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# Pure-state $N$ -representability in current-spin-density-functional theory

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## Abstract

This paper is concerned with the pure-state  $N$ -representability problem for systems under a magnetic field. Necessary and sufficient conditions are given for a spin-density  $2 \times 2$  matrix  $R$  to be representable by a Slater determinant. We also provide sufficient conditions on the paramagnetic current  $\mathbf{j}$  for the pair  $(R, \mathbf{j})$  to be Slater-representable in the case where the number of electrons  $N$  is greater than 12. The case  $N < 12$  is left open.

## 1 Introduction

The density-functional theory (DFT), first developed by Hohenberg and Kohn [1], then further developed and formalized mathematically by Levy [2], Valone [3] and Lieb [4], states that the ground state energy and density of a non-magnetic electronic system can be obtained by minimizing some functional of the density only, over the set of all admissible densities. Characterizing this set is called the  *$N$ -representability problem*. More precisely, as the so-called constrained search method leading to DFT can be performed either with  $N$ -electron wave functions [2, 4], or with  $N$ -body density matrices [3, 4], the  $N$ -representability problem can be recast as follows: *What is the set of electronic densities that come from an admissible  $N$ -electron wave function?* (pure-state  $N$ -representability) and *What is the set of electronic densities that come from an admissible  $N$ -electron density matrix?* (mixed-state  $N$ -representability) This question was answered by Gilbert [5], Harriman [6] and Lieb [4] (see also Remark 1).

For a system subjected to a magnetic field, the energy of the ground state can be obtained by a minimization over the set of pairs  $(R, \mathbf{j})$ , where  $R$  denotes the  $2 \times 2$  spin-density matrix [7] (from which we recover the standard electronic density  $\rho$  and the spin angular momentum density  $\mathbf{m}$ ) and  $\mathbf{j}$  the paramagnetic current [8]. This has led to several density-based theories, that come from several different approximations. In spin-density-functional theory (SDFT), one is only interested in spin effects, hence the paramagnetic term is neglected. The SDFT energy functional of the system therefore only depends on the spin-density  $R$ . The  $N$ -representability problem in SDFT are therefore: *What is the set of spin-densities that come from an admissible  $N$ -electron wave function?* (pure-state representability) and *What is the set of spin-densities that come from an admissible  $N$ -body density matrices?* (mixed-state representability). This question was left open in the pioneering work by von Barth and Hedin [9], and was answered recently in the mixed-case setting [7]. In parallel, in current-density-functional theory (CDFT), one is only interested in magnetic orbital effects, and spin effects are neglected [10]. In this case, the CDFT energy functional of the system only depends on  $\rho$  and  $\mathbf{j}$ , and we need a characterization of the set of pure-state and mixed-state  $N$ -representable pairs  $(\rho, \mathbf{j})$ . Such a characterization was given recently by Hellgren, Kvaal and Helgaker in the mixed-state setting [11], and by Lieb and Schrader in the pure-state setting, when the number of electrons is greater than 4 [12]. In the latter article, the authors rely on the so-called Lazarev-Lieb orthogonalization process [13] (see also Lemma 5) in order to

orthogonalize the Slater orbitals.

The purpose of this article is to give an answer to the  $N$ -representability problem in the current-spin-density-functional theory (CSDFT): *What is the set of pairs  $(R, \mathbf{j})$  that come from an admissible  $N$ -electron wave-function?* (pure-state) and *What is the set of pairs  $(R, \mathbf{j})$  that come from an admissible  $N$ -body density-matrix?* (mixed-state). We will answer the question in the mixed-state setting for all  $N \in \mathbb{N}^*$ , and in the pure-state setting when  $N \geq 12$  by combining the results in [7] and in [12]. In the process, we will answer the  $N$ -representability problem for SDFT for all  $N \in \mathbb{N}^*$  in the pure-state setting. The proof relies on the Lazarev-Lieb orthogonalization process. In particular, our method does not give an upper-bound for the kinetic energy of the wave-function in terms of the previous quantities (we refer to [13, 14] for more details). We leave open the case  $N < 12$  for pure-state CSDFT representability.

The article is organized as follows. In Section 2, we recall briefly what are the sets of interest. We present our main results in Section 3, the proofs of which are given in Section 4.

## 2 The different Slater-state, pure-state and mixed-state sets

We recall in this section the definition of Slater-states, pure-states and mixed-states. We denote by  $L^p(\mathbb{R}^3)$ ,  $H^1(\mathbb{R}^3)$ ,  $C^\infty(\mathbb{R}^3)$ , ... the spaces of *real-valued*  $L^p$ ,  $H^1$ ,  $C^\infty$ , ... functions on  $\mathbb{R}^3$ , and by  $L^p(\mathbb{R}^3, \mathbb{C}^d)$ ,  $H^1(\mathbb{R}^3, \mathbb{C}^d)$ ,  $C^\infty(\mathbb{R}^3, \mathbb{C}^d)$ , ... the spaces of  $\mathbb{C}^d$ -*valued*  $L^p$ ,  $H^1$ ,  $C^\infty$  functions on  $\mathbb{R}^3$ . We will also make the identification  $L^p(\mathbb{R}^3, \mathbb{C}^d) \equiv (L^p(\mathbb{R}^3, \mathbb{C}))^d$  (and so on). The one-electron state space is

$$L^2(\mathbb{R}^3, \mathbb{C}^2) \equiv \left\{ \Phi = (\phi^\uparrow, \phi^\downarrow)^T, \|\Phi\|_{L^2} := \int_{\mathbb{R}^3} |\phi^\uparrow|^2 + |\phi^\downarrow|^2 < \infty \right\},$$

endowed with the natural scalar product  $\langle \Phi_1 | \Phi_2 \rangle := \int_{\mathbb{R}^3} (\overline{\phi_1^\uparrow} \phi_2^\uparrow + \overline{\phi_1^\downarrow} \phi_2^\downarrow)$ . The Hilbert space for  $N$ -electrons is the fermionic space  $\Lambda_{i=1}^N L^2(\mathbb{R}^3, \mathbb{C}^2)$  which is the set of wave-functions  $\Psi \in L^2((\mathbb{R}^3, \mathbb{C}^2)^N)$  satisfying the Pauli-principle: for all permutations  $p$  of  $\{1, \dots, N\}$ ,

$$\Psi(\mathbf{r}_{p(1)}, s_{p(1)}, \dots, \mathbf{r}_{p(N)}, s_{p(N)}) = \varepsilon(p) \Psi(\mathbf{r}_1, s_1, \dots, \mathbf{r}_N, s_N),$$

where  $\varepsilon(p)$  denotes the parity of the permutation  $p$ ,  $\mathbf{r}_k \in \mathbb{R}^3$  the position of the  $k$ -th electron, and  $s_k \in \{\uparrow, \downarrow\}$  its spin. The set of admissible wave-functions, also called the set of pure-states, is the set of normalized wave-function with finite kinetic energy

$$\mathcal{W}_N^{\text{pure}} := \left\{ \Psi \in \bigwedge_{i=1}^N L^2(\mathbb{R}^3, \mathbb{C}^2), \|\nabla \Psi\|_{L^2}^2 < \infty, \|\Psi\|_{L^2(\mathbb{R}^{3N})}^2 = 1 \right\}$$

where  $\nabla$  is the gradient with respect to the  $3N$  position variables. A special case of wave-functions is given by Slater determinants: let  $\Phi_1, \Phi_2, \dots, \Phi_N$  be a set of orthonormal functions in  $L^2(\mathbb{R}^3, \mathbb{C}^2)$ , the Slater determinant generated by  $(\Phi_1, \dots, \Phi_N)$  is (we denote by  $\mathbf{x}_k := (\mathbf{r}_k, s_k)$  the  $k$ -th spatial-spin component)

$$\mathcal{S}[\Phi_1, \dots, \Phi_N](\mathbf{x}_1, \dots, \mathbf{x}_N) := \frac{1}{\sqrt{N!}} \det(\Phi_i(\mathbf{x}_j))_{1 \leq i, j \leq N}.$$

The subset of  $\mathcal{W}_N^{\text{pure}}$  consisting of all finite energy Slater determinants is noted  $\mathcal{W}_N^{\text{Slater}}$ . It holds that  $\mathcal{W}_1^{\text{Slater}} = \mathcal{W}_1^{\text{pure}}$ , and,  $\mathcal{W}_N^{\text{Slater}} \subsetneq \mathcal{W}_N^{\text{pure}}$  for  $N \geq 2$ .

For a wave-function  $\Psi \in \mathcal{W}_N^{\text{pure}}$ , we define the corresponding  $N$ -body density matrix  $\Gamma_\Psi = |\Psi\rangle\langle\Psi|$ , which corresponds to the projection on  $\{\mathbb{C}\Psi\}$  in  $\Lambda_{i=1}^N L^2(\mathbb{R}^3, \mathbb{C}^2)$ . The set of pure-state (resp. Slater-state)  $N$ -body density matrices is

$$\mathcal{G}_N^{\text{pure}} := \{\Gamma_\Psi, \Psi \in \mathcal{W}_N^{\text{pure}}\} \text{ resp. } \mathcal{G}_N^{\text{Slater}} := \{\Gamma_\Psi, \Psi \in \mathcal{W}_N^{\text{Slater}}\}.$$

It holds that  $G_1^{\text{Slater}} = G_1^{\text{pure}}$  and that  $G_N^{\text{Slater}} \not\subseteq G_N^{\text{pure}}$  for  $N \geq 2$ . The set of mixed-state  $N$ -body density matrices  $G_N^{\text{mixed}}$  is defined as the convex hull of  $G_N^{\text{pure}}$ :

$$G_N^{\text{mixed}} = \left\{ \sum_{k=1}^{\infty} n_k |\Psi_k\rangle\langle\Psi_k|, 0 \leq n_k \leq 1, \sum_{k=1}^{\infty} n_k = 1, \Psi_k \in \mathcal{W}_N^{\text{pure}} \right\}.$$

It is also the convex hull of  $G_N^{\text{Slater}}$ . The kernel of an operator  $\Gamma \in G_N^{\text{mixed}}$  will be denoted by

$$\Gamma(\mathbf{r}_1, s_1, \dots, \mathbf{r}_N, s_N; \mathbf{r}'_1, s'_1, \dots, \mathbf{r}'_N, s'_N).$$

The quantities of interest in density-functional theory are the spin-density  $2 \times 2$  matrix, and the paramagnetic-current. For  $\Gamma \in G_N^{\text{mixed}}$ , the associated spin-density  $2 \times 2$  matrix is the  $2 \times 2$  hermitian function-valued matrix

$$R_{\Gamma}(\mathbf{r}) := \begin{pmatrix} \rho_{\Gamma}^{\uparrow\uparrow} & \rho_{\Gamma}^{\uparrow\downarrow} \\ \rho_{\Gamma}^{\downarrow\uparrow} & \rho_{\Gamma}^{\downarrow\downarrow} \end{pmatrix}(\mathbf{r}),$$

where, for  $\alpha, \beta \in \{\uparrow, \downarrow\}^2$ ,

$$\rho_{\Gamma}^{\alpha\beta}(\mathbf{r}) := N \sum_{\vec{s} \in \{\uparrow, \downarrow\}^{(N-1)}} \int_{\mathbb{R}^{3(N-1)}} \Gamma(\mathbf{r}, \alpha, \vec{\mathbf{z}}, \vec{s}; \mathbf{r}, \beta, \vec{\mathbf{z}}, \vec{s}) d\vec{\mathbf{z}}.$$

In the case where  $\Gamma$  comes from a Slater determinant  $\mathcal{S}[\Phi_1, \dots, \Phi_N]$ , we get

$$R_{\Gamma}(\mathbf{r}) = \sum_{k=1}^N \begin{pmatrix} |\phi_k^{\uparrow}|^2 & \phi_k^{\uparrow} \overline{\phi_k^{\downarrow}} \\ \phi_k^{\downarrow} \phi_k^{\uparrow} & |\phi_k^{\downarrow}|^2 \end{pmatrix}. \quad (1)$$

The total electronic density is  $\rho_{\Gamma} = \rho_{\Gamma}^{\uparrow\uparrow} + \rho_{\Gamma}^{\downarrow\downarrow}$ , and the spin angular momentum density is  $\mathbf{m}_{\Gamma} = \text{tr}_{\mathbb{C}^2}[\sigma R_{\Gamma}]$ , where

$$\sigma := (\sigma_x, \sigma_y, \sigma_z) = \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right)$$

contains the Pauli-matrices. Note that the pair  $(\rho_{\Gamma}, \mathbf{m}_{\Gamma})$  contains the same information as  $R_{\Gamma}$ , hence the  $N$ -representability problem for the matrix  $R$  is the same as the one for the pair  $(\rho, \mathbf{m})$ . However, as noticed in [7], it is more natural mathematically speaking to work with  $R_{\Gamma}$ . The Slater-state, pure-state and mixed-state sets of spin-density  $2 \times 2$  matrices are respectively defined by

$$\begin{aligned} \mathcal{J}_N^{\text{Slater}} &:= \{R_{\Gamma}, \Gamma \in G_N^{\text{Slater}}\}, \\ \mathcal{J}_N^{\text{pure}} &:= \{R_{\Gamma}, \Gamma \in G_N^{\text{pure}}\}, \\ \mathcal{J}_N^{\text{mixed}} &:= \{R_{\Gamma}, \Gamma \in G_N^{\text{mixed}}\}. \end{aligned}$$

Since the map  $\Gamma \mapsto R_{\Gamma}$  is linear, it holds that  $\mathcal{J}_N^{\text{Slater}} \subset \mathcal{J}_N^{\text{pure}} \subset \mathcal{J}_N^{\text{mixed}}$ , that  $\mathcal{J}_N^{\text{mixed}}$  is convex, and is the convex hull of both  $\mathcal{J}_N^{\text{Slater}}$  and  $\mathcal{J}_N^{\text{pure}}$ .

For a  $N$ -body density matrix  $\Gamma \in G_N^{\text{mixed}}$ , we define the associated paramagnetic current  $\mathbf{j}_{\Gamma} = \mathbf{j}_{\Gamma}^{\uparrow} + \mathbf{j}_{\Gamma}^{\downarrow}$  with

$$\mathbf{j}_{\Gamma}^{\alpha} = \text{Im} \left( N \sum_{\vec{s} \in \{\uparrow, \downarrow\}^{N-1}} \int_{\mathbb{R}^{3(N-1)}} \nabla_{\mathbf{r}'} \Gamma(\mathbf{r}, \alpha, \vec{\mathbf{z}}, \vec{s}; \mathbf{r}', \alpha, \vec{\mathbf{z}}, \vec{s}) \Big|_{\mathbf{r}'=\mathbf{r}} d\vec{\mathbf{z}} \right).$$

In the case where  $\Gamma$  comes from a Slater determinant  $\mathcal{S}[\Phi_1, \dots, \Phi_N]$ , we get

$$\mathbf{j}_{\Gamma} = \sum_{k=1}^N \text{Im} \left( \overline{\phi_k^{\downarrow}} \nabla \phi_k^{\uparrow} + \overline{\phi_k^{\uparrow}} \nabla \phi_k^{\downarrow} \right). \quad (2)$$

Note that while only the total paramagnetic current  $\mathbf{j}$  appears in the theory of C(S)DFT, the pair  $(\mathbf{j}^{\uparrow}, \mathbf{j}^{\downarrow})$  is sometimes used to design accurate current-density functionals (see [8] for instance). In this article however, we will only focus on the representability of  $\mathbf{j}$ .

### 3 Main results

#### 3.1 Representability in SDFT

Our first result concerns the characterization of  $\mathcal{J}_N^{\text{Slater}}$ ,  $\mathcal{J}_N^{\text{pure}}$  and  $\mathcal{J}_N^{\text{mixed}}$ . For this purpose, we introduce

$$\mathcal{C}_N := \left\{ R \in \mathcal{M}_{2 \times 2}(L^1(\mathbb{R}^3, \mathbb{C})), R^* = R, R \geq 0, \int_{\mathbb{R}^3} \text{tr}_{\mathbb{C}^2} [R] = N, \sqrt{R} \in \mathcal{M}_{2 \times 2}(H^1(\mathbb{R}^3, \mathbb{C})) \right\}, \quad (3)$$

and  $\mathcal{C}_N^0 := \{R \in \mathcal{C}_N, \det R \equiv 0\}$ . The following characterization of  $\mathcal{C}_N$  was proved in [7].

**Lemma 1.** *A function-valued matrix  $R = \begin{pmatrix} \rho^\uparrow & \sigma \\ \bar{\sigma} & \rho^\downarrow \end{pmatrix}$  is in  $\mathcal{C}_N$  iff its coefficients satisfy*

$$\begin{cases} \rho^{\uparrow/\downarrow} \geq 0, & \rho^\uparrow \rho^\downarrow - |\sigma|^2 \geq 0, & \int \rho^\uparrow + \int \rho^\downarrow = N, \\ \sqrt{\rho^{\uparrow/\downarrow}} \in H^1(\mathbb{R}^3), & \sigma, \sqrt{\det(R)} \in W^{1,3/2}(\mathbb{R}^3), \\ |\nabla \sigma|^2 \rho^{-1} \in L^1(\mathbb{R}^3), \\ \left| \nabla \sqrt{\det(R)} \right|^2 \rho^{-1} \in L^1(\mathbb{R}^3). \end{cases} \quad (4)$$

The complete answer for  $N$ -representability in SDFT is given by the following theorem, whose proof is given in Section 4.1.

**Theorem 1.**

*Case  $N = 1$ : It holds that*

$$\mathcal{J}_1^{\text{Slater}} = \mathcal{J}_1^{\text{pure}} = \mathcal{C}_1^0 \quad \text{and} \quad \mathcal{J}_1^{\text{mixed}} = \mathcal{C}_1.$$

*Case  $N \geq 2$ : For all  $N \geq 2$ , it holds that*

$$\mathcal{J}_N^{\text{Slater}} = \mathcal{J}_N^{\text{pure}} = \mathcal{J}_N^{\text{mixed}} = \mathcal{C}_N.$$

Note that the equality  $\mathcal{J}_N^{\text{mixed}} = \mathcal{C}_N^{\text{mixed}}$  for all  $N \in \mathbb{N}^*$  was already proven in [7].

**Remark 1.** *Gilbert [5], Harriman [6] and Lieb [4] proved that the  $N$ -representability set for the total electronic density  $\rho$  is the same for Slater-states, pure-states and mixed-states, and is characterized by*

$$\mathcal{I}_N := \left\{ \rho \in L^1(\mathbb{R}^3), \rho \geq 0, \int_{\mathbb{R}^3} \rho = N, \sqrt{\rho} \in H^1(\mathbb{R}^3) \right\}. \quad (5)$$

Comparing (5) and (3), we see that our theorem is a natural extension of the previous result.

#### 3.2 Representability in CSDF

We first recall some classical necessary conditions for a pair  $(R, \mathbf{j})$  to be  $N$ -representable (we refer to [11, 12] for the proof). In the sequel, we will denote by  $\rho^\uparrow := \rho^{\uparrow\uparrow}$ ,  $\rho^\downarrow := \rho^{\downarrow\downarrow}$  and  $\sigma := \rho^{\uparrow\downarrow}$  the elements of a matrix  $R$ , so that  $R = \begin{pmatrix} \rho^\uparrow & \sigma \\ \bar{\sigma} & \rho^\downarrow \end{pmatrix}$ , and by  $\rho = \rho^\uparrow + \rho^\downarrow$  the associated total electronic density.

**Lemma 2.** *If a pair  $(R, \mathbf{j})$  is representable by a mixed-state  $N$ -body density matrix, then*

$$\begin{cases} R \in \mathcal{C}_N \\ \|\mathbf{j}\|^2 / \rho \in L^1(\mathbb{R}^3). \end{cases} \quad (6)$$

From the second condition of (6), it must hold that the support of  $\mathbf{j}$  is contained in the support of  $\rho$ . The vector  $\mathbf{v} := \rho^{-1}\mathbf{j}$  is called the velocity field, and  $\mathbf{w} := \mathbf{curl}(\mathbf{v})$  is called the vorticity.

Let us first consider the pure-state setting. Recall that in the spin-less setting, in the case  $N = 1$ , a pair  $(\rho, \mathbf{j})$  representable by a single orbital generally satisfies (provided that the phases of the orbital are globally well-defined) the curl-free condition  $\mathbf{curl}(\rho^{-1}\mathbf{j}) = \mathbf{0}$  (see [12, 11]). This is no longer the case when spin is considered, as is shown in the following Lemma, whose proof is postponed until Section 4.2.

**Lemma 3** (CSDFT, case  $N = 1$ ). *Let  $\Phi = (\phi^\uparrow, \phi^\downarrow)^T \in \mathcal{W}_1^{\text{Slater}}$  be such that both  $\phi^\uparrow$  and  $\phi^\downarrow$  have well-defined global phases in  $C^1(\mathbb{R})$ . Then, the associated pair  $(R, \mathbf{j})$  satisfies  $R \in \mathcal{C}_1^0$ ,  $|\mathbf{j}|^2/\rho \in L^1(\mathbb{R}^3)$ , and the two curl-free conditions*

$$\mathbf{curl}\left(\frac{\mathbf{j}}{\rho} - \frac{\text{Im}(\bar{\sigma}\nabla\sigma)}{\rho\rho^\downarrow}\right) = \mathbf{0}, \quad \mathbf{curl}\left(\frac{\mathbf{j}}{\rho} + \frac{\text{Im}(\bar{\sigma}\nabla\sigma)}{\rho\rho^\uparrow}\right) = \mathbf{0}. \quad (7)$$

**Remark 2.** *If we write  $\sigma = |\sigma|e^{i\tau}$ , then,  $|\sigma|^2 = \rho^\uparrow\rho^\downarrow$ , and*

$$\text{Im}(\bar{\sigma}\nabla\sigma) = |\sigma|^2\nabla\tau = \rho^\uparrow\rho^\downarrow\nabla\tau. \quad (8)$$

*In particular, it holds that*

$$\mathbf{curl}\left(\frac{\text{Im}(\bar{\sigma}\nabla\sigma)}{\rho\rho^\downarrow} + \frac{\text{Im}(\bar{\sigma}\nabla\sigma)}{\rho\rho^\uparrow}\right) = \mathbf{curl}(\nabla\tau) = \mathbf{0},$$

*so that one of the equalities in (7) implies the other one.*

**Remark 3.** *We recover the traditional result in the spin-less case, where  $\sigma \equiv 0$ .*

In the case  $N > 1$ , things are very different. In [12], the authors gave a rigorous proof for the representability of the pair  $(\rho, \mathbf{j})$  by a Slater determinant (of orbitals having well-defined global phases) whenever  $N \geq 4$  under a mild condition (see equation (9) below). By adapting their proof to our case, we are able to ensure representability of a pair  $(R, \mathbf{j})$  by a Slater determinant for  $N \geq 12$  under the same mild condition (see Section 4.3 for the proof).

**Theorem 2** (CSDFT, case  $N \geq 12$ ).

*A sufficient set of conditions for a pair  $(R, \mathbf{j})$  to be representable by a Slater determinant is*

- $R \in \mathcal{C}_N$  with  $N \geq 12$  and  $\mathbf{j}$  satisfies  $|\mathbf{j}|^2/\rho \in L^1(\mathbb{R}^3)$
- there exists  $\delta > 0$  such that,

$$\sup_{\mathbf{r} \in \mathbb{R}^3} f(\mathbf{r})^{(1+\delta)/2} |\mathbf{w}(\mathbf{r})| < \infty, \quad \sup_{\mathbf{r} \in \mathbb{R}^3} f(\mathbf{r})^{(1+\delta)/2} |\nabla\mathbf{w}(\mathbf{r})| < \infty, \quad (9)$$

*where  $\mathbf{w} := \mathbf{curl}(\rho^{-2}\mathbf{j})$  is the vorticity, and*

$$f(\mathbf{r}) := (1 + (r_1)^2)(1 + (r_2)^2)(1 + (r_3)^2).$$

**Remark 4.** *The conditions (9) are the ones found in [12]. The authors conjectured that this condition "can be considerably loosened".*

**Remark 5.** *We were only able to prove this theorem for  $N \geq 12$ . In [12], the authors proved that conditions (9) were not sufficient for  $N = 2$ . We do not know whether conditions (9) are sufficient in the case  $3 \leq N \leq 11$ .*

Let us finally turn to the mixed-state case. We notice that if  $(R, \mathbf{j})$  is representable by a Slater determinant  $\mathcal{S}[\Phi_1, \dots, \Phi_N]$ , then, for all  $k \in \mathbb{N}^*$ , the pair  $(k/N)(R, \mathbf{j})$  is mixed-state representable, where  $N$  is the number of orbitals (simply take the uniform convex combination of the pairs represented by  $\mathcal{S}[\Phi_1]$ ,  $\mathcal{S}[\Phi_2]$ , etc.). In particular, from Theorem 2, we deduce the following corollary.

**Corollary 1** (CSDFT, case mixed-state).

A sufficient set of conditions for a pair  $(R, \mathbf{j})$  to be mixed-state representable is  $R \in \mathcal{C}_N^0$  for some  $N \in \mathbb{N}^*$ ,  $\mathbf{j}$  satisfies  $|\mathbf{j}|^2/\rho \in L^1(\mathbb{R}^3)$ , and (9) holds for some  $\delta > 0$ .

In [11], the authors provide different sufficient conditions than (9) for a pair  $(\rho, \mathbf{j})$  to be mixed-state representable, where  $\rho$  is the electronic density. They proved that if

$$(1 + |\cdot|^2)\rho|\nabla(\rho^{-1}\mathbf{j})|^2 \in L^1(\mathbb{R}^3),$$

then the pair  $(\rho, \mathbf{j})$  is mixed-state representable. Their proof can be straightforwardly adapted for the representability of the pair  $(R, \mathbf{j})$ , so that similar results hold. The details are omitted here for the sake of brevity.

## 4 Proofs

### 4.1 Proof of Theorem 1

The mixed-state case was already proved in [7]. We focus on the pure-state representability.

**Case  $N = 1$**

The fact that  $\mathcal{J}_1^{\text{Slater}} = \mathcal{J}_1^{\text{pure}}$  simply comes from the fact that  $G_1^{\text{Slater}} = G_1^{\text{pure}}$ . To prove  $\mathcal{J}_1^{\text{Slater}} \subset \mathcal{C}_1^0$ , we let  $R \in \mathcal{J}_1^{\text{Slater}}$  be represented by  $\Phi = (\phi^\uparrow, \phi^\downarrow)^T \in H^1(\mathbb{R}^3, \mathbb{C}^2)$ , so that

$$R = \begin{pmatrix} |\phi^\uparrow|^2 & \phi^\uparrow \overline{\phi^\downarrow} \\ \phi^\downarrow \overline{\phi^\uparrow} & |\phi^\downarrow|^2 \end{pmatrix}.$$

Since  $R \in \mathcal{J}_1^{\text{Slater}} \subset \mathcal{J}_1^{\text{mixed}} = \mathcal{C}_1$  and  $\det R \equiv 0$ , we deduce  $R \in \mathcal{C}_1^0$ .

We now prove that  $\mathcal{C}_1^0 \subset \mathcal{J}_1^{\text{Slater}}$ . Let  $R = \begin{pmatrix} \rho^\uparrow & \sigma \\ \overline{\sigma} & \rho^\downarrow \end{pmatrix} \in \mathcal{C}_1^0$ . From  $\det R \equiv 0$  and Lemma 1, we get

$$\begin{cases} \rho^{\uparrow/4} \geq 0, & \rho^\uparrow \rho^\downarrow = |\sigma|^2, & \int_{\mathbb{R}^3} \rho^\uparrow + \int_{\mathbb{R}^3} \rho^\downarrow = 1, \\ \sqrt{\rho^{\uparrow/4}} \in H^1(\mathbb{R}^3), & \sigma \in W^{1,3/2}(\mathbb{R}^3), \\ |\nabla \sigma|^2 / \rho \in L^1(\mathbb{R}^3). \end{cases} \quad (10)$$

There are two natural choices that we would like to make for a representing orbital, namely

$$\Phi_1 = \left( \sqrt{\rho^\uparrow}, \frac{\overline{\sigma}}{\sqrt{\rho^\uparrow}} \right)^T \quad \text{and} \quad \Phi_2 = \left( \frac{\sigma}{\sqrt{\rho^\downarrow}}, \sqrt{\rho^\downarrow} \right)^T. \quad (11)$$

Unfortunately, it is not guaranteed that these orbitals are indeed in  $H^1(\mathbb{R}^3, \mathbb{C}^2)$ . It is the case only if  $|\nabla \sigma|^2 / \rho^\downarrow$  is in  $L^1(\mathbb{R}^3)$  for  $\Phi_1$ , and if  $|\nabla \sigma|^2 / \rho^\uparrow$  is in  $L^1(\mathbb{R}^3)$  for  $\Phi_2$ . Due to (10), we know that  $|\nabla \sigma|^2 / \rho \in L^1(\mathbb{R}^3)$ . The idea is therefore to interpolate between these two orbitals, taking  $\Phi_1$  in regions where  $\rho^\uparrow \gg \rho^\downarrow$ , and  $\Phi_2$  in regions where  $\rho^\downarrow \gg \rho^\uparrow$ . This is done via the following process.

Let  $\chi \in C^\infty(\mathbb{R})$  be a non-decreasing function such that  $0 \leq \chi \leq 1$ ,  $\chi(x) = 0$  if  $x \leq 1/2$  and  $\chi(x) = 1$  if  $x \geq 1$ . We write  $\sigma = \alpha + i\beta$  where  $\alpha$  is the real-part of  $\sigma$ , and  $\beta$  is its imaginary part. We introduce

$$\begin{aligned} \lambda_1 &:= \frac{\sqrt{\alpha^2 + \chi^2(\rho^\uparrow/\rho^\downarrow)\beta^2}}{\sqrt{\rho^\uparrow}}, & \mu_1 &:= \sqrt{1 - \chi^2(\rho^\uparrow/\rho^\downarrow)} \frac{\beta}{\sqrt{\rho^\downarrow}}, \\ \lambda_2 &:= \frac{\alpha\lambda_1 + \beta\mu_1}{\rho^\uparrow}, & \mu_2 &:= \frac{\beta\lambda_1 - \alpha\mu_1}{\rho^\uparrow}, \end{aligned}$$

and we set

$$\phi^\uparrow := \lambda_1 + i\mu_1 \quad \text{and} \quad \phi^\downarrow := \lambda_2 + i\mu_2.$$

Let us prove that  $\Phi$  represents  $R$  and that  $\Phi := (\phi^\uparrow, \phi^\downarrow) \in \mathcal{W}_1^{\text{Slater}}$ . First, an easy calculation shows that

$$\begin{aligned} |\phi^\uparrow|^2 &= \lambda_1^2 + \mu_1^2 = \frac{\alpha^2 + \chi^2 \beta^2 + (1 - \chi^2) \beta^2}{\rho^\downarrow} = \frac{|\sigma|^2}{\rho^\downarrow} = \rho^\uparrow, \\ |\phi^\downarrow|^2 &= \frac{(\alpha^2 + \beta^2)(\lambda_1^2 + \mu_1^2)}{(\rho^\uparrow)^2} = \frac{|\sigma|^2}{\rho^\uparrow} = \rho^\downarrow, \\ \operatorname{Re} \left( \phi^\uparrow \overline{\phi^\downarrow} \right) &= \lambda_1 \lambda_2 - \mu_1 \mu_2 = \frac{\alpha(\lambda_1^2 + \mu_1^2)}{\rho^\uparrow} = \alpha, \\ \operatorname{Im} \left( \phi^\uparrow \overline{\phi^\downarrow} \right) &= \lambda_1 \mu_2 + \lambda_2 \mu_1 = \frac{\beta(\lambda_1^2 + \mu_1^2)}{\sqrt{\rho^\uparrow}} = \beta, \end{aligned}$$

so that  $\Phi \in L^2(\mathbb{R}^3, \mathbb{C}^2)$  with  $\|\Phi\| = 1$ , and  $\Phi$  represents  $R$ . To prove that  $\Phi \in \mathcal{W}_1^{\text{Slater}}$ , we need to check that  $\lambda_1, \lambda_2, \mu_1$  and  $\mu_2$  are in  $H^1(\mathbb{R}^3)$ . For  $\lambda_1$ , we choose another non-increasing function  $\xi \in C^\infty(\mathbb{R})$  such that  $0 \leq \xi \leq 1$ ,  $\xi(x) = 0$  for  $x \leq 1$ , and  $\xi(x) = 1$  for  $x \geq 2$ . Note that  $(1 - \chi)\xi \equiv 0$ . It holds that

$$\nabla \lambda_1 = (1 - \xi^2(\rho^\uparrow/\rho^\downarrow)) \nabla \lambda_1 + \xi^2(\rho^\uparrow/\rho^\downarrow) \nabla \lambda_1. \quad (12)$$

The second term in the right-hand side of (12) is non-null only if  $\rho^\uparrow \geq \rho^\downarrow$ , so that  $\chi(\rho^\uparrow/\rho^\downarrow) = 1$  on this part. In particular, from the equality  $\rho^\uparrow \rho^\downarrow = |\sigma|^2$ , we get

$$\xi^2(\rho^\uparrow/\rho^\downarrow) \lambda_1 = \xi^2(\rho^\uparrow/\rho^\downarrow) \frac{|\sigma|}{\sqrt{\rho^\downarrow}} = \xi^2(\rho^\uparrow/\rho^\downarrow) \sqrt{\rho^\uparrow},$$

and similarly,

$$\xi^2(\rho^\uparrow/\rho^\downarrow) \nabla \lambda_1 = \xi^2(\rho^\uparrow/\rho^\downarrow) \nabla \sqrt{\rho^\uparrow},$$

which is in  $L^2(\mathbb{R}^3)$  according to (10). On the other hand, the first term in the right-hand side of (12) is non-null only if  $\rho^\uparrow \leq 2\rho^\downarrow$ , so that  $(1/3)\rho \leq \rho^\downarrow$  on this part. In particular, from the following point-wise estimate

$$|\nabla \sqrt{f+g}| \leq |\nabla \sqrt{f}| + |\nabla \sqrt{g}|,$$

which is valid almost everywhere whenever  $f, g \geq 0$ , the inequality  $(a+b)^2 \leq 2(a^2 + b^2)$ , and the fact that  $\alpha^2 + \chi^2 \beta^2 \leq |\sigma|^2$ , we get on this part (we write  $\chi$  for  $\chi(\rho^\uparrow/\rho^\downarrow)$ )

$$\begin{aligned} |\nabla \lambda_1|^2 &= \left| \frac{\sqrt{\rho^\downarrow} \nabla \sqrt{\alpha^2 + \chi^2 \beta^2} - \sqrt{\alpha^2 + \chi^2 \beta^2} \nabla \sqrt{\rho^\downarrow}}{\rho^\downarrow} \right|^2 \\ &\leq 2 \left( \frac{|\nabla \sqrt{\alpha^2 + \chi^2 \beta^2}|^2}{\rho^\downarrow} + \frac{(\alpha^2 + \chi^2 \beta^2)}{(\rho^\downarrow)^2} |\nabla \sqrt{\rho^\downarrow}|^2 \right) \\ &\leq 2 \left( \frac{|\nabla \alpha|^2}{\rho^\downarrow} + \frac{2 \left| \nabla \chi \frac{\rho^\downarrow \nabla \rho^\uparrow - \rho^\uparrow \nabla \rho^\downarrow}{(\rho^\downarrow)^2} \right|^2 \beta^2}{\rho^\downarrow} + \right. \\ &\quad \left. + \frac{2\chi^2 |\nabla \beta|^2}{\rho^\downarrow} + \frac{2|\sigma|^2}{(\rho^\downarrow)^2} |\nabla \sqrt{\rho^\downarrow}|^2 \right). \end{aligned}$$

We finally use the inequality  $(\rho^\downarrow)^{-1} \leq (3/\rho)$ , and the inequality  $|\sigma|^2/(\rho^\downarrow)^2 = \rho^\uparrow/\rho^\downarrow \leq 2$  and get

$$\begin{aligned} |\nabla \lambda_1|^2 &\leq C \left( \frac{|\nabla \alpha|^2}{\rho} + \|\nabla \chi\|_{L^\infty}^2 \left( \frac{|\nabla \rho^\uparrow|^2}{\rho^\uparrow} + \frac{|\nabla \rho^\downarrow|^2}{\rho^\downarrow} \right) \right. \\ &\quad \left. + \frac{|\nabla \beta|^2}{\rho} + |\nabla \sqrt{\rho^\downarrow}|^2 \right). \end{aligned}$$

The right-hand side is in  $L^1(\mathbb{R}^3)$  according to (10). Hence,  $(1 - \xi^2(\rho^\uparrow/\rho^\downarrow)) |\nabla \lambda_1| \in L^2(\mathbb{R}^3)$ , and finally  $\lambda_1 \in H^1(\mathbb{R}^3)$ .

The other cases are treated similarly, observing that,



- whenever  $\rho^\uparrow \geq \rho^\downarrow$ , then  $\chi = 1$ , and  $\Phi = \Phi_1$  where  $\Phi_1$  was defined in (11). We then control  $(\rho^\uparrow)^{-1}$  with the inequality  $(\rho^\uparrow)^{-1} \leq 2\rho^{-1}$  ;
- whenever  $\rho^\uparrow \leq \rho^\downarrow/2$ , then  $\chi = 0$ ,  $\Phi = \Phi_2$ . We control  $(\rho^\downarrow)^{-1}$  with the inequality  $(\rho^\downarrow)^{-1} \leq \frac{3}{2}\rho^{-1}$  ;
- whenever  $\rho^\downarrow/2 \leq \rho^\uparrow \leq \rho^\downarrow$ , then both  $(\rho^\uparrow)^{-1}$  and  $\rho^\downarrow$  are controlled via  $(\rho^\uparrow)^{-1} \leq 3\rho^{-1}$  and  $(\rho^\downarrow)^{-1} \leq 2\rho^{-1}$ .

**Case  $N \geq 2$ .**

Since  $\mathcal{J}_N^{\text{Slater}} \subset \mathcal{J}_N^{\text{pure}} \subset \mathcal{J}_N^{\text{mixed}} = \mathcal{C}_N$ , it is enough to prove that  $\mathcal{C}_N \subset \mathcal{J}_N^{\text{Slater}}$ . We start with a key lemma.

**Lemma 4.** *For all  $M, N \in \mathbb{N}^2$ , it holds that  $\mathcal{J}_{N+M}^{\text{Slater}} = \mathcal{J}_N^{\text{Slater}} + \mathcal{J}_M^{\text{Slater}}$ .*

*Proof of Lemma 4.* The case  $\mathcal{J}_{N+M}^{\text{Slater}} \subset \mathcal{J}_N^{\text{Slater}} + \mathcal{J}_M^{\text{Slater}}$  is trivial: if  $R \in \mathcal{J}_{N+M}^{\text{Slater}}$  is represented by the Slater determinant  $\mathcal{S}[\Phi_1, \dots, \Phi_{N+M}]$ , then, by denoting by  $R_1$  (resp.  $R_2$ ) the spin-density matrix associated to the Slater determinant  $\mathcal{S}[\Phi_1, \dots, \Phi_N]$  (resp.  $\mathcal{S}[\Phi_{N+1}, \dots, \Phi_{N+1}]$ ), it holds  $R = R_1 + R_2$  (see Equation (1) for instance), with  $R_1 \in \mathcal{J}_N^{\text{Slater}}$  and  $R_2 \in \mathcal{J}_M^{\text{Slater}}$ .

The converse is more involving, and requires an orthogonalization step. Let  $R_1 \in \mathcal{J}_N^{\text{Slater}}$  be represented by the Slater determinant  $\mathcal{S}[\Phi_1, \dots, \Phi_N]$ , and  $R_2 \in \mathcal{J}_M^{\text{Slater}}$  be represented by the Slater determinant  $\mathcal{S}[\tilde{\Phi}_1, \dots, \tilde{\Phi}_M]$ . We cannot directly consider the Slater determinant  $\mathcal{S}[\Phi_1, \dots, \Phi_N, \tilde{\Phi}_1, \dots, \tilde{\Phi}_M]$ , for  $(\Phi_1, \dots, \Phi_N)$  is not orthogonal to  $(\tilde{\Phi}_1, \dots, \tilde{\Phi}_M)$ .

We recall the following lemma, which is a smooth version of the Hobby-Rice theorem [15] (see also [16]), and that was proved by Lazarev and Lieb in [13] (see also [12]).

**Lemma 5.** *For all  $N \in \mathbb{N}^*$ , and for all  $(f_1, \dots, f_N) \in L^1(\mathbb{R}^3, \mathbb{C})$ , there exists a function  $u \in C^\infty(\mathbb{R}^3)$ , with bounded derivatives, such that*

$$\forall 1 \leq k \leq N, \quad \int_{\mathbb{R}^3} f_k e^{iu} = 0.$$

Moreover,  $u$  can be chosen to vary in the  $r_1$  direction only.

We now modify the phases of  $\tilde{\Phi}_1, \dots, \tilde{\Phi}_M$  as follows. First, we choose  $\tilde{u}_1$  as in Lemma 5 such that,

$$\forall 1 \leq k \leq N, \quad \int_{\mathbb{R}^3} (\overline{\phi_k^\uparrow \tilde{\phi}_1^\uparrow} + \overline{\phi_k^\downarrow \tilde{\phi}_1^\downarrow}) e^{i\tilde{u}_1} = 0,$$

and we set  $\Phi_{N+1} = \tilde{\Phi}_1 e^{i\tilde{u}_1}$ . Note that, by construction,  $\Phi_{N+1}$  is normalized, in  $H^1(\mathbb{R}^3, \mathbb{C}^2)$ , and orthogonal to  $(\Phi_1, \dots, \Phi_N)$ . We then construct  $\tilde{u}_2$  as in Lemma 5 such that

$$\forall 1 \leq k \leq N+1, \quad \int_{\mathbb{R}^3} (\overline{\phi_k^\uparrow \tilde{\phi}_2^\uparrow} + \overline{\phi_k^\downarrow \tilde{\phi}_2^\downarrow}) e^{i\tilde{u}_2} = 0,$$

and we set  $\Phi_{N+2} = \tilde{\Phi}_2 e^{i\tilde{u}_2}$ . We continue this process for  $3 \leq k \leq M$  and construct  $\Phi_{N+k} = \tilde{\Phi}_k e^{i\tilde{u}_k}$ . We thus obtain an orthonormal family  $(\Phi_1, \dots, \Phi_{N+M})$ . By noticing that the spin-density matrix of the Slater determinant  $\mathcal{S}[\tilde{\Phi}_1, \dots, \tilde{\Phi}_M]$  is the same as the one of  $\mathcal{S}[\Phi_{N+1}, \dots, \Phi_{N+M}]$  (the phases cancel out), we obtain that  $R = R_1 + R_2$ , where  $R$  is the spin-density matrix represented by  $\mathcal{S}[\Phi_1, \dots, \Phi_{N+M}]$ . The result follows.  $\square$

We now prove that  $\mathcal{C}_N \subset \mathcal{J}_N^{\text{Slater}}$ . We start with the case  $N = 2$ .

**Case  $N = 2$ .**

Let  $R = \begin{pmatrix} \rho^\uparrow & \sigma \\ \bar{\sigma} & \rho^\downarrow \end{pmatrix} \in \mathcal{C}_2$ . We write  $\sqrt{R} = \begin{pmatrix} r^\uparrow & s \\ \bar{s} & r^\downarrow \end{pmatrix}$ , with  $r^\uparrow, r^\downarrow \in (H^1(\mathbb{R}^3))^2$  and  $s$  in  $H^1(\mathbb{R}^3, \mathbb{C})$ . Let

$$R^\uparrow := \begin{pmatrix} |r^\uparrow|^2 & sr^\uparrow \\ \bar{s}r^\uparrow & |s|^2 \end{pmatrix} \quad \text{and} \quad R^\downarrow := \begin{pmatrix} |s|^2 & sr^\downarrow \\ \bar{s}r^\downarrow & |r^\downarrow|^2 \end{pmatrix}. \quad (13)$$

It is easy to check  $R = R^\dagger + R^\downarrow$ , that  $R^{\uparrow/\downarrow}$  are hermitian, of null determinant, and  $\sqrt{R^{\uparrow/\downarrow}} \in \mathcal{M}_{2 \times 2}(H^1(\mathbb{R}^3, \mathbb{C}))$ . However, it may hold that  $\int_{\mathbb{R}^3} \text{tr}_{\mathbb{C}^2}[R^\dagger] \notin \mathbb{N}^*$ , so that  $R^\dagger$  is not in  $\mathcal{C}_M^0$  for some  $M \in \mathbb{N}^*$ .

The case  $R^\dagger = 0$  or  $R^\downarrow = 0$  are trivial. Let us suppose that, for  $\alpha \in \{\uparrow, \downarrow\}$ ,  $m^\alpha := \int_{\mathbb{R}^3} \rho_{R^\alpha} \neq 0$ . In this case, the matrices  $\widetilde{R}^\alpha = (m^\alpha)^{-1} R^\alpha$  are in  $\mathcal{C}_1^0$ , hence are representable by a single orbital, due to the first statement of Theorem 1. Let  $\widetilde{\Phi} = (\widetilde{\phi}_1^\uparrow, \widetilde{\phi}_1^\downarrow)^T \in H^1(\mathbb{R}^3, \mathbb{C}^2)$  and  $\widetilde{\Phi}_2 = (\widetilde{\phi}_2^\uparrow, \widetilde{\phi}_2^\downarrow)^T \in H^1(\mathbb{R}^3, \mathbb{C}^2)$  be normalized orbitals that represent respectively  $\widetilde{R}^\dagger$  and  $\widetilde{R}^\downarrow$ . It holds

$$\widetilde{\Phi}_1 \widetilde{\Phi}_1^* = \widetilde{R}^\dagger = (m^\dagger)^{-1} R^\dagger \quad \text{and} \quad \widetilde{\Phi}_2 \widetilde{\Phi}_2^* = \widetilde{R}^\downarrow = (m^\downarrow)^{-1} R^\downarrow.$$

From the Lazarev-Lieb orthogonalization process (see Lemma 5), there exists a function  $u \in C^\infty(\mathbb{R})$  with bounded derivatives such that

$$\langle \widetilde{\Phi}_1 | \widetilde{\Phi}_2 e^{iu} \rangle = \int_{\mathbb{R}^3} \left( \overline{\psi_1^\uparrow} \psi_2^\uparrow + \overline{\psi_1^\downarrow} \psi_2^\downarrow \right) e^{iu} = 0. \quad (14)$$

Once this function is chosen, there exists a function  $v \in C^\infty(\mathbb{R})$  with bounded derivatives such that

$$\langle \widetilde{\Phi}_1 | \widetilde{\Phi}_1 e^{iv} \rangle = \langle \widetilde{\Phi}_1 | \widetilde{\Phi}_2 e^{i(u+v)} \rangle = \langle \widetilde{\Phi}_2 e^{iu} | \widetilde{\Phi}_1 e^{iv} \rangle = \langle \widetilde{\Phi}_2 | \widetilde{\Phi}_2 e^{iv} \rangle = 0. \quad (15)$$

We finally set

$$\Phi_1 := \frac{1}{\sqrt{2}} \left( \sqrt{m^\dagger} \widetilde{\Phi}_1 + \sqrt{m^\downarrow} \widetilde{\Phi}_2 e^{iu} \right)$$

and

$$\Phi_2 := \frac{1}{\sqrt{2}} \left( \sqrt{m^\dagger} \widetilde{\Phi}_1 - \sqrt{m^\downarrow} \widetilde{\Phi}_2 e^{iu} \right) e^{iv}.$$

From (14), we deduce  $\|\Phi_1\|^2 = \|\Phi_2\|^2 = 1$ , so that both  $\Phi_1$  and  $\Phi_2$  are normalized. Also, from (15), we get  $\langle \Phi_1 | \Phi_2 \rangle = 0$ , hence  $\{\Phi_1, \Phi_2\}$  is orthonormal. As  $\widetilde{\Phi}_1$  and  $\widetilde{\Phi}_2$  are in  $H^1(\mathbb{R}^3, \mathbb{C}^2)$ , and  $u$  and  $v$  have bounded derivatives,  $\Phi_1$  and  $\Phi_2$  are in  $H^1(\mathbb{R}^3, \mathbb{C}^2)$ . Finally, it holds that

$$\begin{aligned} \Phi_1 \Phi_1^* + \Phi_2 \Phi_2^* &= \\ &= \frac{1}{2} \left( m^\dagger \widetilde{\Phi}_1 \widetilde{\Phi}_1^* + m^\downarrow \widetilde{\Phi}_2 \widetilde{\Phi}_2^* + 2\sqrt{m^\dagger m^\downarrow} \text{Re} \left( \widetilde{\Phi}_1 \widetilde{\Phi}_2^* e^{-iu} \right) \right. \\ &\quad \left. + m^\dagger \widetilde{\Phi}_1 \widetilde{\Phi}_1^* + m^\downarrow \widetilde{\Phi}_2 \widetilde{\Phi}_2^* - 2\sqrt{m^\dagger m^\downarrow} \text{Re} \left( \widetilde{\Phi}_1 \widetilde{\Phi}_2^* e^{-iu} \right) \right) \\ &= m^\dagger \widetilde{\Phi}_1 \widetilde{\Phi}_1^* + m^\downarrow \widetilde{\Phi}_2 \widetilde{\Phi}_2^* = R. \end{aligned}$$

We deduce that the Slater determinant  $\mathcal{S}[\Phi_1, \Phi_2]$  represents  $R$ , so that  $R \in \mathcal{J}_2^{\text{Slater}}$ . Altogether,  $\mathcal{C}_2 \subset \mathcal{J}_2^{\text{Slater}}$ , and therefore  $\mathcal{C}_2 = \mathcal{J}_2^{\text{Slater}}$ .

**Case  $N > 2$ .**

We proceed by induction. Let  $R \in \mathcal{C}_{N+1}$  with  $N \geq 2$ , and suppose  $\mathcal{C}_N = \mathcal{J}_N^{\text{Slater}}$ . We use again the decomposition (13) and write  $R = R^\dagger + R^\downarrow$ , where  $R^{\uparrow/\downarrow}$  are two null-determinant hermitian matrices. For  $\alpha \in \{\uparrow, \downarrow\}$ , we note  $m^\alpha := \int_{\mathbb{R}^3} \rho_{R^\alpha}$ . Since  $m^\dagger + m^\downarrow = N + 1 \geq 3$ , at least  $m^\dagger$  or  $m^\downarrow$  is greater than 1. Let us suppose without loss of generality that  $m^\dagger \geq 1$ . We then write  $R = R_1 + R_2$  with

$$R_1 := (m^\dagger)^{-1} R^\dagger \quad \text{and} \quad R_2 := \left( (1 - (m^\dagger)^{-1}) R^\dagger + m^\downarrow R^\downarrow \right).$$

It holds that  $R_1 \in \mathcal{C}_1^0 = \mathcal{J}_1^{\text{Slater}}$  and  $R_2 \in \mathcal{C}_N = \mathcal{J}_N^{\text{Slater}}$  (by induction). Together with Lemma 4, we deduce that  $R \in \mathcal{J}_{N+1}^{\text{Slater}}$ . The result follows.

## 4.2 Proof of Lemma 3

Let  $\Phi = (\phi^\uparrow, \phi^\downarrow) \in H^1(\mathbb{R}^3, \mathbb{C}^2)$  having well-defined global phases in  $C^1(\mathbb{R})$ , and let  $(R, \mathbf{j})$  be the associated spin-density matrix and paramagnetic current. It holds

$$R = \begin{pmatrix} \rho^\uparrow & \sigma \\ \bar{\sigma} & \rho^\downarrow \end{pmatrix} = \begin{pmatrix} |\phi^\uparrow|^2 & \phi^\uparrow \bar{\phi}^\downarrow \\ \phi^\downarrow \phi^\uparrow & |\phi^\downarrow|^2 \end{pmatrix}.$$

For  $\alpha \in \{\uparrow, \downarrow\}$ , we let  $\tau^\alpha$  be the phase of  $\phi^\alpha$ , so that  $\phi^\alpha = \sqrt{\rho^\alpha} e^{i\tau^\alpha}$ . Setting  $\tau = \tau^\uparrow - \tau^\downarrow$ , we obtain  $\sigma = |\sigma| e^{i\tau} = \sqrt{\rho^\uparrow \rho^\downarrow} e^{i\tau}$ . The paramagnetic current is

$$\mathbf{j} = \rho^\uparrow \nabla \tau^\uparrow + \rho^\downarrow \nabla \tau^\downarrow = \rho \nabla \tau^\downarrow + \rho^\uparrow \nabla \tau = \rho \nabla \tau^\uparrow - \rho^\downarrow \nabla \tau.$$

In particular, using (8),

$$\frac{\mathbf{j}}{\rho} - \frac{\text{Im}(\bar{\sigma} \nabla \sigma)}{\rho \rho^\downarrow} = \frac{\mathbf{j} - \rho^\downarrow \nabla \tau}{\rho} = \nabla \tau^\downarrow \quad (16)$$

is curl-free, and so is

$$\frac{\mathbf{j}}{\rho} + \frac{\text{Im}(\bar{\sigma} \nabla \sigma)}{\rho \rho^\uparrow} = \nabla \tau^\uparrow. \quad (17)$$

## 4.3 Proof of Theorem 2

We break the proof in several steps.

**Step 1: Any  $R \in \mathcal{C}_N$  can be written as  $R = R_1 + R_2 + R_3$  with  $R_k \in \mathcal{C}_{N_k}^0$ ,  $N_k \geq 4$ .**

Let  $R = \begin{pmatrix} \rho^\uparrow & \sigma \\ \bar{\sigma} & \rho^\downarrow \end{pmatrix} \in \mathcal{C}_N$ , with  $N \geq 12$ . We write  $\sqrt{R} = \begin{pmatrix} r^\uparrow & s \\ \bar{s} & r^\downarrow \end{pmatrix}$ , with  $r^\uparrow, r^\downarrow \in (H^1(\mathbb{R}^3))^2$  and  $s$  in  $H^1(\mathbb{R}^3, \mathbb{C})$ . We write  $R = R^\uparrow + R^\downarrow$  where  $R^{\uparrow/\downarrow}$  were defined in (13). As in the proof of Theorem 1 for the case  $N = 2$ ,  $R^{\uparrow/\downarrow}$  are hermitian, of null determinant, and  $\sqrt{R^{\uparrow/\downarrow}} \in \mathcal{M}_{2 \times 2}(H^1(\mathbb{R}^3, \mathbb{C}))$ . However, it may hold that  $\int \text{tr}_{\mathbb{C}^2}[R^\uparrow] \notin \mathbb{N}^*$ , so that  $R^\uparrow$  is not in  $\mathcal{C}_M^0$  for some  $M \in \mathbb{N}^*$ . In order to handle this difficulty, we will distribute the mass of  $R^\uparrow$  and  $R^\downarrow$  into three density-matrices. More specifically, let us suppose without loss of generality that  $\int \text{tr}_{\mathbb{C}^2}[R^\uparrow] \geq \int \text{tr}_{\mathbb{C}^2}[R^\downarrow]$ . We set

$$\begin{aligned} R_1 &= (1 - \xi_1)R^\uparrow + \xi_2 R^\downarrow \\ R_2 &= \xi_1(1 - \xi_3)R^\uparrow \\ R_3 &= (1 - \xi_2)R^\downarrow + \xi_3 R^\uparrow, \end{aligned} \quad (18)$$

where  $\xi_1, \xi_2, \xi_3$  are suitable non-decreasing functions in  $C^\infty(\mathbb{R}^3)$ , that depends only on (say)  $r_1$ , such that, for  $1 \leq k \leq 3$ ,  $0 \leq \xi_k \leq 1$ . We will choose them of the form  $\xi_k(\mathbf{r}) = 0$  for  $x_1 < \alpha_k$  and  $\xi_k(\mathbf{r}) = 1$  for all  $x_1 \geq \beta_k > \alpha_k$ , and such that

$$(1 - \xi_1)\xi_2 = (1 - \xi_2)\xi_3 = (1 - \xi_1)\xi_3 = 0. \quad (19)$$

Finally, these functions are tuned so that  $\int \text{tr}_{\mathbb{C}^2}(R_k) \in \mathbb{N}^*$  and  $\int \text{tr}_{\mathbb{C}^2}(R_k) \geq 4$  for all  $1 \leq k \leq 3$  (see Figure 1 for an example of such a triplet  $(\xi_1, \xi_2, \xi_3)$ ). Although it is not difficult to convince oneself that such functions  $\xi_k$  exist, we provide a full proof of this fact in the Appendix.

From (19), it holds that for all  $1 \leq k \leq 3$ ,  $R_k \in \mathcal{C}_{N_k}^0$ , and that  $R_1 + R_2 + R_3 = R^\uparrow + R^\downarrow = R$ .

**Step 2 : The pair  $(R_1, \mathbf{j}_1)$  is representable by a Slater determinant.**

In order to simplify the notation, we introduce the total densities of  $R^\uparrow$  and  $R^\downarrow$ :

$$f^\uparrow := |r^\uparrow|^2 + |s|^2 \quad \text{and} \quad f^\downarrow := |r^\downarrow|^2 + |s|^2.$$

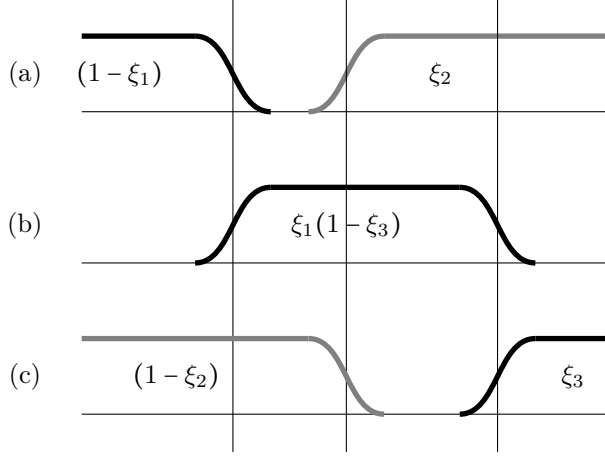


Figure 1: Weights of the matrices  $R^\dagger$  (black) and  $R^\leftarrow$  (gray) in (a)  $R_1 = (1 - \xi_1)R^\dagger + \xi_2 R^\leftarrow$ , (b)  $R_2 = \xi_2(1 - \xi_3)R^\dagger$  and  $R_3 = (1 - \xi_2)R^\dagger + \xi_3 R^\leftarrow$ .

Recall that  $\rho = f^\dagger + f^\leftarrow$ . We consider the previous decomposition  $R = R_1 + R_2 + R_3$ , and we decompose  $\mathbf{j}$  in a similar fashion. More specifically, we write  $\mathbf{j} = \mathbf{j}_1 + \mathbf{j}_2 + \mathbf{j}_3$  with

$$\begin{aligned} \mathbf{j}_1 &= (1 - \xi_1) \left( \frac{f^\dagger}{\rho} \mathbf{j} - \text{Im}(\bar{s} \nabla s) \right) + \xi_2 \left( \frac{f^\leftarrow}{\rho} \mathbf{j} + \text{Im}(\bar{s} \nabla s) \right), \\ \mathbf{j}_2 &= \xi_1(1 - \xi_3) \left( \frac{f^\dagger}{\rho} \mathbf{j} - \text{Im}(\bar{s} \nabla s) \right), \\ \mathbf{j}_3 &= (1 - \xi_2) \left( \frac{f^\leftarrow}{\rho} \mathbf{j} + \text{Im}(\bar{s} \nabla s) \right) + \xi_3 \left( \frac{f^\dagger}{\rho} \mathbf{j} - \text{Im}(\bar{s} \nabla s) \right). \end{aligned} \quad (20)$$

Let us show that the pair  $(R_1, \mathbf{j}_1)$  is representable. Following [12], we introduce

$$\xi(x) = \frac{1}{m} \int_{-\infty}^x \frac{1}{(1+y^2)^{(1+\delta)/2}} dy,$$

where  $\delta$  is the one in (9), and  $m$  is a constant chosen such that  $\xi(\infty) = 1$ . We then introduce

$$\begin{aligned} \eta_{1,1}(\mathbf{r}) &= \frac{2}{N} \xi(\mathbf{r} + \alpha), \\ \eta_{1,2}(\mathbf{r}) &= \frac{2}{N-1} \xi(x_1 + \beta)(1 - \eta_1(\mathbf{r})), \\ \eta_{1,3}(\mathbf{r}) &= \frac{2}{N-2} \xi(x_2 + \gamma)(1 - \eta_1(\mathbf{r}) - \eta_2(\mathbf{r})), \\ \eta_{1,k}(\mathbf{r}) &= \frac{1}{N-3} (1 - \eta_1(\mathbf{r}) - \eta_2(\mathbf{r}) - \eta_3(\mathbf{r})) \quad \text{for } 4 \leq k \leq N, \end{aligned} \quad (21)$$

where  $\alpha, \beta, \gamma$  are tuned so that, if  $\rho_1 := \text{tr}_{\mathbb{C}^2} R_1$  denotes the total density of  $R_1$ ,

$$\forall 1 \leq k \leq N_k, \quad \int_{\mathbb{R}^3} \eta_{1,k} \rho_1 = 1. \quad (22)$$

It can be checked (see [12]) that  $\eta_{1,k} \geq 0$  and  $\sum_{k=1}^N \eta_{1,k} = 1$ . We seek orbitals of the form

$$\Phi_{1,k} := \sqrt{\eta_{1,k}} \left( \sqrt{(1 - \xi_1)} \begin{pmatrix} r^\dagger \\ s \end{pmatrix} + \sqrt{\xi_2} \begin{pmatrix} s \\ r^\leftarrow \end{pmatrix} \right) e^{iu_{1,k}}, \quad 1 \leq k \leq N_1,$$

and where the phases  $u_{1,k}$  will be chosen carefully later. From (19), we recall that  $(1 - \xi_1)\xi_2 = 0$ , so that, by construction,  $\Phi_{1,k}$  is normalized, and

$$\Phi_{1,k}\Phi_{1,k}^* = \eta_{1,k}R_1.$$

Let us suppose for now that the phases  $u_{1,k}$  are chosen so that the orbitals are orthogonal. This will indeed be achieved thanks to the Lazarev-Lieb orthogonalization process (see Lemma 5). Then,  $\Psi_1 := \mathcal{S}[\Phi_{1,1}, \dots, \Phi_{1,N}]$  indeed represents the spin-density matrix  $R_1$ . The paramagnetic current of  $\Psi$  is (we recall that  $r^\uparrow$  and  $r^\downarrow$  are real-valued, and we write  $s = |s|e^{i\tau}$  for simplicity)

$$\begin{aligned} \mathbf{j}_\Psi &= \sum_{k=1}^{N_1} \eta_{1,k}(1 - \xi_1) (|r^\uparrow|^2 \nabla u_{1,k} + |s|^2 \nabla(-\tau + u_{1,k})) + \\ &\quad + \sum_{k=1}^{N_1} \eta_{1,k}\xi_2 (|s|^2 \nabla(\tau + u_{1,k}) + |r^\downarrow|^2 \nabla u_{1,k}) \\ &= ((1 - \xi_1)f^\uparrow + \xi_2 f^\downarrow) \left( \sum_{k=1}^{N_1} \eta_{1,k} \nabla u_{1,k} \right) + (\xi_2 - (1 - \xi_1)) |s|^2 \nabla \tau. \end{aligned}$$

Since  $|s|^2 \nabla \tau = \text{Im}(\bar{s} \nabla s)$ , this current is equal to the target current  $\mathbf{j}_1$  defined in (20) if and only if

$$\rho_1 \frac{\mathbf{j}}{\rho} = \rho_1 \sum_{k=1}^{N_1} \eta_k \nabla u_{1,k}. \quad (23)$$

In [12], Lieb and Schrader provided an explicit solution of this system when  $N_1 \geq 4$ . We do not repeat the proof, but emphasize on the fact that because condition (9) is satisfied by hypothesis, the phases  $u_{1,k}$  can be chosen to be functions of  $r_1$  only, and to have bounded derivatives. In particular, the functions  $\Phi_{1,k}$  are in  $H^1(\mathbb{R}^3, \mathbb{C}^2)$ . Also, as their proof relies on the Lazarev-Lieb orthogonalization process, it is possible to choose the phases  $u_{1,k}$  so that the functions  $\Phi_{1,k}$  are orthogonal, and orthogonal to a finite-dimensional subspace of  $L^2(\mathbb{R}^3, \mathbb{C}^2)$ .

Altogether, we proved that the pair  $(R_1, \mathbf{j}_1)$  is representable by the Slater determinant  $\mathcal{S}[\Phi_{1,1}, \dots, \Phi_{1,N_1}]$ .

### Step 3: Representability of $(R_2, \mathbf{j}_2)$ and $(R_3, \mathbf{j}_3)$ , and finally of $(R, \mathbf{j})$ .

In order to represent the pair  $(R_2, \mathbf{j}_2)$ , we first construct the functions  $\eta_{2,k}$  for  $1 \leq k \leq N_2$  of the form (21) so that (22) holds for  $\rho_2 := \text{tr}_{\mathbb{C}^2} R_2$ . We then seek orbitals of the form

$$\Phi_{2,k} := \sqrt{\eta_{2,k}\xi_1(1 - \xi_3)} \begin{pmatrix} r^\uparrow \\ s \end{pmatrix} e^{iu_{2,k}}, \quad \text{for } 1 \leq k \leq N_2.$$

Reasoning as above, the Slater determinant of these orbitals represents the pair  $(R_2, \mathbf{j}_2)$  if and only if

$$\rho_2 \frac{\mathbf{j}_2}{\rho} = \rho_2 \sum_{k=1}^{N_2} \eta_{2,k} \nabla u_{2,k}.$$

Again, due to the fact that  $N_2 \geq 4$ , this equation admits a solution. Moreover, it is possible to choose the phases  $u_{2,k}$  so that the functions  $\Phi_{2,k}$  are orthogonal to the previously constructed  $\Phi_{1,k}$ .

We repeat again this argument for the pair  $(R_3, \mathbf{j}_3)$ . Once the new set of functions  $\eta_{3,k}$  is constructed, we seek orbitals of the form

$$\Phi_{3,k} := \sqrt{\eta_{3,k}} \left( \sqrt{(1 - \xi_2)} \begin{pmatrix} s \\ r^\downarrow \end{pmatrix} + \sqrt{\xi_3} \begin{pmatrix} r^\uparrow \\ s \end{pmatrix} \right) e^{iu_{3,k}}$$

and construct the phases so that the functions  $\Phi_{3,k}$  are orthogonal to the functions  $\Phi_{1,k}$  and  $\Phi_{2,k}$ .

Altogether, the pair  $(R, \mathbf{j})$  is represented by the (finite energy) Slater determinant  $\mathcal{S}[\Phi_{1,1}, \dots, \Phi_{1,N_1}, \Phi_{2,1}, \dots, \Phi_{2,N_2}, \Phi_{3,1}, \dots, \Phi_{3,N_3}]$  which concludes the proof.

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## 5 Appendix

We explain in this section how to construct three functions  $\xi_1, \xi_2, \xi_3 \in (C^\infty(\mathbb{R}))^3$  like in Figure 1. In order to simplify the notation, we introduce

$$\begin{aligned} f(r) &:= \iint_{\mathbb{R} \times \mathbb{R}} \operatorname{tr}_{\mathbb{C}^2}(R^\downarrow)(r, r_2, r_3) \, dr_2 dr_3, \\ g(r) &:= \iint_{\mathbb{R} \times \mathbb{R}} \operatorname{tr}_{\mathbb{C}^2}(R^\uparrow)(r, y, z) \, dr_2 dr_3, \end{aligned}$$

where  $R^\uparrow, R^\downarrow$  were defined in (13). We denote by

$$F(\alpha) = \int_{-\infty}^{\alpha} f(x) dx \quad \text{and} \quad G(\alpha) = \int_{-\infty}^{\alpha} g(x) dx,$$

and finally  $\mathcal{F} = F(\infty) = \int_{\mathbb{R}} f$  and  $\mathcal{G} = G(\infty) = \int_{\mathbb{R}} g$ . Note that  $F$  and  $G$  are continuous non-decreasing functions going from 0 to  $\mathcal{F}$  (respectively  $\mathcal{G}$ ), and that it holds  $\mathcal{F} + \mathcal{G} = N$ . Let us suppose without loss of generality that  $\mathcal{F} \leq \mathcal{G}$ , so that  $0 \leq \mathcal{F} \leq N/2 \leq \mathcal{G} \leq N$ . If  $\mathcal{F} = 0$ , then  $R^\downarrow = 0$  and we can choose  $R_1 = R_2 = (4/N)R^\uparrow \in \mathcal{C}_4^0$  and  $R_3 = (N-8)/NR^\uparrow \in \mathcal{C}_{N-8}^0$ . Since  $N \geq 12$ , it holds  $N-8 \geq 4$ , so that this is the desired decomposition. We now consider the case where  $\mathcal{F} \neq 0$ .

In order to keep the notation simple, we will only study the case  $\mathcal{F} < 8$  (the case  $\mathcal{F} > 8$  is similar by replacing the integer 4 by a greater integer  $M$  such that  $\mathcal{F} < 2M < N-4$  in the sequel). We seek for  $\alpha$  such that

$$\begin{cases} \int_{-\infty}^{\alpha} f(x) dx < 4 & \text{and} & \int_{-\infty}^{\alpha} f(x) + \int_{\alpha}^{\infty} g(x) > 4, \\ \int_{\alpha}^{\infty} f(x) dx < 4 & \text{and} & \int_{-\infty}^{\alpha} g(x) dx + \int_{\alpha}^{\infty} f(x) dx > 4, \end{cases}$$

or equivalently

$$\begin{cases} F(\alpha) < 4 & \text{and} & F(\alpha) + \mathcal{G} - G(\alpha) > 4, \\ \mathcal{F} - F(\alpha) < 4 & \text{and} & \mathcal{F} - F(\alpha) + G(\alpha) > 4, \end{cases}$$

that is

$$\mathcal{F} - 4 < F(\alpha) < 4, \quad \text{and} \quad F(\alpha) + 4 - \mathcal{F} < G(\alpha) < F(\alpha) + \mathcal{G} - 4. \quad (24)$$

Let  $\alpha_{(\mathcal{F}-4)}$  be such that  $F(\alpha_{(\mathcal{F}-4)}) = \mathcal{F} - 4$  (with  $\alpha_{(\mathcal{F}-4)} = -\infty$  if  $\mathcal{F} \leq 4$ ), and  $\alpha_{(4)}$  be such that  $F(\alpha_{(4)}) = 4$  (with  $\alpha_{(4)} = +\infty$  if  $\mathcal{F} \leq 4$ ). As  $F$  is continuous non-decreasing, the first equation of (24) is satisfied whenever  $\alpha_{(\mathcal{F}-4)} < \alpha < \alpha_{(4)}$ .

The function  $[\alpha_{(\mathcal{F}-4)}, \alpha_{(4)}] \ni \alpha \mapsto m(\alpha) := F(\alpha) + 4 - \mathcal{F}$  goes continuously and non-decreasingly from 0 to  $8 - \mathcal{F}$ , and the function  $[\alpha_{(\mathcal{F}-4)}, \alpha_{(4)}] \ni \alpha \mapsto M(\alpha) := F(\alpha) + \mathcal{G} - 4$  goes continuously and non-decreasingly from  $N - 8$  to  $\mathcal{G}$  between  $\alpha_{(\mathcal{F}-4)}$  and  $\alpha_{(4)}$ . In particular, since  $G(\alpha)$  goes continuously and non-decreasingly from 0 to  $\mathcal{G}$ , only three cases may happen:

- There exists  $\alpha_0 \in (\alpha_{(\mathcal{F}-4)}, \alpha_{(4)})$  such that  $m(\alpha_0) < G(\alpha_0) < M(\alpha_0)$ . In this case, (24) holds for  $\alpha = \alpha_0$ . By continuity, there exists  $\varepsilon > 0$  such that

$$\begin{cases} F(\alpha + \varepsilon) < 4, \\ F(\alpha) + \mathcal{G} - G(\alpha + \varepsilon) > 4, \\ G(\alpha) + \mathcal{F} - F(\alpha + \varepsilon) > 4. \end{cases}$$

Let  $\xi_2 \in C^\infty(\mathbb{R})$  be a non-decreasing function such that  $\xi_2(x) = 0$  for  $x < \alpha$  and  $\xi_2(x) = 1$  for  $x > \alpha + \varepsilon$ . Then, as  $0 \leq \xi_2 \leq 1$ , it holds that:

$$\int_{\mathbb{R}} (1 - \xi_2)f \leq F(\alpha + \varepsilon) < 4$$

and

$$\int_{\mathbb{R}} (1 - \xi_2)f + \int_{\alpha + \varepsilon}^{\infty} g \geq F(\alpha) + \mathcal{G} - G(\alpha + \varepsilon) > 4.$$

We deduce that there exists a non-decreasing function  $\xi_3 \in C^\infty(\mathbb{R})$  such that  $\xi_3(x) = 0$  for  $x < \alpha + \varepsilon$ , and such that

$$\int_{\mathbb{R}} (1 - \xi_2)f + \xi_3g = 4.$$

Note that  $(1 - \xi_2)\xi_3 = 0$ . On the other hand, from

$$\begin{cases} \int_{\mathbb{R}} \xi_2f \leq \mathcal{F} - F(\alpha) < 4 \\ \int_{\mathbb{R}} \xi_2f + \int_{-\infty}^{\alpha} g \geq \mathcal{F} - F(\alpha + \varepsilon) + G(\alpha) > 4, \end{cases}$$

we deduce that there exists a non-decreasing function  $\xi_1 \in C^\infty(\mathbb{R})$  such that  $\xi_1(x) = 1$  for  $x > \alpha$ ,

$$\int_{\mathbb{R}} (1 - \xi_1)g + \xi_2f = 4.$$

and  $(1 - \xi_1)\xi_2 = (1 - \xi_1)\xi_3 = 0$ . Finally, we set

$$\begin{aligned} R_1 &= (1 - \xi_1)R^\dagger + \xi_2R^\downarrow \\ R_2 &= \xi_1(1 - \xi_3)R^\dagger \\ R_3 &= (1 - \xi_2)R^\downarrow + \xi_3R^\dagger. \end{aligned}$$

By construction,  $R = R^\dagger + R^\downarrow = R_1 + R_2 + R_3$ ,  $R_1 \in \mathcal{C}_4^0$  and  $R_3 \in \mathcal{C}_4^0$ . We deduce that  $R_4 \in \mathcal{C}_{N-8}^0$ . Together with the fact that  $N \geq 12$ , this leads to the desire decomposition.

- For all  $\alpha \in (\alpha_{(\mathcal{F}-4)}, \alpha_{(4)})$ , it holds  $G(\alpha) < m(\alpha)$ . Note that this may only happen if  $m(\alpha_{(4)}) > 0$ , or  $\mathcal{F} < 4$ , so that  $\mathcal{G} > N - 4 \geq 8$ . It holds  $G(\alpha_{(\mathcal{F}-4)}) = 0$ , so that  $g(r)$  is null for  $r < \alpha_{(\mathcal{F}-4)}$ . Let  $\alpha_0$  be such that  $\alpha_{(\mathcal{F}-4)} < \alpha_0 < \alpha_{(4)}$ . As

$$\int_{\mathbb{R}} f = \mathcal{F} > 4 \quad \text{and} \quad \int_{\alpha_0}^{\infty} = \mathcal{F} - F(\alpha_0) < 4,$$

there exists a non-decreasing function  $\xi_1 \in C^\infty(\mathbb{R})$  satisfying  $\xi_1(x) = 1$  for  $x \geq \alpha_0$  and such that

$$\int_{\mathbb{R}} \xi_1f = 4.$$

Now, since  $G(\alpha_{(4)}) < m(\alpha_{(4)}) = 8 - \mathcal{F}$ , it holds that

$$\begin{cases} \int_{\mathbb{R}} (1 - \xi_1)f \leq F(\alpha_{(4)}) = 4 \\ \int_{\mathbb{R}} (1 - \xi_1)f + \int_{\alpha_0}^{\infty} g \geq F(\alpha_{(\mathcal{F}-4)}) + \mathcal{G} - G(\alpha_{(4)}) > 4. \end{cases}$$

There exists a non-decreasing function  $\xi_2 \in C^\infty(\mathbb{R})$  satisfying  $\xi_2(x) = 0$  for  $x \leq \alpha_0$  and such that

$$\int_{\mathbb{R}} (1 - \xi_1)f + \xi_2g = 4.$$

Note that  $(1 - \xi_1)\xi_2 = 0$ . Finally, we set

$$\begin{aligned} R_1 &= \xi_1R^\downarrow \\ R_2 &= (1 - \xi_2)R^\dagger \\ R_3 &= \xi_2R^\dagger + (1 - \xi_1)R^\downarrow. \end{aligned}$$

By construction, it holds that  $R = R_1 + R_2 + R_3$ , and that  $R_1 \in \mathcal{C}_4^0$  and  $R_3 \in \mathcal{C}_4^0$ . We deduce  $R_2 \in \mathcal{C}_{N-8}^0$ , and the result follows.

- For all  $\alpha \in (\alpha_{(\mathcal{F}-4)}, \alpha_{(4)})$ , it holds  $\mathcal{G}(\alpha) > M(\alpha)$ . This case is similar than the previous one.

## References