Lipschitz-Killing curvatures and polar images

Nicolas Dutertre

LIPSCHITZ-KILLING CURVATURES AND POLAR IMAGES

NICOLAS DUTERTRE

Abstract. We relate the Lipschitz-Killing measures of a definable set $X \subset \mathbb{R}^n$ in an o-minimal structure to the volumes of generic polar images. For smooth submanifolds of $\mathbb{R}^n$, such results were established by Langevin and Shifrin [17]. Then we give infinitesimal versions of these results. As a corollary, we obtain a relation between the polar invariants of Comte and Merle [7] and the densities of generic polar images.

1. Introduction

Let $M \subset \mathbb{R}^n$ be a smooth compact submanifold of dimension d_M. To each $x \in M$, we can associate a sequence of curvatures $K_0(x), \ldots, K_{d_M}(x)$ called the Lipschitz-Killing-Weyl curvatures. Let us recall their definition. We denote by S_{N_xM} the unit sphere of N_xM, the normal space to M at x. For $i \in \{0, \ldots, d_M\}$, the i-th Lipschitz-Killing-Weyl curvature $K_i(x)$ is defined by

$$K_i(x) = \int_{S_{N_xM}} \sigma_i(II_{x,v})dv,$$

where $II_{x,v}$ is the second fundamental form on M at x in the direction v and $\sigma_i(II_{x,v})$ is the i-th elementary symmetric function of its eigenvalues. The second fundamental form $II_{x,v}$ is defined on T_xM as follows:

$$II_{x,v}(W_1, W_2) = -\langle \nabla_{W_1} W, W_2 \rangle,$$

for W_1 and W_2 in T_xM, where ∇ is the covariant differentiation in $\mathbb{R}^n$ and W is a local extension of v normal to M. Note that $K_i = 0$ if i is odd. These curvatures are important objects: the integrals $\int_M K_i(x)dx$ appear in Weyl's tube formula [34] and the integral $\int_M K_{d_M}(x)dx$ in the Gauss-Bonnet formula [1, 11].

In [17], Langevin and Shifrin explained how to compute the integrals $\int_M K_i(x)dx$, which have a differential geometric definition, by methods of differential topology. Let us explain briefly their work. Let q be an even integer in $\{0, \ldots, d_M\}$, let $G_n^{d_M-q+1}$be the Grassmann manifold of $(d_M - q + 1)$-dimensional linear spaces in $\mathbb{R}^n$ and for $P \in G_n^{d_M-q+1}$, let $\pi_P^M : M \to P$ be the restriction to M of the orthogonal projection on P. Generically the discriminant Δ_P^M of π_P^M, also called the polar image of π_P^M, is almost everywhere a $(d_M - q)$-dimensional submanifold of P. With each regular point y in Δ_P^M, we can associate an integer $\alpha(\pi_P^M, y)$, which is very roughly speaking a kind of Morse index.

Mathematics Subject Classification (2010) : 14B05, 53C65, 58K05.
Keywords: Lipschitz-Killing curvatures, stratified Morse theory, polar varieties, polar images, o-minimal sets, generic projections, fold points.

Langevin and Shifrin showed that

$$\int_M K_q(x)dx = \text{cst}\frac{1}{g_n^{d_M-q+1}}\int_{G_n^{d_M-q+1}}\int_{\Delta_P^M}\alpha(\pi_P^M, y)dydP,$$

where $g_n^{d_M-q+1}$ is the volume of $G_n^{d_M-q+1}$ and "cst" means a universal constant that depends only on $d_M - q$ and n. For $q = d_M$, this formula is exactly the exchange formula or exchange principle (already proved in [16]) which relates the integral $\int_M K_{d_M}(x)dx$ to the Morse indices of critical points of generic orthogonal projections onto lines.

Our first aim is to extend Langevin and Shifrin's results to a large class of non-smooth objects, namely the class of definable sets in an o-minimal structure. Definable sets are a generalization of semi-algebraic sets and global subanalytic sets, we refer the reader to classical references [33, 32, 8, 21] for basic definitions and results on this topics. The study of the geometric properties of these objects was initiated by Fu [12], who developed integral geometry for compact subanalytic sets. Using the technology of the normal cycle, he associated with every compact subanalytic set X of $\mathbb{R}^n$ a sequence of curvature measures

$$\Lambda_0(X,-),\ldots,\Lambda_n(X,-),$$

called the Lipschitz-Killing measures. In [5] (see also [2, 3]), Bröcker and Kuppe gave a geometric characterization of these measures using stratified Morse theory, in the more general setting of definable sets.

Let us describe now how we adapt the technics of Langevin and Shifrin to the singular definable case. We consider a compact definable set $X \subset \mathbb{R}^n$ and assume that it is equipped with a finite definable Whitney stratification $\{S_a\}_{a\in A}$. Let S be a stratum of X of dimension $d_S < n$ and let U be an open subset of X. As in the smooth case, for $q \in \{0,\ldots,d_S\}$ and P generic in G_n^{q+1}, the discriminant $\Delta_P^{S\cap U}$ of the restriction to $S \cap U$ of the orthogonal projection on P is almost everywhere a smooth hypersurface in P, and to almost all y in $\Delta_P^{S\cap U}$, we can assign an index $\alpha(\pi_P^S, y)$, defined by means of stratified Morse theory. Then we set (see Definition 3.17)

$$L_q(X,S,U) = \text{cst}\int_{G_n^{q+1}}\int_{\Delta_P^{S\cap U}}\alpha(\pi_P^S, y)dydP.$$

For $d_S < q \leq n$, we set $L_q(X,S,U) = 0$. If $d_S = n$, we set $L_q(X,S,U) = 0$ if $q < n$ and $L_n(X,S,U) = \text{vol}(S \cap U)$. We define the polar lengths of X (see Definition 3.18) by

$$L_q(X,U) = \sum_{a\in A} L_q(X,S_a,U) \text{ for } q \in \{0,\ldots,n\}.$$

In Theorem 3.20, we establish a connection between the Lipschitz-Killing measures and the polar lengths, namely we prove that for any open subset U of X and for $q \in \{0,\ldots,n\}$,

$$\Lambda_q(X,U) = L_q(X,U).$$

Our second goal is to give infinitesimal versions of the previous equalities. We consider $(X,0) \subset (\mathbb{R}^n,0)$ a germ of a closed definable set equipped with a finite definable Whitney stratification. We define localized versions of the polar lengths that we denote by $L_k^{\text{loc}}(X,0)$, $k = 0,\ldots,n$. Roughly speaking, these localized polar

lengths are mean-values over Grassmann manifolds of weighted sums of densities of polar images. We prove that (see Theorem 4.7)

$$\lim_{\epsilon\to 0}\frac{\Lambda(X, X\cap B_\epsilon)}{b_k\epsilon^k} = L_k^{\mathrm{loc}}(X,0),$$

for $k\in\{0,\dots,n\}$. As a consequence, we obtain in Corollary 4.8 a relation between the $L_k^{\mathrm{loc}}(X,0)$'s and the polar invariants of Comte and Merle [7]. This implies that the $L_k^{\mathrm{loc}}(X,0)$'s are continuous along the strata of a Verdier stratification of X. This result should be related to Teissier's famous result on the constancy of polar multiplicities along the strata of a Whitney stratification of a complex analytic set [29].

In complex analytic and algebraic geometry, polar and relative polar varieties and their relations with curvatures, characteristic classes and equisingularity problems have been widely studied by many authors since the 70's. We cannot give here a complete list of all the interesting papers published on these subjects and we apologize for this. The study of real polar varieties and real equisingularity, suggested by Trotman [30], was started by Comte in [6]. It was continued in [7], [31] and [25]. We hope that the present paper will provide new interesting developments in this theory.

Throughout the paper, we will use the following notations and conventions (some of them have already appeared in this introduction):

- s_k is the volume of unit sphere S^k of dimension k and b_k is the volume of the unit ball B^k of dimension k,
- $\beta(n,k)=\frac{\Gamma(\frac{k+1}{2})\Gamma(\frac{n-k+1}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{n+1}{2})}$ where Γ is the Euler function,
- for $k\in\{0,\dots,n\}$, G_n^k is the Grassmann manifold of k-dimension linear spaces in $\mathbb{R}^n$ equipped with the $O(n)$-invariant density (see for instance [27], p.200), g_n^k is its volume,
- A_n^k is the affine grassmanian of k-dimensional affine spaces in $\mathbb{R}^n$,
- if P is a linear subspace of $\mathbb{R}^n$, S_P is the unit sphere in P, G_P^k is the Grassmann manifold of k-dimensional linear spaces in P, A_P^k is the affine grassmanian of k-dimensional affine spaces in P, $P^\perp$ is the orthogonal space to P, $\pi_P:\mathbb{R}^n\to P$ is the orthogonal projection on P and for any subset $A\subset\mathbb{R}^n$, $\pi_P^A: A\to P$ is its restriction to A,
- for $v\in\mathbb{R}^n$, the function $v^*:\mathbb{R}^n\to\mathbb{R}$ is defined by $v^*(y)=\langle v,y\rangle$,
- in $\mathbb{R}^n$, $B_\epsilon(x)$ is the closed ball of radius ϵ centered at x and $S_\epsilon(x)$ is the sphere of radius ϵ centered at x, if $x=0$, we simply write B_ϵ and S_ϵ,
- if $X\subset\mathbb{R}^n$, $\mathrm{Sing}(X)$ is the singular set of X, $\overline{X}$ is its topological closure, $\mathring{X}$ its topological interior, $\mathrm{Fr}(X)=\overline{X}\setminus X$ its frontier,
- when it makes sense, $\mathrm{vol}(X)$ means the volume of the set X and d_X its dimension,
- if $v_1,\dots,v_k$ are vectors in $\mathbb{R}^n$, $[v_1,\dots,v_k]$ is the linear space spanned by $v_1,\dots,v_k$,
- a universal constant that we do not want to specify will be denoted by "cst",
- the word "smooth" means of class C^3 at least,
- if $f(x,y)=f(x_1,\dots,x_n,y_1,\dots,y_m)$ is a smooth function, ∇f is its gradient and $\nabla_x f$ is its gradient with respect to the variables $x_1,\dots,x_n$.

The paper is organized as follows. Section 2 contains a summary of results on stratified mappings, stratified More theory and the definitions of the Lipschitz-Killing measures and the polar invariants. In Section 3, we study generic polar varieties and images and prove the relation between the Lipschitz-Killing measures and the polar lengths. Section 4 deals with the local situation and contains infinitesimal versions of the results of Section 3.

The author thanks David Mond and Terry Gaffney for interesting discussions on double points of fold singularities.

2. Stratified mappings, stratified Morse theory and Lipschitz-Killing curvatures

In this section, we present different mathematical objects, notions and results that we will use in the next sections.

2.1. **Stratified mappings.** In this subsection, we recall well-known facts on mappings defined on stratified sets and on their critical points and values.

Let $X \subset \mathbb{R}^n$ be a compact definable set equipped with a finite definable Whitney stratification $\mathcal{S} = \{S_a\}_{a\in A}$. The fact that such a stratification exists is due to Loi [20]. Recently Nguyen, Trivedi and Trotman [24] gave another proof of this result. Let $f : \mathbb{R}^n \to \mathbb{R}^k$ be a smooth definable mapping and let $f_{|X} : X \to \mathbb{R}^k$ be its restriction to X. We assume further that f is a submersion in an open neighborhood of X.

A point x in X is a (stratified) critical point of $f_{|X}$ if x is a critical point of $f_{|S}$, where S is the stratum that contains x. This means that

$$\dim(T_xS + T_xf^{-1}(f(x))) < n,$$

or equivalently

$$\dim(T_xS \cap T_xf^{-1}(f(x))) > d_S - k.$$

Note that if $d_S < k$, then all the points of S are critical. If x is not a critical point of $f_{|X}$, we say that x is a regular point of $f_{|X}$. We denote by Σ_f^S the set of critical points of $f_{|S}$ and we set $\Sigma_f^X = \cup_{a\in A}\Sigma_f^{S_a}$.

Lemma 2.1. *The set Σ_f^X is a compact definable set of X.*

Proof. It is enough to prove that it is closed. Let $(x_m)_{m\in\mathbb{N}}$ be a sequence of points in Σ_f^X that tends to x. We can assume that $(x_m)_{m\in\mathbb{N}}$ is included in Σ_f^S. If x belongs to S then x belongs to $\Sigma_f^S \subset \Sigma_f^X$. If x belongs to $S' \subset \mathrm{Fr}(S)$ then, by Whitney condition (a), there exists a vector space T of dimension d_S such that $\dim(T + T_xf^{-1}(f(x))) < n$ and such that $T_xS' \subset T$. Therefore $x \in \Sigma_f^{S'}$. □

Let $y \in \mathbb{R}^k$. We say that y is a critical value of $f_{|X}$ if $f_{|X}^{-1}(y)$ contains a critical point of $f_{|X}$. Otherwise we say that y is a regular value of $f_{|X}$. Note that if y is a regular value of $f_{|X}$, then $f^{-1}(y) \cap X$ is a Whitney stratified set (see for instance [26]).

Lemma 2.2. *The set of regular values of $f_{|X}$ is an open definable and dense subset of $\mathbb{R}^k$.*

Proof. This set is equal to $\mathbb{R}^k \setminus f(\Sigma_f^X)$. By Bertini-Sard's theorem (see [4]), each set $f(\Sigma_f^S)$ is a definable set of dimension less or equal to $k-1$. Hence $f(\Sigma_f^X)$ is a compact definable set of dimension less or equal to $k-1$. □

2.2. **Stratified Morse theory.** Let M be a smooth riemannian manifold. Let X be a Whitney-stratified set of M. Let x be a point in X and let $S(x)$ be the stratum that contains x. A generalized tangent space at x is a limit of a sequence of tangent spaces $(T_{y_m}S_1)_{m\in\mathbb{N}}$, where S_1 is a stratum distinct from $S(x)$ such that $x \in \overline{S_1}$ and $(y_m)_{m\in\mathbb{N}}$ is a sequence of points in S_1 tending to x.

A Morse function $f : X \to \mathbb{R}$ is the restriction of a smooth function $\tilde{f} : M \to \mathbb{R}$ such that the following conditions hold:

(1) f is proper and the critical values of f are distinct i.e., if x and y are two distinct critical points of f then $f(x) \neq f(y)$,
(2) for each stratum S of X, the critical points of $f_{|S}$ are non-degenerate,
(3) at each critical point x, the differential $D\tilde{f}(x)$ does not annihilate any generalized tangent space at x.

Let $f : X \to \mathbb{R}$ be a Morse function, S a stratum of X and x a critical point of $f_{|S}$. Let us write $c = f(x)$. The local Morse data for f at x is the pair

$$\left(B_\epsilon^M(x) \cap f^{-1}([c-\delta, c+\delta]), B_\epsilon^M(x) \cap f^{-1}(c-\delta)\right),$$

where $0 < \delta \ll \epsilon \ll 1$ and $B_\epsilon^M(x)$ is the d_M-dimensional closed ball centered at x and of radius ϵ in M. This definition is justified by the following property (see [13] or [14]). There exists an open subset A in $\mathbb{R}^+ \times \mathbb{R}^+$ such that

(1) the closure $\overline{A}$ of A in $\mathbb{R}^2$ contains an interval $]0, \epsilon[$, $\epsilon > 0$, such that for all $a \in]0, \epsilon[$, the set $\{b \in \mathbb{R}^+ \mid (a, b) \in A\}$ contains an open interval $]0, \delta(a)[$ with $\delta(a) > 0$,
(2) for all $(\epsilon, \delta) \in A$, the above pairs of spaces are homeomorphic.

The local Morse data are Morse data in the sense that $f^{-1}(]-\infty, c+\delta])$ is homeomorphic to the space one gets by attaching $B_\epsilon^M(x) \cap f^{-1}([c-\delta, c+\delta])$ at $f^{-1}(]-\infty, c-\delta])$ along $B_\epsilon^M(x) \cap f^{-1}(c-\delta)$ (see [13], I 3.5).

If x belongs to a stratum of dimension d_X, the local Morse data at x are homeomorphic to the classical Morse data $(B^\lambda \times B^{d_X-\lambda}, \partial B^\lambda \times B^{d_X-\lambda})$ where λ is the Morse index of f at x.

If x belongs to a zero-dimensional stratum then $B_\epsilon^M(x) \cap f^{-1}([c-\delta, c+\delta])$ has the structure of a cone (see [13], I 3.11).

If x lies in a stratum S with $0 < d_S < d_X$, then one can consider the classical Morse data of $f_{|S}$ at x. We will call them tangential Morse data and denote them by $(P_{\mathrm{tg}}, Q_{\mathrm{tg}})$. One may choose a normal slice of N at x, that is a closed submanifold of M of dimension $d_M - d_S$, which intersects S in x transversally. We define the normal Morse data $(P_{\mathrm{nor}}, Q_{\mathrm{nor}})$ at x to be the local Morse data of $f_{|X\cap N}$ at x. Goresky and Mac-Pherson proved that

- the normal Morse data are well defined, that is to say they are independent of the Riemannian metric and the choice of the normal slice ([13], I 3.6),
- the local Morse data (P, Q) of f at x are the product of the tangential and the normal Morse data ([13], I 3.7):

$$(P, Q) \simeq (P_{\mathrm{tg}} \times P_{\mathrm{nor}}, P_{\mathrm{tg}} \times Q_{\mathrm{nor}} \cup P_{\mathrm{nor}} \times Q_{\mathrm{tg}}).$$

This implies that $P = B_\epsilon^M(x) \cap f^{-1}([c-\delta, c+\delta])$ has the structure of a cone.

Definition 2.3. Let $x \in X$ be a critical point of the Morse function $f : X \to \mathbb{R}$. Let (P, Q) be the local Morse data of f at x. The Euler-Poincaré characteristic

$\chi(P,Q) = 1 - \chi(Q)$ is called the stratified Morse index of f at x and is denoted by $\mathrm{ind}(f,X,x)$. If $x \in X$ is not a critical point of f, we set $\mathrm{ind}(f,X,x) = 0$.

If (P_{tg}, Q_{tg}) and (P_{nor}, Q_{nor}) are the tangential and normal Morse data, then one has

$$\chi(P,Q) = \chi(P_{tg}, Q_{tg}) \cdot \chi(P_{nor}, Q_{nor}).$$

Thus we can write

$$\mathrm{ind}(f,X,x) = \mathrm{ind}_{\mathrm{tg}}(f,X,x) \cdot \mathrm{ind}_{\mathrm{nor}}(f,X,x),$$

where $\mathrm{ind}_{\mathrm{tg}}(f,X,x)$ is called the tangential Morse index and $\mathrm{ind}_{\mathrm{nor}}(f,X,x)$ the normal Morse index. Note that if x belongs to a zero-dimensional stratum then $\mathrm{ind}_{\mathrm{tg}}(f,X,x) = 1$. If x belongs a stratum of dimension d_X, then $\mathrm{ind}_{\mathrm{tg}}(f,X,x) = (-1)^{\lambda}$ and $\mathrm{ind}_{\mathrm{nor}}(f,X,x) = 1$, where λ is the Morse index of f at x.

The following theorem relates the Euler characteristic of X to the indices of the critical points of a Morse function.

Theorem 2.4. *Let $X \subset M$ be a compact Whitney-stratified set and let $f : X \to \mathbb{R}$ be a Morse function. One has*

$$\chi(X) = \sum_{x \in X} \mathrm{ind}(f,X,x).$$

2.3. **Lipschitz-Killing measures of definable sets.** In this subsection, we present the Lipschitz-Killing measures of a definable set in an o-minimal structure. We describe Bröcker and Kuppe's approach [5].

Let $X \subset \mathbb{R}^n$ be a compact definable set equipped with a finite definable Whitney stratification $\mathcal{S} = \{S_a\}_{a \in A}$.

Let us fix a stratum S. For $k \in \{0, \ldots, d_S\}$, let $\lambda_k^S : S \to \mathbb{R}$ be defined by

$$\lambda_k^S(x) = \frac{1}{s_{n-k-1}} \int_{S_{T_xS^{\perp}}} \mathrm{ind}_{\mathrm{nor}}(v^*, X, x) \sigma_{d_S-k}(II_{x,v}) dv,$$

where $II_{x,v}$ is the second fundamental form on S in the direction of v and where $\sigma_{d_S-k}(II_{x,v})$ is the (d_S-k)-th elementary symmetric function of its eigenvalues. The index $\mathrm{ind}_{\mathrm{nor}}(v^*, X, x)$ is defined as follows:

$$\mathrm{ind}_{\mathrm{nor}}(v^*, X, x) = 1 - \chi\Big(X \cap N_x \cap B_\epsilon(x) \cap \{v^* = v^*(x) - \delta\}\Big),$$

where $0 < \delta \ll \epsilon \ll 1$ and N_x is a normal (definable) slice to S at x in $\mathbb{R}^n$. Since we work in the definable setting, this index is well-defined thanks to Hardt's theorem [15, 8]. Furthermore when $v^*_{|X}$ has a stratified Morse critical point at x, it coincides with the normal Morse index at x of a function $f : \mathbb{R}^n \to \mathbb{R}$ such that $f_{|X}$ has a stratified Morse critical point at x and $\nabla f(x) = v$. For $k \in \{d_S + 1, \ldots, n\}$, we set $\lambda_k^S(x) = 0$.

If S has dimension n then for all $x \in S$, we put $\lambda_0^S(x) = \cdots = \lambda_{n-1}^S(x) = 0$ and $\lambda_n^S(x) = 1$. If S has dimension 0 then we set

$$\lambda_0^S(x) = \frac{1}{s_{n-1}} \int_{S^{n-1}} \mathrm{ind}_{\mathrm{nor}}(v^*, X, x) dv,$$

and $\lambda_k^S(x) = 0$ if $k > 0$.

Definition 2.5. For every Borel set $U \subset X$ and for every $k \in \{0, \dots, n\}$, we define $\Lambda_k(X, U)$ by

$$\Lambda_k(X, U) = \sum_{a \in A} \int_{S_a \cap U} \lambda_k^{S_a}(x) dx.$$

These measures $\Lambda_k(X, -)$ are called the Lipschitz-Killing measures of X. Note that for any Borel set U of X, we have

$$\Lambda_{d_X+1}(X, U) = \cdots = \Lambda_n(X, U) = 0,$$

and $\Lambda_{d_X}(X, U) = \mathcal{L}_{d_X}(U)$, where $\mathcal{L}_{d_X}$ is the d_X-th dimensional Lebesgue measure in $\mathbb{R}^n$. If X is smooth then for $k \in \{0, \dots, d_X\}$, $\Lambda_k(X, U)$ is equal to

$$\frac{1}{s_{n-k-1}} \int_U K_{d_X-k}(x) dx.$$

As in the smooth case, the measure $\Lambda_0(X, -)$ satisfies an exchange formula (see [5]).

Proposition 2.6. *For every Borel set $U \subset X$, we have*

$$\Lambda_0(X, U) = \frac{1}{s_{n-1}} \int_{S^{n-1}} \sum_{x \in X} \mathrm{ind}(v^*, X, x) dv.$$

In the next section, we will generalize this exchange formula to the other measures $\Lambda_k(X, -)$, $k \geq 1$.

The Lipschitz-Killing measures satisfy the kinematic formula (see [12, 5, 2]). We will need a particular case of this formula, namely the linear kinematic formula.

Proposition 2.7. *Let U be an open subset of X. For $k \in \{0, \dots, n\}$, we have*

$$\Lambda_{n-k}(X, U) = \mathrm{cst} \int_{A_n^k} \Lambda_0(X \cap E, X \cap E \cap U) dE.$$

Proof. See [5], Corollary 8.5. □

In [9], we gave a localized version of this equality. Let $(X, 0) \subset (\mathbb{R}^n, 0)$ be the germ of a closed definable set. Let $H \in G_n^{n-k}$, $k \in \{1, \dots, n\}$, and let v be an element in $S_{H^\perp}$. For $\delta > 0$, we denote by $H_{v,\delta}$ the $(n-k)$-dimensional affine space $H + \delta v$ and we set

$$\beta_0(H, v) = \lim_{\epsilon \to 0} \lim_{\delta \to 0} \Lambda_0(H_{\delta,v} \cap X, H_{\delta,v} \cap X \cap B_\epsilon).$$

Then we set

$$\beta_0(H) = \frac{1}{s_{k-1}} \int_{S_{H^\perp}} \beta_0(H, v) dv.$$

Theorem 2.8. *For $k \in \{1, \dots, n\}$, we have*

$$\lim_{\epsilon \to 0} \frac{\Lambda_k(X, X \cap B_\epsilon)}{b_k \epsilon^k} = \frac{1}{g_n^{n-k}} \int_{G_n^{n-k}} \beta_0(H) dH.$$

Proof. See [9], Theorem 5.5. □

In [9], we also established a relation between the limits $\lim_{\epsilon \to 0} \frac{\Lambda_k(X, X \cap B_\epsilon)}{b_k \epsilon^k}$ and the polar invariants introduced by Comte and Merle in [7]. These polar invariants are real versions of the vanishing Euler characteristics of Lê and Teissier (see [18],

Proposition 6.1.8 or [19], (3.1.2)). They can be defined as follows. Let $H \in G_n^{n-k}$, $k \in \{1, \ldots, n\}$, and let v be an element in $S_{H^\perp}$. We set

$$\alpha_0(H, v) = \lim_{\epsilon \to 0} \lim_{\delta \to 0} \chi(H_{\delta, v} \cap X \cap B_\epsilon),$$

$$\alpha_0(H) = \frac{1}{s_{k-1}} \int_{S_{H^\perp}} \alpha_0(H, v) dv,$$

and then

$$\sigma_k(X, 0) = \frac{1}{g_n^{n-k}} \int_{G_n^{n-k}} \alpha_0(H) dH.$$

Moreover, we put $\sigma_0(X, 0) = 1$.

Theorem 2.9. *For $k \in \{0, \ldots, n-1\}$, we have*

$$\lim_{\epsilon \to 0} \frac{\Lambda_k(X, X \cap B_\epsilon)}{b_k \epsilon^k} = \sigma_k(X, 0) - \sigma_{k+1}(X, 0).$$

Furthermore, we have

$$\lim_{\epsilon \to 0} \frac{\Lambda_n(X, X \cap B_\epsilon)}{b_n \epsilon^n} = \sigma_n(X, 0).$$

Proof. See [9], Theorem 5.6. □

3. Curvature measures and volume of polar images

In this section, we relate the curvature measures of definable sets to the volumes of polar images of generic projections. We start recalling the definition of polar varieties and polar images.

Definition 3.1. Let $P \in G_n^k$, $k = 1, \ldots, n$, and let $M \subset \mathbb{R}^n$ be a smooth submanifold. The polar variety (or polar set) Σ_P^M is the set of critical points of π_P^M, i.e.,

$$\Sigma_P^M = \left\{ x \in M \mid \dim(T_x M \cap P^\perp) \geq d_M - k + 1 \right\},$$

if $k \leq d_M$. If $k > d_M$, we set $\Sigma_P^M = M$.

The polar image of π_P^M is the set Δ_P^M defined by $\Delta_P^M = \pi_P^M(\Sigma_P^M)$.

Let $X \subset \mathbb{R}^n$ be a compact definable set in an o-minimal structure. We equip it with a definable Whitney stratification $\mathcal{S}$. The following statements describe the structure of Σ_P^S and Δ_P^S for a stratum S of X.

The next two lemmas can be proved using the machinery of modular submanifolds of multi-jet spaces developed by Mather and the fact the Thom-Boardman manifolds are modular (see [22], Theorem 2). The proofs we present below are more elementary and enable us to stay in the definable setting.

Lemma 3.2. *Let S be a stratum of X such that $d_S < n$. If $k \leq d_S + 1$ then for almost all $P \in G_n^k$, Σ_P^S is either empty or a $(k-1)$-dimensional definable set. Moreover the set*

$$\Sigma_P^{'S} = \left\{ x \in \Sigma_P^S \mid \dim(T_x S \cap P^\perp) \geq d_S - k + 2 \right\},$$

is a definable subset of Σ_P^S of dimension at most $k-2$ and $\Sigma_P^S \setminus \Sigma_P^{'S}$ is smooth.

Proof. We treat first the case $k \leq d_S$. Let $\mathcal{V} \subset (\mathbb{R}^n)^k$ be the open subset consisting of the k-tuples $(v_1, \ldots, v_k)$ such that $(v_1, \ldots, v_k)$ has rank k. Let M be the following definable set:

$$M = \Big\{(x, v_1, \ldots, v_k) \in \mathbb{R}^n \times \mathcal{V} \mid x \in S \text{ and } \dim(T_x S \cap P^\perp) \geq d_s - k + 1 \\ \text{where } P = [v_1, \ldots, v_k]\Big\}.$$

The set M is the finite disjoint union of the sets M_i, $i = 1, \ldots, k$, where

$$M_i = \Big\{(x, v_1, \ldots, v_k) \in \mathbb{R}^n \times \mathcal{V} \mid x \in S \text{ and } \dim(T_x S \cap P^\perp) = d_s - k + i \\ \text{where } P = [v_1, \ldots, v_k]\Big\}.$$

The set M_1 is a definable manifold of dimension $nk + k - 1$. To see this, we can assume that locally S is given by

$$S = \{x \in \mathbb{R}^n \mid f_1(x) = \cdots = f_{c_S}(x) = 0\},$$

where $c_S = n - d_S$ and $f_1, \ldots, f_{c_S}$ are smooth definable functions. From now on, we set $V = (v_1, \ldots, v_k)$. Then (x, V) is in M_1 if and only if the matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_{c_S}}{\partial x_1}(x) & \cdots & \frac{\partial f_{c_S}}{\partial x_n}(x) \\ v_1^1 & \cdots & v_1^n \\ \vdots & \ddots & \vdots \\ v_k^1 & \cdots & v_k^n \end{pmatrix},$$

has rank $c_S + k - 1$, where we use the notation $v_i = (v_i^1, \ldots, v_i^n) \in \mathbb{R}^n$. It is an easy exercise of linear algebra to see that we can assume that the matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_{c_S}}{\partial x_1}(x) & \cdots & \frac{\partial f_{c_S}}{\partial x_n}(x) \\ v_1^1 & \cdots & v_1^n \\ \vdots & \ddots & \vdots \\ v_{k-1}^1 & \cdots & v_{k-1}^n \end{pmatrix},$$

has rank $c_S + k - 1$ and so that the minor

$$\begin{vmatrix} \frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_{c_S+k-1}}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_{c_S}}{\partial x_1}(x) & \cdots & \frac{\partial f_{c_S}}{\partial x_{c_S+k-1}}(x) \\ v_1^1 & \cdots & v_1^{c_S+k-1} \\ \vdots & \ddots & \vdots \\ v_{k-1}^1 & \cdots & v_{k-1}^{c_S+k-1} \end{vmatrix},$$

does not vanish. Therefore the matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_{c_S}}{\partial x_1}(x) & \cdots & \frac{\partial f_{c_S}}{\partial x_n}(x) \\ v_1^1 & \cdots & v_1^n \\ \vdots & \ddots & \vdots \\ v_{k-1}^1 & \cdots & v_{k-1}^n \end{pmatrix},$$

has rank $c_S + k - 1$ if and only if the following minors:

$$m_j = \begin{vmatrix} \frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_{c_S+k-1}}(x) & \frac{\partial f_1}{\partial x_j}(x) \\ \vdots & \ddots & \vdots & \vdots \\ \frac{\partial f_{c_S}}{\partial x_1}(x) & \cdots & \frac{\partial f_{c_S}}{\partial x_{c_S+k-1}}(x) & \frac{\partial f_{c_S}}{\partial x_j} v \\ v_1^1 & \cdots & v_1^{c_S+k-1} & v_1^j \\ \vdots & \ddots & \vdots & \vdots \\ v_k^1 & \cdots & v_k^{c_S+k-1} & v_k^j \end{vmatrix},$$

$j = c_S + k, \ldots, n$, vanish. Hence M_1 is defined locally by $c_S + d_S - k + 1 = n - k + 1$ equations. Looking at the partial derivatives of the m_j's with respect to the variables v_k^j, $j = c_S + k, \ldots, n$, it easy to see that the gradient vectors of the functions defining M_1 are linearly independent, because the above minor of size $c_S + k - 1$ is not zero. Hence M_1 is a submanifold of dimension $nk + k - 1$. Now let us consider the projection $\pi_1 : M_1 \to \mathcal{V}, (x, V) \mapsto V$. Bertini-Sard's theorem implies that the set D_{π_1} of critical values of π_1 is a definable set of dimension less than nk. Hence for all $V \in \mathcal{V} \setminus D_{\pi_1}$, $\pi_1^{-1}(V)$ is a smooth definable set of dimension $k - 1$ (possibly empty). But $\pi_1^{-1}(V)$ is exactly the set

$$\left\{ x \in \Sigma_P^S \mid \dim(T_x S \cap P^\perp) = d_S - k + 1 \right\},$$

where $V = (v_1, \ldots, v_k)$ and $P = [v_1, \ldots, v_k]$. Similarly we can show that the sets M_i, $i \geq 2$, are definable manifolds of dimension strictly less than $nk + k - 1$ and we can consider the projections $\pi_i : M_i \to \mathcal{V}, (x, V) \mapsto V$. As above, for all $V \in \mathcal{V} \setminus D_{\pi_i}$, where D_{π_i} is the discriminant of π_i, the set

$$\left\{ x \in \Sigma_P^S \mid \dim(T_x S \cap P^\perp) = d_S - k + i \right\},$$

is a smooth definable set of dimension less or equal to $k-2$. Let us denote by $\mathbf{D}$ the union of the discriminants D_{π_i}. Then for all V in $\mathcal{V} \setminus \mathbf{D}$, Σ_P^S is a $(k-1)$-dimensional definable set, $\Sigma_P'^S$ is a definable subset of Σ_P^S of dimension at most $k-2$ and $\Sigma_P^S \setminus \Sigma_P'^S$ is smooth, where $V = (v_1, \ldots, v_k)$ and $P = [v_1, \ldots, v_k]$.

The same proof works if $k = d_S + 1$. □

When $k \leq d_S$, let $\Sigma_P^{S,0}$ be set of fold points of π_P^S and when $k = d_S + 1$, let $\Sigma_P^{S,0}$ be the set of regular points of π_P^S. The following lemma gives a more precise description of Σ_P^S.

Lemma 3.3. *Let S be a stratum of X such that $d_S < n$. If $k \leq d_S + 1$ then for almost all $P \in G_n^k$, $\Sigma_P^{S,0}$ is a smooth definable set of dimension $k-1$ and Σ_P^S admits the following decomposition:*

$$\Sigma_P^S = \Sigma_P^{S,0} \sqcup Z_P^S,$$

where Z_P^S is a closed definable set of dimension strictly less than $k-1$.

Proof. Let us treat first the case $k \leq d_S$. We recall that x is a fold point of π_P^S if $\dim(T_xS \cap P^\perp) = d_S - k + 1$ and $\dim(T_x\Sigma_P^S \cap P^\perp) = 0$. We keep the notations of the previous lemma. Let N be the following definable set:

$$N = \Big\{(x, v_1, \ldots, v_k) \in \mathbb{R}^n \times (\mathcal{V} \setminus \overline{\mathbf{D}}) \mid x \in S,\ \dim(T_xS \cap P^\perp) = d_s - k + 1 \\ \text{and } \dim(T_x\Sigma_P^S \cap P^\perp) \geq 1 \text{ where } P = [v_1, \ldots, v_k]\Big\}.$$

The set N is the finite union of the sets N_i, $i = 1, \ldots, k-1$, where

$$N_i = \Big\{(x, v_1, \ldots, v_k) \in \mathbb{R}^n \times (\mathcal{V} \setminus \overline{\mathbf{D}}) \mid x \in S,\ \dim(T_xS \cap P^\perp) = d_s - k + 1 \\ \text{and } \dim(T_x\Sigma_P^S \cap P^\perp) = i \text{ where } P = [v_1, \ldots, v_k]\Big\}.$$

The set N_1 is a definable submanifold of dimension $nk + k - 2$. To see this, let (x, V) be a point in M_1. Since $\dim(T_xS \cap P^\perp) = d_S - k + 1$, by the proof of Lemma 3.2 we can assume that around (x, V), M_1 is defined by the vanishing of smooth definable functions $f_1, \ldots, f_{c_S}$ and $m_{c_S+k}, \ldots, m_n$, where $f_1, \ldots, f_{c_S}$ depend on x, $m_{c_S+k}, \ldots, m_n$ depend on x and V and where

$$\operatorname{rank}(\nabla f_1(x), \ldots, \nabla f_{c_S}(x), v_1, \ldots, v_{k-1}) = c_S + k - 1.$$

Furthermore since $V \notin D_{\pi_1}$, Σ_P^S is defined locally at x by the vanishing of $f_1, \ldots, f_{c_S}$ and $m_{c_S+k}(-, V), \ldots, m_n(-, V)$ and the following gradient vectors:

$$\nabla f_1, \ldots, \nabla f_{c_S}, \nabla_x m_{c_S+k}(-, V), \ldots, \nabla_x m_n(-, V),$$

are linearly independent. So we see that (x, V) belongs to N_1 if and only if the matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_{c_S}}{\partial x_1}(x) & \cdots & \frac{\partial f_{c_S}}{\partial x_n}(x) \\ \frac{\partial m_{c_S+k}}{\partial x_1}(x) & \cdots & \frac{\partial m_{c_S+k}}{\partial x_n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial m_n}{\partial x_1}(x) & \cdots & \frac{\partial m_n}{\partial x_n}(x) \\ v_1^1 & \cdots & v_1^n \\ \vdots & \ddots & \vdots \\ v_k^1 & \cdots & v_k^n \end{pmatrix},$$

has rank $n-1$. Since $\operatorname{rank}(\nabla f_1(x),\dots,\nabla f_{c_S}(x),v_1,\dots,v_{k-1}) = c_S+k-1$, this condition is equivalent to the fact that the matrix

$$C = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_{c_S}}{\partial x_1}(x) & \cdots & \frac{\partial f_{c_S}}{\partial x_n}(x) \\ \frac{\partial m_{c_S+k}}{\partial x_1}(x) & \cdots & \frac{\partial m_{c_S+k}}{\partial x_n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial m_n}{\partial x_1}(x) & \cdots & \frac{\partial m_n}{\partial x_n}(x) \\ v_1^1 & \cdots & v_1^n \\ \vdots & \ddots & \vdots \\ v_{k-1}^1 & \cdots & v_{k-1}^n \end{pmatrix},$$

has rank $n-1$. As in Lemma 3.2, we can assume that the minor

$$\begin{vmatrix} \frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_{n-1}}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_{c_S}}{\partial x_1}(x) & \cdots & \frac{\partial f_{c_S}}{\partial x_{n-1}}(x) \\ \frac{\partial m_{c_S+k}}{\partial x_1}(x) & \cdots & \frac{\partial m_{c_S+k}}{\partial x_{n-1}}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial m_n}{\partial x_1}(x) & \cdots & \frac{\partial m_n}{\partial x_{n-1}}(x) \\ v_1^1 & \cdots & v_1^{n-1} \\ \vdots & \ddots & \vdots \\ v_{k-2}^1 & \cdots & v_{k-2}^{n-1} \end{vmatrix},$$

does not vanish. The set N_1 is defined locally by the vanishing of the f_i's, the m_j's and the determinant of C. Because the above $(n-1)\times(n-1)$-minor is not zero, we see that the gradient vectors of the functions defining N_1 have rank $n-k+2$ and N_1 is a submanifold of dimension $nk+k-2$. In the same way, the sets N_i, $i\geq 2$, have dimension less than $nk+k-2$ and N is a definable set of dimension at most $nk+k-2$. Let us consider the projection $\rho : N \to \mathcal{V}\setminus\overline{\mathbf{D}}, (x,V)\mapsto V$. Let $\mathbf{D'}$ be the union of the discriminants of the restrictions of ρ to the N_i's. It is a definable set of dimension strictly less than nk. Hence for all $V\in\mathcal{V}\setminus(\overline{\mathbf{D}}\cup\mathbf{D'})$, $\rho^{-1}(V)$ is a definable set of dimension $k-2$ (possibly empty). But $\rho^{-1}(V)$ is exactly $\Sigma_P^S\setminus(\Sigma_P^{S,0}\cup\Sigma_P'^S)$, where $V=(v_1,\dots,v_k)$ and $P=[v_1,\dots,v_k]$. We take $Z_P^S=\Sigma_P'^S\cup\rho^{-1}(V)$. It is closed because $\Sigma_P'^S$ is closed and $\overline{\rho^{-1}(V)}\subset\rho^{-1}(V)\cup\Sigma_P'^S$.

For $k=d_S+1$, the statement is just a reformulation of Lemma 3.2. □

Lemma 3.4. *Let S be a stratum of X such that $d_S<n$. If $k\leq d_S+1$ then for almost all $P\in G_n^k$, Δ_P^S is a definable set of dimension $k-1$ (or empty).*

Proof. The set Δ_P^S is definable as the projection of a definable set. Moreover $\dim(\Delta_P^S)\leq\dim(\Sigma_P^S)=k-1$. The projection $\pi^S_{P|\Sigma_P^{S,0}}:\Sigma_P^{S,0}\to\Delta_P^S$ is a local diffeomorphism because for all $x\in\Sigma_P^{S,0}$, $T_x\Sigma_P^S\cap P^\perp=\{0\}$. Let x be a point in $\Sigma_P^{S,0}$. Then there exists a neighborhood U_x of x included in $\Sigma_P^{S,0}$ on which $\pi^S_{P|\Sigma_P^{S,0}}$

is a diffeomorphism and so dim $\pi_P^S(U_x) = k-1$. But $\pi_P^S(U_x)$ is included in Δ_P^S. Therefore the dimension of Δ_P^S is greater or equal to $k-1$. □

Let $D(\pi^S_{P|\Sigma_P^{S,0}})$ be the set of double-points of $\pi^S_{P|\Sigma_P^{S,0}}$, i.e.,

$$D(\pi^S_{P|\Sigma_P^{S,0}}) = \left\{ x \in \Sigma_P^{S,0} \mid \exists y \in \Sigma_P^{S,0} \text{ with } x \neq y \text{ and } \pi_P(x) = \pi_P(y) \right\}.$$

Lemma 3.5. *Let S be a stratum of X such that $d_S < n$. If $k \leq d_S + 1$ then for almost all $P \in G_n^k$, $D(\pi^S_{P|\Sigma_P^{S,0}})$ is a definable set of dimension at most $k-2$ (or empty).*

Proof. The set $D(\pi^S_{P|\Sigma_P^{S,0}})$ is clearly definable. Let us treat first the case $k \leq d_S$. We keep the notations of the previous lemmas. Let L be the following definable set:

$$L = \Big\{ (x, y, v_1, \ldots, v_k) \in \mathbb{R}^n \times \mathbb{R}^n \times (\mathcal{V} \setminus \overline{\mathbf{D} \cup \mathbf{D}'}) \mid x \in \Sigma_P^{S,0}, y \in \Sigma_P^{S,0}, x \neq y \\ \text{and } \langle v_j, x \rangle = \langle v_j, y \rangle \text{ for all } j \in \{1, \ldots, k\} \text{ where } P = [v_1, \ldots, v_k] \Big\}.$$

This set is a definable submanifold of dimension $nk + k - 2$. Let us explain this assertion. As explained in Lemma 3.2, if $x \in \Sigma_P^{S,0}$ then there exist smooth definable functions $f_1, \ldots, f_{c_S}$ and $m_{c_S+k}(-, V), \ldots, m_n(-, V)$, where $V = (v_1, \ldots, v_k)$, such that in a neighborhood of x, $\Sigma_P^{S,0}$ is the set

$$\{f_1 = \cdots = f_{c_S} = m_{c_S+k}(-, V) = \cdots = m_n(-, V) = 0\},$$

and the gradient vectors of these functions are linearly independent. Furthermore, since x is a fold point,

$$\operatorname{rank}(\nabla f_1(x), \ldots, \nabla f_{c_S}(x), \nabla_x m_{c_S+k}(x, V), \ldots, \nabla_x m_n(x, V), v_1, \ldots, v_k) = n.$$

Similarly, there exist smooth definable functions $g_1, \ldots, g_{c_S}$ and $\tilde{m}_{c_S+k}(-, V), \ldots, \tilde{m}_n(-, V)$ such that in a neighborhood of y, $\Sigma_P^{S,0}$ is the set

$$\{g_1 = \cdots = g_{c_S} = \tilde{m}_{c_S+k}(-, V) = \cdots = \tilde{m}_n(-, V) = 0\},$$

and the gradient vectors of these functions are linearly independent. We also have that

$$\operatorname{rank}(\nabla g_1(y), \ldots, \nabla g_{c_S}(y), \nabla_y \tilde{m}_{c_S+k}(y, V), \ldots, \nabla_y \tilde{m}_n(y, V), v_1, \ldots, v_k) = n.$$

Therefore L is locally given by the vanishing of the functions

$$f_1, \ldots, f_{c_S}, m_{c_S+k}, \ldots, m_n, g_1, \ldots, g_{c_S}, \tilde{m}_{c_S+k}, \ldots, \tilde{m}_n,$$

and

$$\langle v_1, x - y \rangle, \ldots, \langle v_k, x - y \rangle.$$

Let us remark that the f_i's depend on x, the m_j's on x and V, the g_i's on y and the $\tilde{m}_j$'s on y and V. We will show that the gradient vectors of these functions have rank $2n - k + 2$. As in the proof of Lemma 3.2, we can assume that v_k belongs to $[\nabla f_1(x), \ldots, \nabla f_{c_S}(x), v_1, \ldots, v_{k-1}]$. Since $x \neq y$, there is $j \in \{1, \ldots, n\}$ such that

$x_j - y_j \neq 0$. Let us consider the following matrix J of size $(2n-k+2)\times(2n+1)$:

$$J = \begin{pmatrix} \nabla f_1(x) & \mathbf{0} & 0 \\ \vdots & \vdots & \vdots \\ \nabla f_{c_S}(x) & \mathbf{0} & 0 \\ \nabla_x m_{c_S+k}(x,V) & \mathbf{0} & * \\ \vdots & \vdots & \vdots \\ \nabla_x m_n(x,V) & \mathbf{0} & * \\ \mathbf{0} & \nabla g_1(y) & 0 \\ \vdots & \vdots & \vdots \\ \mathbf{0} & \nabla g_{c_S}(y) & 0 \\ \mathbf{0} & \nabla_y \tilde{m}_{c_S}(y,V) & * \\ \vdots & \vdots & \vdots \\ \mathbf{0} & \nabla_y \tilde{m}_n(y,V) & * \\ v_1 & -v_1 & 0 \\ \vdots & \vdots & \vdots \\ v_{k-1} & -v_{k-1} & 0 \\ v_k & -v_k & x_j - y_j \end{pmatrix},$$

where $\mathbf{0} = (0,\dots,0)$ in $\mathbb{R}^n$. It is a submatrix of the matrix whose lines are the gradient vectors of the f_i's, the m_j's, the g_i's, the $\tilde{m}_j$'s and the k functions $\langle v_i, x-y\rangle$. Since v_k belongs to $[\nabla f_1(x),\dots,\nabla f_{n-d_S}(x), v_1,\dots,v_{k-1}]$, J has the same rank as the matrix $\tilde{J}$ given by

$$\tilde{J} = \begin{pmatrix} \nabla f_1(x) & \mathbf{0} & 0 \\ \vdots & \vdots & \vdots \\ \nabla f_{c_S}(x) & \mathbf{0} & 0 \\ \nabla_x m_{c_S+k}(x,V) & \mathbf{0} & * \\ \vdots & \vdots & \vdots \\ \nabla_x m_n(x,V) & \mathbf{0} & * \\ \mathbf{0} & \nabla g_1(y) & 0 \\ \vdots & \vdots & \vdots \\ \mathbf{0} & \nabla g_{c_S}(y) & 0 \\ \mathbf{0} & \nabla_y \tilde{m}_{c_S}(y,V) & * \\ \vdots & \vdots & \vdots \\ \mathbf{0} & \nabla_y \tilde{m}_n(y,V) & * \\ v_1 & -v_1 & 0 \\ \vdots & \vdots & \vdots \\ v_{k-1} & -v_{k-1} & 0 \\ \mathbf{0} & w & x_j - y_j \end{pmatrix},$$

where $w = (w_1,\dots,w_n)$ is a linear combination of the $-v_i$'s and thus belongs to P. Let us denote the lines of $\tilde{J}$ by

$$\mathbf{F}_1,\dots,\mathbf{F}_{c_S},\mathbf{M}_{c_S+k},\dots,\mathbf{M}_n,\mathbf{G}_1,\dots,\mathbf{G}_{c_S},\tilde{\mathbf{M}}_{c_S+k},\dots,\tilde{\mathbf{M}}_n,$$

and

$$\mathbf{V}_1,\dots,\mathbf{V}_k,$$

and show that they are linearly independent. So let us suppose that

$$\sum_i \alpha_i \mathbf{F}_i + \sum_j \beta_j \mathbf{M}_j + \sum_i \gamma_i \mathbf{G}_i + \sum_j \delta_j \tilde{\mathbf{M}}_j + \sum_i \xi_i \mathbf{V}_i = 0.$$

Since

$$\operatorname{rank}(\nabla f_1(x),\dots,\nabla f_{c_S}(x),\nabla_x m_{c_S+k}(x,V),\dots,\nabla_x m_n(x,V),v_1,\dots,v_{k-1}) = n,$$

we see that

$$\alpha_1 = \cdots = \alpha_{c_S} = \beta_{c_S+k} = \cdots = \beta_n = \xi_1 = \cdots = \xi_{k-1} = 0.$$

We are left with the equality

$$\sum_i \gamma_i \mathbf{G}_i + \sum_j \delta_j \tilde{\mathbf{M}}_j + \xi_k \mathbf{V}_k = 0,$$

which implies that $\xi_k w$ belongs to

$$[\nabla g_1(y), \dots, \nabla g_{c_S}(y), \nabla_y \tilde{m}_{c_S+k}(y, V), \dots, \nabla_y \tilde{m}_n(y, V)].$$

But y is a fold point so $\dim(T_y S \cap P^\perp) = d_S - k + 1$ and $\dim(T_y \Sigma_P^S \cap P^\perp) = 0$. Therefore, $\dim(N_y S \cap P) = \dim(N_y \Sigma_P^S \cap P) = 1$ and $N_y S \cap P = N_y \Sigma_P^S \cap P$. So the vector $\xi_k w$ belongs to $[\nabla g_1(y), \dots, \nabla g_{c_S}(y)]$, and $\delta_{c_S+k} = \cdots = \delta_n = 0$. Since $x_j - y_j \neq 0$, we find that $\xi_k = 0$ and finally that $\gamma_1 = \cdots = \gamma_{c_S} = 0$. We conclude that the gradient vectors of the functions which define L locally at (x, y, V) are linearly independent and that L has dimension $nk + k - 2$. As in the previous statements, we see that for almost all $(v_1, \dots, v_k)$ in $\mathcal{V} \setminus \overline{\mathbf{D} \cup \mathbf{D}'}$, the set

$$\Big\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid x \in \Sigma_P^{S,0}, y \in \Sigma_P^{S,0}, x \neq y \\ \text{and } \pi_P^S(x) = \pi_P^S(y) \text{ where } P = [v_1, \dots, v_k]\Big\},$$

has dimension at most $k - 2$. Therefore $D(\pi^S_{P|\Sigma_P^{S,0}})$ has dimension at most $k - 2$ because it is the image of this set by the projection $(x, y) \mapsto x$.

Let us treat the case $k = d_S + 1$. Here we recall that $\Sigma_P^{S,0}$ is the set of regular point of π_P^S. Let Q be the following definable set:

$$Q = \Big\{(x, y, v_1, \dots, v_k) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{V} \mid x \in S, y \in S, x \neq y \\ \text{and } \langle v_j, x\rangle = \langle v_j, y\rangle \text{ for all } j \in \{1, \dots, k\} \text{ where } P = [v_1, \dots, v_k]\Big\}.$$

This set is a definable submanifold of dimension $nk + k - 2$. The proof of this fact is an in the previous case. Locally at a point (x, y, V), Q is given by the vanishing of functions $f_1, \dots, f_{c_S}, g_1, \dots, g_{c_S}$ and the functions $\langle v_1, x - y\rangle, \dots, \langle v_k, x - y\rangle$. As before the f_i's depend only on x and the g_i's only on y. It is not difficult to see that the gradient vectors of these functions are linearly independent because $x - y \neq 0$. Therefore for almost all V in $\mathcal{V}$, the set

$$\Big\{(x, y) \in S \times S \mid x \neq y \text{ and } \pi_P^S(x) = \pi_P^S(y) \text{ where } P = [v_1, \dots, v_k]\Big\},$$

has dimension at most $k - 2$ and so has $D(\pi^S_{P|\Sigma_P^{S,0}})$. □

Remark 3.6. As noticed to the author by Terry Gaffney, we can also use Mather's technology of modular submanifolds to prove the above lemma. Double points of fold singularities form a contact class and a Thom-Boardmann manifold. Hence by [22], pages 233 and 234, they form a modular submanifold. Therefore by Theorem 1 in [22], for almost all $P \in G_n^k$, π_P^S is transverse with respect to this submodular manifold. Our proof avoids the notions of modular manifolds and contact classes and enables us to stay in the definable setting.

For each stratum S and each $P \in G_n^k$, $k \leq d_S + 1$, let $\Sigma^S_{P,\lim}$ be the following set:

$$\Sigma^S_{P,\lim} = \big\{x \in \bar{S} \setminus S \mid \exists (x_m)_{m\in\mathbb{N}} \text{ in } S \text{ such that } x_m \to x, T_{x_m} S \to T \\ \text{and } \dim(T \cap P^\perp) \geq d_S - k + 1\big\}.$$

Proposition 3.7. *Let S be a stratum of X. If $k \leq d_S + 1$ then for almost all $P \in G_n^k$, $\Sigma^S_{P,\lim}$ is a definable set of dimension at most $k-2$ (or empty).*

Proof. We treat first the case $d_S < n$ and $k \leq d_S$. In Lemma 3.2, we proved that the following definable set M:

$$M = \Big\{(x, v_1, \ldots, v_k) \in \mathbb{R}^n \times \mathcal{V} \mid x \in S \text{ and } \dim\,(T_xS \cap P^\perp) \geq d_s - k + 1 \\ \text{where } P = [v_1, \ldots, v_k]\Big\}.$$

had dimension $nk + k - 1$.

If $x \in \Sigma^S_{P,\lim}$ then there exists a sequence $(x_m)_{m\in\mathbb{N}}$ in S such that $x_m \to x$, $T_{x_m} \to T$ and $\dim(T \cap P^\perp) \geq d_S - k + 1$. This implies that there exists a sequence $(P_m)_{m\in\mathbb{N}}$ in G_n^k such that $P_m \to P$ and $\dim(T_{x_m}S \cap P_m^\perp) \geq d_S - k + 1$, because we can find a subspace L_m in $T_{x_m}S$ such that $L_m \to T \cap P^\perp$. Conversely if $T_{x_m}S \to T$, $P_m \to P$ and $\dim(T_{x_m}S \cap P_m^\perp) \geq d_S - k + 1$ then $\dim(T \cap P^\perp) \geq d_S - k + 1$. Therefore $x \in \Sigma^S_{P,\lim}$ if and only if $(x, v_1, \ldots, v_k) \in \overline{M} \setminus M$ where $P = [v_1, \ldots, v_k]$.

The set $\overline{M} \setminus M$ is definable of dimension $nk + k - 2$. We can conclude that for almost all $V = (v_1, \ldots, v_k)$ in $\mathcal{V}$, $\pi^{-1}(V) \cap \overline{M} \setminus M$ has dimension at most $k-2$, where π is the projection $(x, V) \mapsto V$.

If $d_S < n$ and $k = d_S + 1$ then $\Sigma^S_{P,\lim}$ is just $\overline{S} \setminus S$, which has dimension at most $d_S - 1 = k - 2$.

If $d_S = n$, the result is obvious because $\Sigma^S_{P,\lim}$ is empty. □

Let us fix now a stratum S such that $d_S < n$ and consider the strata that contain S in their frontier. We denote them by $S_1, \ldots, S_r$. Note that for $i = 1, \ldots, r$, $\Sigma^{S_i}_{P,\lim} \cap S \subset \Sigma^S_P$ by Whitney condition (a) (see the proof of Lemma 2.1). Applying the previous results, we see that for almost all $P \in G_n^k$, the set T^S_P defined by

$$T^S_P = Z^S_P \bigcup \overline{D(\pi^S_{P|\Sigma^{S,0}_P})} \bigcup \overline{(\cup_{i=1}^r \Sigma^{S_i}_{P,\lim}) \cap S},$$

is a closed definable set of dimension at most $k-2$.

From now on, we fix such a generic P in G_n^k, $k = 1, \ldots, d_S + 1$. Let y be a point in $\Delta^S_P \setminus (\pi^S_P(T^S_P) \cup \mathrm{Sing}(\Delta^S_P))$. Since y does not belong to $\pi(\overline{D(\pi^S_{P|\Sigma^{S,0}_P})})$, there is a unique x in

$$\Sigma^{S,0}_P \setminus \left[\overline{D(\pi^S_{P|\Sigma^{S,0}_P})} \bigcup \overline{(\cup_{i=1}^r \Sigma^{S_i}_{P,\lim}) \cap S}\right],$$

such that $\pi^S_P(x) = y$. Let u be a unit vector in P not belonging to $T_y\Delta^S_P$. Let $Q_u = P^\perp \oplus u$ and let $Q_{u,x}$ be the affine space through x parallel to Q_u. Since $T_x\Sigma^S_P \oplus P^\perp$ has dimension $n-1$ and u does not belong to this vector space, $Q_{u,x}$ intersects Σ^S_P and S transversally at x.

Lemma 3.8. *For any unit vector u in $P \setminus T_y\Delta^S_P$, the point x is a non-degenerate critical point of $u^*_{|Q_{u,x}\cap S}$. Furthermore if ν is the unit normal vector to Δ^S_P at y in P such that $\langle u, \nu\rangle > 0$, then $u^*_{|Q_{u,x}\cap S}$ and $\nu^*_{|Q_{\nu,x}\cap S}$ have the same tangential Morse index.*

Proof. Let us treat first the case $k \leq d_S$. It is clear that x is a critical point of $u^*_{|Q_{x,u}\cap S}$ because $(P^\perp \oplus u) \cap T_xS = P^\perp \cap T_xS$. We keep the notations of the previous proofs. For simplicity, we assume that $y = 0$ in P.

Note that ν is a normal vector to S at x because $T_xS = T_x\Sigma_P^S \oplus (T_xS \cap P^\perp)$. Let $(v_1, \dots, v_{k-1})$ be an orthonormal basis of $T_y\Delta_P^S$ such that $(\nu, v_1, \dots, v_{k-1})$ is a positive orthonormal basis of P. Then $Q_{\nu,x} \cap S$ is the manifold

$$\{f_1 = \cdots = f_{c_S} = v_1^* = \cdots = v_{k-1}^* = 0\},$$

in a neighborhood of x. We can assume that the following minor:

$$m^\nu = \frac{\partial(v_1^*, \dots, v_{k-1}^*, f_1, \dots, f_{c_S})}{\partial(x_1, \dots, x_{k-1}, x_k, \dots, x_{c_S+k-1})},$$

does not vanish at x. Therefore x is a critical point of $\nu^*_{|Q_{\nu,x}\cap S}$ if and only if the following minors:

$$m_j^\nu = \frac{\partial(\nu^*, v_1^*, \dots, v_{k-1}^*, f_1, \dots, f_{c_S})}{\partial(x_1, \dots, x_{k-1}, x_k, \dots, x_{c_S+k-1}, x_j)},$$

$j = c_S + k, \dots, n$, vanish at x. But as explained in the previous lemmas, Σ_P^S is the set $\{f_1 = \cdots = f_{c_S} = m^\nu_{c_S+k} = \cdots = m^\nu_n = 0\}$ in a neighborhood of x. Since $\{v_1^* = \cdots = v_{k-1}^* = 0\}$ intersects Σ_P^S transversally at x, the following $n \times n$ minor:

$$\frac{\partial(v_1^*, \dots, v_{k-1}^*, f_1, \dots, f_{c_S}, m^\nu_{c_S+k}, \cdots, m^\nu_n)}{\partial(x_1, \dots, x_n)},$$

does not vanish at x. By the computations of Szafraniec ([28], pages 248-250), this exactly means that $\nu^*_{|Q_{\nu,x}\cap S}$ has a Morse critical point. Moreover the tangential index of $\nu^*_{|Q_{\nu,x}\cap S}$ at x is the sign of

$$(-1)^{(c_S+k-1)(d_S-k+1)}(m^\nu(x))^{d_S+k+2} \times \frac{\partial(v_1^*, \dots, v_{k-1}^*, f_1, \dots, f_{c_S}, m^\nu_{c_S+k}, \cdots, m^\nu_n)}{\partial(x_1, \dots, x_n)}(x).$$

Let $(w_1, \dots, w_{k-1})$ be an orthonormal basis of u^* such that $(u, w_1, \dots, w_{k-1})$ is a positive orthonormal basis of P. There exists a positive orthogonal $k \times k$ matrix $A = (a_{ij})$ such that

$$\begin{cases} u = a_{11}\nu + \cdots + a_{1k}v_{k-1} \\ \vdots \\ w_{k-1} = a_{k1}\nu + \cdots + a_{kk}v_{k-1}, \end{cases}$$

where $a_{11} > 0$. Then $Q_{u,x} \cap S$ is the manifold

$$\{f_1 = \cdots = f_{c_S} = w_1^* = \cdots = w_{k-1}^* = 0\},$$

in a neighborhood of x. Let m^u be the following minor:

$$m^u = \frac{\partial(w_1^*, \dots, w_{k-1}^*, f_1, \dots, f_{c_S})}{\partial(x_1, \dots, x_{k-1}, x_k, \dots, x_{c_S+k-1})}.$$

Since ν belongs to $[\nabla f_1(x), \dots, \nabla f_{c_S}(x)]$ and A is orthogonal, it is not difficult to see that $m^u(x) = a_{11}m^\nu(x)$ and thus does not vanish. For $j = c_S + k, \dots, n$, let m_j^u be the following minor:

$$m_j^u = \frac{\partial(u^*, w_1^*, \dots, w_{k-1}^*, f_1, \dots, f_{c_S})}{\partial(x_1, \dots, x_{k-1}, x_k, \dots, x_{c_S+k-1}, x_j)}.$$

Since A is orthogonal, m_j^u is equal to m_j^ν and since ν belongs to $[\nabla f_1(x), \dots, \nabla f_{c_S}(x)]$,

$$\frac{\partial(w_1^*, \dots, w_{k-1}^*, f_1, \dots, f_{c_S}, m^u_{c_S+k}, \cdots, m^u_n)}{\partial(x_1, \dots, x_n)}(x) = a_{11}\frac{\partial(v_1^*, \dots, v_{k-1}^*, f_1, \dots, f_{c_S}, m^\nu_{c_S+k}, \cdots, m^\nu_n)}{\partial(x_1, \dots, x_n)}(x).$$

We can conclude that $u^*_{|Q_{u,x}\cap S}$ has a Morse critical point at x and that the two functions have the same tangential Morse index because $a_{11} > 0$.

The case $k = d_S + 1$ is easy because in this situation,

$$Q_{u,x} \cap S = Q_{\nu,x} \cap S = \{x\},$$

in a neighborhood of x and the tangential Morse index is 1. □

By Whitney condition (a), $Q_{u,x}$ intersects also the strata that contains x in their frontier transversally in a neighborhood of x and hence, $Q_{u,x} \cap X$ is Whitney stratified around x.

Lemma 3.9. *For any unit vector u in $P \backslash T_y\Delta_P^S$, the function $u^*_{|Q_{u,x}\cap X}$ is a stratified Morse function in a neighborhood of x, with a critical point at x.*

Proof. We already know that $u^*_{|Q_{u,x}\cap S}$ has a Morse critical point at x. The fact that x does not belong to $\cup_{i=1}^r \Sigma^{S_i}_{P,\lim}$ implies that $P^\perp$ intersects the strata $S_1, \dots, S_r$ transversally in a neighborhood of x. Since $Q_{u,x}$ is also transverse to these strata around x, $u^*_{|Q_{u,x}\cap X}$ has no critical point on these strata.

Let us check now that u is not perpendicular to any limit tangent space at x in $Q_{u,x} \cap X$. If it is not the case then there is a sequence of points $(x_m)_{m\in\mathbb{N}}$ in a stratum S_i such that $x_m \to x$, and $T_{x_m}S_i$ tends to T where T satisfies the following condition: $u \perp (T \cap Q_{u,x})$ in $Q_{u,x}$. This implies that $T \cap P^\perp = T \cap Q_{u,x}$ and that $\dim(T \cap P^\perp) \geq d_{S_i} - k + 1$, which contradicts the fact that x is not in $\Sigma^{S_i}_{P,\lim}$. □

Lemma 3.10. *Let u be a unit vector in $P \setminus T_y\Delta_P^S$ and let ν be the unit normal vector to Δ_P^S at y in P such that $\langle u, \nu\rangle > 0$, then $u^*_{|Q_{u,x}\cap X}$ and $\nu^*_{|Q_{\nu,x}\cap X}$ have the same stratified Morse index.*

Proof. Let us study the behavior of $\pi_P^X : X \to P$ in the neighborhood of x. Since $P^\perp$ is transverse to the strata $S_1, \dots, S_r$ in a neighborhood of x and since $\pi^S_{P|\Sigma_P^{S,0}}$ is a local diffeomorphism, we see that $\pi_P^{-1}(y)$ intersects $X \setminus \{x\}$ transversally in a neighborhood of x. By the Curve Selection Lemma applied to the stratified space $\pi_P^{-1}(y) \cap X$, the function ω_x is a submersion on $\pi_P^{-1}(y) \cap X \setminus \{x\}$, where $\omega_x(z) = \|z - x\|^2$. Hence, for $\epsilon > 0$ small enough so that $S_\epsilon(x)$ is transverse to X in the stratified sense, there exists a small neighborhood U_ϵ of y in P such that for all y' in U_ϵ, $\pi_P^{-1}(y')$ is transverse to $X \cap S_\epsilon(x)$ by Lemma 2.2.

Now in a neighborhood of y, the discriminant of π_P^X is exactly Δ_P^S, so by the Thom-Mather lemma and shrinking U_ϵ if necessary, we can say that if y_1 and y_2 lie in the same component of $U_\epsilon \setminus \Delta_P^S$ then $\pi_P^{-1}(y_1) \cap X \cap B_\epsilon(x)$ and $\pi_P^{-1}(y_2) \cap X \cap B_\epsilon(x)$ are homeomorphic and $\chi(\pi_P^{-1}(y_1) \cap X \cap B_\epsilon(x)) = \chi(\pi_P^{-1}(y_2) \cap X \cap B_\epsilon(x))$. The Morse index of $u^*_{|Q_{u,x}\cap X}$ at x is

$$1 - \chi(Q_{u,x} \cap X \cap \{u^* = u(x) - \delta'\} \cap B_{\epsilon'}(x)),$$

where $0 < \delta' \ll \epsilon' \ll 1$, that is

$$1 - \chi(\pi_P^{-1}(y - \delta' u) \cap X \cap B_{\epsilon'}(x)).$$

Similarly the Morse index of $\nu^*_{|Q_{\nu,x} \cap X}$ at x is

$$1 - \chi(\pi_P^{-1}(y - \delta" \nu) \cap X \cap B_{\epsilon"}(x)),$$

where $0 < \delta" \ll \epsilon" \ll 1$. We can choose ϵ small enough so that the Morse index of $u^*_{|Q_{u,x} \cap X}$ at x is

$$1 - \chi(\pi_P^{-1}(y - \delta' u) \cap X \cap B_{\epsilon}(x)).$$

where $0 < \delta' \ll \epsilon$ and $y - \delta' u \in U_\epsilon \setminus \Delta_P^S$, and the Morse index of $\nu^*_{|Q_{\nu,x} \cap X}$ at x is

$$1 - \chi(\pi_P^{-1}(y - \delta" \nu) \cap X \cap B_{\epsilon}(x)),$$

where $0 < \delta" \ll \epsilon$ and $y - \delta" \nu \in U_\epsilon \setminus \Delta_P^S$. But, if δ' and $\delta"$ are small enough, then $y - \delta' u$ and $y - \delta" \nu$ lie in the same component of $U_\epsilon \setminus \Delta_P^S$, since $\langle u, \nu \rangle > 0$. □

Corollary 3.11. *Let u be a unit vector in $P \setminus T_y \Delta_P^S$ and let ν be the unit normal vector to $T_y \Delta_P^S$ in P such that $\langle u, \nu \rangle > 0$, then $u^*_{|Q_{u,x} \cap X}$ and $\nu^*_{|Q_{\nu,x} \cap X}$ have the same normal Morse index at x.*

Proof. Combine Lemma 3.8, Lemma 3.10 and the results of Subsection 2.2. □

Definition 3.12. For P generic in the sense of Lemmas 3.2, 3.3, 3.5 and Proposition 3.7 and for $y \in \Delta_P^S \setminus (\pi_P^S(T_P^S) \cap \mathrm{Sing}(\Delta_P^S))$, we define

$$\alpha(\pi_P^S, y) = \frac{1}{2}\Big(\mathrm{ind}(\nu^*, X \cap Q_{\nu,x}, x) + \mathrm{ind}(-\nu^*, X \cap Q_{\nu,x}, x)\Big),$$

where ν is a unit normal vector to $T_y \Delta_P^S$ and $y = \pi_P^S(x)$.

Note that by Lemma 3.10, if u is a unit vector in $P \setminus T_y \Delta_P^S$, then

$$\alpha(\pi_P^S, y) = \frac{1}{2}\Big(\mathrm{ind}(u^*, X \cap Q_{u,x}, x) + \mathrm{ind}(-u^*, X \cap Q_{u,x}, x)\Big).$$

Lemma 3.13. *There exists a closed definable set $T_P'^S \subset \Sigma_P^S$ of dimension less or equal to $k-2$ such that for x in $\Sigma_P^S \setminus (T_P^S \cup T_P'^S)$, there exists a neighborhood U_x of x in $\Sigma_P^S \setminus (T_P^S \cup T_P'^S)$ such that $\alpha(\pi_P^S, \pi_P^S(x)) = \alpha(\pi_P^S, \pi_P^S(x'))$ for all $x' \in U_x$.*

Proof. Let $d : \mathbb{R}^n \to \mathbb{R}$ be the distance function to Σ_P^S. It is a continuous definable function and moreover there exists an open definable neighborhood $\mathcal{U}$ of $\Sigma_P^S \setminus T_P^S$ such that d is smooth on $\mathcal{U} \setminus (\Sigma_P^S \setminus T_P^S)$. By the Curve Selection Lemma, we can also assume that d is a (stratified) submersion on $X \cap [\mathcal{U} \setminus (\Sigma_P^S \setminus T_P^S)]$.

Let x be a point in $\Sigma_P^S \setminus T_P^S$ and let ν be a unit normal vector to Δ_P^S at y in P where $\pi_P^S(x) = y$. Since x is an isolated point in $Q_{\nu,x} \cap \Sigma_P^S$, the function $d_{|Q_{\nu,x}} : Q_{\nu,x} \to \mathbb{R}$ is a distance function to x and therefore

$$\mathrm{ind}(\nu^*, X \cap Q_{\nu,x}, x) = 1 - \chi\left(\pi_P^{-1}(y - \delta\nu) \cap X \cap d^{-1}([0, \epsilon])\right),$$

where $0 < \delta \ll \epsilon \ll 1$. As explained in Lemma 3.10, there exists a neighborhood W of y in P such that for all $y_1 \in W \setminus \Delta_P^S$, $\pi_P^{-1}(y_1)$ intersects X transversally in a neighborhood of x. Moreover there exists $\epsilon_x > 0$ and a neighborhood Ω of x in $\Sigma_P^S \setminus T_P^S$ such that for all z in Ω, $\pi_P^{-1}(\pi_P(z))$ intersects $X \setminus \{z\}$ transversally in $\{0 < d \leq \epsilon_x\}$. If it is not the case, we can find a sequence of points $(z_m)_{m \in \mathbb{N}}$ in $\Sigma_P^S \setminus T_P^S$ that tends to x and a sequence of non-negative real numbers $(\epsilon_m)_{m \in \mathbb{N}}$

that tends to 0 such that $\pi_P^{-1}(\pi_P(z_m))$ does not intersect $X \setminus \{z_m\}$ transversally in $\{0 < d \leq \epsilon_m\}$. This implies that x belongs to $\cup_{i=1}^{r} \Sigma_{P,\lim}^{S_i}$, which is excluded.

As in Lemma 3.10, there exists $\epsilon'_x > 0$ such that for $0 < \epsilon \leq \epsilon'_x$, $\pi_P^{-1}(y)$ intersects $X \cap d^{-1}(\epsilon)$ transversally. We are going to explain that for x generic, we can choose ϵ'_x uniformly in a neighborhood of x. Let us consider the mapping

$$\begin{array}{rccc} (d, \pi_P^X) & : & (X \cap \mathcal{U}) \setminus \Sigma_P^X & \to & \mathbb{R} \times P \\ & & x & \mapsto & (d(x), \pi_P^X(x)), \end{array}$$

where Σ_P^X is the union of the $\Sigma_P^{S'}$'s, for S' a stratum of X. Let us call $\mathbf{D}$ its discriminant. It is a definable set of dimension at most k. In $\mathbb{R} \times P$, let π_2 be the projection $(r, y) \mapsto y$. For y' in $\Delta_P^S \setminus \pi_P^S(T_P^S)$, $\pi_2^{-1}(y') \cap \mathbf{D}$ is included in the set

$$\left\{(r, y') \mid r \text{ critical value of } d_{|\pi_P^{-1}(y') \cap [(X \cap \mathcal{U}) \setminus \Sigma_P^X]}\right\},$$

and so $\pi_2^{-1}(y') \cap \mathbf{D}$ has dimension zero. Therefore the set $\pi_2^{-1}(\Delta_P^S \setminus \pi_P^S(T_P^S)) \cap \mathbf{D}$ has dimension at most $k-1$ and the set

$$K_P^S = \overline{\pi_2^{-1}(\Delta_P^S \setminus \pi_P^S(T_P^S)) \cap \mathbf{D}} \cap (\Delta_P^S \setminus \pi_P^S(T_P^S)),$$

has dimension at most $k-2$. Let us set $T_P'^S = \overline{(\pi_P^S)^{-1}(K_P^S)}$. Since $\pi_{P|\Sigma_P^S \setminus T_P^S}^S$ is a local diffeomorphism, $\dim(T_P'^S) \leq k-2$.

Now we suppose that x belongs to $\Sigma_P^S \setminus (T_P^S \cup T_P'^S)$. Then there exist a neighborhood $\Omega' \subset \Omega$ of x in $\Sigma_P^S \setminus (T_P^S \cup T_P'^S)$ and $0 < \epsilon'_x \leq \epsilon_x$ such that for $0 < \epsilon \leq \epsilon'_x$ and $z \in \Omega'$, $\pi_P^{-1}(\pi_P(z))$ intersects $X \cap d^{-1}(\epsilon)$ transversally. If it is not the case, we can find a sequence of points $(z_m)_{m \in \mathbb{N}}$ in Ω that tends to x and a sequence of non-negative real numbers $(\epsilon_m)_{m \in \mathbb{N}}$ that tends to 0 such that $\pi_P^{-1}(\pi_P(z_m))$ does not intersect $X \cap d^{-1}(\epsilon_m)$ transversally. But $(\epsilon_m, \pi_p(z_m))$ belongs to $\pi_2^{-1}(\Delta_P^S \setminus \pi_P^S(T_P^S)) \cap \mathbf{D}$ and so $y = \pi_P^S(x)$ belongs to K_P^S, which is excluded.

Let Ω" be an open neighborhood of x such that $\Omega" \subsetneq \Omega'$. Then for all $0 < \epsilon \leq \epsilon'_x$, there exists $\delta_\epsilon > 0$ such that for $z \in \overline{\Omega"}$ and $0 < \delta \leq \delta_\epsilon$, $\pi_P^{-1}(\pi_P(z) - \delta\nu(z))$ intersects $X \cap d^{-1}(\epsilon)$ transversally, where $\nu(z)$ is the unit normal vector to Δ_P^S at $\pi_P(z)$ in P that lies in the same component as ν. If it is not true, then we can find ϵ with $0 < \epsilon \leq \epsilon'$, a sequence of points $(z_m)_{m \in \mathbb{N}}$ in $\overline{\Omega"}$ that tends to z and a sequence of non-negative real numbers $(\delta_m)_{m \in \mathbb{N}}$ that tends to 0 such that $\pi_P^{-1}(\pi_P(z_m) - \delta_m \nu(z_m))$ does not intersect $X \cap d^{-1}(\epsilon)$ transversally. But $\pi_P^{-1}(\pi_P(z))$ intersects $X \cap d^{-1}(\epsilon)$ transversally and so for w in a neighborhood of $\pi_P(z)$ in P, $\pi_P^{-1}(w)$ intersects $X \cap d^{-1}(\epsilon)$ transversally by Lemma 2.2.

We take $U_x = \Omega" \cap \pi_P^{-1}(W) \cap \Sigma_P^S$ and conclude using the Thom-Mather isotopy lemma as in Lemma 3.10. □

Remark 3.14. We have proved that the index $\mathrm{ind}(\nu^*, X \cap Q_{\nu,x}, x)$ was locally constant. But since the tangential index $\mathrm{ind}_{\mathrm{tg}}(\nu^*, X \cap Q_{\nu,x}, x)$ is locally constant, the normal index $\mathrm{ind}_{\mathrm{nor}}(\nu^*, X \cap Q_{\nu,x}, x)$ is locally constant along $\Sigma_P^S \setminus (T_P^S \cup T_P'^S)$ as well.

Lemma 3.15. *There exists a closed definable set $W_P^S \subset \overline{\Delta_P^S}$ with $\dim(W_P^S) < k-1$ such that $\Delta_P^S \setminus W_P^S$ is smooth of dimension $k-1$ and such that the following function in $y \mapsto \alpha(\pi_P^S, y)$ is defined and constant on each connected component of $\Delta_P^S \setminus W_P^S$.*

Proof. Let

$$W_P^S = \overline{\mathrm{Sing}(\Delta_P^S) \cup \pi_P^S(T_P^S \cup T_P'^S) \cup \pi_P^S(\mathrm{Fr}(\Sigma_P^S))}.$$

By the previous results, W_P^S is a definable set of dimension strictly less than $k-1$. Moreover, W_P^S is a closed set in P, hence $\Delta_P^S \setminus W_P^S$ is an open set in Δ_P^S. The set $\Delta_P^S \setminus W_P^S$ is a smooth $(k-1)$-dimensional manifold for $\mathrm{Sing}(\Delta_P^S) \subset W_P^S$ and the function of the statement is well-defined because $\pi_P^S(T_P^S) \cup \mathrm{Sing}(\Delta_P^S) \subset W_P^S$.

Let y be a point in $\Delta_P^S \setminus W_P^S$ and let x be the point in $\Sigma_P^S \setminus (T_P^S \cup T_P'^S)$ such that $\pi_P^S(x) = y$. We can choose an open neighborhood $U_x \subset \Sigma_P^S \setminus (T_P^S \cup T_P'^S)$ such that $\pi_{P|U_x}^S$ is a diffeomorphism and such that

$$\alpha(\pi_P^S, y) = \alpha(\pi_P^S, \pi_P^S(x')),$$

for each x' in U_x by Lemma 3.13. Let $V \subsetneq U_x$ be a smaller open neighborhood of x in $\Sigma_P^S \setminus (T_P^S \cup T_P'^S)$. Let A be the following set:

$$A = \overline{\Sigma_P^S} \setminus V.$$

It is a compact subset of $\overline{\Sigma_P^S}$, hence $\pi_P^S(A)$ is compact in $\overline{\Delta_P^S}$. The point y does not belong to $\overline{\pi_P^S(A)}$ because y does not belong neither to $\pi_P^S(\mathrm{Fr}(\Sigma_P^S))$ nor to $\pi_P^S(T_P^S \cup T_P'^S)$. Hence there exists an open neighborhood $\mathcal{U}$ of y in $\overline{\Delta_P^S}$ which does not intersect $\pi_P^S(A)$. The function $y' \mapsto \alpha(\pi_P^S, y')$ is constant on $\mathcal{U}' = \mathcal{U} \cap (\Delta_P^S \setminus W_P^S)$. Note that $(\pi_P^S)^{-1}(\mathcal{U}')$ is included U_x. □

Let U be an open subset of X. Recall that $\Delta_P^{S\cap U} = \pi_P^S(\Sigma_P^{S\cap U}) = \pi_P^S(\Sigma_P^S \cap U)$.

Lemma 3.16. *The set $\Delta_P^{S\cap U} \setminus W_P^S$ is an open subset of $\Delta_P^S \setminus W_P^S$.*

Proof. The inclusion is clear. Now let y be a point in $\Delta_P^{S\cap U} \setminus W_P^S$. There exists a unique point x in $(\Sigma_P^S \cap U) \setminus (T_P^S \cup T_P'^S)$ such that $\pi_P^S(x) = y$. We can find a small open neighborhood U_x of x in Σ_P^S such that U_x is included in $(\Sigma_P^S \cap U) \setminus (T_P^S \cup T_P'^S)$. As above, there is an open neighborhood $\mathcal{U}'$ of y in $\Delta_P^S \setminus W_P^S$ such that $(\pi_P^S)^{-1}(\mathcal{U}')$ is included in U_x. □

Definition 3.17. Let U be an open subset of X and let S be a stratum of X. If $d_S < n$, for each $q \in \{0, \ldots, d_S\}$ and for $P \in G_n^{q+1}$ generic in the sense of Lemmas 3.2, 3.3, 3.5 and Proposition 3.7, we set

$$m_{S,q}(P, U) = \int_{\Delta_P^{S\cap U}} \alpha(\pi_P^S, y) dy.$$

We define

$$L_q(X, S, U) = \frac{\beta(1, n-q)}{\beta(q+1, n) g_n^{q+1}} \int_{G_n^{q+1}} m_{S,q}(P, U) dP.$$

For $d_S < q \leq n$, we set $L_q(X, S, U) = 0$.

If $d_S = n$, we set $L_q(X, S, U) = 0$ if $q < n$ and $L_n(X, S, U) = \mathrm{vol}(S \cap U)$.

We note that $m_{S,q}(P, U)$ is well-defined because the definable set W_P^S has dimension at most $k-2$ and the index $\alpha(\pi_P^S, y)$ is defined for y in $\Delta_P^{S\cap U} \setminus W_P^S \subset \Delta_P^S \setminus W_P^S$.

We are in position to define the polar lengths (or polar measures) of a definable stratified set.

Definition 3.18. Let $X \subset \mathbb{R}^n$ be a compact definable set equipped with a finite definable stratification $X = \sqcup_{a\in A} S_a$. Let U be an open subset of X. For each $q \in \{0, \ldots, n\}$, we set

$$L_q(X, U) = \sum_{a\in A} L_q(X, S_a, U).$$

First we remark that $L_0(X,U) = \Lambda_0(X,U)$ since

$$L_0(X,U) = \frac{1}{g_n^1}\int_{G_n^1}\left(\sum_{x\in X}\frac{1}{2}\Big(\mathrm{ind}(v^*,X,x)+\mathrm{ind}(-v^*,X,x)\Big)\right)dL,$$

where v is a unit vector such that $L=[v]$. This can be rewritten

$$L_0(X,U) = \frac{1}{s_{n-1}}\int_{S^{n-1}}\left(\sum_{x\in X}\frac{1}{2}\Big(\mathrm{ind}(v^*,X,x)+\mathrm{ind}(-v^*,X,x)\Big)\right)dv.$$

The right-hand side of this equality coincides with the right-hand side of the equality of Proposition 2.6.

The polar lengths satisfy the linear kinematic formula.

Proposition 3.19. *For $q\in\{0,\dots,n\}$, we have*

$$L_q(X,U) = \mathrm{cst}\int_{A_n^{n-q}} L_0(X\cap L, X\cap L\cap U)dL.$$

Proof. We assume first that $d_X < n$. If $q > d_X$ then the result is clear because almost all L in A_n^{n-q} does not intersect X. Let us fix $q\le d_X$. We have

$$L_q(X,U) = \sum_{a\ |\ d_{S_a}\ge q} L_q(X,S_a,U).$$

Let us a fix a stratum S such that $d_S\ge q$. We have

$$L_q(X,S,U) = \frac{\beta(1,n-q)}{\beta(q+1,n)g_n^{q+1}}\int_{G_n^{q+1}}\int_{\Delta_P^{S\cap U}}\alpha(\pi_P^S,y)dydP.$$

It is clear that

$$\int_{\Delta_P^{S\cap U}}\alpha(\pi_P^S,y)dy = \int_{\Delta_P^{S\cap U}\setminus W_p^S}\alpha(\pi_P^S,y)dy.$$

Let us decompose $\Delta_P^S\setminus W_P^S$ into the finite union of its connected components, i.e. $\Delta_P^S\setminus W_P^S = \cup Y_j^{P,S}$. For each j, let us denote by $\lambda_j^{P,S}$ the value of $\alpha(\pi_P^S,y)$ on $Y_j^{P,S}$. We can write

$$\int_{\Delta_P^{S\cap U}}\alpha(\pi_P^S,y)dy = \sum_j \lambda_j^{P,S}\cdot\mathrm{vol}(Y_j^{P,S}\cap\Delta_P^{S\cap U}).$$

The Cauchy-Crofton formula [10] in P gives

$$\mathrm{vol}(Y_j^{P,S}\cap\Delta_P^{S\cap U}) = \mathrm{cst}\int_{A_P^1}\#(Y_j^{P,S}\cap\Delta_P^{S\cap U}\cap l)dl,$$

and so

$$L_q(X,S,U) = \mathrm{cst}\int_{G_n^{q+1}}\left[\int_{A_P^1}\sum_j\lambda_j^{P,S}\cdot\#(Y_j^{P,S}\cap\Delta_P^{S\cap U}\cap l)dl\right]dP.$$

Let l be a generic line in A_P^1 which intersects the $Y_j^{P,S}$'s transversally and does not meet W_P^S. Let us write $(\cup_j Y_j^{P,S})\cap l = \{y_1,\dots,y_t\}$. Then the affine space L, defined by $L = P^\perp\oplus l$ intersects S transversally. To see this, let x be a point in $S\cap L$. If $x\notin\cup_{k=1}^t(\pi^S_{P|\Sigma_P^S})^{-1}(y_k)$ then $P^\perp$ intersects S transversally at x and so L intersects S transversally at x. If $x\in\cup_{k=1}^t(\pi^S_{P|\Sigma_P^S})^{-1}(y_k)$ then $P^\perp\oplus l$ is exactly the affine space $Q_{u,x}$, where u is a unit vector in the direction of l. Furthermore

the set $\cup_{k=1}^{t}(\pi^S_{P|\Sigma^S_P})^{-1}(y_k)$ is exactly the set of critical points of $u^* : S\cap L \to \mathbb{R}$. Let us denote by Γ^S_U the set of these critical points lying in $S\cap U$. We get

$$\sum_j \lambda_j^{P,S}\cdot \#(Y_j^{P,S}\cap \Delta_P^{S\cap U}\cap l) = \sum_{x\in\Gamma^S_U}\alpha(\pi^S_P,\pi^S_P(x)),$$

and

$$L_q(X,S,U) = \mathrm{cst}\int_{G_n^{q+1}}\int_{A^1_P}\sum_{x\in\Gamma^S_U}\alpha(\pi^S_P,\pi^S_P(x))dldP.$$

Let $\mathcal{F}$ be the flag variety of pairs (P,l), $P\in G_n^{q+1}$ and $l\in A^1_P$. The mapping $(P,l)\mapsto(L,l)$, where $L=P^\perp\oplus l$, enables us to identify $\mathcal{F}$ with the flag variety of pairs (L,l), where $L\in A_n^{n-q}$ and $l\in G^1_L$. Therefore we can write

$$L_q(X,S,U) = \mathrm{cst}\int_{A_n^{n-q}}\left(\int_{G^1_L}\sum_{x\in\Gamma^S_U}\alpha(\pi^S_P,\pi^S_P(x))dl\right)dL,$$

where u is a unit vector vector in l and $P=L^\perp\oplus l$. But for almost all u unit vector in L, $u^*_{|X\cap L}$ is a Morse function (see [5], Lemma 3.5), and by Lemma 3.10,

$$\alpha(\pi^S_P,\pi^S_P(x)) = \frac{1}{2}\Big(\mathrm{ind}(u^*,X\cap L,x)+\mathrm{ind}(-u^*,X\cap L,x)\Big).$$

Hence by the definition of L_0, we can conclude that

$$L_q(X,S,U) = \mathrm{cst}\int_{A_n^{n-q}} L_0(X\cap L, S\cap L, X\cap L\cap U)dL,$$

and so

$$L_q(X,U) = \sum_{a\ |\ d_{S_a}\geq q}\mathrm{cst}\int_{A_n^{n-q}} L_0(X\cap L, S_a\cap L, X\cap L\cap U)dL =$$

$$\mathrm{cst}\sum_a\int_{A_n^{n-q}} L_0(X\cap L, S_a\cap L, X\cap L\cap U)dL,$$

because if $d_{S_a}<q$ then generically L does not meet S_a. Finally we can write

$$L_q(X,U) = \mathrm{cst}\int_{A_n^{n-q}} L_0(X\cap L, X\cap L\cap U)dL.$$

If $d_X=n$ then for $q<n$

$$L_q(X,U) = \sum_{a\ |\ d_{S_a}<n} L_q(X,S_a,U) =$$

$$\mathrm{cst}\sum_{a\ |\ d_{S_a}<n}\int_{A_n^{n-q}} L_0(X\cap L, S_a\cap L, X\cap L\cap U)dL.$$

But if S is a stratum of dimension n then $L_0(X\cap L, S\cap L, X\cap L\cap U)=0$ because $S\cap L\cap U$ is an open subset of the affine space L, on which any u^* has no critical point. Furthermore

$$L_n(X,U) = \sum_{a\ |\ d_{S_a}=n} L_n(X,S,U) = \sum_{a\ |\ d_{S_a}=n}\mathrm{vol}(S_a\cap U) =$$

$$\sum_{a\ |\ d_{S_a}=n}\int_{S_a\cap U} dx = \sum_{a\ |\ d_{S_a}=n}\int_{\mathbb{R}^n} L_0(X\cap\{x\}, S_a\cap\{x\}, X\cap\{x\}\cap U)dx.$$

□

Theorem 3.20. *For $q \in \{0, \dots, n\}$, $L_q(X,U) = \Lambda_q(X,U)$. The polar lengths $L_q(X,U)$ do not depend on the stratification of X.*

Proof. First let us recall that $L_0(X,U) = \Lambda_0(X,U)$. By the linear kinematic formula for Λ_q (see Proposition 2.7) and the previous proposition, we see that L_q and Λ_q are proportional. Let us prove that they coincide on q-dimensional smooth sets. It is true by definition for n-dimensional smooth sets. Let S be a smooth compact definable manifold of dimension $q < n$ and let U be an open subset of S. By the Cauchy-Crofton formula [10], we have

$$\mathrm{vol}(S \cap U) = \frac{1}{\beta(q,n) g_n^q} \int_{G_n^q} \left[\int_H \left(\# \pi_H^{S \cap U} \right)^{-1} (z) dz \right] dH,$$

and

$$\mathrm{vol}(S \cap U) = \frac{1}{\beta(q,n) g_n^q} \int_{G_n^q} \left[\int_H \frac{1}{g_{n-q}^1} \int_{G^1_{H^\perp}} \left(\# \pi_H^{S \cap U} \right)^{-1} (z) dl dz \right] dH.$$

Let $P \in G_n^{q+1}$ such that $H \subset P$ and let π_H^P be the orthogonal projection on H restricted to P. We have $\pi_H^{S\cap U} = \pi_H^P \circ \pi_P^{S \cap U}$ and so

$$\#(\pi_H^{S\cap U})^{-1}(z) = \# \left\{ (\pi_P^{S\cap U})^{-1} \left[(\pi_H^P)^{-1}(z) \right] \right\},$$

i.e.,

$$\#(\pi_H^{S\cap U})^{-1}(z) = \sum_{y \in (\pi_H^P)^{-1}(z) \cap \Delta_P^{S\cap U}} \#(\pi_P^{S\cap U})^{-1}(y).$$

Writing $P = H \oplus l$, where $l \in G^1_{H^\perp}$, we can identify the flag variety of pairs (P,H), $P \in G_n^{q+1}$ and $H \in G_P^q$, with the flag variety (H,l), $H \in G_n^q$ and $l \in G^1_{H^\perp}$. Therefore, we can write

$$\mathrm{vol}(S \cap U) = \frac{1}{\beta(q,n) g_n^q g_{n-q}^1} \int_{G_n^{q+1}} \int_{G_P^q} \int_H \sum_{y \in (\pi_H^P)^{-1}(z) \cap \Delta_P^{S\cap U}} \#(\pi_P^{S\cap U})^{-1}(y) dz dH dP.$$

Now we know that for P generic in G_n^{q+1}, there is a definable set W_P^S of dimension strictly less than q such that $\#(\pi_P^S)^{-1}(y)$ is equal to 1 on $\Delta_P^S \setminus W_P^S$. Therefore we have

$$\sum_{y \in (\pi_H^P)^{-1}(z) \cap \Delta_P^{S\cap U}} \#(\pi_P^{S\cap U})^{-1}(y) = \# \left\{ (\pi_H^P)^{-1}(z) \cap \Delta_P^{S\cap U} \right\}.$$

Applying the Cauchy-Crofton formula to compute $\mathrm{vol}(\Delta_P^{S\cap U})$, we find that

$$\mathrm{vol}(S \cap U) = \frac{\beta(q,q+1) g_{q+1}^q}{\beta(q,n) g_n^1 g_{n-q}^1} \int_{G_n^{q+1}} \mathrm{vol}(\Delta_P^{S\cap U}) dP,$$

which we can rewrite

$$\mathrm{vol}(S \cap U) = \frac{\beta(1,n-q)}{\beta(q+1,n) g_n^{q+1}} \int_{G_n^{q+1}} \int_{\Delta_P^{S\cap U}} \alpha(\pi_P^S, y) dy dP = L_q(S,U),$$

because here $\alpha(\pi_P^S, y) = 1$. □

Let us remark that this last equality can be stated in a nicer way. Let $X \subset \mathbb{R}^n$ be a compact definable set and let $U \subset X$ be an open subset. If $d_X \leq n-1$ then

$$\mathrm{vol}(X \cap U) = \frac{\beta(1, n-d_X)}{\beta(d_X+1, n) g_n^{d_X+1}} \int_{G_n^{d_X+1}} \mathrm{vol}(\pi_P^X(X \cap U)) dP.$$

Of course, this equality is trivial if $d_X = n-1$.

4. Limits of Lipschitz-Killing curvatures and density of polar images

In this section, we give formulas which can be viewed as infinitesimal versions of the formulas established in the previous section.

Let $(X,0) \subset (\mathbb{R}^n, 0)$ be the germ at the origin of a closed definable set. We assume that X is equipped with a finite definable stratification $\{S_a\}_{a=0}^t$, where S_0 is the stratum that contains 0 and 0 belongs to the frontier of each S_a, $a = 1, \ldots, t$.

As in the previous section, for every stratum S of X, Σ_P^S will denote the polar variety of π_P^S, where $P \in G_n^k$, $k = 1, \ldots, n$. As explained in [7], the projection of a definable germ does not define necessarily a germ. But, generically, it defines a definable germ thanks to the following proposition.

Proposition 4.1. *Let $k \in \{1, \ldots, n\}$. For almost all $P \in G_n^k$, there exists an open neighborhood U of 0 in $\mathbb{R}^n$ such that*

$$(U \cap \overline{\Sigma_P^S} \cap P^\perp) \setminus \{0\} = \emptyset.$$

Proof. See [7], Proposition 2.2. □

Corollary 4.2. *Let $(X,0) \subset (\mathbb{R}^n, 0)$ be the germ of a closed definable set. Let $k \in \{1, \ldots, n\}$. Then for almost all $P \in G_n^k$, there exits $\epsilon_P > 0$ such that for all $\epsilon \in]0, \epsilon_P]$, there exists $\eta_\epsilon > 0$ such that*

$$B_{\eta_\epsilon}^P \cap \pi_P^X(X \cap B_\epsilon) = B_{\eta_\epsilon}^P \cap \pi_P^X(X \cap B_{\epsilon_P}),$$

where $B_{\eta_\epsilon}^P$ is the ball of radius η_ϵ centered at the origin in P. In other words, a generic projection of the germ $(X,0)$ defines a germ at the origin in P.

Proof. See [7], Proposition 2.3. □

Corollary 4.3. *Let S be a stratum of X. Then for almost all $P \in G_n^k$, there exists $\epsilon_P > 0$ such that for all $\epsilon \in]0, \epsilon_P]$, there exists η_ϵ such that*

$$B_{\eta_\epsilon}^P \cap \pi_P^X(\overline{\Sigma_P^S} \cap B_\epsilon) = B_{\eta_\epsilon}^P \cap \pi_P^X(\overline{\Sigma_P^S} \cap B_{\epsilon_P}),$$

and

$$B_{\eta_\epsilon}^P \cap \pi_P^X(\mathrm{Fr}(\overline{\Sigma_P^S} \cap B_\epsilon)) = B_{\eta_\epsilon}^P \cap \pi_P^X(\mathrm{Fr}(\overline{\Sigma_P^S} \cap B_{\epsilon_P})).$$

Proof. See [7], Proposition 2.4. □

For each stratum S of X and for P generic in G_n^k, $k = 1, \ldots, d_S + 1$, we recall that T_P^S is the following closed definable subset of Σ_P^S of dimension at most $k-2$:

$$T_P^S = Z_P^S \bigcup \overline{D(\pi^S_{P|\Sigma_P^{S,0}})} \bigcup \overline{(\cup_i \Sigma^{S_i}_{P,\lim}) \cap S},$$

where the strata S_i contain S in their frontier, and that $T_P'^S$ is the closed definable subset of Σ_P^S of dimension at most $k-2$ defined in Lemma 3.13. As above, we have:

Corollary 4.4. *Let S be a stratum of X and let $k \in \{1, \ldots, d_S + 1\}$. Then for almost all $P \in G_n^k$, there exists $\epsilon_P > 0$ such that for all $\epsilon \in]0, \epsilon_P]$, there exists η_ϵ such that*

$$B_{\eta_\epsilon}^P \cap \pi_P^X(\overline{T_P^S \cup T_P'^S} \cap B_\epsilon^n) = B_{\eta_\epsilon}^P \cap \pi_P^X(\overline{T_P^S \cup T_P'^S} \cap B_{\epsilon_P}^n).$$

For $\epsilon > 0$, let us denote by $\Delta_P^{S,\epsilon}$ the definable set $\pi_P^S(\overline{\Sigma_P^S} \cap B_\epsilon^n)$. By the results of Section 2, generically, $\Delta_P^{S,\epsilon}$ is a definable subset of P of dimension $k-1$. Let $W_P^{S,\epsilon}$ be the following closed definable set:

$$W_P^{S,\epsilon} = \overline{\mathrm{Sing}(\Delta_P^{S,\epsilon}) \cup \pi_P^S((T_P^S \cup T_P'^S) \cap B_\epsilon^n) \cup \pi_P^S(\mathrm{Fr}(\overline{\Sigma_P^S} \cap B_\epsilon^n))}.$$

As in Section 3, $W_P^{S,\epsilon}$ is a closed definable set included in $\Delta_P^{S,\epsilon}$ of dimension $k-2$. Let us consider the connected components of $\Delta_P^{S,\epsilon} \setminus W_P^{S,\epsilon}$ that contain 0 in their closure. We denote them by $Y_1^{P,S,\epsilon}, \ldots, Y_{r_{P,S}}^{P,S,\epsilon}$. As in Section 2, on each $Y_j^{P,S,\epsilon}$ the function $y \mapsto \alpha(\pi_P^S, y)$ is constant. Furthermore, since by Corollaries 4.3 and 4.4 the sets $\Delta_P^{S,\epsilon}$ and $W_P^{S,\epsilon}$ define germs, the sets $Y_1^{P,S,\epsilon}, \ldots, Y_{r_P}^{P,S,\epsilon}$ define germs as well.

From now on, we will denote by Δ_P^S the germ defined by the $\Delta_P^{S,\epsilon}$'s and for $j \in \{1, \ldots, r_{P,S}\}$, by $Y_j^{P,S}$ the germ defined by the $Y_j^{P,S,\epsilon}$'s. To each $Y_j^{P,S}$, we can associate the following integer

$$\lambda_j^{P,S} = \lim_{\epsilon \to 0} \lim_{\substack{y \to 0 \\ y \in Y_j^{P,S}}} \alpha(\pi_P^S, y).$$

This integer does not depend on ϵ nor on y, provided they are small enough. We are now in position to define the localizations of the polar lengths.

Definition 4.5. Let S be a stratum of X such that $0 \in \overline{S}$. If $d_S < n$ then for each $q \in \{0, \ldots, d_S\}$, we set

$$L_q^{\mathrm{loc}}(X, S, 0) = \frac{1}{g_n^{q+1}} \int_{G_n^{q+1}} \left[\sum_{j=1}^{r_{P,S}} \lambda_j^{P,S} \cdot \Theta_q(Y_j^{P,S}, 0) \right] dP.$$

For $q > d_S$, we set $L_q^{\mathrm{loc}}(X, S, 0) = 0$.

If $d_S = n$ then we set $L_q^{\mathrm{loc}}(X, S, 0) = 0$ for $q < n$ and $L_n^{\mathrm{loc}}(X, S, 0) = \Theta_n(S, 0)$.

Definition 4.6. Let $(X, 0)$ be the germ of a closed definable set and let $\{S_a\}_{a=0}^t$ be a definable stratification of X such that $0 \in \overline{S_a}$ for $\alpha \in \{0, \ldots, t\}$. For each $q \in \{0, \ldots, n\}$, we set

$$L_q^{\mathrm{loc}}(X, 0) = \sum_{a=0}^{t} L_q^{\mathrm{loc}}(X, S_a, 0).$$

The following theorem relates the limits $\lim_{\epsilon \to 0} \frac{\Lambda_k(X, X \cap B_\epsilon^n)}{b_k \epsilon^k}$ to the polar lengths. It can be viewed as an infinitesimal version of Theorem 3.20.

Theorem 4.7. *Let $(X, 0) \subset (\mathbb{R}^n, 0)$ be the germ of a closed definable set. For $k \in \{0, \ldots, n\}$, we have*

$$\lim_{\epsilon \to 0} \frac{\Lambda_k(X, X \cap B_\epsilon)}{b_k \epsilon^k} = L_k^{\mathrm{loc}}(X, 0).$$

Proof. Let us assume first that $d_X < n$. If $k > d_X$ then the result is obvious by the definitions of the Λ_k's and L_k^{loc}'s. Let us fix $k \leq d_X$. In this case,

$$L_k^{\mathrm{loc}}(X,0) = \sum_{a \ | \ d_{S_a} \geq k} L_k^{\mathrm{loc}}(X, S_a, 0).$$

Let us fix a stratum S such that $d_S \geq k$. We know that

$$L_k^{\mathrm{loc}}(X,S,0) = \frac{1}{g_n^{k+1}} \int_{G_n^{k+1}} \left[\sum_{j=1}^{r_{P,S}} \lambda_j^{P,S} \cdot \Theta_k(Y_j^{P,S}, 0) \right] dP.$$

We apply the Cauchy-Crofton formula for the density due to Comte [6]. Actually we will use here a bit different version of this formula, namely we will apply Theorem 2.8.

For $P \in G_n^{k+1}$, let $L \in G_P^1$ and $L^{\perp P}$ be the orthogonal complement of L in P. For each v in $S_{L^{\perp P}}$ and each $\delta > 0$, let $L_{\delta,v}$ be the affine line $L + \delta v$. We set

$$\beta_{0,j}(L,v) = \lim_{\epsilon \to 0} \lim_{\delta \to 0} \# L_{\delta,v} \cap Y_j^{P,S} \cap B_\epsilon^P.$$

By this theorem, we know that

$$\Theta_k(Y_j^{P,S}, 0) = \frac{1}{g_{k+1}^1} \int_{G_P^1} \left(\frac{1}{s_{k-1}} \int_{S_{L^{\perp P}}} \beta_{0,j}(L,v) dv \right) dL.$$

In fact, Theorem 2.8 is proved in [9] for germs of closed sets. But, it holds also here because $\Theta_k(Y_j^{P,S}, 0) = \Theta_k(\overline{Y_j^{P,S}}, 0)$ and generically $\# L_{\delta,v} \cap \mathrm{Fr}(\overline{Y_j^{P,S}}) = 0$. Then we find

$$L_k^{\mathrm{loc}}(X,S,0) = \frac{1}{g_n^{k+1}} \int_{G_n^{k+1}} \frac{1}{g_{k+1}^1} \int_{G_P^1} \frac{1}{s_{k-1}} \int_{S_{L^{\perp P}}} I^S(P,L,v) dv dL dP,$$

where $I^S(P,L,v) = \sum_{j=1}^{r_{P,S}} \lambda_j^{P,S} \beta_{0,j}(L,v)$. Since the flag variety of pairs (P,L), $P \in G_n^{k+1}$ and $L \in G_P^1$, is isomorphic to the flag variety of pairs (H,L), $H \in G_n^{n-k}$ and $L \in G_H^1$, by the mapping $(P,L) \mapsto (H,L)$ where $H = P^\perp \oplus L$, we can write

$$L_k^{\mathrm{loc}}(X,S,0) = \frac{1}{g_n^{k+1} g_{k+1}^1} \int_{G_n^{n-k}} \frac{1}{s_{k-1}} \int_{S_{H^\perp}} \left(\int_{G_H^1} I^S(H,L,v) dH \right) dv dP,$$

because $H^\perp = L^{\perp P}$, and where $I^S(H,L,v) = I^S(P,L,v)$. As in the global case, we see that the integer $I^S(H,L,v)$ admits the following description:

$$I^S(H,L,v) = \lim_{\epsilon \to 0} \lim_{\delta \to 0} \sum_{x \in \Gamma_{\delta,\epsilon}^S} \alpha(\pi_P^S, \pi_P^S(x)),$$

where $0 < |\delta| \ll \epsilon \ll 1$ and $\Gamma_{\delta,\epsilon}^S$ is the set of critical points of $u^* : S \cap H_{\delta,v} \cap \mathring{B}_\epsilon \to \mathbb{R}$, u being a unit vector of the line L.

Let us choose a sequence $(\epsilon_m)_{m \in \mathbb{N}}$ such that $\lim_{m \to +\infty} \epsilon_m = 0$ and for each $m \in \mathbb{N}$, a sequence and $(\delta_p^m)_{p \in \mathbb{N}}$ such that $\lim_{p \to +\infty} \delta_p^m = 0$. By [5] Lemma 3.5, for almost all L in G_H^1, $u^*_{|X \cap H_{\delta_p^m, v} \cap B_{\epsilon_m}}$ is a Morse function. Since a countable union of sets of measure zero has measure zero, for almost all L in G_H^1, the function

$u^*_{|X\cap H_{\delta_p^m,v}\cap B_{\epsilon_m}}$ is a Morse function for every $(m,p)\in\mathbb{N}^2$. Hence by Lemma 3.10, for every $x\in\Gamma^S_{\delta_p^m,\epsilon^m}$, we have

$$\alpha(\pi_P^S,\pi_P^S(x))=\frac{1}{2}\left(\mathrm{ind}(u^*,X\cap H_{\delta_p^m,v},x)+\mathrm{ind}(-u^*,X\cap H_{\delta_p^m,v},x)\right).$$

Therefore we find that

$$\int_{G^1_H} I^S(H,L,v)dL=$$

$$\int_{G^1_H}\lim_{m\to+\infty}\lim_{p\to\infty}\sum_{x\in\Lambda^S_{\delta_p^m,\epsilon_m}}\frac{1}{2}\left(\mathrm{ind}(u^*,X\cap H_{\delta_p^m,v},x)+\mathrm{ind}(-u^*,X\cap H_{\delta_p^m,v},x)\right)dL=$$

$$g^1_{n-k}\lim_{m\to+\infty}\lim_{p\to\infty}\Lambda_0(X\cap H_{\delta_p^m,v},S\cap H_{\delta_p^m,v}\cap B_{\epsilon_m}).$$

Summing over all the strata S_a such that $d_{S_a}\geq k$ leads to

$$L^{\mathrm{loc}}_k(X,0)=\frac{g^1_{n-k}}{g^{k+1}_n g^1_{k+1}s_{k-1}}\int_{S_{H^\perp}}\lim_{m\to+\infty}\lim_{p\to\infty}\Lambda_0(X\cap H_{\delta_p^m,v},X\cap H_{\delta_p^m,v}\cap B_{\epsilon_m})dvdH=$$

$$\frac{1}{g_n^{n-k}}\int_{G_n^{n-k}}\frac{1}{s_{k-1}}\int_{S_{H^\perp}}\lim_{\epsilon\to0}\lim_{\delta\to0}\Lambda_0(X\cap H_{\delta,v},X\cap H_{\delta,v}\cap B_\epsilon)dvdH.$$

Theorem 2.8 implies that

$$L^{\mathrm{loc}}_k(X,0)=\lim_{\epsilon\to0}\frac{\Lambda_k(X,X\cap B_\epsilon)}{b_k\epsilon_k}.$$

If $d_X=n$ and $k<n$ then the proof works in the same way because the strata of dimension n have no contribution. If $k=n$ then the formula is obvious because both expressions are equal to the density of $(X,0)$. □

Applying Theorem 2.9, we obtain the following relation between the polar invariants and the $L^{\mathrm{loc}}_k(X,0)$'s.

Corollary 4.8. *Let $(X,0)$ be the germ of a closed definable set. For $k\in\{0,\dots,n-1\}$, we have*

$$\sigma_k(X,0)-\sigma_{k+1}(X,0)=L^{\mathrm{loc}}_k(X,0).$$

Furthermore, we have

$$\sigma_n(X,0)=L^{\mathrm{loc}}_n(X,0).$$

□

Let us refine the above results. By Definitions 4.5 and 4.6, we know that $L^{\mathrm{loc}}_k(X,0)=0$ for $k>d_X$. Furthermore, it is explained in [7] that for $k\in\{1,\dots,d_{S_0}\}$ and P generic in G^k_n, the projection π^X_P is a submersion in a neighborhood of 0. Hence, generically the polar varieties and the polar images are empty near the origin and so $L^{\mathrm{loc}}_k(X,0)=0$ for $k\in\{0,\dots,d_{S_0}-1\}$. Let us focus now on the localized polar length $L^{\mathrm{loc}}_{d_{S_0}}(X,0)$. If $d_S>d_{S_0}$, then generically $\mathrm{Fr}(\Delta^S_P)$ has dimension $d_{S_0}-1$ and 0 is not in the closure of Δ^S_P, so $L^{\mathrm{loc}}_{d_{S_0}}(X,S,0)=0$. Combining this with Definition 4.6 and Corollary 4.8, we find that

$$L^{\mathrm{loc}}_{d_{S_0}}(X,S_0,0)=1-\sigma_{d_{S_0}+1}(X,0),$$

because $\sigma_{d_{S_0}}(X,0)=1$ (see [7], Remarque 2.9).

If $d_{S_0} < k \leq d$ then S_0 has no contribution in the computation of $L_k^{\mathrm{loc}}(X,0)$ and so $L_k^{\mathrm{loc}}(X,0) = \sum_{a=1}^t L_k^{\mathrm{loc}}(X,S_a,0)$. Similarly the strata of dimension strictly less than d_X have no contribution in the computation of $L_{d_X}^{\mathrm{loc}}(X,0) = \Theta_{d_X}(X,0)$. Moreover, if S is a stratum of dimension d_X, the indices $\lambda_j^{P,S}$ that appear in the definition of $L_{d_X}^{\mathrm{loc}}(X,0)$ are generically equal to 1. Therefore we can restate Corollary 4.8.

Corollary 4.9. *Let $(X,0)$ be the germ of a closed definable set, equipped with a finite definable Whitney stratification $\{S_a\}_{a=0}^t$, where S_0 is the stratum that contains 0 and 0 belongs to the frontier of each S_a, $a = 1,\ldots,t$. We have*

$$L_{d_{S_0}}^{\mathrm{loc}}(X,0) = L_{d_{S_0}}^{\mathrm{loc}}(X,S_0,0) = 1 - \sigma_{d_{S_0}+1}(X,0).$$

For $d_{S_0} < k < d_X$, we have

$$\sigma_k(X,0) - \sigma_{k+1}(X,0) = \sum_{a=1}^t L_k^{\mathrm{loc}}(X,S_a,0).$$

Furthermore if $d_X < n$, then we have

$$\Theta_{d_X}(X,0) = L_{d_X}^{\mathrm{loc}}(X,0) = \frac{1}{g_n^{d_X+1}} \int_{G_n^{d_X+1}} \Theta_{d_X}(\pi_P^X(X),0)dP.$$

Of course, this last equality is trivial if $d_X = n-1$.

In [18], Theorem 6.1.9 and [19], Theorem 4.1.1, Lê and Teissier proved a relation between the vanishing Euler characteristics and the multiplicities of the polar varieties of a complex analytic germ. More precisely, if $(X,0) \subset (\mathbb{C}^n,0)$ is a complex analytic germ, they showed that the difference of two consecutive vanishing Euler characteristics of X at 0 is equal to a weighted sum of polar multiplicities, the sum been taken over the strata that contain 0 in their frontier. Let us describe the weights that appear in this sum. If S_0 denotes the stratum that contains 0 and S_a is a stratum such that $0 \in \mathrm{Fr}(S_a)$, then the weight attached to S_a is

$$(-1)^{d_{S_a}-d_{S_0}-1} \cdot (1 - \chi(\mathrm{Lk}^{\mathbb{C}}(X,S_a)),$$

where $\mathrm{Lk}^{\mathbb{C}}(X,S_a)$ is the complex link of the stratum S_a in X (see for instance [13], II 2.2, for the definition of the complex link). The first term of this product is actually a complex tangential Morse index and the second term is a complex normal Morse index. In the equalities

$$\sigma_k(X,0) - \sigma_{k+1}(X,0) = \sum_{a=1}^t L_k^{\mathrm{loc}}(X,S_a,0),\ d_{S_0} < k < d_X,$$

the weights $\lambda_j^{P,S}$ are half-sums of two stratified Morse indices, which are also decomposed into the product of a tangential Morse index and a normal Morse index. Therefore, and since the polar invariants are real versions of the vanishing Euler characteristics, we can view the equalities

$$\sigma_k(X,0) - \sigma_{k+1}(X,0) = \sum_{a=1}^t L_k^{\mathrm{loc}}(X,S_a,0),\ d_{S_0} < k < d_X,$$

as real versions of Lê and Teissier's results.

In [7], Theorems 4.9 and 4.10, Comte and Merle showed that the polar invariants were continuous along the strata of a (w)-stratification of a closed subanalytic set.

Recently, Nguyen and Valette [25] extended this result for a (b)-stratification and for a closed set X, definable in a polynomially bounded structure. They even proved that the polar invariants were locally lipschitz along the strata of a (w)-stratification of X. Note that these results are not true if the structure is not polynomially bounded. A counter-example to the continuity of the density along the strata of a (b)-stratified set, definable in a non-polynomially bounded structure, was found recently by Trotman and Valette (see [23]). As a straightforward consequence of these results and Corollary 4.8, we see that that localized polar lengths are continuous along the strata of a (b)-stratification of a closed definable set, and locally lipschitz if the stratification is (w)-regular. Let us examine these last results in the case of two strata.

So let S be stratum that contains S_0 in its frontier and such that the depth of S_0 in $\overline{S}$ is one. This means that there is no stratum S' such that $S_0 \subset \mathrm{Fr}(S')$ and $S' \subset \mathrm{Fr}(S)$. Adapting the notations introduced before Definition 4.5 in an obvious way, we have that for $k \in \{d_{S_0}+1, \ldots, d_S\}$ and $z \in S_0$,

$$L_k^{\mathrm{loc}}(\overline{S}, z) = \frac{1}{g_n^{k+1}} \int_{G_n^{k+1}} \sum_{j=1}^{r_{P,S}(z)} \lambda_j^{P,S}(z) \Theta\Big(Y_j^{P,S}(z), z\Big) dP,$$

where the $Y_j^{P,S}(z)$'s are defined in the same way as the $Y_j^{P,S}$ were defined, replacing the ball B_ϵ with the ball $B_\epsilon(z)$. The index $\lambda_j^{P,S}(z)$ was defined like that:

$$\lambda_j^{P,S}(z) = \lim_{\epsilon \to 0} \lim_{\substack{y \to z \\ y \in Y_j^{P,S,\epsilon}(z)}} \alpha(\pi_P^S, y),$$

with

$$\alpha(\pi_P^S, y) = \frac{1}{2}\Big(\mathrm{ind}(\nu^*, \overline{S} \cap Q_{\nu,x}, x) + \mathrm{ind}(-\nu^*, \overline{S} \cap Q_{\nu,x}, x)\Big),$$

where ν is a unit normal vector to $T_y\Delta_P^S$ in P and $y = \pi_P^S(x)$. But since S is a top stratum in $\overline{S}$, the indices $\mathrm{ind}(\nu^*, \overline{S} \cap Q_{\nu,x}, x)$ and $\mathrm{ind}(-\nu^*, \overline{S} \cap Q_{\nu,x}, x)$ are actually tangential Morse indices. They are equal if $\dim(S \cap Q_{\nu,x})$ is even and opposite if $\dim(S \cap Q_{\nu,x})$ is odd. The dimension of $S \cap Q_{\nu,x}$ is $d_S - k$, so $L_k^{\mathrm{loc}}(\overline{S}, z) = 0$ if $d_S - k$ is odd and if $d_S - k$ is even,

$$L_k^{\mathrm{loc}}(\overline{S}, z) = \frac{1}{g_n^{k+1}} \int_{G_n^{k+1}} \sum_{j=1}^{r_{P,S}(z)} \lambda_j^{P,S}(z) \Theta\Big(Y_j^{P,S}(z), z\Big) dP,$$

where $\lambda_j^{P,S}(z) \in \{-1, +1\}$. Hence in this situation, the localized polar lengths $L_k^{\mathrm{loc}}(\overline{S}, -)$ are somehow real counterparts of the complex polar multiplicities.

Proposition 4.10. *Assume that the depth of S_0 in $\overline{S}$ is one. In this case, the localized polar lengths $L_k^{\mathrm{loc}}(\overline{S}, -)$, $k \in \{d_{S_0}+1, \ldots, d_S\}$ and $d_S - k$ even, are continuous along S_0 if the stratification is (b)-regular and locally lipschitz if the stratification is (w)-regular.*

Proof. By Corollary 4.9, for $d_{S_0} < k < d_S$ we have

$$\sigma_k(\overline{S}, z) - \sigma_{k+1}(\overline{S}, z) = L_k^{\mathrm{loc}}(\overline{S}, z) = L_k^{\mathrm{loc}}(\overline{S}, S, z),$$

and $\Theta_{d_S}(\overline{S}, z) = L_{d_S}^{\mathrm{loc}}(\overline{S}, S, z) = L_{d_S}^{\mathrm{loc}}(\overline{S}, z)$. Then we apply the results of [25]. □

It would be interesting to see if this result still holds without the assumption on the depth of S_0 in $\overline{S}$.

References

[1] Allendoerfer, C. B.: The Euler number of a Riemann manifold, *Amer. J. Math.* **62** (1940), 243–248.

[2] Bernig, A. and Bröcker, L.: Lipschitz-Killing invariants, *Math. Nachr.* **245** (2002), 5–25.

[3] Bernig, A. and Bröcker, L.: Courbures intrinsèques dans les catégories analytico-géométriques, *Ann. Inst. Fourier (Grenoble)* **53**(6) (2003), 1897–1924.

[4] Bochnak, J., Coste, M. and Roy, M.F.: *Real algebraic geometry*, Ergeb. Math. Grenzgeb. (3) **36**, Springer-Verlag, Berlin, 1998.

[5] Bröcker, L. and Kuppe, M.: Integral geometry of tame sets, *Geometriae Dedicata* **82** (2000), 285–323.

[6] Comte, G.: Equisingularité réelle: nombres de Lelong et images polaires, *Ann. Sci. École Norm. Sup (4)* **33**(6) (2000), 757–788.

[7] Comte, G. and Merle, M.: Equisingularité réelle II : invariants locaux et conditions de régularité, *Ann. Sci. Éc. Norm. Supér. (4)* **41** (2008), no. 2, 221–269.

[8] Coste, M.: An introduction to o-minimal geometry, *Dip. Mat. Univ. Pisa, Dottorato di Ricerca in Matematica*, Istituti Editoriali e Poligrafici Internazionali, Pisa, 2000.

[9] Dutertre, N.: Stratified critical points on the real Milnor fibre and integral-geometric formulas, *J. Singul.* **13** (2015), 87–106.

[10] Federer, H.: The (φ, k) rectifiable subsets of n-space, *Trans. Amer. Math. Soc.* **62** (1947), 114–192.

[11] Fenchel, W.: On total curvatures of Riemannian manifolds: I., *J. London Math. Soc.* **15**, (1940). 15–22.

[12] Fu, J.H.G.: Curvature measures of subanalytic sets, *Amer. J. Math.* **116**, no. 4 (1994), 819–880.

[13] Goresky, M. and Mac-Pherson, R.: *Stratified Morse theory*, Springer-Verlag, Berlin, 1988.

[14] Hamm, H. : On stratified Morse theory, *Topology* **38** (1999), 427–438.

[15] Hardt, R.M.: Topological properties of subanalytic sets, *Trans. Amer. Math. Soc.* **211** (1975), 57–70.

[16] Langevin, R.: Courbures, feuilletages et surfaces, *Publications Mathématiques d'Orsay 80,* **3**, Université de Paris-Sud, Département de Mathématiques, Orsay 1980.

[17] Langevin, R. and Shifrin, T.: Polar varieties and integral geometry, *Amer. J. Math.* **104** (3) (1982), 553-605.

[18] Lê, D. T. and Teissier, B.: Variétés polaires locales et classes de Chern des variétés singulières, *Ann. of Math.* **114** (1981), 457-491.

[19] Lê, D. T. and Teissier, B.: Cycles évanescents, sections planes et conditions de Whitney II, Singularities, Part 2 (Arcata, Calif., 1981), 65-103, *Proc. Sympos. Pure Math.* **40**, Amer. Math. Soc., Providence, RI, 1983.

[20] Loi, T. L.: Verdier and strict Thom stratifications in o-minimal structures, *Illinois J. Math.* **42** (1998), no. 2, 347–356.

[21] Loi, T. L.: Lecture 1: o-minimal structures, *The Japanese-Australian Workshop on Real and Complex Singularities–JARCS III*, 19–30, Proc. Centre Math. Appl. Austral. Nat. Univ., **43**, Austral. Nat. Univ., Canberra, 2010.

[22] Mather, J. N.: Generic projections, *Ann. of Math.* **98** (2) (1973), 226–245.

[23] Nguyen, X. V. N.: Structure métrique et géométrie des ensembles définissables dans des structures o-minimales, Thèse de Doctorat (2015), Aix-Marseille Université.

[24] Nguyen, X. V. N., Trivedi, S. and Trotman, D.: A geometric proof of the existence of definable Whitney stratifications, *Illinois J. Math.* **58** (2014), no. 2, 381–389.

[25] Nguyen, X. V. N. and Valette, G.: Whitney stratifications and the continuity of local Lipschitz Killing curvatures, *preprint* arXiv:1507.01262.

[26] Orro, P. and Trotman, D.: Regularity of the transverse intersection of two regular stratifications, *Real and complex singularities*, 298–304, London Math. Soc. Lecture Note Ser., **380**, Cambridge Univ. Press, Cambridge, 2010.

[27] Santaló, L.A.: *Integral geometry and geometric probability*, Encyclopedia of Mathematics and its Applications, Vol 1., Addison-Wesley Publishing Co., Reading, Mass., 1976.

[28] Szafraniec, Z.: A formula for the Euler characteristic of a real algebraic manifold, *Manuscripta Math.* **85** (1994), no. 3–4, 345–360.
[29] Teissier, B.: Variétés polaires, II. Multiplicités polaires, sections planes, et conditions de Whitney, *Algebraic geometry (La Rábida, 1981)*, 314–491, Lecture Notes in Math., **961**, Springer, Berlin, 1982.
[30] Trotman, D.: Espaces stratifiés réels, *Stratifications, singularities and differential equations, II (Marseille, 1990 ; Honolulu, HI, 1990)*, 93–107, Travaux en Cours, **55**, Hermann, Paris, 1997.
[31] Valette, G.: Volume, Whitney conditions and Lelong number, *Ann. Polon. Math.* **93** (2008), no. 1, 1–16.
[32] van den Dries, L.: Tame topology and o-minimal structures, *London Mathematical Society Lecture Note Series*, **248**, Cambridge University Press, Cambridge, 1998.
[33] van den Dries, L. and Miller, C.: Geometric categories and o-minimal structures, *Duke Math. J.* **84** (1996), no. 2, 497–540.
[34] Weyl, H.: On the Volume of Tubes, *Amer. J. Math.* **61** (1939), no. 2, 461–472.

Aix-Marseille Université, CNRS, Centrale Marseille, I2M, UMR 7373, 13453 Marseille, France.
E-mail address: nicolas.dutertre@univ-amu.fr