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| Quang-Huy Nguyen. Sharp Strichartz estimates for water waves systems. 2015. hal-01238605

HAL Id: hal-01238605

<https://hal.science/hal-01238605>

Preprint submitted on 6 Dec 2015

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Sharp Strichartz estimates for water waves systems

Quang-Huy Nguyen

ABSTRACT. We prove for reasonably smooth solutions the optimal Strichartz estimates for pure gravity waves and the semi-classical Strichartz estimates for gravity-capillary waves; for both 2D and 3D waves. Here, by optimal we mean the gains of regularity obtained for the linearized systems. Our proofs combine the paradifferential reductions of Alazard-Burq-Zuily [1, 3] with a dispersive estimate using a localized wave package type parametrix of Koch-Tataru [19].

1. Introduction

Water waves systems govern the dynamic of an interface between a fluid domain and the vacuum. It is well-known that these systems are dispersive, i.e., waves at different frequency propagate at different speed. For approximate models of water waves in certain regimes such as Kadomtsev-Petviashvili equations, Korteweg-de Vries equations, Schrodinger equations, wave equations,...dispersive properties have been extensively studied. For the fully nonlinear system of water waves, dispersive properties are however less understood.

On the one hand, in global dynamic, dispersive properties have been considered in establishing the existence of global (or almost global) solutions for small, localized, smooth data by the works of Wu [25, 26], Germain-Masmoudi-Shatah [12, 13], Ionescu-Pusateri [17, 18], Alazard-Delort [5], Ifrim-Tataru [15, 16]. On the other hand, in local dynamic, dispersive properties and more precisely Strichartz estimates have been exploited in proving the existence of local-in-time solutions with rough, generic data, initiated by the work of Alazard-Burq-Zuily [4] and then followed by de Poyferré-Nguyen [10, 11]. Prior to these, a Strichartz estimate was proved for 2D gravity-capillary waves by Christianson-Hur-Staffilani [9]. Unlike the case of semilinear Schrodinger (wave) equations, water waves systems are quasilinear in nature and thus how much regularity one can gain in Strichartz estimates depends also on the smoothness of solutions under consideration. In other words, in term of dispersive analysis (for generic solutions), the nonlinear systems are not obviously dictated by their linearizations. In fact, the Strichartz estimates proved in [4], [11] are non optimal compared to the linearized systems. We address in this paper the following problem:

At which level of regularity, solutions to the fully nonlinear systems of water waves obey the same Strichartz estimates as its linearization?

Remark first that due to the systematic use of symbolic calculus in the framework of semi-classical analysis in [4], [11] we were not able to reach sharp Strichartz estimates by simply adapting their method to the case of sufficiently smooth solutions.

The author was supported in part by Agence Nationale de la Recherche project ANAÉ ANR-13-BS01-0010-03.

In this paper, we choose to work on the The Zakharov-Craig-Sulem formulation of water waves, which is recalled now.

1.1. The The Zakharov-Craig-Sulem formulation of water waves. We consider an incompressible inviscid fluid with unit density moving in a time-dependent domain

$$\Omega = \{(t, x, y) \in [0, T] \times \mathbf{R}^d \times \mathbf{R} : (x, y) \in \Omega_t\}$$

where each Ω_t is a domain located underneath a free surface

$$\Sigma_t = \{(x, y) \times \mathbf{R}^d \times \mathbf{R} : y = \eta(t, x)\}$$

and above a fixed bottom $\Gamma = \partial\Omega_t \setminus \Sigma_t$. We make the following assumption on the domain: *Assumption (H)*

Each Ω_t is the intersection of the haft space

$$\Omega_{1,t} = \{(x, y) \times \mathbf{R}^d \times \mathbf{R} : y = \eta(t, x)\}$$

and an open connected set Ω_2 containing a fixed strip around Σ_t , i.e., there exists $h > 0$ such that for all $t \in [0, T]$

$$\{(x, y) \times \mathbf{R}^d \times \mathbf{R} : \eta(x) - h \leq y \leq \eta(t, x)\} \subset \Omega_2.$$

Assume that the velocity field v admits a potential $\phi : \Omega \rightarrow \mathbf{R}$, i.e, $v = \nabla\phi$. Using the Zakharov formulation, we introduce the trace of ϕ on the free surface

$$\psi(t, x) = \phi(t, x, \eta(t, x)).$$

Then $\phi(t, x, y)$ is the unique variational solution of

$$(1.1) \quad \Delta\phi = 0 \text{ in } \Omega_t, \quad \phi(t, x, \eta(t, x)) = \psi(t, x).$$

The Dirichlet-Neumann operator is then defined by

$$\begin{aligned} G(\eta)\psi &= \sqrt{1 + |\nabla_x \eta|^2} \left(\frac{\partial\phi}{\partial n} \Big|_{\Sigma} \right) \\ &= (\partial_y \phi)(t, x, \eta(t, x)) - \nabla_x \eta(t, x) \cdot (\nabla_x \phi)(t, x, \eta(t, x)). \end{aligned}$$

The *gravity-capillary waves* (see [20]) problem consists in solving the following system of (η, ψ) :

$$(1.2) \quad \begin{cases} \partial_t \eta = G(\eta)\psi, \\ \partial_t \psi + g\eta - \sigma H(\eta) + \frac{1}{2} |\nabla_x \psi|^2 - \frac{1}{2} \frac{(\nabla_x \eta \cdot \nabla_x \psi + G(\eta)\psi)^2}{1 + |\nabla_x \eta|^2} = 0, \end{cases}$$

where σ is the surface tension coefficient and $H(\eta)$ is the mean curvature of the free surface:

$$H(\eta) = \operatorname{div} \left(\frac{\nabla \eta}{\sqrt{1 + |\nabla \eta|^2}} \right).$$

In the regime of large wavelengths, one can discard the effect of surface tension by taking $\sigma = 0$ in the system (1.3) to obtain the system of *pure gravity water waves*

$$(1.3) \quad \begin{cases} \partial_t \eta = G(\eta)\psi, \\ \partial_t \psi + g\eta + \frac{1}{2} |\nabla_x \psi|^2 - \frac{1}{2} \frac{(\nabla_x \eta \cdot \nabla_x \psi + G(\eta)\psi)^2}{1 + |\nabla_x \eta|^2} = 0, \end{cases}$$

The physical dimensions are $d = 1, 2$. For terminologies, when $d = 1$ (respectively $d = 2$) we call (1.2), (1.3) the 2D (respectively 3D) waves systems. It is important

to introduce the vertical and horizontal components of the trace of the velocity on Σ , which can be expressed in terms of η and ψ :

$$(1.4) \quad B = (v_y)|_\Sigma = \frac{\nabla_x \eta \cdot \nabla_x \psi + G(\eta)\psi}{1 + |\nabla_x \eta|^2}, \quad V = (v_x)|_\Sigma = \nabla_x \psi - B \nabla_x \eta.$$

We recall also that the Taylor coefficient defined by $a = -\frac{\partial P}{\partial y} \Big|_\Sigma$ can be defined in terms of η, ψ, B, V only (see §4.2 in [3]).

1.2. Known results and main theorems.

1.2.1. *Pure gravity water waves.* For the system (1.3) of pure gravity water waves, the only existent Strichartz estimate, to our knowledge, is [4] where the authors proved Strichartz estimates for rough solutions with a gain of

$$(1.5) \quad \mu = \frac{1}{24} - \text{ when } d = 1, \quad \mu = \frac{1}{12} - \text{ when } d \geq 2$$

The starting point of this result is the symmetrization of (1.2) into a quasilinear paradifferential equation of the following form (see Appendix A for the paradifferential calculus theory and Theorem 2.2 below for a precise reduction statement)

$$(1.6) \quad (\partial_t + T_V \cdot \nabla + iT_\gamma)u = f \in L_t^\infty H_x^s, \quad s > 1 + \frac{d}{2},$$

where γ is a symbol of order $\frac{1}{2}$.

Let us now look at the linearization of (1.3) (take $g = 1$ and infinite depth) around the rest state $(0, 0)$:

$$\begin{cases} \partial_t \eta - |D_x| \psi = 0, \\ \partial_t \psi + \eta = 0 \end{cases}$$

which is equivalent to, after imposing $u := \eta + i|D_x|^{\frac{1}{2}}\psi$,

$$(1.7) \quad \partial_t u + i|D_x|^{\frac{1}{2}}u = 0.$$

For this Schrodinger-type dispersive equation we can prove classically the Strichartz estimates

$$(1.8) \quad \|u\|_{L^p W^{s-\frac{d}{2}+\mu_{opt}, \infty}} \leq C(s, d) \|u|_{t=0}\|_{H^s}, \quad \begin{cases} \mu_{opt} = \frac{1}{8}, p = 4 & \text{if } d = 1, \\ \mu_{opt} = \frac{1}{4}-, p = 2 & \text{if } d \geq 2, \end{cases}$$

from which the estimates for the original unknowns η, ψ can be recovered. Our first result states that the fully nonlinear system (1.3) enjoys Strichartz estimates with the same gain as in (1.10), for solutions slightly smoother than the energy threshold in [3].

NOTATION 1.1. Denote

$$\begin{aligned} \mathcal{H}^s &= H^{s+\frac{1}{2}}(\mathbf{R}^d) \times H^{s+\frac{1}{2}}(\mathbf{R}^d) \times H^s(\mathbf{R}^d) \times H^s(\mathbf{R}^d), \\ \mathcal{W}^s &= W^{r+\frac{1}{2}, \infty}(\mathbf{R}^d) \times W^{r+\frac{1}{2}, \infty}(\mathbf{R}^d) \times W^{r, \infty}(\mathbf{R}^d) \times W^{r, \infty}(\mathbf{R}^d). \end{aligned}$$

THEOREM 1.2. *Let $d = 1, 2$ and consider a solution (η, ψ) of (1.3) on the time interval $I = [0, T]$, $T < +\infty$ such that Ω_t satisfies H_t for every $t \in [0, T]$ and*

$$(\eta, \psi, B, V) \in C([0, T]; \mathcal{H}^s).$$

(see Theorem 1.2, [3]). Define

$$\begin{cases} s(d) = \frac{5}{3} + \frac{d}{2}, \mu_{opt}(d) = \frac{1}{8}, p(d) = 4 & \text{if } d = 1, \\ s(d) = 2 + \frac{d}{2}, \mu_{opt}(d) = \frac{1}{4}-, p(d) = 2 & \text{if } d \geq 2. \end{cases}$$

Then for any $s > s(d)$ we have

$$(\eta, \psi, B, V) \in L^{p(d)}(I; \mathcal{W}^{s-\frac{d}{2}+\mu_{opt}(d), \infty}).$$

1.2.2. *Gravity-capillary waves.* Let us now look at the linearization of (1.2) (with infinite depth) around the rest state $(0, 0)$,

$$\begin{cases} \partial_t \eta - |D_x| \psi = 0, \\ \partial_t \psi - \Delta \eta = 0 \end{cases}$$

or equivalently, with $\Phi = |D_x|^{\frac{1}{2}} \eta + i\psi$

$$(1.9) \quad \partial_t \Phi + i|D_x|^{\frac{3}{2}} \Phi = 0,$$

for which one can easily prove the following Strichartz estimates

$$(1.10) \quad \|\Phi\|_{L^p W^{s-\frac{d}{2}+\mu_{opt}, \infty}} \leq C(s, d) \|\Phi|_{t=0}\|_{H^s}, \quad \begin{cases} \mu_{opt} = \frac{3}{8}, p = 4 & \text{if } d = 1, \\ \mu_{opt} = \frac{3}{4}, p = 2 & \text{if } d \geq 2. \end{cases}$$

Turning to the nonlinear case, in high dimensions ($d \geq 2$) the geometry can be non trivial and hence trapping can occur. As a consequence, natural dispersive estimates expected are the one constructed at small time scales which are tailored to the frequencies. The propagator $e^{-it|D_x|^{\frac{3}{2}}}$ has the speed of propagation at order $|\xi|^{\frac{1}{2}}$. Hence, for time $|t| < |\xi|^{-\frac{1}{2}}$, we expect no problem due to the global geometry. This leads to the so called *semi-classical Strichartz estimate*. This terminology appeared in [8] for a study of the Schrödinger equations on compact manifolds. To realize this heuristic argument, one multiplies both side of (1.9) by $h^{\frac{3}{2}}$ with $h = 2^{-j}$, $j \in \mathbf{N}$ and make a change of temporal variables $t = h^{\frac{1}{2}}\sigma$, $u(\sigma, x) = \Phi(h^{\frac{1}{2}}\sigma, x)$ to derive the semi-classical equation

$$(1.11) \quad h\partial_\sigma u + |hD_x|^{\frac{3}{2}} u = 0.$$

Then the optimal dispersive estimates for (1.11) implies the semi-classical Strichartz estimates for (1.9) with a lost of $\frac{1}{8}$ derivatives when $d = 1$ and $\frac{1}{4}$ derivatives when $d \geq 2$.

In [1] it was proved that if

$$(1.12) \quad (\eta, \psi) \in C([0, T]; H^{s+\frac{1}{2}} \times H^s), \quad s > 2 + \frac{d}{2}$$

then system (1.2) can be symmetrized into a single equation analogous to its linearization (1.9):

$$(1.13) \quad (\partial_t + T_V \cdot \nabla + iT_\gamma) u = f \in L^\infty H^s, \quad \gamma \in \Gamma^{\frac{3}{2}};$$

from which the local-wellposedness was obtain at this regularity level-(1.12). Using this reduction, Alazard-Burq-Zuily [2] established, for 2D waves, the semi-classical Strichartz estimate at the threshold (1.12) and the classical (optimal) Strichartz estimate when $s > 5 + \frac{1}{2}$. We remark that in [1], the semi-classical gain is achieved due to the fact that after a *para change of variables*, the highest order term $T_\gamma u$ in (1.13) is converted into the simple Fourier multiplier $|D_x|^{\frac{3}{2}}$. Unfortunately, such a reduction can not work for the 3D case and hence, semi-classical Strichartz estimate in this case is much more difficult to establish, especially at the regularity level (1.12). In the present paper, we want to investigate the semi-classical Strichartz estimate for (1.2) when $d \geq 2$, assuming that the solution is slightly smoother than (1.12) (1/2 derivatives). Our second result reads as follows.

THEOREM 1.3. *Let $d \geq 2$ and $0 < T < \infty$. Consider a solution (η, ψ) of (1.2) on the time interval $I = [0, T]$ such that Ω_t satisfies $H(t)$ for every $t \in [0, T]$ and*

$$(\eta, \psi) \in C([0, T]; H^{s+\frac{1}{2}}(\mathbf{R}^d) \times H^s(\mathbf{R}^d)).$$

If $s > \frac{5}{2} + \frac{d}{2}$ then for every $\varepsilon > 0$, there holds

$$(\eta, \psi) \in L^2([0, T]; W^{s+1-\varepsilon-\frac{d}{2}}(\mathbf{R}^d) \times W^{s+\frac{1}{2}-\varepsilon-\frac{d}{2}, \infty}(\mathbf{R}^d)).$$

REMARK 1.4. Our proof of Theorem 1.3 works equally for the 2D waves ($d = 1$), when $(\eta, \psi) \in C([0, T]; H^{s+\frac{1}{2}}(\mathbf{R}) \times H^s(\mathbf{R}))$ with $s > \frac{5}{2} + \frac{1}{2}$. On the other hand, using the paracomposition reduction in [2] we can indeed improve the preceding regularity to $s > 2 + \frac{1}{2}$, which is the same as Theorem 1.1 in [2].

1.2.3. *On the proof of the main results.* In [4, 11] the authors worked completely in the semi-classical formulism and proved dispersive estimate using the approximation WKB method. This allowed the authors to prove Strichartz estimates with nontrivial gains even for very rough backgrounds. However, we emphasize that with this method, we were not able to reach the classical or semi-classical level as in Theorem 1.2 and 1.3. The dispersive estimates for principally normal pseudodifferential operators in [19] require more regularity (C^2) of the symbols to control the Hamiltonian flow and apply the FBI transform technique. This indeed allows us to obtain sharp dispersive estimates when the characteristic set of the symbol has maximal nonvanishing curvatures.

For the proof of our main results, we shall combine the para-differential reduction in the works of Alazard-Burq-Zuily with the phase transform method in the work [19] of Koch-Tataru. Notice that the later works effectively for operators of order 1, after renormalizing. For gravity-capillary waves (see (1.13)) the dispersive term has order $\frac{3}{2}$ and thus the semi-classical time-scale brings it to the one of order 1 and hence leads to the semi-classical Strichartz estimate in Theorem 1.3. For the pure gravity waves (1.6), one observes that the dispersive term iT_γ has order $\frac{1}{2}$ which is lower than the transport term $T_V \cdot \nabla$. Here, we follow [4], suppressing this transport term by straightening the vector field $\partial_t + T_V \cdot \nabla$ and then make another change of spatial variables to convert it to an operator of order 1. However, the new symbol then is not in the standard form $p(x, \xi)$ to apply phase transforms and other technical issues appear. The proof of Theorem 1.2 thus requires much more care.

2. Preliminaries

2.1. Symmetrization of system (1.3). We first recall the symmetrization of system (1.3) to a single quasilinear equation, performed in [1]. This reduction requires the following symbols:

- Symbols of the Dirichlet-Neumann operator

$$\lambda^{(1)} := \sqrt{(1 + |\nabla\eta|^2)|\xi|^2 - (\nabla\eta \cdot \xi)^2},$$

$$\lambda^{(0)} := \frac{1 + |\nabla\eta|^2}{2\lambda^{(1)}} \left\{ \operatorname{div}(\alpha^{(1)} \nabla\eta) + i\partial_\xi \lambda^{(1)} \cdot \nabla \alpha^{(1)} \right\}, \quad \alpha^{(1)} := \frac{\lambda^{(1)} + i\nabla\eta \cdot \xi}{1 + |\nabla\eta|^2}.$$

- Symbols of the mean-curvature operator:

$$\ell^{(2)} := (1 + |\nabla\eta|^2)^{-\frac{1}{2}} \left(|\xi|^2 - \frac{(\nabla\eta \cdot \xi)^2}{1 + |\nabla\eta|^2} \right), \quad \ell^{(1)} := -\frac{i}{2}(\partial_x \cdot \partial_\xi) \ell^{(2)};$$

- Symbols using for symmetrization

$$q := (1 + (\nabla_x \eta)^2)^{-\frac{1}{2}}, \quad p = (1 + (\nabla_x \eta)^2)^{-\frac{5}{4}} |\xi|^{\frac{1}{2}} + p^{(-\frac{1}{2})},$$

where $p^{(-\frac{1}{2})} := F(\nabla_x \eta, \xi) \partial_x^\alpha \eta$, with $|\alpha| = 2$ and $F \in C^\infty(\mathbf{R}^d \times \mathbf{R}^d \setminus \{0\}; \mathbf{C})$ is homogeneous of order $-\frac{1}{2}$ in ξ .

- Symbols in the symmetrized equation:

$$\gamma := \sqrt{l^{(2)} \lambda^{(1)}} = \left(\frac{|\xi|^2 (1 + |\nabla \eta|^2) - (\nabla \eta \cdot \xi)^2}{1 + |\nabla \eta|^2} \right)^{\frac{3}{4}},$$

$$\omega := -\frac{i}{2} (\partial_\xi \cdot \partial_x) \sqrt{l^{(2)} \lambda^{(1)}}, \quad \omega_1 := \sqrt{\frac{l^{(2)} \Re \lambda^{(0)}}{\lambda^{(1)}} \frac{1}{2}}.$$

Then with the *good-unknown* $U := \psi - T_B \eta$, it was proved in [1] that

THEOREM 2.1 ([1, Corollary 4.9]). *Let $s > 2 + \frac{d}{2}$ and $(\eta, \psi) \in C^0([0, T]; H^{s+\frac{1}{2}} \times H^s)$ be a solution to the gravity-capillary wave system (1.2) and satisfies condition (H_t) for every $t \in [0, T]$. The complex-valued function $u := T_p \eta + iT_q U$ then solves the following para-differential equation*

$$(2.1) \quad \partial_t u + T_V \cdot \nabla u + iT_{\gamma+\omega+\omega_1} u = f,$$

where, there exists a nondecreasing function $\mathcal{F} : \mathbf{R}^+ \times \mathbf{R}^+ \rightarrow \mathbf{R}^+$, in dependent of (η, ψ) , such that

$$(2.2) \quad \|f\|_{L^\infty([0, T]; H^s)} \leq \mathcal{F} \left(\|(\eta, \psi)\|_{L^\infty([0, T]; H^{s+\frac{1}{2}} \times H^s)} \right).$$

2.2. Symmetrization of system (1.2). Define first the principle symbol of the Dirichlet-Neumann operator

$$\lambda = \left((1 + |\nabla \eta|^2) |\xi|^2 - (\xi \cdot \nabla \eta)^2 \right)^{\frac{1}{2}}.$$

Next, set $\zeta = \nabla \eta$ and introduce

$$U_s := \langle D_x \rangle^s V + T_\zeta \langle D_x \rangle^s B, \quad \zeta_s := \langle D_x \rangle^s \zeta.$$

THEOREM 2.2 ([3, Proposition 4.10]). *Let $s > 1 + \frac{d}{2}$ and $(\eta, \psi) \in C^0([0, T]; H^{s+\frac{1}{2}} \times H^s)$ be a solution to the pure gravity water wave system (1.3) such that condition (H_t) is fulfilled for every $t \in [0, T]$ and the velocity trace*

$$(B, V) \in C^0([0, T]; H^{s+\frac{1}{2}} \times H^s);$$

and there exists $c_0 > 0$ such that $a(t, x) \geq c_0$. Then the complex-valued function

$$u := \langle D_x \rangle^{-s} (U_s - iT_{\sqrt{a/\lambda}} \zeta_s)$$

solves the following para-differential equation

$$(2.3) \quad \partial_t u + T_V \cdot \nabla u + iT_\gamma u = f,$$

where $\gamma = \sqrt{a\lambda}$ and

$$\|a - g\|_{L^\infty([0, T]; H^{s-\frac{1}{2}})} + \|f\|_{L^\infty([0, T]; H^s)} \leq \mathcal{F} \left(\|(\eta, \psi)(t)\|_{H^{s+\frac{1}{2}}}, \|(V, B)(t)\|_{H^s} \right).$$

REMARK 2.3. The change of variables $(\eta, \psi) \mapsto u$ in Theorem 2.1 and $(\eta, \psi, B, V) \mapsto u$ in Theorem 2.2 are essentially "invertible" in the sense that one can recover Sobolev estimates and Hölder estimates for (η, ψ, B, V) from those for u by virtue of the symbolic calculus for para-differential operators contained in Theorem A.4.

2.3. Para and pseudo differential operators. Since the para-differential setting is not suitable for proving dispersive estimates, we shall change it into the pseudo-differential one, whose standard definitions are recalled here.

DEFINITION 2.4. 1. For any $m \in \mathbf{R}$, $0 \leq \delta_1, \delta_2, \rho \leq 1$ we denote by $S_{\rho, \delta_1, \delta_2}^m$ the class of all symbols $a(x, y, \xi) : (\mathbf{R}^d)^3 \rightarrow \mathbf{C}$ satisfying

$$\left| \partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma a(x, y, \xi) \right| \leq C_{\alpha, \beta, \gamma} (1 + |\xi|)^{m + \delta_1 |\alpha| + \delta_2 |\beta| - \rho |\gamma|}.$$

The corresponding pseudo-differential operator is defined by

$$\text{Op}(a)u(x) = \int_{\mathbf{R}^d} e^{i(x-y)\xi} a(x, y, \xi) u(y) dy d\xi.$$

When $a : (\mathbf{R}^d)^2 \rightarrow \mathbf{C}$ we consider it as a symbol in $S_{\rho, \delta_1, 0}^m$ that does not depend on y and rename $S_{\rho, \delta_1, 0}^m \equiv S_{\rho, \delta_1}^m$.

2. For any symbol $a(x, \xi) \in S_{\rho, \delta}^m$ the Weyl quantization $\text{Op}^w(a) \equiv a^w(x, D_x)$ is defined by $\text{Op}^w(a)u(x) = \text{Op}(b)u(x)$ with $b(x, y, \xi) := a(\frac{x+y}{2}, \xi) \in S_{\rho, \delta, \delta}^m$.

We shall later need to transform the operators $\text{Op}(a)$ to $\text{Op}^w(a)$. This is done by means of the following result, which can be easily deduce from [24], Proposition 0.3.A.

PROPOSITION 2.5. For any symbol $a \in S_{\rho, \delta}^m$ with $m \in \mathbf{R}$, $0 \leq \delta < \rho \leq 1$ there exists a symbol $b \in S_{\rho, \delta}^m$ such that $\text{Op}^w(a) = \text{Op}(b)$. Moreover, we have the following asymptotic expansion in the sense of symbolic calculus:

$$b(x, \xi) \sim \sum_{|\alpha| \geq 0} \frac{(-i)^{|\alpha|}}{\alpha! 2^{|\alpha|}} \partial_x^\alpha \cdot \partial_\xi^\alpha a(x, \xi).$$

Remark that for all α , $\partial_x^\alpha \cdot \partial_\xi^\alpha a(x, \xi) \in S_{\rho, \delta}^{m - (\rho - \delta)|\alpha|}$.

Now, let $a \in \Gamma_r^m$, $r > 0$ be a para-differential symbol (see Definition A.2) and define

$$(2.4) \quad \forall j \in \mathbf{Z}, \forall \delta > 0, \quad S_{j\delta}(a)(x, \xi) = \psi(2^{-j\delta} D_x) a(x, \xi)$$

the spatial regularization of the symbol a , where ψ is the Littlewood-Paley function defined in (A.1). We first prove a Bernstein's type inequality for $S_{j\delta}(a)$.

LEMMA 2.6. If $a \in \Gamma_\rho^m$ then for all $\alpha, \beta \in \mathbf{N}^d$, $|\alpha| \geq \rho$, there exists a constant $C_{\alpha, \beta}$ such that for all $(x, \xi) \in \mathbf{R}^{2d}$

$$|\partial_x^\alpha \partial_\xi^\beta S_{j\delta}(a)(x, \xi)| \leq C_{\alpha, \beta} 2^{j(|\alpha| - \rho)} \|\partial_\xi^\beta a(\cdot, \xi)\|_{W^{\rho, \infty}(\mathbf{R}^d)}.$$

PROOF. If $|\alpha| = \rho$ the estimate is obvious by writing $\partial_x^\alpha \partial_\xi^\beta S_{j\delta}(a)$ as a convolution of $\partial_x^\alpha \partial_\xi^\beta a$ with a kernel. Considering now $|\alpha| > \rho$. Recall the dyadic partition of unity (A.2): $1 = \sum_{k=0}^{\infty} \Delta_k$ where each Δ_k is spectrally supported in the annulus $\{2^{k-1} \leq |\xi| \leq 2^{k+1}\}$. Using this partition, we can write

$$\partial_x^\alpha \partial_\xi^\beta S_{j\delta}(a)(x, \xi) = \sum_{k=0}^{+\infty} \Delta_k \partial_x^\alpha \psi(2^{-j\delta} D_x) \partial_\xi^\beta a(x, \xi) := \sum_{k=0}^{+\infty} u_k$$

If $\frac{1}{2}2^k \geq 2^{j\delta}$ then $\Delta_k \psi(2^{-j\delta} D_x) = 0$. Therefore

$$\partial_x^\alpha \partial_\xi^\beta S_{j\delta}(a)(x, \xi) = \sum_{k=0}^{2+[j\delta]} u_k.$$

Now, introducing $\varphi_1(\xi) \in C_c^\infty(\mathbf{R}^d)$, supported in $\{\frac{1}{3} \leq |\xi| \leq 3\}$ one has

$$u_k = 2^{k|\alpha|} \varphi_1(2^{-k} D_x) \psi(2^{-j\delta} D_x) \Delta_k \partial_\xi^\beta a(x, \xi).$$

Consequently,

$$\|u_k\|_{L^\infty(\mathbf{R}^d)} \leq 2^{k|\alpha|} \|\Delta_k D_\xi^\beta a(\cdot, \xi)\|_{L^\infty(\mathbf{R}^d)} \leq C 2^{k|\alpha|} 2^{-k\rho} \|\partial_\xi^\beta a(\cdot, \xi)\|_{W^{\rho, \infty}(\mathbf{R}^d)}.$$

It follows that

$$\|\partial_x^\alpha \partial_\xi^\beta S_{j\delta}(a)(x, \xi)\|_{L^\infty(\mathbf{R}^d)} \leq C \sum_{k=0}^{2+[j\delta]} 2^{k(|\alpha|-\rho)} \|D_\xi^\beta a(\cdot, \xi)\|_{W^{\rho, \infty}(\mathbf{R}^d)}.$$

Finally, since $|\alpha| - \rho > 0$ we deduce that

$$\|\partial_x^\alpha \partial_\xi^\beta S_{j\delta}(a)(x, \xi)\|_{L^\infty(\mathbf{R}^d)} \leq C 2^{j\delta(|\alpha|-\rho)} \|\partial_\xi^\beta a(\cdot, \xi)\|_{W^{\rho, \infty}(\mathbf{R}^d)},$$

which concludes the proof. \square

We show in the next Proposition that after localizing a distribution u in frequency, one can go from para-differential operators to pseudo-differential operators when acting on u .

PROPOSITION 2.7. *For every $j \in \mathbf{N}^*$, define*

$$R_j u := T_a \Delta_j u - S_{j-3}(a)(x, D_x) \Delta_j u.$$

Then the spectrum of $R_j u$ is contained in an annulus $\{c_1^{-1} 2^{j-1} \leq |\xi| \leq c_1 2^{j+1}\}$ and for every $\mu \in \mathbf{R}$ we have

$$\|R_j u\|_{H^{\mu-m+r}(\mathbf{R}^d)} \leq C M_r^m(a) \|u\|_{H^\mu(\mathbf{R}^d)}$$

where the constants $c_1, C > 0$ are independent of a, u, j .

PROOF. Recall first the definition (A.5) of $T_a u$, where we have $\varrho = 1$ on the support of φ_j for any $j \geq 1$, so

$$R_j u = T_a \Delta_j u - S_{j-3}(a)(x, D_x) \varrho(D_x) \Delta_j u.$$

In the following proof, we shall use the presentation of Métivier [22] on pseudodifferential and paradifferential operators. To be compatible with [22] we also abuse notations: by Γ_r^m we denote the class of symbols a satisfying the growth condition (A.3) for any $\xi \in \mathbf{R}^d$ and by M_0^m the semi-norm (A.4) where the supremum is taken over $\xi \in \mathbf{R}^d$.

1. By definition (A.5) it holds that $T_a v = \text{Op}(\sigma_a \varrho) v$, where $\text{Op}(\sigma_a \varrho)$ denotes the classical pseudodifferential operator with symbol

$$\sigma_a(x, \xi) \varrho(\xi) = \chi(D_x, \xi) a(x, \xi) \varrho(\xi).$$

Hence $R_j u = \text{Op}(a_j) u$ with

$$a_j(x, \xi) = \sigma_a(x, \xi) \varrho(\xi) \varphi_j(\xi) - S_{j-3}(a)(x, \xi) \varrho(\xi) \varphi_j(\xi).$$

Now, we write

$$a_j = (\sigma_a \varrho \varphi_j - a \varrho \varphi_j) + (a \varrho \varphi_j - S_{j-3}(a) \varrho \varphi_j) = a_j^1 + a_j^2.$$

Applying Proposition 5.8(ii) in [22] gives $a_j^1 \in \Gamma_0^{m-r}$ and (remark that $(\varphi_j)_j$ is bounded in Γ_r^0)

$$M_0^{m-r}(a_j^1) \leq C M_r^m(a \varrho \varphi_j) \leq C M_r^m(a \varrho).$$

On the other hand, if we denote $b = a\rho\varphi$ then $a_j^2(x, \xi) = b(x, \xi) - \psi_{j-3}(D_x, \xi)b(x, \xi)$. Taking into account the fact that $\text{supp } \varphi_j \subset B(0, C2^j)$ we may estimate

$$\begin{aligned} |a_j^2(x, \xi)| &\leq \sum_{k \geq j-2} |\Delta_j b(x, \xi)| \leq \sum_{k \geq j-2} 2^{-kr} \|b(\cdot, \xi)\|_{W^{r, \infty}} \\ &\leq C2^{-jr} \|b(\cdot, \xi)\|_{W^{r, \infty}} = C2^{-jr} |\varphi_j(\xi)| \|a(\cdot, \xi)\varrho(\xi)\|_{W^{r, \infty}} \\ &\leq C(1 + |\xi|)^{m-r} M_r^m(a\rho), \quad \forall \xi \in \mathbf{R}^d. \end{aligned}$$

By the same method for estimating $|\partial_\xi^\alpha a_j^2|$ we obtain that $a_j^2 \in \Gamma_0^{m-r}$ and hence $a_j \in \Gamma_0^{m-r}$; moreover

$$M_0^{m-r}(a_j) \leq CM_r^m(a\rho).$$

2. Property (A.7) implies in particular that

$$\mathfrak{F}_x(\sigma_a)(\eta, \xi) = 0 \text{ for } |\eta| \geq \varepsilon_2(1 + |\xi|),$$

here we denote \mathfrak{F}_x the Fourier transform with respect to the partial variable x . On the other hand, by definition of the smoothing operator

$$\mathfrak{F}_x S_{j-3}(a)(x, \xi)\varrho(\xi)\varphi_j(\xi) = \psi(2^{-(j-3)}\eta)\mathfrak{F}_x a(\eta, \xi)\varrho(\xi)\varphi(2^{-j}\xi)$$

which is vanishing if $|\eta| \geq \frac{1}{2}(1 + |\xi|)$. Indeed, if either $|\xi| > 2^{j+1}$ or $|\xi| \leq 2^{j-1}$ then $\varphi(2^{-j}\xi) = 0$. Considering $2^{j-1} < |\xi| \leq 2^{j+1}$ then $|\eta| \geq \frac{1}{2}(1 + |\xi|) > 2^{j-2}$ and thus $\psi(2^{-(j-3)}\eta) = 0$. We have proved the existence of $0 < \varepsilon < 1$ such that

$$(2.5) \quad \mathfrak{F}_x a_j(\eta, \xi) = 0 \text{ for } |\eta| \geq \varepsilon(1 + |\xi|).$$

3. By the spectral property (2.5) one can use the Bernstein's inequalities (see Corollary 4.1.7, [22]) to prove that a_j is a pseudodifferential symbol in the class $S_{1,1}^{m-r}$. Then, applying Theorem 4.3.5 in [22] we conclude that

$$\|R_j u\|_{H^{\mu-m+r}(\mathbf{R}^d)} = \|\text{Op}(a_j)u\|_{H^{\mu-m+r}(\mathbf{R}^d)} \leq CM_0^{m-r}(a_j) \|u\|_{H^\mu(\mathbf{R}^d)}.$$

Finally, the Fourier transform of $R_j u$ reads

$$\mathfrak{F}(R_j u)(\xi) = \int_{\mathbf{R}^d} \mathfrak{F}_x(a_j)(\xi - \eta, \eta)\hat{u}(\eta)d\eta.$$

Using the spectral localization property (2.5) and the fact that $\mathfrak{F}_x(a_j)(\xi - \eta, \eta)$ contains the factor $\varphi_j(\eta)$ we conclude that the spectrum of $R_j u$ is contained in an annulus of size 2^j as claimed. \square

2.4. A result of Koch-Tataru. In this paragraph, we recall the dispersive estimates proved by Koch-Tataru [19] based on the technique of FBI transform on phase space. These estimates were established for the following class of operators.

DEFINITION 2.8. For $\lambda > 1$, $m \in \mathbf{R}$ and $k = 0, 1, \dots$ consider classes of symbols $p : T^*\mathbf{R}^d \rightarrow \mathbf{C}$, denoted by $\lambda^m S_\lambda^k$, which satisfy

$$(2.6) \quad \begin{aligned} \left| \partial_x^\alpha \partial_\xi^\beta p(x, \xi) \right| &\leq c_{\alpha, \beta} \lambda^{m-|\beta|}, \quad |\alpha| \leq k, \\ \left| \partial_x^\alpha \partial_\xi^\beta p(x, \xi) \right| &\leq c_{\alpha, \beta} \lambda^{m + \frac{|\alpha|-k}{2} - |\beta|}, \quad |\alpha| \geq k. \end{aligned}$$

The mentioned result reads

PROPOSITION 2.9 ([19, Proposition 4.7]). Let $p(\sigma, x, \xi) \in \lambda S_\lambda^2$ be a real symbol in (x, ξ) , uniformly in $\sigma \in [0, 1]$. Assume that p satisfies the following curvature condition

(A) for each $(\sigma, x, \xi) \in [0, 1] \times \mathbf{R}^d \times \mathcal{C}_\lambda$, $|\det \partial_\xi^2 p| \gtrsim \lambda^{-d}$, where $\mathcal{C}_\lambda = \{c^{-1}\lambda \leq |\xi| \leq$

$c\lambda\}$.

Denote by $S(\sigma, \sigma_0)$ the flow maps of $D_\sigma + \text{Op}^w(p)$. Then for any $\chi \in S_\lambda^0$ such that for all $x \in \mathbf{R}^d$, $\chi(x, \cdot)$ compactly supported in $\mathcal{C}'_\lambda = \{c'^{-1}\lambda \leq |\xi| \leq c'\lambda\}$ with $1 < c' < c$, we have

$$\|S(\sigma, \sigma_0)(\chi(x, D_x)v_0)\|_{L^\infty} \lesssim \lambda^{\frac{d}{2}} |\sigma - \sigma_0|^{\frac{-d}{2}} \|v_0\|_{L^1}, \quad \forall \sigma, \sigma_0 \in [0, 1].$$

REMARK 2.10. In the statement of Proposition 4.7, [19], condition **(A)** is stated for $(x, \xi) \in B_\lambda := \{|x| \leq 1, |\xi| \leq \lambda\}$ and correspondingly, χ is supported in B ; in addition, the usual quantization $\chi(x, D_x)$ is replaced by the Weyl quantization χ^w . However, one can inspect easily its proof to see that if **(A)** is fulfilled globally in x then we have the above variant.

2.5. Remarks on the symbolic calculus for $\lambda^m S_\lambda^k$. Let $a \in \lambda^m S_m^k$, $k \geq 0$ and suppose that on the support of $a(x, \xi)$, $\lambda^{-1}|\xi| \sim 1$. It follows by definition of $\lambda^m S_\lambda^k$ that $a \in S_{1, \frac{1}{2}}^m$. Observe however that when $k \geq 1$, a behaves better than a general symbol in $S_{1, \frac{1}{2}}^m$. In this paragraph we present some enhanced properties of $\lambda^m S_\lambda^k$ with $k \geq 1$.

First, we are concerned with the relation between the the Weyl quantization and the usual quantization. According to Proposition 2.5, for $a \in S_{1, \frac{1}{2}}^m$ there holds

$$\text{Op}^w(a) - \text{Op}(a) = \text{Op}(r), \quad r \in S_{1, \frac{1}{2}}^{m-\frac{1}{2}}.$$

In fact, we have

$$\text{Op}^w(a) = \text{Op}(\tilde{a}), \quad \tilde{a}(x, y, \xi) = a\left(\frac{x+y}{2}, \xi\right)$$

and

$$\tilde{a}(x, y, \xi) = a\left(x + \frac{y-x}{2}, \xi\right) = a(x, \xi) + \frac{1}{2} \int_0^1 (\partial_x a)\left(x + s\frac{y-x}{2}, \xi\right) ds (y-x).$$

It follows that

$$(2.7) \quad \text{Op}^w(a) - \text{Op}(a) = \text{Op}(r),$$

with

$$(2.8) \quad r(x, y, \xi) = -\frac{i}{2} \int_0^1 (\partial_\xi \partial_x a)\left(x + s\frac{y-x}{2}, \xi\right) ds.$$

We now show that in fact r is of order $m-1$ as in the case $a \in S_{1,0}^m$.

LEMMA 2.11. *Let $a \in \lambda^m S_\lambda^k$, $k \geq 1$ satisfying $\lambda^{-1}|\xi| \sim 1$ on the support of $a(x, \xi)$. Then we have the relation (2.7)-(2.8) with $r \in S_{1, \frac{1}{2}, \frac{1}{2}}^{m-1}$.*

PROOF. For any $\alpha, \beta, \nu \in \mathbf{N}^d$ we have

$$|\partial_x^\alpha \partial_y^\beta \partial_\xi^\nu r(x, y, \xi)| \leq \begin{cases} C_{\alpha, \beta, \nu} \lambda^{m-|\nu|-1} 1_{\lambda^{-1}|\xi| \sim 1}(\xi), & \text{if } |\alpha| + |\beta| + 1 \leq k, \\ C_{\alpha, \beta, \nu} \lambda^{m + \frac{|\alpha|+1+|\beta|-k}{2} - |\nu|-1} 1_{\lambda^{-1}|\xi| \sim 1}(\xi), & \text{if } |\alpha| + |\beta| + 1 \geq k. \end{cases}$$

Since $k \geq 1$,

$$\frac{|\alpha| + 1 + |\beta| - k}{2} \leq \frac{|\alpha| + |\beta|}{2}$$

Consequently, it holds that

$$|\partial_x^\alpha \partial_y^\beta \partial_\xi^\nu r(x, y, \xi)| \leq C_{\alpha, \beta, \nu} (1 + |\xi|)^{(m-1) + \frac{|\alpha|+|\beta|}{2} - |\nu|}, \quad \forall \alpha, \beta, \nu \in \mathbf{N}^d;$$

in other words, $r \in S_{1, \frac{1}{2}, \frac{1}{2}}^{m-1}$. □

For the composition rule we prove

LEMMA 2.12. *Let $p \in S_{1,0}^n$ and $a \in \lambda^m S_\lambda^k$, $k \geq 1$ satisfying $\lambda^{-1}|\xi| \sim 1$ on the support of $a(x, \xi)$. Then we have*

$$\text{Op}(p) \circ \text{Op}(a) - \text{Op}(pa) = \text{Op}(r)$$

with $r \in S_{1,\frac{1}{2}}^{n+m-1}$.

PROOF. According to Proposition 0.3.C [24], one has $\text{Op}(p) \circ \text{Op}(a) = \text{Op}(b)$ with

$$b \sim \sum_{|\alpha| \geq 0} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha p(x, \xi) \partial_x^\alpha a(x, \xi)$$

in the sense of symbol asymptotic. The general term in the above expansion belongs to $S_{1,\frac{1}{2}}^{n+m-\frac{|\alpha|}{2}}$, hence

$$(b - pa) - \sum_{|\alpha|=1} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha p(x, \xi) \partial_x^\alpha a(x, \xi) \in S_{1,\frac{1}{2}}^{n+m-1}.$$

It then suffices to show that $c_\alpha := \partial_\xi^\alpha p(x, \xi) \partial_x^\alpha a(x, \xi) \in S_{1,\frac{1}{2}}^{n+m-1}$ for $|\alpha| = 1$ or again, $\partial_x^\alpha a(x, \xi) \in S_{1,\frac{1}{2}}^m$ for $|\alpha| = 1$. The later follows along the same lines as in the proof of Lemma 2.11. \square

In the same spirit we have the following result on adjoint operators, taking into account Proposition 0.3.B [24].

LEMMA 2.13. *Let $a \in \lambda^m S_\lambda^k$, $k \geq 1$ satisfying $\lambda^{-1}|\xi| \sim 1$ on the support of $a(x, \xi)$. Then we have*

$$\text{Op}^*(a) - \text{Op}(\bar{a}) = \text{Op}(r)$$

with $r \in S_{1,\frac{1}{2}}^{m-1}$ and \bar{a} is the complex conjugate of a .

NOTATION 2.14. Throughout this article, we write $A \lesssim B$ if there exists a constant $C > 0$ such that $A \leq CB$, where C may depend on the coefficients of the equations under consideration. If the constant C involved has some explicit dependency, say, on some quantity μ , we emphasize it by denoting $A \lesssim_\mu B$.

3. Proof of Theorem 1.3

Throughout this section, the dimension d is greater than or equal 2 and $s > \frac{5}{2} + \frac{d}{2}$ is a Sobolev index.

3.1. Littlewood-Paley reduction. We shall prove Strichartz estimate for solution u to (2.1), which is a quasilinear para-differential equation with time-dependent coefficients. Remark that since

$$(\eta, \psi) \in C^0([0, T]; H^{s+\frac{1}{2}} \times H^s),$$

we have

$$V \in C^0([0, T]; H^{s-1}), \quad \gamma(\cdot, \xi) \in C^0([0, T]; H^{s-\frac{1}{2}}).$$

The first step in our proof consists in localizing (2.1) at frequency 2^j using the Littlewood-Paley decomposition (cf. Definition A.1 1.). For every $j \geq 0$, the dyadic piece $\Delta_j u$ solves

$$(3.1) \quad (\partial_t + T_V \cdot \nabla + iT_{\gamma+\omega}) \Delta_j u = F_j,$$

with

$$(3.2) \quad F_j^1 := \Delta_j f - i\Delta_j(T_{\omega_1}u) + i[T_\gamma, \Delta_j] + i[T_\omega, \Delta_j]u + [T_V, \Delta_j] \cdot \nabla u.$$

Remark that for each $j \geq 1$, $\Delta_j u$ is spectrally supported in $\{2^{j-1} \leq |\xi| \leq 2^{j+1}\}$. In views of Proposition 2.7 and the fact that $\gamma \in \Gamma_{\frac{3}{2}}^{\frac{3}{2}}$, $\omega \in \Gamma_{\frac{1}{2}}^{\frac{1}{2}}$ and $V \cdot \xi \in \Gamma_1^1$, equation (3.1) can be rewritten as

$$(3.3) \quad (\partial_t + S_{j-3}(V) \cdot \nabla + iS_{j-3}(\gamma)(x, D_x)) \Delta_j u = F_j^2,$$

with

$$(3.4) \quad F_j^2 := F_j^1 + R_j,$$

$R_j u$ is spectrally supported in an annulus $\{c_1^{-1}2^{j-1} \leq |\xi| \leq c_1 2^{j+1}\}$ and satisfies

$$(3.5) \quad \|R_j\|_{H^s} \lesssim \|u\|_{H^s}.$$

Next, we use (2.4) to smooth out the symbols by $\delta = \frac{1}{2}$ derivatives.

Now, let $\frac{1}{2} < c_1 < c_2 < c_3$, $\mathcal{C}^k := \{(2c_k)^{-1} \leq |\xi| \leq 2c_k\}$, $k = 1, 2, 3$ and

$$\tilde{\varphi} \in C^\infty, \text{ supp } \tilde{\varphi} \subset \mathcal{C}^3, \tilde{\varphi} \equiv 1 \text{ on } \mathcal{C}^2.$$

Then, equation (3.3) is equivalent to

$$(3.6) \quad \begin{aligned} \mathcal{L}_j \Delta_j u := & \left(\partial_t + S_{(j-3)\frac{1}{2}}(V) \cdot \nabla \tilde{\varphi}(2^{-j} D_x) \right. \\ & \left. + iS_{(j-3)\frac{1}{2}}(\gamma)(x, D_x) \tilde{\varphi}(2^{-j} D_x) + iS_{(j-3)\frac{1}{2}}(\omega)(x, D_x) \tilde{\varphi}(2^{-j} D_x) \right) \Delta_j u = F_j, \end{aligned}$$

with

$$(3.7) \quad \begin{aligned} F_j = F_j^2 + F_j^3 := & F_j^2 + i \left(S_{(j-3)\frac{1}{2}}\gamma(x, D_x) - S_{j-3}\gamma(x, D_x) \right) \Delta_j u \\ & + i \left(S_{(j-3)\frac{1}{2}}\omega(x, D_x) - S_{j-3}\omega(x, D_x) \right) \Delta_j u + \left(S_{(j-3)\frac{1}{2}}(V) - S_{j-3}(V) \right) \cdot \nabla \Delta_j u. \end{aligned}$$

3.2. Semi-classical time scale. Observe that the highest order operator on the left-hand side of (3.3) has order $\frac{3}{2}$, which does not match the result in [19] that we want to apply. Therefore, we reduce it to an operator of order 1 by multiplying both side by $h^{\frac{1}{2}}$, where

$$h := 2^{-j},$$

then making a change of temporal variables $t = h^{\frac{1}{2}}\sigma$. For this purpose, let us reset the symbols in this new time scale:

$$(3.8) \quad \Gamma_h(\sigma, x, \xi) = h^{\frac{1}{2}} S_{(j-3)\frac{1}{2}}(\gamma)(h^{\frac{1}{2}}\sigma, x, \xi) \tilde{\varphi}(h\xi),$$

$$(3.9) \quad \Omega_h(\sigma, x, \xi) = h^{\frac{1}{2}} S_{(j-3)\frac{1}{2}}(\omega)(h^{\frac{1}{2}}\sigma, x, \xi) \tilde{\varphi}(h\xi),$$

$$(3.10) \quad V_h(\sigma, x, \xi) = h^{\frac{1}{2}} S_{(j-3)\frac{1}{2}}(V)(h^{\frac{1}{2}}\sigma, x) \cdot \xi \tilde{\varphi}(h\xi).$$

Next, set

$$w_h(\sigma, x) = \Delta_j u(h^{\frac{1}{2}}\sigma, x), \quad G_h(\sigma, x) = -ih^{\frac{1}{2}} F_j(h^{\frac{1}{2}}\sigma, x).$$

Equation (3.6) is then equivalent to

$$(3.11) \quad (D_t + \Gamma_h(\sigma, x, D_x) + \Omega_h(\sigma, x, D_x) + V_h(\sigma, x, D_x)) w_h(\sigma, x) = G_h(\sigma, x).$$

In what follows, we shall prove the classical Strichartz estimate for (3.11), from which the semi-classical Strichartz estimate for (3.6) follows.

We now replace the pseudo-differential operators in (3.11) by the corresponding Weyl operators using Proposition 2.5. Noticing that $\omega = \frac{i}{2}\gamma$, we set

$$(3.12) \quad \begin{aligned} \text{Op}^w(\Gamma_h) &= \text{Op}(\Gamma_h) + \text{Op}(\Omega_h) + R_h^1, \\ \text{Op}^w(V_h) &= \text{Op}(V_h) + R_h^2. \end{aligned}$$

Inserting these identities into (3.11) leads to

$$(3.13) \quad \begin{aligned} L_h w_h(\sigma, x) &:= (D_t + \Gamma_h^w(\sigma, x, D_x) + V_h^w(\sigma, x, D_x)) w_h(\sigma, x) \\ &= G_h(\sigma, x) + R_h^1(\sigma, x) w_h(\sigma, x) + R_h^2(\sigma, x) w_h(\sigma, x). \end{aligned}$$

3.3. Classical Strichartz estimate for (3.11) (\Leftrightarrow (3.13)). In this step, we shall show that Theorem 2.9 can be applied to the real symbol

$$p_h := \Gamma_h + V_h.$$

Set $\lambda = h^{-1} = 2^j$. First, the characteristic set of γ has maximal (d) nonvanishing curvatures:

PROPOSITION 3.1. *Let \mathcal{C} be a fixed annulus in \mathbf{R}^d .*

1. *There exists an absolute constant $C_d > 0$ such that with $c_0 = C_d(1 + \|\nabla\eta\|_{L^\infty(I \times \mathbf{R}^d)})$ we have*

$$(3.14) \quad \sup_{(t,x,\xi) \in I \times \mathbf{R}^d \times \mathcal{C}} |\det \partial_\xi^2 \gamma(t, x, \xi)| \geq c_0.$$

2. *For any $0 < \delta \leq 1$ there exists $j_0 \in \mathbf{N}$ large enough such that*

$$(3.15) \quad \sup_{(t,x,\xi) \in I \times \mathbf{R}^d \times \mathcal{C}} |\det \partial_\xi^2 S_{j\delta}(\gamma)(t, x, \xi)| \geq c_0.$$

PROOF. 1. For the proof of part 1, we refer to Corollary 4.7, [4]. Part 2. is a consequence of part 1. because $S_{j\delta}(\gamma)$ is a small perturbation of γ when j is large enough (see for instance Proposition 4.5, [4]). \square

LEMMA 3.2. 1. *We have $\Gamma_h \in \lambda S_\lambda^2$, $V_h \in \lambda^{\frac{3}{4}} S_\lambda^2$ and hence $p_h \in \lambda S_\lambda^2$.*

2. *There exists $h_0 > 0$ small enough such that for $0 < h \leq h_0$, the symbol p_h satisfies condition (A) in Theorem 2.9 with $\mathcal{C}_\lambda = \lambda \mathcal{C}^2$.*

PROOF. 1. Remark that since γ and V are respectively $W^{2,\infty}$ and $W^{\frac{3}{2},\infty}$ in x , assertion 1. then follows easily from Lemma 2.6 and the fact that on the support of $\tilde{\varphi}(h\xi)$ we have $|\xi| \sim \lambda$.

2. Let $\xi \in \lambda \mathcal{C}^2$. We have $\tilde{\varphi}(\tilde{h}\xi) = 1$, hence $\partial_\xi^2 V_h$ vanishes and since γ is homogeneous of order $\frac{3}{2}$ in ξ ,

$$\partial_\xi^2 \Gamma_h(\sigma, x, \xi) = \partial_\xi^2 \left(h^{\frac{1}{2}} S_{(j-3)\frac{1}{2}}(\gamma)(h^{\frac{1}{2}}\sigma, x, \xi) \right) = h \left(\partial_\xi^2 S_{(j-3)\frac{1}{2}}(\gamma) \right) (h^{\frac{1}{2}}\sigma, x, h\xi).$$

Therefore, condition (A) is verified by virtue of (3.15). \square

Calling $S_h(\sigma, \sigma_0)$ the flow map of the evolution operator $L_h = D_t + \text{Op}^w(p_h)$ (see (3.13)), we have

LEMMA 3.3. *If v_h^0 is spectrally supported in $\lambda \mathcal{C}^1 = \{(2c_1)^{-1}h^{-1} \leq |\xi| \leq 2c_1 h^{-1}\}$ then (i)*

$$\|S_h(\sigma, \sigma_0)v_h^0\|_{L^\infty(\mathbf{R}^d)} \lesssim h^{-\frac{d}{2}} |\sigma - \sigma_0|^{-\frac{d}{2}} \|v_h^0\|_{L^1(\mathbf{R}^d)},$$

for all $\sigma, \sigma_0 \in [0, 1]$ and $0 < h \leq h_0$.
(ii) with $q > 2$ and $\frac{2}{q} = \frac{d}{2} - \frac{d}{r}$,

$$\|S_h(\cdot, 0)v_h^0\|_{L^q([0,1], L^r)} \lesssim h^{\frac{1}{q}} \|v_h^0\|_{L^2}.$$

PROOF. By Lemma 3.2, (i) is a direct consequence of Proposition 2.9 if one chooses

$$\chi(\xi) \in C^\infty, \text{ supp } \chi \subset \{(2c_{1,2})^{-1}\lambda \leq |\xi| \leq 2c_{1,2}\lambda\}, \quad c_1 < c_{1,2} < c_2, \quad \chi \equiv 1 \text{ in } \mathcal{C}^1.$$

For (ii) we remark that since $\text{Op}^w(\Gamma_h)$ and $\text{Op}^w(V_h)$ are self-adjoint, $S_h(\sigma, \sigma_0)$ is isometric in L^2 . This combining with the dispersive estimate (i) and a standard TT^* argument (see the abstract semi-classical Strichartz estimate in Theorem 10.7, [27]) yields (ii). \square

LEMMA 3.4. For any $\mu \in \mathbf{R}$, the operators $S_h(\sigma, \tau)$ are bounded on $H^\mu(\mathbf{R}^d)$ uniformly in $t, s \in I$.

PROOF. This result bases on a standard energy estimate. However, the proof requires more care since we are not working on standard operators of classes $S_{1,0}^m$. Without loss of generality we assume $\tau = 0$ and let $f(t, x)$ be a solution of

$$(\partial_t + i \text{Op}^w(p_h)) f(t, x) = 0, \quad f(0, x) = f_0(x).$$

We first apply Lemma 2.11 to obtain

$$\text{Op}^w(p_h) = \text{Op}(p_h) + \text{Op}(r_h), \quad r_h \in S_{1, \frac{1}{2}, \frac{1}{2}}^0.$$

Then f solves the problem

$$(\partial_t + i \text{Op}(p_h) + i \text{Op}(r_h)) f(t, z) = 0, \quad f(0, x) = f_0(x).$$

Let $\mu \in \mathbf{R}$ and set $f^\mu := \langle D_z \rangle^\mu f$ then

$$\frac{d}{dt} \|f^\mu\|_{L^2}^2 = -i \langle (\text{Op}(p_h) - \text{Op}^*(p_h)) f^\mu, f^\mu \rangle + 2\Re \langle F, f^\mu \rangle$$

where

$$F := i [\text{Op}(p_h), \langle D_z \rangle^\mu] f - i \langle D_z \rangle^\mu \text{Op}(r_h) f.$$

According to Lemma 2.12 one has $[\text{Op}(p_h), \langle D_z \rangle^\mu] \in S_{1, \frac{1}{2}}^\mu$. This combining with the fact that $r_h \in S_{1, \frac{1}{2}, \frac{1}{2}}^0$ gives

$$\|F\|_{L^2} \lesssim \|f\|_{H^\mu}.$$

On the other hand, since p_h is real, Lemma 2.13 implies

$$\text{Op}(p_h) - \text{Op}^*(p_h) \in S_{1, \frac{1}{2}}^0.$$

Consequently, we have

$$\|(\text{Op}(p_h^0) - \text{Op}^*(p_h^0)) f^\mu\|_{L^2} \lesssim \|f^\mu\|_{L^2}.$$

Finally, we conclude by Gronwall's inequality that

$$\|f(t)\|_{H^\mu} \lesssim \|f_0\|_{H^\mu}, \quad \forall t \in I.$$

\square

PROPOSITION 3.5. If w_h is a solution to (3.11) with data $w_h(0) = w_h^0$ and

$$\text{supp } \widehat{w}_h, \text{ supp } \widehat{w}_h^0 \subset \lambda \mathcal{C}^1$$

then we have for all $\varepsilon > 0$

$$\|w_h\|_{L^{2+\varepsilon}([0,1], W^{s-\frac{d}{2}+\frac{1}{2}-\varepsilon, \infty})} \lesssim_\varepsilon \|w_h^0\|_{H^s} + \|G_h\|_{L^1([0,1], H^s)} + h^{\frac{1}{2}} \|w_h\|_{L^1([0,1], H^s)}.$$

PROOF. To simplify notations, let us write $S_h(\sigma, \tau) = S(\sigma, \tau)$. If w_h is a solution to (3.11) and it is also a solution to (3.13). By Duhamel's formula, there holds

$$w_h(\sigma, 0) = S_h(\sigma, 0)w_h^0 + \int_0^\sigma S(\sigma, \tau)[G_h(\tau)]d\tau + \int_0^\sigma S(\sigma, \tau)[(R_h^1 w_h + R_h^2 w_h)(\tau)]d\tau.$$

Let us call (I) and (II), respectively, the first and the second integral on the right-hand side. Choosing c_1 large enough such that $\text{supp } \widehat{G}_h \subset \lambda\mathcal{C}^1$, Lemma 3.3 (ii) gives

$$\|(I)\|_{L^q([0,1], L^r)} \lesssim h^{-\frac{1}{q}} \|G_h\|_{L^1([0,1]; L^2)}$$

For (II) we set

$$b_h^\alpha = \frac{(-i)^{|\alpha|}}{\alpha!2^{|\alpha|}} \partial_x^\alpha \partial_\xi^\alpha \Gamma_h(\sigma, x, \xi), \quad |\alpha| \geq 2; \quad c_h^\alpha = \frac{(-i)^{|\alpha|}}{\alpha!2^{|\alpha|}} \partial_x^\alpha \partial_\xi^\alpha V_h(\sigma, x, \xi), \quad |\alpha| \geq 1.$$

For each $|\alpha| \geq 2$, since γ is $W^{2,\infty}$ in x we have by applying Lemma 2.6

$$\|\partial_x^\mu \partial_\xi^\nu (\partial_x^\alpha \partial_\xi^\alpha \Gamma_h)\| \lesssim (1 + |\xi|)^{1 + \frac{|\mu| + |\alpha| - 2}{2} - (|\nu| + |\alpha|)} \lesssim (1 + |\xi|)^{-\frac{|\alpha|}{2} + \frac{|\mu|}{2} - |\nu|},$$

hence $b_h^\alpha \in S_{1, \frac{1}{2}}^{-\frac{|\alpha|}{2}}$. Similarly, it holds that $c_h^\alpha \in S_{1, \frac{1}{2}}^{-\frac{|\alpha|}{2}}$ for $V \in W^{1,\infty}$. Taking $q > 2$ and $\frac{2}{q} = \frac{d}{2} - \frac{d}{r}$, we claim that uniformly in $\tau \in [0, 1]$

$$(3.16) \quad \|S(\sigma, \tau)[(R_h^1 w_h)(\tau)]\|_{L_\sigma^q L_x^r} \lesssim h^{-\frac{1}{q} + 1} \|w_h(\tau)\|_{L_x^2}.$$

Indeed, by the asymptotic expansion in Proposition 2.5 it holds that

$$R_h^1 = \sum_{|\alpha|=2}^{N-1} \text{Op}(b_h^\alpha) + \text{Op}(r_h^N), \quad r_h^N \in S_{1, \frac{1}{2}}^{1-\frac{N}{2}}, \quad \forall N \geq 3.$$

If we choose c_1 large enough then each $\text{Op}(b_h^\alpha w_h)(\tau)$ is spectrally supported in $\lambda\mathcal{C}^1$ (and so is w_h) so that Lemma 3.3 (ii) can be applied to derive

$$(3.17) \quad \|S(\sigma, \tau)[\text{Op}(b_h^\alpha)w_h(\tau)]\|_{L_\sigma^q L_x^r} \leq h^{-\frac{1}{q}} \|\text{Op}(b_h^\alpha)w_h\|_{L_x^2} \lesssim h^{-\frac{1}{q} + \frac{|\alpha|}{2}} \|w_h(\tau)\|_{L_x^2}.$$

For $\text{Op}(r_h^N)$ one uses the Sobolev embedding $H^{\frac{d}{2}} \hookrightarrow L^r$, $\forall r \in [2, +\infty)$ to estimate

$$\|S(\sigma, \tau)[\text{Op}(r_h^N)w_h(\tau)]\|_{L_x^r} \lesssim \|S(\sigma, \tau)[\text{Op}(r_h^N)w_h(\tau)]\|_{H_x^{\frac{d}{2}}}.$$

On the other hand, we know from Lemma 3.4 that $S(\sigma, \tau)$ is bounded from H^μ to H^μ uniformly in $\sigma, \tau \in [0, 1]$ for all $\mu \in \mathbf{R}$. Hence

$$(3.18) \quad \|S(\sigma, \tau)[\text{Op}(r_h^N)w_h(\tau)]\|_{L_x^r} \lesssim h^{-1 + \frac{N}{2} - \frac{d}{2}} \|w_h(\tau)\|_{L_x^2}.$$

Choosing $N = N(d)$ large enough, we conclude the claim (3.16) from (3.17) and (3.18).

In the same way, we obtain the following estimate for R_h^2 (which is also uniformly in $\tau \in [0, 1]$)

$$\|S(\sigma, \tau)[(R_h^2 w_h)(\tau)]\|_{L_\sigma^q L_x^r} \lesssim h^{-\frac{1}{q} + \frac{1}{2}} \|w_h(\tau)\|_{L_x^2}.$$

Putting together the estimates above leads to

$$\|w_h\|_{L^q L^r} \leq h^{-\frac{1}{q}} \left(\|w_h^0\|_{L^2} + \|G_h\|_{L^1([0,1]; L^2)} + h^{\frac{1}{2}} \|w_h\|_{L^1 L^2} \right).$$

Taking $q = 2 + \varepsilon$ then $h^{-\frac{1}{q}} \leq h^{-\frac{1}{2}}$. We multiply both sides by h^{-s} and use the frequency localization of w_h , w_h^0 to get

$$\|w_h\|_{L^{2+\varepsilon} W^{s-\frac{1}{2}, r}} \lesssim \varepsilon \|w_h^0\|_{H^s} + \|G_h\|_{L^1 H^s} + h^{\frac{1}{2}} \|w_h\|_{L^1 H^s}.$$

Finally $s - \frac{1}{2} = (\frac{d}{2} - 1 + \varepsilon) + (s - \frac{d}{2} + \frac{1}{2} - \varepsilon)$ and $\frac{d}{2} - 1 + \varepsilon > \frac{d}{2} - 1 + \frac{\varepsilon}{2 + \varepsilon} = \frac{d}{r}$. Consequently, the Sobolev embedding $W^{s - \frac{1}{2}, r} \hookrightarrow W^{s - \frac{d}{2} + \frac{1}{2} - \varepsilon, \infty}$ concludes the proof. \square

3.4. Semi-classical Strichartz estimate for (3.6). From the preceding Proposition, one deduces the corresponding Strichartz estimate for $u_j \equiv \Delta_j u$ as a solution of (3.6) via the change of temporal variables $w_h(\sigma, x) = u_j(h^{\frac{1}{2}}\sigma, x)$ as follows.

COROLLARY 3.6. *If u_j is solution to (3.6), i.e., $\mathcal{L}_j u_j = F_j$ with data u_j^0 and u_j, u_j^0, F_j are spectrally supported in $2^j \mathcal{C}^1$ then u_j satisfies with $I_j = [0, 2^{-\frac{j}{2}}] = [0, h^{\frac{1}{2}}]$ and $\varepsilon > 0$*

$$\|u_j\|_{L^{2+\varepsilon}(I_j; W^{s - \frac{d}{2} + \frac{3}{4} - \varepsilon, \infty})} \lesssim_{\varepsilon} \|u_j^0\|_{H^s} + \|F_j\|_{L^1(I_j; H^s)} + \|u_j\|_{L^1 H^s}.$$

The next step consists in gluing the estimates on small time scales above to derive an estimate on the whole interval of time $[0, 1]$ to the price of loosing $\frac{1}{4}$ derivatives.

COROLLARY 3.7. *If u_j is solution to $\mathcal{L}_j u_j = F_j$ with data u_j^0 and u_j, u_j^0, F_j are spectrally supported in \mathcal{C}_h^1 then u_j satisfies with $I = [0, T]$ and $\varepsilon > 0$*

$$\|u_j\|_{L^2(I; W^{s - \frac{d}{2} + \frac{1}{2} - \varepsilon, \infty})} \lesssim_{\varepsilon} \|F_j\|_{L^2(I; H^{s - \frac{1}{2}})} + \|u_j\|_{L^\infty(I, H^s)}.$$

PROOF. Let $\chi \in C_0^\infty(0, 2)$ equal to one on $[\frac{1}{2}, \frac{3}{2}]$. For $0 \leq k \leq [Th^{-\frac{1}{2}}] - 2$ define

$$I_{j,k} = [kh^{\frac{1}{2}}, (k+2)h^{\frac{1}{2}}], \quad \chi_{j,k}(t) = \chi\left(\frac{t - kh^{\frac{1}{2}}}{h^{\frac{1}{2}}}\right), \quad u_{j,k} = \chi_{j,k}(t)u_j.$$

Then each $u_{j,k}$ is a solution to

$$\mathcal{L}_j u_{j,k} = \chi_{j,k} F_j + h^{-\frac{1}{2}} \chi' \left(\frac{t - kh^{\frac{1}{2}}}{h^{\frac{1}{2}}} \right) u_j, \quad u_{j,k}(t = kh^{\frac{1}{2}}) = 0,$$

from which we deduce by virtue of Corollary 3.6

$$\|u_{j,k}\|_{L^2(I_{j,k}; W^{s - \frac{d}{2} + \frac{3}{4} - \varepsilon, \infty})} \lesssim_{\varepsilon} \|F_j\|_{L^1(I_{j,k}; H^s)} + h^{-\frac{1}{2}} \|u_j\|_{L^1(I_{j,k}; H^s)}.$$

Notice that $\chi_{j,k} = 1$ on $\left((k + \frac{1}{2})h^{\frac{1}{2}}, (k + \frac{3}{2})h^{\frac{1}{2}}\right)$ we get

$$\begin{aligned} \|u_{j,k}\|_{L^2((k + \frac{1}{2})h^{\frac{1}{2}}, (k + \frac{3}{2})h^{\frac{1}{2}}); W^{s - \frac{d}{2} + \frac{3}{4} - \varepsilon, \infty})} &\lesssim_{\varepsilon} \|F_j\|_{L^1(I_{j,k}; H^s)} + h^{-\frac{1}{2}} \|u_j\|_{L^1(I_{j,k}; H^s)} \\ &\lesssim_{\varepsilon} h^{\frac{1}{4}} \|F_j\|_{L^2(I_{j,k}; H^s)} + h^{-\frac{1}{4}} \|u_j\|_{L^2(I_{j,k}; H^s)}. \end{aligned}$$

Squaring both sides of the above inequality and then summing in $0 \leq k \leq [Th^{-\frac{1}{2}}] - 2 =: N_h$ yields with $J_j := [\frac{1}{2}h^{\frac{1}{2}}, (N_h - \frac{1}{2})h^{\frac{1}{2}}]$

$$\|u_j\|_{L^2(J_j; W^{s - \frac{d}{2} + \frac{3}{4} - \varepsilon, \infty})} \lesssim_{\varepsilon} h^{\frac{1}{4}} \|F_j\|_{L^2(I; H^s)} + h^{-\frac{1}{4}} \|u_j\|_{L^2(I; H^s)}$$

or, equivalently after multiplying by $h^{\frac{1}{4}}$

$$(3.19) \quad \|u_j\|_{L^2(J_j; W^{s - \frac{d}{2} + \frac{1}{2} - \varepsilon, \infty})} \lesssim_{\varepsilon} \|F_j\|_{L^2(I; H^{s - \frac{1}{2}})} + \|u_j\|_{L^2(I; H^s)}.$$

On the other hand, on $J = [0, \frac{1}{2}h^{\frac{1}{2}}]$ one can apply Corollary 3.6 and Hölder's inequality to have

$$(3.20) \quad \begin{aligned} \|u_j\|_{L^2(J; W^{s-\frac{d}{2}+\frac{1}{2}-\varepsilon, \infty})} &\lesssim h^{\frac{1}{4}} \|u_j\|_{L^2(J; W^{s-\frac{d}{2}+\frac{3}{4}-\frac{\varepsilon}{2}, \infty})} \\ &\lesssim_{\varepsilon} h^{\frac{1}{4}} \|u_j^0\|_{H^s} + h^{\frac{1}{2}} \|F_j\|_{L^2(J; H^s)} + h^{\frac{1}{2}} \|u_j\|_{L^2(J; H^s)} \\ &\lesssim_{\varepsilon} \|u_j\|_{L^\infty(I; H^{s-\frac{1}{4}})} + \|F_j\|_{L^2(I; H^{s-\frac{1}{2}})} + \|u_j\|_{L^2(I; H^{s-\frac{1}{2}})}. \end{aligned}$$

Likewise, we have (3.20) with $J = [(N_h - \frac{1}{2})h^{\frac{1}{2}}, T]$ and thus (3.19) concludes that

$$\|u_j\|_{L^2(I; W^{s-\frac{d}{2}+\frac{1}{2}-\varepsilon, \infty})} \lesssim_{\varepsilon} \|F_j\|_{L^2(I; H^{s-\frac{1}{2}})} + \|u_j\|_{L^\infty(I; H^s)}.$$

□

In the finally step, we shall glue the estimates for $\Delta_j u$ over different frequency regimes to obtain an estimate for u , from which the corresponding estimates for (η, ψ) follow.

3.5. Concluding the proof of Theorem 1.3. If u is a solution to (2.1) with data u^0 then by (3.6), the dyadic piece $\Delta_j u$ is a solution to $L_j \Delta_j u = F_j$ with F_j given by (3.7). Applying Corollary 3.7 one gets

$$(3.21) \quad \|\Delta_j u\|_{L^2(I; W^{s-\frac{d}{2}+\frac{1}{2}-\varepsilon, \infty})} \lesssim_{\varepsilon} \|F_j\|_{L^2(I; H^{s-\frac{1}{2}})} + \|\Delta_j u\|_{L^\infty(I; H^s)}.$$

Recall that $F_j = F_j^1 + R_j + F_j^3$ where F_j^k are given in (3.2), (3.7) and from (3.5) that

$$\|R_j\|_{H^s} \leq \|u\|_{H^s}.$$

Using the symbolic calculus Theorem A.4 one obtains without any difficulty that

$$\|F_j^1\|_{L^2 H^{s-\frac{1}{2}}} \lesssim \|u\|_{L^2(I; H^s)}.$$

For F_j^3 we consider for example

$$\begin{aligned} A_j &:= S_{(j-3)\frac{1}{2}} \gamma(x, D_x) \Delta_j u - S_{j-3} \gamma(x, D_x) \Delta_j u \\ &= \left(S_{(j-3)\frac{1}{2}} \gamma(x, D_x) \Delta_j u - \gamma(x, D_x) \Delta_j u \right) + (\gamma(x, D_x) \Delta_j u - S_{j-3} \gamma(x, D_x) \Delta_j u) \\ &= A_j^1 + A_j^2. \end{aligned}$$

More generally, let $a \in \Gamma_\rho^m$, homogeneous of degree m in ξ . Using the spherical harmonic decomposition we can assume $a(x, \xi) = b(x)c(\xi)$ with $b \in W^{\rho, \infty}$ and c is homogeneous of order m . Then with $S_{\delta_j}(a)(x, \xi) = S_{\delta_j}(b)(x)c(\xi)$ one has

$$(3.22) \quad \|(S_{\delta_j}(a)(x, D_x) - a(x, D_x))v\|_{L^2} \leq \|S_{\delta_j}(b) - b\|_{L^\infty} \|c(D_x)v\|_{L^2} \lesssim 2^{-\delta_j \rho} \|v\|_{H^m}.$$

Since $\gamma \in \Gamma_{\frac{3}{2}}^{\frac{3}{2}}$ is homogeneous of degree $\frac{3}{2}$ in ξ , the $H^{s-\frac{1}{2}}$ -norm of A_j^1 now can be bounded by

$$2^{j(s-\frac{1}{2})-\frac{1}{2}j^2+\frac{3}{2}j} \|\Delta_j u\|_{L^2} \lesssim \|u\|_{H^s}$$

while

$$\|A_j^2\|_{H^{s-\frac{1}{2}}} \lesssim 2^{j(s-\frac{1}{2})-2j+\frac{3}{2}j} \|\Delta_j u\|_{L^2} \lesssim \|u\|_{H^{s-1}}.$$

Similarly, we have

$$\|F_j^3\|_{L^2 H^{s-\frac{1}{2}}} \lesssim \|u\|_{L^2(I; H^s)} + \|f\|_{L^2(I; H^s)}.$$

The estimate (3.21) then implies

$$\|\Delta_j u\|_{L^2(I; W^{s-\frac{d}{2}+\frac{1}{2}-\varepsilon, \infty})} \lesssim_\varepsilon \|u\|_{L^\infty(I; H^s)} + \|f\|_{L^2(I; H^{s-\frac{1}{2}})}.$$

Gluing these estimates together one derives

$$(3.23) \quad \begin{aligned} \|u\|_{L^2(I; W^{s-\frac{d}{2}+\frac{1}{2}-2\varepsilon, \infty})} &\leq \sum_j 2^{-j\varepsilon} \|\Delta_j u\|_{L^2(I; W^{s-\frac{d}{2}+\frac{1}{2}-\varepsilon, \infty})} \\ &\lesssim_\varepsilon \|u\|_{L^\infty(I; H^s)} + \|f\|_{L^2(I; H^{s-\frac{1}{2}})}. \end{aligned}$$

Recall that $u = T_p \eta + iT_q(\psi - T_B \eta)$. From (3.23) one can use the symbolic calculus for para-differential operators in Theorem A.4 to recover the corresponding estimates for (η, ψ) (cf. [1], [2]):

$$\|\eta\|_{L^2(I; W^{s-\frac{d}{2}+1-2\varepsilon, \infty})} + \|\psi\|_{L^2(I; W^{s-\frac{d}{2}+\frac{1}{2}-2\varepsilon, \infty})} \lesssim \mathcal{F}_\varepsilon \left(\|(\eta, \psi)\|_{L^\infty([0, T]; H^{s+\frac{1}{2}} \times H^s)} \right).$$

The proof of Theorem 1.3 is complete.

4. Proof of Theorem 1.2

We consider three parameters $\delta \in (0, 1)$, $r_0 \in [0, 1]$, $r_1 \in [0, \frac{1}{2}]$, which shall be determined later and assume that

$$(4.1) \quad V \in L^\infty(I; W^{1+r_0, \infty}(\mathbf{R}^d)), \quad \gamma(\cdot, \xi) \in L^\infty(I; W^{\frac{1}{2}+r_1, \infty}(\mathbf{R}^d)).$$

4.1. Littlewood-Paley reduction. For every $j \geq 0$, the dyadic piece $\Delta_j u$ solves the equation

$$(4.2) \quad (\partial_t + T_V \cdot \nabla + iT_\gamma) \Delta_j u = F_j,$$

where

$$(4.3) \quad F_j^1 := \Delta_j f + i[T_\gamma, \Delta_j] + [T_V, \Delta_j] \cdot \nabla u.$$

In view of Proposition 2.7 one has

$$(4.4) \quad (\partial_t + S_{j-3}(V) \cdot \nabla + iS_{j-3}(\gamma)(x, D_x)) \Delta_j u = F_j^2,$$

with

$$(4.5) \quad F_j^2 := F_j^1 + R_j,$$

$R_j u$ is spectrally supported in an annulus $\{c_1^{-1}2^{j-1} \leq |\xi| \leq c_1 2^{j+1}\}$ and

$$(4.6) \quad \|R_j\|_{H^s} \lesssim \|u\|_{H^s}.$$

Now, let $\frac{1}{2} < c_1 < \dots < c_5$, $\mathcal{C}^k := \{(2c_k)^{-1} \leq |\xi| \leq 2c_k\}$, $k = \overline{1, 5}$ and

$$\tilde{\varphi} \in C^\infty, \quad \text{supp } \tilde{\varphi} \subset \mathcal{C}^5, \quad \tilde{\varphi} \equiv 1 \text{ on } \mathcal{C}^4.$$

Equation (4.4) is then equivalent to

$$(4.7) \quad (\partial_t + S_{(j-3)\delta}(V) \cdot \nabla + iS_{(j-3)\delta}(\gamma)(x, D_x) \tilde{\varphi}(2^{-j} D_x)) \Delta_j u = F_j,$$

with

$$(4.8) \quad \begin{aligned} F_j = F_j^2 + F_j^3 := F_j^2 + i(S_{(j-3)\delta}\gamma(x, D_x) - S_{j-3}\gamma(x, D_x)) \Delta_j u \\ + (S_{(j-3)\delta}(V) - S_{j-3}(V)) \cdot \nabla \Delta_j u. \end{aligned}$$

Let us define the operator corresponding to the homogeneous problem of (4.7)

$$(4.9) \quad \mathcal{L}_j := \partial_t + S_{(j-3)\delta}(V) \cdot \nabla + iS_{(j-3)\delta}(\gamma)(x, D_x) \tilde{\varphi}(2^{-j} D_x).$$

To prove Strichartz estimate for $\Delta_j u$ as a solution to (4.7), we shall first establish a "pseudo dispersive estimate" for \mathcal{L}_j . Set

$$h := 2^{-j}, \quad \tilde{h} := h^{\frac{1}{2}}.$$

4.2. Straightening the vector field. Following [4] we straighten the vector field $\partial_t + S_{(j-3)\delta} \cdot \nabla$ by considering the system

$$(4.10) \quad \begin{cases} \dot{X}_k(s) = S_{(j-3)\delta}(V_k)(s, X(s)), & 1 \leq k \leq d, \\ X_k(0) = x_k. \end{cases} \quad X = (X_1, \dots, X_d)$$

Since $V \in L^\infty([0, T]; L_x^\infty)$, system (4.10) has unique solution on $I = [0, T]$, which shall be denoted for simplicity by $X(s, x; h)$, or even $X(s, x)$. Estimates for the flow $s \mapsto X(s, \cdot)$ is given in the next Proposition.

PROPOSITION 4.1. *For fixed (s, h) the map $x \mapsto X(s; x, h)$ belongs to $C^\infty(\mathbf{R}^d, \mathbf{R}^d)$. Moreover, for all $(s, h) \in I \times (0, 1]$ we have*

$$(4.11) \quad \|(\partial_x X)(s, \cdot; h) - Id\|_{L^\infty(\mathbf{R}^d)} \leq \mathcal{F}(\|V\|_{L^\infty([0, T]; W^{1, \infty})})|s|,$$

$$(4.12) \quad \|(\partial_x^\alpha X)(s, \cdot; h)\|_{L^\infty(\mathbf{R}^d)} \leq \alpha \mathcal{F}(\|V\|_{L^\infty([0, T]; W^{1+r_0, \infty})}) h^{-\delta(|\alpha| - (1+r_0))} |s|, \quad |\alpha| \geq 2.$$

PROOF. Here we follow the poof Proposition 4.10, [4]. The improvement is due to the following estimate (by applying Lemma 2.6)

$$(4.13) \quad \left\| \partial_x^\beta S_{j\delta}(V)(s) \right\|_{L^\infty(\mathbf{R}^d)} \leq C_\beta h^{-\delta(|\beta| - 1 - r_0)} \|V(s)\|_{W^{1+r_0, \infty}}, \quad \forall |\beta| \geq 2.$$

(i) To prove (4.11) we differentiate the system with respect to x_l to obtain

$$\begin{cases} \frac{\partial \dot{X}_k}{\partial x_l}(s) = \sum_{q=1}^d S_{j\delta} \left(\frac{\partial V_k}{\partial x_q} \right)(s, X(s)) \frac{\partial X_q}{\partial x_l}(s) \\ \frac{\partial X_k}{\partial x_l}(0) = \delta_{kl} \end{cases}$$

from which we deduce

$$(4.14) \quad \frac{\partial X_k}{\partial x_l}(s) = \delta_{kl} + \int_0^s \sum_{q=1}^d S_{j\delta} \left(\frac{\partial V_k}{\partial x_q} \right)(\sigma, X(\sigma)) \frac{\partial X_q}{\partial x_l}(\sigma) d\sigma.$$

Setting $|\nabla X| = \sum_{k,l=1}^d \left| \frac{\partial X_k}{\partial x_l} \right|$ we obtain from (4.14)

$$|\nabla X(s)| \leq C_d + \int_0^s |\nabla V(\sigma, X(\sigma))| |\nabla X(\sigma)| d\sigma.$$

The Gronwall inequality implies that

$$(4.15) \quad |\nabla X(s)| \leq \mathcal{F}(\|V\|_{W^{1, \infty}}) \quad \forall s \in I.$$

Coming back to (4.14) and using (4.15) lead to

$$\left| \frac{\partial X}{\partial x}(s) - Id \right| \leq \mathcal{F}(\|V\|_{W^{1, \infty}}) \int_0^s \|\nabla V(\sigma, \cdot)\|_{L^\infty(\mathbf{R}^d)} d\sigma \leq \mathcal{F}_1(\|V\|_{W^{1, \infty}}) |s|.$$

(ii) We shall prove (4.12) for $|\alpha| = 2$ first and then prove by induction on $|\alpha|$ that the estimate

$$\|(\partial_x^\alpha X)(s; \cdot, h)\|_{L^\infty(\mathbf{R}^d)} \leq \mathcal{F}_\alpha(\|V\|_{W^{1+r_0, \infty}}) h^{-\delta(|\alpha| - 1 - r_0)}$$

for $2 \leq |\alpha| \leq k$ implies (4.12) for $|\alpha| = k + 1$.

Differentiating $|\alpha|$ times ($|\alpha| \geq 2$) the system (4.10) and using the Faa-di-Bruno formula we obtain

$$(4.16) \quad \frac{d}{ds}(\partial_x^\alpha X)(s) = S_{j\delta}(\nabla V)(s, X(s))\partial_x^\alpha X + (1)$$

where the term (1) is a finite linear combination of terms of the form

$$A_\beta(s, x) = \partial_x^\beta(S_{j\delta}(V))(s, X(s)) \prod_{i=1}^q (\partial_x^{L_i} X(s))^{K_i}$$

where

$$2 \leq |\beta| \leq |\alpha|, \quad 1 \leq q \leq |\alpha|, \quad \sum_{i=1}^q |K_i|L_i = \alpha, \quad \sum_{i=1}^q K_i = \beta.$$

1. When $|\alpha| = 2$, we have

$$A_\beta(s, x) = \partial_x^\beta(S_{j\delta}(V))(s, X(s)) \prod_{i=1}^q (\partial_x^{L_i} X(s))^{K_i}$$

with $|L_i| = 1$ and $|\beta| = |\alpha| = 2$. It then follows from (i) that

$$\left| \prod_{i=1}^q (\partial_x^{L_i} X(s))^{K_i} \right| \leq \mathcal{F}(\|V\|_{W^{1,\infty}}).$$

On the other hand, we have by (4.13)

$$\left\| \partial_x^\beta S_{j\delta}(V)(s) \right\|_{L^\infty(\mathbf{R}^d)} \leq Ch^{-\delta(|\alpha|-1-r_0)} \|V(s)\|_{W^{1+r_0,\infty}}.$$

Consequently, it holds that

$$\|(1)\|_{L^\infty(\mathbf{R}^d)} \leq h^{-\delta(|\alpha|-1-r_0)} \mathcal{F}(\|V\|_{W^{1+r_0,\infty}}),$$

from which we obtain (4.12) for $|\alpha| = 2$ by using (4.16) and Gronwall's inequality.

2. Assuming now that (4.12) holds with $|\alpha| = k \geq 2$, we shall prove it for $|\alpha| = k + 1$. Indeed, from (4.11) and the induction hypothesis it holds for any $1 \leq |\nu| \leq k$ that

$$\|(\partial_x^\nu X)(s, \cdot; h)\|_{L^\infty(\mathbf{R}^d)} \leq \alpha \mathcal{F}(\|V\|_{L^\infty([0,T]; W^{1+r_0,\infty})}) h^{-\delta(|\nu|-1)} |s|.$$

Therefore, with $1 \leq |L_i| < |\alpha|$ we can write in view of the estimate (4.13)

$$\begin{aligned} \|A_\beta(s, \cdot)\|_{L^\infty(\mathbf{R}^d)} &\leq \left\| \partial_x^\beta(S_{j\delta}(V))(s, \cdot) \right\|_{L^\infty(\mathbf{R}^d)} \prod_{i=1}^q \left\| \partial_x^{L_i} X(s, \cdot) \right\|_{L^\infty(\mathbf{R}^d)}^{K_i} \\ &\leq Ch^{-\delta(|\beta|-1-r_0)} \|V(s, \cdot)\|_{W^{1+r_0,\infty}} h^{-\delta \sum_{i=1}^q |K_i|(|L_i|-1)} \mathcal{F}(\|V\|_{W^{1+r_0,\infty}}) \\ &\leq h^{-\delta(|\alpha|-1-r_0)} \mathcal{F}(\|V\|_{W^{1+r_0,\infty}}) \|V(s, \cdot)\|_{W^{1+r_0,\infty}}. \end{aligned}$$

As before, we conclude by (4.16) and Gronwall's inequality. \square

In view of (4.11) the mapping $x \mapsto X(s, x; h)$ is a C^∞ -diffeomorphism when $0 \leq s \leq T_0$ small enough. This restriction of T is harmless for one can iterate the final estimate over time intervals of length T_0 which depends only on $\|V\|_{L^\infty([0,T]; W^{1,\infty})}$.

Now, in (4.9) we first make the change of spatial variables

$$(4.17) \quad v_h(t, y) = u_j(t, X(t, y; h))$$

so that

$$(4.18) \quad (\partial_t + S_{(j-3)\delta}(V) \cdot \nabla) u_j(t, X(t, y; h)) = \partial_t v_h(t, y).$$

Denoting

$$(4.19) \quad q_h(x, \xi) := S_{(j-3)\delta}(\gamma)(x, \xi) \tilde{\varphi}(h\xi),$$

let us compute this dispersive term after the above change of variables. To this end, one sets

$$(4.20) \quad \begin{aligned} H(y, y') &= \int_0^1 \frac{\partial X}{\partial x}(\lambda y + (1-\lambda)y') d\lambda, \quad M(y, y') = ({}^t H(y, y'))^{-1} \\ M_0(y) &= \left({}^t \left(\frac{\partial X}{\partial x}(y) \right) \right)^{-1}, \quad J(y, y') = \left| \det \left(\frac{\partial X}{\partial x}(y') \right) \right| \left| \det M(y, y') \right|. \end{aligned}$$

Then,

$$(\text{Op}(q_h)u_j) \circ X(y) = (2\pi)^{-d} \iint e^{i(X(y)-x') \cdot \eta} q_h(X(y), \eta) u_j(x') dx' d\eta.$$

Now, we make two changes of variables $x' = X(y')$, then $\eta = M(y, y')\xi$ to obtain

$$(\text{Op}(q_h)u_j) \circ X(y) = (2\pi)^{-d} \iint e^{i(y-y') \cdot \zeta} q_h(X(y), M(y, y')\zeta) J(y, y') v_h(y') dy' d\zeta.$$

Observe that the pseudo-differential operator above is still of order $\frac{1}{2}$. To change its order to 1, we need to make in addition another change of spatial variables

$$(4.21) \quad y = h^{\frac{1}{2}}z, \quad y' = \tilde{h}z', \quad w_h(z') = v_h(\tilde{h}z'), \quad \xi = \tilde{h}\zeta$$

so that

$$(4.22) \quad \begin{aligned} (\text{Op}(q_h)u_j) \circ X(y) &= (2\pi)^{-d} \iint e^{i(z-z') \cdot \xi} q_h(X(\tilde{h}z), M(\tilde{h}z, \tilde{h}z')\tilde{h}^{-1}\xi) \times \\ &\quad \times J(\tilde{h}z, \tilde{h}z') w_h(z') dz' d\xi. \end{aligned}$$

Summing up, with

$$(4.23) \quad p_h(z, z', \xi) := q_h(X(\tilde{h}z), M(\tilde{h}z, \tilde{h}z')\tilde{h}^{-1}\xi) J(\tilde{h}z, \tilde{h}z'), \quad w_h(t, z) = u_j(t, X(t, \tilde{h}z))$$

it holds that

$$(\text{Op}(q_h)u_j) \circ X(\tilde{h}z) = \text{Op}(p_h)w_h(z),$$

which combines with (4.18) and (4.9) yields

$$(4.24) \quad (\mathcal{L}_j u_j)(t, X(t, \tilde{h}z)) = (\partial_t + i \text{Op}(p_h)) w_h(t, z), \quad w_h(t, z) = u_j(t, X(t, \tilde{h}z)).$$

We have transformed the operator \mathcal{L}_j of order $\frac{1}{2}$ into the right-hand side of (4.24), which has order 1.

4.3. Approximation of the symbol p_h . Observe that p_h in (4.23) depends on (z, z', ξ) which is not in the standard form to use the phase space transform in [19]. Therefore, one need to approximate p_h by some symbol depend only on (z, ξ) . A general result can be found in Proposition 0.3A [24]. However, here we need the difference between p_h and its approximation to be more regular, so we have to inspect more carefully the smoothness of p_h . To do this, one writes as for (2.7)-(2.8)

$$\begin{aligned} p_h(z, z', \xi) &= p_h(z, z, \xi) + \int_0^1 \partial_{z'} p_h(z, z + s(z' - z), \xi) ds(z' - z) \\ &:= p_h^0(z, \xi) + r_h^0(z, z', \xi)(z' - z), \end{aligned}$$

where

$$(4.25) \quad p_h^0(z, \xi) = p_h(z, z, \xi) = q_h(X(\tilde{h}z), M_0(\tilde{h}z)\tilde{h}^{-1}\xi).$$

On the other hand,

$$\text{Op}(r_h^0(z' - z))w(z) = -i \text{Op}(r_h)w(z)$$

with

$$r_h(z, z', \xi) = \int_0^1 \partial_\xi \partial_{z'} p_h(z, z + s(z' - z), \xi) ds.$$

To simplify notations, we denote $[z, z']_s = z + s(z' - z)$ so that

$$(4.26) \quad r_h(z, z', \xi) = \int_0^1 \partial_\xi \partial_{z'} q_h(X(\tilde{h}z), M(\tilde{h}z, \tilde{h}[z, z']_s)\tilde{h}^{-1}\xi) J(\tilde{h}z, \tilde{h}[z, z']_s) ds.$$

In conclusion, $\text{Op}(p_h) = \text{Op}(p_h^0) - i \text{Op}(r_h)$.

4.3.1. *The symbol p_h^0 .* First, Lemma 4.1 implies directly the estimates for M and J .

LEMMA 4.2. *It holds for $(\alpha, \alpha') \in (\mathbf{N}^d)^2$ that*

$$|\partial_z^\alpha \partial_{z'}^{\alpha'} M(z, z')| + |\partial_z^\alpha \partial_{z'}^{\alpha'} J(z, z')| \lesssim_{\alpha, \alpha'} \begin{cases} 1, & \text{if } |\alpha| + |\alpha'| = 0, \\ \tilde{h}^{-2\delta(|\alpha| + |\alpha'| - r_0)}, & \text{if } |\alpha| + |\alpha'| \geq 1. \end{cases}$$

On the other hand, by Bernstein's inequalities and the fact that on the support of $\tilde{\varphi}(h\xi)$, $|\xi| \sim \tilde{h}^{-2}$, we can estimate the derivatives of q_h given by (4.19) as follows.

LEMMA 4.3. *We have for $(\alpha, \beta) \in (\mathbf{N}^d)^2$*

$$\left| \partial_x^\alpha \partial_\xi^\beta q_h(x, \xi) \right| \lesssim_{\alpha, \beta} \begin{cases} \tilde{h}^{-1+2|\beta|}, & \text{if } |\alpha| = 0, \\ \tilde{h}^{-1-2\delta(|\alpha| - (\frac{1}{2} + r_1)) + 2|\beta|}, & \text{if } |\alpha| \geq 1. \end{cases}$$

We now prove study the regularity of the symbol p_h^0 .

PROPOSITION 4.4. *Choosing r_0, r_1 satisfying*

$$(4.27) \quad 2\delta(1 - r_0) \leq 1, \quad 2\delta(2 - r_0) \leq 2, \quad 2\delta\left(\frac{1}{2} - r_1\right) \leq 1, \quad 2\delta\left(\frac{3}{2} - r_1\right) \leq 2$$

then the symbol p_h^0 verifies

(i) *for all $(\alpha, \beta) \in \mathbf{N}^d$, $|\alpha| \leq 2$*

$$(4.28) \quad \left| \partial_z^\alpha \partial_\xi^\beta p_h^0(z, \xi) \right| \lesssim_{\alpha, \beta} \tilde{h}^{-1+|\beta|} 1_{\tilde{h}|\xi| \sim 1}(\xi),$$

(ii) *for all $(\alpha, \beta) \in \mathbf{N}^d$, $|\alpha| \geq 3$*

$$(4.29) \quad \left| \partial_z^\alpha \partial_\xi^\beta p_h^0(z, \xi) \right| \lesssim_{\alpha, \beta} \tilde{h}^{-1-(2\delta-1)(|\alpha|-2)+|\beta|} 1_{\tilde{h}|\xi| \sim 1}(\xi).$$

PROOF. To simplify notations, we shall denote in this proof $q \equiv q_h$.

(i) Observe that (4.28) is trivial when $\alpha = 0$. The argument below is independent of the dimension, however let us further simplify the notations by writing as if $d = 1$. For $|\alpha| = 1$, we compute

$$(4.30) \quad \partial_z^\alpha p^0(z, \xi) = q_x(X(\tilde{h}z), M_0(\tilde{h}z)\tilde{h}^{-1}\xi)\tilde{h}X'(\tilde{h}z)$$

$$(4.31) \quad + q_\xi(X(\tilde{h}z), M_0(\tilde{h}z)\tilde{h}^{-1}\xi)(\tilde{h}^{-1}\xi)\tilde{h}M'_0(\tilde{h}z).$$

For $|\alpha| = 2$, we have

$$(4.32) \quad \partial_z^\alpha p_h^0(z, \xi) = q_{xx}(\cdots) \tilde{h}^2 (X')^2 + 2q_{x\xi}(\cdots) X' M_0' \tilde{h} \xi + q_{\xi\xi}(\cdots) (M_0')^2 \xi^2,$$

$$(4.33) \quad + q_x(\cdots) \tilde{h}^2 X'' + q_\xi(\cdots) (\tilde{h} \xi) M_0''.$$

Remark that on the support of $p^0(z, \xi)$, $\tilde{h}|\xi| \sim 1$, using Proposition 4.1, Lemmas 4.2 and 4.3 one deduces easily that

$$\begin{aligned} |\partial_z^\alpha p^0(z, \xi)| &\lesssim_\alpha \tilde{h}^{-1-2\delta(\frac{1}{2}-r_1)+1} + \tilde{h}^{-1+2-1-2\delta(1-r_0)}, \quad |\alpha| = 1, \\ |\partial_z^\alpha p^0(z, \xi)| &\lesssim_\alpha \tilde{h}^{-1-2\delta(\frac{3}{2}-r_1)+2} + \tilde{h}^{-1-2\delta(\frac{1}{2}-r_1)+2-2\delta(1-r_0)} + \tilde{h}^{-1+4-4\delta(1-r_0)-2} \\ &\quad + \tilde{h}^{-1-2\delta(\frac{1}{2}-r_1)+2-2\delta(1-r_0)} + \tilde{h}^{-1+2-2\delta(2-r_0)}, \quad |\alpha| = 2. \end{aligned}$$

Under conditions (4.27), it follows that

$$|\partial_z^\alpha p^0(z, \xi)| \lesssim_\alpha \tilde{h}^{-1} 1_{\tilde{h}|\xi| \sim 1}(\xi), \quad |\alpha| \leq 2.$$

To obtain (i) it remains to estimate $\partial_\xi^\beta (\partial_z^\alpha p_h^0)$ for $|\alpha| \leq 2$ and $\beta \in \mathbf{N}^d$. From the explicit expressions (4.30), (4.32) of $\partial_z^\alpha p_h^0$ above, we see that there are two possibilities when differentiating in ξ . A possibility is that a derivative falls down to q . This makes appear the factor $M_0(\tilde{h}z)\tilde{h}^{-1}$ while we gain \tilde{h}^2 when differentiating q in ξ (by Lemma 4.3), we thus gain \tilde{h} . Another possibility is that a derivative falls down to ξ^ν , $\nu = 1, 2$, which results in $\nu\xi^{\nu-1}$. Since $\xi \sim \tilde{h}^{-1}$ on the support of p_h^0 one deduces that $\xi^{\nu-1} \sim \xi^\nu \tilde{h}$, which means that we still gain \tilde{h} . Therefore, in both cases we gain \tilde{h} when differentiating once in ξ and thus (4.28) follows.

(ii) As just explained above, it suffices to prove (4.29) with $\beta = 0$. From the formula (4.32), the proof of (4.29) reduces to showing for $|\alpha| \geq 0$

$$|\partial_z^\alpha A_j(z, \xi, h)| \lesssim_{\alpha, \beta} \tilde{h}^{-1-(2\delta-1)|\alpha|} 1_{\tilde{h}|\xi| \sim 1}(\xi), \quad j = \overline{1, 5}$$

with

$$\begin{cases} A_1 = q_{x\xi}(X(\tilde{h}z), M_0(\tilde{h}z)\tilde{h}^{-1}\xi)\tilde{h}^{-1}, \\ A_2 = q_{xx}(X(\tilde{h}z), M_0(\tilde{h}z)\tilde{h}^{-1}\xi)\tilde{h}^2, \\ A_3 = q_{\xi\xi}(X(\tilde{h}z), M_0(\tilde{h}z)\tilde{h}^{-1}\xi)\tilde{h}^{-4}, \\ A_4 = q_x(X(\tilde{h}z), M_0(\tilde{h}z)\tilde{h}^{-1}\xi)\tilde{h}, \\ A_5 = q_\xi(X(\tilde{h}z), M_0(\tilde{h}z)\tilde{h}^{-1}\xi)\tilde{h}^{-2} \end{cases}$$

and

$$|\partial_z^\alpha B_j(z, h)| \lesssim_{\alpha, \beta} \tilde{h}^{-(2\delta-1)|\alpha|}, \quad j = \overline{1, 4}$$

with

$$\begin{cases} B_1 = X'(\tilde{h}z), \\ B_2 = \tilde{h}M_0'(\tilde{h}z), \\ B_3 = \tilde{h}X''(\tilde{h}z), \\ B_4 = \tilde{h}^2M''(\tilde{h}z). \end{cases}$$

1. B_j . It follows from Lemma (4.11) that

$$|\partial_z^\alpha B_1| = |\partial_z^\alpha X'(\tilde{h}z)| = \tilde{h}^{|\alpha|} |(\partial_x^{\alpha+1} X)(\tilde{h}z)| \lesssim \tilde{h}^{|\alpha|-2\delta|\alpha|} \lesssim \tilde{h}^{-1-(2\delta-1)|\alpha|}.$$

On the other hand, (4.12) and the condition $2\delta(1-r_0) \leq 1$ imply

$$|\partial_z^\alpha B_3| = \tilde{h} |\partial_z^\alpha X''(\tilde{h}z)| = \tilde{h}^{1+|\alpha|} |(\partial_x^{\alpha+2} X)(\tilde{h}z)| \lesssim \tilde{h}^{1+|\alpha|-2\delta(|\alpha|+1-r_0)} \lesssim \tilde{h}^{-(2\delta-1)|\alpha|}.$$

Remark that $M'_0(\tilde{h}z)$ is as smooth as $X''(\tilde{h}z)$, the preceding estimate also holds for B_2 . For B_4 , we apply (4.12) and use the condition $2\delta(2-r_0) \leq 2$ to estimate

$$|\partial_z^\alpha B_4| = \tilde{h}^2 |\partial_z^\alpha M'_0(\tilde{h}z)| = \tilde{h}^{2+|\alpha|} |(\partial_x^{\alpha+2} M_0)(\tilde{h}z)| \lesssim \tilde{h}^{|\alpha|+2-2\delta(|\alpha|+2-r_0)} \lesssim \tilde{h}^{-(2\delta-1)|\alpha|}.$$

2. A_1 . For $\alpha = 0$, Lemma 4.3 gives

$$|A_1| \lesssim \tilde{h}^{-1-2\delta(\frac{1}{2}-r_1)+2-1} \lesssim \tilde{h}^{-1}$$

since $2\delta(\frac{1}{2}-r_1) \leq 1$. Considering now $|\alpha| \geq 1$, we write using the Faa-di-Bruno formula that $\partial_z^\alpha A_1$ is a linear combination of terms of the form

$$C_1 = \tilde{h}^{|\alpha|-1} (\partial_x^{a+1} \partial_\xi^{b+1} q)(\dots) \prod_{j=1}^r ((\partial_x^{L_j} X)(\tilde{h}z))^{P_j} ((\partial_x^{L_j} M_0)(\tilde{h}z) \tilde{h}^{-1} \xi)^{Q_j}$$

where $1 \leq |a| + |b| \leq |\alpha|$, $|L_j| \geq 1$, $\forall j = \overline{1, r}$ and

$$\sum_{j=1}^r P_j = a, \quad \sum_{j=1}^r Q_j = b, \quad \sum_{j=1}^r (|P_j| + |Q_j|) L_j = \alpha.$$

According to Lemma 4.3,

$$|(\partial_x^{a+1} \partial_\xi^{b+1} q)(\dots)| \lesssim \tilde{h}^{-1-2\delta(|a|+\frac{1}{2}-r_1)+2(|b|+1)}.$$

On the other hand, since $|L_j| \geq 1$ Lemmas 4.3, 4.2 allow us to estimate the product appearing in C_1 as follows

$$\begin{aligned} |\prod_{j=1}^r| &\lesssim \tilde{h}^K, \quad K = \sum_{j=1}^r \left(-2\delta(|L_j| - 1)|P_j| - 2\delta(|L_j| - r_0)|P_j| \right) - 2 \sum_{j=1}^r |Q_j| \\ &= -2\delta|\alpha| + 2\delta|a| + 2\delta r_0 |b| - 2|b|. \end{aligned}$$

Therefore, $|C_1| \lesssim \tilde{h}^L$ with

$$\begin{aligned} L &= |\alpha| - 1 - 1 - 2\delta(|a| + \frac{1}{2} - r_1) + 2(|b| + 1) - 2\delta|\alpha| + 2\delta|a| + 2\delta r_0 |b| - 2|b| \\ &\geq -1 - (2\delta - 1)|\alpha| + 1 - 2\delta(\frac{1}{2} - r_1) \\ &\geq -1 - (2\delta - 1)|\alpha|, \end{aligned}$$

where we have used again the condition that $2\delta(\frac{1}{2}-r_1) \leq 1$. The proof for A_1 is complete.

3. A_2, A_3, A_4, A_5 . The estimate for these terms can be derived along the same lines as for A_3 , where one need to make use of the condition $2\delta(\frac{3}{2}-r_1) \leq 2$ for A_2 and the condition $2\delta(\frac{1}{2}-r_1) \leq 1$ for A_4 . \square

From now on, we always assume condition (4.27) for r_0 and r_1 .

4.3.2. *The symbol r_h .* The next lemma provides the order of r_h and shows that it decays in ξ faster than in (z, z') which shall be important in our ‘‘pseudo-dispersive estimates’’ in paragraph 4.4.

LEMMA 4.5. *For any $(\alpha, \alpha', \xi) \in (\mathbf{N}^d)^3$*

$$\left| \partial_z^\alpha \partial_{z'}^{\alpha'} \partial_\xi^\beta r_h(z, z', \xi) \right| \lesssim_{\alpha, \alpha', \beta} \tilde{h}^{1-2\delta(1-r_0)-(2\delta-1)(|\alpha|+|\alpha'|)+|\beta|} 1_{\{\tilde{h}|\xi| \sim 1\}}(\xi).$$

Consequently, we have $r_h \in S_{1, (2\delta-1), (2\delta-1)}^{-1+2\delta(1-r_0)}$.

PROOF. Recall the definition (4.26) of r_h , on the support of this symbol we have $|\xi| \sim \tilde{h}^{-1}$. In this proof, all the estimates are uniform in $s \in [0, 1]$. It follows from Lemma 4.2 that

$$\forall(\alpha, \alpha') \in (\mathbf{N}^d)^2, \quad \left| \partial_z^\alpha \partial_{z'}^{\alpha'} J(\tilde{h}z, \tilde{h}[z, z']_s) \right| \lesssim_{\alpha, \alpha'} \tilde{h}^{-(2\delta-1)(|\alpha|+|\alpha'|)}.$$

Next, setting

$$\tilde{q}(x, \xi) = S_{(j-3)\delta}(\gamma)(x, \xi) \tilde{\varphi}(\xi)$$

we see that

$$q(X(\tilde{h}z), M(\tilde{h}z, \tilde{h}[z, z']_s) \tilde{h}^{-1}\xi) = \tilde{h}^{-1} \tilde{q}(X(\tilde{h}z), M(\tilde{h}z, \tilde{h}[z, z']_s) \tilde{h}\xi).$$

The proof of this lemma then boils down to showing for any $(\alpha, \alpha', \beta) \in (\mathbf{N}^d)^3$,

$$(4.34) \quad \left| \partial_\xi^\beta \partial_z^\alpha \partial_{z'}^{\alpha'} \partial_\xi \partial_{z'} \tilde{q}(X(z), M(z, [z, z']_s)\xi) \right| \lesssim_{\alpha, \alpha', \beta} \tilde{h}^{-2\delta(|\alpha|+|\alpha'|)-2\delta(1-r_0)}.$$

We compute

$$\Xi := \partial_\xi \partial_{z'} \tilde{q}(X(z), M(z, [z, z']_s)\xi) = s \tilde{q}_{\xi\xi}(\cdots) M_{z'} \xi M + s \tilde{q}_\xi(\cdots) M_{z'},$$

which is bounded by $\tilde{h}^{-2\delta(1-r_0)}$ in view of Lemma 4.2 and the fact that $|\xi| \sim 1$ on the support of \tilde{q} . For the same reason we see that taking ξ -derivatives of Ξ is harmless (notice that M is bounded), so we only need to prove (4.34) for $|\beta| = 0$. Indeed, one has again by Lemma 4.2

$$(4.35) \quad \left| \left(\partial_z^\alpha \partial_{z'}^{\alpha'} M \right) (\cdot) \right| + \left| \left(\partial_z^\alpha \partial_{z'}^{\alpha'} M_{z'} \right) (\cdot) \right| \lesssim \tilde{h}^{-2\delta(|\alpha|+|\alpha'|)-2\delta(1-r_0)}$$

On the other hand, using the Faa-di-Bruno formula (as in the proof of Proposition 4.4) we can prove that

$$\left| \partial_z^\alpha \partial_{z'}^{\alpha'} (\partial_\xi^\gamma \tilde{q})(X(z), M(z, [z, z']_s)\xi) \right| \lesssim \tilde{h}^{-2\delta(|\alpha|+|\alpha'|)},$$

from which we conclude the proof. \square

In view of equation (4.24) we have proved that

$$(4.36) \quad (\mathcal{L}_j u_j)(t, X(t, \tilde{h}z)) = (\partial_t + i \text{Op}(p_h^0)) w_h(t, z) - i \text{Op}(r_h) w_h(t, z)$$

via the relation $w_h(t, z) = u_j(t, X(t, \tilde{h}z))$.

4.4. A “pseudo dispersive estimate” for \mathcal{L}_j . In this step, we shall show that Theorem 2.9 can be applied to the evolution operator

$$L_h := D_t + \text{Op}^w(p_h^0).$$

Henceforth, we set

$$\delta = \frac{3}{4}, \quad \lambda = \tilde{h}^{-1}.$$

Proposition 4.4 then shows that p_h^0 belongs to λS_λ^2 . Using Lemma 2.11 to replace $\text{Op}(p_h^0)$ in (4.36) by $\text{Op}^w(p_h)$ we have

$$(4.37) \quad \text{Op}^w(p_h^0) = \text{Op}(p_h^0) + \text{Op}(r'_h)$$

with $r'_h \in S_{1, \frac{1}{2}, \frac{1}{2}}^0$. On the other hand, since $2\delta(1-r_0) \leq 1$, Lemma 4.5 combining with (4.36), (4.37) leads to

PROPOSITION 4.6. *There holds for some symbol $r_h^1 \in S_{1, \frac{1}{2}, \frac{1}{2}}^0$ that*

$$(4.38) \quad \frac{1}{i} (\mathcal{L}_j u_j)(t, X(t, \tilde{h}z)) = (D_t + \text{Op}^w(p_h^0)) w_h(t, z) + \text{Op}(r_h^1) w_h(t, z).$$

Next, we recall the following proposition, which shows that the characteristic set of p_h^0 has d nonvanishing curvatures.

PROPOSITION 4.7 ([4, Proposition 4.16]). *Let \mathcal{C} be a fixed annulus in \mathbf{R}^d . For any $0 < \delta < 1$ there exist $m_0 > 0$, $h_0 > 0$ such that*

$$\sup_{(t,x,\xi,h) \in I \times \mathbf{R}^d \times (\mathcal{C} \times (0, h_0])} |\det \partial_\xi^2 S_{\delta j}(\gamma)(t, x, \xi)| \geq m_0.$$

Calling \mathcal{S}_j , S_h are, respectively, the propagator of \mathcal{L}_j , L_h , we are now in position to apply Theorem 2.9 to derive dispersive estimates for S_h .

PROPOSITION 4.8. *For any symbol $\chi \in S_\lambda^0$ satisfying for all $z \in \mathbf{R}^d$, $\text{supp } \chi(z, \cdot) \subset \lambda \mathcal{C}^2$ we have*

$$(4.39) \quad \|S_h(t, t_0)(\chi(z, D_z)f)\|_{L^\infty} \lesssim \tilde{h}^{-\frac{d}{2}} |t - t_0|^{-\frac{d}{2}} \|f\|_{L^1},$$

for all $t, t_0 \in [0, 1]$ and $0 < \tilde{h} \leq \tilde{h}_0$.

If in addition, $\chi(z, D_z) : L^2 \rightarrow L^2$ then for any $r \in [2, \infty]$ there holds by interpolation

$$(4.40) \quad \|S_h(t, t_0)(\chi(z, D_z)f)\|_{L^r} \lesssim \tilde{h}^{-d(\frac{1}{2} - \frac{1}{r})} |t - t_0|^{-d(\frac{1}{2} - \frac{1}{r})} \|f\|_{L^{r'}}.$$

where r' is the conjugate exponent of r , i.e., $\frac{1}{r} + \frac{1}{r'} = 1$.

PROOF. We have seen that $p_h^0 \in \lambda S_\lambda^2$. On the other hand, $\tilde{\varphi} \equiv 1$ in \mathcal{C}^4 , Proposition 4.7 then gives

$$\sup_{(t,x,\xi,h) \in I \times \mathbf{R}^d \times \mathcal{C}^4 \times (0, h_0]} |\det (\partial_\xi^2 S_{\delta j}(\gamma)(t, x, \xi) \tilde{\varphi}(\xi))| \gtrsim 1.$$

Remark that (4.11) implies $|M_0(y)| \geq c_0$ for all $y \in \mathbf{R}^d$ (by choosing T small enough as explained after Proposition 4.1). Consequently,

$$\sup_{(t,x,\xi,h) \in I \times \mathbf{R}^d \times (\lambda \mathcal{C}^3) \times (0, h_0]} |\det \partial_\xi^2 p_h^0| \gtrsim \lambda^{-d}$$

if $c_3 < c_4$ is chosen appropriately. In other word, condition (A) in Theorem 2.9 is fulfilled with $c = c_3$ and thus the Proposition follows. \square

Let φ_1 be a smooth function verifying

$$\text{supp } \varphi_1 \subset \{(2c_2)^{-1} \leq |\xi| \leq 2c_2\}, \quad \varphi_1 \equiv 1 \text{ in } \{(2c_1)^{-1} \leq |\xi| \leq 2c_1\}.$$

LEMMA 4.9. *For $f(t, z) = g(t, X(t, \tilde{h}z))$ we have*

$$(4.41) \quad (\varphi_1(hD_x)g)(t, X(t, \tilde{h}z)) = \varphi_h^*(z, D_z)f(t, z) - i \text{Op}(r_h^2)f(t, z),$$

with

$$(4.42) \quad \varphi_h^*(z, \xi) = \varphi_1(M_0(\tilde{h}z)\tilde{h}\xi),$$

$$(4.43) \quad r_h^2(z, z', \xi) = \int_0^1 \partial_\xi \partial_{z'} \varphi_1(M(\tilde{h}z, \tilde{h}[z, z']_s)\tilde{h}\xi) J(\tilde{h}z, \tilde{h}[z, z']_s) ds.$$

Moreover, for every $(\alpha, \alpha', \xi) \in (\mathbf{N}^d)^3$ there hold

$$(4.44) \quad \left| \partial_z^\alpha \partial_\xi^\beta \varphi_h^*(z, \xi) \right| \lesssim_{\alpha, \beta} \tilde{h}^{-(2\delta-1)+|\beta|} \mathbf{1}_{\{\tilde{h}|\xi| \sim 1\}},$$

$$(4.45) \quad \left| \partial_z^\alpha \partial_{z'}^{\alpha'} \partial_\xi^\beta r_h^2(z, z', \xi) \right| \lesssim_{\alpha, \alpha', \beta} \tilde{h}^{2-2\delta(1-r_0)-(2\delta-1)(|\alpha|+|\alpha'|)+|\beta|} \mathbf{1}_{\{\tilde{h}|\xi| \sim 1\}}.$$

PROOF. The formulas (4.41) and (4.42), (4.43) are derived along the same lines as in paragraph 4.2, 4.3 where we performed the change of variables $x = X(t, \tilde{h}z)$ to derive (4.36).

1. Proof of (4.44).

Observe first that

$$\partial_{\xi}^{\beta} \varphi_h^*(z, \xi) = (\partial^{\gamma} \varphi_1)(M_0(\tilde{h}z)\tilde{h}\xi) (M_0(\tilde{h}z))^{\gamma} \tilde{h}^{|\beta|}$$

where $|\gamma| = |\beta|$. Next, Lemma 4.2 implies for every $\alpha \in \mathbf{N}^d$,

$$\left| \partial_z^{\alpha} (\partial^{\gamma} \varphi_1)(M_0(\tilde{h}z)\tilde{h}\xi) \right| + \left| \partial_z^{\alpha} (M_0(\tilde{h}z))^{\gamma} \right| \lesssim \tilde{h}^{-(2\delta-1)|\alpha|},$$

and thus (4.44) follows.

2. For (4.45) one proceeds exactly as in the proof of Lemma 4.5. \square

COROLLARY 4.10. *If g is spectrally supported in the annulus $\frac{1}{h}\mathcal{C}^1$ then for any $r \in [2, \infty]$ we have*

$$(4.46) \quad \left\| S_h(t, t_0)(g \circ X(t_0, \tilde{h}z)) \right\|_{L^r} \lesssim \tilde{h}^{-d(\frac{2}{r}-\frac{1}{2})} |t - t_0|^{-d(\frac{1}{2}-\frac{1}{r})} \|g\|_{L^{r'}} \\ + \left\| S(t, t_0) \text{Op}(r_h^2)(g \circ X(t_0, \tilde{h}z)) \right\|_{L^r},$$

for all $t, t_0 \in [0, 1]$ and $0 < \tilde{h} \leq \tilde{h}_0$.

PROOF. We first apply the identity (4.41) at $t = t_0$ and notice that $\varphi_1(h\xi) = 1$ if $\xi \in \frac{1}{h}\mathcal{C}^1$ to have

$$g \circ X(t_0, \tilde{h}z) = \varphi_h^*(z, D_z)(g \circ X(t_0, \tilde{h}z))(z) - i \text{Op}(r_h^2)(g \circ X(t_0, \tilde{h}z))(z).$$

The estimate (4.44) implies that $\varphi_h^* \in S_{\lambda}^0 \cap S_{1, \frac{1}{2}}^0$ ($\lambda = \tilde{h}^{-1}$), so the estimate (4.40) applied to $\chi = \varphi_h^*$ and $f(z) = g \circ X(t_0, \tilde{h}z)$ gives for $r \in [2, \infty]$

$$\left\| S_h(t, t_0)(g \circ X(t_0, \tilde{h}z)) \right\|_{L^r} \lesssim \tilde{h}^{-d(\frac{1}{2}-\frac{1}{r})} |t - t_0|^{-d(\frac{1}{2}-\frac{1}{r})} \left\| g \circ X(t_0, \tilde{h}z) \right\|_{L^{r'}} \\ + \left\| S(t, t_0) \text{Op}(r_h^2)(g \circ X(t_0, \tilde{h}z)) \right\|_{L^r}.$$

Finally, since X is Lipschitz we have

$$\left\| g \circ X(t_0, \tilde{h}z) \right\|_{L^{r'}} \lesssim \tilde{h}^{-\frac{d}{r'}} \|g\|_{L^{r'}},$$

from which we conclude the proof. \square

To control the right-hand side of the estimate in the preceding Corollary, we use the following Lemma, whose proof is identical to that of Lemma 3.4.

LEMMA 4.11. *For any $\mu \in \mathbf{R}$, the operators $S_h(t, s)$, $\mathcal{S}_j(t, s)$ are bounded on $H^{\mu}(\mathbf{R}^d)$ uniformly in $t, s \in I$.*

Henceforth, we choose $\varepsilon_0 > 0$ arbitrarily small and

$$(4.47) \quad r_1 = \frac{1}{6}; \quad r_0 = \frac{2}{3} + \varepsilon_0 \text{ when } d = 1, \quad r_0 = 1 \text{ when } d = 2.$$

so that

$$2\delta(1 - r_0) = \frac{1}{2} - \frac{3}{2}\varepsilon_0, \quad 2\delta(2 - r_0) = 2 - \frac{3}{2}\varepsilon_0 \quad \text{when } d = 1,$$

$$2\delta(1 - r_0) = 0, \quad 2\delta(2 - r_0) = \frac{3}{2} \quad \text{when } d = 2$$

and (4.27) is fulfilled.

The next proposition proves what we called “pseudo-dispersive estimates” above.

PROPOSITION 4.12. *If $\mathcal{L}_j u_j(t, x) = 0$ and $u_j(t)$, $t \in [0, T]$ are spectrally supported in $\frac{1}{h}\mathcal{C}^1$ then for any $t_0 \in [0, T]$ and $r \in [2, \infty]$ we have*

$$\|u_j(t)\|_{L^r} \lesssim \tilde{h}^{-d(\frac{2}{r}-\frac{1}{2})} |t-t_0|^{-d(\frac{1}{2}-\frac{1}{r})} \|u_j(t_0)\|_{L^{r'}} + h^{-\frac{d}{4}} \|u_j(t_0)\|_{L^2}$$

where $r = \infty$ when $d = 1$ and $r \in [2, \infty)$ when $d = 2$.

PROOF. First, equation (4.38) implies that $w_h(t, z) := u_j(t, X(t, \tilde{h}z))$ satisfies

$$w_h(t) = S_h(t, t_0)w_h(t_0) - \int_{t_0}^t S(t, s) (\text{Op}_h(r_h^1)) w_h(s) ds =: (1) + (2).$$

Applying Corollary 4.10 to $g(x) = u_j(t_0, x)$ and then using the Sobolev embeddings

$$H^{\frac{d}{2}+\varepsilon} \hookrightarrow L^\infty, \quad H^{\frac{d}{2}} \hookrightarrow L^r, \quad \forall r \in [2, \infty)$$

together with Lemma 4.11 one gets

$$\|(1)\|_{L^r} \lesssim \tilde{h}^{-d(\frac{2}{r}-\frac{1}{2})} |t-t_0|^{-d(\frac{1}{2}-\frac{1}{r})} \|u_j(t_0)\|_{L^1} + \|\text{Op}(r_h^2)w_h(t_0)\|_{H^{\frac{d}{2}+\varepsilon_r}}$$

where

$$\varepsilon_r = 0, \text{ if } r \in [2, \infty), \quad \varepsilon_r = \frac{3}{2}\varepsilon_0, \text{ if } r = \infty.$$

The estimate (4.45) gives $r_h^2 \in S_{1, \frac{1}{2}, \frac{1}{2}}^{-2+2\delta(1-r_0)}$, hence

$$\|\text{Op}(r_h^2)w_h(t_0)\|_{H^{\frac{d}{2}+\varepsilon_r}} \lesssim \|w_h(t_0)\|_{H^{\frac{d}{2}+\varepsilon_r-2+2\delta(1-r_0)}}.$$

Similarly, since $r_h^1 \in S_{1, \frac{1}{2}, \frac{1}{2}}^{-1+2\delta(1-r_0)}$ one deduces with the aid of Lemma 4.11

$$\|(2)\|_{L^r} \lesssim \int_{t_0}^t \|w_h(s)\|_{H^{\frac{d}{2}+\varepsilon_r-1+2\delta(1-r_0)}} ds.$$

Putting together the estimates above, one obtains

$$\|w_h(t)\|_{L^r} \lesssim \tilde{h}^{-d(\frac{2}{r}-\frac{1}{2})} |t-t_0|^{-d(\frac{1}{2}-\frac{1}{r})} \|u_j(t_0)\|_{L^{r'}} + \int_{t_0}^t \|w_h(s)\|_{H^{\frac{d}{2}+\varepsilon_r-1+2\delta(1-r_0)}} ds.$$

When $d = 1$, $r = \infty$, $\frac{d}{2} + \varepsilon_r - 1 + 2\delta(1-r_0) = 0$ and since $X(t, \cdot) \in W^{1, \infty}(\mathbf{R}^d)$ we have for all $s \in [t_0, t]$

$$\|w_h(s)\|_{H^{\frac{d}{2}+\varepsilon_r-1+2\delta(1-r_0)}} \lesssim \tilde{h}^{-\frac{d}{2}} \|u_j(s)\|_{L^2}.$$

When $d = 2$, $r \in [2, \infty)$, $\frac{d}{2} + \varepsilon_r - 1 + 2\delta(1-r_0) = 0$ and thus by Lemma 4.11 it holds for all $s \in [t_0, t]$ that

$$\|w_h(s)\|_{H^{\frac{d}{2}+\varepsilon_r-1+2\delta(1-r_0)}} ds \lesssim \tilde{h}^{-\frac{d}{2}} \|u_j(s)\|_{L^2} ds \lesssim \tilde{h}^{-\frac{d}{2}} \|u_j(s)\|_{L^2}.$$

On the other hand, by Lemma (4.11), $\|u_j(s)\|_{L^2} \lesssim \|u_j(t_0)\|_{L^2}$ for all $s \in [t_0, t]$. Finally, noticing that

$$\|u_j(t)\|_{L^r} \lesssim \tilde{h}^{\frac{1}{r}} \|w_h\|_{L^r} \lesssim \|w_h\|_{L^r}$$

we conclude the proof. \square

REMARK 4.13. Strictly speaking, the preceding estimate is not a standard dispersive estimate since it does not show decay in time on the right hand side. The appearance of the non-decaying term $h^{-\frac{d}{4}} \|u_j(t_0)\|_{L^2}$ is however harmless for the purpose of proving Strichartz estimate in the next paragraph.

4.5. Strichartz estimates.

PROPOSITION 4.14. *Suppose that $\mathcal{L}_j u_j(t, x) = 0$ and $u_j(t)$, $t \in I := [0, T]$ is spectrally supported in $\frac{1}{h}\mathcal{C}^1$. Then,*

(i) *when $d = 1$ we have*

$$(4.48) \quad \|u_j\|_{L^4(I; L^\infty)} \lesssim h^{-\frac{3}{8}} \|u_j|_{t=0}\|_{L^2},$$

(ii) *when $d = 2$ we have with $q > 2$, $\frac{2}{q} + \frac{2}{r} = 1$*

$$(4.49) \quad \|u_j\|_{L^q(I; L^r)} \lesssim h^{\frac{1}{4} - \frac{1}{r'}} \|u_j|_{t=0}\|_{L^2};$$

consequently, for any $s_0 \in \mathbf{R}$ and $\varepsilon > 0$

$$(4.50) \quad \|u_j\|_{L^{2+\varepsilon}(I; W^{s_0 - \frac{3}{4} - \varepsilon, \infty})} \lesssim \|u_j|_{t=0}\|_{H^{s_0}}.$$

PROOF. For the two estimates (4.48), (4.49), using the TT^* argument, one need to show that

$$K := \int_I S(t, s) ds : L^{q'}(I; L^{r'}) \rightarrow L^q(I, L^r)$$

with norm bounded by h^M where $M = -\frac{3}{4}$ when $d = 1$ and $M = \frac{1}{2} - \frac{2}{r'}$ when $d = 2$. Moreover, since u_j is spectrally supported in $\frac{1}{h}\mathcal{C}^1$, it suffices to prove

$$(4.51) \quad \|Kf\|_{L^q(I, L^r)} \lesssim h^M \|f\|_{L^{q'}(I; L^{r'})}$$

for every f spectrally supported in $\frac{1}{h}\mathcal{C}^1$.

In view of the "pseudo dispersive estimate" in Proposition 4.12,

$$\|K(t)f\|_{L^r} \lesssim (1) + (2), \quad \text{with}$$

$$(1) = h^{-d(\frac{1}{r'} - \frac{1}{4})} \int_I |t - s|^{-d(\frac{1}{2} - \frac{1}{r'})} \|f(s)\|_{L^{r'}} ds,$$

$$(2) = h^{-\frac{d}{4}} \int_I \|f(s)\|_{L^2} ds.$$

(i) $d = 1$, $(q, r) = (4, \infty)$. By the Hardy-Littlewood-Sobolev inequality, $\|(1)\|_{L_t^q}$ is bounded by the right-hand side of (4.51). On the other hand, (2) can be estimated using Sobolev embedding as

$$(2) \lesssim h^{-\frac{d}{4}} h^{-\frac{d}{2}} \|f(s)\|_{L^1} ds \lesssim h^{-\frac{3d}{4}} \|f\|_{L^1 L^1},$$

which concludes the proof of (4.48).

(ii) $d = 2$, $q > 2$, $\frac{2}{q} + \frac{2}{r} = 1$. Again, the Hardy-Littlewood-Sobolev inequality yields

$$\|(1)\|_{L^q} \lesssim h^{-d(\frac{1}{r'} - \frac{1}{4})} \|f\|_{L^{q'}(I; L^{r'})}.$$

For (2) one uses the embedding $L^{r'} \hookrightarrow L^2$, $r' \in [1, 2)$

$$\|f(s)\|_{L^2} \lesssim h^{\frac{d}{2} - \frac{d}{r'}} \|f(s)\|_{L^{r'}},$$

which implies

$$(2) \lesssim h^{\frac{d}{4} - \frac{d}{r'}} \|f\|_{L^1 L^{r'}}.$$

The estimate (4.49) then follows.

Now, for any $\varepsilon > 0$, let $q = 2 + \varepsilon$ then

$$\frac{2}{r} = \frac{\varepsilon}{\varepsilon + 2}, \quad \frac{1}{4} - \frac{1}{r'} = -\frac{3}{4} + \frac{\varepsilon}{2(\varepsilon + 2)}.$$

Multiplying both sides of (4.49) by h^{-s_0} yields

$$(4.52) \quad \|u_j\|_{L^q(I; W^{s_0 + \frac{3}{4} - \frac{\varepsilon}{2(\varepsilon+2)}, r})} \lesssim \|u_j|_{t=0}\|_{H^{s_0}}.$$

Writing $s_0 - \frac{3}{4} + \frac{\varepsilon}{2(\varepsilon+2)} = a + b$, $a = s_0 - \frac{3}{4} - \varepsilon$, $b = \varepsilon + \frac{\varepsilon}{2(\varepsilon+2)} > \frac{d}{r}$ we obtain (4.50) from (4.52) via the Sobolev embedding $W^{a+b, r}(\mathbf{R}^d) \hookrightarrow W^{a, \infty}$ if $b > \frac{d}{r}$. \square

Recall from (4.1) and (4.47) that we have required

$$(4.53) \quad V \in L^\infty(I; W^{\rho, \infty}(\mathbf{R}^d)), \quad \gamma(\cdot, \xi) \in L^\infty(I; W^{\frac{2}{3}, \infty}(\mathbf{R}^d))$$

where $\rho = \frac{5}{3} + \varepsilon_0$, $\varepsilon_0 > 0$ when $d = 1$ and $\rho = 2$ when $d = 2$.

THEOREM 4.15. *Let $s_0 \in \mathbf{R}$, $I = [0, T]$, $T \in (0, +\infty)$ and $u \in L^\infty(I, H^{s_0}(\mathbf{R}^d))$ be a solution to the problem*

$$\partial_t u + T_V \cdot \nabla u + iT_\gamma u = f, \quad u|_{t=0} = u^0$$

where $\gamma = \sqrt{\lambda a}$ as defined in Theorem 2.2.

1. When $d = 1$, if for some $\varepsilon_0 > 0$, $V \in L^\infty(I, W^{\frac{5}{3} + \varepsilon_0, \infty}(\mathbf{R}^d))$, $\eta \in L^\infty(I, W^{\frac{5}{3}, \infty}(\mathbf{R}^d))$ then

$$\|u\|_{L^4(I; W^{s_0 - \frac{3}{8}, \infty})} \lesssim \|u^0\|_{H^{s_0}} + \|f\|_{L^1(I, H^{s_0})}.$$

2. When $d = 2$, if $V \in L^\infty(I, W^{2, \infty}(\mathbf{R}^d))$, $\eta \in L^\infty(I, W^{\frac{5}{3}, \infty}(\mathbf{R}^d))$ then for every $\varepsilon > 0$

$$\|u\|_{L^{2+\varepsilon}(I; W^{s_0 - \frac{3}{4} - \varepsilon, \infty})} \lesssim_\varepsilon \|u^0\|_{H^{s_0}} + \|f\|_{L^1(I, H^{s_0})}.$$

In the above estimates, the dependent constants depend on a finite number of semi-norms of the symbols V and γ .

PROOF. If u is a solution to (2.3) with data u^0 then by (4.7), the dyadic piece $\Delta_j u$ is a solution to $\mathcal{L}_j \Delta_j u = F_j$ with F_j given by (4.8) spectrally supported in $\frac{1}{h}\mathcal{C}^1$ if c_1 large enough. Under the regularity of V , γ given in 1. and 2., condition (4.53) is fulfilled. Using Duhamel's formula and applying the Strichartz estimates in Proposition 4.14 we deduce that

$$(4.54) \quad \|\Delta_j u\|_{L^q(I; W^{s_0 - \frac{q}{2} + \mu, \infty})} \lesssim \|\Delta_j u^0\|_{H^{s_0}} + \|F_j\|_{L^1(I; H^{s_0})},$$

where

$$q = 4, \quad \mu = \frac{1}{8}, \quad \text{when } d = 1; \quad q = 2 + \varepsilon, \quad \mu = \frac{1}{4} - \varepsilon, \quad \text{when } d = 2.$$

We are left with the estimate for $F_j = F_j^1 + R_j + F_j^3$ where F_j^k are given by (4.3), (4.8). Defining

$$\tilde{\Delta}_j = \sum_{|k-j| \leq 3} \Delta_k,$$

it follows from (4.5) that

$$\|R_j\|_{H^{s_0}} \lesssim \|\tilde{\Delta}_j u\|_{H^{s_0}}.$$

Using the symbolic calculus Theorem A.4 one obtains without any difficulty that

$$\|F_j^1\|_{L^1(I; H^{s_0})} \lesssim \|\tilde{\Delta}_j u\|_{L^1(I; H^{s_0})}.$$

For F_j^3 we use (3.22) to obtain that: if

$$V \in L^\infty(I, W^{\frac{4}{3}, \infty}(\mathbf{R}^d)), \quad \eta \in L^\infty(I, W^{\frac{5}{3}, \infty}(\mathbf{R}^d))$$

then

$$(4.55) \quad \|F_j^3\|_{L^1(I; H^{s_0})} \lesssim \|\Delta_j u\|_{L^1(I; H^{s_0})} + \|\Delta_j f\|_{L^1(I; H^{s_0})}.$$

Finally, putting together the above estimates we conclude from (4.54) that

$$\begin{aligned} \|u\|_{L^q(I; W^{s_0 - \frac{d}{2} + \mu, \infty})} &\leq \sum_j \|\Delta_j u\|_{L^q(I; W^{s_0 - \frac{d}{2} + \mu, \infty})} \\ &\lesssim \|u^0\|_{H^{s_0}} + \|u\|_{L^1(I; H^{s_0})} + \|f\|_{L^1(I; H^{s_0})} \\ &\lesssim \|u^0\|_{H^{s_0}} + \|f\|_{L^1(I; H^{s_0})}, \end{aligned}$$

where in the last inequality, we have used the energy estimate

$$\|u\|_{L^\infty(I; H^{s_0})} \lesssim \|u^0\|_{H^{s_0}} + \|f\|_{L^1(I; H^{s_0})}.$$

The proof is complete. \square

According to Remark 2.3, after having the estimate for u -solution to (2.3) one can use the symbolic calculus to obtain the desired estimates for the original solution (η, ψ, B, V) as stated in Theorem 1.2.

Acknowledgment. This work was partially supported by the labex LMH through the grant no ANR-11-LABX-0056-LMH in the "Programme des Investissements d'Avenir". I sincerely thank Prof. Nicolas Burq and Prof. Claude Zuily for many helpful discussions and encouragements. I thank Prof. Daniel Tataru for suggesting me his joint work [19] with Herbert Koch, with whom I shared an interesting discussion.

Appendix A.

DEFINITION A.1. 1. (Littlewood-Paley decomposition) Let $\psi \in C_0^\infty(\mathbf{R}^d)$ be such that

$$(A.1) \quad \psi(\theta) = 1 \quad \text{for } |\theta| \leq 1, \quad \psi(\theta) = 0 \quad \text{for } |\theta| > 2.$$

Then we define $\chi(\theta, \eta) = \sum_{k=0}^{+\infty} \psi_{k-3}(\theta) \varphi_k(\eta)$, where

$$\psi_k(\theta) = \kappa(2^{-k}\theta) \quad \text{for } k \in \mathbf{Z}, \quad \varphi_0 = \kappa_0, \quad \text{and} \quad \varphi_k = \psi_k - \psi_{k-1} \quad \text{for } k \geq 1.$$

Given a temperate distribution u and an integer k in \mathbf{N} we also introduce $S_k u$ and $\Delta_k u$ by $S_k u = \psi_k(D_x)u$ and $\Delta_k u = S_k u - S_{k-1} u$ for $k \geq 1$ and $\Delta_0 u = S_0 u$. Then we have the formal decomposition

$$(A.2) \quad u = \sum_{k=0}^{\infty} \Delta_k u.$$

2. (Hölder spaces) For $k \in \mathbf{N}$, we denote by $W^{k, \infty}(\mathbf{R}^d)$ the usual Sobolev spaces. For $\rho = k + \sigma$, $k \in \mathbf{N}$, $\sigma \in (0, 1)$ denote by $W^{\rho, \infty}(\mathbf{R}^d)$ the space of functions whose derivatives up to order k are bounded and uniformly Hölder continuous with exponent σ .

Let us review notations and results about Bony's paradifferential calculus (cf. [7], [22]).

DEFINITION A.2. 1. (Symbols) Given $\rho \in [0, \infty)$ and $m \in \mathbf{R}$, $\Gamma_\rho^m(\mathbf{R}^d)$ denotes the space of locally bounded functions $a(x, \xi)$ on $\mathbf{R}^d \times (\mathbf{R}^d \setminus 0)$, which are C^∞ with respect to ξ for $\xi \neq 0$ and such that, for all $\alpha \in \mathbf{N}^d$ and all $\xi \neq 0$, the function $x \mapsto \partial_\xi^\alpha a(x, \xi)$ belongs to $W^{\rho, \infty}(\mathbf{R}^d)$ and there exists a constant C_α such that,

$$(A.3) \quad \forall |\xi| \geq \frac{1}{2}, \quad \|\partial_\xi^\alpha a(\cdot, \xi)\|_{W^{\rho, \infty}(\mathbf{R}^d)} \leq C_\alpha (1 + |\xi|)^{m - |\alpha|}.$$

Let $a \in \Gamma_\rho^m(\mathbf{R}^d)$, we define the semi-norm

$$(A.4) \quad M_\rho^m(a) = \sup_{|\alpha| \leq 2(d+2)+\rho} \sup_{|\xi| \geq 1/2} \left\| (1 + |\xi|)^{|\alpha|-m} \partial_\xi^\alpha a(\cdot, \xi) \right\|_{W^{\rho, \infty}(\mathbf{R}^d)}.$$

2. (Paradifferential operators) Given a symbol a , we define the paradifferential operator T_a by

$$(A.5) \quad \widehat{T_a u}(\xi) = (2\pi)^{-d} \int \chi(\xi - \eta, \eta) \widehat{a}(\xi - \eta, \eta) \rho(\eta) \widehat{u}(\eta) d\eta,$$

where $\widehat{a}(\theta, \xi) = \int e^{-ix \cdot \theta} a(x, \xi) dx$ is the Fourier transform of a with respect to the first variable; χ and ρ are two fixed C^∞ functions such that:

$$(A.6) \quad \rho(\eta) = 0 \quad \text{for } |\eta| \leq \frac{1}{5}, \quad \rho(\eta) = 1 \quad \text{for } |\eta| \geq \frac{1}{4},$$

and $\chi(\theta, \eta)$ is defined by $\chi(\theta, \eta) = \sum_{k=0}^{+\infty} \kappa_{k-3}(\theta) \varphi_k(\eta)$.

We remark that the cut-off function χ in the preceding definition has the following properties for some $0 < \varepsilon_1 < \varepsilon_2 < 1$

$$(A.7) \quad \begin{cases} \chi(\eta, \xi) = 1, & \text{for } |\eta| \leq \varepsilon_1(1 + |\xi|), \\ \chi(\eta, \xi) = 0, & \text{for } |\eta| \geq \varepsilon_2(1 + |\xi|). \end{cases}$$

DEFINITION A.3. Let $m \in \mathbf{R}$. An operator T is said to be of order m if, for all $\mu \in \mathbf{R}$, it is bounded from H^μ to $H^{\mu-m}$.

Symbolic calculus for paradifferential operators is summarized in the following theorem.

THEOREM A.4. (Symbolic calculus) Let $m \in \mathbf{R}$ and $\rho \in [0, \infty)$.

(i) If $a \in \Gamma_0^m(\mathbf{R}^d)$, then T_a is of order m . Moreover, for all $\mu \in \mathbf{R}$ there exists a constant K such that

$$(A.8) \quad \|T_a\|_{H^\mu \rightarrow H^{\mu-m}} \leq K M_0^m(a).$$

(ii) If $a \in \Gamma_\rho^m(\mathbf{R}^d)$, $b \in \Gamma_\rho^{m'}(\mathbf{R}^d)$ then $T_a T_b - T_{a\sharp b}$ is of order $m + m' - \rho$ with

$$a\sharp b := \sum_{|\alpha| < \rho} \frac{(-i)^\alpha}{\alpha!} \partial_\xi^\alpha a(x, \xi) \partial_x^\alpha b(x, \xi).$$

Moreover, for all $\mu \in \mathbf{R}$ there exists a constant K such that

$$(A.9) \quad \|T_a T_b - T_{a\sharp b}\|_{H^\mu \rightarrow H^{\mu-m-m'+\rho}} \leq K M_\rho^m(a) M_0^{m'}(b) + K M_0^m(a) M_\rho^{m'}(b).$$

(iii) Let $a \in \Gamma_\rho^m(\mathbf{R}^d)$. Denote by $(T_a)^*$ the adjoint operator of T_a and by \bar{a} the complex conjugate of a . Then $(T_a)^* - T_{\bar{a}}$ is of order $m - \rho$ with

$$b := \sum_{|\alpha| < \rho} \frac{(-i)^\alpha}{\alpha!} \partial_\xi^\alpha \partial_x^\alpha \bar{a}(x, \xi).$$

Moreover, for all μ there exists a constant K such that

$$(A.10) \quad \|(T_a)^* - T_{\bar{a}}\|_{H^\mu \rightarrow H^{\mu-m+\rho}} \leq K M_\rho^m(a).$$

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