Numerical time integration schemes for nonsmooth multibody systems in the event-driven framework

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Motivations

High-fidelity dynamical simulation of mechanisms

Nonsmooth multi-body systems with

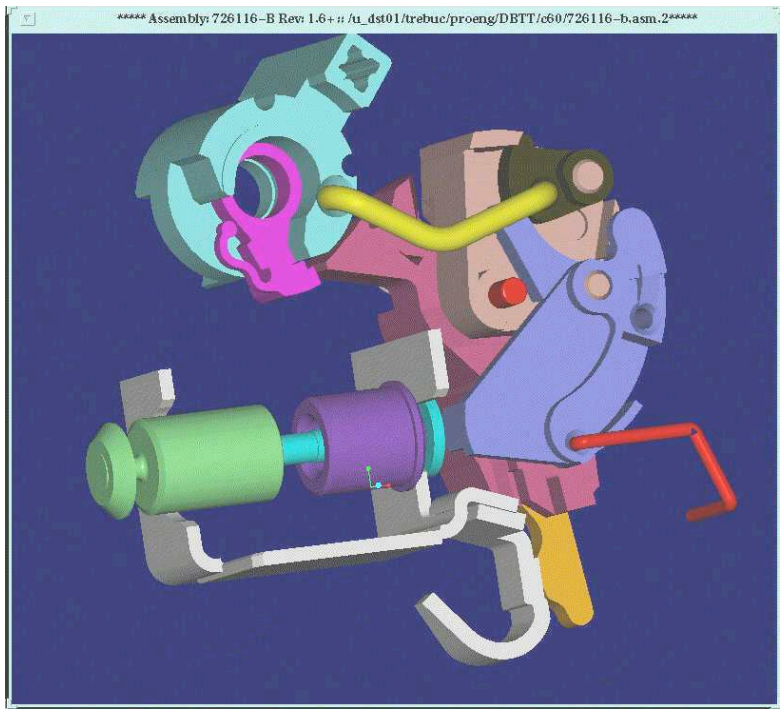
- Signorini unilateral contact,
- Coulomb friction,
- Newton (or Poisson) impact law,
- clearances in joints.

Industrial context

- Real CAD geometries with edge discontinuities
- Robustness w.r.t large number of events:
  contact activation and deactivation
  finite accumulation of impacts
  stick/slip transitions.

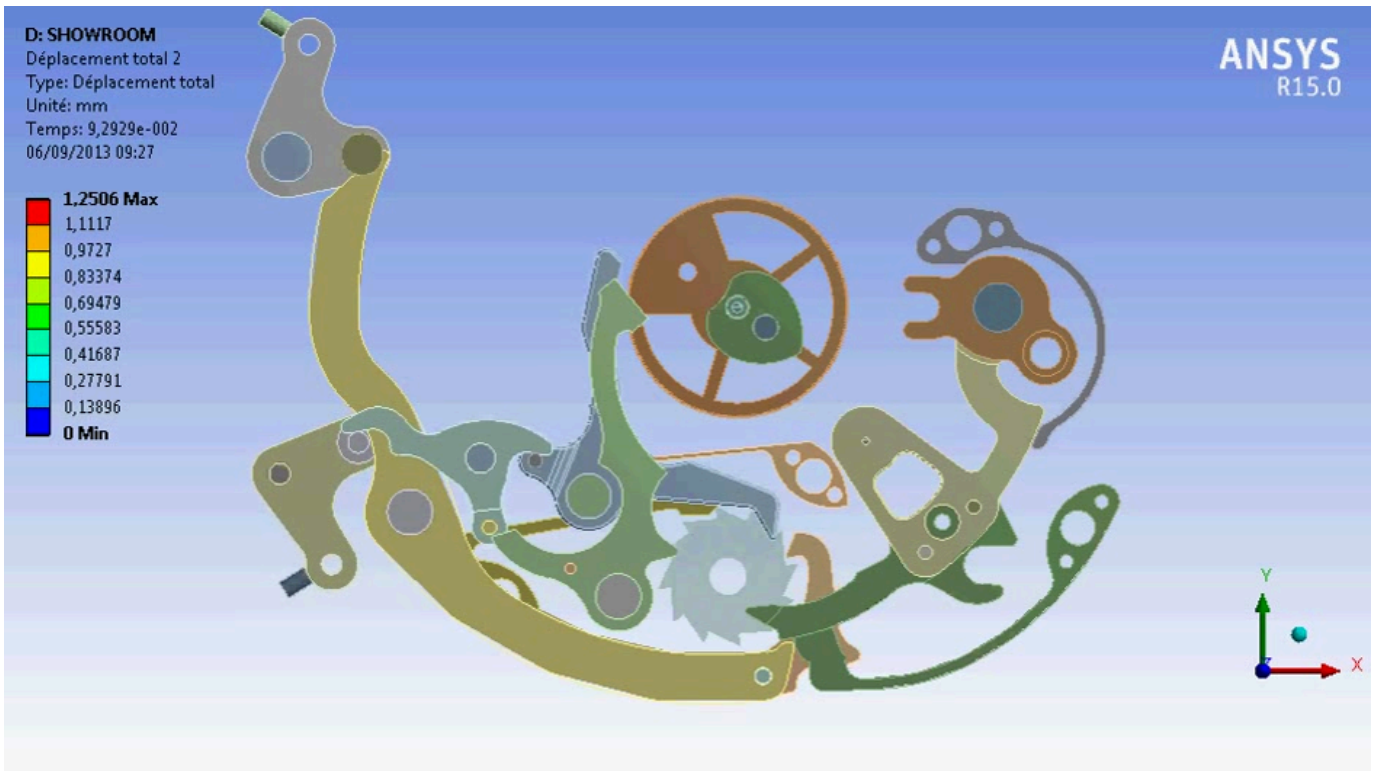
Motivations

Simulation of Circuit breakers (INRIA/Schneider Electric)



Motivations

Simulation of watch chronograph mechanism (INRIA/ANSYS)



Objectives

- Time–integration methods in an event–driven framework
- Review of D.A.E. integrators with various indices (from 1 to 3).
- Standard comparisons on academical examples
- Performance profiles on industrial benchmarks

Modeling framework

Signorini unilateral contact and impact law

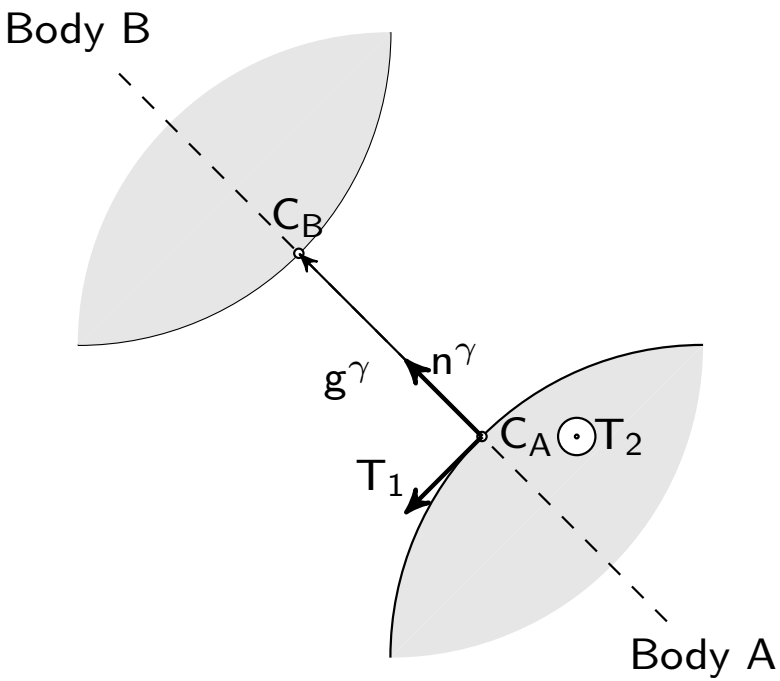


Figure: Signed distance between two bodies A and B at contact γ

Unilateral contact law :

$$0 \leqslant g(q) \perp \lambda \geqslant 0. \qquad (1)$$

Newton Impact law:

$$\text{if } g(q) \leqslant 0, \text{ then } U^+ = -eU^- \qquad (2)$$

U : normal relative velocity ($U = \dot{g}$)
e : kinetic coefficient of restitution

Modeling framework (cont.)

Equations of motion

$$\begin{cases} \dot{q} = v \\ M(q)\dot{v} = F(q, v, t) + G^T(q)\lambda \\ g^\alpha(q) = 0,\ \alpha \in \mathbb{B} \\ 0 \leqslant g^\beta(q) \perp \lambda^\beta \geqslant 0,\ \beta \in \mathbb{U}, \\ \text{if } g^\beta(q) \leqslant 0, \text{ then } U^{\beta,+} = -eU^{\beta,-} \end{cases} \tag{3}$$

- $g(q) \in \mathbb{R}^m$: vector of constraints
- $\mathbb{B} \subset \mathbb{N}$ index set of bilateral constraints
- $\mathbb{U} \subset \mathbb{N}$ index set of unilateral constraints
- $G(q) = \nabla^T g(q) \in \mathbb{R}^{m \times n}$ Jacobian matrix of the constraints
- $\lambda \in \mathbb{R}^m$ is the Lagrange multipliers vector associated to the constraints.

Modeling framework (cont.)

Index sets of active constraints

The set of all constraints is denoted by $I_0 = \mathbb{B} \cup \mathbb{U}$.
Closed contacts index set:

$$I_1 = \{\gamma \in I_0,\ g^\gamma(q) = 0\}$$

Closed contacts index set for a non trivial period of time:

$$I_2 = \{\gamma \in I_0,\ g^\gamma(q) = 0,\ \dot{g}^\gamma(q) = 0\}$$

Position based constraints : index-3 differential algebraic equation.

On the period over which I_2 is constant, we solve

$$\begin{cases} \dot{q} = v \\ M(q)\dot{v} = F(q, v, t) + G^T(q)\lambda \\ g^\gamma(q) = 0,\ \gamma \in I_2. \end{cases} \tag{4}$$

Modeling framework (cont.)

Lower index differential algebraic equation

Velocity based constraints : index-2 differential algebraic equation.

If the constraint $g(\cdot)$ is differentiated once with respect to time, one obtains the following index-2 DAE

$$\begin{cases} \dot{q} = v \\ M(q)\dot{v} = F(q, v, t) + G^T(q)\lambda \\ G^\gamma(q)v = 0,\ \gamma \in I_2. \end{cases} \tag{5}$$

Acceleration based constraints : index-1 differential algebraic equation.

If $g(\cdot)$ is differentiated twice, one gets the index-1 DAE

$$\begin{cases} \dot{q} = v \\ M(q)\dot{v} = F(q, v, t) + G^T(q)\lambda \\ G^\gamma(q)\dot{v} + \dfrac{dG^\gamma(q)}{dt}v = 0,\ \gamma \in I_2. \end{cases} \tag{6}$$

Time–stepping schemes

Principle of nonsmooth event capturing methods

1. A unique formulation of the dynamics is considered. For instance, a dynamics in terms of measures.

$$\begin{cases} mdu = dr \\ \dot{q} = u^+ \\ 0 \leqslant dr \perp \dot{u}^+ \geqslant 0 \text{ if } q \leqslant 0 \end{cases} \tag{7}$$

2. The time-integration is based on a consistent approximation of the equations in terms of measures. For instance,

$$\int_{]t_k,t_{k+1}]} du = \int_{]t_k,t_{k+1}]} du = (u^+(t_{k+1}) - u^+(t_k)) \approx (u_{k+1} - u_k) \tag{8}$$

3. Consistent approximation of measure inclusion.

$$-dr \in N_{K(t)}(u^+(t)) \tag{9}$$

$$\rightarrow \quad \begin{cases} p_{k+1} \approx \int_{]t_k,t_{k+1}]} dr \\ p_{k+1} \in N_{K(t)}(u_{k+1}) \end{cases} \tag{10}$$

Event-driven schemes

Principle of nonsmooth event tracking methods

Time-decomposition of the dynamics in

- *modes*, time-intervals in which the dynamics is smooth (I_1 and I_2 invariant),
- discrete *events*, times where the dynamics is nonsmooth (changes in I_1 and/or I_2).

Comments

On the numerical point of view, we need

- detect events with for instance root-finding procedure.
  - Dichotomy and interval arithmetic
  - Newton procedure for C^2 function and polynomials
- solve the non smooth dynamics at events with a reinitialization rule of the state,
- integrate the smooth dynamics between two events with any DAE solvers associated with a given index formulation.

Comparison

Numerical time–integration methods for Nonsmooth Multibody systems (NSMBS):

Nonsmooth event capturing methods (Time–stepping methods)

- ⊕ robust, stable and proof of convergence
- ⊕ low kinematic level for the constraints
- ⊕ able to deal with finite accumulation
- ⊖ very low order of accuracy even in free flight motions

Nonsmooth event tracking methods (Event–driven methods)

- ⊕ higher order accuracy integration of free flight motions
- ⊖ no proof of convergence
- ⊖ sensitivity to numerical thresholds
- ⊖ reformulation of constraints at higher kinematic levels.
- ⊖ unable to deal with finite accumulation

Index-1 DAE integrators

Using the acceleration based constraints, we have to solve

$$\begin{bmatrix} I & 0 & 0 \\ 0 & M(q) & -G^T(q) \\ 0 & G(q) & 0 \end{bmatrix} \begin{pmatrix} \dot{q} \\ \dot{v} \\ \lambda \end{pmatrix} = \begin{pmatrix} v \\ F(q,v,t) \\ -\frac{dG(q)}{dt} v \end{pmatrix} \tag{11}$$

The Lagrange multipliers $\lambda(v,q,t)$ can be obtained for a given q and v by solving

$$\left[G(q) M^{-1}(q) G^T(q) \right] \lambda(v,q,t) = - \left[G(q) M^{-1} F(q,v,t) + \frac{dG(q)}{dt} v \right] \tag{12}$$

The following index-1 DAE

$$\begin{bmatrix} I & 0 \\ 0 & M(q) \end{bmatrix} \begin{pmatrix} \dot{q} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} v \\ F(q,v,t) + G^T(q)\, \lambda(v,q,t) \end{pmatrix} \tag{13}$$

can be numerically solved by a any solver for ODE.
We use in the work embedded 4/5 order Runge–Kutta-Fehlberg (RKF45) method.

Index-2 DAE integrators

Half-explicit method of order 5 (HEM5) [Brasey and Hairer, 1993]

8 stages $T_i = t_n + c_i h$

$$\begin{cases} M(Q_i)\dot{V}_i = F(Q_i, V_i, T_i) + G^T(Q_i)\Lambda_i \\ \dot{Q}_i = V_i \\ G(Q_i)V_i = 0, \end{cases} \tag{14}$$

At each stage, we solve $Q_i = q_n + h\sum_{j<i} a_{ij}\dot{Q}_j$, $V_i = v_n + h\sum_{j<i} a_{ij}\dot{V}_j$.

$$\begin{pmatrix} M(Q_i) & -G^T(Q_i) \\ G(Q_{i+1}) & 0 \end{pmatrix} \begin{pmatrix} \dot{V}_i \\ \Lambda_i \end{pmatrix} = \begin{pmatrix} F(Q_i, V_i, T_i) \\ r_i \end{pmatrix}, \tag{15}$$

where $r_i = -\dfrac{G(Q_{i+1})}{h a_{i+1,i}}(v_n + h\sum_{j=1}^{i-1} a_{i+1,j}\dot{V}_j)$.

Comments

- Exact enforcement velocity constraints $G(Q_i)V_i = 0\,, \forall i = 1\ldots 8.$
- Λ_i is NOT an approximation of $\lambda(T_i)$
- non symmetric matrix solver.

Index-2 DAE integrators

Partitioned half-explicit method of order 5/6 (PHEM56) [Murua, 1997]

6 stages

$$\begin{cases} \dot{Q}_i = V_i \\ M(Q_i, \tau_i)\dot{V}_i = F(Q_i, V_i, \tau_i) + G^T(\bar{Q}_i, \tau_i)\Lambda_i \\ G(\bar{Q}_i, \bar{\tau}_i)\bar{V}_i = 0, \end{cases} \tag{14}$$

where

$$\begin{cases} Q_i = q_n + h\sum_{j<i} a_{ij} V_j, & V_i = v_n + h\sum_{j<i} a_{ij}\dot{V}_j \\ \bar{Q}_i = q_n + h\sum_{j\leqslant i} \bar{a}_{ij} V_j, & \bar{V}_i = v_n + h\sum_{j\leqslant i} \bar{a}_{ij}\dot{V}_j \\ \tau_i = t_n + c_i h, & \bar{\tau}_i = t_n + \bar{c}_i h. \end{cases} \tag{15}$$

At each stage, we solve

$$\begin{pmatrix} M(Q_i, \tau_i) & -G^T(Q_i, \tau_i) \\ G(\bar{Q}_i, \bar{\tau}_i) & 0 \end{pmatrix} \begin{pmatrix} \dot{V}_i \\ \Lambda_i \end{pmatrix} = \begin{pmatrix} F(Q_i, V_i, \tau_i) \\ r_i \end{pmatrix} \tag{16}$$

with $r_i = -\dfrac{G(\bar{Q}_i, \bar{\tau}_i)}{h\bar{a}_{i,i}}(v_n + h\sum_{j=1}^{i-1} \bar{a}_{i,j}\dot{V}_j)$.

Index-3 DAE integrators

The generalized-α cheme. (Géradin & Cardona 1989, Brüls & Arnold 2007)

Collocation of the dynamics at time t_{n+1}

$$M\ddot{q}_{n+1} = F(q_{n+1}, \dot{q}_{n+1}, t_{n+1}) + G^T(q_{n+1})\lambda_{n+1} \tag{17}$$

α-schemes approximations:

$$\begin{cases} q_{n+1} = q_n + h\dot{q}_n + h^2(\frac{1}{2} - \beta)a_n + h^2\beta a_{n+1} \\ \dot{q}_{n+1} = \dot{q}_n + h(1-\gamma) + h\gamma a_{n+1} \\ (1-\alpha_m)a_{n+1} + \alpha_m a_n = (1-\alpha_f)\ddot{q}_{n+1} + \alpha_f \ddot{q}_n. \end{cases} \tag{18}$$

Newton's iterations to reduce the dynamical and the constraint residuals

$$\begin{cases} R_q = M(q,t)\ddot{q} - F(q,\dot{q},t) - G^T(q)\lambda \\ R_\lambda = g(q) \qquad \text{or} \qquad R_\lambda = G(q)\dot{q} \end{cases} \tag{19}$$

Comments

- Enforcement of the constraints at the Newton tolerance:

$$g(q_{n+1}) = 0 \quad \text{or} \quad G(q_{n+1})\dot{q}_{n+1}$$

Time step size control strategy

General formula

$$h_{\text{opt}} = \text{safe}\left(\frac{\text{tol}\ h}{\text{err}}\right)^{1/p} h. \tag{20}$$

with err a practical error estimation, and tol the user defined tolerance.

- For the generalized-α scheme ($p = 2$) (Géradin & Cardona 1989) :

$$\text{err} = q_{n+1} - q_n - \frac{1}{h}\dot{q}_n - \frac{h^2}{3}\ddot{q}_n - \frac{h^2}{6}\ddot{q}_{n+1} + \mathcal{O}(h^4). \tag{21}$$

- For the HEM5 scheme ($p = 5$):

$$\begin{aligned} \text{err}_1 &= \|q_{n+1} - Q_8\|_s = O(h^4), \\ \text{err}_2 &= \|q_{n+1} - q_n - h(\tfrac{5}{2}Q_7 - \tfrac{3}{2}Q_8)\|_s = O(h^3), \\ \text{err} &= \frac{{\text{err}_1}^2}{\text{err}_1 + 0.01\,\text{err}_2} = O(h^5) \end{aligned} \tag{22}$$

- For the RK-Fehlberg scheme: $\text{err} = \|y_{5^{th}order} - y_{4^{th}order}\|$ and $p = 4$.

Projection on the constrained manifolds

1. Projection on position constraint:
   Q_n the position obtained at time t_n. The projected position q_n is obtained by

$$\begin{cases} M(Q_n)(q_n - Q_n) + G^T(Q_n)\Lambda = 0 \\ g(q_n) = 0 \end{cases} \tag{23}$$

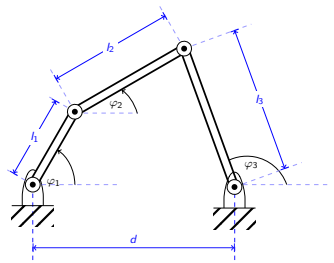
2. Projection on velocity constraint:
   V_n the velocity obtained at time t_n. The projected velocity v_n is obtained by

$$\begin{cases} M(Q_n)(v_n - V_n) + G^T(q_n)\Lambda = 0 \\ G(q_n)v_n = 0. \end{cases} \tag{24}$$

Similar projection techniques can be found in [Shampine, 1986, Hairer and Wanner, 2002, Eich, 1993, Rheinboldt and Simeon, 1995].

Academic Examples

Four-bar linkage

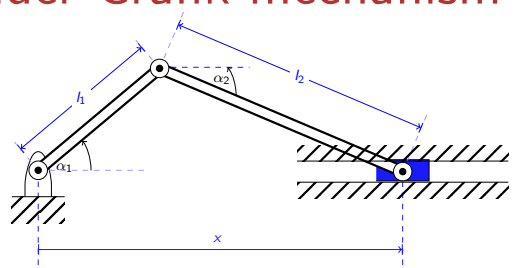


$$q = [\varphi_1,\ \varphi_2,\ \varphi_3]^T$$

$$g_1(q) := l_1\cos(\varphi_1)+l_2\cos(\varphi_2)-l_3\cos(\varphi_3)-d_1 = 0$$

$$g_2(q) := l_1\sin(\varphi_1)+l_2\sin(\varphi_2)-l_3\sin(\varphi_3) = 0$$

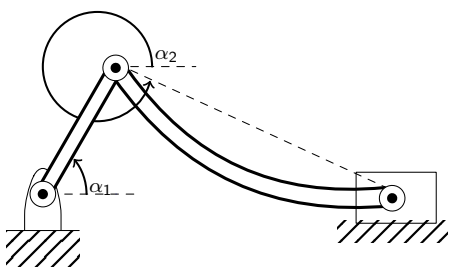
Slider-Crank mechanism



$$q = [\alpha_1,\ \alpha_2]^T$$

$$g_1(q) := l_1\sin(\alpha_1)+l_2\sin(\alpha_2) = 0$$

Flexible slider-Crank mechanism (Simeon (1994))



$$q = [\alpha_1,\ \alpha_2,\ x,\ q_1,\ q_2,\ q_3,\ q_4]^T$$

$$g_1(q) := l_1\sin(\alpha_1)+l_2\sin(\alpha_2)+q_4\sin\alpha_2 = 0$$

$$g_2(q) := x-l_1\cos(\alpha_1)-l_2\cos(\alpha_2)-q_4\cos(\alpha_2) = 0$$

$$g_3(q) := \alpha_1 - \Omega t = 0$$

Numerical simulation settings

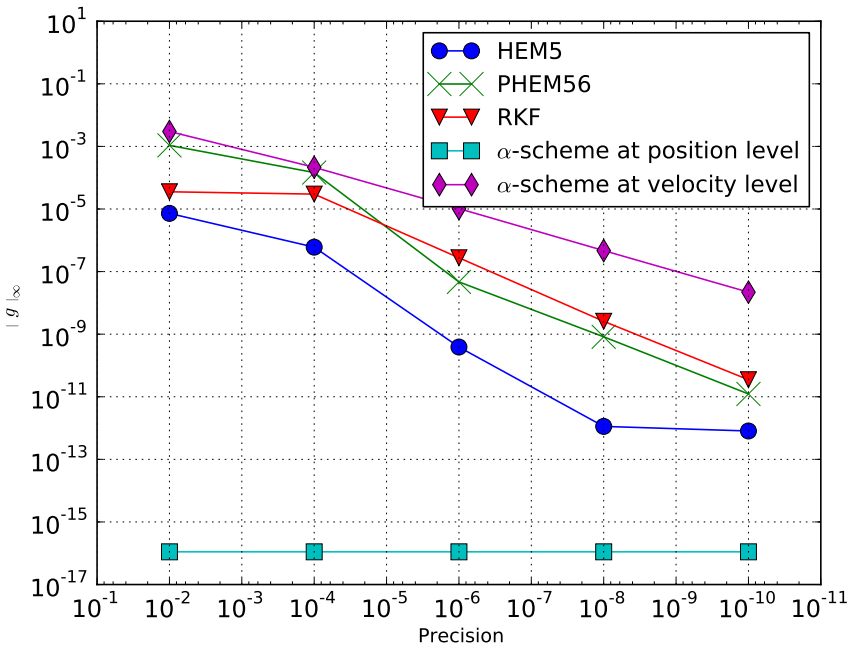
- Solvers : RKF45, HEM5, PHEM56, Index-2 and index-3 generalized-α schemes.

Table: Parameters for time step control

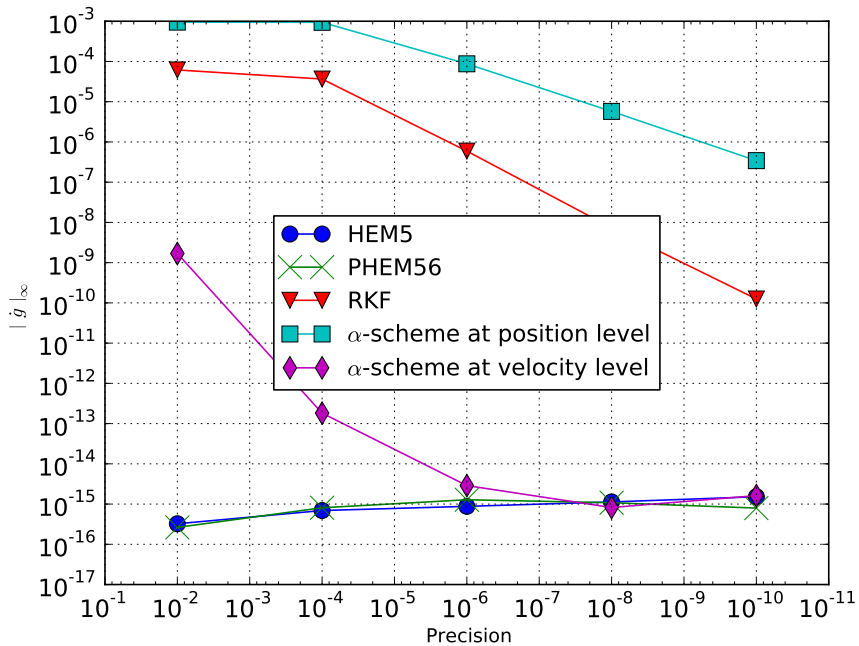
Integration tolerance (tol)	Minimum time step	Tolerance of Newton's loop	Maximum drift of g and $\dot{g}$	safety factor (s)
$[10^{-10}, 10^{-2}]$(*)	10^{-6}s	10^{-10}	10^{-2}	0.9

(*) We vary the value of tol to compare the computational effort and the drift of the constraints

Slider-crank mechanism. Violation of the constraints



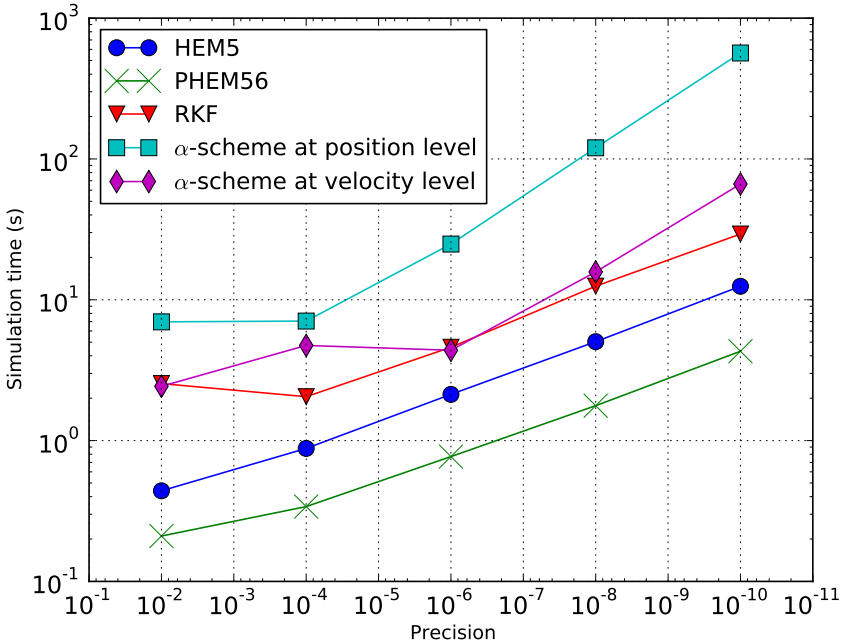
(a) maximum of violation of the position constraint



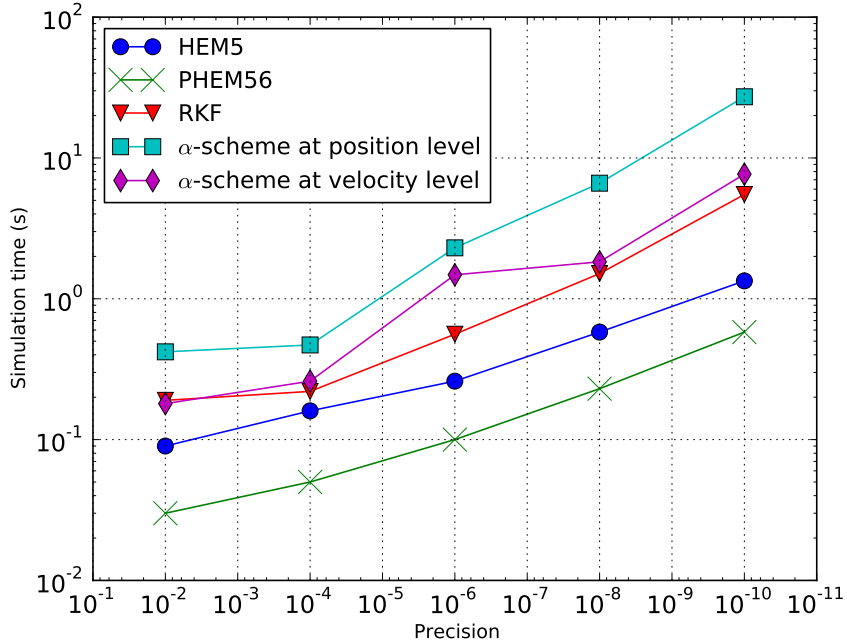
(b) maximum of violation of the velocity constraint

Figure: Slider crank: simulation characteristics

Work-Precision diagrams



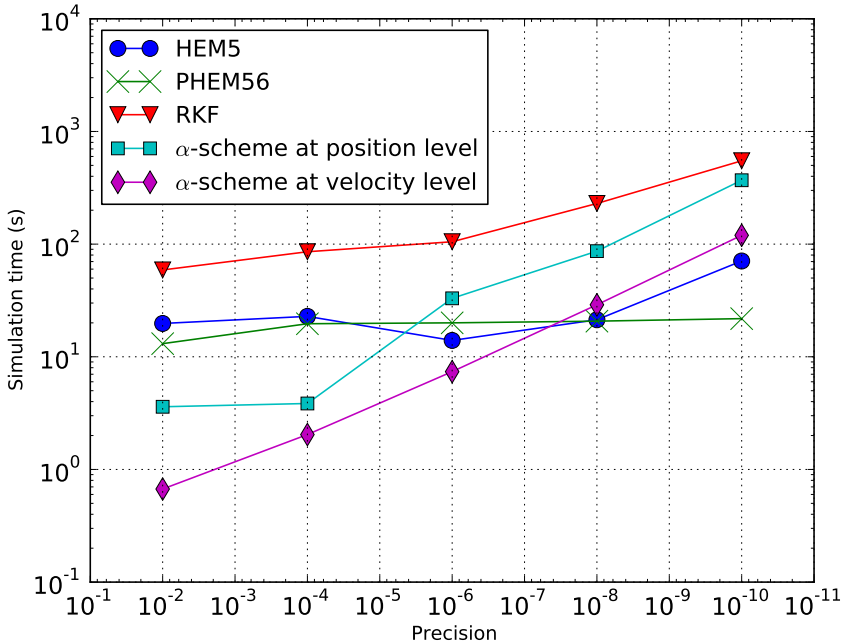
(a) Four-bar linkage



(b) Slider-crank

Figure: Work/Precision diagrams for the four-bar linkage, the slider-crank, and the flexible slider-crank

Work-Precision diagrams



(a) Flexible slider-crank

Figure: Work/Precision diagrams for the four-bar linkage, the slider-crank, and the flexible slider-crank

Industrial Examples

Industrial benchmark libraries

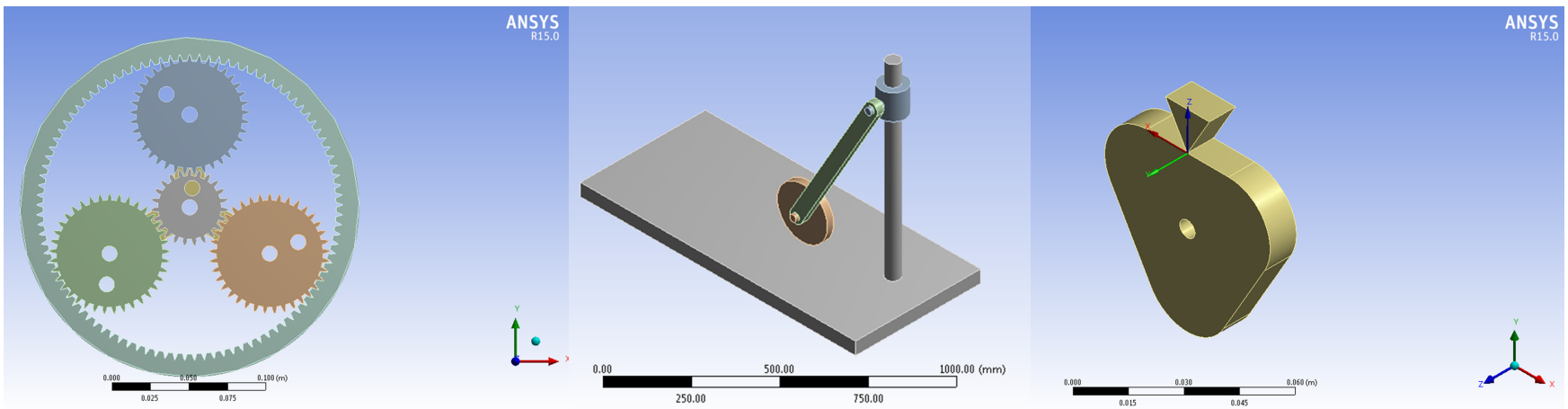
- Non regression tests of ANSYS Rigid Body Solver RDB.
- Implemented solvers in ANSYS RDB : RK4, HEM5, index-2 generalized-α scheme
- Coordinate projection on the constraints at velocity (if needed) and position levels
- User required accuracy : $\mathrm{tol} = 10^{-4}$

Table: Characteristics of the sets of problems

		# of DOF		# of joints eq.		# of contacts
Set Id.	# of problems in the set	Max	Min	Max	Min	Max
1	43	19	1	38	2	-
2	25	18	2	22	3	-
3	21	11	2	15	2	7
4	9	31	5	15	4	1

Industrial Examples

Illustrations of the first set



(a) Epicyclic gear train

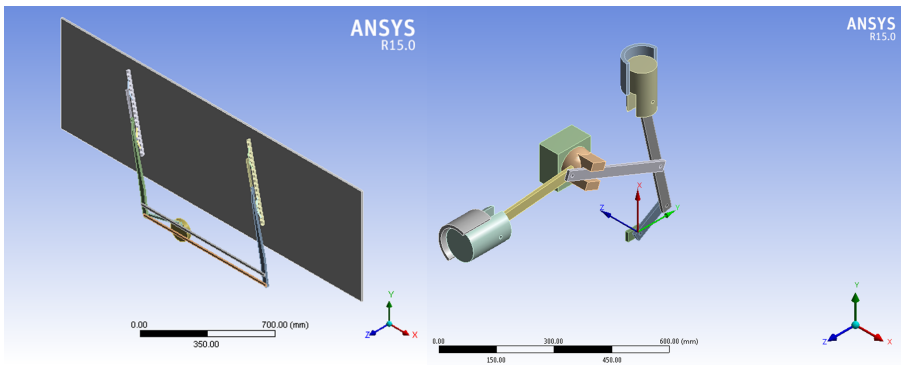
(b) Rotating disk attached to a rod with a collar

(c) A cam mechanism

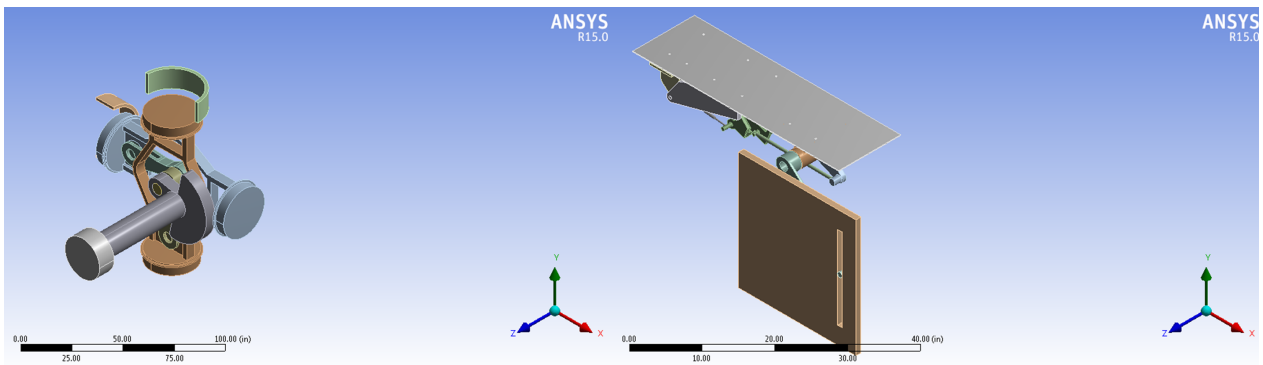
Figure: Examples from the first set

Industrial Examples

Illustrations of the second set



(a) Windshield wiper (b) Trunnion mechanism

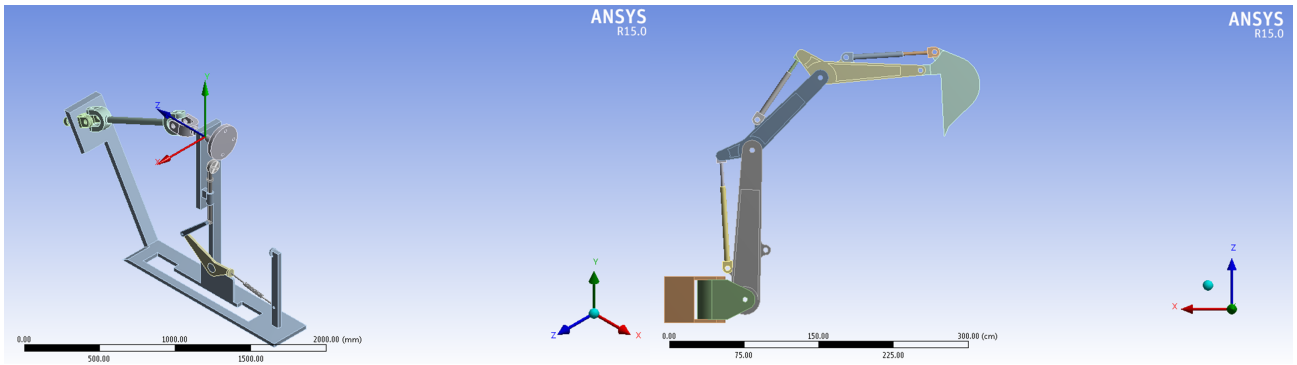


(c) Air piston (d) Subway door

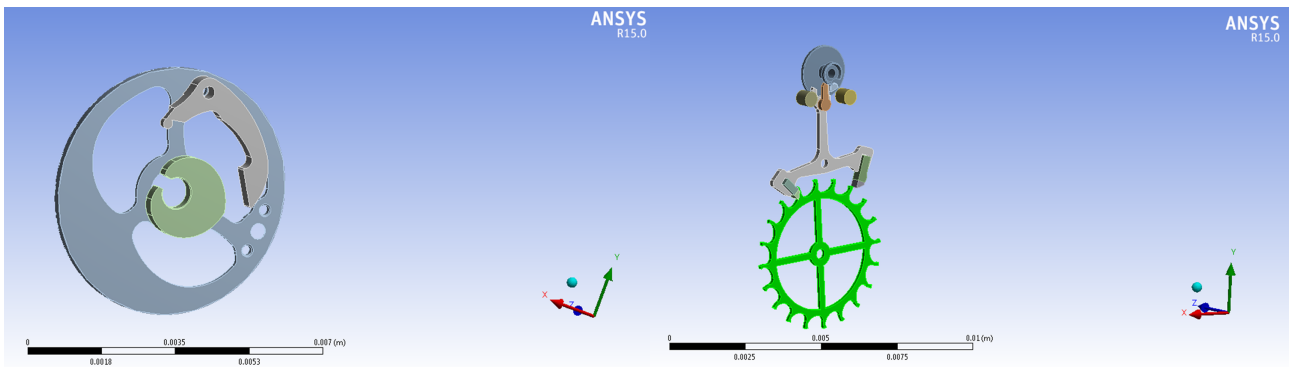
Figure: Examples from the second set

Industrial Examples

Illustrations of the third set



(a) Press machine (b) An excavator model

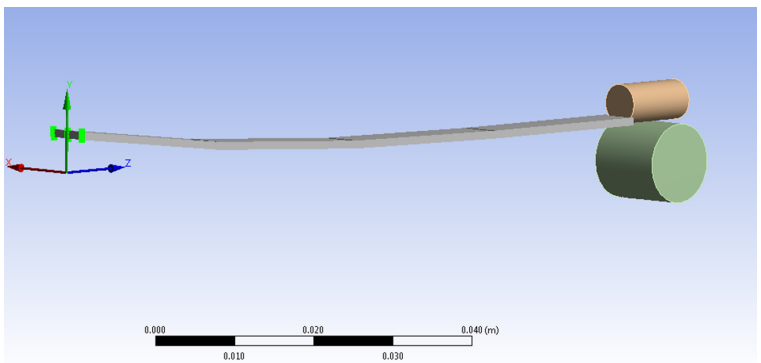


(c) A watch mechanism sub-assembly (d) Escapement of a mechanical watch

Figure: Examples from the third set

Industrial Examples

Illustrations of the fourth set



(a) Beam under gravity in contact with a cylinder

Figure: Example from the fourth set

Performance profiles [Dolan and Moré, 2002]

- Given a set of problems $\mathcal{P}$
- Given a set of solvers $\mathcal{S}$
- A performance measure for each problem with a solver $t_{p,s}$ (cpu time, flops, ...)
- Compute the performance ratio

$$\tau_{p,s} = \frac{t_{p,s}}{\min\limits_{s \in \mathcal{S}} t_{p,s}} \geqslant 1 \tag{25}$$

- Compute the performance profile $\rho_s(\tau) : [1, +\infty] \to [0, 1]$ for each solver $s \in \mathcal{S}$

$$\rho_s(\tau) = \frac{1}{|\mathcal{P}|} \big|\{p \in \mathcal{P} \mid \tau_{p,s} \leqslant \tau\}\big| \tag{26}$$

The value of $\rho_s(1)$ is the probability that the solver s will win over the rest of the solvers.

Performance profiles on first and second sets

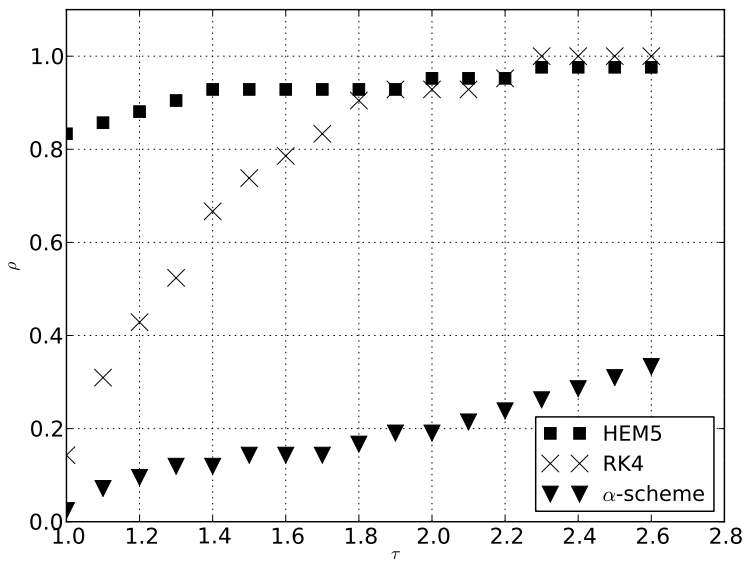


Figure: Performance profile of the first set

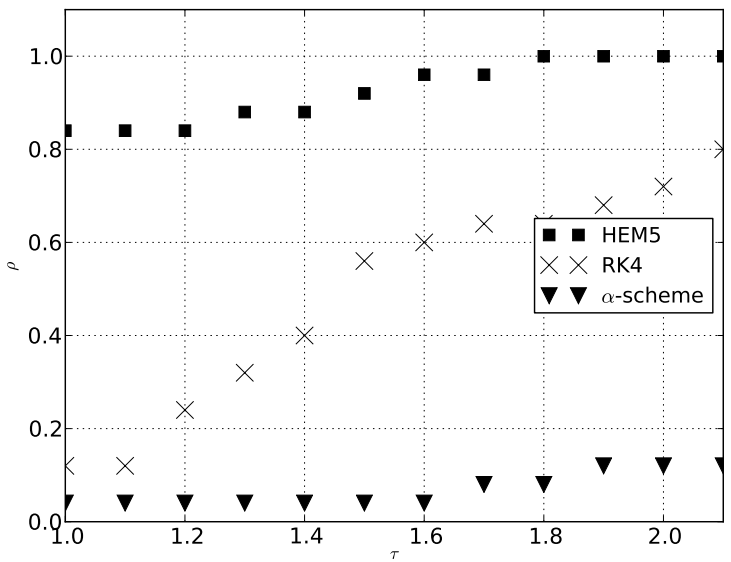


Figure: Performance profile of the second set

Performance profiles on third and fourth sets

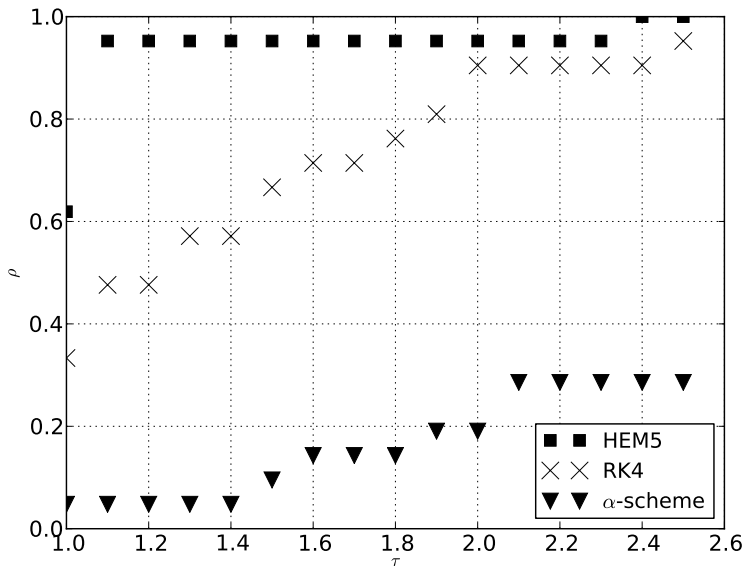


Figure: Performance profile of the third set

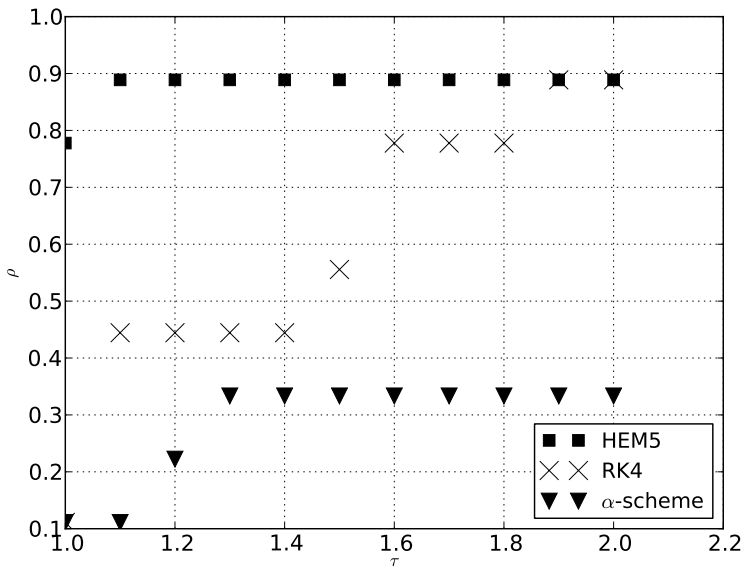
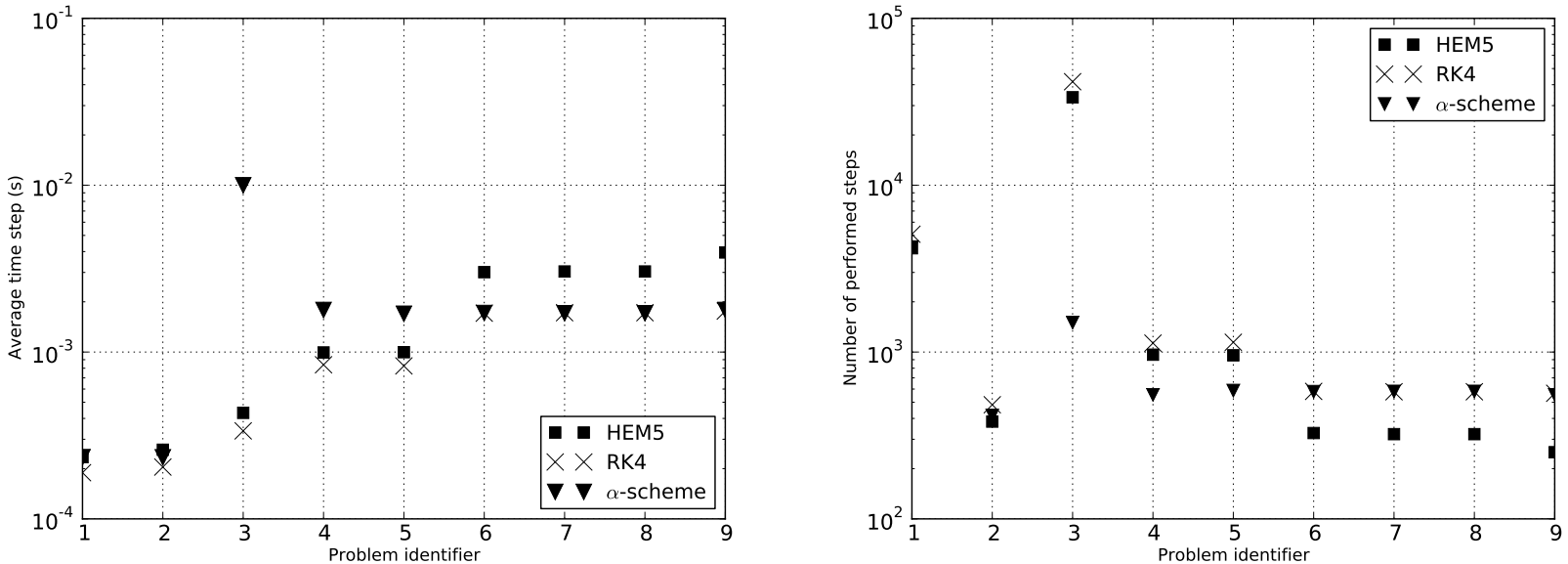


Figure: Performance profile of the fourth set

Details on the fourth set



(a) $h_{average}$ (b) # steps

Figure: Average time step, number of iterations of the fourth set

Comments

- The set is mainly composed of flexible linear beams examples.
- Problem 3 is different : a stiff problem.

A stiff problem: rotor mechanism

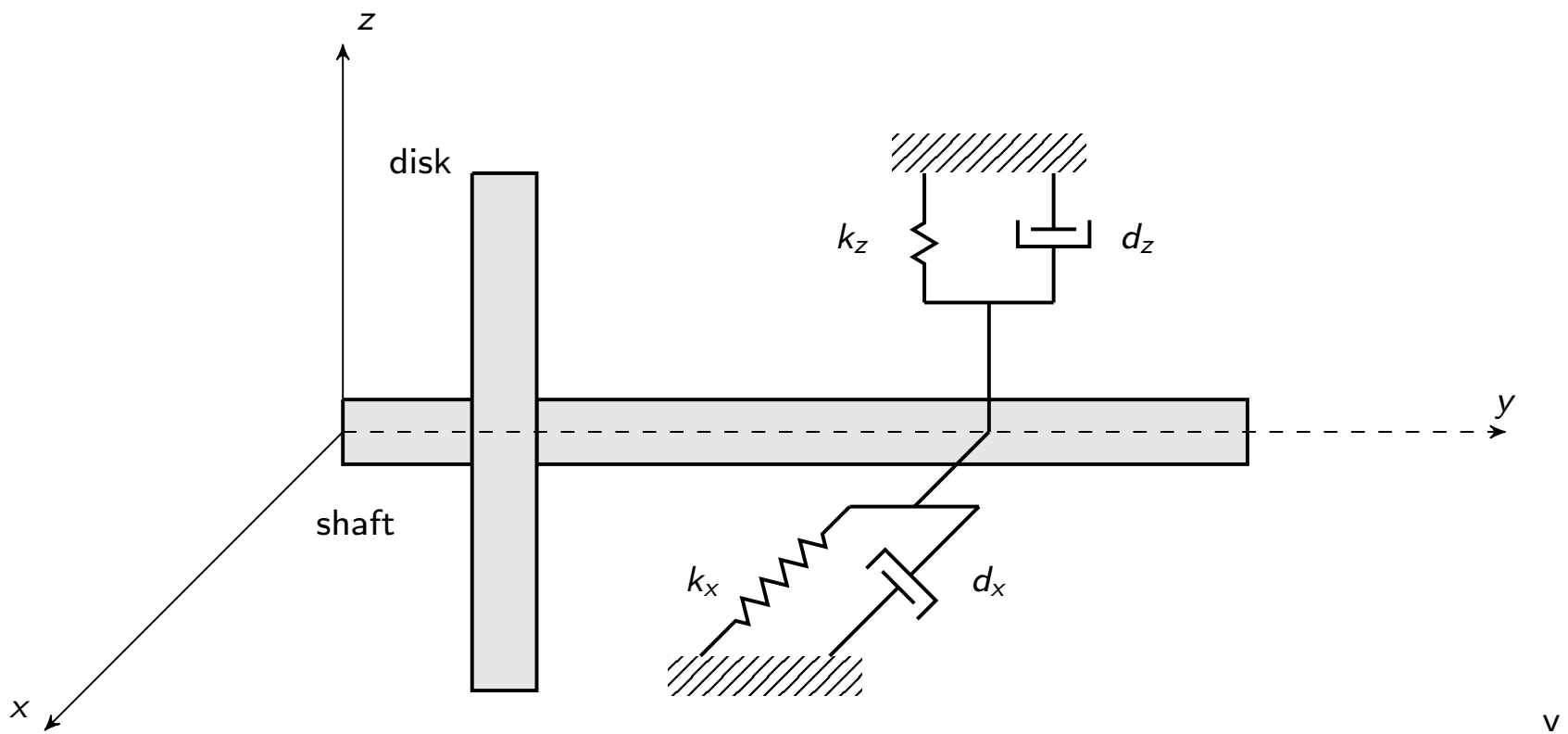


Figure: Rotor mechanism

A stiff problem: rotor mechanism

Table: Eccentrically suspended rotating beam: average time step size, simulation time, number of accepted steps and number of rejected steps for different tolerances.

method	tolerance	average h	t_s	accepted	rejected
HEM5	10^{-2}	$4.22\ 10^{-4}$	548.98	34123	1381
	10^{-4}	$4.20\ 10^{-4}$	549.70	34364	1393
	10^{-6}	$2.40\ 10^{-4}$	985.79	62255	179
RK4	10^{-2}	$3.38\ 10^{-4}$	384.13	41428	2903
	10^{-4}	$3.34\ 10^{-4}$	376.89	41700	2825
	10^{-6}	$3.37\ 10^{-4}$	398.29	41562	2876
α-scheme, $\rho_\infty = 0.99$	10^{-2}	$9.61\ 10^{-2}$	10.55	156	0
	10^{-4}	$8.01\ 10^{-3}$	86.32	1869	3
	10^{-6}	$2.35\ 10^{-3}$	264.80	6371	3
α-scheme, $\rho_\infty = 0.8$	10^{-2}	$9.61\ 10^{-2}$	10.69	156	0
	10^{-4}	$3.16\ 10^{-2}$	24.42	474	0
	10^{-6}	$6.15\ 10^{-3}$	109.83	2437	0
α-scheme, $\rho_\infty = 0.5$	10^{-2}	$9.61\ 10^{-2}$	11.29	156	0
	10^{-4}	$2.74\ 10^{-2}$	28.72	538	0
	10^{-6}	$5.50\ 10^{-3}$	123.09	2714	0

Conclusions

- Computational effort:
  Half explicit solvers (HEM5 and PHEM56) outperforms the other solvers.
- Drift :
  Index 2 DAE solvers are the best compromise for the constraints drift by controlling drift at the accelerarion and position level at order 1
- Stiff Dynamics:
  Fully implicit solvers (generalized-α) are required to efficiently integrate the dynamics.
- Implementation effort:
  Half explicit solvers (HEM5 and PHEM56) requires more effort to be implemented:
  - Non symmetric linear system solvers
  - High sensitivity to rank deficiency of the active constraints.

Thank you for your attention.

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