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SHARP REGULARITY AND CAUCHY PROBLEM OF THE SPATIALLY HOMOGENEOUS BOLTZMANN EQUATION WITH DEBYE-YUKAWA POTENTIAL

LÉO GLANGETAS, HAO-GUANG LI

ABSTRACT. In this paper, we study the Cauchy problem for the linear spatially homogeneous Boltzmann equation with Debye-Yukawa potential. Using the spectral decomposition of the linear operator, we prove that, for an initial datum in the sense of distribution which contains the dual of the Sobolev spaces, there exists a unique solution which belongs to a more regular Sobolev space for any positive time. We also study the sharp regularity of the solution.

1. INTRODUCTION AND MAIN RESULTS

In this work, we consider the spatially homogeneous Boltzmann equation

$$\frac{\partial f}{\partial t} = Q(f, f) \tag{1.1}$$

where $f = f(t, v)$ is the density distribution function depending only on two variables $t \geq 0$ and $v \in \mathbb{R}^3$. The Boltzmann bilinear collision operator is given by

$$Q(g, f)(v) = \int_{\mathbb{R}^3} \int_{S^2} B(v - v_*, \sigma) (g(v'_*)f(v') - g(v_*)f(v)) dv_* d\sigma,$$

where for $\sigma \in \mathbb{S}^2$, the symbols v'_* and v' are abbreviations for the expressions,

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma,$$

which are obtained in such a way that collision preserves momentum and kinetic energy, namely

$$v'_* + v' = v + v_*, \quad |v'_*|^2 + |v'|^2 = |v|^2 + |v_*|^2.$$

For monatomic gas, the collision cross section $B(v - v_*, \sigma)$ is a non-negative function which depends only on $|v - v_*|$ and $\cos \theta$ which is defined through the scalar product in \mathbb{R}^3 by

$$\cos \theta = \frac{v - v_*}{|v - v_*|} \cdot \sigma.$$

Without loss of generality, we may assume that $B(v - v_*, \sigma)$ is supported on the set $\cos \theta \geq 0$, i.e. where $0 \leq \theta \leq \frac{\pi}{2}$. See for example [12], [26] for more explanations about the support of θ . For physical models, the collision cross section usually takes the form

$$B(v - v_*, \sigma) = \Phi(|v - v_*|)b(\cos \theta),$$

with a kinetic factor

$$\Phi(|v - v_*|) = |v - v_*|^\gamma, \quad \gamma \in]-3, +\infty[.$$

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The molecules are said to be Maxwellian when the parameter $\gamma = 0$.

Except for the hard sphere model, the function $b(\cos \theta)$ has a singularity at $\theta = 0$. For instance, in the important model case of the inverse-power potentials,

$$\phi(\rho) = \frac{1}{\rho^r}, \text{ with } r > 1,$$

with ρ being the distance between two interacting particles in the physical 3-dimensional space \mathbb{R}^3 ,

$$b(\cos \theta) \sin \theta \approx K\theta^{-1-\frac{2}{r}}, \text{ as } \theta \rightarrow 0^+,$$

The notation $a \approx b$ means that there exist positive constants $C_2 > C_1 > 0$, such that

$$C_1 a \leq b \leq C_2 a.$$

Notice that the Boltzmann collision operator is not well defined for the case when $r = 1$ corresponding to the Coulomb potential.

If the inter-molecule potential satisfies the Debye-Yukawa type potentials where the potential function is given by

$$\phi(\rho) = \frac{1}{\rho e^{\rho^s}}, \text{ with } s > 0,$$

then the collision cross section has a singularity in the following form

$$b(\cos \theta) \approx \theta^{-2} (\log \theta^{-1})^{\frac{2}{s}-1}, \text{ when } \theta \rightarrow 0^+, \text{ with } s > 0. \quad (1.2)$$

This explicit formula was first appeared in the Appendix in [19]. In some sense, the Debye-Yukawa type potentials is a model between the Coulomb potential corresponding to $s = 0$ and the inverse-power potential. For further details on the physics background and the derivation of the Boltzmann equation, we refer the reader to the extensive expositions [3], [26].

We linearize the Boltzmann equation near the absolute Maxwellian distribution

$$\mu(v) = (2\pi)^{-\frac{3}{2}} e^{-\frac{|v|^2}{2}}.$$

Let $f(t, v) = \mu(v) + \sqrt{\mu}(v)g(t, v)$. Plugging this expression into (1.1), we have

$$\frac{\partial g}{\partial t} + \mathcal{L}[g] = \mathbf{\Gamma}(g, g)$$

with

$$\mathbf{\Gamma}(g, h) = \frac{1}{\sqrt{\mu}} Q(\sqrt{\mu}g, \sqrt{\mu}h), \quad \mathcal{L}(g) = -\frac{1}{\sqrt{\mu}} [Q(\sqrt{\mu}g, \mu) + Q(\mu, \sqrt{\mu}g)].$$

Then the Cauchy problem (1.1) can be re-written in the form

$$\begin{cases} \partial_t g + \mathcal{L}(g) = \mathbf{\Gamma}(g, g), \\ g|_{t=0} = g_0. \end{cases}$$

In the present work, we consider the linearized Cauchy problem

$$\begin{cases} \partial_t g + \mathcal{L}(g) = 0, \\ g|_{t=0} = g_0. \end{cases} \quad (1.3)$$

In the case of the inverse-power potential with $r > 1$, the regularity of the Boltzmann equation has been studied by numerous papers. It is well known that the non-cutoff spatially homogeneous Boltzmann equation enjoys an $\mathcal{S}(\mathbb{R}^3)$ -regularizing effect for the weak solutions to the Cauchy problem (see [5, 19]). Regarding the

Gevrey regularity, Ukai showed in [25] that the Cauchy problem for the Boltzmann equation has a unique local solution in Gevrey classes. Then, Desvillettes, Furioli and Terraneo proved in [4] the propagation of Gevrey regularity for solutions of the Boltzmann equation with Maxwellian molecules. For mild singularities, Morimoto and Ukai proved in [18] the Gevrey regularity of smooth Maxwellian decay solutions to the Cauchy problem of the spatially homogeneous Boltzmann equation with a modified kinetic factor. See also [28] for the non-modified case. On the other hand, Lekrine and Xu proved in [11] the property of Gevrey smoothing effect for the weak solutions to the Cauchy problem associated to the radially symmetric spatially homogeneous Boltzmann equation with Maxwellian molecules for $r > 2$. This result was then completed by Gnangetas and Najeme who established in [7] the analytic smoothing effect in the case when $1 < r < 2$. In [12, 15], the solutions of the Cauchy problem (1.3) for linearized non-cutoff Boltzmann equation belong to the symmetric Gelfand-Shilov spaces $S_{r/2}^{\nu}(\mathbb{R}^3)$ for any positive time and

$$\|e^{c\mathcal{H}^{\frac{1}{r}}}\|g(t)\|_{L^2} \leq C\|g_0\|_{L^2},$$

where

$$\mathcal{H} = -\Delta + \frac{|v|^2}{4}.$$

The Gelfand-Shilov space $S_{\nu}^{\nu}(\mathbb{R}^3)$ with $\nu \geq \frac{1}{2}$ can be identify with

$$S_{\nu}^{\nu}(\mathbb{R}^3) = \left\{ f \in C^{\infty}(\mathbb{R}^3); \exists \tau > 0, \|e^{\tau \mathcal{H}^{\frac{1}{2\nu}}} f\|_{L^2} < +\infty \right\}.$$

This space can also be characterized as is the space of smooth functions $f \in C^{+\infty}(\mathbb{R}^3)$ satisfying (see Appendix 5):

$$\exists A > 0, C > 0, \sup_{v \in \mathbb{R}^3} |v^{\beta} \partial_v^{\alpha} f(v)| \leq CA^{|\alpha|+|\beta|} (\alpha!)^{\nu} (\beta!)^{\nu}, \quad \forall \alpha, \beta \in \mathbb{N}^3.$$

The linear Boltzmann operator \mathcal{L} is shown to be diagonal in the basis of $L^2(\mathbb{R}^3)$ and this property has been used in [14] and [8] to prove that the Cauchy problem to the non-cutoff spatially homogeneous Boltzmann equation with the small initial datum $g_0 \in L^2(\mathbb{R}^3)$ has a global solution, which belongs to the Gelfand-Shilov class $S_{r/2}^{\nu}(\mathbb{R}^3)$.

The initial value problem in a space of probability measures defined via the Fourier transform has been studied in [2] and [17]. Recently, the case of initial datum in the sense of distribution, which contains the dual space of a Gelfand-Shilov class for the linear case has been studied in [15].

In this paper, we consider the collision kernel in the Maxwellian molecules case and the angular function b satisfying the Debye-Yukawa potential (1.2) for some $s > 0$. For convenience, we denote

$$\beta(\theta) = 2\pi b(\cos \theta) \sin \theta. \quad (1.4)$$

We study the smoothing effect for the Cauchy problem (1.3) associated to the non-cutoff spatially homogeneous Boltzmann equation with Debye-Yukawa potential (1.2). The singularity of the collision kernel b endows the linearized Boltzmann operator \mathcal{L} with the logarithmic Gelfand-Shilov regularity property (see Proposition 2.1). The logarithmic regularity theory was first introduced in [16] on the hypoellipticity of the infinitely degenerate elliptic operator and was developed in

[20],[21] on the logarithmic Sobolev estimates. Recently, for $0 < s < 2$, the initial datum $f_0 \geq 0$ and

$$\int_{\mathbb{R}^3} f_0(v)(1 + |v|^2 + \log(1 + f_0(v)))dv < +\infty,$$

Morimoto, Ukai, Xu and Yang in [19] show that the weak solution of the Cauchy problem (1.1) satisfying

$$\sup_{t \in [0, T]} \int_{\mathbb{R}^3} f(t, v)(1 + |v|^2 + \log(1 + f(t, v)))dv < +\infty$$

enjoys a $H^{+\infty}(\mathbb{R}^3)$ smoothing effect property.

In order to precise the regularity of the solution of the Cauchy problem, we introduce some functional spaces. The linear operator \mathcal{L} is nonnegative ([12, 13, 14]), with the null space

$$\mathcal{N} = \text{span} \left\{ \sqrt{\mu}, \sqrt{\mu}v_1, \sqrt{\mu}v_2, \sqrt{\mu}v_3, \sqrt{\mu}|v|^2 \right\}.$$

Denote by \mathbf{P} the orthoprojection from $L^2(\mathbb{R}^3)$ into \mathcal{N} . Then

$$(\mathcal{L}g, g) = 0 \Leftrightarrow g = \mathbf{P}g$$

and the operator $\mathcal{L} + \mathbf{P}$ is a positive self-adjoint operator. We define

$$D(\mathcal{L}) = \bigcup_{\tau > 0} D_\tau(\mathcal{L})$$

with

$$D_\tau(\mathcal{L}) = \left\{ u \in C^\infty(\mathbb{R}^3), \sum_{k=0}^{+\infty} \tau^k (k!)^{-1} \|(\mathcal{L} + \mathbf{P})^{\frac{k}{2}} u\|_{L^2}^2 < +\infty \right\},$$

which is a Banach space with the norm

$$\|u\|_{D_\tau(\mathcal{L})}^2 = \sum_{k=0}^{+\infty} \tau^k (k!)^{-1} \|(\mathcal{L} + \mathbf{P})^{\frac{k}{2}} u\|_{L^2}^2.$$

Analogously, we define

$$D^+(\mathcal{L}) = \bigcup_{\tau > 0} D_\tau^+(\mathcal{L})$$

where

$$D_\tau^+(\mathcal{L}) = \left\{ u \in C^\infty(\mathbb{R}^3), \sum_{k=0}^{+\infty} \tau^k (k!)^{-1} \|(\mathcal{L} + \mathbf{P})^{\frac{k+1}{2}} u\|_{L^2}^2 < +\infty \right\}$$

with the norm

$$\|u\|_{D_\tau^+(\mathcal{L})}^2 = \sum_{k=0}^{+\infty} \tau^k (k!)^{-1} \|(\mathcal{L} + \mathbf{P})^{\frac{k+1}{2}} u\|_{L^2}^2.$$

The distribution spaces $(D(\mathcal{L}))'$, $(D^+(\mathcal{L}))'$ are defined as follows

$$(D(\mathcal{L}))' = \bigcup_{\tau > 0} (D_\tau(\mathcal{L}))', \quad (D^+(\mathcal{L}))' = \bigcup_{\tau > 0} (D_\tau^+(\mathcal{L}))'$$

where $(D_\tau(\mathcal{L}))'$, $(D_\tau^+(\mathcal{L}))'$ are the dual spaces of $D_\tau(\mathcal{L})$ and $D_\tau^+(\mathcal{L})$. Now we begin to present our result. Firstly, we give the definition of the weak solution of the Cauchy problem (1.3).

Definition 1.1. For any $g_0 \in (D(\mathcal{L}))'$, $T > 0$, $g(t, v)$ is a weak solution of the Cauchy problem (1.3) if $\mathbf{P}g \equiv \mathbf{P}g_0$,

$$\begin{aligned} g &\in L^\infty([0, T], (D(\mathcal{L}))') \cap H^1([0, T], (D^+(\mathcal{L}))'), \\ \mathcal{L}^{\frac{1}{2}}g &\in L^2([0, T], (D(\mathcal{L}))'), \end{aligned} \quad (1.5)$$

and for any $\phi \in C^1([0, T], C_0^\infty(\mathbb{R}^3))$ we have

$$\begin{aligned} \forall t \in [0, T], \quad &\langle g(t), \phi(t) \rangle - \langle g_0, \phi(0) \rangle \\ &= \int_0^t \langle g, \partial_\tau \phi \rangle d\tau - \int_0^t \langle g, \mathcal{L}\phi \rangle d\tau. \end{aligned} \quad (1.6)$$

In the main theorem, we consider the initial distribution data case, which is given in the following.

Theorem 1.1. Assume that the Debye-Yukawa potential $b(\cdot)$ is given in (1.2) with $s > 0$ and $g_0 \in (D(\mathcal{L}))'$. Therefore the Cauchy problem (1.3) admits a unique weak solution. Moreover there exist $t_0 > 0$ and $c_0 > 0$ such that for all $t \geq t_0/c_0$

$$\begin{aligned} \|e^{-t_0(\log(e+\mathcal{H}))^{\frac{2}{3}}} g_0\|_{L^2} &< +\infty, \\ \|e^{tc_0(\log(\mathcal{H}+e))^{\frac{2}{3}}} (\mathbf{I} - \mathbf{P})g(t)\|_{L^2} &\leq e^{-\frac{1}{4}\lambda_{2,0}t} \|e^{-t_0(\log(e+\mathcal{H}))^{\frac{2}{3}}} (\mathbf{I} - \mathbf{P})g_0\|_{L^2} \end{aligned}$$

where $\mathcal{H} = -\Delta + \frac{|v|^2}{4}$,

$$\lambda_{2,0} = \int_0^{\pi/4} \beta(\theta)(1 - \sin^4 \theta - \cos^4 \theta) d\theta > 0.$$

In particular, for all $t > t_0/c_0$, the solution $g(t)$ belongs to $L^2(\mathbb{R}^3)$.

Remark 1.1. The regularizing effect $g(t) \in L^2(\mathbb{R}^3)$ has usually a positive delay time. For example, consider some $\tau_0 > 0$ and the initial data $g_0 = \sum_{n \geq 1} \frac{1}{n} e^{\tau_0 \lambda_{n,0}} \phi_{n,0,0}$. It is easy to check that $g(t) \in L^2(\mathbb{R}^3)$ for $t \geq \tau_0$ but $g(t) \notin L^2(\mathbb{R}^3)$ for $t < \tau_0$.

In order to precise the regularizing effect in the Sobolev spaces, it is convenient to consider the symmetric weighted Sobolev space $Q^{2\tau}(\mathbb{R}^3)$ introduced by Shubin [24] with norm

$$\|u\|_{Q^{2\tau}(\mathbb{R}^3)} = \left\| \left(-\Delta + \frac{|v|^2}{4} + e \right)^\tau u \right\|_{L^2}.$$

Theorem 1.2. Regularizing effect for an initial data in $L^2(\mathbb{R}^3)$.

Assume that the Debye-Yukawa potential $b(\cdot)$ is given in (1.2) with $s > 0$ and $g_0 \in L^2(\mathbb{R}^3)$. Therefore the Cauchy problem (1.3) admits a unique weak solution. Moreover there exists $c_0 > 0$ such that :

1) Case $0 < s \leq 2$.

$$\forall t > 0, \quad \left\| (e + \mathcal{H})^{c_0 t} (\mathbf{I} - \mathbf{P})g(t) \right\|_{L^2} \leq e^{-\lambda_{2,0}t} \|(\mathbf{I} - \mathbf{P})g_0\|_{L^2(\mathbb{R}^3)}. \quad (1.7)$$

This shows that $g(t)$ belongs to the Sobolev space $Q^{ct}(\mathbb{R}^3)$ for any time $t > 0$.

2) Case $0 < s < 2$. there exists a constant $c_s > 0$ such that for any $t > 0$, one has

$$\forall k \geq 0, \quad \|(\mathbf{I} - \mathbf{P})g(t)\|_{Q^k} \leq e^{-\lambda_{2,0}t} e^{c_s(1/t)^{\frac{s}{2-s}} k^{\frac{2}{2-s}}} \|(\mathbf{I} - \mathbf{P})g_0\|_{L^2(\mathbb{R}^3)}. \quad (1.8)$$

Remark 1.2. *Comments on the regularizing effect. When the singularity of the collision cross section (1.2) for θ near 0 become smoother (that is when the real s increases), the regularizing effect become weaker, and disappears in the context of the Sobolev spaces when $s > 2$:*

- Case $0 < s < 2$. The solution $g(t) \in \cap_{k \geq 0} Q^k(\mathbb{R}^3)$ for each positive time.
- Case $s = 2$. The regularizing effect in $Q^k(\mathbb{R}^3)$ has usually a positive time delay. For example, consider the initial data $g_0 = \sum_{n \geq 2} \frac{1}{n^{\frac{1}{2}} \log n} \varphi_{n,0,0}$ where $\{\varphi_{n,l,m}(v)\}$ constitutes an orthonormal basis of $L^2(\mathbb{R}^3)$, which is given in Section 2. We can check that there exists $t_k > 0$ such that $g(t) \notin Q^k(\mathbb{R}^3)$ for $0 \leq t < t_k$.
- Case $s > 2$. There is no regularizing effect in the Sobolev space. Consider any real numbers $0 < \tau < \tau'$ and $g_0 = \sum_{n \geq 2} \frac{1}{n^{\frac{\tau+1}{2}} \log n} \varphi_{n,0,0}$ where $\varphi_{n,0,0}$ is given in Section 2. We can check that for $t \geq 0$ the solution $g(t)$ stays in the space $Q^\tau(\mathbb{R}^3)$, but never belongs to $Q^{\tau'}(\mathbb{R}^3)$. However, there is a very slight regularizing effect and the Boltzmann equation remains irreversible.

Remark 1.3. *We think that the non-linear case is similar to the linear case, but the proofs are more technical. This work is a first step to study the non-linear case.*

The rest of the paper is arranged as follows. In Section 2, we introduce the spectral analysis of the linear Boltzmann operator and in Section 3 we precise some properties of the distribution spaces. The proof of the main Theorems 1.1-1.2 will be presented in Section 4, where we construct a sequence of solutions of the Cauchy problem (1.3) with initial datum equal to the projection of g_0 on an increasing sequence of finite dimensional subspaces, which converges to the solution of the Cauchy problem. In the appendix, we present some spectral properties of the functional spaces used in this paper and the proof of some technical lemmas.

2. THE PRELIMINARY RESULTS

We first recall the spectral decomposition of linear Boltzmann operator. In the cutoff case, that is, when $b(\cos \theta) \sin \theta \in L^1([0, \frac{\pi}{2}])$, it was shown in [27] that

$$\mathcal{L}(\varphi_{n,l,m}) = \lambda_{n,l} \varphi_{n,l,m}, \quad n, l \in \mathbb{N}, \quad m \in \mathbb{Z}, \quad |m| \leq l$$

where

$$\lambda_{n,l} = \int_0^{\frac{\pi}{4}} \beta(\theta) \left(1 + \delta_{n,0} \delta_{l,0} - (\sin \theta)^{2n+l} P_l(\sin \theta) - (\cos \theta)^{2n+l} P_l(\cos \theta) \right) d\theta. \quad (2.1)$$

This diagonalization of the linearized Boltzmann operator with Maxwellian molecules holds as well in the non-cutoff case, (see [1, 3, 6, 12, 13]).

The eigenfunctions are

$$\varphi_{n,l,m}(v) = \left(\frac{n!}{\sqrt{2} \Gamma(n+l+3/2)} \right)^{1/2} \left(\frac{|v|}{\sqrt{2}} \right)^l e^{-\frac{|v|^2}{4}} L_n^{(l+1/2)} \left(\frac{|v|^2}{2} \right) Y_l^m \left(\frac{v}{|v|} \right)$$

where $\Gamma(\cdot)$ is the standard Gamma function: for any $x > 0$,

$$\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt.$$

The l^{th} -Legendre polynomial P_l and the Laguerre polynomial $L_n^{(\alpha)}$ of order α , degree n (see [23]) read,

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l, \text{ where } |x| \leq 1;$$

$$L_n^{(\alpha)}(x) = \sum_{r=0}^n (-1)^{n-r} \frac{\Gamma(\alpha + n + 1)}{r!(n-r)!\Gamma(\alpha + n - r + 1)} x^{n-r}.$$

For any unit vector $\sigma = (\cos \theta, \sin \theta \cos \phi, \sin \theta \sin \phi)$ with $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$, the orthonormal basis of spherical harmonics $Y_l^m(\sigma)$ is

$$Y_l^m(\sigma) = N_{l,m} P_l^{|m|}(\cos \theta) e^{im\phi}, \quad |m| \leq l,$$

where the normalisation factor is given by

$$N_{l,m} = \sqrt{\frac{2l+1}{4\pi} \cdot \frac{(l-|m|)!}{(l+|m|)!}}$$

and $P_l^{|m|}$ is the associated Legendre functions of the first kind of order l and degree $|m|$ with

$$P_l^{|m|}(x) = (1-x^2)^{\frac{|m|}{2}} \left(\frac{d}{dx} \right)^{|m|} P_l(x).$$

We recall from Lemma 7.2 in [8] that

$$\sqrt{\widehat{\mu}} \varphi_{n,l,m}(\xi) = (-i)^l (2\pi)^{\frac{3}{4}} \left(\frac{1}{\sqrt{2}n! \Gamma(n+l+\frac{3}{2})} \right)^{\frac{1}{2}} \left(\frac{|\xi|}{\sqrt{2}} \right)^{2n+l} e^{-\frac{|\xi|^2}{2}} Y_l^m \left(\frac{\xi}{|\xi|} \right).$$

We can extend the spectral decomposition to the Debye-Yukawa potential case. The family $(Y_l^m(\sigma))_{l \geq 0, |m| \leq l}$ constitutes an orthonormal basis of the space $L^2(\mathbb{S}^2, d\sigma)$ with $d\sigma$ being the surface measure on \mathbb{S}^2 (see [10], [22]). Noting that $\{\varphi_{n,l,m}(v)\}$ constitutes an orthonormal basis of $L^2(\mathbb{R}^3)$ composed of eigenvectors of the harmonic oscillator (see[1], [13])

$$\mathcal{H}(\varphi_{n,l,m}) = (2n + l + \frac{3}{2}) \varphi_{n,l,m}.$$

As a special case, $\{\varphi_{n,0,0}(v)\}$ constitutes an orthonormal basis of $L_{rad}^2(\mathbb{R}^3)$, the radially symmetric function space (see [14]). We have for suitable functions g

$$\mathcal{L}(g) = \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=-l}^l \lambda_{n,l} g_{n,l,m} \varphi_{n,l,m} \quad (2.2)$$

where $g_{n,l,m} = (g, \varphi_{n,l,m})_{L^2(\mathbb{R}^3)}$ and

$$\mathcal{H}(g) = \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=-l}^l (2n + l + \frac{3}{2}) g_{n,l,m} \varphi_{n,l,m}.$$

Using this spectral decomposition, for any $s > 0$, the definitions of $(\log(\mathcal{H} + e))^s$, $e^{c(\log(\mathcal{H}+e))^s}$ and $e^{c\mathcal{L}}$ are then classical.

Remark 2.1. *It is trivial to obtain from (2.1) that $\lambda_{0,0} = \lambda_{1,0} = \lambda_{0,1} = 0$ and the others are strictly positive. Thus the null space of the linear Boltzmann operator \mathcal{L} is*

$$\mathcal{N} = \text{span} \left\{ \sqrt{\mu}, \sqrt{\mu}v_1, \sqrt{\mu}v_2, \sqrt{\mu}v_3, \sqrt{\mu}|v|^2 \right\}.$$

We have for $(n, l) \in \mathbb{N}^2$

$$\begin{aligned} \lambda_{n,l} &= 0 & \text{if } n+l \leq 1, \\ \lambda_{n,l} &> 0 & \text{otherwise.} \end{aligned}$$

We derive the following estimate of $\lambda_{n,l}$ defined in (2.1).

Proposition 2.1 (Spectral estimates of the linearized Boltzmann operator \mathcal{L}). *Let collision the kernel b satisfies the Debye-Yukawa potential condition (1.2). Then, there exists a positive constant c_0 such that for any $n, l \in \mathbb{N}$, $n+l \geq 2$, we have*

$$c_0 (\log(2n+l+e))^{\frac{2}{s}} \leq \lambda_{n,l} \leq \frac{1}{c_0} (\log(2n+l+e))^{\frac{2}{s}}, \quad (2.3)$$

where e is the Natural constant.

Proof. From Remark 2.1, for any $n, l \in \mathbb{N}$ and $n+l \geq 2$ we have $\lambda_{n,l} > 0$. Therefore we need only to consider the case $2n+l \rightarrow +\infty$. We have from (1.4) for $\theta \in]0, \frac{\pi}{4}]$

$$\beta(\theta) \approx (\sin \theta)^{-1} (\log(\sin \theta)^{-1})^{\frac{2}{s}-1}.$$

From (2.1) and putting $x = \sin \theta$, $\lambda_{n,l}$ can be decomposed as follows

$$\begin{aligned} \lambda_{n,l} &= \int_0^{\frac{\pi}{4}} \beta(\theta) (1 - P_l(\sin \theta)(\sin \theta)^{2n+l} - P_l(\cos \theta)(\cos \theta)^{2n+l}) d\theta \\ &\approx \int_0^{\frac{\sqrt{2}}{2}} (\log x^{-1})^{\frac{2}{s}-1} (1 - P_l(x) x^{2n+l} - P_l(\sqrt{1-x^2})(1-x^2)^{\frac{2n+l}{2}}) \frac{dx}{x} \\ &\approx \int_0^{\frac{\sqrt{2}}{2}} (\log x^{-1})^{\frac{2}{s}-1} (1 - (1-x^2)^{\frac{2n+l}{2}}) \frac{dx}{x} \\ &\quad - \int_0^{\frac{\sqrt{2}}{2}} (\log x^{-1})^{\frac{2}{s}-1} P_l(x) x^{2n+l} \frac{dx}{x} \\ &\quad + \int_0^{\frac{\sqrt{2}}{2}} (\log x^{-1})^{\frac{2}{s}-1} (1 - P_l(\sqrt{1-x^2})) (1-x^2)^{\frac{2n+l}{2}} \frac{dx}{x} \\ &= A_1 + A_2 + A_3. \end{aligned} \quad (2.4)$$

We first estimate A_1 . Setting $y = x \sqrt{2n+l}$, we decompose it in two parts

$$\begin{aligned} A_1 &= \int_0^{\frac{\sqrt{2}}{2} \sqrt{2n+l}} (\log \frac{\sqrt{2n+l}}{y})^{\frac{2}{s}-1} (1 - (1 - \frac{y^2}{2n+l})^{\frac{2n+l}{2}}) \frac{dy}{y} \\ &= \int_1^{\frac{\sqrt{2}}{2} \sqrt{2n+l}} + \int_0^1 = A_{11} + A_{12}. \end{aligned}$$

The main term is A_{11} . Putting $z = \log \frac{\sqrt{2n+l}}{y}$, we get when $2n+l \rightarrow \infty$

$$\begin{aligned} A_{11} &= \int_{\log \sqrt{2}}^{\log \sqrt{2n+l}} z^{\frac{2}{s}-1} (1 - (1 - e^{-2z})^{\frac{2n+l}{2}}) dz \\ &= \left(\int_{\log \sqrt{2}}^{\log \sqrt{2n+l}} z^{\frac{2}{s}-1} dz \right) (1 + O(\frac{1}{2})^{\frac{2n+l}{2}}) \\ &\approx \frac{s}{2} (\log \sqrt{2n+l})^{\frac{2}{s}}. \end{aligned}$$

We now check that the other term A_{12} has a lower order (A_2 and A_3 will have also a lower order). We decompose the term A_{12} as follows:

$$\begin{aligned} A_{12} &= \int_0^1 \left(\log \sqrt{2n+l} + \log \frac{1}{y} \right)^{\frac{2}{s}-1} \left(1 - \left(1 - \frac{y^2}{2n+l} \right)^{\frac{2n+l}{2}} \right) \frac{dy}{y} \\ &= (\log \sqrt{2n+l})^{\frac{2}{s}-1} \int_0^1 g_{2n+l}(y) dy \end{aligned}$$

where

$$g_k(y) = \left(1 + \frac{1}{\log \sqrt{k}} \log \frac{1}{y} \right)^{\frac{2}{s}-1} \left(1 - \left(1 - \frac{y^2}{k} \right)^{\frac{k}{2}} \right) \frac{1}{y}.$$

It is easy to check that, uniformly for $y \in]0, 1]$ and $k \geq 2$, we have

$$g_k(y) = \max\left(1, \left(\log \frac{1}{y}\right)^{\frac{2}{s}-1}\right) O(y) \quad \text{and} \quad g_k(y) \xrightarrow[k \rightarrow \infty]{} \left(1 - e^{-\frac{1}{2}y^2}\right) \frac{1}{y}.$$

From the dominated convergence theorem we get

$$\begin{aligned} A_{12} &\approx (\log \sqrt{2n+l})^{\frac{2}{s}-1} \int_0^1 \left(1 - e^{-\frac{1}{2}y^2}\right) \frac{1}{y} dy \\ &\lesssim (\log \sqrt{2n+l})^{\frac{2}{s}-1}, \end{aligned}$$

where we use the fact that

$$\int_0^1 \left(1 - e^{-\frac{1}{2}y^2}\right) \frac{1}{y} dy < +\infty.$$

We estimate the second term A_2 . From the classical inequality $|P_l| \leq 1$ on $[-1, 1]$,

$$\begin{aligned} |A_2| &\leq \int_0^{\frac{\sqrt{2}}{2}} (\log x^{-1})^{\frac{2}{s}-1} |P_l(x)| x^{2n+l} \frac{dx}{x} \\ &\leq \left(\frac{\sqrt{2}}{2}\right)^{2n+l-1} \int_0^{\frac{\sqrt{2}}{2}} (\log x^{-1})^{\frac{2}{s}-1} dx \\ &\leq \left(\frac{\sqrt{2}}{2}\right)^{2n+l-1} \int_{\sqrt{2}}^{+\infty} x^{\frac{2}{s}-1} e^{-x} dx \leq \Gamma\left(\frac{2}{s}\right). \end{aligned}$$

We estimate the third term A_3 . We divide A_3 into two parts for $l \geq 2$

$$\begin{aligned} A_3 &= \int_0^{\frac{\sqrt{2}}{2}} (\log x^{-1})^{\frac{2}{s}-1} \left(1 - P_l(\sqrt{1-x^2})\right) (1-x^2)^{\frac{2n+l}{2}} \frac{dx}{x} \\ &= \int_{\frac{1}{7}}^{\frac{\sqrt{2}}{2}} + \int_0^{\frac{1}{7}} = A_{31} + A_{32}. \end{aligned}$$

For the first part A_{31} , since $|P_l| \leq 1$ on $[-1, 1]$, we can estimate as follows

$$\begin{aligned} 0 \leq A_{31} &\leq \int_{\frac{1}{7}}^{\frac{\sqrt{2}}{2}} (\log x^{-1})^{\frac{2}{s}-1} \frac{dx}{x} \\ &\leq \frac{s}{2} \left((\log l)^{\frac{2}{s}} - (\log \sqrt{2})^{\frac{2}{s}} \right). \end{aligned}$$

For the second part A_{32} , setting $y = lx$, we get

$$0 \leq A_{32} \leq \int_0^1 \left(\log \frac{l}{y}\right)^{\frac{2}{s}-1} \left(1 - P_l(\sqrt{1 - y^2/l^2})\right) \frac{dy}{y}.$$

By Lemma 2.3 in [15] (for a proof, see lemma 5.1 in the appendix), we have

$$1 - P_l\left(\cos \frac{\theta}{l}\right) = O(\theta^2)$$

uniformly for $l \geq 1$ and $\theta \in [0, \frac{\pi}{2}]$. We then deduce that for $l \geq 2$

$$\begin{aligned} A_{32} &\lesssim \int_0^1 \left(\log \frac{l}{y}\right)^{\frac{2}{s}-1} y dy \\ &\lesssim (\log l)^{\frac{2}{s}-1} \int_0^1 \left(1 + \frac{1}{\log l} \log \frac{1}{y}\right)^{\frac{2}{s}-1} y dy \\ &\lesssim (\log l)^{\frac{2}{s}-1}. \end{aligned}$$

It follows from the above estimate of A_1, A_2, A_3 and (2.4)

$$(\log(2n + l + e))^{\frac{2}{s}} \lesssim \lambda_{n,l} = A_1 + A_2 + A_3 \lesssim (\log(2n + l + e))^{\frac{2}{s}}.$$

This concludes the proof of (2.3). \square

3. PROPERTIES OF SOME DISTRIBUTION SPACES

In order to give some more precise descriptions on the regularity of the linearized Boltzmann operator \mathcal{L} , we introduce the following Sobolev-type spaces: for any real numbers $\tau > 0$ and $\nu > 0$,

$$E_\nu^\tau(\mathbb{R}^3) = \left\{ u \in C^\infty(\mathbb{R}^3), \|e^{\tau(\log(e+\mathcal{H}))^{\frac{2}{\nu}}} u\|_{L^2}^2 < +\infty \right\},$$

which is a Banach space with the norm

$$\|u\|_{E_\nu^\tau(\mathbb{R}^3)}^2 = \|e^{\tau(\log(e+\mathcal{H}))^{\frac{2}{\nu}}} u\|_{L^2}^2.$$

In the case $\nu = 2$, the space $E_2^\tau(\mathbb{R}^3)$ is equivalent to the symmetric weighted Sobolev space $Q^{2\tau}(\mathbb{R}^3)$ with the norm (see (2.1) in [9] or 25.3 of Ch. IV in [24])

$$\|u\|_{Q^{2\tau}(\mathbb{R}^3)} = \left\| (e + \mathcal{H})^\tau u \right\|_{L^2}.$$

In the case $0 < \nu < 2$ and $\tau > 0$, one can verify that $E_\nu^\tau(\mathbb{R}^3)$ is an intermediate space between the symmetric weighted Sobolev spaces and the Gelfand-Shilov spaces : For all $\nu_1 \geq \frac{1}{2}$,

$$S_{\nu_1}^{\nu_1}(\mathbb{R}^3) \subset E_\nu^\tau(\mathbb{R}^3) \subset \bigcap_{k \geq 0} Q^k(\mathbb{R}^3).$$

Moreover, we have the following property of this space: There exists a constant $C = C_\nu > 0$ such that (see Proposition 5.1)

$$\forall k \geq 1, \forall \tau > 0, \|u\|_{Q^k(\mathbb{R}^3)} \leq e^{C(\frac{1}{\tau})^{\frac{\nu}{2-\nu}}} k^{\frac{2}{2-\nu}} \|u\|_{E_\nu^\tau(\mathbb{R}^3)}.$$

We denote the dual space of the Sobolev-type space $E_\nu^\tau(\mathbb{R}^3)$ by

$$\left(E_\nu^\tau(\mathbb{R}^3)\right)' = E_\nu^{-\tau}(\mathbb{R}^3).$$

Obviously, for $\nu > 0$ fixed, when $\tau_2 > \tau_1$, one has $E_\nu^{\tau_2}(\mathbb{R}^3) \subset E_\nu^{\tau_1}(\mathbb{R}^3)$; for τ fixed, when $\nu_2 > \nu_1 > 0$, one has $E_{\nu_1}^\tau(\mathbb{R}^3) \subset E_{\nu_2}^\tau(\mathbb{R}^3)$. Then we have the following inclusions.

Theorem 3.1. *Assume the cross section kernel b of the linearized Boltzmann operator \mathcal{L} defined in (1.2) with $s > 0$. For $\tau > 0$, there exist constants $\tau_2 > \tau_1 > 0$, such that*

$$E_s^{\tau\tau_2}(\mathbb{R}^3) \subset D_\tau(\mathcal{L}) \subset E_s^{\tau\tau_1}(\mathbb{R}^3).$$

In particular, when $0 < s \leq 2$, for $u \in D_1(\mathcal{L})$,

$$\|u\|_{\mathcal{O}^{2\tau_1}} = \|(e + \mathcal{H})^{\tau_1} u\|_{L^2} \leq \|e^{\tau_1(\log(e+\mathcal{H}))} u\|_{L^2} < +\infty. \quad (3.1)$$

Remark 3.1. *By the conjugation property, we have*

$$E_s^{-\tau\tau_1}(\mathbb{R}^3) \subset (D_\tau(\mathcal{L}))' \subset E_s^{-\tau\tau_2}(\mathbb{R}^3).$$

Using the spectral decomposition, we give some another expressions of the norm of the spaces $D_\tau(\mathcal{L})$, $(D_\tau(\mathcal{L}))'$ and $E_s^\tau(\mathbb{R}^3)$.

Proposition 3.1. *Let us define for $n, l \in \mathbb{N}$*

$$\tilde{\lambda}_{n,l} = \begin{cases} 1 & n+l \leq 1, \\ \lambda_{n,l} & n+l \geq 2. \end{cases} \quad (3.2)$$

(1) *For $\tau > 0$ and $u \in D_\tau(\mathcal{L})$, let $\langle u, \varphi_{n,l,m} \rangle$ be the inner product in $L^2(\mathbb{R}^3)$. Therefore the sequence $\{\langle u, \varphi_{n,l,m} \rangle\}$ satisfies*

$$\|u\|_{D_\tau(\mathcal{L})}^2 = \sum_{n=0}^{+\infty} \sum_{l=0}^{+\infty} \sum_{|m| \leq l} e^{\tau \tilde{\lambda}_{n,l}} \langle u, \varphi_{n,l,m} \rangle^2 < +\infty.$$

(2) *For $\tau > 0$ and $T \in (D_\tau(\mathcal{L}))'$, let $\langle T, \varphi_{n,l,m} \rangle$ be the inner product in sense of distribution. Analogously the sequence $\{\langle T, \varphi_{n,l,m} \rangle\}$ satisfies*

$$\|u\|_{(D_\tau(\mathcal{L}))'}^2 = \sum_{n=0}^{+\infty} \sum_{l=0}^{+\infty} \sum_{|m| \leq l} e^{-\tau \tilde{\lambda}_{n,l}} \langle T, \varphi_{n,l,m} \rangle^2 < +\infty.$$

(3) *For $\tau \in \mathbb{R}$, $s > 0$, let $u \in E_s^\tau(\mathbb{R}^3)$. Therefore the sequence $\{\langle u, \varphi_{n,l,m} \rangle\}$ satisfies*

$$\|u\|_{E_s^\tau(\mathbb{R}^3)}^2 = \sum_{n=0}^{+\infty} \sum_{l=0}^{+\infty} \sum_{|m| \leq l} e^{2\tau(\log(2n+l+\frac{3}{2}+e))} \langle u, \varphi_{n,l,m} \rangle^2 < +\infty.$$

Proof. The proof of part (1) is direct: From the spectral decomposition (2.2)

$$\begin{aligned} \|u\|_{D_\tau(\mathcal{L})}^2 &= \sum_{k=0}^{+\infty} \tau^k (k!)^{-1} \|(\mathcal{L} + \mathbf{P})^{\frac{k}{2}} u\|_{L^2}^2 \\ &= \sum_{k=0}^{+\infty} \tau^k (k!)^{-1} \sum_{n=0}^{+\infty} \sum_{l=0}^{+\infty} \sum_{|m| \leq l} \tilde{\lambda}_{n,l}^k \langle u, \varphi_{n,l,m} \rangle^2. \end{aligned}$$

Now we prove part (2): From the above characterization of $D_\tau(\mathcal{L})$, we find that $\varphi_{n,l,m} \in D_\tau(\mathcal{L})$. Then for any $T \in (D_\tau(\mathcal{L}))'$, $\langle T, \varphi_{n,l,m} \rangle$ is well-defined. We construct the following smooth function

$$u_N(v) = \frac{1}{C_N} \sum_{\substack{2n+l \leq N \\ n \geq 0, l \geq 0}} \sum_{|m| \leq l} e^{-\tau \tilde{\lambda}_{n,l}} \langle T, \varphi_{n,l,m} \rangle \varphi_{n,l,m}(v),$$

where

$$C_N = \sqrt{\sum_{\substack{2n+l \leq N \\ n \geq 0, l \geq 0}} \sum_{|m| \leq l} e^{-\tau \tilde{\lambda}_{n,l}} \langle T, \varphi_{n,l,m} \rangle^2}.$$

Applying the result of (1), we find that

$$\|u_N\|_{D_\tau(\mathcal{L})} = 1.$$

From the definition of $(D_\tau(\mathcal{L}))'$, one can verify

$$\|T\|_{(D_\tau(\mathcal{L}))'} \geq \langle T, u_N \rangle = \sqrt{\sum_{\substack{2n+l \leq N \\ n \geq 0, l \geq 0}} \sum_{|m| \leq l} e^{-\tau \tilde{\lambda}_{n,l}} \langle T, \varphi_{n,l,m} \rangle^2}.$$

Passing N to $+\infty$, we have

$$\|T\|_{(D_\tau(\mathcal{L}))'} \geq \sqrt{\sum_{n \geq 0} \sum_{l \geq 0} \sum_{|m| \leq l} e^{-\tau \tilde{\lambda}_{n,l}} \langle T, \varphi_{n,l,m} \rangle^2}. \quad (3.3)$$

On the other hand, for any $u \in D_\tau(\mathcal{L})$ with $\|u\|_{D_\tau(\mathcal{L})} = 1$, we define a series

$$u_N = \sum_{\substack{2n+l \leq N \\ n \geq 0, l \geq 0}} \sum_{|m| \leq l} (u, \varphi_{n,l,m}) \varphi_{n,l,m}.$$

Then $u_N \rightarrow u \in D_\tau(\mathcal{L})$ as $N \rightarrow +\infty$ with $\|u_N\|_{D_\tau(\mathcal{L})} \leq 1$. Therefore,

$$\begin{aligned} |\langle T, u_N \rangle| &= \left| \sum_{\substack{2n+l \leq N \\ n \geq 0, l \geq 0}} \sum_{|m| \leq l} (u, \varphi_{n,l,m}) \langle T, \varphi_{n,l,m} \rangle \right| \\ &\leq \left(\sum_{\substack{2n+l \leq N \\ n \geq 0, l \geq 0}} \sum_{|m| \leq l} e^{-\tau \tilde{\lambda}_{n,l}} \langle T, \varphi_{n,l,m} \rangle^2 \right)^{\frac{1}{2}} \|u_N\|_{D_\tau(\mathcal{L})} \\ &\leq \left(\sum_{\substack{2n+l \leq N \\ n \geq 0, l \geq 0}} \sum_{|m| \leq l} e^{-\tau \tilde{\lambda}_{n,l}} \langle T, \varphi_{n,l,m} \rangle^2 \right)^{\frac{1}{2}}. \end{aligned}$$

By continuity,

$$|\langle T, u \rangle| = \lim_{N \rightarrow +\infty} |\langle T, u_N \rangle|$$

and we obtain

$$\|T\|_{(D_\tau(\mathcal{L}))'} \leq \left(\sum_{n \geq 0} \sum_{l \geq 0} \sum_{|m| \leq l} e^{-\tau \tilde{\lambda}_{n,l}} \langle T, \varphi_{n,l,m} \rangle^2 \right)^{1/2}. \quad (3.4)$$

Part (2) follows from (3.3) and (3.4).

For the part (3), note that $\{\varphi_{n,l,m}(v)\}$ constitutes an orthonormal basis of $L^2(\mathbb{R}^3)$ composed of eigenvectors of the harmonic oscillator \mathcal{H} , then

$$e^{\tau(\log(e+\mathcal{H}))^{\frac{2}{3}}} \varphi_{n,l,m} = e^{\tau(\log(2n+l+\frac{3}{2}+e))^{\frac{2}{3}}} \varphi_{n,l,m}.$$

This ends the proof of Proposition 3.1. \square

Remark 3.2. For $\tau > 0$ and $u \in D_\tau^+(\mathcal{L})$, let $(u, \varphi_{n,l,m})$ be the inner product in $L^2(\mathbb{R}^3)$. Therefore the sequence $\{(u, \varphi_{n,l,m})\}$ satisfies

$$\|u\|_{D_\tau^+(\mathcal{L})}^2 = \sum_{n=0}^{+\infty} \sum_{l=0}^{+\infty} \sum_{|m| \leq l} \tilde{\lambda}_{n,l} e^{\tau \tilde{\lambda}_{n,l}} (u, \varphi_{n,l,m})^2 < +\infty.$$

For $\tau > 0$ and $T \in (D_\tau^+(\mathcal{L}))'$, let $\langle T, \varphi_{n,l,m} \rangle$ be the inner product in sense of distribution. Analogously the sequence $\{\langle T, \varphi_{n,l,m} \rangle\}$ satisfies

$$\|u\|_{(D_\tau^+(\mathcal{L}))'}^2 = \sum_{n=0}^{+\infty} \sum_{l=0}^{+\infty} \sum_{|m| \leq l} \frac{1}{\tilde{\lambda}_{n,l}} e^{-\tau \tilde{\lambda}_{n,l}} \langle T, \varphi_{n,l,m} \rangle^2 < +\infty.$$

We are prepared to prove Theorem 3.1.

Proof of Theorem 3.1: Applying Proposition 2.1, for $\tau > 0$, there exist constant $\tau_2 > \tau_1 > 0$, such that

$$\begin{aligned} \sum_{n=0}^{+\infty} \sum_{l=0}^{+\infty} \sum_{|m| \leq l} e^{2\tau_1(\log(2n+l+\frac{3}{2}+e))} (u, \varphi_{n,l,m})^2 &\leq \sum_{n=0}^{+\infty} \sum_{l=0}^{+\infty} \sum_{|m| \leq l} e^{\tau \tilde{\lambda}_{n,l}} (u, \varphi_{n,l,m})^2 \\ &\leq \sum_{n=0}^{+\infty} \sum_{l=0}^{+\infty} \sum_{|m| \leq l} e^{2\tau_2(\log(2n+l+\frac{3}{2}+e))} (u, \varphi_{n,l,m})^2. \end{aligned}$$

From Proposition 3.1, we have

$$E_s^{\tau_2}(\mathbb{R}^3) \subset D_\tau(\mathcal{L}) \subset E_s^{\tau_1}(\mathbb{R}^3).$$

In addition, when $0 < s \leq 2$, for $u \in D_1(\mathcal{L})$,

$$e^{2\tau_1(\log(2n+l+e))} \geq e^{2\tau_1(\log(2n+l+\frac{3}{2}+e))} = (2n+l+\frac{3}{2}+e)^{2\tau_1}.$$

Therefore

$$\begin{aligned} \|(e + \mathcal{H})^{\tau_1} u\|_{L^2}^2 &= \sum_{n=0}^{+\infty} \sum_{l=0}^{+\infty} \sum_{|m| \leq l} (2n+l+\frac{3}{2}+e)^{2\tau_1} (u, \varphi_{n,l,m})^2 \\ &\leq \sum_{n=0}^{+\infty} \sum_{l=0}^{+\infty} \sum_{|m| \leq l} e^{2\tau_1(\log(2n+l+\frac{3}{2}+e))} (u, \varphi_{n,l,m})^2 \\ &= \|u\|_{E_s^{\tau_1}}^2 < +\infty. \end{aligned}$$

This ends the proof of Theorem 3.1. \square

4. PROOF OF THEOREMS 1.1-1.2

Now we are prepared to prove Theorem 1.1.

Proof of Theorem 1.1. We proceed to treat the proof by the following four steps.

Step 1. Construction of an auxiliary function g_N with initial datum which approximates $g_0 \in (D_\tau(\mathcal{L}))'$.

For $g_0 \in (D_\tau(\mathcal{L}))'$, $\tilde{\lambda}_{n,l}$ defined in (3.2), we obtain from Proposition 3.1

$$\sum_{n=0}^{+\infty} \sum_{l=0}^{+\infty} \sum_{|m| \leq l} e^{-\tau \tilde{\lambda}_{n,l}} \langle g_0, \varphi_{n,l,m} \rangle^2 = \|g_0\|_{(D_\tau(\mathcal{L}))'}^2 < +\infty. \quad (4.1)$$

For all $n, l \in \mathbb{N}$, $m \in \{-l, \dots, l\}$, we consider the Cauchy problem associated with the ODEs

$$\begin{cases} \partial_t a_{n,l,m}(t) + \lambda_{n,l} a_{n,l,m}(t) = 0, \\ a_{n,l,m}(0) = \langle g_0, \varphi_{n,l,m} \rangle. \end{cases}$$

Direct calculation shows that $a_{n,l,m}(t) = e^{-\lambda_{n,l}t} \langle g_0, \varphi_{n,l,m} \rangle$. Let us now fix some positive integer $N \geq 3$ and define the following function $g_N : [0, +\infty[\times \mathbb{R}^3 \rightarrow L^2(\mathbb{R}^3)$ by

$$g_N = \sum_{\substack{2n+l \leq N \\ n \geq 0, l \geq 0}} \sum_{|m| \leq l} e^{-\lambda_{n,l}t} \langle g_0, \varphi_{n,l,m} \rangle \varphi_{n,l,m}.$$

Then g_N satisfies

$$\begin{cases} \partial_t g_N + \mathcal{L}g_N = 0, \\ g_N(0) = \sum_{\substack{2n+l \leq N \\ n \geq 0, l \geq 0}} \langle g_0, \varphi_{n,l,m} \rangle \varphi_{n,l,m}. \end{cases} \quad (4.2)$$

Step 2. Existence of the solution to the Cauchy problem (1.3).

It is obvious that,

$$\mathbf{P}g_N \equiv \mathbf{P}g_0.$$

For $N \in \mathbb{N}$ big enough and for any $P \in \mathbb{N}^+$,

$$\begin{aligned} \forall t > 0, \|g_{N+P} - g_N\|_{(D_\tau(\mathcal{L}))'}^2 &= \sum_{\substack{N+1 \leq 2n+l \leq N+P \\ n \geq 0, l \geq 0}} \sum_{|m| \leq l} e^{-2\lambda_{n,l}t} e^{-\lambda_{n,l}\tau} |\langle g_0, \varphi_{n,l,m} \rangle|^2 \\ &\leq \sum_{\substack{N+1 \leq 2n+l \leq N+P \\ n \geq 0, l \geq 0}} \sum_{|m| \leq l} e^{-\lambda_{n,l}\tau} |\langle g_0, \varphi_{n,l,m} \rangle|^2 \rightarrow 0. \end{aligned}$$

Let us fix $T > 0$, $N \geq 3$ big enough. Using the estimate in Proposition 2.1, we can check that

$$\begin{aligned} &\int_0^T \|g_{N+P} - g_N\|_{(D_\tau^+(\mathcal{L}))'}^2 dt \\ &= \sum_{\substack{N+1 \leq 2n+l \leq N+P \\ n \geq 0, l \geq 0}} \sum_{|m| \leq l} \lambda_{n,l}^{-1} e^{-\lambda_{n,l}\tau} |\langle g_0, \varphi_{n,l,m} \rangle|^2 \frac{1}{2\lambda_{n,l}} (1 - e^{-2\lambda_{n,l}T}) \\ &\leq \sum_{\substack{N+1 \leq 2n+l \leq N+P \\ n \geq 0, l \geq 0}} \sum_{|m| \leq l} e^{-\lambda_{n,l}\tau} |\langle g_0, \varphi_{n,l,m} \rangle|^2 \rightarrow 0; \end{aligned}$$

$$\begin{aligned} &\int_0^T \|\partial_t g_{N+P} - \partial_t g_N\|_{(D_\tau^+(\mathcal{L}))'}^2 dt \\ &= \sum_{\substack{N+1 \leq 2n+l \leq N+P \\ n \geq 0, l \geq 0}} \sum_{|m| \leq l} e^{-\lambda_{n,l}\tau} |\langle g_0, \varphi_{n,l,m} \rangle|^2 \frac{\lambda_{n,l}}{2\lambda_{n,l}} (1 - e^{-2\lambda_{n,l}T}) \\ &\leq \frac{1}{2} \sum_{\substack{N+1 \leq 2n+l \leq N+P \\ n \geq 0, l \geq 0}} \sum_{|m| \leq l} e^{-\lambda_{n,l}\tau} |\langle g_0, \varphi_{n,l,m} \rangle|^2 \rightarrow 0; \end{aligned}$$

$$\begin{aligned} &\int_0^T \|\mathcal{L}^{\frac{1}{2}} g_{N+P} - \mathcal{L}^{\frac{1}{2}} g_N\|_{(D_\tau(\mathcal{L}))'}^2 dt \\ &\leq \frac{1}{2} \sum_{\substack{N+1 \leq 2n+l \leq N+P \\ n \geq 0, l \geq 0}} \sum_{|m| \leq l} e^{-\lambda_{n,l}\tau} |\langle g_0, \varphi_{n,l,m} \rangle|^2 \rightarrow 0. \end{aligned}$$

Therefore, for $T > 0$ fixed,

$$\begin{aligned} \{g_N(t)\} & \text{ is a Cauchy sequence in } (D_\tau(\mathcal{L}))', \quad \forall t \in [0, T]; \\ \{g_N\} & \text{ is a Cauchy sequence in } H^1([0, T], (D_\tau^+(\mathcal{L}))'); \\ \{\mathcal{L}^{\frac{1}{2}}g_N\} & \text{ is a Cauchy sequence in } L^2([0, T], (D_\tau(\mathcal{L}))'). \end{aligned}$$

Then there exists a function $g \in L^\infty([0, T], (D_\tau(\mathcal{L}))') \cap H^1([0, T], (D_\tau^+(\mathcal{L}))')$, $\mathbf{P}g \equiv \mathbf{P}g_0$ and $\mathcal{L}^{\frac{1}{2}}g \in L^2([0, T], (D_\tau(\mathcal{L}))')$, such that

$$\begin{aligned} \forall t > 0, \quad g_N(t) & \rightarrow g(t) \text{ in } (D_\tau(\mathcal{L}))', \\ g_N & \rightarrow g \text{ in } H^1([0, T], (D_\tau^+(\mathcal{L}))'), \\ \mathcal{L}^{\frac{1}{2}}g_N & \rightarrow \mathcal{L}^{\frac{1}{2}}g \text{ in } L^2([0, T], (D_\tau(\mathcal{L}))'). \end{aligned}$$

By Sobolev embedding theorem

$$H^1([0, T], (D_\tau^+(\mathcal{L}))') \hookrightarrow C([0, T], (D_\tau^+(\mathcal{L}))'),$$

we have

$$g \in C([0, T], (D_\tau^+(\mathcal{L}))').$$

Now we prove g is the desired weak solution of Cauchy problem (1.3). For any test function $\phi \in C^1([0, T], C_0^\infty(\mathbb{R}^3))$ and $0 < t < T$, recalled from (4.2), we have

$$\begin{aligned} & \langle g_N(t), \phi(t) \rangle - \langle g_N(0), \phi(0) \rangle \\ &= \int_0^t \langle g_N(\tau), \partial_\tau \phi \rangle d\tau - \int_0^t \langle g_N(\tau), \mathcal{L}\phi(\tau) \rangle d\tau. \end{aligned}$$

Passing to the limit as $N \rightarrow +\infty$, we get

$$\begin{aligned} & \langle g(t), \phi(t) \rangle - \langle g(0), \phi(0) \rangle \\ &= \int_0^t \langle g, \partial_\tau \phi \rangle d\tau - \int_0^t \langle g, \mathcal{L}\phi \rangle d\tau. \end{aligned} \tag{4.3}$$

Besides, from (4.1), we see that

$$g_N(0) \rightarrow g(0) \text{ in } (D_\tau(\mathcal{L}))'.$$

Henceforth, we obtain

$$\langle g(0), \phi(0) \rangle = \langle g_0, \phi(0) \rangle.$$

Substituting the above result into (4.3), (1.6) follows.

Step 3. Uniqueness of the solution to the Cauchy problem (1.3).

Assume that \tilde{g} is another solution satisfies (1.5) and (1.6). Denote

$$h(t) = g(t) - \tilde{g}(t).$$

For $T > 0$, for any $\phi \in C^1([0, T], C_0^\infty(\mathbb{R}^3))$ and $0 < t < T$, we have $h(0) = 0$ and

$$\langle h(t), \phi(t) \rangle = \int_0^t \langle h, \partial_\tau \phi \rangle d\tau - \int_0^t \langle h, \mathcal{L}\phi(\tau) \rangle d\tau. \tag{4.4}$$

We define a smooth function

$$\phi(t) = \sum_{\substack{2n+l \leq N \\ n \geq 0, l \geq 0}} e^{-2\tau \tilde{\lambda}_{n,l}} \langle h(t), \varphi_{n,l,m} \rangle \varphi_{n,l,m}.$$

Substituted into (4.4) as a test function, we have

$$\sum_{\substack{2n+l \leq N \\ n \geq 0, l \geq 0}} e^{-2\tau \tilde{\lambda}_{n,l}} \langle h(t), \varphi_{n,l,m} \rangle^2 \leq 0.$$

Passing $N \rightarrow +\infty$, we have

$$\|h(t)\|_{(D_\tau(\mathcal{L}))'}^2 \leq 0.$$

Thus $h = 0$ in $L^2(\mathbb{R}^3)$.

Step 4. Regularity of the solution with the initial data $g_0 \in E_s^{-t_0}(\mathbb{R}^3)$.

Using Theorem 3.1, we have

$$D_{t_0/\tau_1}(\mathcal{L}) \subset E_s^{t_0}(\mathbb{R}^3) \quad \text{and} \quad E_s^{-t_0}(\mathbb{R}^3) \subset (D_{t_0/\tau_1}(\mathcal{L}))'$$

where τ_1 is given by Theorem 3.1. Namely,

$$\|e^{-t_0} (\log(e+\mathcal{H}))^{\frac{2}{s}} g_0\|_{L^2} < +\infty.$$

Recalling that $g_0 = \mathbf{P}g_0 + (\mathbf{I} - \mathbf{P})g_0$, we have

$$(\mathbf{I} - \mathbf{P})g_N(t) = \sum_{\substack{2n+l \leq N \\ n \geq 0, l \geq 0}} e^{-\lambda_{n,l}t} \langle (\mathbf{I} - \mathbf{P})g_0, \varphi_{n,l,m} \rangle \varphi_{n,l,m}.$$

Then for any $T > 0$,

$$(\mathbf{I} - \mathbf{P})g_N(t) \rightarrow g \quad \text{in } L^\infty([0, T], (D_{t_0/\tau_1}(\mathcal{L}))').$$

Moreover, from Proposition 2.1, there exists a constant $c > 0$ such that for any $t > t_0/c$

$$\|e^{ct(\log(\mathcal{H}+e))^{\frac{2}{s}}} (\mathbf{I} - \mathbf{P})g_N\|_{L^2(\mathbb{R}^3)} \leq e^{-\frac{1}{4}\lambda_{2,0}t} \|(\mathbf{I} - \mathbf{P})g_0\|_{E_s^{-t_0}(\mathbb{R}^3)},$$

where we have used the following estimate of the eigenvalue (see Part 4.3 of [8])

$$0 < \lambda_{2,0} \leq \lambda_{n,l}.$$

By the lower continuity, we have

$$\begin{aligned} & \|e^{ct(\log(\mathcal{H}+e))^{\frac{2}{s}}} (\mathbf{I} - \mathbf{P})g\|_{L^2(\mathbb{R}^3)} \\ & \leq \liminf_{N \rightarrow +\infty} \|e^{ct(\log(\mathcal{H}+e))^{\frac{2}{s}}} (\mathbf{I} - \mathbf{P})g_N\|_{L^2(\mathbb{R}^3)} \\ & \leq e^{-\frac{1}{4}\lambda_{2,0}t} \|(\mathbf{I} - \mathbf{P})g_0\|_{E_s^{-t_0}(\mathbb{R}^3)}. \end{aligned} \tag{4.5}$$

This concludes the proof of Theorem 1.1. \square

We now give the proof of Theorem 1.2

Proof. Consider that $g_0 \in L^2(\mathbb{R}^3)$. For $0 < s \leq 2$, by using the formula (3.1) in Theorem 3.1 and the formula (4.5), we obtain,

$$\|(\mathcal{H} + e)^{ct} (\mathbf{I} - \mathbf{P})g\|_{L^2(\mathbb{R}^3)} \leq e^{-\frac{1}{4}\lambda_{2,0}t} \|(\mathbf{I} - \mathbf{P})g_0\|_{L^2(\mathbb{R}^3)}.$$

This is the formula (1.7).

For the part 2) of Theorem 1.2, i.e, in the case $0 < s < 2$, the formula (1.8) follows from Proposition 5.1 and formula (4.5). The proof of Theorem 1.2 is completed. \square

Now we study the regularizing effect of the Boltzmann equation with the initial data in Sobolev space, which provides a detailed exposition of Remark 1.2.

Recalled that $\{\varphi_{n,l,m}(v)\}_{n,l \in \mathbb{N}, |m| \leq l}$ constitutes an orthonormal basis of $L^2(\mathbb{R}^3)$ composed of eigenvectors of the harmonic oscillator (see [1], [13])

$$\mathcal{H}(\varphi_{n,l,m}) = (2n + l + \frac{3}{2})\varphi_{n,l,m}.$$

For $\tau > 0$ (see (5.1) in the Appendix),

$$\|u\|_{Q^\tau(\mathbb{R}^3)}^2 = \|(\mathcal{H} + e)^{\frac{\tau}{2}} u\|_{L^2}^2 = \sum_{n=0}^{+\infty} \sum_{l=0}^{+\infty} \sum_{|m| \leq l} (2n + l + \frac{3}{2} + e)^\tau (u, \varphi_{n,l,m})_{L^2}^2.$$

We consider the example in Remark 1.2.

In the case $s = 2$, the regularizing effect in $Q^k(\mathbb{R}^3)$ has usually a positive time delay.

Example 4.1. Consider the initial data $g_0 = \sum_{n \geq 2} \frac{1}{n^{\frac{1}{2}} \log n} \phi_{n,0,0}$, then

$$\|g_0\|_{L^2(\mathbb{R}^3)}^2 = \sum_{n=2}^{+\infty} \frac{1}{n(\log n)^2} < +\infty.$$

This shows that $g_0 \in L^2(\mathbb{R}^3)$. The solution of the Cauchy problem (1.3) can be written as

$$g(t) = \sum_{n=2}^{+\infty} e^{-t\lambda_{n,0}} \frac{1}{n^{\frac{1}{2}} \log n} \phi_{n,0,0}.$$

By using Proposition 2.1, for $s = 2$, we have

$$\lambda_{n,0} \approx \log(2n + e).$$

Then

$$\begin{aligned} \|g(t)\|_{Q^k(\mathbb{R}^3)}^2 &= \sum_{n=2}^{+\infty} (2n + \frac{3}{2} + e)^k e^{-2t\lambda_{n,0}} \frac{1}{n(\log n)^2} \\ &\approx \sum_{n=2}^{+\infty} (2n + \frac{3}{2} + e)^k \frac{1}{n^{1+2t}(\log n)^2}, \end{aligned}$$

which is convergent when $2t \geq k$, i.e. $t \geq \frac{k}{2}$.

We can check that there exists $t_k = \frac{k}{2}$ such that $g(t) \notin Q^k(\mathbb{R}^3)$ for $0 \leq t < t_k$.

In the case $s > 2$. We prove that there is no regularizing effect in the Sobolev space.

Example 4.2. Consider any real numbers $0 < \tau < \tau'$ and $g_0 = \sum_{n \geq 2} \frac{1}{n^{\frac{\tau+1}{2}} \log n} \phi_{n,0,0}$.

Therefore we have

$$\begin{aligned} \|g_0\|_{Q^\tau(\mathbb{R}^3)}^2 &= \sum_{n=2}^{+\infty} (2n + \frac{3}{2} + e)^\tau \frac{1}{n^{\tau+1}(\log n)^2} \\ &\approx \sum_{n=2}^{+\infty} \frac{1}{n(\log n)^2} < +\infty. \end{aligned}$$

This means that $g_0 \in Q^\tau(\mathbb{R}^3)$.

However, the solution of the Cauchy problem (1.3) can be written as

$$g(t) = \sum_{n=2}^{+\infty} e^{-t\lambda_{n,0}} \frac{1}{n^{\frac{\tau+1}{2}} \log n} \phi_{n,0,0}.$$

It is easy to prove that $g(t) \in Q^\tau(\mathbb{R}^3)$. Now we prove $g(t) \notin Q^{\tau'}(\mathbb{R}^3)$.

In the case $s > 2$, by using the Proposition 2.1, we obtain

$$\lambda_{n,0} \approx (\log(2n+e))^{\frac{2}{s}}.$$

Therefore, there exists a constant $c_0 > 0$ such that

$$\begin{aligned} \|g(t)\|_{Q^{\tau'}(\mathbb{R}^3)}^2 &= \sum_{n=2}^{+\infty} (2n + \frac{3}{2} + e)^{\tau'} e^{-2t\lambda_{n,0}} \frac{1}{n^{\tau+1} (\log n)^2} \\ &\leq \sum_{n=2}^{+\infty} \frac{n^{\tau'-\tau}}{e^{2tc_0(\log(2n+e))^{\frac{2}{s}}} n (\log n)^2} \\ &= \sum_{n=2}^{+\infty} e^{\frac{\tau'-\tau}{2} \log(n) - 2tc_0(\log(2n+e))^{\frac{2}{s}}} \frac{1}{n^{1-\frac{\tau'-\tau}{2}} (\log n)^2}. \end{aligned}$$

Considering the condition that $\tau' > \tau$ and $s > 2$, one can verify that

$$\frac{\tau' - \tau}{2} \log(n) - 2tc_0(\log(2n+e))^{\frac{2}{s}} \rightarrow +\infty, \text{ as } n \rightarrow +\infty.$$

Since

$$\sum_{n=2}^{+\infty} \frac{1}{n^{1-\frac{\tau'-\tau}{2}} (\log n)^2} = +\infty,$$

we obtain

$$\|g(t)\|_{Q^{\tau'}(\mathbb{R}^3)}^2 = +\infty.$$

We conclude

$$g(t) \notin Q^{\tau'}(\mathbb{R}^3).$$

We can check that for $t \geq 0$ the solution $g(t)$ stays in the space $\in Q^\tau(\mathbb{R}^3)$, but never belongs to $Q^{\tau'}(\mathbb{R}^3)$.

5. APPENDIX

We present in this section some spectral properties of the functional spaces used in this paper and the proof of lemma 5.1.

The symmetric Gelfand-Shilov space $S_v^\nu(\mathbb{R}^3)$ can be characterized through the decomposition into the Hermite basis $\{H_\alpha\}_{\alpha \in \mathbb{N}^3}$ and the harmonic oscillator $\mathcal{H} = -\Delta + \frac{|v|^2}{4}$. For more details, see Theorem 2.1 in the book [9].

$$\begin{aligned} f \in S_v^\nu(\mathbb{R}^3) &\Leftrightarrow f \in C^\infty(\mathbb{R}^3), \exists \tau > 0, \|e^{\tau \mathcal{H}^{\frac{1}{2\nu}}} f\|_{L^2} < +\infty; \\ &\Leftrightarrow f \in L^2(\mathbb{R}^3), \exists \epsilon_0 > 0, \left\| \left(l e^{\epsilon_0 |\alpha|^{\frac{1}{2\nu}}} (f H_\alpha)_{L^2} \right)_{\alpha \in \mathbb{N}^3} \right\|_{l^2} < +\infty; \\ &\Leftrightarrow \exists C > 0, A > 0, \left\| \left(-\Delta + \frac{|v|^2}{4} \right)^{\frac{k}{2}} f \right\|_{L^2(\mathbb{R}^3)} \leq A C^k (k!)^\nu, \quad k \in \mathbb{N} \end{aligned}$$

where

$$H_\alpha(v) = H_{\alpha_1}(v_1) H_{\alpha_2}(v_2) H_{\alpha_3}(v_3), \quad \alpha \in \mathbb{N}^3,$$

and for $x \in \mathbb{R}$,

$$H_n(x) = \frac{(-1)^n}{\sqrt{2^n n! \pi}} e^{\frac{x^2}{2}} \frac{d^n}{dx^n} (e^{-x^2}) = \frac{1}{\sqrt{2^n n! \pi}} \left(x - \frac{d}{dx}\right)^n (e^{-\frac{x^2}{2}}).$$

For the harmonic oscillator $\mathcal{H} = -\Delta + \frac{|v|^2}{4}$ of 3-dimension and $s > 0$, we have

$$\mathcal{H}^{\frac{k}{2}} H_\alpha = (\lambda_\alpha)^{\frac{k}{2}} H_\alpha, \quad \lambda_\alpha = \sum_{j=1}^3 \left(\alpha_j + \frac{1}{2}\right), \quad k \in \mathbb{N}, \alpha \in \mathbb{N}^3.$$

The symmetric weighted Sobolev space $Q^{2\tau}(\mathbb{R}^3)$ can be also characterized through the decomposition into the Hermite basis :

$$\begin{aligned} f \in Q^{2\tau}(\mathbb{R}^3) &\Leftrightarrow f \in L^2(\mathbb{R}^3), \quad \left\| (e + \mathcal{H})^\tau f \right\|_{L^2} < +\infty; \\ &\Leftrightarrow f \in L^2(\mathbb{R}^3), \quad \left\| (|\alpha|^\tau (f, H_\alpha)_{L^2})_{\alpha \in \mathbb{N}^3} \right\|_{\ell^2} < +\infty. \end{aligned} \quad (5.1)$$

Concerning the Sobolev-type space $E_v^\tau(\mathbb{R}^3)$ for $v > 0$ introduced in part 1, we have the following property :

Proposition 5.1. *Let $0 < v < 2$ and $\tau > 0$. There exists a constant $C = C_v$ such that, for any $f \in E_v^\tau(\mathbb{R}^3)$,*

$$\forall k \geq 1, \quad \left\| \left(-\Delta + \frac{|v|^2}{4}\right)^{\frac{k}{2}} f \right\|_{L^2} \leq e^{C \left(\frac{1}{\tau}\right)^{\frac{v}{2-v}} k^{\frac{2}{2-v}}} \|f\|_{E_v^\tau}.$$

Proof. We expand f in the Hermite basis : noting $f_\alpha = (f, H_\alpha)_{L^2}$, we get

$$\sum_{\alpha} e^{2\tau(\log(e+\lambda_\alpha))} |f_\alpha|^2 = \|f\|_{E_v^\tau}^2.$$

We rephrase the previous identity as follows

$$\sum_{\alpha \in \mathbb{N}^3} h_{\tau,k}(e + \lambda_\alpha) (e + \lambda_\alpha)^k |f_\alpha|^2 = \|f\|_{E_v^\tau}^2$$

where

$$h_{\tau,k}(x) = \frac{e^{2\tau(\log x)^{\frac{2}{v}}}}{x^k}.$$

It is easy to check that

$$\forall x \geq 1, \quad h_{\tau,k}(x) \geq e^{-\frac{2-v}{2} \left(\frac{v}{4\tau}\right)^{\frac{v}{2-v}} k^{\frac{2}{2-v}}}. \quad (5.2)$$

Indeed, using Young's inequality

$$xy \leq \frac{1}{p} x^p + \frac{1}{q} y^q, \quad \text{where } \frac{1}{p} + \frac{1}{q} = 1,$$

with $p = \frac{2}{2-v}$, $q = \frac{2}{v}$, we obtain

$$k \log x \leq \frac{2-v}{2} \left[\left(\frac{v}{4\tau}\right)^{\frac{v}{2-v}} k\right]^{\frac{2}{2-v}} + 2\tau [\log x]^{\frac{2}{v}},$$

and (5.2) follows immediately. Therefore we deduce that

$$\begin{aligned} \|f\|_{E_v^\tau}^2 &= \sum_{\alpha \in \mathbb{N}^3} h_{\tau,k}(e + \lambda_\alpha) (e + \lambda_\alpha)^k |f_\alpha|^2 \\ &\geq e^{-\frac{2-v}{2} \left(\frac{v}{4\tau}\right)^{\frac{v}{2-v}} k^{\frac{2}{2-v}}} \sum_{\alpha \in \mathbb{N}^3} (e + \lambda_\alpha)^k |f_\alpha|^2 \end{aligned}$$

and using (5.1) we conclude the proof. \square

We now give a proof of Lemma 2.3 in [15].

Lemma 5.1. ([15]) *We have*

$$1 - P_l\left(\cos \frac{\theta}{l}\right) = O(\theta^2)$$

uniformly for $l \geq 1$ and $\theta \in [0, \frac{\pi}{2}]$.

Proof. We recall that the Legendre polynomial $w(t) = P_l(\cos t)$ satisfies

$$\frac{1}{\sin t} \frac{d}{dt} \left(\sin t \frac{dw}{dt} \right) + l(l+1)w = 0, \quad w(0) = 1, \quad |w| \leq 1.$$

Let us define the new function for $\theta \in [0, \frac{\pi}{2}]$

$$u(\theta) = 1 - P_l\left(\cos \frac{\theta}{l}\right).$$

The function u is solution of the differential equation

$$\frac{d}{d\theta} \left(\sin\left(\frac{\theta}{l}\right) \frac{du}{d\theta} \right) = \frac{l+1}{l} \sin\left(\frac{\theta}{l}\right) u = O\left(\frac{\theta}{l}\right).$$

We integrate on the interval $[0, \theta]$ and we get

$$\sin\left(\frac{\theta}{l}\right) \frac{du}{d\theta} = O\left(\frac{\theta^2}{l}\right).$$

Therefore

$$\frac{du}{d\theta} = O(\theta).$$

Since $u(0) = 0$, another integration finishes the proof of the estimate. \square

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