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Admissible coadjoint orbits for compact Lie groups.

Paul-Emile PARADAN* and Michèle VERGNE†

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1 Introduction

Let K be a real Lie group with Lie algebra $\mathfrak{k}$. The notion of admissible coadjoint orbit was introduced by M. Duflo [2] in order to study the unitary dual of K. This notion is also important in the work of D.Vogan (see [5]). Indeed, quoting him, "the orbit method provides some light in a very dark room". In this article, we restrict ourselves to the case where K is a compact connected Lie group. In this case, everything seems clear, admissible orbits are very easy to describe as well as $\hat{K}$. Nevertheless, we discovered some remarkable properties of admissible coadjoint orbits which (we believe) were unnoticed before. Furthermore, as we briefly explain below, these properties are fundamental in the study of the equivariant index of twisted Dirac operators on general K-manifolds. It thus became necessary to return to this subject. Let us describe our motivations, and some of our results.

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Choose a K-invariant scalar product on $\mathfrak{k}^*$, a Cartan subalgebra, and a system of positive roots and denote by $\|\rho^K\|$ the norm of $\rho^K = \frac{1}{2}\sum_{\alpha>0}\alpha$.

Coadjoint orbits can be grouped in different subsets that we call Dixmier sheets. If a coadjoint orbit $K\xi$ varies in a Dixmier sheet, the stabilizer subgroup K_ξ of $\xi \in \mathfrak{k}^*$ remains in a fixed conjugacy class. The coadjoint orbits of maximal dimension form the regular sheet. Furthermore, there is a map $\mathrm{Q}_K^{\mathrm{spin}}$ (see Definition 3.8) associating to an admissible coadjoint orbit $\mathcal{P}$ a virtual representation $\mathrm{Q}_K^{\mathrm{spin}}(\mathcal{P})$ of K. By this correspondence, regular admissible coadjoint orbits parameterize the set $\widehat{K}$ of classes of unitary irreducible representations of K. For example, the coadjoint orbit of ρ^K is regular admissible and parameterize the trivial representation of K.

We associate to a Dixmier sheet S a subset I_S of $\widehat{K}$, a positive number $\|\rho_S\|$ and a conjugacy class $\mathfrak{s}_S$ of semi-simple subalgebras of $\mathfrak{k}$: the set I_S is the set of irreducible representations of K of the form $\mathrm{Q}_K^{\mathrm{spin}}(\mathcal{P})$ where $\mathcal{P}$ is an admissible orbit belonging to S, $\mathfrak{s}_S$ is the conjugacy class of the semi-simple part of the infinitesimal stabilizer $\mathfrak{k}_\xi$ for any orbit $K\xi$ in S, and $\|\rho_S\|$ is the norm of the ρ-element for the subgroup K_ξ of $\mathfrak{k}$ (thus $\|\rho_S\| = 0$ exactly when S is the regular sheet).

Let $\mathcal{S}_\mathfrak{k}$ be the set of conjugacy classes of the subalgebras $\mathfrak{s}_S$ where S varies over all Dixmier sheets. The set $\mathcal{S}_\mathfrak{k}$ and the particular subset I_S of $\hat{K}$ associated to a Diximier sheet S occur in the following theorem, that we prove in [4] and announced in [3].

Assume K is acting on a connected Spin manifold M (or more generally a Spin^c manifold). Let L be a K-equivariant line bundle on M. Consider the twisted Dirac operator D_L and its equivariant index $Q(M, L)$. Then the following holds.

• If the semi-simple part $\mathfrak{s}$ of the generic infinitesimal stabilizer $\mathfrak{k}_M$ of the action of K in M does not belong to $\mathcal{S}_\mathfrak{k}$, then $Q(M, L) = \{0\}$ for any equivariant line bundle L on M.

• If $\mathfrak{s}$ is equal to $\mathfrak{s}_S$, then $Q(M, L)$ is a virtual sum of irreducible representations of K belonging to I_S.

The proof of this result uses an inequality on distance between coadjoint orbits, that we call the magical inequality. We prove this inequality in this article (Theorem 4.3). In this introduction, let us just state two of its consequences.

Theorem 1.1 *Let $\mathcal{O}$ be a regular admissible coadjoint orbit, and let $\mathcal{P}$ be a coadjoint orbit in a Dixmier sheet S. Then the distance between $\mathcal{O}$ and $\mathcal{P}$ is greater or equal to $\|\rho_S\|$.*

Assume that $\mathcal{P}$ is an admissible coadjoint orbit. We prove in Theorem 5.1 that $\mathrm{Q}_K^{\mathrm{spin}}(\mathcal{P})$ is either 0, or irreducible. If $\mathrm{Q}_K^{\mathrm{spin}}(\mathcal{P})$ is irreducible, then $\mathrm{Q}_K^{\mathrm{spin}}(\mathcal{P}) = \mathrm{Q}_K^{\mathrm{spin}}(\mathcal{O})$ for a unique admissible regular orbit $\mathcal{O}$. We then say that $\mathcal{P}$ is an ancestor of $\mathcal{O}$. We prove the following theorem (Theorem 5.4).

Theorem 1.2 *Let $\mathcal{O}$ be a regular admissible orbit. The ancestors of $\mathcal{O}$ in a Dixmier sheet S are all the elements in S at distance $\|\rho_S\|$ of $\mathcal{O}$.*

In general $\mathcal{O}$ has a unique ancestor, $\mathcal{O}$ itself. But for example the orbit of ρ^K has 2^r ancestors, where r is the rank of $[K, K]$.

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Notations

Throughout the paper :

- K denotes a compact connected Lie group with Lie algebra $\mathfrak{k}$.
- T is a maximal torus in K with Lie algebra $\mathfrak{t}$.
- $\Lambda \subset \mathfrak{t}^*$ is the weight lattice of T : every $\mu \in \Lambda$ defines a 1-dimensional T-representation, denoted $\mathbb{C}_\mu$, where $t = \exp(X)$ acts by $t^\mu := e^{i\langle \mu, X\rangle}$.
- If $\mathfrak{h}$ is a subalgebra of $\mathfrak{k}$, we denote by $(\mathfrak{h})$ its conjugacy class.
- We fix a K-invariant Euclidean inner product $(\cdot, \cdot)$ on $\mathfrak{k}$. This allows us to identify $\mathfrak{k}$ and $\mathfrak{k}^*$ when needed.

  We denote by $\langle \cdot, \cdot \rangle$ the natural duality between $\mathfrak{k}$ and $\mathfrak{k}^*$.
- We denote by $R(K)$ the representation ring of K : an element $E \in R(K)$ can be represented as finite sum $E = \sum_{\lambda \in \widehat{K}} \mathrm{m}_\lambda V_\lambda^K$, with $\mathrm{m}_\lambda \in \mathbb{Z}$. Here we have denoted by V_λ^K the irreducible representation of K indexed by λ.
- We denote by $\hat{R}(K)$ the space of $\mathbb{Z}$-valued functions on $\hat{K}$. An element $E \in \hat{R}(K)$ can be represented as an infinite sum $E = \sum_{\lambda \in \widehat{K}} \mathrm{m}(\lambda) V_\lambda^K$, with $\mathrm{m}(\lambda) \in \mathbb{Z}$. An element of $\hat{R}(K)$ will be called a virtual representation of K.

- If H is a closed subgroup of K, the induction map $\mathrm{Ind}_H^K : \hat{R}(H) \to \hat{R}(K)$ is the dual of the restriction morphism $R(K) \to R(H)$. It is given by the Frobenius reciprocity formula: the multiplicity $\mathrm{m}(\lambda)$ of the irreducible representation V_λ^K in $\mathrm{Ind}_H^K V_\mu^H$ is the multiplicity of V_μ^H in the restriction of the representation V_λ^K to H.

- If E is a complex representation space of H, we denote simply by $\bigwedge^\bullet E$ the virtual representation of H which is the difference of the representations of H in $\bigwedge^{\mathrm{even}} E$ and $\bigwedge^{\mathrm{odd}} E$.

- The stabilizer of $\xi \in \mathfrak{k}^*$ is denoted by K_ξ. It is a connected subgroup of K. The Lie algebra of K_ξ is denoted by $\mathfrak{k}_\xi$. The element $\xi \in \mathfrak{k}^*$ is called regular if K_ξ is a Cartan subgroup of K.

- In general, the notation $\mathcal{P}$ is for any coadjoint orbit, while the notation $\mathcal{O}$ is reserved for regular coadjoint orbits.

2 Coadjoint orbits and Dixmier sheets

Let K be a compact connected Lie group with Lie algebra $\mathfrak{k}$.

We first define the ρ-orbit. Let T be a Cartan subgroup of K. Then $\mathfrak{t}^*$ is imbedded in $\mathfrak{k}^*$ as the subspace of T-invariant elements. Roots will be considered as elements of $\mathfrak{t}^*$ (the corresponding character of T being $e^{i\alpha}$). Choose a system of positive roots $\Delta^+ \subset \mathfrak{t}^*$, and let $\rho^K = \frac{1}{2}\sum_{\alpha>0} \alpha$. The definition of ρ^K requires the choice of a Cartan subgroup T and of a positive root system. However a different choice leads to a conjugate element. Thus we can make the following definition.

Definition 2.1 *We denote by $o(\mathfrak{k})$ the coadjoint orbit of $\rho^K \in \mathfrak{k}^*$. We call $o(\mathfrak{k})$ the ρ-orbit. We denote by $\|\rho^K\|$ the norm of any point on $o(\mathfrak{k})$.*

Let $\mathcal{H}_\mathfrak{k}$ be the set of conjugacy classes of the reductive algebras $\mathfrak{k}_\xi, \xi \in \mathfrak{k}^*$. The set $\mathcal{H}_\mathfrak{k}$ contains the conjugacy class $(\mathfrak{t})$ formed by the Cartan subalgebras. It contains also $(\mathfrak{k})$ (stabilizer of 0).

Remark 2.2 *If $\mathfrak{h} = \mathfrak{k}_\xi$, then $\mathfrak{h}_\mathbb{C}$ is the Levi subalgebra of the parabolic subalgebra determined by ξ. Parabolics are classified by subsets of simple roots. Thus there are 2^r conjugacy classes of parabolics if r is the rank of $[\mathfrak{k}, \mathfrak{k}]$. However, different conjugacy classes of parabolics might give rise to the same conjugacy class of Levi subalgebras (as seen immediately for type A_n).*

We group the coadjoint orbits according to the conjugacy class $(\mathfrak{h}) \in \mathcal{H}_{\mathfrak{k}}$ of the stabilizer.

Definition 2.3 *The Dixmier sheet $\mathfrak{k}^*_{(\mathfrak{h})}$ is the set of orbits $K\xi$ with $\mathfrak{k}_\xi$ conjugated to $\mathfrak{h}$.*

We also say that an element of $\mathfrak{k}^*_{(\mathfrak{h})}$ is a coadjoint orbit of type $(\mathfrak{h})$.

If $(\mathfrak{h}) = (\mathfrak{t})$, the corresponding Dixmier sheet is the set of regular coadjoint orbits.

If $(\mathfrak{h}) = (\mathfrak{k})$, the corresponding Dixmier sheet is the set of 0-dimensional orbits, that is the orbits $\{\xi\}$ of the elements ξ of $\mathfrak{k}^*$ vanishing on $[\mathfrak{k}, \mathfrak{k}]$.

We denote by $\mathcal{S}_{\mathfrak{k}}$ the set of conjugacy classes of the semi-simple parts $[\mathfrak{h}, \mathfrak{h}]$ of the elements $(\mathfrak{h}) \in \mathcal{H}_{\mathfrak{k}}$.

Lemma 2.4 *The map $(\mathfrak{h}) \to ([\mathfrak{h}, \mathfrak{h}])$ induces a bijection between $\mathcal{H}_{\mathfrak{k}}$ and $\mathcal{S}_{\mathfrak{k}}$.*

Proof. Assume that $[\mathfrak{h}, \mathfrak{h}] = [\mathfrak{h}', \mathfrak{h}'] = \mathfrak{s}$. Let N be the normalizer of $\mathfrak{s}$, and $\mathfrak{n}$ its Lie algebra. Thus $\mathfrak{h}$ and $\mathfrak{h}'$ are contained in $\mathfrak{n}$. Let $\mathfrak{t}, \mathfrak{t}'$ be Cartan subalgebras of $\mathfrak{h}, \mathfrak{h}'$. Then $\mathfrak{t}$ and $\mathfrak{t}'$ are conjugated by an element of N. As $\mathfrak{h} = \mathfrak{s} + \mathfrak{t}$, we see that $\mathfrak{h}$ is conjugated to $\mathfrak{h}'$. $\square$

The connected Lie subgroup with Lie algebra $\mathfrak{h}$ is denoted H. Thus if $\mathfrak{h} = \mathfrak{k}_\xi$, then $H = K_\xi$. We write $\mathfrak{h} = \mathfrak{z} \oplus [\mathfrak{h}, \mathfrak{h}]$ where $\mathfrak{z}$ is the center and $[\mathfrak{h}, \mathfrak{h}]$ is the semi-simple part of $\mathfrak{h}$. Thus $\mathfrak{h}^* = \mathfrak{z}^* \oplus [\mathfrak{h}, \mathfrak{h}]^*$ and $\mathfrak{z}^*$ is the set of elements in $\mathfrak{h}^*$ vanishing on the semi-simple part of $\mathfrak{h}$. We write $\mathfrak{k} = \mathfrak{h} \oplus [\mathfrak{z}, \mathfrak{k}]$, so we embed $\mathfrak{h}^*$ in $\mathfrak{k}^*$ as a H-invariant subspace, that is we consider an element $\xi \in \mathfrak{h}^*$ also as an element of $\mathfrak{k}^*$ vanishing on $[\mathfrak{z}, \mathfrak{k}]$.

We consider the ρ-orbit $o(\mathfrak{h})$ in $\mathfrak{h}^*$. Remark that $o(\mathfrak{h})$ is contained in $[\mathfrak{h}, \mathfrak{h}]^*$. We denote by $\|\rho^H\|$ the norm of any point in $o(\mathfrak{h})$.

If S is the Dixmier sheet associated to $(\mathfrak{h})$, we denote by $\|\rho^S\|$ the norm of ρ^H.

3 Admissible coadjoint orbits

We will be interested in admissible coadjoint orbits. This notion is defined in [2] for any real Lie group. Let us now describe the set of admissible coadjoint orbits in concrete terms when K is a compact connected Lie group.

Consider $\xi \in \mathfrak{k}^*$. We have $\langle \xi, [\mathfrak{k}_\xi, \mathfrak{k}_\xi] \rangle = 0$. If $i\theta : \mathfrak{k}_\xi \to i\mathbb{R}$ is the differential of a character of K_ξ, we denote by $\mathbb{C}_\theta$ the corresponding 1-dimensional representation of K_ξ, and by $[\mathbb{C}_\theta] = K \times_{K_\xi} \mathbb{C}_\theta$ the corresponding line bundle over the coadjoint orbit $K\xi \subset \mathfrak{k}^*$.

Definition 3.1 *An element $\xi \in \mathfrak{k}^*$ is integral if $i\xi$ is the differential of a 1-dimensional representation of K_ξ.*

This notion of integrability is invariant under the coadjoint action, so a coadjoint orbit is called integral when any of its element is integral.

The following notion of admissibility is necessary for defining the spin quantization $\mathrm{Q}_K^{\mathrm{spin}}(\mathcal{P})$ of an orbit $\mathcal{P}$.

Let $K\xi$ be a coadjoint orbit. The quotient space $\mathfrak{k}/\mathfrak{k}_\xi$ is equipped with the symplectic form $\Omega_\xi(\bar{X}, \bar{Y}) := \langle \xi, [X, Y] \rangle$, and with a unique K_ξ-invariant complex structure J_ξ such that $\Omega_\xi(-, J_\xi -)$ is a scalar product. We denote by $\mathfrak{q}^\xi$ the space $\mathfrak{k}/\mathfrak{k}_\xi$ **considered as a complex vector space** via the complex structure J_ξ. Any element $X \in \mathfrak{k}_\xi$ defines a complex linear map $\mathrm{ad}(X) : \mathfrak{q}^\xi \to \mathfrak{q}^\xi$.

Definition 3.2 *For any $\xi \in \mathfrak{k}^*$, we denote by $\rho(\xi)$ the element of $\mathfrak{k}_\xi^*$ such that*

$$\langle \rho(\xi), X \rangle = \frac{1}{2i} \mathbf{Tr}_{\mathfrak{q}^\xi} \mathrm{ad}(X), \quad X \in \mathfrak{k}_\xi.$$

Note that $\rho(\xi)$ vanishes on $[\mathfrak{k}_\xi, \mathfrak{k}_\xi]$. As explained in the preceding section, we consider also $\rho(\xi)$ as an element of $\mathfrak{k}^*$.

Notice that $2\rho(\xi)$ is integral since $2i\rho(\xi)$ is the differential of the character $k \in K_\xi \mapsto \det_{\mathfrak{q}^\xi}(k)$.

Let $\mathfrak{t}$ be a Cartan subalgebra and $\xi \in \mathfrak{t}^*$. Then if Δ is the root system with respect to $\mathfrak{t}$, we have $\rho(\xi) = \frac{1}{2} \sum_{\alpha \in \Delta, (\alpha, \xi) > 0} \alpha$.

Definition 3.3 *We say that the coadjoint orbit $K\xi$ is admissible if $i(\xi - \rho(\xi))$ is the differential of a 1-dimensional representation of K_ξ.*

Remark that if the coadjoint orbit $K\xi$ is admissible, then 2ξ is integral.

It is easy to see that, for a compact connected Lie group K, this definition of admissibility is equivalent to Duflo definition via the metaplectic correction.

In particular the orbit $o(\mathfrak{k})$ is admissible. Indeed if $\xi = \rho^K$, then $\xi - \rho(\xi) = 0$.

Remark 3.4 *When the group K is simply connected, then a regular admissible orbit $K\xi$ is integral. This is not the case when K is not simply connected. For example the orbit $o(\mathfrak{k})$ (which is regular admissible) is not integral when $K = SO(3)$. However, even if the group K is simply connected, non regular admissible coadjoint orbits are not necessary integral. See the example 3.7 below.*

Definition 3.5 *We denote by $\mathcal{A}((\mathfrak{h}))$ the set of admissible orbits of type $(\mathfrak{h})$.*

We now define the shift of an orbit. If $\mu \in \mathfrak{k}^*$, recall that $o(\mathfrak{k}_\mu) \subset \mathfrak{k}_\mu^*$ is the coadjoint orbit of the ρ-element for the group K_μ.

Definition 3.6 *To any coadjoint orbit $\mathcal{P} \subset \mathfrak{k}^*$, we associate the coadjoint orbit $s(\mathcal{P}) \subset \mathfrak{k}^*$ which is defined as follows : if $\mathcal{P} = K\mu$, take $s(\mathcal{P}) = K\xi$ with $\xi \in \mu + o(\mathfrak{k}_\mu)$. We call $s(\mathcal{P})$ the shift of the orbit $\mathcal{P}$.*

If $\mathcal{O}$ is regular, $s(\mathcal{O}) = \mathcal{O}$, and if $\mathcal{P} = \{0\}$, then $s(\mathcal{P}) = o(\mathfrak{k})$.

Beware that we have two elements of $\mathfrak{h}^* = [\mathfrak{h}, \mathfrak{h}]^* \oplus \mathfrak{z}^*$ associated to an element $\mu \in \mathfrak{k}^*$ such that $\mathfrak{k}_\mu = \mathfrak{h}$: the element $\rho(\mu) \in \mathfrak{z}^*$ (defined canonically) and the element $\rho^{K_\mu} \in [\mathfrak{h}, \mathfrak{h}]^*$ (defined up to H-conjugacy). Remark that $\rho(\mu) + \rho^{K_\mu}$ is conjugated to ρ^K. More concretely, if we choose a Cartan subgroup T of H with Lie algebra $\mathfrak{t}$, and a positive root system Δ^+ for roots of $\mathfrak{k}$ with respect to $\mathfrak{t}$, then, when μ is dominant,

$$\rho(\mu) = \frac{1}{2} \sum_{\alpha \in \Delta^+, (\alpha,\mu)>0} \alpha$$

and

$$\rho^{K_\mu} = \frac{1}{2} \sum_{\alpha \in \Delta^+, (\alpha,\mu)=0} \alpha.$$

The orbit $K\mu$ is admissible if $\mu - \rho(\mu)$ is in the weight lattice of T. The shift of the orbit $K\mu$ is $K(\mu + \rho^{K_\mu})$.

Example 3.7 *Consider the group $K = SU(3)$. Then there are 3 different sheets, determined (for this case) by the dimensions of orbits. The regular sheet consists of coadjoint orbits of dimension 6, The subregular sheet consists of coadjoint orbits of dimension 4, finally we have the orbit $\{0\}$ of dimension 0.*

We now parameterize the subregular sheet. Let $\mathfrak{h}$ be the Lie algebra of $H = S(U(1) \times U(2))$. Then the stabilizer $\mathfrak{k}_f$ of a coadjoint orbit Kf of dimension 4 is conjugated to $\mathfrak{h}$. Let us give representatives of the orbits in $\mathfrak{k}^*_{(\mathfrak{h})}$.

We consider the Cartan subalgebra of diagonal matrices and choose a Weyl chamber. Let ω_1, ω_2 be the two fundamental weights. Thus $\rho^K = \omega_1 + \omega_2$. Let σ_1, σ_2 be the half lines $\mathbb{R}_{>0}\omega_1$, $\mathbb{R}_{>0}\omega_2$.

Then $\mathfrak{k}^*_{(\mathfrak{h})}$ is the set of orbits $K(t\omega_1)$, with $t \neq 0$. Here we have privileged an element f on the orbit $\mathcal{P} \in \mathfrak{k}^*_{(\mathfrak{h})}$ such that $\mathfrak{k}_f = \mathfrak{h}$. We could also privilege

representatives of orbits $\mathcal{P} \in \mathfrak{k}^*_{(\mathfrak{h})}$ belonging to the chosen closed Weyl chamber. As $-\omega_1$ conjugated to ω_2, we obtain the description of $\mathfrak{k}^*_{(\mathfrak{h})}$ as the set of orbits $\{K(t_1\omega_1), K(t_2\omega_2)\}$ with $t_1 > 0$ and $t_2 > 0$ (orbits of points in the boundary of the Weyl chamber, except $\{0\}$).

Let us now describe the set $\mathcal{A}((\mathfrak{h}))$ of admissible coadjoint orbits of type $(\mathfrak{h})$.

We obtain, using the first description, that the set $\mathcal{A}((\mathfrak{h}))$ is equal to the collection of orbits $K \cdot (\frac{1+2n}{2}\omega_1), n \in \mathbb{Z}$ (see Figure 1). Remark that orbits in $\mathcal{A}((\mathfrak{h}))$ are admissible, but not integral.

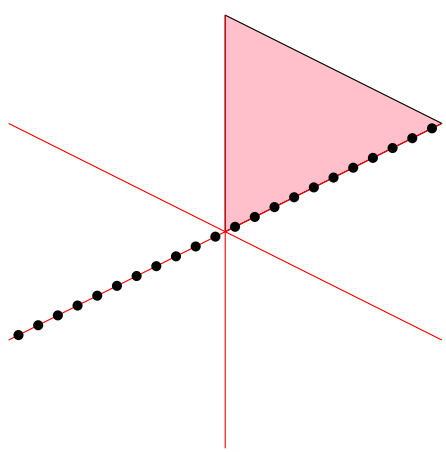

Figure 1: A set of representatives of admissible orbits of type $(\mathfrak{h})$

Using the second description, with representatives in the chosen closed Weyl chamber, we see that the set $\mathcal{A}((\mathfrak{h}))$ is equal to the collection of orbits $K \cdot (\frac{1+2n}{2}\omega_i), n \in \mathbb{Z}_{\geqslant 0}, i = 1, 2$.

We have also depicted in this description the shifted orbits. Remark that the shift of the admissible elements in $\mathcal{A}_{(\mathfrak{h})}$ are admissible, except for the two elements $\frac{1}{2}\omega_1$ and $\frac{1}{2}\omega_2$ with shifts ω_2 and ω_1 respectively, elements which are not admissible.

Finally remark that the orbit $K\rho^K$ is obtained as the shift of the admissible orbits $\{0\}$, $K(\frac{3}{2}\omega_1)$, $K(\frac{3}{2}\omega_2)$, and $K\rho_K$.

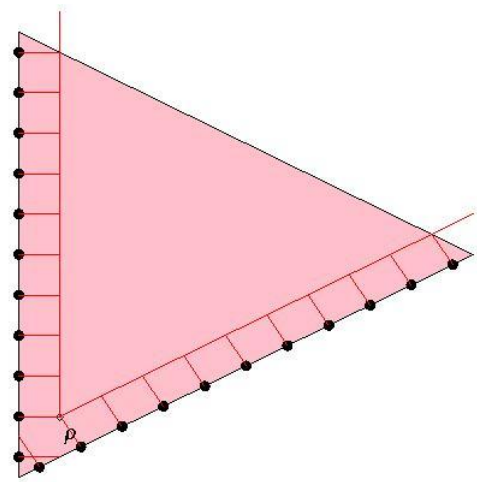

Figure 2: Another set of representatives of admissible orbits of type ($\mathfrak{h}$) and their shifts

We now associate to an admissible coadjoint orbit an element of $R(K)$.

Let $\mathcal{P} = K\xi$ be an admissible coadjoint orbit. We consider the $\mathbb{Z}/2\mathbb{Z}$-complex space $\bigwedge^\bullet \mathfrak{q}^\xi \otimes \mathbb{C}_{\xi-\rho(\xi)} = E^+ \oplus E^-$ with $E^+ = \bigwedge^{\text{even}} \mathfrak{q}^\xi \otimes \mathbb{C}_{\xi-\rho(\xi)}$ and $E^- = \bigwedge^{\text{odd}} \mathfrak{q}^\xi \otimes \mathbb{C}_{\xi-\rho(\xi)}$.

We still denote by $\bigwedge^\bullet \mathfrak{q}^\xi \otimes \mathbb{C}_{\xi-\rho(\xi)}$ the element $E^+ - E^-$ in $R(H)$.

Definition 3.8 *We define $\mathrm{Q}_K^{\mathrm{spin}}(\mathcal{P}) \in R(K)$ by the formula:*

$$\mathrm{Q}_K^{\mathrm{spin}}(\mathcal{P}) = \mathrm{Ind}_{K_\xi}^K \left(\bigwedge^\bullet \mathfrak{q}^\xi \otimes \mathbb{C}_{\xi-\rho(\xi)} \right). \tag{3.1}$$

Thus $\mathrm{Q}_K^{\mathrm{spin}}(\mathcal{P})$ is the virtual representation $\mathrm{Ind}_{K_\xi}^K E^+ - \mathrm{Ind}_{K_\xi}^K E^-$. The fact that $\mathrm{Q}_K^{\mathrm{spin}}(\mathcal{P})$ is a finite signed sum of representations of K is easy to see using Frobenius reciprocity formula.

We now interpret $\mathrm{Q}_K^{\mathrm{spin}}(\mathcal{P})$ as the equivariant index of a twisted Dirac operator on $\mathcal{P}$. Consider the K-equivariant graded complex vector bundle $\mathcal{S}_\mathcal{P} = K \times_{K_\xi} E$. It defines a Spin^c-bundle on $\mathcal{P}$ (see [4]). The determinant line bundle of $\mathcal{S}_\mathcal{P}$ is the line bundle $[\mathbb{C}_{2\xi}]$ and its corresponding moment

map (as defined in [4]) is the canonical injection $\mathcal{P} \to \mathfrak{k}^*$. Consider the Dirac operator $D_{\mathcal{P}}$ associated to this Spin^c-bundle. Then by Atiyah-Bott [1], the equivariant index of $D_{\mathcal{P}}$ is equal to $\mathrm{Ind}_{K_\xi}^K E^+ - \mathrm{Ind}_{K_\xi}^K E^-$. Thus $\mathrm{Q}_K^{\mathrm{spin}}(\mathcal{P})$ coincides with the equivariant index of the twisted Dirac operator $D_{\mathcal{P}}$ (and so belongs to $R(K)$). For this reason $\mathrm{Q}_K^{\mathrm{spin}}(\mathcal{P})$ is called the spin quantization of $\mathcal{P}$.

The following proposition is well known. We will recall its proof in Lemma 4.1 in the next subsection.

Proposition 3.9
- *The map $\mathcal{O} \mapsto \pi_{\mathcal{O}} := \mathrm{Q}_K^{\mathrm{spin}}(\mathcal{O})$ defines a bijection between the set of regular admissible orbits and $\widehat{K}$.*
- $\mathrm{Q}_K^{\mathrm{spin}}(o(\mathfrak{k}))$ *is the trivial representation of K.*

In Section 5, we will describe the virtual representation $\mathrm{Q}_K^{\mathrm{spin}}(\mathcal{P})$ attached to any admissible orbit in terms of regular admissible orbits. It is either equal to 0, or is an element of $\widehat{K}$.

4 The magical inequality

In order to parameterize coadjoint orbits, we choose a Cartan subgroup T of K with Lie algebra $\mathfrak{t}$. Let $\Lambda \subset \mathfrak{t}^*$ be the lattice of weights of T. Let W be the Weyl group. Choose a system of positive roots $\Delta^+ \subset \mathfrak{t}^*$, and let as before

$$\rho^K = \frac{1}{2} \sum_{\alpha > 0} \alpha.$$

If $\alpha \in \mathfrak{t}^*$ is a root, we denote by $H_\alpha \in \mathfrak{t}$ the corresponding coroot (so $\langle \alpha, H_\alpha \rangle = 2$). Then $\langle \rho^K, H_\alpha \rangle = 1$ if and only if α is a simple root.

Define the positive closed Weyl chamber by

$$\mathfrak{t}^*_{\geqslant 0} = \{\xi \in \mathfrak{t}^*;\ \langle \xi, H_\alpha \rangle \geqslant 0 \text{ for all } \alpha > 0\},$$

and we denote by $\Lambda_{\geqslant 0} := \Lambda \cap \mathfrak{t}^*_{\geqslant 0}$ the set of dominant weights. Any coadjoint orbit $\mathcal{P}$ of K is of the form $\mathcal{P} = K\xi$ with $\{\xi\} = \mathcal{P} \cap \mathfrak{t}^*_{\geqslant 0}$.

We index the set $\widehat{K}$ of classes of finite dimensional irreducible representations of K by the set $\rho^K + \Lambda_{\geqslant 0}$. The irreducible representation π_λ corresponding to $\lambda \in \rho^K + \Lambda_{\geqslant 0}$ is the irreducible representation with infinitesimal character λ. Its highest weight is $\lambda - \rho^K$. The representation

π_{ρ^K} is the trivial representation of K. The Weyl character formula for the representation π_λ is, for $X \in \mathfrak{t}$,

$$\mathbf{Tr}\, \pi_\lambda(e^X) = \frac{\sum_{w\in W} \epsilon(w) e^{i\langle w\lambda, X\rangle}}{\prod_{\alpha>0} e^{i\langle \alpha, X\rangle/2} - e^{-i\langle \alpha, X\rangle/2}}.$$

For any $\mu \in \mathfrak{t}^*$, recall its element $\rho(\mu) \in \mathfrak{k}^*$ (Definition 3.2).

Lemma 4.1 *Let $\lambda \in \mathfrak{t}^*_{\geqslant 0}$ be a regular admissible element of $\mathfrak{k}^*$. Then*

1. $\rho(\lambda) = \rho^K$.
2. $\lambda \in \rho^K + \Lambda_{\geqslant 0}$.
3. $\mathrm{Q}_K^{\mathrm{spin}}(K\lambda) = \pi_\lambda$.

Proof. Let $\lambda \in \mathfrak{t}^*_{\geqslant 0}$ be regular and admissible, then $\rho(\lambda) = \rho^K$, so $\lambda \in \{\rho^K + \Lambda\} \cap \mathfrak{t}^*_{>0}$. If α is a simple root, then **the integer** $\langle \lambda - \rho^K, H_\alpha\rangle = \langle \mu, H_\alpha\rangle - 1$ is non negative, as $\langle \lambda, H_\alpha\rangle > 0$. So $\lambda - \rho^K$ is a dominant weight.

Let $\mathcal{O} = K\lambda$. Weyl character formula coincide with Atiyah-Bott-Lefschetz formula [1] for the index of the Dirac operator $D_{\mathcal{O}}$. Thus we obtain Lemma 4.1 and Proposition 3.9. $\square$

The positive Weyl chamber $\mathfrak{t}^*_{\geqslant 0}$ is the cone determined by the equations $\langle \lambda, H_\alpha\rangle \geqslant 0$ for the simple roots $\alpha \geqslant 0$. We denote by $\mathcal{F}_{\mathfrak{k}}$ the set of the relative interiors of the faces of $\mathfrak{t}^*_{\geqslant 0}$. Thus the set $\mathcal{F}_{\mathfrak{k}}$ is parameterized by subsets of the simple roots and has 2^r elements, r being the rank of $[\mathfrak{k}, \mathfrak{k}]$. Thus $\mathfrak{t}^*_{\geqslant 0} = \coprod_{\sigma\in\mathcal{F}_{\mathfrak{k}}} \sigma$, and we denote by $\mathfrak{t}^*_{>0} \in \mathcal{F}_{\mathfrak{k}}$ the interior of $\mathfrak{t}^*_{\geqslant 0}$.

Let $\sigma \in \mathcal{F}_{\mathfrak{k}}$. Thus $\mathbb{R}\sigma$, the linear span of σ, is the subspace determined by $\langle \lambda, H_\alpha\rangle = 0$ where the α varies over a subset of the simple roots.

The stabilizer K_ξ does not depend of the choice of the point $\xi \in \sigma$: we denote it by K_σ. The map $\sigma \to \mathfrak{k}_\sigma$ induces a surjective map from $\mathcal{F}_{\mathfrak{k}}$ to $\mathcal{H}_{\mathfrak{k}}$. This map may not be injective: in Example 3.7, σ_1 and σ_2 leads to the same element of $\mathcal{H}_{\mathfrak{k}}$, as the corresponding groups $K_{\sigma_1} = S(U(1)\times U(2))$ and $K_{\sigma_2} = S(U(2)\times U(1))$ are conjugated.

For $\sigma \in \mathcal{F}_{\mathfrak{k}}$, we have the decomposition $\mathfrak{k}_\sigma = [\mathfrak{k}_\sigma, \mathfrak{k}_\sigma] \oplus \mathfrak{z}(\mathfrak{k}_\sigma)$ with dual decomposition $\mathfrak{k}^*_\sigma = [\mathfrak{k}_\sigma, \mathfrak{k}_\sigma]^* \oplus \mathbb{R}\sigma$. Let

$$\rho^{K_\sigma} := \frac{1}{2} \sum_{\substack{\alpha>0 \\ (\alpha,\sigma)=0}} \alpha$$

be the ρ-element of the group K_σ associated to the positive root system $\{\alpha > 0, (\alpha, \sigma) = 0\}$ for K_σ. Then

$$\rho^K - \rho^{K_\sigma} = \frac{1}{2} \sum_{\substack{\alpha > 0 \\ (\alpha, \sigma) > 0}} \alpha,$$

and for any $\mu \in \sigma$, the element $\rho(\mu) \in \mathfrak{k}^*$ is equal to $\rho^K - \rho^{K_\sigma}$. In particular, $\rho^K - \rho^{K_\sigma}$ vanishes on $[\mathfrak{k}_\sigma, \mathfrak{k}_\sigma]$, so $\rho^K - \rho^{K_\sigma} \in \mathbb{R}\sigma$, while $\rho^{K_\sigma} \in [\mathfrak{k}_\sigma, \mathfrak{k}_\sigma]^*$. The decomposition $\rho^K = (\rho^K - \rho^{K_\sigma}) + \rho^{K_\sigma}$ is an orthogonal decomposition.

Figure 3 shows this orthogonal decomposition of ρ for the case $SU(3)$.

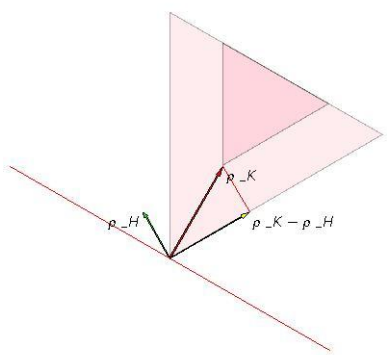

Figure 3: Orthogonal decomposition of ρ_K

The subset $\rho_K + \mathfrak{t}^*_{\geqslant 0}$ of the positive Weyl chamber will be called the shifted Weyl chamber. It is determined by the inequalities $\langle \lambda, H_\alpha \rangle \geqslant 1$ for any simple root $\alpha \geqslant 0$, and thus $\langle \lambda, H_\alpha \rangle \geqslant 1$ for any positive root. Let us list some immediate properties of the shifted Weyl chamber.

Proposition 4.2 *1. If $\lambda \in \rho^K + \mathfrak{t}^*_{\geqslant 0}$, then $(\lambda, \lambda) \geqslant (\lambda, \rho^K) \geqslant (\rho^K, \rho^K)$. The equality $(\lambda, \lambda) = (\lambda, \rho^K)$ holds only if $\lambda = \rho^K$.*

2. Let $\sigma \in \mathcal{F}_{\mathfrak{k}}$.

- *The orthogonal projection of $\xi \in \mathfrak{t}^*_{>0}$ onto $\mathbb{R}\sigma$ belongs to σ.*
- *We have $\rho^K - \rho^{K_\sigma} \in \sigma$ for any $\sigma \in \mathcal{F}_{\mathfrak{k}}$.*

3. For any $(\mathfrak{h}) \in \mathcal{H}_{\mathfrak{k}}$, $\|\rho^K\| \geqslant \|\rho^H\|$, and $\|\rho^K\| = \|\rho^H\|$ only if $H = K$.

Proof. If $\lambda = \rho^K + c$, with $c \in \mathfrak{t}^*_{\geqslant 0}$, inequalities $(\lambda, \lambda) \geqslant (\lambda, \rho^K) \geqslant (\rho^K, \rho^K)$ follows from the fact that (λ, c) and (ρ^K, c) are non negative, as the scalar product of two elements of $\mathfrak{t}^*_{\geqslant 0}$ is non negative. Equality $(\lambda, \lambda) = (\lambda, \rho^K)$ implies $\lambda = \rho^K$.

The second point follows from the fact that the dual cone to $\mathfrak{t}^*_{\geqslant 0}$ is generated by the simple roots α_i, and $(\alpha_i, \alpha_j) \leqslant 0$, if $i \neq j$.

We have the orthogonal decomposition $\rho^K = \rho^{K_\sigma} + (\rho^K - \rho^{K_\sigma})$: hence $\rho^K - \rho^{K_\sigma}$, which is the orthogonal projection of ρ^K on $\mathbb{R}\sigma$, belongs to σ.

For the third point, we might choose H conjugated to K_σ, so $\|\rho^K\|^2 = \|\rho^{K_\sigma}\|^2 + \|\rho^K - \rho^{K_\sigma}\|^2$. $\square$

The following theorem analyze the distance of a point in $\mathfrak{t}^*$ to the shifted Weyl chamber. It is illustrated in Figure 4 in the case $SU(3)$.

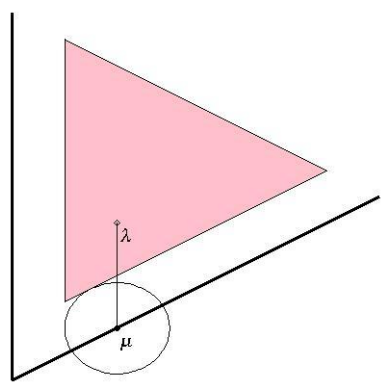

Figure 4: Distance of a singular element μ to an element λ in the shifted Weyl chamber

Theorem 4.3 • *If $\lambda \in \rho^K + \mathfrak{t}^*_{\geqslant 0}$ and $\mu \in \mathfrak{t}^*$, then:*

$$\|\lambda - \mu\|^2 \geqslant \frac{1}{2} \sum_{\substack{\alpha > 0 \\ (\alpha, \mu) = 0}} (\lambda, \alpha). \tag{4.2}$$

• *If $\lambda \in \rho^K + \mathfrak{t}^*_{\geqslant 0}$ and $\mu \in \mathfrak{t}^*$, then:*

$$\frac{1}{2} \sum_{\substack{\alpha > 0 \\ (\alpha, \mu) = 0}} (\lambda, \alpha) \geqslant \|\rho^{K_\mu}\|^2. \tag{4.3}$$

• *If one of the inequalities (4.2) or (4.3) is an equality, then μ belongs to $\mathfrak{t}^*_{\geqslant 0}$, and $\lambda = \mu + \rho^{K_\sigma}$ where $\sigma \in \mathcal{F}_{\mathfrak{k}}$ is the stratum of $\mathfrak{t}^*_{\geqslant 0}$ containing μ. Thus the two inequalities (4.2) or (4.3) are equalities. In particular, $\lambda - \rho(\lambda) = \mu - \rho(\mu)$.*

Proof. Let $\mathfrak{k}_\mu$ be the centralizer of μ and let $\mathfrak{z}$ be the center of $\mathfrak{k}_\mu$. Consider the orthogonal decomposition $\mathfrak{t}^* = \mathfrak{z}^* \oplus \mathfrak{a}^*$ where $\mathfrak{a}$ is a Cartan subalgebra for $[\mathfrak{k}_\mu, \mathfrak{k}_\mu]$, that is $\mathfrak{a} = \sum_{(\alpha,\mu)=0} \mathbb{R}H_\alpha$. Let $\rho^{K_\mu} \in \mathfrak{a}^*$ be the ρ element for the system $\Delta^1_+ = \{\alpha > 0, (\alpha, \mu) = 0\}$ of $[\mathfrak{k}_\mu, \mathfrak{k}_\mu]$.

Let us write $\lambda = \rho^K + c$, with c dominant, and decompose $\rho^K = p_0 + p_1$, $c = c_0 + c_1$, with $p_0, c_0 \in \mathfrak{z}^*$, $p_1, c_1 \in \mathfrak{a}^*$. Thus $\lambda = \lambda_0 + \lambda_1$, with $\lambda_0 \in \mathfrak{z}^*$ and $\lambda_1 = p_1 + c_1$. Now p_1 belongs to the shifted Weyl chamber in $\mathfrak{a}^*$. Indeed, for any $\alpha > 0$ such that $(\alpha, \mu) = 0$, we have $\langle p_1, H_\alpha \rangle = \langle \rho^K, H_\alpha \rangle \geqslant 1$. Similarly c_1 is dominant for the system Δ^1_+.

As $\mu \in \mathfrak{z}^*$, we have $\|\lambda - \mu\|^2 = \|\lambda_0 - \mu\|^2 + \|p_1 + c_1\|^2$. Using the first point of 4.2, we obtain

$$\|\lambda - \mu\|^2 = \|\lambda_0 - \mu\|^2 + \|p_1 + c_1\|^2 \geqslant (p_1 + c_1, \rho^{K_\mu}) \geqslant \|\rho^{K_\mu}\|^2.$$

As

$$(p_1 + c_1, \rho^{K_\mu}) = (\lambda, \rho^{K_\mu}) = \frac{1}{2} \sum_{\substack{\alpha > 0 \\ (\alpha,\mu)=0}} (\lambda, \alpha)$$

we obtain Inequalities (4.2) and (4.3).

If the inequality $\|\lambda - \mu\|^2 \geqslant (p_1 + c_1, \rho^{K_\mu})$ is an equality, then

$$\|\lambda_0 - \mu\|^2 + \big(\|p_1 + c_1\|^2 - (p_1 + c_1, \rho^{K_\mu})\big) = 0.$$

Both terms of the left hand side are non negative. Thus, using again the first point of Proposition 4.2, we see that necessarily $c_1 = 0$, $p_1 = \rho^{K_\mu}$, and $\lambda_0 = \mu$. Thus for roots $\alpha \in \Delta^1_+$, $\langle \rho^{K_\mu}, H_\alpha \rangle = \langle \rho^K, H_\alpha \rangle$. As ρ^{K_μ} takes value 1 on simple roots for K_μ, it follows that the set S_1 of simple roots for the system Δ^1_+ is contained in the set of simple roots for Δ^+. As $\mathfrak{a} = \oplus_{\alpha \in S_1} \mathbb{R}H_\alpha$, the orthogonal $\mathfrak{z}$ of $\mathfrak{a}$ is $\mathbb{R}\sigma$ for the face σ of $\mathfrak{t}^*$ orthogonal to the subset S_1 of simple roots. We then have $K_\mu = K_\sigma$. Furthermore, $\lambda = \mu + \rho^{K_\sigma}$. Thus μ is the projection of λ on $\mathbb{R}\sigma$, so $\mu \in \sigma \subset \mathfrak{t}^*_{\geqslant 0}$. As $\rho(\lambda) = \rho^K$, and $\rho(\mu) = \rho^K - \rho^{K_\sigma}$, we obtain $\lambda - \rho(\lambda) = \mu - \rho(\mu)$. So all assertions are proved. □

In this article, we will only use the following obvious corollary of Inequalities (4.2) and (4.3). However, in our application [3], we will need both inequalities.

Corollary 4.4 *(The magical inequality) If $\lambda \in \rho^K + \mathfrak{t}^*_{\geqslant 0}$ and $\mu \in \mathfrak{t}^*$, then:*

$$\|\lambda - \mu\| \geqslant \|\rho^{K_\mu}\|. \tag{4.4}$$

The equality holds if and only if μ is dominant and $\lambda - \rho(\lambda) = \mu - \rho(\mu)$.

Let us give some consequences of the magical inequality (4.4). Let us define the notion of very regular element.

Definition 4.5 *Let $\lambda \in \mathfrak{t}^*$ be a regular element. Then λ determines a closed positive Weyl chamber $C_\lambda \subset \mathfrak{t}^*_\lambda$. We say that λ is very regular if $\lambda \in \rho(\lambda) + C_\lambda$.*

In other words, a very regular element is an element which is conjugated to an element of the shifted Weyl chamber. Note that regular admissible elements are very regular.

Corollary 4.6 *Let λ, μ be two elements of $\mathfrak{t}^*$. Assume that λ is very regular, then $\|\lambda - \mu\| \geqslant \|\rho^{K_\mu}\|$. The equality holds if and only if $\mu \in C_\lambda$ and $\lambda - \rho(\lambda) = \mu - \rho(\mu)$.*

Proof. Consider the minimum of $\|\lambda - \mu'\|^2$ when μ' varies in $K\mu$. Using the differential, we see that a point where the minimum is reached is in the Cartan subalgebra determined by λ. We can then conclude by Corollary (4.4). $\square$

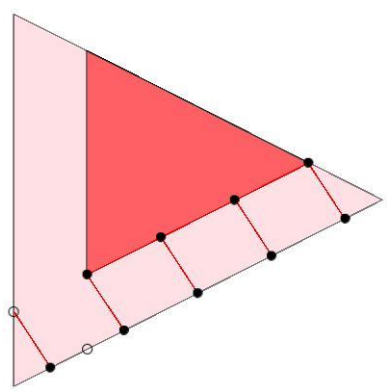

Figure 5: Shifts of admissible orbits

As consequence, we obtain Theorem 1.1 stated in the introduction.

Corollary 4.7 *The distance between a regular admissible coadjoint orbit $\mathcal{O}$ and an orbit $\mathcal{P}$ in a Dixmier sheet S is greater or equal to $\|\rho_S\|$.*

Let us now study the admissible coadjoint orbits and their shifts. We see that if μ is in σ, its shift is $s(K\mu) = K(\mu + \rho^{K_\sigma})$. Furthermore, $\mu \in \sigma$ is admissible if and only if $\mu + \rho^{K_\sigma} \in \rho^K + \Lambda$.

Theorem 4.8 below says in particular that if a shift $\mu \to \mu + \rho^{K_\sigma}$ of an admissible element $\mu \in \sigma$ is regular, then it is in the shifted Weyl chamber $\rho_K + \mathfrak{t}^*_{\geqslant 0}$.

Figure 5 illustrate this fact in the case $SU(3)$. In this figure, the two points ω_1, ω_2 (which are not admissible) are represented by the empty circles. The shift of the admissible element $\frac{1}{2}\omega_1$ is ω_2, and so is singular and is not admissible.

Theorem 4.8 *Let σ be a relative interior of a face of $\mathfrak{t}^*_{\geqslant 0}$.*

1. *Let $\mu \in \sigma$. If $\mu + \rho^{K_\sigma}$ is regular admissible, then μ is admissible.*
2. *If $\mu \in \sigma$ is admissible, and $\mu + \rho^{K_\sigma}$ is regular, then $\mu + \rho^{K_\sigma}$ is regular admissible. Moreover $\mu + \rho^{K_\sigma} \in \rho^K + (\Lambda_{\geqslant 0} \cap \overline{\sigma})$. In particular $\mu + \rho^{K_\sigma}$ is in the shifted Weyl chamber.*
3. *If $\mu \in \sigma$ is admissible, we have*

$$\mathrm{Q}_K^{\mathrm{spin}}(K\mu) = \begin{cases} 0 & \text{if } \mu + \rho^{K_\sigma} \text{ is singular,} \\ \pi_{\mu+\rho^{K_\sigma}} & \text{if } \mu + \rho^{K_\sigma} \text{ is regular.} \end{cases}$$

Proof. Let us prove the two first points.

Let $\mu \in \sigma$ such that $\lambda = \mu + \rho^{K_\sigma}$ is regular admissible. Thus $\|\lambda - \mu\|^2 = \|\rho^{K_\mu}\|^2$. The point λ being regular and admissible, λ is very regular. We use the magical inequality. The equality $\|\lambda - \mu\|^2 = \|\rho^{K_\mu}\|^2$ implies $\lambda - \rho(\lambda) = \mu - \rho(\mu) = \mu - (\rho^K - \rho^{K_\sigma})$. Thus $\rho(\lambda) = \rho^K$, so $\lambda \in \mathfrak{t}^*_{\geqslant 0}$. Furthermore μ is admissible.

Conversely, we see that if μ is admissible, and $\lambda = \mu + \rho^{K_\sigma}$ is regular, then λ is regular admissible. We have seen that $\lambda \in \mathfrak{t}^*_{\geqslant 0}$. As λ is regular admissible, it is in $\rho^K + \mathfrak{t}^*_{\geqslant 0}$. The element $\lambda - \rho^K = \mu - (\rho^K - \rho^{K_\sigma})$ is in $\mathbb{R}\sigma$. As it is dominant, it is in $\overline{\sigma}$.

Let us prove the last point. Let $\mathfrak{q}^\mu$ be the complex space $\mathfrak{k}/\mathfrak{k}_\mu$ equipped with the complex structure J_μ. The equivariant index Θ of the Dirac operator $D_{K\mu}$ associated to the Spin^c-bundle $\mathcal{S}_{K\mu} = K \times_{K_\mu} (\bigwedge^\bullet \mathfrak{q}^\mu \otimes \mathbb{C}_{\mu - \rho^K + \rho^{K_\sigma}})$ is given by Atiyah-Bott fixed point formula: for $X \in \mathfrak{t}$, $\Theta(e^X) = \sum_{w \in W/W_\mu} w \cdot$

$\frac{e^{i\langle\mu,X\rangle}}{\prod_{\langle\alpha,\mu\rangle>0} e^{i\langle\alpha,X\rangle/2}-e^{-i\langle\alpha,X\rangle/2}}$. Here W_μ, the stabilizer of μ in W, is equal to the Weyl group of the group K_σ. Using $\sum_{w\in W_\sigma}\epsilon(w)e^{w\rho^{K_\sigma}} = \prod_{\alpha>0,\langle\alpha,\sigma\rangle=0}(e^{\alpha/2}-e^{-\alpha/2})$, we obtain

$$\Theta(e^X) = \frac{\sum_{w\in W}\epsilon(w)e^{i\langle w(\mu+\rho^{K_\sigma}),X\rangle}}{\prod_{\alpha>0} e^{i\langle\alpha,X\rangle/2}-e^{-i\langle\alpha,X\rangle/2}}. \tag{4.5}$$

If $\mu+\rho^{K_\sigma}$ is singular, Θ is equal to zero. If $\mu+\rho^{K_\sigma}$ is regular, thanks to the first point, $\mu+\rho^{K_\sigma}$ is in $\rho^K+\Lambda_{\geqslant 0}$, so $\Theta=\pi_{\mu+\rho^{K_\sigma}}$. $\square$

We proved that if $\mu\in\sigma$ is admissible **and** its shift $\mu+\rho^{K_\sigma}$ is regular, then $\mu+\rho^{K_\sigma}$ is admissible and dominant. However, as illustrated in the following examples, if $\mu+\rho^{K_\sigma}$ is singular,

- $\mu+\rho^{K_\sigma}$ is not necessarily dominant,
- $\mu+\rho^{K_\sigma}$ might be admissible or not admissible.

Example 4.9 *Consider the group $K = U(7)$ and the face $\sigma=\{\lambda_1>\lambda_2=\lambda_3=\lambda_4=\lambda_5=\lambda_6>\lambda_7\}$ of the Weyl chamber. Here $\rho^K=(3,2,1,0,-1,-2,-3)$ and $\rho^{K_\sigma}=(0,2,1,0,-1,-2,0)$. Then $\mu:=(1,0,0,0,0,0,-1)$ is an admissible element of σ, but its shift $\mu+\rho^{K_\sigma}=(1,2,1,0,-1,-2,0)$ is singular, not admissible and not dominant.*

Example 4.10 *Consider the group $K=U(5)$ and the face $\sigma=\{\lambda_1=\lambda_2>\lambda_3>\lambda_4=\lambda_5\}$ of the Weyl chamber. Then $\mu=(1/2,1/2,0,-1/2,-1/2)$ is an admissible element of σ. Its shift $\mu+\rho^{K_\sigma}=(1,0,0,0,-1)$ is singular and admissible.*

For application to the equivariant index of Dirac operators on general Spin^c K-manifolds, we reformulate Inequality (4.2) independently of a choice of a Weyl chamber using normalized traces.

Definition 4.11 *Let N be a real vector space and $b:N\to N$ a linear transformation, such that $-b^2$ is diagonalizable with non negative eigenvalues, and let $|b|=\sqrt{-b^2}$. We denote by $\mathbf{nTr}_N|b|=\frac{1}{2}\mathbf{Tr}_N|b|$, that is half of the trace of the action of $|b|$ in the real vector space N. We call $\mathbf{nTr}_N|b|$ the normalized trace of b.*

If N is an Euclidean space and b a skew-symmetric transformation of N, then $-b^2$ is diagonalizable with non negative eigenvalues.

For any $b\in\mathfrak{k}$ and $\mu\in\mathfrak{k}^*$ fixed by b, we may consider the action $\mathrm{ad}(b):\mathfrak{k}_\mu\to\mathfrak{k}_\mu$. The corresponding normalized trace $\mathbf{nTr}_{\mathfrak{k}_\mu}|\mathrm{ad}(b)|$ is denoted simply by $\mathbf{nTr}_{\mathfrak{k}_\mu}|b|$.

Proposition 4.12 *Let $b \in \mathfrak{k}$ and denote by β the corresponding element in $\mathfrak{k}^*$. Let λ, μ be elements of $\mathfrak{k}^*$ fixed by b. Assume that λ is very regular and that $\mu - \lambda = \beta$. Then*

$$\|\beta\|^2 \geqslant \frac{1}{2}\mathbf{nTr}_{\mathfrak{k}_\mu}|b|.$$

If the equality holds, then μ belongs to the positive Weyl chamber C_λ and

1. *$\lambda - \rho(\lambda) = \mu - \rho(\mu)$, hence λ is admissible if and only if μ is admissible,*
2. *$s(K\mu) = K\lambda$.*

Proof. Indeed, as λ is fixed by b, we see that β belong to $\mathfrak{k}^*_\lambda$. We may assume that $\mathfrak{k}^*_\lambda = \mathfrak{t}^*$. Thus β, λ and $\mu = \lambda - \beta$ belong to $\mathfrak{t}^*$. The element λ is a very regular element of $\mathfrak{t}^*$. Note that the element of $\mathfrak{k}$ corresponding to μ acts trivially on $\mathfrak{k}_\mu$. So the inequality of Proposition 4.12 is a restatement of Inequality 4.2. If the equality holds, we apply Theorem 4.3 and we obtain the proposition. $\square$

5 Admissible coadjoint orbits and associated representations

In this section, we give some more information on the map $\mathcal{P} \to \mathrm{Q}_K^{\mathrm{spin}}(\mathcal{P})$. The following proposition follows almost immediately from Theorem 4.8.

Theorem 5.1 *Let $\mathcal{P}$ be an admissible orbit.*

- *$\mathcal{P}^* := -\mathcal{P}$ is also admissible and $\mathrm{Q}_K^{\mathrm{spin}}(\mathcal{P}^*) = \mathrm{Q}_K^{\mathrm{spin}}(\mathcal{P})^*$.*
- *If $s(\mathcal{P})$ is not regular, then $\mathrm{Q}_K^{\mathrm{spin}}(\mathcal{P}) = 0$.*
- *If $s(\mathcal{P})$ is regular, then $s(\mathcal{P})$ is a regular admissible coadjoint orbit and $\mathrm{Q}_K^{\mathrm{spin}}(\mathcal{P}) = \mathrm{Q}_K^{\mathrm{spin}}(s(\mathcal{P})) = \pi_{s(\mathcal{P})}$.*

For the remaining part of this section, we fix a conjugacy class $(\mathfrak{h})$. Let $\mathfrak{k}^*_{(\mathfrak{h})}$ be the Dixmier sheet determined by $(\mathfrak{h})$.

Definition 5.2 *Let $\mathcal{O} \subset \mathfrak{k}^*$ be a regular orbit. A K-orbit $\mathcal{P}$ in is called a $(\mathfrak{h})$-ancestor of $\mathcal{O}$ if $\mathcal{P} \in \mathfrak{k}^*_{(\mathfrak{h})}$ and $s(\mathcal{P}) = \mathcal{O}$.*

Lemma 5.3 *If $\mathcal{P} \in \mathfrak{k}^*_{(\mathfrak{h})}$ and $s(\mathcal{P})$ is regular, then $s(\mathcal{P})$ is admissible if and only $\mathcal{P}$ is admissible.*

Proof. Let $\mathcal{O} = s(\mathcal{P})$. Assume that $\mathcal{O}$ is admissible. We may assume that $\mathcal{O} = K\lambda$ with $\lambda \in \rho^K + \mathfrak{t}^*_{\geqslant 0}$ regular admissible, and $\mathcal{P} = K\mu$, with $\mu \in \mathfrak{t}^*_{\geqslant 0}$. Let σ be the stratum of $\mathfrak{t}^*_{\geqslant 0}$ containing μ. By Theorem 4.12, $\lambda = \mu + \rho^{K_\sigma}$. We have $\mu - \rho(\mu) = \lambda - \rho^K$, so μ is admissible. The converse is proved the same way. $\square$

Theorem 5.4 *Let $\mathcal{O} \subset \mathfrak{k}^*$ be a regular admissible orbit.*

- *If $\mathcal{P}$ is a $(\mathfrak{h})$-ancestor of $\mathcal{O}$, then $\mathcal{P}$ is admissible and at distance $\|\rho^H\|$ of $\mathcal{O}$.*
- *If $\mathcal{P}$ is an element in $\mathfrak{k}^*_{(\mathfrak{h})}$ at distance $\|\rho^H\|$ of $\mathcal{O}$, then $\mathcal{P}$ is admissible, $s(\mathcal{P}) = \mathcal{O}$ and $\mathrm{Q}^{\mathrm{spin}}_K(\mathcal{P}) = \mathrm{Q}^{\mathrm{spin}}_K(\mathcal{O})$.*

Proof. We need only to prove the second point. Assume that the distance between $\mathcal{O}$ and $\mathcal{P}$ is equal to $\|\rho^H\|$. We may assume that $\mathcal{O} = K\lambda$ with $\lambda \in \rho^K + \mathfrak{t}^*_{\geqslant 0}$ regular admissible. We write $\mathcal{P} = K\mu$, with μ a point in $\mathfrak{t}^*$, such that $\|\lambda - \mu\|^2 = \|\rho^{K_\mu}\|$. This implies that μ belongs to $\mathfrak{t}^*_{\geqslant 0}$, and that $s(\mathcal{P}) = \mathcal{O}$. So $\mathrm{Q}^{\mathrm{spin}}_K(\mathcal{P}) = \mathrm{Q}^{\mathrm{spin}}_K(\mathcal{O})$. $\square$

Example 5.5 *Let us go back to Example 3.7 for the group $K = SU(3)$. We see that the orbit of ρ^K has one ancestor (itself) in the regular sheet, two ancestors in the subregular sheet, and one ancestor 0 in the sheet $\{0\}$.*

In general an orbit $\mathcal{O} = K\mu$ has only one ancestor, that is itself. Only the orbits $\mathcal{O}$ belonging to the boundary of the shifted Weyl chamber might have lower dimensional ancestors. For example the orbits $\mathcal{P}_\sigma$ of the orthogonal projections of ρ^K on the 2^r linear spaces $\mathbb{R}\sigma$ ($\sigma \in \mathcal{F}_{\mathfrak{k}}$) are ancestors of $o(\mathfrak{k})$. For all these orbits $\mathcal{P}_\sigma$, the representation $\mathrm{Q}^{\mathrm{spin}}_K(\mathcal{P}_\sigma)$ is the trivial representation of K. As the number of sheets is usually less than 2^r, some of the ancestors of ρ^K lie in the same sheet.

Finally we end this article by an induction formula relating H-admissible coadjoint orbits to K-admissible coadjoint orbits.

Consider the open subset $\mathfrak{h}^*_0 := \{\xi \in \mathfrak{h}^* \mid K_\xi \subset H\}$. Equivalently, the element ξ, identified to an element of $\mathfrak{h}$, is such that the transformation $\mathrm{ad}(\xi)$ is invertible on $\mathfrak{q} := \mathfrak{k}/\mathfrak{h}$, so it determines a complex structure J_ξ on $\mathfrak{q}$. We see that the complex structure J_ξ depends only of the connected component C of $\mathfrak{h}^*_0$ containing ξ. We denote it by J_C (remark that J_C is

H-invariant). We denote $\mathfrak{q}^C$ the complex H-module $(\mathfrak{q}, J_C)$, and ρ_C the element of $\mathfrak{z}^*$ defined by the relation

$$\langle \rho_C, X\rangle = \frac{1}{2i}\mathbf{Tr}_{\mathfrak{q}^C}\mathrm{ad}(X), \quad X \in \mathfrak{h}.$$

We define the holomorphic induction map $\mathrm{Hol}_H^K : R(H) \to R(K)$ by the relation

$$\mathrm{Hol}_H^K(V) = \mathrm{Ind}_H^K\left(\bigwedge\nolimits^\bullet \mathfrak{q}^C \otimes V\right).$$

The following proposition explains the interaction between the holomorphic induction map Hol_H^K and the spin quantization procedure.

Proposition 5.6 *Let $H\mu$ be an admissible orbit for H.*

- *If $\mu + \rho_C \notin \mathfrak{h}_0^*$, then $\mathrm{Hol}_H^K(\mathrm{Q}_H^{\mathrm{spin}}(H\mu)) = 0$.*
- *If $\mu + \rho_C \in \mathfrak{h}_0^*$, then $\mu + \rho_C$ is K-admissible and*

$$\mathrm{Hol}_H^K(\mathrm{Q}_H^{\mathrm{spin}}(H\mu)) = \epsilon_{C'}^C\, \mathrm{Q}_K^{\mathrm{spin}}(K(\mu + \rho_C))$$

where C' is the connected component of $\mathfrak{h}_0^*$ containing $\mu + \rho_C$, and $\epsilon_{C'}^C$ is the ratio of the orientation $o(J_C)$ and $o(J_{C'})$ on $\mathfrak{q}$.

Proof. Let $I_\mu^C = \mathrm{Ind}_H^K\left(\bigwedge^\bullet(\mathfrak{k}/\mathfrak{h})^C \otimes \mathrm{Q}_H^{\mathrm{spin}}(H\mu)\right)$. By definition $\mathrm{Q}_H^{\mathrm{spin}}(H\mu) = \mathrm{Ind}_{H_\mu}^H\left(\bigwedge^\bullet(\mathfrak{h}/\mathfrak{h}_\mu)^\mu \otimes \mathbb{C}_{\mu-\rho^H(\mu)}\right)$.

Assume first that $\mu' := \mu + \rho_C \in \mathfrak{h}_0^*$: let C' be the connected component of $\mathfrak{h}_0^*$ containing μ'. As $K_{\mu'} = H_{\mu'} = H_\mu$, we have

$$I_\mu^C = \mathrm{Ind}_{K_{\mu'}}^K\left(\bigwedge\nolimits^\bullet(\mathfrak{k}/\mathfrak{h})^C \otimes \bigwedge\nolimits^\bullet(\mathfrak{h}/\mathfrak{h}_{\mu'})^{\mu'} \otimes \mathbb{C}_{\mu'-\rho_C-\rho^H(\mu')}\right).$$

Now we use the fact that the graded $K_{\mu'}$-module $\bigwedge^\bullet(\mathfrak{k}/\mathfrak{h})^C$ is equal to $\epsilon_{C'}^C \bigwedge^\bullet(\mathfrak{k}/\mathfrak{h})^{C'} \otimes \mathbb{C}_{\rho_C-\rho_{C'}}$. It gives that

$$\begin{aligned} I_\mu^C &= \epsilon_{C'}^C\, \mathrm{Ind}_{K_{\mu'}}^K\left(\bigwedge\nolimits^\bullet(\mathfrak{k}/\mathfrak{h})^{C'} \otimes \bigwedge\nolimits^\bullet(\mathfrak{h}/\mathfrak{h}_{\mu'})^{\mu'} \otimes \mathbb{C}_{\mu'-\rho_{C'}-\rho^H(\mu')}\right) \\ &= \epsilon_{C'}^C\, \mathrm{Ind}_{K_{\mu'}}^K\left(\bigwedge\nolimits^\bullet(\mathfrak{k}/\mathfrak{k}_{\mu'})^{\mu'} \otimes \mathbb{C}_{\mu'-\rho(\mu')}\right) \\ &= \epsilon_{C'}^C\, \mathrm{Q}_K^{\mathrm{spin}}(K\mu'). \end{aligned}$$

Here we have used that $\rho(\mu') = \rho_{C'} + \rho^H(\mu')$.

Assume now that $I_\mu^C \neq 0$. Thus $\mathrm{Q}_H^{\mathrm{spin}}(H\mu)$ must be non zero. Hence we have $\mathrm{Q}_H^{\mathrm{spin}}(H\mu) = \mathrm{Q}_H^{\mathrm{spin}}(H\tilde{\mu})$ where $\tilde{\mu} \in \mu + o(\mathfrak{h}_\mu)$ is an H-admissible and H-regular element.

Consider the maximal torus $T := H_{\tilde{\mu}}$, and a Weyl chamber $\mathcal{C} = \mathfrak{t}^*_{\geqslant 0}$ for K containing $\tilde{\mu}$. Let $J_{\mathcal{C}}$ be the corresponding complex structure on $\mathfrak{k}/\mathfrak{t}$. Let ρ^K be the ρ-element associated to the choice of Weyl chamber. Let C' be the connected component of $\mathfrak{h}_0^*$ that contains the open face $\mathfrak{t}^*_{>0}$.

If we use the relation $\rho^K = \rho_{C'} + \rho^H(\tilde{\mu})$, one has like before

$$\begin{aligned} I_\mu^C &= \mathrm{Ind}_H^K\Big(\bigwedge\nolimits^\bullet(\mathfrak{k}/\mathfrak{h})^C \otimes \mathrm{Q}_H^{\mathrm{spin}}(H\tilde{\mu})\Big) \\ &= \mathrm{Ind}_T^K\Big(\bigwedge\nolimits^\bullet(\mathfrak{k}/\mathfrak{h})^C \otimes \bigwedge\nolimits^\bullet(\mathfrak{h}/\mathfrak{t})^{\tilde{\mu}} \otimes \mathbb{C}_{\tilde{\mu}-\rho^H(\tilde{\mu})}\Big) \\ &= \epsilon_{C'}^C\, \mathrm{Ind}_T^K\Big(\bigwedge\nolimits^\bullet(\mathfrak{k}/\mathfrak{t})^{\mathcal{C}} \otimes \mathbb{C}_{\tilde{\mu}+\rho_C-\rho^K}\Big). \end{aligned}$$

We see then that $I_\mu^C \neq 0$ only if $\lambda := \tilde{\mu} + \rho_C = \mu' + \rho^{H_{\mu'}}$ is a K-regular element.

Here we have $\|\rho^{H_{\mu'}}\| = \|\lambda - \mu'\|$, and on the other hand by the magical inequality we must have $\|\lambda - \mu'\| \geqslant \|\rho^{K_{\mu'}}\|$ since λ is K-regular and admissible. It forces $\|\rho^{K_{\mu'}}\|$ to be equal to $\|\rho^{H_{\mu'}}\|$, and then $K_{\mu'} = H_{\mu'}$: the element $\mu' = \mu + \rho_C$ belongs to $\mathfrak{h}_0^*$.

The proof is completed. $\square$

Proposition 5.6 is a very special case of the formula for equivariant indices of twisted Dirac operators obtained in [3]. Indeed the representation $\mathrm{Hol}_H^K(\mathrm{Q}_H^{\mathrm{spin}}(H\mu))$ is the equivariant index for a Spin^c-bundle on $M = K/H_\mu$. The infinitesimal stabilizer $\mathfrak{k}_M$ is the conjugacy classes of $\mathfrak{h}_\mu = \mathfrak{k}_{\mu+\rho_C}$. It is indeed a representation associated to the Dixmier sheet attached to $(\mathfrak{k}_{\mu+\rho_C})$.

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