iochum

Spectral action in noncommutative geometry: An example

B. Iochum[1,2]

[1]Centre de Physique Théorique [3], CNRS–Luminy, Case 907,13288 Marseille Cedex 9 FRANCE
[2] Also at Université de Provence

E-mail: `iochum@cpt.univ-mrs.fr`

Abstract. This is a report on a joint work [12] with D. Essouabri, C. Levy and A. Sitarz.

The spectral action on noncommutative torus is obtained, using a Chamseddine–Connes formula via computations of zeta functions. The importance of a Diophantine condition is outlined as far as the difficulties to go beyond. Some results on holomorphic continuation of series of holomorphic functions are presented.

1. Introduction

The spectral action introduced by Chamseddine–Connes plays an important role (see [3] and Chamseddine's contribution to this proceedings) in noncommutative geometry. More precisely, given a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ where $\mathcal{A}$ is an algebra acting on the Hilbert space $\mathcal{H}$ and $\mathcal{D}$ is a Dirac-like operator (see [9,21]), they proposed a physical action depending only on the spectrum of the covariant Dirac operator

$$\mathcal{D}_A := \mathcal{D} + \widetilde{A}, \quad \widetilde{A} := A + \epsilon\, JAJ^{-1} \tag{1}$$

where A is a one-form represented on $\mathcal{H}$, so has the decomposition $A = \sum_i a_i[\mathcal{D}, b_i]$, with a_i, $b_i \in \mathcal{A}$, J is a real structure on the triple corresponding to charge conjugation and $\epsilon \in \{1, -1\}$ depending on the dimension of this triple and comes from the commutation relation

$$J\mathcal{D} = \epsilon\, \mathcal{D}J. \tag{2}$$

This action is defined by

$$\mathcal{S}(\mathcal{D}_A, \Phi, \Lambda) := \mathrm{Tr}\left(\Phi(\mathcal{D}_A/\Lambda)\right) \tag{3}$$

where Φ is any even positive cut-off function which could be think as a step function. This means that Φ counts the spectral values of $|\mathcal{D}_A|$ less than the mass scale Λ.

Even if the spectral action on NC-tori has been computed for operators of the form $\mathcal{D} + A$ in [16] and for $\mathcal{D}_A$ in [18], it is interesting to show that this can be also directly obtained from

[3] Unité Mixte de Recherche (UMR 6207) du CNRS et des Universités Aix-Marseille I, Aix-Marseille II et de l'Université du Sud Toulon-Var. Laboratoire affilié à la FRUMAM – FR 229

CPT-P27-2007

the Chamseddine–Connes analysis of [4]. Actually,

$$\mathcal{S}(\mathcal{D}_A,\Phi,\Lambda) \,=\, \sum_{0<k\in Sd^+} \Phi_k\,\Lambda^k \fint |D_A|^{-k} + \Phi(0)\,\zeta_{D_A}(0) + \mathcal{O}(\Lambda^{-1}) \tag{4}$$

where $D_A = \mathcal{D}_A + P_A$, P_A is the projection on $\operatorname{Ker}\mathcal{D}_A$, $\Phi_k = \frac{1}{2}\int_0^\infty \Phi(t)\,t^{k/2-1}\,dt$ and Sd^+ is the strictly positive part of the dimension spectrum of $(\mathcal{A},\mathcal{H},\mathcal{D})$. As we will see, $Sd^+ = \{\,1,2,\cdots,n\,\}$ and $\fint |D_A|^{-n} = \fint |D|^{-n}$. Moreover, the constant term $\zeta_{D_A}(0)$ in (4) can be computed from unperturbed spectral action since it has been proved in [4] that

$$\zeta_{\mathcal{D}+A}(0) - \zeta_{\mathcal{D}}(0) = \sum_{q=1}^{n} \tfrac{(-1)^q}{q} \fint (A\mathcal{D}^{-1})^q, \tag{5}$$

using $\zeta_X(s) = \operatorname{Tr}(|X|^{-s})$. We show that this formula can be extended to non invertible Dirac operators and non invertible perturbations.
All results on spectral action are quite important in physics, especially in quantum field theory and particle physics, where one adds to the effective action some counterterms explicitly given by (5), see for instance [2–4, 6, 14, 16, 18, 20, 22, 25–28].

2. How to compute $\operatorname{Tr}\big(\Phi(\mathcal{D}_A/\Lambda)\big)$?

2.1. Heat kernel approach

Constraints on Φ: Actually, the step functions is not allowed due to Gibbs' phenomenon. So, we can
- use the distributional approach investigated in [13],
- or follow [23] and assume that $\Phi \in C^\infty(\mathbb{R}^+)$ is a Laplace transform of $\hat{\psi}$ in the Schwartz space $\mathcal{S}(\mathbb{R}^+) = \{\,g\in\mathcal{S}(\mathbb{R})\,:\,g(x)=0,\ x\le 0\,\}$. So Φ has analytic extension on the right complex plane and $\Phi(z) = (-1)^m\int_0^\infty e^{-tz}\,t^m\,\hat{\psi}(t)\,dt$, $Re(z)>0$ where $m := \lfloor \frac{n}{2}\rfloor$. In this case, the following is well established

$$\operatorname{Tr}\big(\Phi(\mathcal{D}/\Lambda)\big) = \sum_{k=0}^{d} \Phi_k\,a_k\,\Lambda^{d-k} + \mathcal{O}(\Lambda^{-1}) \tag{6}$$

where a_k are the Seeley–De Witt coefficients (see [19]) and when $n=2m$ is even, Φ_{2k} has the more familiar form:

$$\Phi_{2k} = \begin{cases} \frac{1}{\Gamma(m-k)}\int_0^\infty \Phi(t)\,t^{m-1-k}\,dt, & \text{for } k=0,\cdots,m-1,\\ (-1)^k\,\Phi^{(k-m)}(0), & \text{for } k=m,\cdots,n. \end{cases}$$

For n odd, the coefficients Φ_{2k} have less explicit forms because they involve fractional derivatives of Φ.

2.2. Pseudodifferential operators and Zeta functions approach

One can enter in the field either via the heat kernel as before or via zeta functions or Dixmier traces.
Consider first the commutative case where $\mathcal{A}\in C^\infty(M)$ for a manifold M. When $P\in\Psi DO$ is a pseudodifferential operator of order q

$$\zeta_P(s) = \operatorname{Tr}(P|\mathcal{D}|^{-s})$$

is holomorphic for $\operatorname{Re}(s) > q+d$, with at most poles at integers $k\le q+d$.

The leading residue is

$$\mathrm{Res}_{s=d+q}\,\zeta_P(s) = c_1\,\mathrm{Tr}_{Dix}(P) = c_2 \int_{S^*(M)} \sigma_P$$

(see Guillemin, Wodzicki,..., where in general $\mathcal{D}^2 = \triangle$ is the scalar Laplacian.)
Here is the interest of the abstract setting introduced in [10, 11] where subleading residues are equally obtained: Given a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ of dimension d, which means

$$0 < \mathrm{Tr}_{Dix}(a|\mathcal{D}|^{-d}) < \infty \quad \forall\, 0 \neq a \in \mathcal{A}^+$$

(when $\mathcal{D}$ is non invertible, one replace $\mathcal{D} \to D := \mathcal{D} + P_0$ where P_0 is the projection on Ker $\mathcal{D}$,) one follow the construction of ΨDO on the track of Cordes' characterization of pseudodifferential operators:

$$\begin{aligned} OP^0 &:= \{\, T : t \to e^{it|\mathcal{D}|}\, T\, e^{-it|\mathcal{D}|} \in C^\infty\big(\mathbb{R}, \mathcal{B}(\mathcal{H})\big) \,\}, \\ OP^\alpha &:= \{\, T : T|D|^{-\alpha} \in OP^0 \,\}, \quad \alpha \in \mathbb{R}. \end{aligned}$$

Definition 2.1. *Define $\mathcal{D}(\mathcal{A})$ as the polynomial algebra generated by $\mathcal{A}$, $J\mathcal{A}J^{-1}$, $\mathcal{D}$ and $|\mathcal{D}|$. A pseudodifferential operator is an operator T such that there exists $d \in \mathbb{Z}$ such that for any $N \in \mathbb{N}$, there exist $p \in \mathbb{N}_0$, $P \in \mathcal{D}(\mathcal{A})$ and $R \in OP^{-N}$ (p, P and R may depend on N) such that $P\,D^{-2p} \in OP^d$ and*

$$T = P\,D^{-2p} + R\,.$$

Define $\Psi(\mathcal{A})$ as the set of pseudodifferential operators and $\Psi(\mathcal{A})^k := \Psi(\mathcal{A}) \cap OP^k$.

The idea is to work modulo large N. This nonstandard definition pays attention to the reality operator J and the kernel of $\mathcal{D}$ and allows $\mathcal{D}$ and $|D|^{-1}$ to be pseudodifferential operators. It is more in the spirit of [4].
Note that if A is a 1-form, A and JAJ^{-1} are in $\mathcal{D}(\mathcal{A})$ and moreover $\mathcal{D}(\mathcal{A}) \subseteq \cup_{p\in\mathbb{N}_0} OP^p$. Since $|\mathcal{D}| \in \mathcal{D}(\mathcal{A})$ by definition and P_0 is a pseudodifferential operator, for any $p \in \mathbb{Z}$, $|D|^p$ is a pseudodifferential operator (in OP^p.) Remark also that $\mathcal{D}(\mathcal{A}) \subseteq \Psi(\mathcal{A}) \subseteq \cup_{k\in\mathbb{Z}} OP^k$.
As in [10, 11], $\Psi(\mathcal{A})$ is an algebra: if $T \in \Psi(\mathcal{A})^d$, $T \in \Psi(\mathcal{A})^{d'}$, then $TT' \in \Psi(\mathcal{A})^{d+d'}$.
The noncommutative integral is naturally defined as

$$⨍ T := \mathrm{Res}_{s=0}\, \zeta_D^T(s) = \mathrm{Res}_{s=0}\, \mathrm{Tr}\,\big(T\,|D|^{-s}\big), \quad T \in \Psi DO.$$

which makes sense since $a|D|^{-(d+\epsilon)} \in \mathcal{L}^1(\mathcal{H})$, $\forall a \in \mathcal{A}$, $\forall \epsilon > 0$.
One checks that $⨍$ is a trace on ΨDO when all poles of $Sd(\mathcal{A}, \mathcal{H}, \mathcal{D})$ are simple.
Another important notion is the *dimension spectrum*

$$Sd(\mathcal{A}, \mathcal{H}, D) := \{\,\text{poles of } \mathrm{Tr}\,\big(T|D|^{-s}\big) : T \in \Psi DO^0 \,\}$$

For example, when M is a compact spin^c Riemannian manifold of dimension n, with spectral triple $\mathcal{A} = C^\infty(M)$, $\mathcal{H} = L^2(Spineurs)$ and $\mathcal{D} = -i\gamma^\mu\partial_\mu$, then $Sd(\mathcal{A}, \mathcal{H}, \mathcal{D}) = \{\, n - k : k \in \mathbb{N}_0 \,\}$.

Remark 2.2. *It is rather difficult to compute zeta functions, even when it is a Dixmier-trace. However, it is proved in [1] that for any $0 \le T \in \mathcal{B}(\mathcal{H})$ such that $T^s \in \mathcal{L}^1(\mathcal{H})$ for all $s > 1$, then, if $l = \lim_{\epsilon\to 0}\, \epsilon\, \mathrm{Tr}(T^{1+\epsilon})$ exists, T is Dixmier-traceable and $\mathrm{Tr}_{Dix}(T) = l$.*

Remark 2.3. *When $\mathcal{D}$ (or $\mathcal{D}_A$) are not invertible, one uses in noncommutative integrals D (or D_A) defined by $D = \mathcal{D} + P_0$ (or $D_A = \mathcal{D}_A + P_A$), where the P's are projections on the kernel of the operator. Since $\mathcal{D}$ has a compact resolvent, $\mathcal{D}_A$ has also a compact resolvent and these projections are finite-rank operators.*
Note that $P_A \in OP^{-\infty}$, so is a smoothing operator.
For the kernels, there is a difference between $\mathcal{D}_A$ and $\mathcal{D}+A$: for the noncommutative torus, the inclusion $\operatorname{Ker}\mathcal{D} \subseteq \operatorname{Ker}\mathcal{D}+A$ is not satisfied since A does not preserve $\operatorname{Ker}\mathcal{D}$ contrarily to $\widetilde{A}$.

Using (6), one obtains (4). Moreover, relation (5) proved in [4], can be extended to

Proposition 2.4. *For any selfadjoint one-form A, the constant term in Λ in (6) is*

$$\zeta_{D_A}(0) - \zeta_D(0) = -\int\!\!\!\!\!\!- \log(1+\widetilde{A}D^{-1}) = \sum_{q=1}^{n} \tfrac{(-1)^q}{q} \int\!\!\!\!\!\!- (\widetilde{A}D^{-1})^q. \tag{7}$$

3. The noncommutative torus

3.1. Notations

Let $\mathcal{A}_\Theta := C^\infty(\mathbb{T}^n_\Theta)$ be the smooth noncommutative n-torus associated to a non-zero skew-symmetric deformation matrix $\Theta \in M_n(\mathbb{R})$ (see [7], [24]): $C^\infty(\mathbb{T}^n_\Theta)$ is generated by n unitaries u_i, $i = 1,\dots,n$ subject to the relations $u_i\, u_j = e^{i\Theta_{ij}}\, u_j\, u_i$, and with Schwartz coefficients: an element $a \in \mathcal{A}_\Theta$ can be written as $a = \sum_{k\in\mathbb{Z}^n} a_k\, U_k$, where $\{a_k\} \in \mathcal{S}(\mathbb{Z}^n)$ with the Weyl elements defined by $U_k := e^{-\frac{i}{2}k.\chi k}\, u_1^{k_1}\cdots u_n^{k_n}$, $k \in \mathbb{Z}^n$. Previous relation on the $u's$ reads

$$U_kU_q = e^{-\frac{i}{2}k.\Theta q}\, U_{k+q}, \text{ and } U_kU_q = e^{-ik.\Theta q}\, U_qU_k \tag{8}$$

where χ is the matrix restriction of Θ to its upper triangular part. Thus unitary operators U_k satisfy $U_k^* = U_{-k}$ and $[U_k, U_l] = -2i\sin(\frac{1}{2}k.\Theta l)\, U_{k+l}$.
Let τ be the trace on $\mathcal{A}_\Theta$) defined by $\tau\big(\sum_{k\in\mathbb{Z}^n} a_k\, U_k\big) := a_0$ and $\mathcal{H}_\tau$ be the GNS Hilbert space obtained by completion of $\mathcal{A}_\Theta$ with respect of the norm induced by the scalar product $\langle a,b\rangle := \tau(a^*b)$. On $\mathcal{H}_\tau = \{\,\sum_{k\in\mathbb{Z}^n} a_k\, U_k \;:\; \{a_k\}_k \in l^2(\mathbb{Z}^n)\,\}$, we have the left and right regular representations of $\mathcal{A}_\Theta$ by bounded operators, denoted respectively by $L(.)$ and $R(.)$.
Let also δ_μ, $\mu \in \{\,1,\dots,n\,\}$, be the n (pairwise commuting) canonical derivations, defined by

$$\delta_\mu(U_k) := ik_\mu U_k. \tag{9}$$

$\mathcal{A}_\Theta$ acts on $\mathcal{H} := \mathcal{H}_\tau \otimes \mathbb{C}^{2^m}$ where $m = \lfloor \frac{n}{2} \rfloor$, the square integrable sections of the trivial spin bundle over $\mathbb{T}^n$: each element of $\mathcal{A}_\Theta$ is represented on $\mathcal{H}$ as $L(a) \otimes 1_{2^m}$.
The Tomita conjugation $J_0(a) := a^*$ satisfies $[J_0, \delta_\mu] = 0$ and we define $J := J_0 \otimes C_0$ where C_0 is an operator on $\mathbb{C}^{2^m}$. The Dirac operator is then given by the selfadjoint extension of

$$\mathcal{D} := -i\,\delta_\mu \otimes \gamma^\mu,$$

where we use hermitian Dirac matrices γ. This implies $C_0\gamma^\alpha = -\varepsilon\gamma^\alpha C_0$, and $\mathcal{D}\, U_k \otimes e_i = k_\mu U_k \otimes \gamma^\mu e_i$, where $\{\, e_i \,\}$ is the canonical basis of $\mathbb{C}^{2^m}$. Moreover, $C_0^2 = \pm 1_{2^m}$ depending on the parity of m. Finally, one introduces the chirality (which in the even case is $\chi := id \otimes (-i)^m\gamma^1\cdots\gamma^n$) and this yields that $(\mathcal{A}_\Theta, \mathcal{H}, \mathcal{D}, J, \chi)$ satisfies all axioms of a spectral triple, see [9,21].
For every unitary $u \in \mathcal{A}$, $uu^* = u^*u = U_0$, the perturbed Dirac operator $V_u\mathcal{D}V_u^*$ by the unitary $V_u := \big(L(u)\otimes 1_{2^m}\big)J\big(L(u)\otimes 1_{2^m}\big)J^{-1}$, must satisfy condition (2).
This yields the necessity of a symmetrized covariant Dirac operator: $\mathcal{D}_A := \mathcal{D} + A + \epsilon J\, A\, J^{-1}$ since $V_u\mathcal{D}V_u^* = \mathcal{D}_{L(u)\otimes 1_{2^m}[\mathcal{D}, L(u^*)\otimes 1_{2^m}]}$: in fact, for $a \in \mathcal{A}_\Theta$, using $J_0L(a)J_0^{-1} = R(a^*)$, we get

$$\epsilon J\big(L(a)\otimes\gamma^\alpha\big)J^{-1} = -R(a^*)\otimes\gamma^\alpha.$$

This induces some covariance property for the Dirac operator: one checks that for all $k \in \mathbb{Z}^n$,

$$L(U_k) \otimes 1_{2^m} [\mathcal{D}, L(U_k^*) \otimes 1_{2^m}] = 1 \otimes (-k_\mu \gamma^\mu), \tag{10}$$

so with $C_0 \gamma^\alpha = -\varepsilon \gamma^\alpha C_0$, we get $U_k[\mathcal{D}, U_k^*] + \epsilon J U_k [\mathcal{D}, U_k^*] J^{-1} = 0$ and

$$V_{U_k} \, \mathcal{D} \, V_{U_k}^* = \mathcal{D} = \mathcal{D}_{L(U_k)\otimes 1_{2^m}[\mathcal{D}, L(U_k^*)\otimes 1_{2^m}]}. \tag{11}$$

Moreover, we get the gauge transformation:

$$V_u \mathcal{D}_A V_u^* = \mathcal{D}_{\gamma_u(A)} \tag{12}$$

where the gauged transform one-form of A is

$$\gamma_u(A) := u[\mathcal{D}, u^*] + uAu^*, \tag{13}$$

with the shorthand $L(u) \otimes 1_{2^m} \longrightarrow u$.
As a consequence, the spectral action is gauge invariant: $\mathcal{S}(\mathcal{D}_A, \Phi, \Lambda) = \mathcal{S}(\mathcal{D}_{\gamma_u(A)}, \Phi, \Lambda)$.
An arbitrary selfadjoint one-form A, can be written as

$$A = L(-iA_\alpha) \otimes \gamma^\alpha, \; A_\alpha = -A_\alpha^* \in \mathcal{A}_\Theta, \tag{14}$$

thus

$$\mathcal{D}_A = -i \left(\delta_\alpha + L(A_\alpha) - R(A_\alpha) \right) \otimes \gamma^\alpha. \tag{15}$$

If $\tilde{A}_\alpha := L(A_\alpha) - R(A_\alpha)$, we get $\mathcal{D}_A^2 = -g^{\alpha_1\alpha_2}(\delta_{\alpha_1} + \tilde{A}_{\alpha_1})(\delta_{\alpha_2} + \tilde{A}_{\alpha_2}) \otimes 1_{2^m} - \frac{1}{2}\Omega_{\alpha_1\alpha_2} \otimes \gamma^{\alpha_1\alpha_2}$ where $\gamma^{\alpha_1\alpha_2} := \frac{1}{2}(\gamma^{\alpha_1}\gamma^{\alpha_2} - \gamma^{\alpha_2}\gamma^{\alpha_1})$, $\Omega_{\alpha_1\alpha_2} := [\delta_{\alpha_1} + \tilde{A}_{\alpha_1}, \delta_{\alpha_2} + \tilde{A}_{\alpha_2}] = L(F_{\alpha_1\alpha_2}) - R(F_{\alpha_1\alpha_2})$ with

$$F_{\alpha_1\alpha_2} := \delta_{\alpha_1}(A_{\alpha_2}) - \delta_{\alpha_2}(A_{\alpha_1}) + [A_{\alpha_1}, A_{\alpha_2}]. \tag{16}$$

In summary,

$$\begin{aligned} \mathcal{D}_A^2 = -\delta^{\alpha_1\alpha_2} \Big(\delta_{\alpha_1} + L(A_{\alpha_1}) - R(A_{\alpha_1}) \Big) \Big(\delta_{\alpha_2} + L(A_{\alpha_2}) - R(A_{\alpha_2}) \Big) \otimes 1_{2^m} \\ -\tfrac{1}{2} \left(L(F_{\alpha_1\alpha_2}) - R(F_{\alpha_1\alpha_2}) \right) \otimes \gamma^{\alpha_1\alpha_2}. \end{aligned} \tag{17}$$

3.2. Spectral action
We first identify the dimension spectrum:

Theorem 3.1. *(i)* $Sd(\mathcal{A}_\Theta, \mathcal{H}, \mathcal{D}) = \{ n - k : n \in \mathbb{N}_0 \}$ *and all poles are simple.*
(ii) $\zeta_D(s) = 2^m \sum_{k \in \mathbb{Z}^n \setminus 0} |k|^{-s} + \dim \text{Ker } \mathcal{D}$. *In particular,* $\zeta_D(0) = 0$.

The reader should notice that

HIC SUNT DRACONES

In fact, (i) *is true when* Θ *is badly approximable.*

Definition 3.2. *(i) Let* $\delta > 0$. *A vector* $a \in \mathbb{R}^n$ *is said to be* $\delta-$*badly approximable if there exists* $c > 0$ *such that* $|q.a - m| \geq c\,|q|^{-\delta}$, $\forall q \in \mathbb{Z}^n \setminus \{ 0 \}$ *and* $\forall m \in \mathbb{Z}$.
We note $\mathcal{BV}(\delta)$ *the set of* $\delta-$*badly approximable vectors and* $\mathcal{BV} := \cup_{\delta>0}\mathcal{BV}(\delta)$ *the set of badly approximable vectors.*
(ii) A matrix $\Theta \in \mathcal{M}_n(\mathbb{R})$ *(real* $n \times n$ *matrices) will be said to said badly approximable if there exists* $u \in \mathbb{Z}^n$ *such that* ${}^t\Theta(u)$ *is a badly approximable vector of* $\mathbb{R}^n$.

Remark 3.3. *A classical result from Diophantine approximation asserts that for all $\delta > n$, the Lebesgue measure of $\mathbb{R}^n \setminus \mathcal{BV}(\delta)$ is zero (i.e almost any element of $\mathbb{R}^n$ is $\delta-$badly approximable.) Let $\Theta \in \mathcal{M}_n(\mathbb{R})$. If its row of index i is a badly approximable vector of $\mathbb{R}^n$ (i.e. if $L_i \in \mathcal{BV}$), then ${}^t\Theta(e_i) \in \mathcal{BV}$ and thus Θ is a badly approximable matrix. It follows that almost any matrix of $\mathcal{M}_n(\mathbb{R}) \approx \mathbb{R}^{n^2}$ is badly approximable.*

This difficulty is due to the presence of J: one must control the holomorphic behavior of few Hurwitz–Epstein Zeta functions.

Theorem 3.4. *Define*

$$f_a : s \in \mathbb{C} \to \sum_{0 \neq k \in \mathbb{Z}^n} \tfrac{P(k)}{||k||^s}\, e^{i2\pi\, k.a}$$

where $a \in \mathbb{R}^n$, $P \in \mathbb{C}[x_1, \dots, x_n]$ is a homogeneous polynomial of degree p and $||k||^2 = \sum_{i=1}^n {k_i}^2$.
(i) When $a \in \mathbb{Z}^n$, f_a has meromorphic extension to the whole complex plane $\mathbb{C}$ and
f_a not entire $\Leftrightarrow \mathrm{Res}_{s=n+p}\, f_a(s) = \int_{u \in S^{n-1}} P(u)\, dS(u) \neq 0$.
(ii) When $a \in \mathbb{R} \setminus \mathbb{Z}^n$, f_a extends holomorphically to the whole complex plane $\mathbb{C}$.
(iii) When Θ is badly approximable, for any integer $q > 0$, the function

$$g(s) := \sum_{l \in (\mathbb{Z}^n)^q} c(l)\, f_{\Theta(\sum_i \epsilon_i l_i)}(s), \qquad \text{with } c(l) \in \mathcal{S}(\mathbb{Z}^n)^q \text{ and } \epsilon_i \in \{1, 0, -1\}.$$

extends meromorphically to $\mathbb{C}$ with only one possible pole at $s = n + p$. If this pole exists, it is simple and

$$\mathrm{Res}_{s=n+p}\, g(s) = c \int_{u \in S^{n-1}} P(u)\, dS(u), \text{ with } c := \sum_{l \in Z} b(l) \text{ where } Z := \{\, l \in (\mathbb{Z}^n)^q : \sum_{i=1}^q \epsilon_i l_i = 0 \,\}.$$

Examples:

$$\mathrm{Res}_{s=0} \sum_{k \in \mathbb{Z}^2} \frac{k_i k_j}{||k||^{s+4}} = \delta_{i,j}\, \pi\,,$$
$$\mathrm{Res}_{s=0} \sum_{k \in \mathbb{Z}^4} \frac{k_i k_j k_l k_m}{||k||^{s+6}} = (\delta_{ij}\delta_{lm} + \delta_{il}\delta_{jm} + \delta_{im}\delta_{jl})\, \tfrac{\pi^2}{12},$$

with other similar results, see [12] for details. This might be helpful for computations in ζ-regularization, multiplicative anomalies or Casimir effect.
The main result is (see (16) for notations)

Theorem 3.5. *Assume $\frac{1}{2\pi}\Theta$ is badly approximable. For any selfadjoint one-form A, the spectral action of the noncommutative torus of dimension n is*
for n=2:

$$\mathcal{S}(\mathcal{D}_A, \Lambda, \Phi) = 4\pi\, \Phi_2\, \Lambda^2 + \mathcal{O}(\Lambda^{-2}),$$

for n=4:

$$\mathcal{S}(\mathcal{D}_A, \Lambda, \Phi) = 8\pi^2\, \Phi_4\, \Lambda^4 - \tfrac{4\pi^2}{3}\, \Phi(0)\, \tau(F_{\mu\nu}F^{\mu\nu}) + \mathcal{O}(\Lambda^{-2}).$$

More generally, $\forall n \geq 1$,

$$\mathcal{S}(\mathcal{D}_A, \Lambda, \phi) = \sum_{k=0}^{n} \Phi_{n-k}\, c_{n-k}(A)\, \Lambda^{n-k} + \mathcal{O}(\Lambda^{-1}) \tag{18}$$

with $c_{n-1}(A) = 0$, $c_{n-k}(A) = 0$ for k odd (n odd $\Rightarrow c_0(A) = 0$.)

Conjecture 3.6. *The ratio* $\big[$*constant term of* $\mathcal{S}(\mathcal{D}_A,\Lambda,\phi)\big]/\big[$*constant term of* $\mathcal{S}(\mathcal{D}+A,\Lambda,\phi)$ *for the commutative torus (i.e.* $\Theta=0$*)*$\big]$ *is independent of* A*. Note, however that,* $\mathcal{D}_A=\mathcal{D}$, $\forall A=A^*$ *in the commutative torus.*

Remark 3.7. *(i) For general spectral triples with simple dimension spectrum,* $⨍$ *can be defined with* $\mathcal{D}$ *or* $\mathcal{D}_A$*:*

$$⨍ P = \mathrm{Res}_{s=0}\ \mathrm{Tr}\,\big(P|D_A|^{-s}\big),\quad \forall P\in\Psi DO.$$

(ii) The top term (cosmological term) is covariance-invariant: $⨍|D_A|^{-n} = ⨍|D|^{-n}$.
(iii) There is no tadpole term in (7) *for the noncommutative torus:* $⨍\widetilde{A}D^{-1}=0$.
(iv) A Diophantine condition was characterized by Connes in [8, Prop. 49] for $n=2$ *in terms of Hochschild cohomology* $H(\mathcal{A}_\Theta,{\mathcal{A}_\Theta}^*)$.

3.3. Beyond Diophantine condition
It is interesting to overcome the Diophantine condition and to consider for instance the case $n=2$, where $\Theta=\theta\big(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\big)$ with $\theta\in\mathbb{R}$.
Let $f:[1,\infty[\to]0,\infty[$ be a continuous function such that $x^2 f(x)$ is non-increasing.

$$\mathcal{K}(f) := \{\,\theta\in\mathbb{R}\,:\,|\theta-\tfrac{p}{q}|<f(q),\ \text{for infinitely many rational numbers } \tfrac{p}{q}\,\}$$

Such θ are termed f-approximable. Note that this is not valid for all rational $\frac{p}{q}$ since $(\theta q)_{q\geq 1}$ are dense in $[0,1]$ when $\theta\notin\mathbb{Q}$.

Lemma 3.8. *Jarnik (1953): For each* f*, there exists an uncountable set* $\mathcal{K}(f)$ *of real numbers* $\frac{1}{2\pi}\theta$*,* f*-approximable but not* $c\,f$*-approximable for any* $0<c<1$.

$\mathcal{K}(f)$ has zero Lebesgue measure if $\sum_{q=1}^{\infty} q\,f(q)$ converges and full Lebesgue measure otherwise.

Consequences: Tuning f, $\exists\, a,\, b\in\mathcal{A}$ such that the correction term

$$\mathrm{Tr}\,\big(L(a)\,R(b)\,e^{-t\mathcal{D}^2}\big) - \big(\text{same term when } \Theta \text{ badly approximable}\big)$$

is not exponentially small, is not $\mathcal{O}(\frac{1}{t})$ like for $\frac{1}{2\pi}\theta\in\mathbb{Q}$, but can be of arbitrary order!
This could have consequences on computations in field theories over noncommutative tori, see [18] for details.
Naturally, all these complications are related to the compactness of the torus, so it is natural to consider the

4. Extension to noncompact manifolds

Let M be a n-dimensional non compact connected complete Riemannian spin^c manifold with bounded curvature and control of its heat kernel $K_t(.,.)$ like

$$\sup_{p\in M}\int_0^\infty t^k\,e^{-t}K_t(p,p)\,dt<\infty,\quad \forall k>\tfrac{n}{2}-1$$
$$\sup_{p\in M}\int_m^\infty \tfrac{e^{-t}}{\sqrt{t}}\,K_t(p,p)\,dt< c\,m^{-(n-1)/2},\quad \forall m\in[0,1]$$

This hypotheses are valid in the following cases (see [17]):
- M has Ricci curvature bounded from below.
- M has a positive injectivity radius and control of isoperimetric constants of balls of a given radius.

- M has a bounded geometry.
Moreover, M is given a smooth isometric proper action α of $\mathbb{R}^l$.
Example: The Moyal planes where $M = \mathbb{R}^{2n}$ and $\alpha =$ translation. A Moyal multiplication can be defined for any $n \times n$-skew-symmetric matrix Θ by

$$f \star_\hbar g(x) := \int_{\mathbb{R}^{2n}\times\mathbb{R}^{2n}} f(x-\Theta u)\, g(x-v)\, e^{-i\, u\cdot v}\, du\, dv.$$

Theorem 4.1. *[17] When $f \in C_c^\infty(M)$, $⨍ L_f |\mathcal{D}|^{-d} = ⨍ M_f |\mathcal{D}|^{-d} = c \int_M f$.*

Remarks:
- When M is compact, α being proper must be periodic.
- When M is non compact and as above, $\mathcal{D}$ has a continuous spectrum but for any $f \in C_c^\infty(M)$, and any $p \geq 2$, $L_f(1+|\mathcal{D}|)^{-2k} \in \mathcal{L}^p(\mathcal{H})$, $\forall k > \frac{n}{2p}$. This particularly fit our purpose since for noncompact spectral triple, we must have by hypothesis

$$a\,(\mathcal{D}-\lambda)^{-1} \text{ is compact } \forall a \in \mathcal{A}. \tag{19}$$

This is indeed the case for Moyal planes:

Theorem 4.2. *[15] Moyal planes are spectral triples.*

To compute a spectral action in the noncompact case, one can use (19) for instance with some spatial localization: Given $\rho \in \mathcal{A}$, the action is defined by

$$\mathcal{S}(\mathcal{D}, \Lambda, \Phi, \rho) := \mathrm{Tr}\,\big(\rho\, \Phi(\mathcal{D}/\Lambda)\big). \tag{20}$$

In this setting,

Theorem 4.3. *[16] $\mathcal{S}(\mathcal{D}, \Lambda, \Phi, \rho)$ has same coefficients as in (18) with replacement $c_k(A) \leftrightarrow \int_{\mathbb{R}^{n/2}} \rho(x)\, c_k(A)(x)\, dx$.*

Remark 4.4. *Definition (20) is not satisfactory since there are too many choices. This question has been investigated in [5] where a dilaton field ϕ is used and the squared Dirac operator $\mathcal{D}^2$ is replaced by $e^{-\phi}\, \mathcal{D}^2\, e^{-\phi}$, $\phi = \phi^* \in Z(\mathcal{A})$. Thus in spectral action, the counting of eigenvalues $N(\Lambda) := \dim\{\, \mathcal{D}^2 \leq \Lambda^2 \,\}$ is replaced by $N(\rho) := \dim\{\, \mathcal{D}^2 \leq \rho^2 \,\}$ where $\rho = e^\phi$.*
A general framework for a good definition of spectral action in the noncompact setting is still lacking.

Acknowledgments
It is a pleasure to thank Driss Essouabri, Victor Gayral, Cyril Levy and Andrzej Sitarz and Joe Várilly for collaborations on which this review is based.

References

[1] A. L. Carey, J. Phillips, A. Rennie and F. A. Sukochev 2006 The local index formula in semifinite von Neumann algebras I: Spectral flow, *Advances in Math.* **202** 415–516.
[2] L. Carminati, B. Iochum and T. Schücker 1999 Noncommutative Yang-Mills and noncommutative relativity: a bridge over troubled water, *Eur. Phys. J.* **C 8** 697–709.
[3] A. Chamseddine and A. Connes 1997 The spectral action principle, *Commun. Math. Phys.* **186** 731–750.
[4] A. Chamseddine and A. Connes 2006 Inner fluctuations of the spectral action, [arXiv:hep-th/0605011].
[5] A. Chamseddine and A. Connes 2006 Scale invariance in the spectral action, *J. Math. Phys.* **47** (2006) 063504.
[6] A. Chamseddine, A. Connes and M. Marcolli 2006 Gravity and the standard model with neutrino mixing, arXiv:hep-th/0610241.
[7] A. Connes 1980 C^*-algèbres et géométrie différentielle, *C. R. Acad. Sci. Paris* **290** 599–604.
[8] A. Connes 1985 Noncommutative differential geometry, *Pub. Math. IHÉS* **39** 257–360.

[9] A. Connes 1994 *Noncommutative Geometry*, Academic Press, London and San Diego.

[10] A. Connes 1995 Geometry from the spectral point of view, *Lett. Math. Phys.* **34** 203–238.

[11] A. Connes and H. Moscovici 1995 The local index formula in noncommutative geometry, *Geom. And Funct. Anal.* **5** 174–243.

[12] D. Essouabri, B. Iochum, C. Levy and A. Sitarz 2007 Spectral action on noncommutative torus, arXiv:0704.0564 [hep-th].

[13] R. Estrada, J. M. Gracia-Bondía and J. C. Várilly 1998 On summability of distributions and spectral geometry, *Commun. Math. Phys.* **191** 219–248.

[14] V. Gayral 2005 Heat-kernel approach to UV/IR Mixing on isospectral deformation manifolds, *Ann. H. Poincaré* **6** 991–1023.

[15] V. Gayral, J. M. Gracia-Bondía, B. Iochum and J. C. Várilly 2004 Moyal planes are spectral triples, *Commun. Math. Phys.* **246** 569–623.

[16] V. Gayral and B. Iochum 2005 The spectral action for Moyal plane, *J. Math. Phys.* **46** no. 4, 043503, 17 pp.

[17] V. Gayral, B. Iochum and J. C. Várilly 2006 Dixmier traces on noncompact isospectral deformations, *J. Funct. Anal.* **237** 507–539.

[18] V. Gayral, B. Iochum and D. Vassilevich 2007 Heat kernel and number theory on NC-torus, *Commun. Math. Phys.* to appear, arXiv:hep-th/0607078.

[19] P. B. Gilkey 2004 *Asymptotic Formulae in Spectral Geometry*, Chapman & Hall/CRC, Boca Raton, FL.

[20] A. de Goursac, J.-C. Wallet and R. Wulkenhaar 2007 Noncommutative induced gauge theory, arXiv:hep-th/0703075.

[21] J. M. Gracia-Bondía, J. C. Várilly and H. Figueroa 2001 *Elements of Noncommutative Geometry*, Birkhäuser Advanced Texts, Birkhäuser, Boston.

[22] M. Knecht and T. Schücker 2006 Spectral action and big desert, *Phys. Lett.* B **640** 272-277

[23] R. Nest, E. Vogt and W. Werner, Spectral action and the Connes–Chamseddine model, p. 109-132 in *Noncommutative Geometry and the Standard Model of Elementary Particle Physics*, F. Scheck, H. Upmeier and W. Werner (Eds.) 2002 Lecture Notes in Phys., **596**, Springer, Berlin.

[24] M. A. Rieffel 1981 C^*-algebras associated with irrational rotations, *Pac. J. Math.* **93** 415–429.

[25] A. Strelchenko 2007 Heat kernel of non-minimal gauge field kinetic operators on Moyal plane, *Int. J. Mod. Phys.* A **22** 181–202.

[26] D. V. Vassilevich 2004 Non-commutative heat kernel, *Lett. Math. Phys.* **67** 185–194.

[27] D. V. Vassilevich 2005 Heat kernel, effective action and anomalies in noncommutative theories, *J. High Energy Phys.* JHEP0508(2005)085.

[28] D. V. Vassilevich 2007 Induced Chern–Simons action on noncommutative torus, arXiv:hep-th/0701017.