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Inverse problems for a two by two reaction-diffusion system using a Carleman estimate with one observation

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Abstract. For a two by two reaction-diffusion system on a bounded domain we give a simultaneous stability result for one coefficient and for the initial conditions. The key ingredient is a global Carleman-type estimate with a single observation acting on a subdomain.

Keywords : Inverse problem, Reaction-diffusion system, Carleman estimate

AMS classification scheme numbers: 35K05, 35K57, 35R30.

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1. Introduction

This paper is devoted to the simultaneous identification of one coefficient and the initial conditions in a reaction-diffusion system using the least number of observations as possible.
Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of $\mathbb{R}^n$ with $n \leq 3$. We denote by n the outward unit normal to Ω on $\Gamma = \partial\Omega$ assumed to be of class $\mathcal{C}^1$. Let $T > 0$ and $t_0 \in (0, T)$. We shall use the following notations $Q_0 = \Omega \times (0, T)$, $Q = \Omega \times (t_0, T)$, $\Sigma = \Gamma \times (t_0, T)$ and $\Sigma_0 = \Gamma \times (0, T)$. We consider the following reaction-diffusion system which arises for instance in mathematical biology:

$$\begin{cases} \partial_t u = \Delta u + a(x)u + b(x)v & \text{in} \quad Q_0, \\ \partial_t v = \Delta v + c(x)u + d(x)v & \text{in} \quad Q_0, \\ u(t,x) = g(t,x), \ v(t,x) = h(t,x) & \text{on} \quad \Sigma_0, \\ u(0,x) = u_0 \quad \text{and} \ \ v(0,x) = v_0 & \text{in} \quad \Omega, \end{cases} \tag{1}$$

Throughout this paper, let us consider the following set

$$\Lambda(R) = \{\Phi \in L^\infty(\Omega); \ \|\Phi\|_{L^\infty(\Omega)} \leqslant R\},$$

where R is a given positive constant.
If we assume that $(u_0, \ v_0)$ belongs to $(H^2(\Omega))^2$ and g, h are sufficiently regular (e.g. $\exists \ \epsilon > 0$ such that $g, h \in H^1(t_0, T, H^{2+\varepsilon}(\partial\Omega)) \cap H^2(t_0, T, H^\varepsilon(\partial\Omega))$), then (1) admits a solution in $H^1(t_0, T, H^2(\Omega))$ (see [10]). We will later use this regularity result.
We also assume that

$$\begin{cases} a, \ b, \ c, \ d \ \in \Lambda(R), \\ \text{There exist } r > 0, \ c_0 > 0 \text{ such that} \\ \widetilde{u}_0 \geq 0, \ \widetilde{v}_0 \geq r, \ c \geq c_0, \ \widetilde{b} > 0, \ c + dr \geq 0, \\ g \geq 0 \text{ and } h \geq r. \end{cases}$$

Let ω be a subdomain of Ω. Let (u, v) (resp. $(\widetilde{u}, \ \widetilde{v})$) be solution of (1) associated to $(a, \ b, \ c, \ d, \ u_0, \ v_0)$ (resp. $(a, \ \widetilde{b}, \ c, \ d, \ \widetilde{u}_0, \ \widetilde{v}_0)$) satisfying some regularity and "positivity" properties. We assume that we can measure $\partial_t v$ on ω in the time interval (t_0, T) for some $t_0 \in (0, T)$ and Δu, u and v in Ω at time $T' \in (t_0, T)$. Our main results are

- A stability result for the coefficient $b(x)$ (or $a(x)$):
  For $\widetilde{u}_0$, $\widetilde{v}_0$ in $H^2(\Omega)$ there exists a constant

  $$C = C(\Omega, \omega, c_0, t_0, T, r, R) > 0$$

  such that

  $$|b - \widetilde{b}|^2_{L^2(\Omega)} \leq C|\partial_t v - \partial_t \widetilde{v}|^2_{L^2((t_0,T)\times\omega)} + C|\Delta u(T', \cdot) - \Delta\widetilde{u}(T', \cdot)|^2_{L^2(\Omega)}$$
  $$+C|u(T', \cdot) - \widetilde{u}(T', \cdot)|^2_{L^2(\Omega)} + C|v(T', \cdot) - \widetilde{v}(T', \cdot)|^2_{L^2(\Omega)}.$$

- A stability estimate for the initial conditions u_0, v_0:
  For u_0, v_0, $\widetilde{u}_0$, $\widetilde{v}_0$ in $H^4(\Omega)$ there exists a constant

$$C = C(\Omega, \omega, c_0, t_0, T, r, R) > 0$$

  such that

$$|u_0 - \widetilde{u}_0|^2_{L^2(\Omega)} + |v_0 - \widetilde{v}_0|^2_{L^2(\Omega)} \leq \frac{C}{|\log E|},$$

  where $E = |\partial_t v - \partial_t \widetilde{v}|^2_{L^2((t_0,T)\times\omega)} + |u(T',\cdot) - \widetilde{u}(T',\cdot)|^2_{H^2(\Omega)} + |v(T',\cdot) - \widetilde{v}(T',\cdot)|^2_{H^2(\Omega)}$.

The key ingredient to these stability results is a global Carleman estimate for a two by two system with one observation. Controllability for such parabolic systems has been studied in [1]. The Carleman estimate obtained in [1] cannot be used to solve the inverse problem of identification of one coefficient and initial conditions because of the weight functions which are different in the left and right hand side of their estimate. We establish a new Carleman estimate with one observation involving the same weight function in the left and right hand side. Concerning the stability of the initial conditions we use an extension of the logarithmic convexity method (see [7]).
The simultaneous reconstruction of one coefficient and initial conditions from the measurement of one solution v over $(t_0, T) \times \omega$ and some measurement at fixed time T' is an essential aspect of our result. In the perspective of numerical reconstruction, such problems are ill-posed. Stability results are thus of importance.
Inverse problems for parabolic equations are well studied (see [4], [8], [13]). A recent book of Klibanov and Timonov [11] is devoted to the Carleman estimates applied to inverse coefficient problems. In our knowledge, there is no work about inverse problems for coupled parabolic systems.
The used method allows us to give a stability result for the coefficient $a(x)$ adapting assumption 3.1. On the other hand, since we only measure $\partial_t v$ on ω, we cannot obtain such stability results for the coefficients $c(x)$ or $d(x)$ of the second equation of (1). For the reconstruction of two coefficients the problem is more complicated. We obtain partial results with restrictive assumptions on the coefficients $a(x)$, $b(x)$, $c(x)$ and $d(x)$. In order to avoid such assumptions, we think it is necessary to use other methods such as those used in [9].
Our paper is organized as follows. In Section 2, we derive a global Carleman estimate for system (1) with one observation, i.e. the measurement of one solution v over $(t_0, T) \times \omega$. In Section 3, we prove a stability result for the coefficient $b(x)$ when one of the solutions $\widetilde{v}$ is in a particular class of solutions with some regularity and "positivity" properties. In Section 4, we prove a stability result for the initial conditions.

2. Carleman estimate

We prove here a Carleman-type estimate with a single observation acting on a subdomain ω of Ω in the right-hand side of the estimate. Let us introduce the following notations:

let $\omega' \Subset \omega$ and let $\widetilde{\beta}$ be a $\mathcal{C}^2(\Omega)$ function such that

$$\widetilde{\beta} > 0, \text{ in } \Omega, \ \widetilde{\beta} = 0 \text{ on } \partial\Omega, \ \min\{|\nabla\widetilde{\beta}(x)|, \ x \in \overline{\Omega \setminus \omega'}\} > 0 \quad \text{and} \quad \partial_n\widetilde{\beta} < 0 \text{ on } \partial\Omega.$$

Then, we define $\beta = \widetilde{\beta} + K$ with $K = m\|\widetilde{\beta}\|_\infty$ and $m > 1$. For $\lambda > 0$ and $t \in (t_0, T)$, we define the following weight functions

$$\varphi(x,t) = \frac{e^{\lambda\beta(x)}}{(t-t_0)(T-t)}, \qquad \eta(x,t) = \frac{e^{2\lambda K} - e^{\lambda\beta(x)}}{(t-t_0)(T-t)}.$$

If we set $\psi = e^{-s\eta}q$, we also introduce the following operators

$$\begin{aligned} M_1\psi &= -\Delta\psi - s^2\lambda^2|\nabla\beta|^2\varphi^2\psi + s(\partial_t\eta)\psi, \\ M_2\psi &= \partial_t\psi + 2s\lambda\varphi\nabla\beta.\nabla\psi + 2s\lambda^2\varphi|\nabla\beta|^2\psi. \end{aligned}$$

Then the following result holds (see [5]).

Theorem 2.1 *There exist $\lambda_0 = \lambda_0(\Omega,\omega) \geq 1$, $s_0 = s_0(\lambda_0, T) > 1$ and a positive constant $C_0 = C_0(\Omega,\omega,T)$ such that, for any $\lambda \geq \lambda_0$ and any $s \geq s_0$, the next inequality holds:*

$$\begin{aligned} &\|M_1(e^{-s\eta}q)\|^2_{L^2(Q)} + \|M_2(e^{-s\eta}q)\|^2_{L^2(Q)} \\ &+s\lambda^2 \iint_Q e^{-2s\eta}\varphi|\nabla q|^2 \ dx \ dt + s^3\lambda^4 \iint_Q e^{-2s\eta}\varphi^3|q|^2 \ dx \ dt \\ &\leq C_0 \left[s^3\lambda^4 \int_{t_0}^T\int_\omega e^{-2s\eta}\varphi^3|q|^2 \ dx \ dt + \iint_Q e^{-2s\eta} \ |\partial_t q - \Delta q|^2 \ dx \ dt \right], \end{aligned} \tag{2}$$

for all $q \in H^1(t_0, T, H^2(\overline{\Omega}))$ with $q = 0$ on Σ.

From the above theorem we have also the following result (see [5] and [6]).

Proposition 2.2 *There exist $\lambda_0 = \lambda_0(\Omega,\omega) \geq 1$, $s_0 = s_0(\lambda_0, T) > 1$ and a positive constant $C_0 = C_0(\Omega,\omega,T)$ such that, for any $\lambda \geq \lambda_0$ and any $s \geq s_0$, the next inequality holds:*

$$\begin{aligned} &s^{-1} \iint_Q e^{-2s\eta}\varphi^{-1}(|\partial_t q|^2 + |\Delta q|^2) \ dx \ dt \\ &+s\lambda^2 \iint_Q e^{-2s\eta}\varphi|\nabla q|^2 \ dx \ dt + s^3\lambda^4 \iint_Q e^{-2s\eta}\varphi^3|q|^2 \ dx \ dt \\ &\leq C_0 \left[s^3\lambda^4 \int_{t_0}^T\int_\omega e^{-2s\eta}\varphi^3|q|^2 \ dx \ dt + \iint_Q e^{-2s\eta} \ |\partial_t q - \Delta q|^2 \ dx \ dt \right], \end{aligned} \tag{3}$$

for all $q \in H^1(t_0, T, H^2(\overline{\Omega}))$ with $q = 0$ on Σ.

We consider the solutions (u, v) and $(\widetilde{u}, \widetilde{v})$ to the following systems

$$\begin{cases} \partial_t u = \Delta u + au + bv & \text{in} \quad Q_0, \\ \partial_t v = \Delta v + cu + dv & \text{in} \quad Q_0, \\ u(t,x) = g(t,x), \ v(t,x) = h(t,x) & \text{on} \quad \Sigma_0, \\ u(0,x) = u_0 \ \text{ and } \ v(0,x) = v_0 & \text{in} \quad \Omega, \end{cases} \tag{4}$$

and

$$\begin{cases} \partial_t \widetilde{u} = \Delta \widetilde{u} + a\widetilde{u} + \widetilde{b}\widetilde{v} & \text{in} \quad Q_0, \\ \partial_t \widetilde{v} = \Delta \widetilde{v} + c\widetilde{u} + d\widetilde{v} & \text{in} \quad Q_0, \\ \widetilde{u}(t,x) = g(t,x),\ \widetilde{v}(t,x) = h(t,x) & \text{on} \quad \Sigma_0, \\ \widetilde{u}(0,x) = \widetilde{u}_0 \ \text{ and } \ \widetilde{v}(0,x) = \widetilde{v}_0 & \text{in} \quad \Omega. \end{cases} \tag{5}$$

We set $U = u - \widetilde{u}$, $V = v - \widetilde{v}$, $y = \partial_t(u - \widetilde{u})$, $z = \partial_t(v - \widetilde{v})$ and $\gamma = b - \widetilde{b}$. Then (y, z) is solution to the following problem

$$\begin{cases} \partial_t y = \Delta y + ay + bz + \gamma \partial_t \widetilde{v} & \text{in} \quad Q_0, \\ \partial_t z = \Delta z + cy + dz & \text{in} \quad Q_0, \\ y(t,x) = z(t,x) = 0 & \text{on} \quad \Sigma_0, \\ y(0,x) = \Delta U(0,x) + aU(0,x) + bV(0,x) + \gamma \widetilde{v}(0,x), & \text{in} \quad \Omega, \\ z(0,x) = \Delta V(0,x) + cU(0,x) + dV(0,x) & \text{in} \quad \Omega. \end{cases} \tag{6}$$

Note that the previous initial conditions are available for all $T' \in (0, T)$. We consider the functional

$$I(q) = s^{-1} \iint_Q e^{-2s\eta} \varphi^{-1} (|\partial_t q|^2 + |\Delta q|^2)\ dx\ dt$$
$$+ s\lambda^2 \iint_Q e^{-2s\eta} \varphi |\nabla q|^2\ dx\ dt + s^3 \lambda^4 \iint_Q e^{-2s\eta} \varphi^3 |q|^2\ dx\ dt.$$

Then using the Carleman estimate (3), the solution $(y,\ z)$ of (6) satisfies

$$I(y) + I(z) \leq C_1 [s^3 \lambda^4 \int_{t_0}^{T} \int_\omega e^{-2s\eta} \varphi^3 |z|^2\ dx\ dt \tag{7}$$
$$+ s^3 \lambda^4 \int_{t_0}^{T} \int_\omega e^{-2s\eta} \varphi^3 |y|^2\ dx\ dt$$
$$+ \iint_Q e^{-2s\eta}\ (|ay|^2 + |bz|^2 + |\gamma \partial_t \widetilde{v}|^2)\ dx\ dt + \iint_Q e^{-2s\eta}\ (|cy|^2 + |dz|^2)\ dx\ dt]$$

Let ξ be a smooth cut-off function satisfying

$$\begin{cases} \xi(x) = 1 & \forall x \in \omega', \\ 0 < \xi(x) \leq 1 & \forall x \in \omega'', \\ \xi(x) = 0 & \forall x \in \mathbb{R}^n \setminus \omega'', \end{cases}$$

where $\omega' \Subset \omega'' \Subset \omega \Subset \Omega$.
We shall to estimate the following three terms

$$I := s^3 \lambda^4 \int_{t_0}^{T} \int_\omega e^{-2s\eta} \varphi^3 |y|^2\ dx\ dt,$$

$$J := \iint_Q e^{-2s\eta}\ |bz|^2\ dx\ dt \ \text{ or } \ \iint_Q e^{-2s\eta}\ |dz|^2\ dx\ dt,$$

$$K := \iint_Q e^{-2s\eta}\ |ay|^2\ dx\ dt \ \text{ or } \ \iint_Q e^{-2s\eta}\ |cy|^2\ dx\ dt.$$

For the first term I, we multiply the second equation of (6) by $s^3\lambda^4 e^{-2s\eta}\xi\varphi^3 y$ and we integrate over $(t_0,T)\times\omega$. We obtain

$$I' := s^3\lambda^4\int_{t_0}^T\!\!\int_\omega c\, e^{-2s\eta}\xi\varphi^3|y|^2\ dx\ dt = s^3\lambda^4\int_{t_0}^T\!\!\int_\omega e^{-2s\eta}\xi\varphi^3(\partial_t z-\Delta z-dz)y\ dx\ dt$$

$$= s^3\lambda^4\int_{t_0}^T\!\!\int_\omega e^{-2s\eta}\xi\varphi^3(\partial_t z)y\ dx\ dt - s^3\lambda^4\int_{t_0}^T\!\!\int_\omega e^{-2s\eta}\xi\varphi^3(\Delta z)y\ dx\ dt$$

$$-s^3\lambda^4\int_{t_0}^T\!\!\int_\omega de^{-2s\eta}\xi\varphi^3 zy\ dx\ dt = I_1+I_2+I_3.$$

By integration by parts with respect to the time variable, the first integral, I_1, can be written as

$$I_1 = -s^3\lambda^4\int_{t_0}^T\!\!\int_\omega e^{-2s\eta}\xi\varphi^3 z(\partial_t y)\ dx\ dt + 2s^4\lambda^4\int_{t_0}^T\!\!\int_\omega e^{-2s\eta}\xi\varphi^3(\partial_t\eta)\ zy\ dx\ dt$$

$$-3s^3\lambda^4\int_{t_0}^T\!\!\int_\omega e^{-2s\eta}\xi\varphi^2(\partial_t\varphi)\, zy\ dx\ dt.$$

We write $I_1 = I_1^1 + I_1^2$ with

$$I_1^1 = -s^3\lambda^4\int_{t_0}^T\!\!\int_\omega e^{-2s\eta}\xi\varphi^3 z(\partial_t y)\ dx\ dt,$$

$$I_1^2 = 2s^4\lambda^4\int_{t_0}^T\!\!\int_\omega e^{-2s\eta}\xi\varphi^3(\partial_t\eta)\ zy\ dx\ dt - 3s^3\lambda^4\int_{t_0}^T\!\!\int_\omega e^{-2s\eta}\xi\varphi^2(\partial_t\varphi)\, zy\ dx\ dt.$$

Using Young inequality, we estimate the two integrals I_1^1 and I_1^2. We have

$$|I_1^1| \le s^3\lambda^4\left[C_\varepsilon s^4\lambda^4\int_{t_0}^T\!\!\int_\omega e^{-2s\eta}\xi^2\varphi^7|z|^2\ dx\ dt + \varepsilon s^{-4}\lambda^{-4}\int_{t_0}^T\!\!\int_\omega e^{-2s\eta}\varphi^{-1}|\partial_t y|^2\ dx\ dt\right]$$

$$\le C_\varepsilon s^7\lambda^8\int_{t_0}^T\!\!\int_\omega e^{-2s\eta}\varphi^7|z|^2\ dx\ dt + \varepsilon s^{-1}\iint_Q e^{-2s\eta}\varphi^{-1}|\partial_t y|^2\ dx\ dt.$$

The last term of the previous inequality can be "absorbed" by the terms in $I(y)$ for ε sufficiently small.

$$|I_1^2| \le Cs^4\lambda^4\left[s\lambda\int_{t_0}^T\!\!\int_\omega e^{-2s\eta}\xi^2\varphi(\varphi^2|\partial_t\eta|^2+|\partial_t\varphi|^2)|z|^2\ dx\ dt\right.$$

$$\left.+s^{-1}\lambda^{-1}\int_{t_0}^T\!\!\int_\omega e^{-2s\eta}\varphi^3|y|^2\ dx\ dt\right]$$

$$\le C\left[s^5\lambda^5\int_{t_0}^T\!\!\int_\omega e^{-2s\eta}\varphi^7|z|^2\ dx\ dt + s^3\lambda^3\iint_Q e^{-2s\eta}\varphi^3|y|^2\ dx\ dt\right].$$

The last inequality holds through the following estimates

$$|\partial_t\varphi| \le C(\Omega,\omega)T\varphi^2, \quad |\partial_t\eta| \le C(\Omega,\omega)T\varphi^2, \quad \varphi \le C(\Omega,\omega)T^4\varphi^3.$$

The last term of the previous inequality can be "absorbed" by the terms in $I(y)$ for s and λ sufficiently large. Finally, we obtain

$$|I_1| \leq Cs^7\lambda^8 \int_{t_0}^{T}\int_{\omega} e^{-2s\eta}\varphi^7|z|^2 \; dx \; dt + \text{"absorbed terms"} \; ,$$

where C is a generic constant which depends on Ω, ω and T.
Integrating by parts the second integral I_2 with respect to the space variable, we obtain

$$I_2 = -s^3\lambda^4 \int_{t_0}^{T}\int_{\omega} \Delta(e^{-2s\eta}\xi\varphi^3 y)z \; dx \; dt.$$

If we denote by $P = e^{-2s\eta}\xi\varphi^3$, then we have

$$I_2 = -s^3\lambda^4 \int_{t_0}^{T}\int_{\omega} (P\Delta y + 2\nabla P\nabla y + y\Delta P)z \; dx \; dt.$$

We compute ∇P and ΔP and we obtain the following estimation for I_2

$$\begin{aligned}
|I_2| \leq s^3\lambda^4\Big[&\varepsilon s^{-4}\lambda^{-4} \int_{t_0}^{T}\int_{\omega} e^{-2s\eta}\varphi^{-1}|\Delta y|^2 \; dx \; dt + C_\varepsilon s^4\lambda^4 \int_{t_0}^{T}\int_{\omega} e^{-2s\eta}\varphi^7|z|^2 \; dx \; dt \\
&+\varepsilon s^{-2}\lambda^{-2} \int_{t_0}^{T}\int_{\omega} e^{-2s\eta}\varphi|\nabla y|^2 \; dx \; dt + C_\varepsilon s^2\lambda^2 \int_{t_0}^{T}\int_{\omega} e^{-2s\eta}\varphi^5|z|^2 \; dx \; dt \\
&+\varepsilon \int_{t_0}^{T}\int_{\omega} e^{-2s\eta}\varphi^3|y|^2 \; dx \; dt + C_\varepsilon \int_{t_0}^{T}\int_{\omega} e^{-2s\eta}\varphi^3|z|^2 \; dx \; dt\Big].
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
|I_2| \leq \varepsilon\Big[&s^{-1} \iint_{Q} e^{-2s\eta}\varphi^{-1}|\Delta y|^2 \; dx \; dt + s\lambda^2 \iint_{Q} e^{-2s\eta}\varphi|\nabla y|^2 \; dx \; dt \\
&+s^3\lambda^4 \iint_{Q} e^{-2s\eta}\varphi^3|y|^2 \; dx \; dt\Big] + C_\varepsilon\Big[s^7\lambda^8 \int_{t_0}^{T}\int_{\omega} e^{-2s\eta}\varphi^7|z|^2 \; dx \; dt \\
&+s^5\lambda^6 \int_{t_0}^{T}\int_{\omega} e^{-2s\eta}\varphi^5|z|^2 \; dx \; dt + s^3\lambda^4 \int_{t_0}^{T}\int_{\omega} e^{-2s\eta}\varphi^3|z|^2 \; dx \; dt\Big].
\end{aligned}$$

The first three integrals of the r.h.s. of the previous inequality can be "absorbed" by the terms in $I(y)$ for ε sufficiently small. Finally, we have

$$|I_2| \leq Cs^7\lambda^8 \int_{t_0}^{T}\int_{\omega} e^{-2s\eta}\varphi^7|z|^2 \; dx \; dt + \text{ "absorbed terms"} \; .$$

For the last integral I_3, we have

$$|I_3| \leq Cs^3\lambda^4 \left[C_\varepsilon \int_{t_0}^{T}\int_{\omega} e^{-2s\eta}\varphi^3|z|^2 \; dx \; dt + \varepsilon \iint_{Q} e^{-2s\eta}\varphi^3|y|^2 \; dx \; dt\right]$$

Finally, if we assume that there exists $c_0 > 0$ such that $c \geq c_0$ in ω, we have thus obtained for λ and s sufficiently large and ε sufficiently small the following estimate:

$$|I| \leq \frac{1}{c_0}|I'| \leq \frac{C}{c_0}s^7\lambda^8 \int_{t_0}^{T}\int_{\omega} e^{-2s\eta}\varphi^7|z|^2 \; dx \; dt.$$

For the integrals J and K, since $a,\ b,\ c,\ d\ \in \Lambda(R)$ and using the estimate

$$1 \leq C(\Omega,\omega)T^6\varphi^3/4,$$

we have

$$|J| \leq C \iint_Q e^{-2s\eta}\ \varphi^3\ |z|^2\ dx\ dt,$$

$$|K| \leq C \iint_Q e^{-2s\eta}\ \varphi^3\ |y|^2\ dx\ dt,$$

and these terms can be "absorbed" by the terms $I(y)$ and $I(z)$ for λ and s sufficiently large. If we now come back to inequality (7), using the estimates for I, J and K, and choosing λ and s sufficiently large and ε sufficiently small, we can thus write

$$I(y)+I(z) \leq C_1[s^3\lambda^4 \int_{t_0}^T\int_\omega e^{-2s\eta}\varphi^3|z|^2\ dx\ dt + s^7\lambda^8 \int_{t_0}^T\int_\omega e^{-2s\eta}\varphi^7|z|^2\ dx\ dt$$
$$+ \iint_Q e^{-2s\eta}\ |\gamma\partial_t\widetilde{v}|^2\ dx\ dt].$$

Observing that

$$\varphi^3 \leq C(\Omega,\omega)T^8\varphi^7,$$

We have thus obtained the fundamental result

Theorem 2.3 *We assume $a,\ b,\ c,\ d\ \in \Lambda(R)$ and that exists $c_0 > 0$ such that $c \geq c_0$ in ω. Then there exist $\lambda_1 = \lambda_1(\Omega,\omega) \geq 1$, $s_1 = s_1(\lambda_1, T) > 1$ and a positive constant $C_1 = C_1(\Omega,\omega,c_0,R,T)$ such that, for any $\lambda \geq \lambda_1$ and any $s \geq s_1$, the following inequality holds:*

$$I(y)+I(z) \leq C_1\left[s^7\lambda^8 \int_{t_0}^T\int_\omega e^{-2s\eta}\varphi^7|z|^2\ dx\ dt + \iint_Q e^{-2s\eta}\ |\gamma\partial_t\widetilde{v}|^2\ dx\ dt\right], \quad (8)$$

for any solution $(y,\ z)$ of (6).

3. Uniqueness and stability estimate with one observation

In this section, we establish, a stability inequality and deduce a uniqueness result for the coefficient b. This inequality (13) estimates the difference between the coefficients b and $\widetilde{b}$ with an upper bound given by some Sobolev norms of the difference between the solutions v, and $\widetilde{v}$ of (4) and (5). Recall that $U = u - \widetilde{u}$, $V = v - \widetilde{v}$, $y = \partial_t(u - \widetilde{u})$, $z = \partial_t(v - \widetilde{v})$, $\gamma = b - \widetilde{b}$ and

$$\begin{cases} \partial_t y = \Delta y + ay + bz + \gamma\partial_t\widetilde{v} & \text{in} \quad Q_0, \\ \partial_t z = \Delta z + cy + dz & \text{in} \quad Q_0, \\ y(t,x) = z(t,x) = 0 & \text{on} \quad \Sigma_0, \\ y(0,x) = \Delta U(0,x) + aU(0,x) + bV(0,x) + \gamma\widetilde{v}(0,x), & \text{in} \quad \Omega, \\ z(0,x) = \Delta V(0,x) + cU(0,x) + dV(0,x) & \text{in} \quad \Omega. \end{cases}$$

The Carleman estimate (8) proved in the previous section will be the key ingredient in the proof of such a stability estimate.

Let $T' = \frac{1}{2}(T + t_0)$ the point for which $\Phi(t) = \frac{1}{(t-t_0)(T-t)}$ has its minimum value. For $(\widetilde{u}, \widetilde{v})$ solutions of (5), we make the following assumption:

Assumption 3.1 *There exist $r > 0$, $c_0 > 0$ such that $\widetilde{b} \geq 0,\ c \geq c_0,\ c + dr \geq 0,\ \widetilde{u}_0 \geq 0,\ \widetilde{v}_0 \geq r,\ g \geq 0$ and $h \geq r$.*

Such assumption allows us to state that the solution $\widetilde{v}$ is such that $|\widetilde{v}(x, T')| \geq r > 0$ in Ω (see [12], theorem 14.7 p.200). Furthermore if we assume that $\widetilde{u}_0$, $\widetilde{v}_0$ in $H^2(\Omega)$,the solutions of (5) belong to $H^1(t_0, T, H^2(\Omega))$. Then using classical Sobolev imbedding (see [3]), we can write for $n \leq 3$, that $\partial_t \widetilde{v}$ belongs to $L^2(t_0, T, L^\infty(\Omega))$ and we assume that $|\partial_t \widetilde{v}|_{L^2(t_0,T)} \in \Lambda(R)$.
We set $\psi = e^{-s\eta} y$. With the operator

$$M_2\psi = \partial_t\psi + 2s\lambda\varphi\nabla\beta.\nabla\psi + 2s\lambda^2\varphi|\nabla\beta|^2\psi, \tag{9}$$

we introduce, following [2],

$$\mathcal{I} = \Re \int_{t_0}^{T'}\!\!\int_\Omega M_2\psi\ \psi\ dxdt$$

We have the following estimates.

Lemma 3.2 *Let $\lambda \geq \lambda_1$ and $s \geq s_1$ and let $a,\ b,\ c,\ d\ \in \Lambda(R)$. We assume that assumption 3.1 is satisfied then there exists a constant $C = C(\Omega, \omega, T)$ such that*

$$|\mathcal{I}| \leq Cs^{-3/2}\lambda^{-2}\left[s^7\lambda^8 \int_{t_0}^{T}\!\!\int_\omega e^{-2s\eta}\varphi^7|z|^2\ dx\ dt + \int_{t_0}^{T}\!\!\int_\Omega e^{-2s\eta}|\gamma|^2|\partial_t\widetilde{v}|^2 dx\ dt\right].$$

Proof:
Observe that

$$|\mathcal{I}| \leq s^{-3/2}\lambda^{-2}\left(\int_{t_0}^{T'}\!\!\int_\Omega |M_2\psi|^2\ dx\ dt\right)^{1/2}\left(s^3\lambda^4\int_{t_0}^{T'}\!\!\int_\Omega e^{-2s\eta}|y|^2\ dx\ dt\right)^{1/2},$$

thus using Young inequality and the estimate $1 \leq C'T^6\varphi^3$, we obtain

$$|\mathcal{I}| \leq Cs^{-3/2}\lambda^{-2}\left(|M_2\psi|^2_{L^2(Q)} + s^3\lambda^4\int_{t_0}^{T}\!\!\int_\Omega e^{-2s\eta}\varphi^3|y|^2 dxdt\right),$$

which yields the result from Carleman estimate (8).

■

Lemma 3.3 *Let $\lambda \geq \lambda_1$, $s \geq s_1$ and let $a,\ b,\ c,\ d\ \in \Lambda(R)$. Furthermore, we assume that $\widetilde{u}_0$, $\widetilde{v}_0$ in $H^2(\Omega)$ and the assumption 3.1 is satisfied. Then there exists a constant $C = C(\Omega, \omega, T)$ such that*

$$\begin{aligned}
&\int_\Omega e^{-2s\eta(T',x)}\ |\gamma\widetilde{v}(T', x)|^2\ dx \qquad (10)\\
&\leq Cs^{-3/2}\lambda^{-2}\left[s^7\lambda^8\int_{t_0}^{T}\!\!\int_\omega e^{-2s\eta}\varphi^7|z|^2\ dx\ dt + \int_{t_0}^{T}\!\!\int_\Omega e^{-2s\eta}|\gamma|^2|\partial_t\widetilde{v}|^2 dx\ dt\right]\\
&+C\int_\Omega e^{-2s\eta(T',x)}\ |\Delta\ U(T', x) + aU(T', x) + bV(T', x)|^2\ dx.
\end{aligned}$$

Proof:
We evaluate integral $\mathcal{I}$ using (9)

$$\mathcal{I} = \frac{1}{2}\int_{t_0}^{T'}\!\!\int_\Omega \partial_t|\psi|^2\,dxdt + s\lambda\int_{t_0}^{T'}\!\!\int_\Omega \varphi\nabla\beta\cdot\nabla|\psi|^2\,dxdt + 2s\lambda^2\int_{t_0}^{T'}\!\!\int_\Omega \varphi|\nabla\beta|^2|\psi|^2dxdt$$
$$= \frac{1}{2}\int_{t_0}^{T'}\!\!\int_\Omega \partial_t|\psi|^2\,dxdt - s\lambda\int_{t_0}^{T'}\!\!\int_\Omega \nabla\cdot(\varphi\nabla\beta)|\psi|^2\,dxdt + 2s\lambda^2\int_{t_0}^{T'}\!\!\int_\Omega \varphi|\nabla\beta|^2|\psi|^2dxdt,$$

by integration by parts. With an integration by parts w.r.t. t in the first integral, we then obtain

$$\frac{1}{2}\int_\Omega |\psi(T',.)|^2\,dx = \mathcal{I} - s\lambda^2\int_{t_0}^{T'}\!\!\int_\Omega \varphi|\nabla\beta|^2|\psi|^2dxdt + s\lambda\int_{t_0}^{T'}\!\!\int_\Omega \varphi(\Delta\beta)|\psi|^2\,dxdt$$

since $\psi(t_0)=0$ and $\nabla\varphi = \lambda\varphi\nabla\beta$.

Then, we have

$$\int_\Omega (e^{-2s\eta)(T',x)}|y(T',x)|^2\,dx \le 2|\mathcal{I}| + Cs\lambda(\lambda+1)\int_{t_0}^{T'}\!\!\int_\Omega e^{-2s\eta(t,x)}\varphi|y|^2dxdt. \quad (11)$$

Using $\varphi \le \frac{T^4}{4}\varphi^3$ the last term in (11) is overestimated by the left hand side of (8) and this last one is absorbed by the l.h.s. of the inequality obtained in lemma 3.2.
If we now observe that

$$y(T',x) = \Delta U(T',x) + aU(T',x) + bV(T',x) + \gamma\widetilde{v}(T',x),$$

we have

$$|y(T',x)|^2 \ge \frac{1}{2}|\gamma\widetilde{v}(T',x)|^2 - |\Delta U(T',x) + aU(T',x) + bV(T',x)|^2.$$

■

The regularity of the solutions of (5) allows us to write that for $n \le 3$, $\partial_t\widetilde{v}$ is an element of $L^2(t_0,T,L^\infty(\Omega))$. So, from $|\widetilde{v}(x,T')| \ge r > 0$, we have

$$\exists\ k \in L^2(t_0,T), |\partial_t\widetilde{v}(x,t)| \le k(t)|\widetilde{v}(x,T')|,\ \forall x\in\Omega,\ t\in(t_0,T).$$

Hence (10) can be written

$$\int_\Omega (e^{-2s\eta})(T',x)|\gamma|^2|\widetilde{v}(x,T')|^2\,dx$$
$$\le Cs^{-3/2}\lambda^{-2}\int_{t_0}^{T}\int_\Omega e^{-2s\eta}|\gamma|^2|k(t)|^2|\widetilde{v}(x,T')|^2\,dx\,dt$$
$$+Cs^{11/2}\lambda^6\int_{t_0}^{T}\!\!\int_\omega e^{-2s\eta}\varphi^7|z|^2\,dx\,dt$$
$$+C\int_\Omega (e^{-2s\eta)(T',x)}(|\Delta U(T',x)|^2 + |U(T',x)|^2 + |V(T',x)|^2)\,dx.$$

Since $k \in L^2(t_0,T)$ implies that $\int_{t_0}^{T}|k(t)|^2\,dt \le k_0 < +\infty$. For λ large enough, the term $(1 - Cs^{-3/2}\lambda^{-2}k_0)$ can be made positive:

$$1 - Cs^{-3/2}\lambda^{-2}k_0 \ge C_2 > 0.$$

Using the fact that $e^{-2s\eta(t,x)} \leq e^{-2s\eta(T',x)}$ $\forall x \in \Omega$, $\forall t \in (t_0, T)$, we deduce that

$$\begin{aligned}
&r^2(1 - Cs^{-3/2}\lambda^{-2}k_0) \int_\Omega (e^{-2s\eta})(T', x)|\gamma|^2 \, dx \\
&\leq Cs^{11/2}\lambda^6 \int_{t_0}^{T}\int_\omega e^{-2s\eta}\varphi^7|z|^2 \, dx \, dt \\
&+C \int_\Omega (e^{-2s\eta)(T',x)}(|\Delta U(T', x)|^2 + |U(T', x)|^2 + |V(T', x)|^2) \, dx,
\end{aligned}$$

where we have also used that $|\widetilde{v}(x, T')| \geq r > 0$ in Ω. Then, by virtue of the properties satisfied by φ and η, we finally obtain

$$\begin{aligned}
&|\gamma|^2_{L^2(\Omega)} \leq \frac{C}{r^2 C_2} s^{11/2}\lambda^6 \int_{t_0}^{T}\int_\omega |z|^2 \, dx \, dt \\
&+\frac{C}{r^2 C_2} \int_\Omega (|\Delta U(T', x)|^2 + |U(T', x)|^2 + |V(T', x)|^2) \, dx.
\end{aligned} \tag{12}$$

With (12), recalling that $U = u - \widetilde{u}$, $V = v - \widetilde{v}$, $y = \partial_t(u - \widetilde{u})$ and $z = \partial_t(v - \widetilde{v})$, we have thus obtained the following stability result.

Theorem 3.4 *Let ω be a subdomain of an open set Ω of $\mathbb{R}^n$, let a, b, c, d $\in \Lambda(R)$. Furthermore, we assume that $\widetilde{u}_0$, $\widetilde{v}_0$ in $H^2(\Omega)$ and the assumption 3.1 is satisfied. Let (u, v), $(\widetilde{u}, \widetilde{v})$ be solutions to (4)-(5). Then there exists a constant C*

$$C = C(\Omega, \omega, c_0, t_0, T, r, R) > 0$$

such that

$$\begin{aligned}
&|b - \widetilde{b}|^2_{L^2(\Omega)} \leq C|\partial_t v - \partial_t \widetilde{v}|^2_{L^2((t_0,T)\times\omega)} + C|\Delta u(T', \cdot) - \Delta\widetilde{u}(T', \cdot)|^2_{L^2(\Omega)} \\
&+C|u(T', \cdot) - \widetilde{u}(T', \cdot)|^2_{L^2(\Omega)} + C|v(T', \cdot) - \widetilde{v}(T', \cdot)|^2_{L^2(\Omega)}.
\end{aligned} \tag{13}$$

Remark 3.5 *If we assume that $u(T', \cdot) = \widetilde{u}(T', \cdot)$ and $v(T', \cdot) = \widetilde{v}(T', \cdot)$ (such an additional assumption is sometimes made, e.g. in [8]), then the stability estimate becomes*

$$|b - \widetilde{b}|^2_{L^2(\Omega)} \leq C|\partial_t v - \partial_t \widetilde{v}|^2_{L^2((t_0,T)\times\omega)}.$$

With Theorem 3.4 we have the following uniqueness result

Corollary 3.6 *Under the same assumptions as in theorem 3.4 and if*

$$(\partial_t v - \partial_t \widetilde{v})(t, x) = 0 \quad in \quad (t_0, T) \times \omega,$$

$$\Delta u(T', x) - \Delta\widetilde{u}(T', x) = 0 \quad in \quad \Omega,$$

$$v(T', x) - \widetilde{v}(T', x) = 0 \quad in \quad \Omega,$$

Then $b = \widetilde{b}$.

4. A uniqueness and stability estimate for the initial conditions

In this section, we use the same method as in [13] to state a stability estimate for the initial conditions u_0, v_0. The idea is to prove logarithmic-convexity inequality. The following method has been used to obtain continuous dependence inequalities in initial value problems. If (y, z) is solution of (6), we introduce (y_1, z_1) and (y_2, z_2) that satisfy

$$\begin{cases} \partial_t y_1 = \Delta y_1 + a y_1 + b z_1 + \gamma \partial_t \widetilde{v} & \text{in} \quad Q_0, \\ \partial_t z_1 = \Delta z_1 + c y_1 + d z_1 & \text{in} \quad Q_0, \\ y_1(t,x) = z_1(t,x) = 0 & \text{on} \quad \Sigma_0, \\ y_1(0,x) = 0, & \text{in} \quad \Omega, \\ z_1(0,x) = 0 & \text{in} \quad \Omega, \end{cases} \tag{14}$$

and

$$\begin{cases} \partial_t y_2 = \Delta y_2 + a y_2 + b z_2 & \text{in} \quad Q_0, \\ \partial_t z_2 = \Delta z_2 + c y_2 + d z_2 & \text{in} \quad Q_0, \\ y_2(t,x) = z_2(t,x) = 0 & \text{on} \quad \Sigma_0, \\ y_2(0,x) = \Delta U(0,x) + a U(0,x) + b V(0,x) + \gamma \widetilde{v}(0,x) & \text{in} \quad \Omega, \\ z_2(0,x) = \Delta V(0,x) + c U(0,x) + d V(0,x) & \text{in} \quad \Omega. \end{cases} \tag{15}$$

Then, we have

$$y = y_1 + y_2 \quad \text{and} \quad z = z_1 + z_2. \tag{16}$$

In a first step, we give an L^2 estimate for (y_1, z_1)

Lemma 4.1 *Let a, b, c, d, $|\partial_t \widetilde{v}|_{L^2(t_0,T)} \in \Lambda(R)$. Then there exists a constant*

$$C = C(t_0, T', R) > 0,$$

such that

$$|y_1(t)|^2_{L^2(\Omega)} + |z_1(t)|^2_{L^2(\Omega)} \leq C|\gamma|^2_{L^2(\Omega)}, \qquad t_0 \leq t \leq T'. \tag{17}$$

Proof:
We multiply the first (resp. the second) equation of (14) by y_1 (resp. by z_1). Then, after integrations by parts with respect to the space variable, we obtain

$$\frac{1}{2}\partial_t \int_\Omega (|y_1|^2 + |z_1|^2)\, dx = -\int_\Omega (|\nabla y_1|^2 + |\nabla z_1|^2)\, dx$$

$$+ \int_\Omega a y_1^2\, dx + \int_\Omega d z_1^2\, dx + \int_\Omega (b+c) y_1 z_1\, dx + \int_\Omega \gamma(\partial_t \widetilde{v}) y_1\, dx.$$

We use Cauchy-Schwarz and Young inequalities and we integrate over (t_0, t) for $t_0 \leq t \leq T'$, and we obtain

$$|y_1(t)|^2_{L^2(\Omega)} + |z_1(t)|^2_{L^2(\Omega)} \leq C_2|\gamma|^2_{L^2(\Omega)} + C_1 \int_{t_0}^t (|y_1(s)|^2_{L^2(\Omega)} + |z_1(s)|^2_{L^2(\Omega)})\, ds.$$

The result follows by a Gronwall inequality.

■

In a second step, we use a logarithmic-convexity inequality for (y_2, z_2)

Lemma 4.2 *Let a, b, c, d $\in \Lambda(R)$ and u_0, v_0, $\widetilde{u}_0$, $\widetilde{v}_0$ in $H^4(\Omega)$. Then there exist constants $M > 0$, $C = C(R) > 0$ and $C_1 = C_1(t_0, T', R) > 0$ such that*

$$|y_2(t)|^2_{L^2(\Omega)} + |z_2(t)|^2_{L^2(\Omega)} \leq C_1 M^{1-\mu(t)}(|y_2(T')|^2_{L^2(\Omega)} + |z_2(T')|^2_{L^2(\Omega)})^{\mu(t)}, \quad (18)$$

for $t_0 \leq t \leq T'$, where $\mu(t) = \dfrac{(e^{-Ct_0} - e^{-Ct})}{(e^{-Ct_0} - e^{-CT'})}$.

Proof:
The proof of this lemma is just an application of Theorem 3.1.3 in [7]. In fact, system (15) can be written in the following form

$$\begin{aligned} &\partial_t W + AW = BW, && \text{in } Q_0, \\ &W(t,x) = 0 && \text{on } \Sigma_0, \\ &W(0,x) = W_0(x) && \text{in } \Omega, \end{aligned}$$

where

$$W = \begin{pmatrix} y_2 \\ z_2 \end{pmatrix}, \quad A = \begin{pmatrix} -\Delta & 0 \\ 0 & -\Delta \end{pmatrix}, \quad B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The operator A is symetric and the solution W satisfies

$$||\partial_t W + AW||_{L^2(\Omega)} \leq \alpha ||W||_{L^2(\Omega)},$$

where $\alpha = ||B||_{L^\infty(\Omega)} < +\infty$ since a, b, c and d are in $\Lambda(R)$. If we assume that u_0, v_0, $\widetilde{u}_0$, $\widetilde{v}_0$ are in $H^4(\Omega)$, the hypothesis of Theorem 3.1.3 in [7] are satisfied, thus we have

$$||W(t)||_{L^2(\Omega)} \leq C_1 ||W(t_0)||^{1-\mu(t)}_{L^2(\Omega)} ||W(T')||^{\mu(t)}_{L^2(\Omega)},$$

with $\mu(t) = \dfrac{(e^{-Ct_0} - e^{-Ct})}{(e^{-Ct_0} - e^{-CT'})}$.
Since $W \in \mathcal{C}(t_0, T, L^2(\Omega))$, we have $||W(t_0)||_{L^2(\Omega)} \leq M^{\frac{1}{2}}$, and the result follows.

■

The two previous lemmas allow us to prove the following

Theorem 4.3 *Let ω be a subdomain of an open set Ω of $\mathbb{R}^n$, let a, b, c, d $\in \Lambda(R)$. Furthermore, we assume that u_0, v_0, $\widetilde{u}_0$, $\widetilde{v}_0$ in $H^4(\Omega)$ and Assumption 3.1 is satisfied. Let (u, v), $(\widetilde{u}, \widetilde{v})$ be solutions to (4)-(5). We set*

$$E = |\partial_t v - \partial_t \widetilde{v}|^2_{L^2((t_0,T)\times\omega)} + |u(T',\cdot) - \widetilde{u}(T',\cdot)|^2_{H^2(\Omega)} + |v(T',\cdot) - \widetilde{v}(T',\cdot)|^2_{H^2(\Omega)}.$$

Then there exists a constant $C = C(\Omega, \omega, c_0, t_0, T, r, R) > 0$ such that

$$|u_0 - \widetilde{u}_0|^2_{L^2(\Omega)} + |v_0 - \widetilde{v}_0|^2_{L^2(\Omega)} \leq \frac{C}{|\log E|}, \quad \textit{for} \quad 0 < E < 1$$

Proof:
Since $\widetilde{v} \in H^1(t_0, T, H^2(\Omega))$, we have $\widetilde{v} \in L^\infty(\Omega)$. In view of (16), inequalities (17), (18) imply

$$\begin{aligned}
&|y(t,\cdot)|^2_{L^2(\Omega)} \leq 2(|y_1(t,\cdot)|^2_{L^2(\Omega)} + |y_2(t,\cdot)|^2_{L^2(\Omega)}) \\
&\leq C_1 \left[|\gamma|^2_{L^2(\Omega)} + M^{1-\mu(t)}(|y_2(T',\cdot)|^2_{L^2(\Omega)} + |z_2(T',\cdot)|^2_{L^2(\Omega)})^{\mu(t)} \right].
\end{aligned}$$

Now, with (16), we write $y_2 = y - y_1$ and $z_2 = z - z_1$. Inequality (17) gives us an estimation of $|y_1(T',\cdot)|^2_{L^2(\Omega)}$ and $|z_1(T',\cdot)|^2_{L^2(\Omega)}$ in terms of $|\gamma|^2_{L^2(\Omega)}$. Then the definition of y and z in (6) gives us an estimation of $|y(T',\cdot)|^2_{L^2(\Omega)}$ and $|z(T',\cdot)|^2_{L^2(\Omega)}$ in terms of $|U(T',\cdot)|^2_{H^2(\Omega)}$ and $|V(T',\cdot)|^2_{H^2(\Omega)}$. Finally, we obtain

$$|y(t,\cdot)|^2_{L^2(\Omega)} \leq C_1 \left[|\gamma|^2_{L^2(\Omega)} + M_2(|\gamma|^2_{L^2(\Omega)} + |U(T',\cdot)|^2_{H^2(\Omega)} + |V(T',\cdot)|^2_{H^2(\Omega)})^{\mu(t)} \right],$$

(a similar estimate is obtained for $|z(t,\cdot)|^2$). If we use (13), the last estimate yields

$$\begin{aligned}
&|u_0 - \widetilde{u}_0|^2_{L^2(\Omega)} + |v_0 - \widetilde{v}_0|^2_{L^2(\Omega)} = |U(t_0,\cdot)|^2_{L^2(\Omega)} + |V(t_0,\cdot)|^2_{L^2(\Omega)} \\
&= |-\int_{t_0}^{T'} y(s,\cdot)\, ds + U(T',\cdot)|^2_{L^2(\Omega)} + |-\int_{t_0}^{T'} z(s,\cdot)\, ds + V(T',\cdot)|^2_{L^2(\Omega)} \\
&\leq M_3 \int_{t_0}^{T'} E^{\mu(s)}\, ds + C_2 E \leq C_3 \frac{E-1}{\log E} + C_4 E \leq \frac{C}{|\log E|}.
\end{aligned}$$

■

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