



HAL
open science

Financial Models with Defaultable Numéraires

Travis Fisher, Sergio Pulido, Johannes Ruf

► **To cite this version:**

Travis Fisher, Sergio Pulido, Johannes Ruf. Financial Models with Defaultable Numéraires. 2016.
hal-01240736v2

HAL Id: hal-01240736

<https://hal.science/hal-01240736v2>

Preprint submitted on 28 Oct 2016 (v2), last revised 8 Aug 2019 (v4)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Financial Models with Defaultable Numéraires*

Travis Fisher[†]

Sergio Pulido[‡]

Johannes Ruf[§]

July 29, 2016

Abstract

Financial models are studied where each asset may potentially lose value relative to any other. To this end, the paradigm of a pre-determined numéraire is abandoned in favour of a symmetrical point of view where all assets have equal priority. This approach yields novel versions of the Fundamental Theorems of Asset Pricing, which clarify and extend non-classical pricing formulas used in the financial community. Furthermore, conditioning on non-devaluation, each asset can serve as proper numéraire and a classical no-arbitrage condition can be formulated. It is shown when and how these local conditions can be aggregated to a global no-arbitrage condition.

Keywords: Defaultable numéraire; Devaluation; Fundamental Theorem of Asset Pricing; Non-classical pricing formulas.

AMS Subject Classification (2010): 60G48, 60H99, 91B24, 91B25, 91B70, 91G20, 91G40.

JEL Classification: D53, G13.

1 Introduction

Classical models of financial markets are built of a family of stochastic processes describing the random dynamics throughout time of the underlying assets' prices in units of a pre-specified numéraire. Such a numéraire, often also interpreted as a money market account, is an asset that cannot devalue. In this paper we cover the case when there are multiple financial assets, any of which may potentially lose all value relative to the others. Thus, none of these assets can serve as a proper numéraire. We shift away from having a pre-determined numéraire to a more symmetrical point of view that does not prioritize any of the assets. The symmetry not only improves the aesthetics of the no-arbitrage theory, but also clarifies non-classical pricing formulas for contingent claims written on these assets.

Pricing models for contingent claims that allow for the devaluation of the underlying assets are ample. For example, they appear naturally in credit risk. In the terminology introduced by [Schönbucher \(2003\)](#),

*We thank Monique Jeanblanc, Martin Larsson, Martin Schweizer, and Hideyuki Takada for helpful discussions. The research of S.P. benefited from the support of the Chair Markets in Transition (Fédération Bancaire Française) and the project ANR 11-LABX-0019. The research of S.P. has also received funding from the European Research Council under the European Union's Seventh Framework Programme (FP/2007-2013) / ERC Grant Agreement n. 307465-POLYTE. J.R. acknowledges generous support from the Oxford-Man Institute of Quantitative Finance, University of Oxford. The views represented herein are the authors' own views and do not necessarily represent the views of Morgan Stanley or its affiliates and are not a product of Morgan Stanley research.

[†]No affiliation, E-mail: traviswfisher@gmail.com

[‡]Laboratoire de Mathématiques et Modélisation d'Évry (LaMME), Université d'Évry-Val-d'Essonne, ENSIIE, UMR CNRS 8071, IBGBI 23 Boulevard de France, 91037 Évry Cedex, France, Email: sergio.pulidonino@ensiie.fr

[§]Department of Mathematics, University College London, Gower Street, London WC1E 6BT, United Kingdom, E-mail: j.ruf@ucl.ac.uk

2004) such assets are called *defaultable numéraires*.¹ Jarrow and Yu (2001), Collin-Dufresne et al. (2004), and Jamshidian (2004) are further examples of this literature. Financial models for foreign exchange yield another source of assets that might devalue due to the possibility of hyperinflation occurring; see, for example, Câmara and Heston (2008), Carr et al. (2014), and Kardaras (2015).

Another class of models that has drawn much attention involves strict local martingale dynamics for the asset price processes; see, for example, Sin (1998) and Heston et al. (2007). Often such models are particularly chosen as they can be interpreted as bubbles (Protter (2013)) or they are easily analytically tractable (Hulley and Platen (2012), Carr et al. (2013)). Both practitioners (Lewis (2000), Paulot (2013)) and academics (Cox and Hobson (2005), Madan and Yor (2006)) suggest non-classical pricing formulas for contingent claims in such models in order to be consistent with market prices. In this paper, we argue that strict local martingale dynamics are consistent with the interpretation that the corresponding numéraire devaluates. This point of view then allows us to interpret the correction term in the pricing formula of Lewis (2000) as the value of the contingent claim’s payoff in the scenarios where the numéraire devaluates. Thus, the pricing formulas of Lewis (2000), Madan and Yor (2006), Paulot (2013), or Kardaras (2015) arise as special cases of this paper’s framework.

This paper’s contributions can now be summarized in three points:

1. It provides a formulation of the First and Second Fundamental Theorem of Asset Pricing and of the superreplication duality in the case that any asset may devalue with respect to any other. The formulation is symmetric in the sense that none of the assets is prioritized.
2. It provides an interpretation of strict local martingale models, which can arise by fixing a numéraire that has positive probability to default. Non-classical pricing formulas, restoring put-call parity, can then be economically justified and extended.
3. Assume, for the moment, that for each asset there exists a probability measure under which the discounted prices (with the corresponding asset as numéraire) are local martingales (or, even, supermartingales). These measures need not be equivalent. By introducing the notion of *numéraire-consistency*, this paper shows when these measures can be *aggregated* to an arbitrage-free pricing operator that takes all events of devaluations into account.

In Section 2, we introduce the framework. We consider a model for d assets. We have in mind a foreign exchange market, and hence call these assets “currencies,” but really these could represent any asset of non-negative value. We denote the value of one unit of the j -th currency, measured in terms of the i -th currency, as $S_{i,j}$. We model the full matrix $(S_{i,j})_{i,j}$ of these exchange rates. This is redundant, but convenient, because the *matrix of exchange rates* is precisely the concept that gives symmetry to our results. If the j -th currency has devaluated with respect to the i -th currency at time t we have $S_{i,j}(t) = 0$ and $S_{j,i}(t) = \infty$. In this case, the j -th currency cannot be used as a numéraire, and the standard results of mathematical finance in units of this currency do not apply. Nevertheless, considering all currencies simultaneously shall allow us to derive Fundamental Theorems of Asset Pricing with a symmetric formulation.

In Section 3, these versions of the Fundamental Theorems of Asset Pricing are stated and the corresponding superreplication duality is derived. These results cleanly and symmetrically cover the case when there are multiple financial assets, any of which may potentially lose all value relative to the others. The First Fundamental Theorem states that the symmetric version of the condition of *No Free Lunch with Vanishing Risk for allowable trading strategies* holds if and only if there is a *martingale valuation operator*. Hence,

¹The term “defaultable numéraire” sometimes appears in the credit risk literature with a different meaning, namely to describe assets with strictly positive but not adapted price processes; for example, Bielecki et al. (2004) use this definition; see also Brigo (2005). Here, however, we follow the convention of Schönbucher (2003, 2004), where a defaultable numéraire is an asset whose price has positive probability to become zero.

in this framework, the dual objects are no longer local martingale measures for the prices quoted in terms of the pre-specified numéraire, but martingale valuation operators. These operators, which are defined in an axiomatic and economically meaningful way, provide in a *vectorized fashion* the prices of contingent claims quoted in terms of all the currencies.

In Section 4, martingale valuation operators are related to families of *numéraire-consistent* probability measures. Each of these measures corresponds, in a certain sense, to fixing a specific currency as the underlying numéraire. We call *disaggregation* the step that constructs this family of *numéraire-consistent* probability measures from a martingale valuation operator. We call *aggregation* the reverse step, namely taking a possibly non-equivalent family of probability measures, corresponding to the different currencies as numéraires, and constructing a martingale valuation operator from it. Embedding a strict local martingale model in a family of numéraire-consistent probability measures and then aggregating this family to a martingale valuation operator yields the non-classical pricing formulas of Lewis (2000), Madan and Yor (2006), Paulot (2013), and Carr et al. (2014). This point of view has two advantages. First of all, it yields generic pricing formulas for any kind of contingent claim. These formulas are consistent with the above-mentioned non-classical pricing formulas, which are usually only provided for specific claims. Secondly, it gives an economic interpretation to the lack of martingale property as the possibility of a default of the underlying numéraire.

Section 5 contains the proofs of the main results. The symmetric approach, insisting in quoting prices in terms of the primitive underlying assets and not giving priority to any of them, leads in a natural way to consider the basket asset – the portfolio consisting of one unit of each currency – as a proper numéraire. The proofs of the main results are based on this observation – see also Delbaen and Shirakawa (1996) and, most importantly, Yan (1998).

We point out the recent work of Tehranchi (2014), who considers an economy where prices quoted in terms of a given non-traded currency are not necessarily positive. Relative prices between the assets are not studied. Instead, Tehranchi (2014) focuses on different arbitrage concepts taking into consideration that the agent might not be able to substitute today’s consumption by tomorrow’s consumption.

Notation

Throughout the paper we fix a deterministic time horizon $T > 0$ and consider an economy with $d \in \mathbb{N}$ traded assets, called “currencies.” To reduce notation, we shall use the generic letter t for time and abstain from using the qualifier “ $\in [0, T]$.” We shall also use the generic letters i, j, k for the currencies and again abstain from using the qualifier “ $\in \{1, \dots, d\}$.” For example, we shall write “ \sum_i ” to denote “ $\sum_{i=1}^d$.” When introducing a process $X = (X(t))_{t \in [0, T]}$, we usually omit “ $= (X(t))_{t \in [0, T]}$.” If $v \in \mathbb{R}^d$, we understand inequalities of the form $v \geq 0$ componentwise. For a matrix $\Gamma \in \mathbb{R}^{d \times d}$, we shall denote by Γ_i the i -th row of Γ . Moreover, we use the convention $\inf \emptyset = \infty$ and we denote the cardinality of a countable set A by $|A|$. Furthermore, we emphasize that a product xy of two numbers $x, y \in [0, \infty]$ is always defined except if either (a) $x = 0$ and $y = \infty$ or (b) $x = \infty$ and $y = 0$.

We fix a filtered space $(\Omega, \mathcal{F}(T), (\mathcal{F}(t))_t)$, where the filtration $(\mathcal{F}(t))_t$ is assumed to be right-continuous and $\mathcal{F}(0)$ to be trivial. In the absence of a probability measure, all statements involving random variables or events are supposed to hold pathwise for all $\omega \in \Omega$. For an event $A \in \mathcal{F}(T)$, we set $\mathbf{1}_A(\omega) \times \infty$ and $\mathbf{1}_A(\omega) \times (-\infty)$ to ∞ and $-\infty$, respectively, for all $\omega \in A$ and to 0 for all $\omega \notin A$. Let us now consider a probability measure \mathbb{Q} on $(\Omega, \mathcal{F}(T))$. We write $\mathbb{E}^{\mathbb{Q}}$ for the corresponding expectation operator and $\mathbb{E}_t^{\mathbb{Q}}$ for the conditional expectation operator, given $\mathcal{F}(t)$, for each t . If $Y = (Y_i)_i$ is a d -dimensional process we say that Y is a \mathbb{Q} -(semi / super) martingale if Y_i is a \mathbb{Q} -(semi / super) martingale for each i . For a real-valued

semimartingale X with $X(0) = 0$ we write $\mathcal{E}(X)$ to denote its stochastic exponential; that is,

$$\mathcal{E}(X) = e^{X - [X]^c/2} \prod_{s \leq \cdot} (1 + \Delta X_s) e^{-\Delta X_s},$$

where $\Delta X = X - X_-$ and $[X]^c$ denotes the continuous part of the quadratic variation of X .

2 Framework

This section introduces the concept of *exchange matrices* to represent prices of the underlying currencies and the related concept of *value vectors* to describe prices of contingent claims with the currencies as underlying. Then, in Subsection 2.2, we define trading strategies and the no-arbitrage condition of *No Free Lunch with Vanishing Risk*. This is straightforward but necessary since we have not assumed that any currency is a proper numéraire. In Subsection 2.3, we define *martingale valuation operators*, which will play the role of risk-neutral probability measures. Subsection 2.4 puts the concepts of this section in the context of the market with the basket as numéraire.

2.1 Exchange matrices and value vectors

We put ourselves in an economy that is characterized by the price processes of d currencies relative to each other via an $[0, \infty]^{d \times d}$ -valued, right-continuous, $(\mathcal{F}(t))_t$ -adapted process $S = (S_{i,j})_{i,j}$. Here, the process $S_{i,j}$ denotes the price process of the j -th currency in units of the i -th currency. We also refer to [Večeř \(2011\)](#), where a similar point of view is taken. In order to simplify the analysis below we assume that interest rates are zero. Alternatively, we might interpret $S_{i,j}(t)$ as the price of one unit of the j -th money market in terms of units of the i -th money market at time t , for each i, j , and t .

In order to provide an economic meaning to the matrix-valued process S we shall assume that it satisfies certain consistency conditions. Formally, we assume that $S(t)$ is an exchange matrix for each t , in the sense of the following definition:

Definition 2.1 (Exchange matrix). An exchange matrix is a $d \times d$ -dimensional matrix $s = (s_{i,j})_{i,j}$ taking values in $[0, \infty]^{d \times d}$ with the property that $s_{i,i} = 1$ and $s_{i,j}s_{j,k} = s_{i,k}$ for all i, j, k , whenever the product is defined. \square

Note that the definition implies, in particular, that an exchange matrix s also satisfies that $s_{i,j} = 0$ if and only if $s_{j,i} = \infty$ for all i, j . The consistency conditions of [Definition 2.1](#) guarantee the following: for fixed i, j, k , an investor who wants to exchange units of the i -th currency into units of the k -th currency is indifferent between exchanging directly $s_{i,k}$ units of the i -th currency into the k -th currency or, instead, going the indirect way and first exchanging the appropriate amount of units of the i -th currency into the j -th currency and then exchanging those units into the k -th currency.

As long as no asset has defaulted, that is, as long as all entries in an exchange matrix s are strictly positive, s is said to have the *triangle property*; see, for example, [Barrett \(1979\)](#). The associated properties of such matrices, however, will not be further relevant for us.

For each t , we define the index set of “active currencies”

$$\mathfrak{A}(t) = \left\{ i : \sum_j S_{i,j}(t) < \infty \right\}.$$

If $i \in \mathfrak{A}(t)$ for some t then the i -th currency is not devaluated against any other currency. Note that $S_{i,j}(t) = 0$ for all $i \in \mathfrak{A}(t)$ and $j \notin \mathfrak{A}(t)$, for each t . To wit, if a currency is devaluated with respect

to another “active” currency, the consistency conditions of Definition 2.1 guarantee that that currency is also devaluated with respect to any other “active” currency. For sake of notational simplicity only, we shall assume that $\mathfrak{A}(0) = \{1, \dots, d\}$; that is, at time 0 no currency is devaluated.

Remark 2.2 (Existence of a strong currency). We always have $\mathfrak{A}(t) \neq \emptyset$ for each t . More precisely, if s is an exchange matrix, there exists i such that $s_{i,j} \leq 1$ for all j . To see this, we define, on the set of indices $\{1, \dots, d\}$, a total preorder as follows: $j \preceq k$ if and only if $s_{j,k} \geq 1$, that is, if and only if the k -th currency is “stronger” than the j -th currency. The consistency conditions of Definition 2.1 guarantee that this is a total preorder. Since the set of indices is finite, there exists a (not necessarily unique) maximal index i corresponding to the “strongest” currency. For such an index i we have $s_{i,j} \leq 1$ for all j . \square

We are interested in additional assets in the economy besides the d currencies and in their relative valuation with respect to those currencies. Towards this end, we introduce the notion of value vector:

Definition 2.3 (Value vector for exchange matrix). A value vector for an exchange matrix s is a d -dimensional vector $v = (v_i)_i$ taking values in $[-\infty, \infty]^d$ with the property that $s_{i,j}v_j = v_i$ for all i, j , whenever the product is defined. \square

A value vector encodes the price of an asset in terms of the d currencies. More precisely, the i -th component describes how many units of the i -th currency are required to obtain one unit of that specific asset. The consistency condition in Definition 2.3 guarantees again that an investor who wants a unit of the new asset does not prefer to first exchange her currency into another one in order to obtain that asset.

Remark 2.4 (Value vectors exist and are essentially unique). If s is an exchange matrix, j is a non-devaluated currency, that is, $\sum_i s_{j,i} < \infty$, and $\hat{v} \in [-\infty, \infty] \setminus \{0\}$ denotes the price of an asset in terms of the j -th currency then there exists always a unique value vector $v \in [-\infty, \infty]^d$ with $v_j = \hat{v}$. Indeed, we may always set $v_i = s_{i,j}\hat{v}$ for all i . If $\hat{v} = 0$ then we could set $v_i = 0$ for all i and note that there might exist other value vectors \tilde{v} with $\tilde{v}_j = \hat{v}$. \square

We use the following numéraire-independent notation, introduced for each t , for sets of $\mathcal{F}(t)$ -measurable contingent claims:

$$\mathcal{C}^t = \left\{ C : \begin{array}{l} C \text{ is an } \mathcal{F}(t)\text{-measurable value vector for } S(t) \text{ such that} \\ \text{there exists } K > 0 \text{ with } C_i \geq -K \sum_j S_{i,j}(t) \text{ for all } i \end{array} \right\}; \quad (1)$$

$$\mathcal{D}^t = \mathcal{C}^t \cap (-\mathcal{C}^t).$$

Thus, for each t , the set \mathcal{C}^t corresponds to the family of $\mathcal{F}(t)$ -measurable value vectors whose payoff is bounded from below by a multiple of the basket value, uniformly across all scenarios $\omega \in \Omega$. Similarly, for each t , the set \mathcal{D}^t corresponds to the family of $\mathcal{F}(t)$ -measurable value vectors whose payoff is bounded from below and from above by a multiple of the basket value.

For all i we denote by $I^{(i)}(\cdot)$ the value vector corresponding to the value of one unit of the i -th currency at time t in terms of the other currencies:

$$I^{(i)}(\cdot) = (S_{j,i}(\cdot))_j. \quad (2)$$

Remark 2.5 (Examples of value vectors in \mathcal{D}^t). Note that, for each i and t , the value vector $I^{(i)}(t)$, given in (2), belongs to \mathcal{D}^t . In other words, all value vectors associated to the relative prices of the traded currencies belong to \mathcal{D}^t for each t . This implies, for instance, that also the value vectors corresponding to call and put payoffs with maturity t written on these currencies belong to \mathcal{D}^t . \square

2.2 Dynamic trading and the concept of no-arbitrage

We start by introducing some helpful notation. For an \mathbb{R}^d -valued process $h = (h_i)_i$ we let $V^h = (V_i^h)_i$ denote the process given by

$$V_i^h(t) = \sum_j h_j(t) S_{i,j}(t) \quad (3)$$

for all $i \in \mathfrak{A}(t)$ and t . When $i \notin \mathfrak{A}(t)$, by using Remark 2.2, we can define $V_i^h(t)$ as in Remark 2.4. As already pointed out there, $V^h(t)$ is not necessarily the unique value vector such that (3) holds for all $i \in \mathfrak{A}(t)$. Note that if h is progressively measurable then V^h is also progressively measurable. Here, we interpret $h_i(t)$ as the number of units of the currency an investor holds at time t for each i and $V^h(t)$ as the value of the corresponding position, relative to all d currencies, for each t .

We are interested in continuous, *self-financing* trading and the associated wealth process. These concepts require the notation of stochastic integrals which again require an underlying probability measure along with the semimartingale property of the currencies. Towards this end, we now formulate the precise assumption that allows us to connect *self-financing* trading strategies with the associated wealth processes.

Definition 2.6 (Smg). We say that a probability measure \mathbb{P} on $(\Omega, \mathcal{F}(T))$ satisfies (Smg) if there exists a family $(A_i)_i$ of events with $\bigcup_i A_i = \Omega$ such that $\mathbb{P}(A_i) > 0$ and S_i is a (d -dimensional) \mathbb{P}_i -semimartingale for each i , where $\mathbb{P}_i(\cdot) = \mathbb{P}(\cdot | A_i)$; that is \mathbb{P}_i is the probability measure \mathbb{P} , conditioned on the event A_i . \square

Assume for a moment that we are given a probability measure \mathbb{P} that satisfies (Smg). Under the probability measure \mathbb{P}_i the i -th currency does not devalue against any other currency since S_i is a semimartingale and therefore, in particular, \mathbb{R}^d -valued, for each i . Alternatively, the probability measure \mathbb{P}_i satisfies $\mathbb{P}_i(\bigcap_t \{i \in \mathfrak{A}(t)\}) = 1$. Thus, the i -th currency can be used as a numéraire under the probability measure \mathbb{P}_i . Observe also that \mathbb{P}_i is in general only absolutely continuous with respect to \mathbb{P} for each i but \mathbb{P} and $\sum_i \mathbb{P}_i/d$ are equivalent.

The property (Smg) now allows the introduction of the self-financing property in terms of stochastic integration. To this end, for a probability measure \mathbb{Q} and an \mathbb{R}^d -valued \mathbb{Q} -semimartingale X we write $L(X, \mathbb{Q})$ to denote the space of \mathbb{R}^d -valued predictable processes h such that the (vector) stochastic integral $h \cdot_{\mathbb{Q}} X$ is well-defined, \mathbb{Q} -almost surely.

Definition 2.7 (\mathbb{P} -trading strategy and \mathbb{P} -allowable strategy). Assume that a given probability measure \mathbb{P} on $(\Omega, \mathcal{F}(T))$ satisfies (Smg). A predictable \mathbb{R}^d -valued process h is called a \mathbb{P} -trading strategy if $h \in L(S_i, \mathbb{P}_i)$ and the self-financing condition holds, that is, $V_i^h - V_i^h(0) = h \cdot_{\mathbb{P}_i} S_i$, \mathbb{P}_i -almost surely, for each i .

We say that the \mathbb{P} -trading strategy h is \mathbb{P} -allowable if there exists $\delta > 0$ such that $V_i^h(t) \geq -\delta \sum_j S_{i,j}(t)$ for all i and t , \mathbb{P} -almost surely. \square

Remark 2.8 (Allowability and admissibility). We emphasize that the standard setup, see, for instance, [Delbaen and Schachermayer \(1994\)](#), focuses on the notion of \mathbb{P} -admissible strategies instead of \mathbb{P} -allowable strategies. However, the notion of admissibility depends strongly on a choice of numéraire, while the notion of allowability, studied by [Yan \(1998\)](#), treats all currencies equally important, and thus, is more suited for our approach. See also [Ruf \(2013\)](#) for more comments on this topic. \square

We are now ready to provide an important notion of no-arbitrage in the spirit of [Delbaen and Schachermayer \(1994\)](#) and [Yan \(1998\)](#).

Definition 2.9 (NFLVR for \mathbb{P} -allowable strategies). Assume that a given probability measure \mathbb{P} on $(\Omega, \mathcal{F}(T))$ satisfies (Smg). We say that S satisfies No Free Lunch with Vanishing Risk (NFLVR) for \mathbb{P} -allowable strategies if for any sequence of \mathbb{P} -allowable strategies $(h^{(n)})_{n \in \mathbb{N}}$ with $V^{h^{(n)}}(0) \leq 0$ and such that there exists a sequence of \mathbb{P} -almost surely bounded random variables $(\xi^{(n)})_{n \in \mathbb{N}}$ satisfying

$$V_i^{h^{(n)}}(T) \geq \xi^{(n)} \sum_j S_{i,j}(T)$$

for all $i \in \mathfrak{A}(T)$, \mathbb{P} -almost surely, the following conclusion holds. If there exists a random variable $\xi \geq 0$ such that $\lim_{n \uparrow \infty} \text{ess sup } |\xi^{(n)} - \xi| = 0$ then $\mathbb{P}(\xi = 0) = 1$. \square

We now introduce the notion of an *obvious devaluation* and argue afterwards that such an obvious devaluation cannot occur if the exchange process S satisfies (NFLVR).

Definition 2.10 (NOD). We say that a probability measure \mathbb{P} on $(\Omega, \mathcal{F}(T))$ satisfies No Obvious Devaluations (NOD) if $\mathbb{P}(i \in \mathfrak{A}(T) | \mathcal{F}(\tau)) > 0$ on $\{\tau < \infty\} \cap \{i \in \mathfrak{A}(\tau)\}$, \mathbb{P} -almost surely, for all i and stopping times τ . \square

A probability measure \mathbb{P} that satisfies (NOD) guarantees the following. If at any point of time τ a certain currency i has not yet defaulted then the probability is strictly positive that this currency will not default in the future. Carr et al. (2014) study the case $d = 2$ and also introduce the notion of “no obvious hyperinflations,” seemingly different. However, that paper has an additional standing hypothesis, namely that there are no sudden complete devaluations through a jump (see Definition 4.9 below). Under this condition, their notion of “no obvious hyperinflations” and this paper’s notion (NOD) agree.

Proposition 2.11 ((NOD) holds under no-arbitrage). *If a probability measure \mathbb{P} on $(\Omega, \mathcal{F}(T))$ satisfies (Smg) and S satisfies (NFLVR) for \mathbb{P} -allowable strategies then \mathbb{P} satisfies (NOD).*

Proof. Assume that \mathbb{P} satisfies (Smg) and suppose that there exists i and a stopping time τ such that $\mathbb{P}(i \in \mathfrak{A}(T) | \mathcal{F}(T \wedge \tau)) = 0$ on $\{\tau < \infty\} \cap \{i \in \mathfrak{A}(\tau)\}$ and $\mathbb{P}(\{\tau < \infty\} \cap \{i \in \mathfrak{A}(\tau)\}) > 0$. To wit, at time τ , if the i -th currency has not devaluated, it is sure that it will completely devalue at time T . Consider now the \mathbb{P} -trading strategy h that sells the i -th currency at time τ if this currency is active at that time; that is,

$$h_i = - \frac{\sum_{k \neq i} S_{i,k}(\tau)}{\sum_k S_{i,k}(\tau)} \mathbf{1}_{\tau, \infty} \mathbf{1}_{\{\tau < \infty\} \cap \{i \in \mathfrak{A}(\tau)\}};$$

$$h_j = \frac{1}{\sum_k S_{i,k}(\tau)} \mathbf{1}_{\tau, \infty} \mathbf{1}_{\{\tau < \infty\} \cap \{i \in \mathfrak{A}(\tau)\}} \text{ for all } j \neq i.$$

Clearly, h is \mathbb{P} -allowable and yields a free lunch with vanishing risk in the sense of Definition 2.9. This observation then yields the statement. \square

2.3 Martingale valuation operators

We would like to derive a Fundamental Theorem of Asset Pricing, but, in general, none of the d currencies can serve as a proper numéraire as each currency might completely devalue. To avoid such problems we replace the concept of equivalent local martingale measure with the notion of a martingale valuation operator, in the spirit of Harrison and Pliska (1981) and Biagini and Cont (2006).

Definition 2.12 (Martingale valuation operator). We say that a family of operators $\mathbb{V} = (\mathbb{V}^{r,t})_{r \leq t}$, with

$$\mathbb{V}^{r,t} : \mathcal{D}^t \rightarrow \mathcal{D}^r,$$

is a martingale valuation operator (with respect to S) if the following conditions hold.

- (a) (Positivity) If $C \in \mathcal{D}^T$ and $C \geq 0$ then $\mathbb{V}^{0,T}(C) \geq 0$.
- (b) (Linearity) If H is a bounded $\mathcal{F}(r)$ -measurable random variable and $C, \tilde{C} \in \mathcal{D}^t$ then

$$\mathbb{V}^{r,t} \left(H \mathbf{1}_{\{H \neq 0\}} C + \tilde{C} \right) = H \mathbf{1}_{\{H \neq 0\}} \mathbb{V}^{r,t}(C) + \mathbb{V}^{r,t}(\tilde{C}) \quad (4)$$

for all $r \leq t$, whenever the sums are well-defined.

(c) (Continuity From Below) If $(C^{(n)})_{n \in \mathbb{N}} \subset \mathcal{D}^T$ is a nondecreasing sequence of nonnegative value vectors that converge (path- and componentwise) to a value vector $C \in \mathcal{D}^T$, then $\mathbb{V}^{0,T}(C^{(n)})$ converges to $\mathbb{V}^{0,T}(C)$, as n increases to infinity.

(d) (Time Consistency) For all $r \leq t$ and $C \in \mathcal{D}^T$,

$$\mathbb{V}^{r,t}(\mathbb{V}^{t,T}(C)) = \mathbb{V}^{r,T}(C).$$

(e) (Martingale Property) For all i and t , we have

$$\mathbb{V}^{t,T}(I^{(i)}(T)) = I^{(i)}(t) \mathbf{1}_{\{i \in \mathfrak{A}(t)\}}, \quad (5)$$

with $I^{(i)}$ as in (2).

(f) (Redundancy) For all $r \leq t$ and $C \in \mathcal{D}^t$ with $\sum_i \mathbf{1}_{\{C_i=0\}} > 0$, we have $\mathbb{V}^{r,t}(C) = 0$.

We denote the projection of $\mathbb{V}^{r,t}$ on its i -th component by $\mathbb{V}_i^{r,t}$ for all i . □

Suppose there exists a family of probability measures $(\mathbb{Q}_i)_i$ such that S_i is a \mathbb{Q}_i -martingale for each i . Under certain consistency conditions, given in Definition 4.1 below, a martingale valuation operator \mathbb{V} can then be defined by

$$\mathbb{V}_i^{r,t}(C) = \mathbb{E}_r^{\mathbb{Q}_i}[C_i] \quad (6)$$

for all $C \in \mathcal{D}^t$, i , and $r \leq t$. Vice versa, the results in Section 4 below yield that any martingale valuation operator has a representation similar to (6); however, for a given i , S_i is not necessarily a \mathbb{Q}_i -martingale, in which case a correction term is added to the right-hand side of (6).

The properties of *Positivity* and *Linearity* reflect the corresponding properties of the expectation operator. The indicator appearing in (4) resolves possible conflicts when multiplying zero and infinity; see also the section on notation above. Such a conflict appears whenever, for some scenario $\omega \in \Omega$, some currency has completely devaluated, the contingent claim's payoff $C(\omega)$ is not zero when measured in a strong currency, and $H(\omega) = 0$. *Continuity From Below* corresponds to the monotone convergence theorem and arises from the fact that the family of set functions $(\mathbb{Q}_i)_i$ is not only finitely but also countably additive. *Time Consistency* corresponds to the tower property for conditional expectations. *Martingale Property* reflects the fact that S_i is a \mathbb{Q}_i -martingale for all i if the representation in (6) without a correction term holds. The indicator in (5) is motivated by Remark 2.4. Indeed, if for some i and t the i -th currency has already completely devaluated at time t then its value, measured in terms of a active currency $j \in \mathfrak{A}(t)$, equals zero. The indicator now takes care of the uniqueness issue raised in Remark 2.4 and forces the corresponding value vector to be zero in each component. *Redundancy* assures that an asset that has zero value with respect to some currency in each possible scenario has to have value zero at any earlier time.

As the following remark discusses, *Redundancy* implies in particular that all assets whose values agree on the active currencies have the same value under a martingale valuation operator.

Remark 2.13 (Valuation of essentially equal value vectors). Any martingale valuation operator \mathbb{V} satisfies $\mathbb{V}^{r,t}(C) = \mathbb{V}^{r,t}(\tilde{C})$ whenever $C, \tilde{C} \in \mathcal{D}^t$ and $C_i = \tilde{C}_i$ for all $i \in \mathfrak{A}(t)$ and $r \leq t$. Indeed, in this case either $C = \tilde{C}$ or $C_i = 0 = \tilde{C}_i$ for all $i \in \mathfrak{A}(t)$. Therefore,

$$\begin{aligned} \mathbb{V}^{r,t}(C) &= \mathbb{V}^{r,t}(C \mathbf{1}_{\{C=\tilde{C}\}} + C \mathbf{1}_{\{C \neq \tilde{C}\}}) = \mathbb{V}^{r,t}(\tilde{C} \mathbf{1}_{\{C=\tilde{C}\}}) + \mathbb{V}^{r,t}(C \mathbf{1}_{\{C \neq \tilde{C}\}}) \\ &= \mathbb{V}^{r,t}(\tilde{C} \mathbf{1}_{\{C=\tilde{C}\}}) = \mathbb{V}^{r,t}(\tilde{C}), \end{aligned}$$

by *Linearity* and *Redundancy* of $\mathbb{V}^{r,t}$. □

The following definitions extend the concept of equivalence of probability measures and of almost-sure statements.

Definition 2.14 (Equivalence between martingale valuation operators and probability measures). We say that two martingale valuation operators \mathbb{V} and $\tilde{\mathbb{V}}$ are equivalent and write $\mathbb{V} \sim \tilde{\mathbb{V}}$ if the following equivalence holds for any nonnegative $C \in \mathcal{D}^T$: we have $\mathbb{V}^{0,T}(C) = 0$ if and only if $\tilde{\mathbb{V}}^{0,T}(C) = 0$.

Analogously, we say that a martingale valuation operator \mathbb{V} and a probability measure \mathbb{P} are equivalent and write $\mathbb{P} \sim \mathbb{V}$ or $\mathbb{V} \sim \mathbb{P}$ if the following equivalence holds for any nonnegative $C \in \mathcal{D}^T$: we have $\mathbb{V}^{0,T}(C) = 0$ if and only if $\sum_i \mathbf{1}_{\{C_i=0\}} > 0$, \mathbb{P} -almost surely. \square

Remark 2.15 (Transitivity of equivalence). Let \mathbb{P} and $\hat{\mathbb{P}}$ denote two probability measures and let \mathbb{V} and $\tilde{\mathbb{V}}$ denote two martingale valuation operators. Then $\mathbb{P} \sim \mathbb{V}$ in conjunction with $\mathbb{P} \sim \tilde{\mathbb{V}}$ implies $\mathbb{V} \sim \tilde{\mathbb{V}}$; moreover, $\mathbb{P} \sim \mathbb{V}$ in conjunction with $\hat{\mathbb{P}} \sim \mathbb{V}$ implies $\mathbb{P} \sim \hat{\mathbb{P}}$; and also $\mathbb{P} \sim \hat{\mathbb{P}}$ in conjunction with $\mathbb{P} \sim \mathbb{V}$ implies $\hat{\mathbb{P}} \sim \mathbb{V}$. \square

Definition 2.16 (\mathbb{V} -almost surely). Suppose that \mathbb{V} is a martingale valuation operator. We say that an event A holds \mathbb{V} -almost surely if the contingent claim $C = \mathbf{1}_{\Omega \setminus A} \sum_i I^{(i)}(T)$ satisfies $\mathbb{V}^{0,T}(C) = 0$. \square

To wit, two contingent claims C and \tilde{C} are \mathbb{V} -almost surely equal if the contingent claim \hat{C} , which pays one unit of each currency in the case that the two contingent claims C and \tilde{C} differ, has zero valuation under \mathbb{V} . Moreover if $\mathbb{P} \sim \mathbb{V}$ then an event holds \mathbb{V} -almost surely if and only if it holds \mathbb{P} -almost surely.

To discuss the concept of superreplication below in full generality we make the following observation.

Lemma 2.17 (Extending the domain of a martingale valuation operator). *Fix $r \leq t$ and $C \in \mathcal{C}^t$. Then there exists a nondecreasing sequence $(C^{(n)})_{n \in \mathbb{N}} \subset \mathcal{D}^t$ with $\lim_{n \uparrow \infty} C^{(n)} = C$. Moreover, the limit $\mathbb{V}^{r,t}(C) = \lim_{n \uparrow \infty} \mathbb{V}^{r,t}(C^{(n)})$ exists and is well-defined in the following sense. If $(\tilde{C}^{(n)})_{n \in \mathbb{N}} \subset \mathcal{D}^t$ is another nondecreasing sequence with $\lim_{n \uparrow \infty} \tilde{C}^{(n)} = C$, then $\lim_{n \uparrow \infty} \mathbb{V}^{r,t}(\tilde{C}^{(n)}) = \mathbb{V}^{r,t}(C)$. Thus, $\mathbb{V}^{r,t}$ can be extended to the unique mapping $\mathcal{C}^t \rightarrow \mathcal{C}^r$ such that the family $(\mathbb{V}^{r,t})_{r \leq t}$ satisfies Definition 2.12 with \mathcal{D}^t replaced by \mathcal{C}^t .*

Proof. The first statement is clear. The remaining statements follow directly from Proposition 2.20 below. \square

2.4 Dynamic trading and no-arbitrage in the basket market

In this subsection we consider the financial market where prices are quoted in terms of the basket asset – the portfolio consisting of one unit of each currency. This will clarify the previously introduced concepts of value vectors, dynamic trading, no-arbitrage, and martingale valuation operators.

The value of the basket in terms of the i -th asset equals exactly $\sum_j S_{i,j}$ and the value of the i -th asset in terms of the basket equals its reciprocal. This simple observation allows us to express the relative price process $\bar{S} = (\bar{S}_i)_i$ with respect to the basket numéraire as

$$\bar{S}_i = \frac{1}{\sum_k S_{i,k}}. \quad (7)$$

Observe that in (7) there are no divisions by zero because $\sum_k S_{i,k} \geq S_{i,i} = 1$; hence, $0 \leq \bar{S}_i \leq 1$ for all i . Vice versa, note that we get

$$S_{i,j} = \frac{\bar{S}_j}{\bar{S}_i}, \quad (8)$$

whenever the fraction is well-defined, for all i, j . Thanks to (8), with $i^* \in \mathfrak{A}(t)$, we get

$$\sum_j \bar{S}_j(t) = \sum_{j \in \mathfrak{A}(t)} \frac{S_{i^*,j}(t) \bar{S}_j(t)}{S_{i^*,j}(t)} = \bar{S}_{i^*}(t) \sum_{j \in \mathfrak{A}(t)} S_{i^*,j}(t) = 1$$

for all t ; hence, in the basket market an asset representing a risk-free bond can be replicated.

Next, for any t , given a value vector C with respect to exchange matrix $S(t)$, we define

$$\bar{C} = \frac{1}{|\mathfrak{A}(t)|} \sum_i \bar{S}_i(t) C_i \mathbf{1}_{\{i \in \mathfrak{A}(t)\}}. \quad (9)$$

Since $0 < \bar{S}_i(t) \leq 1$ on $\{i \in \mathfrak{A}(t)\}$ all the multiplications in (9) are well-defined. To understand the definition in (9) better, first note that inserting the value vector $C = I^{(j)}$ of (2) leads exactly to $\bar{C} = \bar{S}_j$, for each j .² Indeed, the formula in (9) transforms, for each active currency $i \in \mathfrak{A}(t)$ the payoff C_i into a payoff measured with respect to the basket, by dividing with the value of the basket. All of the $|\mathfrak{A}(t)|$ values are identical, thus summing them up and dividing by $|\mathfrak{A}(t)|$ does not modify the value, which can be interpreted as the payoff of the corresponding contingent claim, measured in terms of the basket.

We have now translated the setup of this section into a classical setup, with \bar{S} denoting a vector-valued price process, measured in terms of a basket as in Yan (1998), and \bar{C} denoting a one-dimensional random variable, representing the payoff of a contingent claim, again measured in terms of the basket.

The following lemmas contain the main observations regarding dynamic trading in the financial market with price process \bar{S} defined in (7). These lemmas require rigorous arguments. The proofs are included in Appendix A. We recall that for a probability measure \mathbb{Q} and an \mathbb{R}^d -valued \mathbb{Q} -semimartingale X , the notation $L(X, \mathbb{Q})$ denotes the space of \mathbb{R}^d -valued predictable processes h such that the (vector) stochastic integral $h \cdot_{\mathbb{Q}} X$ is well-defined, \mathbb{Q} -almost surely. The next lemma shows that if \mathbb{P} satisfies the property (Smg) of Definition 2.6, then the process \bar{S} is a semimartingale with respect to \mathbb{P} .

Lemma 2.18 (The semimartingale property of the basket). *Suppose that \mathbb{P} satisfies (Smg). Then \bar{S} is a \mathbb{P} - and a \mathbb{P}_k -semimartingale for each k . Moreover, we have $L(\bar{S}, \mathbb{P}) = \bigcap_i L(\bar{S}, \mathbb{P}_i)$, and if $h \in L(\bar{S}, \mathbb{P})$ then $h \cdot_{\mathbb{P}} \bar{S} = h \cdot_{\mathbb{P}_i} \bar{S}$, \mathbb{P}_i -almost surely for each i .*

In terms of dynamic trading, by taking into account the previous lemma, we observe that \mathbb{P} -allowable strategies correspond to \mathbb{P} -admissible strategies with respect to \bar{S} .

Lemma 2.19 (Allowability is equivalent to admissibility with respect to the basket). *Assume that \mathbb{P} satisfies (Smg). Let h be a predictable process. Then h is a \mathbb{P} -trading strategy with respect to the exchange process S if and only if h is a trading strategy with respect to \bar{S} , in the sense that $h \in L(\bar{S}, \mathbb{P})$ and*

$$\overline{V^h} - \overline{V^h(0)} = h \cdot_{\mathbb{P}} \bar{S}, \quad \mathbb{P}\text{-almost surely,}$$

with $\overline{V^h}$ given by (9). Moreover, in this case, h is \mathbb{P} -allowable if and only if h is \mathbb{P} -admissible with respect to \bar{S} in the sense that there exists $\delta > 0$ such that $\overline{V^h} > -\delta$, \mathbb{P} -almost surely.

The axiomatic definition of a martingale valuation operator (see Definition 2.12), which is stated in terms of the primitives of the market and not in terms of the auxiliary price process \bar{S} defined in (7), guarantees that the set of martingale measures \mathbb{Q} for \bar{S} is in one-to-one correspondence with the set of martingale valuation operators \mathbb{V} . More precisely we have the following result.

²Indeed, with $C = I^{(j)}$, (9) becomes, with the help of (8),

$$\bar{C} = \frac{1}{|\mathfrak{A}(t)|} \sum_i \bar{S}_i S_{i,j} \mathbf{1}_{\{i \in \mathfrak{A}(t)\}} = \bar{S}_j \frac{1}{|\mathfrak{A}(t)|} \sum_i \mathbf{1}_{\{i \in \mathfrak{A}(t)\}} = \bar{S}_j$$

for each j .

Proposition 2.20 (Existence of a risk-neutral measure for the basket). *The following two statements hold.*

- (a) *Suppose that \mathbb{V} is a martingale valuation operator. Then there exists a unique probability measure \mathbb{Q} such that*

$$\mathbb{E}_r^{\mathbb{Q}} [C] = \overline{\mathbb{V}^{r,t}(C)} \quad (10)$$

for all $C \in \mathcal{D}^t$ and $r \leq t$. In particular, we have $\mathbb{Q} \sim \mathbb{V}$ and \overline{S} is a \mathbb{Q} -martingale.

- (b) *Suppose that \mathbb{Q} is a probability measure such that \overline{S} is a \mathbb{Q} -martingale. Then there exists a unique martingale valuation operator \mathbb{V} that satisfies (10) for all $C \in \mathcal{D}^t$ and $r \leq t$. In particular, we have $\mathbb{V} \sim \mathbb{Q}$.*

The proof of the proposition is included in Appendix A. The construction of a probability measure in the proof can be seen as a special case (the linear case) when representing an agent's preferences or a risk measure in terms of expectations; see, for instance, Föllmer and Schied (2011). Cassese (2008) is another example, where risk-neutral measures are constructed without an a-priori given reference measure.

3 The Fundamental Theorems of Asset Pricing

3.1 The theorems

In this section, the two Fundamental Theorems of Asset Pricing and some of its consequences are stated. We provide the corresponding proofs in Section 5.

The First Fundamental Theorem of Asset Pricing relates the economic concept of no-arbitrage to the existence of a linear pricing rule, usually formulated in terms of an equivalent local martingale measure. Dybvig and Ross (1987) first used the term Fundamental Theorem of Asset Pricing, but already de Finetti studied these concepts in the context of gambles; see Schervish et al. (2008) for a survey of his original insights.³ The most general version of the First Fundamental Theorem of Asset Pricing, in the presence of a numéraire, is due to Delbaen and Schachermayer (1994, 1998a). The following version, in terms of martingale valuation operators, resembles the original approach in Harrison and Pliska (1981), and more recently the study in Biagini and Cont (2006).

Theorem 3.1 (First Fundamental Theorem of Asset Pricing). *The following implications hold:*

- (a) *If there exists a probability measure \mathbb{P} on $(\Omega, \mathcal{F}(T))$ that satisfies (Smg) and S satisfies (NFLVR) for \mathbb{P} -allowable strategies then there exists a martingale valuation operator $\mathbb{V} \sim \mathbb{P}$.*
- (b) *If there exists a martingale valuation operator \mathbb{V} then there exists a probability measure $\mathbb{P} \sim \mathbb{V}$ that satisfies (Smg) and such that S satisfies (NFLVR) for \mathbb{P} -allowable strategies.*

Corollary 3.2 (First Fundamental Theorem of Asset Pricing in the presence of a reference measure). *Suppose \mathbb{P} is a probability measure on $(\Omega, \mathcal{F}(T))$. Then the following statements are equivalent:*

- (i) *\mathbb{P} satisfies (Smg) and S satisfies (NFLVR) for \mathbb{P} -allowable strategies.*
- (ii) *There exists a martingale valuation operator $\mathbb{V} \sim \mathbb{P}$.*

Proof. Note that if $\widehat{\mathbb{P}} \sim \mathbb{P}$ then \mathbb{P} satisfies (Smg) and S satisfies (NFLVR) for \mathbb{P} -allowable strategies if and only if $\widehat{\mathbb{P}}$ satisfies (Smg) and S satisfies (NFLVR) for $\widehat{\mathbb{P}}$ -allowable strategies. Therefore, the equivalence follows directly from Theorem 3.1 and Remark 2.15. \square

³We thank Marco Frittelli and Marco Maggis for pointing us to Schervish et al. (2008).

The recent papers of [Herdegen \(2014\)](#) and [Herdegen and Schweizer \(2015\)](#), which are closely related to [Delbaen and Schachermayer \(1997\)](#), develop a numéraire-independent theory of arbitrage and obtain also a version of the First Fundamental Theorem of Asset Pricing.

We next have a closer look at the condition in [Corollary 3.2\(i\)](#). Towards this end, we call a predictable process h *simple* if it has the form $h(t) = h^0 1_{\{0\}}(t) + \sum_{n=1}^m h^n 1_{(\tau_{n-1}, \tau_n]}(t)$, where $0 = \tau_{-1} = \tau_0 \leq \dots \leq \tau_m \leq T$ with $m \in \mathbb{N}$ is a finite sequence of stopping times and $h^n \in \mathcal{F}(\tau_{n-1})$ is an \mathbb{R}^d -valued random variable for all $n \in \{0, \dots, m\}$. Note that in the case of simple predictable processes the stochastic integrals in the self-financing condition of [Definition 2.7](#) can be defined in a pathwise sense. Thus, the condition of (NFLVR) for \mathbb{P} -allowable simple strategies can be formulated without the assumption that \mathbb{P} satisfies (Smg). As the following proposition shows, the property (Smg) can then be deduced from the financial condition of (NFLVR) for \mathbb{P} -allowable simple strategies.

Proposition 3.3 ((NFLVR) for simple strategies implies (Smg)). *Let \mathbb{P} be a probability measure on $(\Omega, \mathcal{F}(T))$. Suppose that S satisfies (NFLVR) for \mathbb{P} -allowable simple strategies. Then \mathbb{P} satisfies (Smg).*

To state the Second Fundamental Theorem of Asset Pricing in this paper's framework, we introduce the following concepts.

Definition 3.4 (\mathbb{V} -trading strategies and \mathbb{V} -allowable strategies). Suppose that \mathbb{V} is a martingale valuation operator. By [Theorem 3.1](#), there exists a probability measure $\mathbb{P} \sim \mathbb{V}$ that satisfies (Smg). We say that a predictable process h is a \mathbb{V} -trading strategy if h is a \mathbb{P} -trading strategy. For a \mathbb{V} -trading strategy h , we say that h is \mathbb{V} -allowable if h is \mathbb{P} -allowable. \square

As a consequence of [Remark 2.15](#), the previous definition is independent of the chosen probability measure \mathbb{P} ; see also [Theorem 4.14](#) in [Shiryayev and Cherny \(2002\)](#).

Definition 3.5 (Superreplication strategy, replication strategy, market completeness). Assume that there exists a martingale valuation operator \mathbb{V} . We say that a \mathbb{V} -allowable trading strategy h superreplicates a claim $C \in \mathcal{C}^T$ if $C_i \leq V_i^h(T)$ for all $i \in \mathfrak{A}(T)$, \mathbb{V} -almost surely. We say that a \mathbb{V} -allowable trading strategy h replicates a claim $C \in \mathcal{C}^T$ if

- (a) $V_i^h(T) = C_i$ for all $i \in \mathfrak{A}(T)$, \mathbb{V} -almost surely; and
- (b) for all \mathbb{V} -allowable trading strategies g with $V^g(0) = V^h(0)$ and $V^g(T) \geq V^h(T)$, \mathbb{V} -almost surely, we have $V^g(T) = V^h(T)$, \mathbb{V} -almost surely.

Moreover, we say that the market is complete if for all $C \in \mathcal{D}^T$ there exists a \mathbb{V} -allowable trading strategy h that replicates C . \square

Theorem 3.6 (Second Fundamental Theorem of Asset Pricing). *Suppose that there exists a martingale valuation operator \mathbb{V} . Then the market is complete if and only if \mathbb{V} is the unique martingale valuation operator equivalent to \mathbb{V} .*

We now state the superreplication duality in terms of martingale valuation operators.

Theorem 3.7 (Superreplication duality). *Assume that there exists a martingale valuation operator \mathbb{V} and let $C \in \mathcal{C}^T$. Then we have*

$$\inf \left\{ V^h(0) : h \text{ superreplicates } C \right\} = \sup \left\{ \tilde{\mathbb{V}}^{0,T}(C) : \tilde{\mathbb{V}} \sim \mathbb{V} \text{ is a martingale valuation operator} \right\}, \quad (11)$$

where the sup and the inf are taken componentwise and for each martingale valuation operator $\tilde{\mathbb{V}}$ we consider the extension of [Lemma 2.17](#). Additionally, when the supremum in (11) is finite the infimum is equal to a minimum, that is, there exists a minimal superreplication strategy for C . Moreover, the supremum in (11) is finite and equals to a maximum if and only if C can be replicated by a \mathbb{V} -allowable strategy h .

3.2 Valuation and replication in terms of the basket market

To complement the discussion in Subsection 2.4, we consider again the financial market where the basket asset serves as a numéraire.

As a consequence of Lemma 2.19, the condition of (NFLVR) for \mathbb{P} -allowable strategies holds if and only if the condition of (NFLVR) for \mathbb{P} -admissible strategies is satisfied by the price process \bar{S} , given in (7). Since the semimartingale \bar{S} is bounded, the classical Fundamental Theorem of Asset Pricing (see Delbaen and Schachermayer (1994)) implies that the condition of (NFLVR) for \mathbb{P} -allowable strategies is equivalent to the existence of a measure $\mathbb{Q} \sim \mathbb{P}$ such that \bar{S} is a \mathbb{Q} -martingale. This observation constitutes the basis of the Fundamental Theorems of Asset Pricing and valuation formulas within our context.

Proposition 2.20 puts martingale valuation operators in a one-to-one relationship with martingale measures, and will be the fundamental component in the proof of the First Fundamental Theorem of Asset Pricing (see Theorem 3.1), included in Section 5. The following lemma allows us to relate this paper's Second Fundamental Theorem of Asset Pricing (see Definition 3.5 and Theorem 3.6) and superreplication duality formulas (see Theorem 3.7) with the classical versions of these results. It constitutes the basis of the proofs which are contained in Section 5.

Lemma 3.8 (Superreplication and replication in terms of the basket). *Suppose that \mathbb{V} is a martingale valuation operator and that \mathbb{P} is a probability measure such that $\mathbb{P} \sim \mathbb{V}$. Let h be a \mathbb{V} -allowable trading strategy and $C \in \mathcal{C}^T$. Let \bar{C} and \bar{V}^h be given by (9). Then h superreplicates the contingent claim C if and only if $\bar{C} \leq \bar{V}^h(T)$, \mathbb{P} -almost surely. Moreover, the following statements are equivalent.*

- (i) h replicates the contingent claim C .
- (ii) $\bar{C} = \bar{V}^h(T)$, \mathbb{P} -almost surely, and \bar{V}^h is a \mathbb{Q} -martingale for some probability measure $\mathbb{Q} \sim \mathbb{P}$ such that \bar{S} is a \mathbb{Q} -martingale.

Furthermore, if $C \in \mathcal{D}^T$ then any of the above statements are equivalent to the following.

- (iii) $\bar{C} = \bar{V}^h(T)$, \mathbb{P} -almost surely, and \bar{V}^h is \mathbb{P} -almost surely uniformly bounded.

The proof of Lemma 3.8 can be found in Appendix A.

4 Aggregation and disaggregation of measures

In this section, we investigate how to aggregate risk-neutral measures, each supported on a subset of the set Ω of possible scenarios and relative to one of the d currencies, to a martingale valuation operator. We provide the proofs of the theorems in Section 5. We structure this study in four parts.

Subsection 4.1 argues that the existence of a martingale valuation operator yields a family of d probability measures, which are not necessarily equivalent. However, each of these d measures can be interpreted as a risk-neutral measure with one of the d numéraires fixed. Moreover, the measures are related to each other via a generalized change-of-numéraire formula. This property is called *numéraire-consistency*. We then show that if a family of probability measures is numéraire-consistent they can be “stuck together” to yield a global martingale valuation operator.

Subsection 4.2 compares the results of Subsection 4.1 with the classical valuation formulas in mathematical finance. Subsection 4.3 provides several examples. They illustrate, in particular, how the results of Carr et al. (2014) and Câmara and Heston (2008) are special cases of this paper's setup. In Subsection 4.4 we start with d probability measures, each serving again as a risk-neutral measure for a fixed numéraire. However, this time we do not assume that these measures are numéraire-consistent. We then study conditions such that a martingale valuation operator exists.

4.1 Aggregation with numéraire-consistency and disaggregation

We start by introducing and discussing the following consistency condition.

Definition 4.1 (Numéraire-consistency of probability measures). Suppose that $(\mathbb{Q}_i)_i$ is a family of probability measures. We say that $(\mathbb{Q}_i)_i$ is a numéraire-consistent family of probability measures if for all $A \in \mathcal{F}(t)$ we have

$$\mathbb{E}^{\mathbb{Q}_i}[S_{i,j}(t)\mathbf{1}_A] = S_{i,j}(0)\mathbb{Q}_j(A \cap \{S_{j,i}(t) > 0\}) \quad (12)$$

for all i, j and t . □

Proposition 4.2 (Properties of a numéraire-consistent family of probability measures). *Suppose that $(\mathbb{Q}_i)_i$ is a numéraire-consistent family of probability measures. Then the following statements hold, for each i, j .*

- (a) S_i is a \mathbb{Q}_i -supermartingale; thus, in particular, $\mathbb{Q}_i(\bigcap_t \{i \in \mathfrak{A}(t)\}) = 1$. More precisely, we have

$$\mathbb{E}_r^{\mathbb{Q}_i}[S_{i,j}(t)X] = S_{i,j}(r)\mathbb{E}_r^{\mathbb{Q}_j}[X\mathbf{1}_{\{S_{j,i}(t) > 0\}}], \quad \mathbb{Q}_i\text{-almost surely} \quad (13)$$

for all bounded $\mathcal{F}(t)$ -measurable random variables X and $r \leq t$.

- (b) $S_{i,j}$ is a \mathbb{Q}_i -local martingale if and only if $S_{j,i}$ does not jump to zero under \mathbb{Q}_j .

- (c) For each stopping time τ , $S_{i,j}^\tau$ is a \mathbb{Q}_i -martingale if and only if $\mathbb{Q}_j(S_{j,i}(\tau) > 0) = 1$. Moreover, in this case we have $d\mathbb{Q}_j/d\mathbb{Q}_i|_{\mathcal{F}(\tau)} = S_{j,i}(0)S_{i,j}(\tau)$. In particular, the i -th currency does not completely devalue with respect to the j -th currency, if and only if $S_{i,j}$ is a true \mathbb{Q}_i -martingale.

Note that (13) can be interpreted as a change-of-numéraire formula.

Remark 4.3 (An interpretation for numéraire-consistency). Let $(\mathbb{Q}_i)_i$ be a numéraire-consistent family of probability measures. Then with $w_{i,j} = S_{i,j}(0)/\sum_k S_{i,k}(0) \in (0, 1)$ for all i, j , we have $\sum_j w_{i,j} = 1$ and

$$1 - \frac{1}{\sum_k S_{i,k}(0)} \mathbb{E}^{\mathbb{Q}_i} \left[\sum_j S_{i,j}(T) \right] = \sum_j w_{i,j} \left(1 - S_{j,i}(0) \mathbb{E}^{\mathbb{Q}_i} [S_{i,j}(T)] \right) = \sum_j w_{i,j} \mathbb{Q}_j(S_{j,i}(T) = 0)$$

for all i . Therefore, the normalized expected decrease of the total value of all currencies, measured in terms of the i -th currency, equals to the sum of the weighted probabilities that the i -th currency completely devaluates. The weights correspond exactly to the proportional value of the corresponding currency at time zero. □

We are now ready to relate martingale valuation operators to numéraire-consistent families of probability measures.

Theorem 4.4 (Aggregation and disaggregation). *The following statements hold.*

- (a) Given a martingale valuation operator \mathbb{V} there exists a unique numéraire-consistent family of probability measures $(\mathbb{Q}_i)_i$ such that $(\sum_i \mathbb{Q}_i/d) \sim \mathbb{V}$ and

$$\mathbb{V}_j^{r,t}(C) = \sum_{i=1}^d S_{j,i}(r) \mathbb{E}_r^{\mathbb{Q}_i} \left[\frac{C_i}{|\mathfrak{A}(t)|} \right] \quad (14)$$

for all $r \leq t$, $j \in \mathfrak{A}(r)$, and $C \in \mathcal{D}^t$.

- (b) Given a numéraire-consistent family of probability measures $(\mathbb{Q}_i)_i$ there exists a unique martingale valuation operator $\mathbb{V} \sim (\sum_i \mathbb{Q}_i/d)$ that satisfies (14) for all $r \leq t$, $j \in \mathfrak{A}(r)$, and $C \in \mathcal{D}^t$.

(c) Consider a martingale valuation operator \mathbb{V} and the corresponding numéraire-consistent family of probability measures $(\mathbb{Q}_i)_i$ from (a) and fix $r \leq t$. If a contingent claim $C \in \mathcal{D}^t$ satisfies $\mathbb{V}^{r,t}(C) = \mathbb{V}^{r,t}(C \mathbf{1}_{\{i \in \mathfrak{A}(t)\}})$ for some i , then we have

$$\mathbb{V}_j^{r,t}(C) = S_{j,i}(r) \mathbb{E}_r^{\mathbb{Q}_i}[C_i] \quad (15)$$

for all $j \in \mathfrak{A}(r)$.

Let us first interpret the representation in (14). In order to compute the valuation $\mathbb{V}^{0,T}(C)$ of a contingent claim $C \in \mathcal{D}^T$ under a martingale valuation operator \mathbb{V} one can proceed according to the following steps. First, one replaces the claim C by the claim $\tilde{C} = C/|\mathfrak{A}(t)|$; to wit, one divides the payoff of the contingent claim by the number of active currencies at maturity T . Then, one computes the risk-neutral expectation of this payoff under \mathbb{Q}_i corresponding to fixing the i -th currency as numéraire, for each i . One then converts all these values into one currency (the j -th one in (14)), and adds them up. This then yields $\mathbb{V}^{0,T}(C)$. If the contingent claim C has no payoff in the case that the i -th currency completely devaluates, then (15) holds so that one can compute the valuation $\mathbb{V}^{0,T}(C)$ by only computing the risk-neutral expectation with the i -th currency as numéraire.

In the terminology of Schönbucher (2003, 2004), \mathbb{Q}_i is called a “survival measure” (corresponding to the i -th currency) as it is equivalent to the probability measure \mathbb{P} (corresponding to \mathbb{V} by Theorem 3.1(b)), conditioned on the i -th currency not completely devaluating.

4.2 Comparison with the classical setup

In this subsection we compare the aggregation results of Theorem 4.4 with the classical valuation formulas in mathematical finance. As it is customary in the classical setup, we fix a numéraire; say, the one corresponding to the first row of the exchange matrix. Moreover, we assume that S_1 , the vector of prices quoted with respect to this numéraire, is a \mathbb{P}_1 -semimartingale. In order to clarify this comparison we now consider three different cases.

1. Suppose that there exists a probability measure $\mathbb{Q}_1 \sim \mathbb{P}_1$ such that S_1 is a \mathbb{Q}_1 -martingale and that $S_1 > 0$ under \mathbb{P}_1 . In the terminology of Yan (1998), in this case the market is *fair*. In particular the condition of (NFLVR) for admissible strategies, as defined in Delbaen and Schachermayer (1994), holds with respect to \mathbb{P}_1 . We can define a numéraire-consistent family of probability measures $(\mathbb{Q}_i)_i$ through the change-of-numéraire formula

$$\frac{d\mathbb{Q}_i}{d\mathbb{Q}_1} = S_{i,1}(0) S_{1,i}(T). \quad (16)$$

Indeed, we have

$$\mathbb{E}^{\mathbb{Q}_i}[S_{i,j}(t) \mathbf{1}_A] = S_{i,1}(0) \mathbb{E}^{\mathbb{Q}_1}[S_{1,i}(t) S_{i,j}(t) \mathbf{1}_A] = S_{i,1}(0) \mathbb{E}^{\mathbb{Q}_1}[S_{1,j}(t) \mathbf{1}_A] = S_{i,j}(0) \mathbb{Q}_j(A)$$

for all i, j, t , and $A \in \mathcal{F}(t)$. Observe that $\mathbb{Q}_i \sim \mathbb{Q}_j$ for all i, j and $\sum_i \mathbb{Q}_i \sim \mathbb{P}_1$. Due to Theorem 4.4(c), the associated martingale valuation operator, with respect to any currency i , corresponds to the classical pricing operator arising from taking conditional expectations. This is

$$\mathbb{V}_i^{r,t}(C) = \mathbb{E}_r^{\mathbb{Q}_i}[C_i]$$

for all $i, r \leq t$ and $C \in \mathcal{D}^t$. Hence, in the special case when S_1 is a strictly positive \mathbb{Q}_1 -martingale, the valuation operator completely agrees with classical valuation formulas.

2. Suppose now that there exists a probability measure $\mathbb{Q}_1 \sim \mathbb{P}_1$ such that S_1 is a \mathbb{Q}_1 -martingale but S_1 is not necessarily positive under \mathbb{P}_1 . In this case, the market is also fair in the terminology of [Yan \(1998\)](#). However, the i -th currency is not a classical numéraire, as it does not stay strictly positive. Nevertheless, an interpretation as “defaultable numéraire” is possible and we can again define a family of probability measures thorough [\(16\)](#). We observe that now \mathbb{Q}_i is not necessarily equivalent to \mathbb{P}_1 , but only absolutely continuous with respect to \mathbb{Q}_1 for all i and $\sum_i \mathbb{Q}_i \sim \mathbb{P}_1$. The family of probability measures is numéraire-consistent since

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}_i}[S_{i,j}(t)\mathbf{1}_A] &= S_{i,1}(0)\mathbb{E}^{\mathbb{Q}_1}[S_{1,i}(t)S_{i,j}(t)\mathbf{1}_{A \cap \{S_{i,j}(t) < \infty\}}}] = S_{i,j}(0)\mathbb{Q}_j(A \cap \{S_{i,j}(t) < \infty\}) \\ &= S_{i,j}(0)\mathbb{Q}_j(A \cap \{S_{j,i}(t) > 0\}). \end{aligned}$$

Furthermore, due to [Theorem 4.4\(c\)](#), the associated martingale valuation operator, with respect to the first currency, corresponds to the classical formula

$$\mathbb{V}_1^{r,t}(C) = \mathbb{E}_r^{\mathbb{Q}_1}[C_1]$$

for all $r \leq t$ and $C \in \mathcal{D}^t$. With respect to the other currencies, however, the martingale valuation operator does not necessarily correspond to taking conditional expectations. Indeed, for $i \geq 2$, it is possible that S_i is not a \mathbb{Q}_i -local martingale but only a \mathbb{Q}_i -supermartingale. Hence, the condition of (NFLVR) as in [Delbaen and Schachermayer \(1994\)](#) does not necessarily hold with respect to \mathbb{Q}_i . By using [Theorem 4.4\(c\)](#) and [\(13\)](#), the associated martingale valuation operator can be written as

$$\begin{aligned} \mathbb{V}_j^{r,t}(C) &= S_{j,1}(r)\mathbb{E}_r^{\mathbb{Q}_1}[C_1] = \mathbb{E}_r^{\mathbb{Q}_j}[S_{j,1}(t)C_1] + S_{j,1}(r)\mathbb{E}_r^{\mathbb{Q}_1}[C_1\mathbf{1}_{\{S_{1,j}(t)=0\}}] \\ &= \mathbb{E}_r^{\mathbb{Q}_j}[C_j] + S_{j,1}(r)\mathbb{E}_r^{\mathbb{Q}_1}[C_1\mathbf{1}_{\{S_{1,j}(t)=0\}}] \end{aligned}$$

for all $r \leq t$, $j \in \mathfrak{A}(r)$, and $C \in \mathcal{D}^t$. Therefore, the classical pricing operator is adjusted by a factor depending on the \mathbb{Q}_1 -probability of devaluation of each currency with respect to the first currency. Let us now fix j, t , and $C \in \mathcal{D}^t$ with $C \geq 0$ and consider the correction term

$$F_j(r) = S_{j,1}(r)\mathbb{E}_r^{\mathbb{Q}_1}[C_1\mathbf{1}_{\{S_{1,j}(t)=0\}}]$$

for all $r \leq t$ more closely. Then $F_j(t) = S_{j,1}(t)C_1\mathbf{1}_{\{S_{j,1}(t)=\infty\}} = 0$ under \mathbb{Q}_j . Moreover, for $r_1 \leq r_2 \leq t$ we have

$$\mathbb{E}_{r_1}^{\mathbb{Q}_j}[F_j(r_2)] = S_{j,1}(r_1)\mathbb{E}_{r_1}^{\mathbb{Q}_1}[\mathbf{1}_{\{S_{1,j}(r_2)>0\}}\mathbb{E}_{r_2}^{\mathbb{Q}_1}[C_1\mathbf{1}_{\{S_{1,j}(t)=0\}}]] \leq F_j(r_1)$$

under \mathbb{Q}_j . Hence the correction term F_j is a \mathbb{Q}_j -potential. More precisely, by the same computations F_j is indeed a \mathbb{Q}_j -local martingale, as long as $S_{1,j}$ does not jump to zero under \mathbb{P}_1 . Hence, under this no-jump condition, valuation according to the pricing operator yields a local martingale, whose Riesz decomposition equals risk-neutral expectation plus the correction term F_j .

3. Suppose now that there does not exist a measure $\mathbb{Q}_1 \sim \mathbb{P}_1$ such that S_1 is a \mathbb{Q}_1 -martingale. In this case, according to [Theorem 3.2](#) in [Yan \(1998\)](#), the condition of (NFLVR) for \mathbb{P}_1 -allowable strategies does not hold. Let us assume, nevertheless, that there exists a family of numéraire-consistent probability measures $(\mathbb{Q}_i)_i$. As S_1 is only a \mathbb{Q}_1 -supermartingale, and there does not necessarily exist a probability measure equivalent to \mathbb{P}_1 under which S_1 is a local martingale, the condition of (NFLVR) with respect to admissible strategies, as defined in [Delbaen and Schachermayer \(1994\)](#), could also fail. In contrast, due to [Theorems 3.1](#) and [4.4](#), the condition of (NFLVR) for $(\sum_i \mathbb{Q}_i)$ -allowable strategies holds. This extended model would depart from the classical setup as in this case there exists i such that $\mathbb{Q}_i(S_{1,i}(T) = \infty) > 0$.

The representation in (14) provides an economically meaningful pricing formula in this setup. This is possible even if after fixing the first currency as numéraire the condition of (NFLVR) with respect to \mathbb{P}_1 -allowable strategies (or even admissible strategies) is violated. The key is to consider non-equivalent probability measures that allow for devaluations of the first currency and to take consistently these states of the world into account for arbitrage considerations and valuation of options. As we illustrate in Example 4.5 below, these valuation formulas can be applied to strict local martingale models and yield prices consistent with market conventions such as put-call parity. This valuation framework corrects the deficiencies of the classical conditional expectation approach for pricing. Additionally, Theorem 3.6 provides a dual superreplication result in this framework where the market is not fair in the classical sense.

4.3 Examples

As already pointed out in Lewis (2000), Cox and Hobson (2005), Madan and Yor (2006), and Carr et al. (2014), among others, a strict local martingale measure is not always suitable for pricing purposes because prices computed through expectations with this measure fail to be in accordance with market conventions such as put-call-parity. The works of Lewis (2000) and Madan and Yor (2006) propose ad-hoc correction terms to solve these deficiencies. Similarly to the study in Carr et al. (2014), we recognize that the problems arise from the fact that a strict local martingale measure does not take into account the states of the world where the corresponding currency devaluates. Martingale valuation operators correct this deficiency, and they do it in a symmetric and financially meaningful form.

Example 4.5 (A representation of \mathbb{V} when $d = 2$). Consider an economy with $d = 2$ currencies and assume the existence of a martingale valuation operator \mathbb{V} . In the following we derive a representation of \mathbb{V} . By Theorem 4.4(a), \mathbb{V} corresponds to a numéraire-consistent family of measures $(\mathbb{Q}_1, \mathbb{Q}_2)$. Next, fix two times $r < t$, a contingent claim $C \in \mathcal{D}^t$, and some active currency $j \in \mathfrak{A}(r)$. We then have

$$\begin{aligned}
\mathbb{V}_j^{r,t}(C) &= S_{j,1}(r)\mathbb{E}_r^{\mathbb{Q}_1} \left[\frac{C_1}{|\mathfrak{A}(t)|} \right] + S_{j,2}(r)\mathbb{E}_r^{\mathbb{Q}_2} \left[\frac{C_2}{|\mathfrak{A}(t)|} \right] \\
&= S_{j,1}(r) \left(\mathbb{E}_r^{\mathbb{Q}_1} \left[\frac{C_1}{2} \mathbf{1}_{\{S_{1,2}(t) > 0\}} \right] + \mathbb{E}_r^{\mathbb{Q}_1} [C_1 \mathbf{1}_{\{S_{1,2}(t) = 0\}}] \right) \\
&\quad + S_{j,2}(r) \left(\mathbb{E}_r^{\mathbb{Q}_2} \left[\frac{C_2}{2} \mathbf{1}_{\{S_{1,2}(t) < \infty\}} \right] + \mathbb{E}_r^{\mathbb{Q}_2} [C_2 \mathbf{1}_{\{S_{1,2}(t) = \infty\}}] \right) \\
&= S_{j,1}(r)\mathbb{E}_r^{\mathbb{Q}_1} [C_1] + S_{j,2}(r)\mathbb{E}_r^{\mathbb{Q}_2} [C_2 \mathbf{1}_{\{S_{1,2}(t) = \infty\}}].
\end{aligned} \tag{17}$$

Here we used (13) (applied with $j = 1$, $i = 2$, and $X = C_1/2$) to deduce that

$$S_{j,2}(r)\mathbb{E}_r^{\mathbb{Q}_2} \left[\frac{C_2}{2} \mathbf{1}_{\{S_{1,2}(t) < \infty\}} \right] = S_{j,1}(r)\mathbb{E}_r^{\mathbb{Q}_1} \left[\frac{C_1}{2} \mathbf{1}_{\{S_{1,2}(t) > 0\}} \right].$$

Therefore, in the case $d = 2$, \mathbb{V} corresponds exactly to the pricing formula in Carr et al. (2014), constructed to restore put-call parity in a strict local martingale model. Looking closer at (17), say with $j = 1$, yields that \mathbb{V} can be written as the sum of two terms. The first term is the risk-neutral expectation of the contingent claim if the first currency is chosen as numéraire. The second term can be interpreted as a correction factor. It is a product of the exchange rate, converting units of the second currency into units of the first currency, and another risk-neutral expectation. This time, the risk-neutral expectation is chosen with respect to the second currency as numéraire. It considers the contingent claim on the event when the first currency completely devaluates. In the case when the contingent claim C is a European call (with the first currency chosen as numéraire), this second term corresponds exactly to the ad-hoc correction in Lewis (2000). Thus, (17)

retrieves exactly the pricing formulas in [Lewis \(2000\)](#), [Madan and Yor \(2006\)](#), [Paulot \(2013\)](#), and [Kardaras \(2015\)](#). \square

In the following, we discuss the superreplication duality of [Theorem 3.7](#) and illustrate that one may not argue currency-by-currency in order to compute the minimal superreplication cost.

Example 4.6 (Superreplication duality: a counter-example). Consider again an economy with $d = 2$ currencies. Assume that \mathbb{Q}_1 and $\tilde{\mathbb{Q}}_1$ denote two equivalent probability measures such that $S_{1,2}$ is a strict local \mathbb{Q}_1 -martingale but a true $\tilde{\mathbb{Q}}_1$ -martingale. Such examples exist; see, for instance, [Delbaen and Schachermayer \(1998b\)](#), or [Carr et al. \(2014\)](#) for a finite-horizon example. Let \mathbb{Q}_2 denote another probability measure such that $S_{2,1}$ is a \mathbb{Q}_2 -local martingale and such that $(\mathbb{Q}_1, \mathbb{Q}_2)$ is a numéraire-consistent family. Such a measure can be constructed, for example by the approach pioneered in [Föllmer \(1972\)](#); see also [Perkowski and Ruf \(2015\)](#). In particular, we have $\mathbb{Q}_2(S_{1,2}(T) = \infty) > 0$.

Now, consider the value vector $C = I^{(2)}(T)$ corresponding to one unit of the second currency and defined in [\(2\)](#). The superreplication value vector of this payoff is given by [\(11\)](#) and clearly bounded from above by $(S_{1,2}(0), 1)^T$, as buying and holding the second currency superreplicates C . Having [Example 4.5](#) and in particular [\(17\)](#) in mind, we now consider

$$\begin{aligned} \sup_{\mathbb{Q} \sim \mathbb{Q}_1: S_{1,2} \text{ is a } \mathbb{Q}\text{-local martingale}} \mathbb{E}^{\mathbb{Q}}[C_1] + S_{1,2}(0) & \quad \sup_{\hat{\mathbb{Q}} \sim \mathbb{Q}_2: S_{2,1} \text{ is a } \hat{\mathbb{Q}}\text{-local martingale}} \mathbb{E}^{\hat{\mathbb{Q}}}[C_2 \mathbf{1}_{\{S_{1,2}(T) = \infty\}}] & (18) \\ & \geq \mathbb{E}^{\tilde{\mathbb{Q}}_1}[S_{1,2}(T)] + S_{1,2}(0) \mathbb{E}^{\mathbb{Q}_2}[\mathbf{1}_{\{S_{1,2}(T) = \infty\}}] > S_{1,2}(0). \end{aligned}$$

Hence, the expression in [\(18\)](#) does usually not yield the minimum superreplication price. Thus, for the superreplication formula, the supremum cannot be taken component-wise by looking at each currency as numéraire separately. To conclude, this example illustrates that the supremum in [\(11\)](#) cannot be split into d suprema in [\(14\)](#). \square

We next study the extension of the Black-Scholes-Merton model proposed in [Câmara and Heston \(2008\)](#). They suggest to augment the original Black-Scholes-Merton model by allowing the relative prices to jump to zero and infinity. The jump to zero “adjust[s] the Black-Scholes model for biases related with out-of-the-money put options,” and the jump to infinity “captures the exuberance and the extreme upside potential of the market and leads to a risk-neutral density with more positive skewness and kurtosis than the density implicit in the Black-Scholes model.” [Câmara and Heston \(2008\)](#) then illustrate that such a modification indeed yields an implied volatility which is closer to the ones observed in the market.

Example 4.7 (Black-Scholes with jumps to zero and infinity). We consider again two currencies, that is, $d = 2$. We assume that the relative prices are described through the Black-Scholes model; however, now with the additional feature that the price may either jump to zero or infinity at some exponential time. We introduce the model formally by specifying a probability measure \mathbb{P} on $(\Omega, \mathcal{F}(T))$. Towards this end, suppose that τ_1 and τ_2 are exponentially distributed stopping times with intensity $\lambda_1^{\mathbb{P}}$ and $\lambda_2^{\mathbb{P}}$, respectively, and satisfy $\mathbb{P}(\tau_1 = \tau_2) = 0$. We then set

$$S_{1,2}(t) = S_{1,2}(0) \exp\left(\sigma W(t) - \frac{\sigma^2}{2}t + \mu t\right) \mathbf{1}_{\{t < \tau_1 \wedge \tau_2\}} + \infty \mathbf{1}_{\{\tau_1 \leq t \wedge \tau_2\}},$$

where $\mu, \sigma \in \mathbb{R}$ are constant with $\sigma \neq 0$ and W is a \mathbb{P} -Brownian motion, independent of τ_1 and τ_2 . This yields directly

$$S_{2,1}(t) = S_{2,1}(0) \exp\left(-\sigma W(t) + \frac{\sigma^2}{2}t - \mu t\right) \mathbf{1}_{\{t < \tau_1 \wedge \tau_2\}} + \infty \mathbf{1}_{\{\tau_2 \leq t \wedge \tau_1\}}.$$

Thus, on the event $\{\tau_1 < \tau_2\}$, the first currency devaluates completely at time τ_1 , while on $\{\tau_2 < \tau_1\}$ the second currency devaluates completely at time τ_2 .

We now want to construct a martingale valuation operator. Towards this end, we first construct a numéraire-consistent family of probability measures $(\mathbb{Q}_1, \mathbb{Q}_2)$ and then apply Theorem 4.4(b). In particular, under \mathbb{Q}_1 the process $S_{1,2}$ stays real-valued and is a supermartingale; a similar statement holds for \mathbb{Q}_2 . To start, we define the probability measures \mathbb{P}_1 and \mathbb{P}_2 by

$$\frac{d\mathbb{P}_1}{d\mathbb{P}} = \frac{\mathbf{1}_{\{\tau_1 > \tau_2 \wedge T\}}}{\mathbb{P}(\tau_1 > \tau_2 \wedge T | \tau_2)} = \mathbf{1}_{\{\tau_1 > T \wedge \tau_2\}} e^{\lambda_1^{\mathbb{P}}(T \wedge \tau_2)}; \quad (19)$$

$$\frac{d\mathbb{P}_2}{d\mathbb{P}} = \frac{\mathbf{1}_{\{\tau_2 > \tau_1 \wedge T\}}}{\mathbb{P}(\tau_2 > \tau_1 \wedge T | \tau_1)} = \mathbf{1}_{\{\tau_2 > T \wedge \tau_1\}} e^{\lambda_2^{\mathbb{P}}(T \wedge \tau_1)}. \quad (20)$$

We next fix some, for the moment arbitrary, constants $\mu_1, \mu_2 \in \mathbb{R}$ and $\lambda_1, \lambda_2 > 0$ and define the probability measures \mathbb{Q}_1 and \mathbb{Q}_2 by

$$\frac{d\mathbb{Q}_1}{d\mathbb{P}_1} = \mathcal{E} \left(\left(\frac{\mu_1 - \mu}{\sigma} \right) W \right) (T) e^{(\lambda_2^{\mathbb{P}} - \lambda_2)(T \wedge \tau_2)} \left(\frac{\lambda_2}{\lambda_2^{\mathbb{P}}} \right)^{\mathbf{1}_{\{\tau_2 \leq T\}}}; \quad (21)$$

$$\frac{d\mathbb{Q}_2}{d\mathbb{P}_2} = \mathcal{E} \left(\left(\frac{\mu_2 - \mu + \sigma^2}{\sigma} \right) W \right) (T) e^{(\lambda_1^{\mathbb{P}} - \lambda_1)(T \wedge \tau_1)} \left(\frac{\lambda_1}{\lambda_1^{\mathbb{P}}} \right)^{\mathbf{1}_{\{\tau_1 \leq T\}}}. \quad (22)$$

Then the \mathbb{Q}_1 -intensity of τ_2 equals λ_2 and the \mathbb{Q}_2 -intensity of τ_1 equals λ_1 . Moreover, we get

$$S_{1,2}(t) = S_{1,2}(0) \exp \left(\sigma W_1(t) - \frac{\sigma^2}{2} t + \lambda_2 t \right) \mathbf{1}_{\{t < \tau_2\}} e^{\mu_1 t - \lambda_2 t}, \quad \mathbb{Q}_1\text{-almost surely}; \quad (23)$$

$$S_{2,1}(t) = S_{2,1}(0) \exp \left(\sigma W_2(t) - \frac{\sigma^2}{2} t + \lambda_1 t \right) \mathbf{1}_{\{t < \tau_1\}} e^{-\mu_2 t - \lambda_1 t}, \quad \mathbb{Q}_2\text{-almost surely} \quad (24)$$

for all t , with W_1 a \mathbb{Q}_1 -Brownian Motion independent of τ_2 and W_2 a \mathbb{Q}_2 -Brownian motion independent of τ_1 . It is clear that it is necessary to have $\lambda_1 \geq -\mu_2$ and $\lambda_2 \geq \mu_1$ for the supermartingale property of $S_{1,2}$ and $S_{2,1}$, respectively.

Fix now $t \in [0, T]$ and $A \in \mathcal{F}(t)$. Then, by (21)–(22), (19)–(20), and (23)–(24)

$$\begin{aligned} \mathbb{Q}_1(A \cap \{S_{1,2}(t) > 0\}) &= \mathbb{E}^{\mathbb{P}} \left[\mathcal{E} \left(\left(\frac{\mu_1 - \mu}{\sigma} \right) W \right) (t) e^{(\lambda_1^{\mathbb{P}} + \lambda_2^{\mathbb{P}} - \lambda_2)t} \mathbf{1}_{\{t < \tau_1 \wedge \tau_2\}} \mathbf{1}_A \right]; \\ S_{1,2}(0) \mathbb{E}^{\mathbb{Q}_2}[S_{2,1}(t) \mathbf{1}_A] &= \mathbb{E}^{\mathbb{P}} \left[\mathcal{E} \left(\left(\frac{\mu_2 - \mu}{\sigma} \right) W \right) (t) e^{(\lambda_1^{\mathbb{P}} + \lambda_2^{\mathbb{P}} - \mu_2 - \lambda_1)t} \mathbf{1}_{\{t < \tau_1 \wedge \tau_2\}} \mathbf{1}_A \right]. \end{aligned}$$

This yields that for (12) to hold we need to impose that

$$\lambda_2 - \lambda_1 = \mu_1 = \mu_2.$$

Indeed, this is sufficient for the numéraire-consistency of $(\mathbb{Q}_1, \mathbb{Q}_2)$ since then also, in the same manner,

$$S_{2,1}(0) \mathbb{E}^{\mathbb{Q}_1}[S_{1,2}(t) \mathbf{1}_A] = \mathbb{Q}_2(A \cap \{S_{2,1}(t) > 0\}).$$

Theorem 4.4(b) now yields a martingale valuation operator \mathbb{V} , corresponding to the family $(\mathbb{Q}_1, \mathbb{Q}_2)$.

Consider next an exchange option $C = (C_1, C_2)$ with $C_1 = (S_{1,2}(T) - K)^+$ and $C_2 = (1 - K S_{2,1}(T))^+$, where $K \in \mathbb{R}$. That is, at time T , the option gives the right to swap K units of the first currency into one unit of the second currency. Then the representation of \mathbb{V} in (17) of Example 4.5 yields

$$\begin{aligned} \mathbb{V}_1^{0,T}(C) &= \mathbb{E}^{\mathbb{Q}_1}[(S_{1,2}(T) - K)^+ \mathbf{1}_{\{\tau_2 > T\}}] + S_{1,2}(0) \mathbb{Q}_2(\tau_1 \leq T) \\ &= \mathbb{Q}_1(\tau_2 > T) \mathbb{E}^{\mathbb{Q}_1} \left[\left(S_{1,2}(0) e^{\sigma W_1(T) + (\lambda_2 - \lambda_1 - \sigma^2/2)T} - K \right)^+ \right] + S_{1,2}(0) (1 - e^{-\lambda_1 T}) \end{aligned}$$

$$= e^{-\lambda_1 T} S_{1,2}(0) \Phi(d_1) - K e^{-\lambda_2 T} \Phi(d_2) + S_{1,2}(0)(1 - e^{-\lambda_1 T}), \quad (25)$$

where

$$d_1 = \frac{1}{\sigma\sqrt{T}} \left(\ln \left(\frac{S_{1,2}(0)}{K} \right) + \left(\lambda_2 - \lambda_1 + \frac{\sigma^2}{2} \right) T \right); \quad d_2 = d_1 - \sigma\sqrt{T}$$

and Φ is the standard normal cumulative distribution function. For the last equality in (25), we have used the standard Black-Scholes-Merton formula with interest rate $\lambda_2 - \lambda_1$. This then directly yields also

$$\mathbb{V}_2^{0,T}(C) = e^{-\lambda_2 T} \Phi(d_1) - K S_{2,1}(0) e^{-\lambda_2 T} \Phi(d_2) + 1 - e^{-\lambda_1 T}.$$

The expression in (25) corresponds to formula (16) in [Câmara and Heston \(2008\)](#). That formula has been derived via solving a partial integral differential equation. In contrast, (25) has been derived by a purely probabilistic approach based on equivalent supermartingale measures. Note that the use of martingale valuation operators yields a systematic way to price more complicated, possibly path-dependent contingent claims in the Câmara-Heston framework. Moreover, this example also shows that the Câmara-Heston framework is free of arbitrage, in the sense of [Definition 2.9](#). Due to the presence of a jump to zero and due to the incompleteness of the model this example is not covered by [Carr et al. \(2014\)](#).

We emphasize that this approach is not restricted to the Black-Scholes model. One might take any model, for example the Heston model, and then add a jump to zero and a jump to infinity. Going through the same steps as in this example then yields a martingale valuation operator that corrects deep out-of-the-money puts and call prices. \square

4.4 Aggregation without numéraire-consistency

[Theorem 4.4\(b\)](#) yields that, given a numéraire-consistent family of probability measures $(\mathbb{Q}_i)_i$, there exists a martingale valuation operator, and thus, by [Theorem 3.1](#), S satisfies (NFLVR) for $(\sum_i \mathbb{Q}_i/d)$ -allowable strategies. In practice it might be difficult to decide whether a given family of probability measures $(\mathbb{Q}_i)_i$ is numéraire-consistent. Thus, the question arises, under which conditions the existence of a not necessarily numéraire-consistent family of probability measures yields the existence of a martingale valuation operator. The next theorem provides more easily verifiable conditions such that there exists a martingale valuation operator $\mathbb{V} \sim (\sum_i \mathbb{Q}_i/d)$ for an arbitrary family of probability measures $(\mathbb{Q}_i)_i$.

Theorem 4.8 (Aggregation without numéraire-consistency). *Let $(\mathbb{Q}_i)_i$ be a family of probability measures. Then there exists a martingale valuation operator $\mathbb{V} \sim (\sum_i \mathbb{Q}_i/d)$ if one of the following two conditions is satisfied.*

- (a) S_i is a \mathbb{Q}_i -martingale for each i .
- (b) The following four conditions hold:
 - (i) S_i is a \mathbb{Q}_i -local martingale for each i .
 - (ii) $\sum_i \mathbb{Q}_i/d$ satisfies (NOD); see [Definition 2.10](#).
 - (iii) For each k ,

$$\mathbb{Q}_k |_{\mathcal{F} \cap \{\sum_j S_{k,j}(T) < \infty\}} \sim \left(\sum_i \mathbb{Q}_i/d \right) \Big|_{\mathcal{F} \cap \{\sum_j S_{k,j}(T) < \infty\}}.$$

- (iv) There exist $\epsilon > 0$, $N \in \mathbb{N}$, and predictable times $(T_n)_{n \in \{1, \dots, N\}}$ such that

$$\bigcup_k \left\{ (t, \omega) : \sum_j S_{k,j}(t) = \infty \text{ and } \sum_j S_{k,j}(t-) \leq d + \epsilon \right\} \subset \bigcup_{n=1}^N \llbracket T_n \rrbracket,$$

$(\sum_i \mathbb{Q}_i/d)$ -almost surely.

As Example 4.10 below illustrates, Theorem 4.8(b) is not sufficient for the existence of a martingale valuation operator, in general, without (b)(i), namely that S_i is a \mathbb{Q}_i -local martingale for each i . The condition in Theorem 4.8(b)(ii) states that $\sum_i \mathbb{Q}_i/d$ must satisfy the minimal no-arbitrage condition given by (NOD) — the selling of an active currency does not yield a simple arbitrage strategy. Indeed, Theorem 3.1(b) in conjunction with Proposition 2.11 yield that this condition is necessary. As Example 4.11 below illustrates, the conclusion of Theorem 4.8 is wrong without (b)(ii). Thus, given the other conditions, it is not redundant for the formulation of the theorem. The condition in Theorem 4.8(b)(iii) means that the support of \mathbb{Q}_k is the event $\{\sum_j S_{k,j}(T) < \infty\}$ for each k . The necessity of such a condition is the content of Example 4.12 below.

Theorem 4.8(b)(iv) is a technical condition and we do not know whether it is necessary for the statement of the theorem to hold. This condition allows the k -th currency to devalue suddenly, as long as it is not “strong” in the sense $\sum_j S_{k,j} \leq d + \varepsilon$. If, however, a “strong” currency devaluates suddenly, it only is allowed to do so at a finite number of fixed, predictable times. In particular, any discrete-time model with finitely many time steps satisfies this condition. This condition also holds if $\sum_i \mathbb{Q}_i/d$ satisfies (NSD), in the sense of the following definition.

Definition 4.9 (NSD). We say that a probability measure \mathbb{P} satisfies No Sudden Devaluation (NSD) if $\mathbb{P}(S_{k,j} \text{ jumps to } \infty) = 0$ for all k, j . \square

Under (NSD) no currency devaluates completely against any other currency suddenly. Example 4.11 below illustrates that there exists a probability measure \mathbb{P} that satisfies (NSD) but not (NOD). It is simple to construct an example that satisfies (NOD) but not (NSD).

Example 4.10 (On the necessity of Theorem 4.8(b)(i)). Fix $T = d = 2$ and $\Omega = \{\omega_1, \omega_2\}$ along with $\mathcal{F}(t) = \{\emptyset, \Omega\}$ for all $t < 1$ and $\mathcal{F}(t) = \{\emptyset, \Omega, \{\omega_1\}, \{\omega_2\}\}$ for all $t \geq 1$. Let $S_{1,2}(\omega_1, t) = 1$ and $S_{1,2}(\omega_2, t) \equiv \mathbf{1}_{t < 1}$ for all t . That is, two states of the world are possible; up to time 1 the exchange rate between the two currencies stays constant, and at time one either the second currency devaluates completely or nothing happens, depending on the state of the world. We now let $\mathbb{Q}_1(\{\omega_1\}) = \mathbb{Q}_1(\{\omega_2\}) = 1/2$, and $\mathbb{Q}_2(\{\omega_1\}) = 1$. Then $S_{1,2}$ is a strict \mathbb{Q}_1 -supermartingale and $S_{2,1}$ is a \mathbb{Q}_2 -martingale. Moreover, all conditions in Theorem 4.8(b), apart from (i), are satisfied. However, selling one unit of the second currency and buying one unit of the first currency at time zero yields a nonnegative wealth process that is strictly positive in state ω_2 , which has strictly positive $(\mathbb{Q}_1 + \mathbb{Q}_2)/2$ -probability; thus a clear arbitrage. Thus, by Theorem 3.1, no martingale valuation operator $\mathbb{V} \sim (\mathbb{Q}_1 + \mathbb{Q}_2)/2$ can exist. This illustrates that Theorem 4.8 indeed needs the local martingale property, formulated in (b)(i), in its statement. \square

Example 4.11 (On the necessity of Theorem 4.8(b)(ii)). We slightly modify Example 4.10. Again, fix $T = d = 2$ and assume that $(\Omega, \mathcal{F}, \mathbb{Q}_1)$ supports a Brownian motion B started in zero and stopped when hitting -1 , and an independent $\{0, 1\}$ -distributed random variable X with $\mathbb{Q}_1(X = 0) = \mathbb{Q}_1(X = 1) = 1/2$. Now, let

$$S_{1,2}(t) = 1 + \mathbf{1}_{\{X=1\}} B \left(\tan \left(0 \vee \frac{\pi}{2}(t-1) \right) \right)$$

for all t . Now let $(\mathcal{F}(t))_t$ denote the smallest right-continuous filtration that makes $S_{1,2}$ adapted. Then $S_{1,2}$ is constant before time one and stays constant afterwards with probability 1/2, but moves like a time-changed Brownian motion stopped when hitting zero, otherwise. We now set $\mathbb{Q}_2 = \mathbb{Q}_1(\cdot | \{X = 0\})$ and note that $S_{2,1}$ is a (constant) \mathbb{Q}_2 -martingale. Then the conditions in Theorem 4.8(b)(i), (iii), and (iv) are all satisfied, but as in the previous example, NFLVR for allowable strategies does not hold. Thus, Theorem 4.8(b)(ii) is necessary to make the theorem valid. Note that $(\mathbb{Q}_1 + \mathbb{Q}_2)/2$ satisfies (NSD) but not (NOD) in this example. \square

Example 4.12 (On the necessity of Theorem 4.8(b)(iii)). With $d = 2$ assets again, we now provide an example for a family of local martingale measures $(\mathbb{Q}_1, \mathbb{Q}_2)$ such that $(\mathbb{Q}_1 + \mathbb{Q}_2)/2$ satisfies (NSD) and

(NOD), but no martingale valuation operator $\mathbb{V} \sim (\mathbb{Q}_1 + \mathbb{Q}_2)/2$ exists. Fix $T = 2$ and a filtered probability space $(\Omega, \mathcal{F}(2), (\mathcal{F}(t))_t, \mathbb{Q}_2)$ that supports a three-dimensional Bessel process R starting in one. Next, let τ denote the smallest time that R hits $1/2$; in particular, we have $\mathbb{Q}_2(\tau < T) > 0$ and $\mathbb{Q}_2(\tau = \infty) > 0$. Consider now the process

$$S_{1,2} = 1 + \left(R - \frac{1}{2}\right) \mathbf{1}_{[\tau, \infty[} > 0.$$

With $\mathbb{Q}_1(\cdot) = \mathbb{Q}_2(\cdot | \{\tau = \infty\})$ we have $\mathbb{Q}_1(S_{1,2} = 1) = 1$. Moreover, $S_{2,1}$ is a \mathbb{Q}_2 -local martingale and $\mathfrak{A}(T) = \{1, 2\}$. In particular, $(\mathbb{Q}_1 + \mathbb{Q}_2)/2$ satisfies (NSD) and (NOD). However, Proposition 4.2(c) yields that no numéraire-consistent family of probability measures can exist. Thus, Theorem 4.4(a) yields that no martingale valuation operator $\mathbb{V} \sim (\mathbb{Q}_1 + \mathbb{Q}_2)/2$ exists either. This shows that Theorem 4.8(b) is not correct without the support condition in (b)(iii). \square

5 Proofs

Subsections 5.1 and 5.2 contain the proofs of the statements in Sections 3 and 4, respectively.

5.1 Proofs of Theorems 3.1, 3.6, and 3.7, and of Proposition 3.3

Proof of Theorem 3.1. We first observe that if \mathbb{P} satisfies (Smg) then, due to Lemmas 2.18 and 2.19, the condition of (NFLVR) for \mathbb{P} -allowable strategies is equivalent to the condition that

(*) the \mathbb{P} -semimartingale \bar{S} satisfies (NFLVR) for admissible strategies.

Since \bar{S} is bounded, by Theorem 1.1 in Delbaen and Schachermayer (1994), (*) is equivalent to the condition that

(**) there exists a probability measure $\mathbb{Q} \sim \mathbb{P}$ such that \bar{S} is a \mathbb{Q} -martingale.

Thus, to see (a), note that Proposition 2.20 and Remark 2.15 imply the existence of a martingale valuation operator $\mathbb{V} \sim \mathbb{P}$ if the conditions in (a) hold.

Suppose now that there exists a martingale valuation operator \mathbb{V} . By Proposition 2.20 there exists a probability measure \mathbb{Q} that satisfies (**) above with \mathbb{P} replaced by \mathbb{V} . Thus, to conclude the proof of (b), we only need to argue that the measure \mathbb{Q} satisfies (Smg) with $A_i = \{i \in \mathfrak{A}(T)\}$ for all i . Indeed, $\mathbb{Q}(A_i) > 0$ since $\mathbb{Q}(\bar{S}_i(T) > 0) > 0$ and S_i is a $\mathbb{Q}(\cdot | A_i)$ -semimartingale by (8) for all i . \square

Proof of Proposition 3.3. Suppose S satisfies (NFLVR) for \mathbb{P} -allowable simple strategies. As it can be checked from their proofs, Lemma 2.19 holds for simple predictable strategies without the assumption that \mathbb{P} satisfies (Smg). Therefore, \bar{S} satisfies (NFLVR) for \mathbb{P} -admissible simple strategies. Theorem 7.2 in Delbaen and Schachermayer (1994) now implies that \bar{S} is a \mathbb{P} -semimartingale. We conclude that \mathbb{P} satisfies (Smg) with $A_i = \{i \in \mathfrak{A}(T)\}$ for each i . Indeed, the proof of Proposition 2.11 shows that \mathbb{P} satisfies (NOD) and in particular $\mathbb{P}(A_i) > 0$ for all i . To conclude, S_i is a $\mathbb{P}(\cdot | A_i)$ -semimartingale by (8) on A_i for all i . \square

Proof of Theorem 3.6. By Theorem 3.1(b) there exists a probability measure $\mathbb{P} \sim \mathbb{V}$ that satisfies (Smg) and the exchange process S satisfies (NFLVR) for \mathbb{P} -allowable strategies. The equivalence between (i) and (iii) in Lemma 3.8 implies that the market is complete if and only if the market with traded assets \bar{S} and reference probability measure \mathbb{P} is complete in the sense of Definition 1.15 in Shiryaev and Cherny (2002).

The classical Second Fundamental Theorem of Asset Pricing, see Theorem 1.17 in Shiryaev and Cherny (2002), implies that the market is complete if and only if there exist a unique martingale measure $\mathbb{Q} \sim \mathbb{P}$. Proposition 2.20 and Remark 2.15 allow us to conclude. \square

Proof of Theorem 3.7. By Proposition 2.20 there exists a probability measure $\mathbb{Q} \sim \mathbb{V}$, such that \bar{S} is a \mathbb{Q} -martingale. With the notation of Lemma 3.8, the classical superreplication theorem (see Theorem 5.7 in Delbaen and Schachermayer (1994) and Theorem 3.2 in Kramkov (1996)) shows that

$$\begin{aligned} & \inf\{\overline{V^h(0)} : h \text{ is } \mathbb{P}\text{-admissible and } \bar{C} \leq \overline{V^h(T)}, \mathbb{P}\text{-almost surely}\} \\ & = \sup\{\mathbb{E}^{\tilde{\mathbb{Q}}}[\bar{C}] : \tilde{\mathbb{Q}} \sim \mathbb{Q} \text{ such that } \bar{S} \text{ is a } \tilde{\mathbb{Q}}\text{-martingale}\}. \end{aligned} \quad (26)$$

Recall that Proposition 2.20 yields a relationship between martingale valuation operators and martingale measures in the extended market. This together with Lemmas 2.19 and 3.8 implies (11).

By the same lemmas, Theorem 3.2 in Kramkov (1996) (see also Remark 5.9 in Delbaen and Schachermayer (1994)) guarantees the infimum in (11) to be a minimum if the supremum is finite. Moreover, if the contingent claim C can be replicated by a \mathbb{V} -allowable strategy, then the supremum in (11) is finite and equals to a maximum, due to the equivalence between (i) and (ii) in Lemma 3.8, by virtue of Proposition 2.20. To conclude, let the supremum in (11), and thus, in (26) be finite and equal to a maximum. Then by Théorème 3.2 in Ansel and Stricker (1993), (ii) in Lemma 3.8 holds, and thus the statement follows. \square

5.2 Proofs of Proposition 4.2 and Theorems 4.4 and 4.8

Proof of Proposition 4.2. In the following we argue the three parts of the statement.

(a): Fix i and j and note that (12) yields that $\mathbb{Q}_i(i \notin \mathfrak{A}(t)) = 0$ for all t . Monotone convergence then yields

$$\mathbb{E}^{\mathbb{Q}_i}[S_{i,j}(t)X] = S_{i,j}(0)\mathbb{E}^{\mathbb{Q}_j}[X\mathbf{1}_{\{S_{j,i}(t)>0\}}] \quad (27)$$

for all bounded $\mathcal{F}(t)$ -measurable random variables X and for all t . To show (13), fix a bounded $\mathcal{F}(t)$ -measurable random variable X , $A \in \mathcal{F}(r)$, and $r \leq t$. We then have

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}_i}[S_{i,j}(t)X\mathbf{1}_A] &= \mathbb{E}^{\mathbb{Q}_i}[S_{i,j}(t)X\mathbf{1}_A\mathbf{1}_{\{S_{j,i}(r)>0\}}] = S_{i,j}(0)\mathbb{E}^{\mathbb{Q}_j}[X\mathbf{1}_{\{S_{j,i}(t)>0\}}\mathbf{1}_A\mathbf{1}_{\{S_{j,i}(r)>0\}}] \\ &= S_{i,j}(0)\mathbb{E}^{\mathbb{Q}_j}[\mathbb{E}_r^{\mathbb{Q}_j}[X\mathbf{1}_{\{S_{j,i}(t)>0\}}]\mathbf{1}_A\mathbf{1}_{\{S_{j,i}(r)>0\}}] \\ &= \mathbb{E}^{\mathbb{Q}_i}[S_{i,j}(r)\mathbb{E}_r^{\mathbb{Q}_j}[X\mathbf{1}_{\{S_{j,i}(t)>0\}}]\mathbf{1}_A] \end{aligned}$$

by applying (27) twice, which yields (13). The fact that S_i is a \mathbb{Q}_i -supermartingale follows from (13) with $X = 1$.

(b): Fix i and j . As in Proposition 2.3 in Perkowski and Ruf (2015), we may replace t in (12) by a stopping time τ . With $A = \Omega$, we then have

$$\mathbb{E}^{\mathbb{Q}_i}[S_{i,j}(\tau)] = S_{i,j}(0)\mathbb{Q}_j(S_{j,i}(\tau) > 0)$$

for all stopping times τ . Recall now that $S_{i,j}$ is a \mathbb{Q}^i -supermartingale and localize with a sequence of first crossing times.

(c): The first part follows as in (b). The second statement follows directly from (13). \square

The following lemma shows that numéraire-consistent families of probability measures are in one-to-one correspondence with martingale measures for \bar{S} .

Lemma 5.1 (Numéraire consistency and martingale property for the basket). *The following formulas establish a one-to-one correspondence between numéraire-consistent families of probability measures $(\mathbb{Q}_i)_i$ such that $(\sum_i \mathbb{Q}_i/d) \sim \mathbb{P}$ and probability measures $\mathbb{Q} \sim \mathbb{P}$ such that the process \bar{S} given by (7) is a \mathbb{Q} -martingale.*

$$\mathbb{Q} = \sum_i \bar{S}_i(0)\mathbb{Q}_i; \quad (28)$$

$$\frac{d\mathbb{Q}_i}{d\mathbb{Q}} = \frac{\bar{S}_i(T)}{\bar{S}_i(0)}. \quad (29)$$

Proof. Let $(\mathbb{Q}_i)_i$ be a numéraire-consistent family of probability measures such that $(\sum_i \mathbb{Q}_i/d) \sim \mathbb{P}$. Define the probability measure \mathbb{Q} through (28). It is clear that $\mathbb{Q} \sim \mathbb{P}$. To show that the process \bar{S} is a \mathbb{Q} -martingale we need the following identity:

$$\mathbb{E}^{\mathbb{Q}}[\bar{S}_k(t)\mathbf{1}_A] = \bar{S}_k(0)\mathbb{Q}_k(A) \quad (30)$$

for all k, t and $A \in \mathcal{F}(t)$. Indeed (8) and Proposition 4.2(a) imply

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}[\bar{S}_k(t)\mathbf{1}_A] &= \sum_j \bar{S}_j(0)\mathbb{E}^{\mathbb{Q}_j}[\bar{S}_k(t)\mathbf{1}_A] \\ &= \sum_j \bar{S}_k(0)S_{k,j}(0)\mathbb{E}^{\mathbb{Q}_j}[\bar{S}_k(t)\mathbf{1}_A\mathbf{1}_{\{S_{j,k}(t)>0\}}] \\ &= \sum_j \bar{S}_k(0)\mathbb{E}^{\mathbb{Q}_k}[S_{k,j}(t)\bar{S}_k(t)\mathbf{1}_A] \\ &= \sum_j \bar{S}_k(0)\mathbb{E}^{\mathbb{Q}_k}[\bar{S}_j(t)\mathbf{1}_A] = \bar{S}_k(0)\mathbb{Q}_k(A). \end{aligned}$$

Fixing $A \in \mathcal{F}(r)$ and $r \leq t$ and applying (30) twice yields that \bar{S} is a \mathbb{Q} -martingale.

Let us now fix a probability measure $\mathbb{Q} \sim \mathbb{P}$ such that \bar{S} is a \mathbb{Q} -martingale. Define the probability measures $(\mathbb{Q}_i)_i$ by (29). Since $\sum_i \bar{S}_i(0)\mathbb{Q}_i = \mathbb{Q}$ it is clear that $(\sum_i \mathbb{Q}_i/d) \sim \mathbb{P}$. We now show that the family of probability measures $(\mathbb{Q}_i)_i$ is numéraire-consistent. Indeed, thanks to (8) we have

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}_i}[S_{i,j}(t)\mathbf{1}_A] &= (\bar{S}_i(0))^{-1}\mathbb{E}^{\mathbb{Q}}[S_{i,j}(t)\mathbf{1}_A\bar{S}_i(t)\mathbf{1}_{\{S_{j,i}(t)>0\}}] \\ &= (\bar{S}_i(0))^{-1}\mathbb{E}^{\mathbb{Q}}[\bar{S}_j(t)\mathbf{1}_A\mathbf{1}_{\{S_{j,i}(t)>0\}}] \\ &= (\bar{S}_i(0))^{-1}\bar{S}_j(0)\mathbb{Q}_j(A \cap \{S_{j,i}(t) > 0\}) = S_{i,j}(0)\mathbb{Q}_j(A \cap \{S_{j,i}(t) > 0\}) \end{aligned}$$

for all $i, j, A \in \mathcal{F}(t)$, and t .

The fact that (28) and (29) establish a one-to-one correspondence follows directly from their definition and (30). \square

Proof of Theorem 4.4. First, we argue (a) and (c). Towards this end, let \mathbb{V} be a martingale valuation operator. Recall Proposition 2.20(a) and the unique probability measure \mathbb{Q} satisfying (10), such that \bar{S} is a \mathbb{Q} -martingale. Let $(\mathbb{Q}_i)_i$ be the family of numéraire-consistent measures from Lemma 5.1. Assume now that $\mathbb{V}^{r,t}(C) = \mathbb{V}^{r,t}(C\mathbf{1}_{\{i \in \mathfrak{A}(t)\}})$ for some $C \in \mathcal{D}^t$, $r \leq t$, and i . Next, using (8) yields

$$\begin{aligned} \mathbb{V}_j^{r,t}(C\mathbf{1}_{\{i \in \mathfrak{A}(t)\}}) &= \frac{1}{\bar{S}_j(r)}\mathbb{E}_r^{\mathbb{Q}}[C\mathbf{1}_{\{i \in \mathfrak{A}(t)\}}] = \frac{1}{\bar{S}_j(r)}\mathbb{E}_r^{\mathbb{Q}}[\bar{S}_i(t)\mathbf{1}_{\{\bar{S}_i(t)>0\}}C_i] \\ &= \frac{\bar{S}_i(r)}{\bar{S}_j(r)}\mathbb{E}_r^{\mathbb{Q}_i}[C_i] = S_{j,i}(r)\mathbb{E}_r^{\mathbb{Q}_i}[C_i] \end{aligned}$$

for all $j \in \mathfrak{A}(r)$, using Proposition 4.2(a). This yields (c). Next, fix a general $C \in \mathcal{D}^t$ and $r \leq t$. Remark 2.13 now implies

$$\mathbb{V}^{r,t}(C) = \mathbb{V}^{r,t}\left(\sum_i \frac{C}{|\mathfrak{A}(t)|}\mathbf{1}_{\{i \in \mathfrak{A}(t)\}}\right).$$

Linearity of the martingale valuation operator \mathbb{V} implies (14). The uniqueness of $(\mathbb{Q}_i)_i$ follows from Lemma 5.1.

In order to see (b) argue in the same way and combine Proposition 4.2(a), Lemma 5.1, and Proposition 2.20(b). \square

Proof of Theorem 4.8. Assume there exists a probability measure $\mathbb{Q} \sim (\sum_j \mathbb{Q}_j/d)$ such that $\bar{S}_i = 1/\sum_j S_{i,j}$ is a \mathbb{Q} -martingale. Then Proposition 2.20(b) in conjunction with Remark 2.15 yields the statement. In the following, we argue the existence of such a probability measure \mathbb{Q} if (a) or (b) or hold.

(a): Consider the probability measures $\tilde{\mathbb{Q}}_i$ given by $d\tilde{\mathbb{Q}}_i/d\mathbb{Q}_i = \sum_j S_{j,i}(0)S_{i,j}(T)$ for each i , and $\mathbb{Q} = \sum_i \tilde{\mathbb{Q}}_i/d$. Then we have $\mathbb{Q} \sim (\sum_j \mathbb{Q}_j/d)$. Moreover, \bar{S} is a \mathbb{Q}_i -martingale for each i , thus it is also a \mathbb{Q} -martingale.

(b): We set $\mathbb{P} = \sum_i \mathbb{Q}_i/d$ and fix $\varepsilon > 0$ as in (b)(iv). To prove the statement it suffices to construct a strictly positive \mathbb{P} -martingale Z such that $Z\bar{S}$ is also a \mathbb{P} -martingale. We proceed in several steps.

Step 1: For the construction of Z below, we shall iteratively pick the strongest currency until some time when it is not the strongest anymore, at which point we switch to the new strongest one. To follow this program, define the sequences of stopping times $(\tau_n)_{n \in \mathbb{N}_0}$ and currency identifiers $(i_n)_{n \in \mathbb{N}}$ by $\tau_0 = 0$ and

$$i_n = \arg \min_{i \in \{1, \dots, d\}} \{(\bar{S}_i(\tau_{n-1}))^{-1}\}; \quad (31)$$

$$\tau_n = \inf\{t \in [\tau_{n-1}, T] : (\bar{S}_{i_n}(t))^{-1} > d + \varepsilon\} \quad (32)$$

for all $n \in \mathbb{N}$, where possible conflicts in (31) are solved by lexicographic order.

Step 2: We claim that $\mathbb{P}(\lim_{n \uparrow \infty} \tau_n > T) = 1$. To see this, assume that $\mathbb{P}(\lim_{n \uparrow \infty} \tau_n \leq T) > 0$. Then there exist i and j such that $(\bar{S}_i)^{-1}$ has infinitely many upcrossings from d to $d + \varepsilon$ with strictly positive \mathbb{Q}_j -probability. Next, by a simple localization argument we may assume that $(\bar{S}_j)^{-1}$ is a \mathbb{Q}_j -martingale and we consider the corresponding measure $\hat{\mathbb{Q}}$, given by $d\hat{\mathbb{Q}}/d\mathbb{Q}_j = \bar{S}_j(0)(\bar{S}_j(T))^{-1}$. Note that $\hat{\mathbb{Q}} \sim \mathbb{Q}_j$ and that the process \bar{S}_i is a bounded $\hat{\mathbb{Q}}$ -martingale that has infinitely many downcrossings from $1/d$ to $1/(d + \varepsilon)$ with positive probability. This, however, contradicts the supermartingale convergence theorem, which then yields a contradiction. Thus, the claim holds.

Step 3: Assume that we are given a nonnegative stochastic process Z such that Z^{τ_n} and $Z^{\tau_n} \bar{S}^{\tau_n}$ are \mathbb{P} -martingales for each $n \in \mathbb{N}$, in the notation of (32). We then claim that Z and $Z\bar{S}$ are \mathbb{P} -martingales. To see this, note that Z and $Z\bar{S}$ are \mathbb{P} -local martingales by Step 2. Next, define a sequence of probability measures $(\mathbb{Q}^n)_{n \in \mathbb{N}}$ via $d\mathbb{Q}^n/d\mathbb{P} = Z^{\tau_n}(T)$ and note that \bar{S}^{τ_n} is a \mathbb{Q}^n -martingale satisfying $\bar{S}_{i_n}(\tau_{n-1}) \geq 1/d$ on the event $\{\tau_{n-1} \leq T\}$, where i_n is given in (31). Thus, on $\{\tau_{n-1} \leq T\}$ we have

$$\frac{1}{d} \leq \mathbb{E}^{\mathbb{Q}^n} [\bar{S}_{i_n}(\tau_n) | \mathcal{F}(\tau_{n-1})] \leq 1 - q_n + \frac{q_n}{d + \varepsilon},$$

where $q_n = \mathbb{Q}^n(\tau_n \leq T | \mathcal{F}(\tau_{n-1}))$, for each $n \in \mathbb{N}$. We obtain that

$$q_n \leq \frac{d^2 + \varepsilon d - d - \varepsilon}{d^2 + \varepsilon d - d} = \eta \in (0, 1),$$

which again yields

$$\mathbb{Q}^n(\tau_n \leq T) \leq \mathbb{E}^{\mathbb{Q}^n} [\mathbb{Q}^n(\tau_n \leq T | \mathcal{F}(\tau_{n-1})) \mathbf{1}_{\{\tau_{n-1} \leq T\}}] \leq \eta \mathbb{Q}^n(\tau_{n-1} \leq T) \leq \eta^n$$

for each $n \in \mathbb{N}$, where the last inequality follows by induction. This yields $\lim_{n \uparrow \infty} \mathbb{Q}^n(\tau_n \leq T) = 0$. Now, a simple extension of Lemma III.3.3 in Jacod and Shiryaev (2003), such as the one of Corollary 2.2 in Blanchet and Ruf (2015), yields that Z is a \mathbb{P} -martingale. Since \bar{S} is bounded, also $Z\bar{S}$ is a \mathbb{P} -martingale.

Step 4: We now construct a stochastic process \tilde{Z} that satisfies the assumptions of Step 3. Towards this end, for each i , let Z_i denote the unique \mathbb{P} -martingale such that $d\mathbb{Q}_i/d\mathbb{P} = Z_i(T)$. With the notation of (32), (b)(ii) and (iii) yield that $Z_{i_n}(\tau_{n-1}) > 0$ for each $n \in \mathbb{N}$. This allows us to define the process \tilde{Z} inductively by $\tilde{Z}(0) = 1$ and

$$\tilde{Z}(t) = \tilde{Z}(\tau_{n-1}) \times \frac{(\bar{S}_{i_n}(t))^{-1} \mathbf{1}_{\{Z_{i_n}(t) > 0\}} Z_{i_n}(t)}{(\bar{S}_{i_n}(\tau_{n-1}))^{-1} Z_{i_n}(\tau_{n-1})}$$

for all $t \in (\tau_{n-1}, \tau_n \wedge T]$ and $n \in \mathbb{N}$. Here we have again used the indices $(i_n)_{n \in \mathbb{N}}$ of (31). Since

$$\begin{aligned} \mathbb{E}^{\mathbb{P}}[(\bar{S}_{i_n}(\tau_n))^{-1} \mathbf{1}_{\{Z_{i_n}(\tau_n) > 0\}} Z_{i_n}(\tau_n) | \mathcal{F}(\tau_n)] &= \mathbb{E}^{\mathbb{Q}_{i_n}}[(\bar{S}_{i_n}(\tau_n))^{-1} | \mathcal{F}(\tau_n)] Z_{i_n}(\tau_{n-1}) \\ &= (\bar{S}_{i_n}(\tau_{n-1}))^{-1} Z_{i_n}(\tau_{n-1}) \end{aligned}$$

on $\{\tau_{n-1} \leq T\}$, the process \tilde{Z}^{τ_n} is a \mathbb{P} -martingale for each $n \in \mathbb{N}$. We now fix j and argue that $\bar{S}_j^{\tau_n} \tilde{Z}^{\tau_n}$ is a \mathbb{P} -martingale for each $n \in \mathbb{N}$. Thanks to (8) we have

$$\bar{S}_j(t) \tilde{Z}(t) = \bar{S}_j(\tau_{n-1}) \tilde{Z}(\tau_{n-1}) \times \frac{S_{i_n, j}(t) \mathbf{1}_{\{Z_{i_n}(t) > 0\}} Z_{i_n}(t)}{S_{i_n, j}(\tau_{n-1}) Z_{i_n}(\tau_{n-1})}$$

for all $t \in (\tau_{n-1}, \tau_n \wedge T]$ on $\{\bar{S}_j(\tau_{n-1}) > 0\}$ and $n \in \mathbb{N}$. Since zero is an absorbing state for \bar{S}_j under $\mathbb{P} = \sum_i \mathbb{Q}_i/d$ the same arguments as above yield that $\bar{S}_j^{\tau_n} \tilde{Z}^{\tau_n}$ is a \mathbb{P} -martingale for each $n \in \mathbb{N}$.

Step 5: If \mathbb{P} satisfies (NSD), then \tilde{Z} is strictly positive since $\tilde{Z}_{i_n}(\tau_n) > 0$ for each $n \in \mathbb{N}$, in the notation of (31) and (32). In this case, the proof of (b) is finished. However, under the more general condition in (b)(iv) it cannot be guaranteed that the \mathbb{P} -martingale \tilde{Z} is strictly positive as it might jump to zero on $\bigcup_{n \in \mathbb{N}} \llbracket \tau_n \rrbracket \cap \bigcup_{m \in \{1, \dots, N\}} \llbracket T_m \rrbracket$. To address this issue, we shall modify the construction in *Step 4* at the predictable times $(T_m)_{m \in \{1, \dots, N\}}$ to obtain a strictly positive \mathbb{P} -martingale Z such that also $Z\bar{S}$ is a \mathbb{P} -martingale.

Step 5A: We may assume that $0 < T_m < T_{m+1}$ on $\{T_m < \infty\}$ for all $m \in \{1, \dots, N\}$ and, set, for sake of notational convenience, $T_0 = 0$ and $T_{N+1} = \infty$. In *Step 5B*, we shall construct a family of strictly positive \mathbb{P} -martingales $(Y_m)_{m \in \{1, \dots, N+1\}}$ that satisfy the following two conditions:

- $Y_m = Y_m^{T_m}$ and $Y_m^{T_{m-1}} = 1$; and
- $Y_m \bar{S}^{T_m} - \bar{S}^{T_{m-1}}$ is a \mathbb{P} -martingale for all $m \in \{1, \dots, N+1\}$.

If we have such a family then the process $Z = \prod_{m=1}^{N+1} Y_m$ is a strictly positive \mathbb{P} -martingale and $Z\bar{S}$ a nonnegative \mathbb{P} -martingale. This then concludes the proof.

Step 5B: In order to construct a family of strictly positive \mathbb{P} -martingales $(Y_m)_{m \in \{1, \dots, N+1\}}$ as desired, let us fix some $m \in \{1, \dots, N+1\}$. We first define a process \tilde{Y} by $\tilde{Y}_m = 1$ on $\llbracket 0, T_{m-1} \rrbracket \cap \llbracket 0, \infty \rrbracket$ and then by proceeding exactly as in *Step 4*, but with $\tau_0 = 0$ replaced by $\tau_0 = T_{m-1}$, with \bar{S} replaced by \bar{S}^{T_m} and with Z_i replaced by $Z_i^{T_m}$ for each i . Then \tilde{Y}_m is a nonnegative \mathbb{P} -martingale that satisfies the two conditions of *Step 5A*. Let \tilde{M} now denote the stochastic logarithm of \tilde{Y}_m and M_i the stochastic logarithm of $(\bar{S}_i)^{-1} Z_i$ for each i . Note that, for each i , M_i is only defined up to the first time that $(\bar{S}_i)^{-1} Z_i$ hits zero, see also Appendix A in Larsson and Ruf (2014). Next, define the stochastic process

$$M = \tilde{M} + \left(\frac{1}{|\mathfrak{A}(T_m-)|} \sum_{j \in \mathfrak{A}(T_m-)} \Delta M_j(T_m) - \Delta \tilde{M}(T_m) \right) \mathbf{1}_{\llbracket T_m, \infty \rrbracket};$$

that is, M equals \tilde{M} apart from the modification at time T_m on $\{T_m < \infty\}$, where we replace its jump by the average jumps of the deflators corresponding to the active currencies at this point of time. Then we have $\Delta M > 1$, which implies that its stochastic exponential $Y_m = \mathcal{E}(M)$ is strictly positive. Due to the predictable stopping theorem, Y_m is a \mathbb{P} -martingale, and moreover, the two conditions in *Step 5A* are satisfied. \square

A Proofs of Lemmas 2.18, 2.19, and 3.8 and of Proposition 2.20.

Proof of Lemma 2.18. We first observe that $(\bar{S}_i)_i$ is a \mathbb{P}_j -semimartingale for each j by (8). Since $\sum_j \mathbb{P}_j \sim \mathbb{P}$, Theorems II.2 and II.3 in Protter (2003) yield that $(\bar{S}_i)_i$ is also a \mathbb{P} -semimartingale. Shiryaev and Cherny (2002) prove that $h \in L(\bar{S}, \mathbb{P})$ if and only if $((h\mathbf{1}_{\{h \leq n\}}) \cdot_{\mathbb{P}} \bar{S})_{n \in \mathbb{N}}$ converges in the Émery topology as n tends to infinity; see their remark after Lemma 4.13. This yields $L(\bar{S}, \mathbb{P}) \subset \bigcap_i L(\bar{S}, \mathbb{P}_i)$, and in the same manner, the reverse implication. The last assertion corresponds to Theorem 4.14 in Shiryaev and Cherny (2002). \square

Proof of Lemma 2.19. The process h is a \mathbb{P} -trading strategy with respect to S if and only if $h \in L(S_i, \mathbb{P}_i)$ and

$$V_i^h - V_i^h(0) = h \cdot_{\mathbb{P}_i} S_i, \quad \mathbb{P}_i\text{-almost surely}$$

for all i . Observe that for all i , the semimartingale \bar{S}_i is positive and $\bar{V}^h = \bar{S}_i V_i^h$ under \mathbb{P}_i . Hence, by the change of numéraire theorem (see Geman et al. (1995) and Lemma 4.16 in Pulido (2014)), the process h is a \mathbb{P}_i -trading strategy with respect to S if and only if $h \in L(\bar{S}, \mathbb{P}_i)$ and

$$\bar{V}^h - \bar{V}^h(0) = h \cdot_{\mathbb{P}_i} \bar{S}, \quad \mathbb{P}_i\text{-almost surely}$$

for all i . Lemma 2.18 now implies the first assertion of the statement.

Next, Remark 2.2 and the identity $V_i^h(t) = \bar{V}(t) \sum_j S_{i,j}(t)$ for all $i \in \mathfrak{A}(t)$ and t yield the second assertion of the statement. \square

Proof of Proposition 2.20.

(a): Suppose that \mathbb{V} is a martingale valuation operator and define

$$\mathbb{Q}(A) = \overline{\mathbb{V}^{0,T} \left(\mathbf{1}_A \sum_j I^{(j)}(T) \right)} \quad (33)$$

for all $A \in \mathcal{F}(T)$. This defines a probability measure on $\mathcal{F}(T)$. Indeed, note that *Linearity* and *Martingale Property* of $\mathbb{V}^{0,T}$ yield that

$$\mathbb{Q}(\Omega) = \overline{\sum_j \mathbb{V}^{0,T} (I^{(j)}(T))} = \overline{\sum_j I^{(j)}(0)} = 1.$$

Similarly, *Positivity* and *Linearity* of $\mathbb{V}^{0,T}$ yield $\mathbb{Q}(A) \in [0, 1]$ for all $A \in \mathcal{F}(T)$. *Linearity* of $\mathbb{V}^{0,T}$ then yields that \mathbb{Q} is a finitely additive measure. The sigma additivity of \mathbb{Q} follows from *Continuity From Below* of $\mathbb{V}^{0,T}$, which, in conjunction with *Linearity* of $\mathbb{V}^{0,T}$, also yields

$$\mathbb{E}^{\mathbb{Q}}[X] = \overline{\mathbb{V}^{0,T} \left(X \sum_j I^{(j)}(T) \mathbf{1}_{\{X \neq 0\}} \right)} \quad (34)$$

for all $\mathcal{F}(T)$ -measurable random variables X .

We now fix $r \leq t$, $C \in \mathcal{D}^t$, and $B \in \mathcal{F}(r)$. Then \bar{C} is a bounded $\mathcal{F}(t)$ -measurable random variable. Next, repeatedly applying (34), *Time Consistency*, *Linearity*, *Redundancy*, and *Martingale Property* of \mathbb{V} yields

$$\mathbb{E}^{\mathbb{Q}}[\mathbf{1}_B \bar{C}] = \overline{\mathbb{V}^{0,T} \left(\mathbf{1}_B \bar{C} \sum_j I^{(j)}(T) \mathbf{1}_{\{\bar{C} \neq 0\}} \right)} = \overline{\mathbb{V}^{0,t} \left(\mathbf{1}_B \bar{C} \sum_{j \in \mathfrak{A}(t)} (I^{(j)}(t)) \mathbf{1}_{\{\bar{C} \neq 0\}} \right)} = \overline{\mathbb{V}^{0,t} (\mathbf{1}_B \bar{C})}$$

$$\begin{aligned}
&= \overline{\mathbb{V}^{0,r}(\mathbf{1}_B \overline{\mathbb{V}^{r,t}(C)})} = \overline{\mathbb{V}^{0,r} \left(\mathbf{1}_B \overline{\mathbb{V}^{r,t}(C)} \sum_{j \in \mathfrak{A}(t)} I^{(j)}(r) \mathbf{1}_{\{\overline{\mathbb{V}^{r,t}(C)} \neq 0\}} \right)} \\
&= \overline{\mathbb{V}^{0,T} \left(\mathbf{1}_B \overline{\mathbb{V}^{r,t}(C)} \sum_j I^{(j)}(T) \mathbf{1}_{\{\overline{\mathbb{V}^{r,t}(C)} \neq 0\}} \right)} = \mathbb{E}^{\mathbb{Q}}[\mathbf{1}_B \overline{\mathbb{V}^{r,t}(C)}],
\end{aligned}$$

which in turn yields (10).

The uniqueness of \mathbb{Q} can be argued by using $C = \mathbf{1}_A \sum_j I^{(j)}(T)$, for each $A \in \mathcal{F}(T)$. The property $\mathbb{Q} \sim \mathbb{V}$ follows directly from (10). Using $C = I^{(i)}(T)$ for all i , yields the martingale property of \overline{S} under \mathbb{Q} .

(b): For the converse direction we define \mathbb{V} by

$$\mathbb{V}^{r,t}(C) = \mathbb{E}_r^{\mathbb{Q}}[\overline{C}] \sum_j I^{(j)}(r) \mathbf{1}_{\{\mathbb{E}_r^{\mathbb{Q}}[\overline{C}] \neq 0\}}$$

for all $C \in \mathcal{D}^t$ and $r \leq t$, which is consistent with (10). We first show that \mathbb{V} is a martingale valuation operator. The properties of *Positivity*, *Linearity*, *Continuity From Below*, and *Redundancy* follow from analogous properties of the conditional expectation. The tower property of conditional expectation yields *Time Consistency* of \mathbb{V} . Additionally, for all i and $t \leq T$, since \overline{S}_i is a \mathbb{Q} -martingale, we have

$$\mathbb{V}^{t,T}(I^{(i)}(T)) = \mathbb{E}_t^{\mathbb{Q}}[\overline{S}_i(T)] \sum_j I^{(j)}(t) \mathbf{1}_{\{\mathbb{E}_t^{\mathbb{Q}}[\overline{S}_i(T)] \neq 0\}} = \overline{S}_i(t) \sum_j I^{(j)}(t) \mathbf{1}_{\{i \in \mathfrak{A}(t)\}} = I^{(i)}(t) \mathbf{1}_{\{i \in \mathfrak{A}(t)\}}$$

for all i , which proves *Martingale Property*.

The uniqueness of the martingale valuation operator \mathbb{V} that satisfies (10) follows from Remark 2.4 and *Redundancy* of \mathbb{V} . \square

Proof of Lemma 3.8. As a consequence of Theorem 3.1(b), \mathbb{P} satisfies (Smg) and S satisfies (NFLVR) for allowable strategies. Moreover, by Lemma 2.18 the strategy h is \mathbb{P} -admissible with respect to \overline{S} . Suppose first that h superreplicates $C \in \mathcal{C}^T$. Then we have $\overline{C} \leq \overline{V^h(T)}$, \mathbb{V} -almost surely, and hence, \mathbb{P} -almost surely. Conversely, suppose that $\overline{C} \leq \overline{V^h(T)}$, \mathbb{P} -almost surely. Then we have $C \leq V^h(T)$, \mathbb{V} -almost surely.

To prove the equivalence between (i) and (ii), we consider the following additional statement:

(i') $\overline{C} = \overline{V^h(T)}$, \mathbb{P} -almost surely, and h is \mathbb{P} -maximal with respect to \overline{S} in the following sense: given any \mathbb{P} -admissible strategy g with $\overline{V^h(0)} = \overline{V^g(0)}$ and $\overline{V^h(T)} \leq \overline{V^g(T)}$, \mathbb{P} -almost surely, we have $\overline{V^h(T)} = \overline{V^g(T)}$, \mathbb{P} -almost surely.

The equivalence between (i) and (i') follows, as above. Theorem 13 in Delbaen and Schachermayer (1995) yields the equivalence between (i') and (ii). We now assume that $C \in \mathcal{D}^T$. Then the equivalence between (ii) and (iii) holds, on the one side, because \overline{C} is bounded, and on the other side, because a uniformly bounded local martingale is a martingale. \square

References

- Ansel, J.-P. and Stricker, C. (1993). Couverture des actifs contingents. *Annales de l'Institut Henri Poincaré (B) Probabilités et Statistiques*, 30(2):303–315.
- Barrett, W. W. (1979). A theorem on inverse of tridiagonal matrices. *Linear Algebra and Its Applications*, 27:211–217.

- Biagini, S. and Cont, R. (2006). Model-free representation of pricing rules as conditional expectations. In *Proceedings of the 6th Ritsumeikan International Symposium*, pages 53–66, Singapore. World Scientific.
- Bielecki, T. R., Jeanblanc, M., and Rutkowski, M. (2004). Modeling and valuation of credit risk. In Frittelli, M. and Runggaldier, W. J., editors, *Stochastic Methods in Finance*, pages 27–126. Springer, Berlin.
- Blanchet, J. and Ruf, J. (2015). A weak convergence criterion constructing changes of measure. *Stochastic Models*, forthcoming.
- Brigo, D. (2005). Market models for CDS options and callable floaters. *Risk Magazine*.
- Câmara, A. and Heston, S. L. (2008). Closed-form option pricing formulas with extreme events. *Journal of Futures Markets*, 28(3):213–230.
- Carr, P., Fisher, T., and Ruf, J. (2013). Why are quadratic normal volatility models analytically tractable? *SIAM Journal on Financial Mathematics*, 4:185–202.
- Carr, P., Fisher, T., and Ruf, J. (2014). On the hedging of options on exploding exchange rates. *Finance Stoch.*, 18(1):115–144.
- Cassese, G. (2008). Asset pricing with no exogenous probability measure. *Mathematical Finance*, 18(1):23–54.
- Collin-Dufresne, P., Goldstein, R., and Hugonnier, J. (2004). A general formula for valuing defaultable securities. *Econometrica*, 72(5):1377–1407.
- Cox, A. and Hobson, D. (2005). Local martingales, bubbles and option prices. *Finance and Stochastics*, 9(4):477–492.
- Delbaen, F. and Schachermayer, W. (1994). A general version of the Fundamental Theorem of Asset Pricing. *Mathematische Annalen*, 300(3):463–520.
- Delbaen, F. and Schachermayer, W. (1995). The no-arbitrage property under a change of numéraire. *Stochastics and Stochastic Reports*, 53:213–226.
- Delbaen, F. and Schachermayer, W. (1997). The Banach space of workable contingent claims in arbitrage theory. *Annales de l'Institut Henri Poincaré (B) Probabilités et Statistiques*, 33(1):357–366.
- Delbaen, F. and Schachermayer, W. (1998a). The Fundamental Theorem of Asset Pricing for unbounded stochastic processes. *Mathematische Annalen*, 312(2):215–250.
- Delbaen, F. and Schachermayer, W. (1998b). A simple counterexample to several problems in the theory of asset pricing. *Mathematical Finance*, 8(1):1–11.
- Delbaen, F. and Shirakawa, H. (1996). A note on the no arbitrage condition for international financial markets. *Financial Engineering and the Japanese Markets*, 3(3):239–251.
- Dybvig, P. H. and Ross, S. A. (1987). Arbitrage. In *The New Palgrave: A Dictionary of Economics*, volume 1, pages 100–106. Palgrave MacMillan.
- Föllmer, H. (1972). The exit measure of a supermartingale. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 21:154–166.
- Föllmer, H. and Schied, A. (2011). *Stochastic Finance. An Introduction in Discrete Time*. De Gruyter, Berlin, third edition.
- Geman, H., El Karoui, N., and Rochet, J.-C. (1995). Changes of numéraire, changes of probability measure and option pricing. *J. Appl. Probab.*, 32(2):443–458.
- Harrison, J. M. and Pliska, S. (1981). Martingales and stochastic integrals in the theory of continuous trading. *Stochastic Processes and their Applications*, 11(3):215–260.
- Herdegen, M. (2014). No-arbitrage in a numéraire-independent modelling framework. *Mathematical Finance*, forthcoming.
- Herdegen, M. and Schweizer, M. (2015). Economics-based financial bubbles (and why they imply strict local martingales). Preprint, <http://ssrn.com/abstract=2566815>.
- Heston, S., Loewenstein, M., and Willard, G. (2007). Options and bubbles. *Review of Financial Studies*, 20(2):359–390.
- Hulley, H. and Platen, E. (2012). Hedging for the long run. *Mathematics and Financial Economics*, 6(2):105–124.
- Jacod, J. and Shiryaev, A. N. (2003). *Limit Theorems for Stochastic Processes*. Springer, Berlin, 2nd edition.
- Jamshidian, F. (2004). Valuation of credit default swaps and swaptions. *Finance and Stochastics*, 8(3):343–371.
- Jarrow, R. A. and Yu, F. (2001). Counterparty risk and the pricing of defaultable securities. *Journal of Finance*, 56(5):1765–1799.
- Kardaras, C. (2015). Valuation and parities for exchange options. *SIAM Journal on Financial Mathematics*, 6:140–

- Kramkov, D. (1996). Optional decomposition of supermartingales and hedging contingent claims in incomplete security markets. *Probability Theory and Related Fields*, 105:459–479.
- Larsson, M. and Ruf, J. (2014). Convergence of local supermartingales and Novikov-Kazamaki-type conditions for processes with jumps. Preprint, arXiv:1411.6229.
- Lewis, A. L. (2000). *Option Valuation under Stochastic Volatility*. Finance Press, Newport Beach.
- Madan, D. and Yor, M. (2006). Itô's integrated formula for strict local martingales. In Émery, M. and Yor, M., editors, *Séminaire de Probabilités, XXXIX (Lecture Notes in Mathematics, Volume 1874)*, pages 157–170. Springer, Berlin.
- Paulot, L. (2013). Arbitrage-free pricing before and beyond probabilities. Preprint, arXiv:1310.1102v1.
- Perkowski, N. and Ruf, J. (2015). Supermartingales as Radon-Nikodym densities and related measure extensions. *Ann. Probab.*, 43(6):3133–3176.
- Protter, P. (2013). A mathematical theory of financial bubbles. In Henderson, V. and Sircair, R., editors, *Paris-Princeton Lectures on Mathematical Finance 2013*, pages 1–108. Springer, Cham.
- Protter, P. E. (2003). *Stochastic Integration and Differential Equations*. Springer, New York, 2nd edition.
- Pulido, S. (2014). The Fundamental Theorem of Asset Pricing, the hedging problem and maximal claims in financial markets with short sales prohibitions. *Annals of Applied Probability*, 24(1):54–75.
- Ruf, J. (2013). Negative call prices. *Annals of Finance*, 9:787–794.
- Schervish, M. J., Seidenfeld, T., and Kadane, J. B. (2008). The fundamental theorems of prevision and asset pricing. *International Journal of Approximate Reasoning*, 49(1):148–158.
- Schönbucher, P. J. (2003). A note on survival measures and the pricing of options on credit default swaps. Preprint, retrieved from <http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.14.1824>.
- Schönbucher, P. J. (2004). A measure of survival. *Risk*, 17(8):79–85.
- Shiryaev, A. N. and Cherny, A. S. (2002). Vector stochastic integrals and the Fundamental Theorems of Asset Pricing. *Proceedings of the Steklov Institute of Mathematics*, 237:6–49.
- Sin, C. A. (1998). Complications with stochastic volatility models. *Adv. in Appl. Probab.*, 30(1):256–268.
- Tehranchi, M. R. (2014). Arbitrage theory without a numéraire. Preprint arXiv:1410.2976.
- Večeř, J. (2011). *Stochastic Finance: A Numéraire Approach*. CRC Press.
- Yan, J.-A. (1998). A new look at the Fundamental Theorem of Asset Pricing. *Journal of the Korean Mathematical Society*, 35(3):659–673.