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Kleber Carrapatoso, Stéphane Mischler. Landau equation for very soft and Coulomb potentials near Maxwellians. 2015. hal-01237530v1

HAL Id: hal-01237530

<https://hal.science/hal-01237530v1>

Preprint submitted on 3 Dec 2015 (v1), last revised 7 Jan 2017 (v3)

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LANDAU EQUATION FOR VERY SOFT AND COULOMB POTENTIALS NEAR MAXWELLIANS

K. CARRAPATOSO AND S. MISCHLER

ABSTRACT. This work deals with the Landau equation for very soft and Coulomb potentials near the associated Maxwellian equilibrium. We first investigate the corresponding linearized operator and develop a method to prove stability estimates of its associated semigroup in large functional spaces. We then deduce existence, uniqueness and fast decay of the solutions to the nonlinear equation in a close-to-equilibrium framework. Our result drastically improves the set of initial data compared to the one considered by Guo and Strain who established similar results in [21, 37, 38]. Our functional framework is compatible with the non perturbative frameworks developed by Villani, Desvillettes and co-authors [42, 17, 16, 13], and our main result then makes possible to improve the speed of convergence to the equilibrium established therein.

December 3, 2015

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2010 *Mathematics Subject Classification.* 35B40, 35Q20, 35K67, 47D06, 47H20.

Key words and phrases. Landau equation, existence, uniqueness, stability, semigroup stability, very soft potentials, Coulomb potential, convergence to equilibrium.

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1. INTRODUCTION

1.1. The Landau equation. The Landau equation is a fundamental equation in kinetic theory modelling the evolution of a dilute plasma interacting through binary collisions. We consider here a plasma confined in a torus \mathbb{T}^3 and described by the distribution function $F = F(t, x, v) \geq 0$ of particles which at time $t \geq 0$ and at position $x \in \mathbb{T}^3$, move with the velocity $v \in \mathbb{R}^3$. The evolution of F is governed by the *spatially inhomogeneous* Landau equation

$$(1.1) \quad \begin{cases} \partial_t F + v \cdot \nabla_x F = Q(F, F) \\ F(0, x, v) = F_0(x, v). \end{cases}$$

For a spatially homogeneous plasma, namely when $F = F(t, v)$, the equation simplifies into the *spatially homogeneous* Landau equation

$$(1.2) \quad \begin{cases} \partial_t F = Q(F, F) \\ F(0, v) = F_0(v). \end{cases}$$

The Landau collision operator Q is a bilinear operator acting only on the velocity variable and it is given by

$$(1.3) \quad Q(g, f)(v) = \partial_i \int_{\mathbb{R}^3} a_{ij}(v - v_*) \{g_* \partial_j f - f \partial_j g_*\} dv_*,$$

where here and below we use the convention of implicit summation over repeated indices and the usual shorthand $g_* = g(v_*)$, $\partial_j g_* = \partial_{v_{*j}} g(v_*)$, $f = f(v)$ and $\partial_j f = \partial_{v_j} f(v)$. The matrix-valued function a is nonnegative, symmetric and depends on the interaction between particles. When particles interact by an inverse power law potential, a is given by

$$(1.4) \quad a_{ij}(z) = |z|^{\gamma+2} \left(\delta_{ij} - \frac{z_i z_j}{|z|^2} \right), \quad -3 \leq \gamma \leq 1.$$

In the present article, we shall consider the cases of very soft potentials $\gamma \in (-3, -2)$ and Coulomb potential $\gamma = -3$. It is worth mentioning that the Coulomb potential is the most physically interesting case, and also the most difficult because of the strong singularity in (1.4).

The Landau equation (1.1) (or (1.2)) possesses two fundamental properties (which hold at least formally). On the one hand, it conserves mass, momentum and energy, more precisely

$$(1.5) \quad \frac{d}{dt} \int_{\mathbb{T}^3 \times \mathbb{R}^3} F \varphi dx dv = \int_{\mathbb{T}^3 \times \mathbb{R}^3} \{Q(F, F) - v \cdot \nabla_x f\} \varphi dx dv = 0 \quad \text{for } \varphi(v) = 1, v, |v|^2.$$

On the other hand, the Landau version of the celebrated Boltzmann H -theorem holds: the entropy $H(F) := \int F \log F dx dv$ is non-increasing and the global equilibria are global Maxwellian distributions that are independent of time and position. Hereafter, we normalize the initial data

$$\int_{\mathbb{T}^3 \times \mathbb{R}^3} F_0 dx dv = 1, \quad \int_{\mathbb{T}^3 \times \mathbb{R}^3} F_0 v dx dv = 0, \quad \int_{\mathbb{T}^3 \times \mathbb{R}^3} F_0 |v|^2 dx dv = 3,$$

and therefore we consider the associated global Maxwellian equilibrium

$$\mu(v) = (2\pi)^{-3/2} e^{-|v|^2/2},$$

with same mass, momentum and energy of the initial data (normalizing the volume of the torus to $|\mathbb{T}_x^3| = 1$).

1.2. Main results. Our aim in this work is to study the Landau equation in a close-to-equilibrium framework (or perturbative regime) in large functional spaces and to establish new well-posedness and trend to the equilibrium results.

Let us then introduce the functional framework we will work with. For a given velocity weight function $m = m(v) : \mathbb{R}^3 \rightarrow \mathbb{R}_+$ and exponent $1 \leq p \leq \infty$, we define the associated weighted Lebesgue space $L_v^p(m)$ and weighted Sobolev space $W_v^{1,p}(m)$, through their norms

$$(1.6) \quad \|f\|_{L_v^p(m)} := \|mf\|_{L_v^p}, \quad \|f\|_{W_v^{1,p}(m)} := \|mf\|_{W_v^{1,p}}.$$

Similarly, we define the weighted Sobolev space $W_x^{n,p} L_v^p(m)$, $n \in \mathbb{N}$, associated to the norm

$$(1.7) \quad \|f\|_{W_x^{n,p} L_v^p(m)}^p := \|mf\|_{W_x^{n,p} L_v^p}^p := \sum_{0 \leq j \leq n} \|\nabla_x^j(mf)\|_{L_{x,v}^p}^p,$$

and we adopt the usual notation $H^n = W^{n,2}$.

We make the following assumption on the weight function m :

$$(1.8) \quad \begin{aligned} m = \langle v \rangle^k &:= (1 + |v|^2)^{k/2} \text{ with } k > 2 + 3/2; \\ m = \exp(\kappa \langle v \rangle^s) &\text{ with } s \in (0, 2) \text{ and } \kappa > 0, \text{ or } s = 2 \text{ and } \kappa \in (0, 1/2); \end{aligned}$$

and through the paper we denote $\sigma = 0$ when m is a polynomial weight, and $\sigma = s$ when m is an exponential weight. We associate the decay functions

$$(1.9) \quad \Theta_m(t) = \begin{cases} C \langle t \rangle^{-\frac{k-\ell}{|\gamma|}}, & \text{if } m = \langle v \rangle^k, \\ C e^{-\lambda t^{\frac{s}{|\gamma|}}}, & \text{if } m = e^{\kappa \langle v \rangle^s}, \end{cases}$$

for any constant $\ell \in (2 + 3/2, k)$ and some constants $C, \lambda \in (0, \infty)$. It is worth emphasising that in the polynomial case $m = \langle v \rangle^k$, the notation Θ_m refers to a class of functions (with increasing rate of decay as ℓ tends to $2 + 3/2$), while in the exponential case $m = e^{\kappa \langle v \rangle^s}$, the notation Θ_m stands for a fixed function. We finally introduce the projection operator P_v on the v -direction for any given $v \in \mathbb{R}^3 \setminus \{0\}$ defined by

$$(1.10) \quad P_v \xi = \left(\xi \cdot \frac{v}{|v|} \right) \frac{v}{|v|}, \quad \forall \xi \in \mathbb{R}^3,$$

as well as the anisotropic gradient $\tilde{\nabla}_v f$ of a function f defined by

$$(1.11) \quad \tilde{\nabla}_v f = P_v \nabla_v f + \langle v \rangle (I - P_v) \nabla_v f.$$

Our main result reads as follows.

Theorem 1.1. *For any weight function m satisfying (1.8), there exists $\epsilon_0 > 0$ small enough so that, if $\|F_0 - \mu\|_{H_x^2 L_v^2(m)} \leq \epsilon_0$, there exists a unique global weak solution F to (1.1) such that*

$$(1.12) \quad \begin{aligned} \sup_{t \geq 0} \|F(t) - \mu\|_{H_x^2 L_v^2(m)}^2 &+ \int_0^\infty \|\langle v \rangle^{\frac{\gamma+\sigma}{2}} (F(\tau) - \mu)\|_{H_x^2 L_v^2(m)}^2 d\tau \\ &+ \int_0^\infty \|\langle v \rangle^{\frac{\gamma}{2}} \tilde{\nabla}_v \{m(F(\tau) - \mu)\}\|_{H_x^2 L_v^2}^2 d\tau \leq C \epsilon_0^2. \end{aligned}$$

This solution verifies the decay estimate

$$(1.13) \quad \|F(t) - \mu\|_{H_x^2 L_v^2} \leq \Theta_m(t) \|F_0 - \mu\|_{H_x^2 L_v^2(m)}, \quad \forall t \geq 0.$$

Remark 1.2. For a spatially homogeneous initial datum $F_0 \in L_v^2(m)$, the associated solution $F(t)$ is also a spatially homogeneous function, and thus satisfies the spatially homogeneous Landau equation (1.2). In that spatially homogeneous framework, the H_x^2 regularity is automatically fulfilled, it can be then removed of the corresponding version of Theorem 1.1 which statement thus simplifies accordingly.

Let us briefly comment on known results on the existence, uniqueness and long-time behaviour of solutions to the Landau equation when $-3 \leq \gamma < -2$. For the other cases $-2 \leq \gamma \leq 1$, we refer the reader to the recent work [14] and the references therein.

In the space homogeneous case, existence of solutions has been first addressed by Arsenev-Penskov [2], and next by Villani [42] and Desvillettes [16] who establish existence of global solutions for any initial datum with finite mass, energy and entropy. Uniqueness of strong solutions (which do exist locally in time) has been proved by Fournier-Guérin [19] and Fournier [18]. In a similar framework and for bounded (after regularisation) collision kernel a with $-3 < \gamma < -2$, polynomial convergence to the equilibrium has been obtained by Toscani and Villani [39] by entropy dissipation method. That last result has been recently improved by Desvillettes, He and the first author [13], who prove convergence to equilibrium with algebraic or stretched exponential rate removing the boundedness (unphysical) assumption on the collision kernel a and also considering the Coulomb potential $\gamma = -3$. The space homogeneous version of the results by Guo and Strain presented below also provides well-posedness and accurate rate of convergence to the equilibrium in a perturbative regime in $H_v^s(\mu^{-\theta})$, $\theta \in (1/2, 1)$. It is worth emphasising that even in that simple space homogeneous case, it was the only known result of existence and uniqueness of global (in time) solutions.

In the space inhomogeneous case, existence of global (renormalized with a defect measure) solutions has been established by Alexandre-Villani [1] for any initial datum with finite mass, energy and entropy. Under an additional (unverified) strong uniform in time boundedness assumption, Desvillettes and Villani [17] proved polynomial convergence of the solutions to the equilibrium. On the other hand, in a perturbative regime, Guo [21] proved well-posedness in the high-order Sobolev space with fast decay in velocity $H_{x,v}^s(\mu^{-1/2})$, and Guo and Strain [37, 38] proved stretched exponential convergence to equilibrium in $H_{x,v}^s(\mu^{-\theta})$, $\theta \in (1/2, 1)$.

Our result thus improves the well-posedness theory of Guo [21] to larger spaces $H_x^2 L_v^2(m)$ as well as the convergence to equilibrium of Guo and Strain [37, 38] to larger spaces and with more accurate rate. It is worth emphasising that in the space homogeneous case, our results only require that initial data are bounded (and close) in the Lebesgue space $L_v^2(m)$ (and thus do not require any control on derivatives).

Our result makes possible to improve the speed of convergence to the equilibrium results available in a non perturbative framework in the following way.

Corollary 1.3 (Spatially homogeneous framework). *Consider a nonnegative normalized initial datum $F_0 = F_0(v)$ with finite entropy such that furthermore $F_0 \in L^1(m)$ for an exponential weight function m satisfying (1.8) with $s \in (0, 1/2)$. There exists a global weak solution F to the spatially homogenous Landau equation (1.2) associated to F_0 satisfying*

$$(1.14) \quad \|F(t) - \mu\|_{L_v^2} \lesssim \Theta_m(t), \quad \forall t \geq 0.$$

Estimate (1.14) improves the rate of convergence of order $e^{-\lambda t^{\frac{s}{s+1\gamma}}}$ established in [13], thanks to an entropy method, for the global weak solutions built in [42, 16]. Corollary 1.3 has to be compared with [33] where the optimal speed of convergence to the equilibrium for the spatially homogeneous Boltzmann equation for hard spheres has been established and with [41] where the optimal speed of convergence to the equilibrium for the spatially homogeneous Boltzmann equation for hard potentials has been proved.

Corollary 1.4 (Spatially inhomogeneous framework with a priori bounds). *Let F be a non-negative normalized global strong solution to the spatially inhomogeneous Landau equation (1.1) such that*

$$(1.15) \quad \sup_{t \geq 0} \left(\|F(t)\|_{H_{x,v}^\ell} + \|F(t)\|_{L_{x,v}^1(m)} \right) < +\infty,$$

for some explicit $\ell \geq 3$ large enough and some exponential weight function m satisfying (1.8), and such that the spatial density is uniformly positive on the torus, namely

$$(1.16) \quad \forall t \geq 0, x \in \mathbb{T}^3, \quad \rho(t, x) = \int_{\mathbb{R}^d} f(t, x, v) dv \geq \alpha > 0.$$

Then this solution satisfies

$$(1.17) \quad \|F(t) - \mu\|_{H_x^2 L_v^2} \lesssim \Theta_m(t), \quad \forall t \geq 0.$$

Estimate (1.17) improves the polynomial (of any order) rate of convergence established in [17, Theorem 2] under stronger (of any order) uniform Sobolev norm estimates but weaker (polynomial of any order) velocity moment uniform estimates. Corollary 1.4 has to be compared with [20] where the optimal speed of convergence to the equilibrium for the spatially inhomogeneous Boltzmann equation for hard spheres has been established.

1.3. Overview of the proof. Our main theorem is based on stability estimates (which are not uniformly exponential) for the semigroup corresponding to the linearized operator in large functional spaces, by taking advantage of well-known (or simple extension of well-known) weak coercivity estimates in some small space and using an enlargement trick. We then conclude to our main result by combining these stability estimates (at the linear level) together with some nonlinear estimates for the Landau operator Q and a trapping argument. It is worth mentioning that our method is mostly based on the semigroup stability estimates, what is quite different of the nonlinear energy method of [21, 37, 38].

Let us explain this enlargement trick in more details. We begin with the simpler hypodissipative framework. Let Λ be a linear operator acting on two Banach spaces $E \subset \mathcal{E}$ and suppose that Λ has a spectral gap in the small space E . The extension theory recently introduced in an abstract Banach framework in [33] and developed in [20, 30, 28] (see also [29, 40, 31] for other developments of the factorization approach for the spectral analysis of semigroups in large Banach spaces) establishes that if we can factorise $\Lambda = \mathcal{A} + \mathcal{B}$ where \mathcal{B} is hypodissipative (with respect to \mathcal{E}) and \mathcal{A} is suitably more regular than \mathcal{B} , then Λ generates a C_0 -semigroup $S_\Lambda(t)$ on the large space \mathcal{E} which is also uniformly exponentially stable. This method has been successfully applied to many evolution equations, and in particular to the Landau equation with hard and moderately soft potentials in [11, 12, 14].

In our case (of very soft and Coulomb potentials $\gamma \in [-3, -2)$), the linearized Landau operator Λ does not satisfy any spectral gap inequality but only a weak coercivity estimate on a small space E . We are however able to adapt the extension theory presented above and prove that Λ generates a uniformly bounded C_0 -semigroup $S_\Lambda(t)$ on small and large spaces X , but which is now only strongly stable but not uniformly exponentially stable.

More precisely, on the one hand, the linearized version of the H -Theorem states that (at least) in one Hilbert space E , the linearized Landau operator Λ enjoys a weak spectral gap estimate

$$(1.18) \quad \forall f \in E_1^\Lambda, \quad \langle \Lambda f, f \rangle_E \lesssim -\|\Pi f\|_{E_*}^2, \quad E \text{ not included into } E_*,$$

where E_1^Λ stands for the domain of Λ when acting on the space E , Π denotes the projector onto the orthogonal of $\ker(\Lambda)$, and E_* is a second Hilbert space (in the norm of which we express the weak dissipativity property of Λ in E).

On the other hand, in many Hilbert spaces X , the linearized Landau operator Λ splits as $\Lambda = \mathcal{A} + \mathcal{B}$ where \mathcal{A} is a bounded operator in X and \mathcal{B} is weakly dissipative

$$(1.19) \quad \forall f \in X_1^\Lambda, \quad \langle \mathcal{B}f, f \rangle_X \lesssim -\|f\|_{X_*}^2, \quad X \text{ not included into } X_*,$$

where again X_1^Λ stands for the domain of Λ when acting on the space X and X_* is a second Hilbert space (in the norm of which we express the weak dissipativity property of \mathcal{B} in X).

It is worth emphasizing that from (1.18) or (1.19) we cannot deduce any uniform exponential stability of the semigroups $S_{\mathcal{B}}$ or ΠS_Λ . However, by using (1.19) with several choices of spaces X and using an interpolation argument, we first obtain that $S_{\mathcal{B}}$ is strongly asymptotically stable (but not uniformly exponentially stable). Next, by using an extension trick, we deduce that the same holds for ΠS_Λ . More precisely, for several choices of Hilbert spaces $X \subsetneq X_0$, we have first

$$(1.20) \quad \|\Pi S_\Lambda(t)\|_{X \rightarrow X_0} \leq \Theta(t) \rightarrow 0, \quad \text{as } t \rightarrow \infty,$$

for some polynomial or stretched exponential decay function $\Theta = \Theta_{X, X_0}$, as well as the regularity estimates

$$(1.21) \quad \|\Pi S_\Lambda(t)\|_{X'_* \rightarrow X}, \|\Pi S_\Lambda(t)\|_{X \rightarrow X_*} \in L^2(\mathbb{R}_+),$$

where X'_* is the dual of X_* (with respect to X). Next, for some convenient choice of $\eta, K > 0$, the norm

$$(1.22) \quad \| \| f \| \|_X^2 := \eta \| f \|_X^2 + \int_0^\infty \| S_\Lambda(\tau) f \|_{X_0}^2 d\tau$$

is an equivalent norm in X and Λ satisfies the weak dissipativity estimate

$$(1.23) \quad \forall f \in X_1^\Lambda, \quad \langle \Lambda f, f \rangle_X \leq -K \| \Pi f \|_{X_*}^2,$$

where $\langle \cdot, \cdot \rangle_X$ stands for the duality bracket associated to the $\| \cdot \|_X$ norm.

By choosing X and X_* well adapted for the quadratic Landau operator, we may then establish that for any solution $f = F - \mu$ to the Landau equation, the following a priori estimate holds (for some constant $C > 0$)

$$\frac{d}{dt} \| \Pi f \|_X^2 \lesssim \| \Pi f \|_{X_*}^2 (-K + C \| \Pi f \|_X).$$

Our existence, uniqueness and asymptotic stability results are then immediate consequences of that last differential inequality and of the estimates it provides.

Let us finally discuss the decay issue for non uniformly (exponential) stable semigroups which naturally arises in many contexts. It arises first in statistical physics when coefficients involved are suitably decaying. In [9, 10], for the Boltzmann equation with soft potential of interaction under Grad's cutoff assumption, Cagliuffi had exhibited the explicit semigroup solution to the associated linearized equation and had deduced well-posedness and stability for the nonlinear Boltzmann equation in a perturbative regime. In [27], a similar result is obtained for the critical case of an attractive reversible nearest particle system. More recently, for the Fokker-Planck equation with weak confinement potential and for the spatial homogeneous Landau equation with soft interaction some polynomial and stretch exponential rate of convergence to the equilibrium have been established in [36, 39]. The proofs are based on entropy methods, moments estimates and interpolation arguments. These results for the Fokker-Planck equation are improved in [23] where a similar factorization approach, as introduced in the present paper, is developed.

Independently, inspired by scattering and control theory [24, 4], many results on the decay rate of the energy for damped wave type equations have been established, see for instance [25, 26, 8]. These results are based on the analysis of the absence of poles (resonances) in the neighbourhood of the real axis for the resolvent of the associated operator. They have inspired an

abstract theory for non-uniformly exponentially stable semigroups, and we refer the interested reader to [7, 5, 6] and the references therein.

It is worth emphasizing that in that last abstract theory, one typically obtains some estimate on the semigroup by allowing the loss of (part of) a domain in the control of the trajectory and, roughly speaking, that is related to the absence of pole in bounded neighbourhoods of the real axis and to the control of how the spectrum approaches the imaginary axis at $\pm i\infty$. That is slightly different from the picture arising in the present statistical physics framework, where the estimates do not involve domains norms but norms controlling the confinement of the distribution function and where the continuous spectrum extends up to the origin.

1.4. Notations and definitions. If Λ is a closed linear operator on a Banach space X that generates a semigroup on X , we denote by $S_\Lambda(t)$ its associated semigroup. Moreover, for Banach spaces X and Y , we denote $\mathcal{B}(X, Y)$ the space of bounded linear operators from X to Y , with the associated operator norm $\|\cdot\|_{X \rightarrow Y}$. We say that the generator Λ of a semigroup in a Banach space X is dissipative if

$$\forall f \in X_\Lambda^1, \exists f^* \in J_f, \quad \langle f^*, \Lambda f \rangle_{X', X} \leq 0$$

where $X_\Lambda^1 = D(\Lambda)$ is the domain of Λ and J_f is the dual set $J_f := \{g \in X'; \|g\|_{X'}^2 = \|f\|_X^2 = \langle g, f \rangle_{X', X}\}$. We say that the generator Λ is hypodissipative if it is dissipative for an equivalent norm.

1.5. Structure of the paper. For the sake of clarity, we shall first consider the spatially homogeneous case through Sections 2 to 5, and in the last Section 6 we show how our method can be adapted to the spatially inhomogeneous equation. In Section 2 we introduce a factorization of the (homogeneous) linearized Landau operator $\mathcal{L} = \mathcal{A} + \mathcal{B}$ and prove several properties of the operators \mathcal{A} and \mathcal{B} . Section 3 is devoted to the proof of (non-uniformly exponential) decay estimates in large functional spaces of the semigroup associated to \mathcal{L} (see Theorem 3.4) as well as weakly dissipative properties for \mathcal{L} (see Corollary 3.6), using the method presented above. In Section 4 we prove nonlinear estimates for the Landau operator Q , and then in Section 5 we prove the spatially homogeneous version of Theorem 1.1. Finally, in Section 6, we deal with the inhomogeneous case and prove Theorem 1.1, by following the same program as for the homogeneous case above.

Acknowledgments. The first author is supported by the Fondation Mathématiques Jacques Hadamard. The second author gratefully acknowledges the support of the STAB ANR project (ANR-12-BS01-0019).

2. LINEARIZED OPERATOR

We define the following quantities

$$(2.1) \quad \begin{aligned} b_i(z) &= \partial_j a_{ij}(z) = -2|z|^\gamma z_i, \\ c(z) &= \partial_{ij} a_{ij}(z) = -2(\gamma + 3)|z|^\gamma \quad \text{if } \gamma \in (-3, -2), \\ c(z) &= \partial_{ij} a_{ij}(z) = -8\pi\delta_0 \quad \text{if } \gamma = -3, \end{aligned}$$

from which we are able to rewrite the Landau operator (1.3) into two other forms

$$(2.2) \quad \begin{aligned} Q(g, f) &= \partial_i \{(a_{ij} * g) \partial_j f - (b_i * g) f\} \\ &= (a_{ij} * g) \partial_{ij} f - (c * g) f. \end{aligned}$$

Consider now the variation $f := F - \mu$ and the linearized (homogeneous) Landau operator

$$(2.3) \quad \mathcal{L}f := Q(\mu, f) + Q(f, \mu).$$

We denote

$$(2.4) \quad \bar{a}_{ij} = a_{ij} * \mu, \quad \bar{b}_i = b_i * \mu, \quad \bar{c} = c * \mu,$$

and remark that

$$\begin{aligned} \bar{c}(v) &= -2(\gamma + 3) \int_{v_*} |v - v_*|^\gamma \mu_* \quad \text{when } \gamma \in (-3, -2), \\ \bar{c}(v) &= -8\pi\mu(v) \quad \text{when } \gamma = -3. \end{aligned}$$

2.1. Known results. On the space $E_0 := L_v^2(\mu^{-1/2})$, we classically observe that \mathcal{L} is self-adjoint and verifies $\langle \mathcal{L}f, f \rangle_{E_0} \leq 0$, so that its spectrum satisfies $\Sigma(\mathcal{L}) \subset \mathbb{R}_-$. Moreover, thanks to the conservation laws, there holds

$$\ker(\mathcal{L}) = \text{span}\{\mu, v_1\mu, v_2\mu, v_3\mu, |v|^2\mu\},$$

and the projection Π_0 onto $\ker(\mathcal{L})$ is given by

$$(2.5) \quad \Pi_0(f) = \left(\int f dv \right) \mu + \sum_{j=1}^3 \left(\int v_j f dv \right) v_j \mu + \left(\int \frac{|v|^2 - 3}{6} f dv \right) \frac{|v|^2 - 3}{6} \mu.$$

Several authors have studied weak coercivity estimates for \mathcal{L} on E_0 . Summarising results from [15, 3, 21, 32, 35], for all $-3 \leq \gamma \leq 1$, we have

$$(2.6) \quad \langle \mathcal{L}f, f \rangle_{E_0} \lesssim -\|\langle v \rangle^{\frac{\gamma+2}{2}} \Pi f\|_{E_0}^2 - \|\langle v \rangle^{\frac{\gamma}{2}} \tilde{\nabla}_v \Pi(\mu^{-1/2}f)\|_{L^2}^2, \quad \forall f \in E_0,$$

where we define the projection $\Pi := I - \Pi_0$ onto the orthogonal of $\ker(\mathcal{L})$ and we recall that the anisotropic gradient $\tilde{\nabla}_v$ has been defined in (1.11). Observe that (2.6) does not provide any spectral gap for the operator \mathcal{L} in E_0 in the very soft and Coulomb potential case $-3 \leq \gamma < -2$ we are concerned with in the present work, contrarily to the moderately soft and hard potentials case $-2 \leq \gamma \leq 1$.

2.2. Factorization of the operator. Using the form (2.2) of the operator Q , we decompose the linearized Landau operator as $\mathcal{L} = \mathcal{A}_0 + \mathcal{B}_0$, where we define

$$(2.7) \quad \begin{aligned} \mathcal{A}_0 f &:= Q(f, \mu) = \partial_i \{(a_{ij} * f) \partial_j \mu + (b_i * f) \mu\} = (a_{ij} * f) \partial_{ij} \mu - (c * f) \mu, \\ \mathcal{B}_0 f &:= Q(\mu, f) = \partial_i \{(a_{ij} * \mu) \partial_j f + (b_i * \mu) f\} = (a_{ij} * \mu) \partial_{ij} f - (c * \mu) f. \end{aligned}$$

Consider a smooth nonnegative function $\chi \in C_c^\infty(\mathbb{R}^3)$ such that $0 \leq \chi(v) \leq 1$, $\chi(v) \equiv 1$ for $|v| \leq 1$ and $\chi(v) \equiv 0$ for $|v| > 2$. For any $R \geq 1$, we define $\chi_R(v) := \chi(R^{-1}v)$. Then, we make the final decomposition of the operator \mathcal{L} as $\mathcal{L} = \mathcal{A} + \mathcal{B}$, with

$$(2.8) \quad \mathcal{A} := \mathcal{A}_0 + M\chi_R, \quad \mathcal{B} := \mathcal{B}_0 - M\chi_R,$$

where $M > 0$ and $R \geq 1$ will be chosen later.

2.3. Preliminaries. We introduce some convenient classes of weight functions and we state some preliminaries results that will be useful in the sequel.

We say that a weight function $m : \mathbb{R}^3 \rightarrow \mathbb{R}_+$ is admissible if

- (i) it is a polynomial function, and we write $m = \langle v \rangle^k$, $k > 0$;
- (ii) or if it is an exponential function, that is $m = e^{\kappa \langle v \rangle^s}$ with $\kappa > 0$ and $s \in (0, 2)$, or with $0 < \kappa < 1/2$ and $s = 2$.

We denote $\sigma = 0$ when $m = \langle v \rangle^k$ and $\sigma = s$ when $m = e^{\kappa \langle v \rangle^s}$. For two admissible weight functions m_0 and m_1 , we write $m_0 \prec m_1$ (or $m_1 \succ m_0$) if $\lim_{|v| \rightarrow \infty} \frac{m_0}{m_1}(v) = 0$. Similarly, we write $m_0 \preceq m_1$ (or $m_1 \succeq m_0$) if $m_0 \prec m_1$ or $m_0 = m_1$ (up to a constant).

We finally define the following functions:

$$(2.9) \quad \zeta_m(v) := \frac{1}{2} \frac{1}{m^2} \partial_{ij} (\bar{a}_{ij} m^2) - \bar{c} = \bar{a}_{ij}(v) \frac{\partial_{ij} m}{m} + \bar{a}_{ij}(v) \frac{\partial_i m}{m} \frac{\partial_j m}{m} + 2\bar{b}_i(v) \frac{\partial_i m}{m} - \frac{1}{2} \bar{c}(v),$$

and also

$$(2.10) \quad \tilde{\zeta}_m(v) := \bar{a}_{ij}(v) \frac{\partial_i m}{m} \frac{\partial_j m}{m} + \bar{b}_i(v) \frac{\partial_i m}{m} - \frac{1}{2} \bar{c}(v).$$

We start stating some estimates on the matrix \bar{a}_{ij} . To that purpose, we define

$$\begin{aligned} \ell_1(v) &= \int_{\mathbb{R}^3} \left(1 - \left(\frac{v}{|v|} \cdot \frac{w}{|w|} \right)^2 \right) |w|^{\gamma+2} \mu(v-w) dw, \\ \ell_2(v) &= \int_{\mathbb{R}^3} \left(1 - \frac{1}{2} \left| \frac{v}{|v|} \times \frac{w}{|w|} \right|^2 \right) |w|^{\gamma+2} \mu(v-w) dw, \end{aligned}$$

where \times stands for the vector product in \mathbb{R}^3 , and, for $-3 < \beta < 0$, we define

$$J_\beta(v) := \int_{\mathbb{R}^3} |v-w|^\beta \mu(w) dw.$$

Lemma 2.1. *The following properties hold:*

(a) *The matrix $\bar{a}(v)$ has a simple eigenvalue $\ell_1(v) > 0$ associated with the eigenvector v and a double eigenvalue $\ell_2(v) > 0$ associated with the eigenspace v^\perp . Moreover, when $|v| \rightarrow +\infty$, we have*

$$\ell_1(v) \sim 2\langle v \rangle^\gamma, \quad \ell_2(v) \sim \langle v \rangle^{\gamma+2}.$$

(b) *The function \bar{a}_{ij} is smooth, more precisely for any multi-index $\beta \in \mathbb{N}^3$,*

$$|\partial^\beta \bar{a}_{ij}(v)| \leq C_\beta \langle v \rangle^{\gamma+2-|\beta|}.$$

Moreover, there exists a constant $K > 0$ such that

$$\begin{aligned} \bar{a}_{ij}(v) \xi_i \xi_j &= \ell_1(v) |P_v \xi|^2 + \ell_2(v) |(I - P_v) \xi|^2 \\ &\geq K \{ \langle v \rangle^\gamma |P_v \xi|^2 + \langle v \rangle^{\gamma+2} |(I - P_v) \xi|^2 \}. \end{aligned}$$

(c) *We have*

$$\text{tr}(\bar{a}(v)) = \ell_1(v) + 2\ell_2(v) = 2J_{\gamma+2}(v) \quad \text{and} \quad \bar{b}_i(v) = -\ell_1(v) v_i.$$

(d) *If $|v| > 1$, we have*

$$|\partial^\beta \ell_1(v)| \leq C_\beta \langle v \rangle^{\gamma-|\beta|} \quad \text{and} \quad |\partial^\beta \ell_2(v)| \leq C_\beta \langle v \rangle^{\gamma+2-|\beta|}.$$

(e) *For any $\beta \in (-3, 0)$, there exists some constant $C_\beta > 0$ such that*

$$|J_\beta(v) - \langle v \rangle^\beta| \leq C_\beta \langle v \rangle^{3\beta/2}, \quad \forall v \in \mathbb{R}^3.$$

Proof. Item (a) comes from [15, Propositions 2.3 and 2.4], (b) is [21, Lemma 3], (c) is evident and (d) is [14, Lemma 2.3].

We just then present the proof of (e). On the one hand, for any $v \in \mathbb{R}^3$, we have

$$(2.11) \quad \begin{aligned} J_\beta(v) &= \int_{|v_*| \leq 1} |v_*|^\beta \mu(v_* - v) dv_* + \int_{|v_*| \geq 1} |v_*|^\beta \mu(v - v_*) dv_* \\ &\leq \sup_{|v_*| \leq 1} \mu(v - v_*) \int_{|v_*| \leq 1} |v_*|^\beta dv_* + \int_{|v_*| \geq 1} \mu(v - v_*) dv_* \leq C_1, \end{aligned}$$

since the two terms are clearly bounded uniformly in $v \in \mathbb{R}^3$.

On the other hand, for any $v \in \mathbb{R}^3$, $|v| \geq 1$, and for any $R > 0$, we write

$$J_\beta(v) = \int_{|v_*| \leq R} |v_* - v|^\beta \mu(v_*) dv_* + \int_{|v_*| \geq R} |v_* - v|^\beta \mu(v_*) dv_* = T_1 + T_2.$$

For the second term, we have

$$|T_2| \leq \sqrt{\mu(R)} \int_{|v_*| \geq R} |v_* - v|^\beta \sqrt{\mu(v_*)} dv_* \leq C_2 e^{-R^2/4},$$

where we have used an estimate very similar to (2.11). For the first term and for $|v| > R$, we have

$$\begin{aligned} T_1 &\geq \int_{|v_*| \leq R} (|v| + |v_*|)^\beta \mu(v_*) dv_* \\ &\geq \int_{|v_*| \leq R} (|v| + R)^\beta \mu(v_*) dv_* \geq (|v| + R)^\beta (1 - C_3 e^{-R^2/4}), \end{aligned}$$

and in a similar way, we have

$$T_1 \leq (|v| - R)^\beta (1 - C_3 e^{-R^2/4}).$$

We conclude by making the choice $R := |v|^{1/2}$. \square

Lemma 2.2. *Let m be an admissible weight function such that $m \succ \langle v \rangle^{(\gamma+3)/2}$.*

(1) *If $\sigma = 0$, then*

$$\limsup_{|v| \rightarrow \infty} \zeta_m(v) \langle v \rangle^{-\gamma} = \limsup_{|v| \rightarrow \infty} \tilde{\zeta}_m(v) \langle v \rangle^{-\gamma} \leq 2\{(\gamma + 3)/2 - k\}.$$

(2) *If $\sigma \in (0, 2)$, then*

$$\limsup_{|v| \rightarrow \infty} \zeta_m(v) \langle v \rangle^{-\sigma-\gamma} = \limsup_{|v| \rightarrow \infty} \tilde{\zeta}_m(v) \langle v \rangle^{-\sigma-\gamma} \leq -2\kappa\sigma.$$

(3) *If $\sigma = 2$, then*

$$\begin{aligned} \limsup_{|v| \rightarrow +\infty} \zeta_m(v) \langle v \rangle^{-2-\gamma} &\leq 4\kappa(4\kappa - 1), \\ \limsup_{|v| \rightarrow +\infty} \tilde{\zeta}_m(v) \langle v \rangle^{-2-\gamma} &\leq 4\kappa(2\kappa - 1). \end{aligned}$$

Proof. We introduce the notation

$$\tilde{J}_\gamma(v) = \begin{cases} (\gamma + 3)J_\gamma(v) & \text{if } \gamma \in (-3, -2), \\ 4\pi\mu(v) & \text{if } \gamma = -3, \end{cases}$$

so that $\bar{c} = -2\tilde{J}$. We observe from Lemma 2.1 that, when $|v| \rightarrow +\infty$, we have

$$(2.12) \quad \frac{1}{2}\ell_1(v) \sim \ell_2(v)|v|^{-2} \sim \langle v \rangle^\gamma \quad \text{and} \quad \tilde{J}_\gamma(v) = (\gamma + 3)\langle v \rangle^\gamma + \mathcal{O}(\langle v \rangle^{3\gamma/2}).$$

Step 1. Polynomial weight. Consider $m = \langle v \rangle^k$. From definition (2.1)-(2.4) and Lemma 2.1, we obtain

$$\begin{aligned} \bar{a}_{ij} \frac{\partial_{ij} m}{m} &= (\delta_{ij} \bar{a}_{ij}) k \langle v \rangle^{-2} + (\bar{a}_{ij} v_i v_j) k(k-2) \langle v \rangle^{-4} \\ &= 2\ell_2(v) k \langle v \rangle^{-2} + \ell_1(v) k \langle v \rangle^{-2} + \ell_1(v) k(k-2) |v|^2 \langle v \rangle^{-4}, \end{aligned}$$

Moreover,

$$\bar{a}_{ij} \frac{\partial_i m}{m} \frac{\partial_j m}{m} = (\bar{a}_{ij} v_i v_j) k^2 \langle v \rangle^{-4} = \ell_1(v) k^2 |v|^2 \langle v \rangle^{-4},$$

and also, using the fact that $\bar{b}_i(v) = -\ell_1(v) v_i$ from Lemma 2.1,

$$\bar{b}_i \frac{\partial_i m}{m} = -\ell_1(v) k |v|^2 \langle v \rangle^{-2}.$$

It follows that

$$\begin{aligned}\zeta_m(v) &= 2k\ell_2(v)\langle v \rangle^{-2} + k\ell_1(v)\langle v \rangle^{-2} + k(k-2)\ell_1(v)|v|^2\langle v \rangle^{-4} \\ &\quad + k^2\ell_1(v)|v|^2\langle v \rangle^{-4} - 2k\ell_1(v)|v|^2\langle v \rangle^{-2} + \tilde{J}_\gamma(v),\end{aligned}$$

as well as

$$\tilde{\zeta}_m(v) = k^2\ell_1(v)|v|^2\langle v \rangle^{-4} - k\ell_1(v)|v|^2\langle v \rangle^{-2} + \tilde{J}_\gamma(v).$$

Thanks to (2.12), the dominant terms are of order $\langle v \rangle^\gamma$. We then obtain

$$\limsup_{|v| \rightarrow +\infty} \zeta_m(v) = \limsup_{|v| \rightarrow +\infty} \tilde{\zeta}_m(v)\langle v \rangle^{-\gamma} \leq 2\{(\gamma+3)/2 - k\},$$

from which we conclude the proof of point (1).

Step 2. Exponential weight. For $m = e^{\kappa\langle v \rangle^s}$, we have

$$\begin{aligned}\zeta_m(v) &= 2\kappa s\ell_2(v)\langle v \rangle^{s-2} + \kappa s\ell_1(v)\langle v \rangle^{s-2} + \kappa s(s-2)\ell_1(v)|v|^2\langle v \rangle^{s-4} \\ &\quad + 2\kappa^2 s^2\ell_1(v)|v|^2\langle v \rangle^{2s-4} - 2\kappa s\ell_1(v)|v|^2\langle v \rangle^{s-2} + \tilde{J}_\gamma(v)\end{aligned}$$

and also

$$\tilde{\zeta}_m(v) = -\kappa s\ell_1(v)|v|^2\langle v \rangle^{s-2} + \kappa^2 s^2\ell_1(v)|v|^2\langle v \rangle^{2s-4} + \tilde{J}_\gamma(v).$$

In any cases $0 < s \leq 2$, the dominant terms are of order $\langle v \rangle^{\gamma+s}$, and we easily conclude. \square

We conclude this section with a remark about the weighted spaces we have defined in (1.6). For any admissible weight function m we easily obtain

$$(2.13) \quad \|\langle v \rangle^{(\sigma-1)+} m f\|_{L^2}^2 + \|\nabla_v(mf)\|_{L^2}^2 \sim \|m f\|_{L^2}^2 + \|m \nabla_v f\|_{L^2}^2,$$

so that in particular $\|f\|_{H^1(m)}^2 \sim \|f\|_{L^2(m)}^2 + \|\nabla_v f\|_{L^2(m)}^2$ when $\sigma \in [0, 1]$.

2.4. Dissipative properties of \mathcal{B} . We prove in this section weakly dissipative properties for the operator \mathcal{B} . These estimates are similar to the estimates established in [12, 14] for $-2 \leq \gamma \leq 1$, in which case it is proven that the operator $\mathcal{B} - \alpha$ is dissipative for some $\alpha < 0$.

Lemma 2.3. *Let m be an admissible weight function such that $m \succ \langle v \rangle^{(\gamma+3)/2}$. There exists $M, R > 0$ large enough such that \mathcal{B} is weakly dissipative in $L^2(m)$ in the sense:*

- If $m \prec \mu^{-1/2}$, there holds

$$(2.14) \quad \langle \mathcal{B}f, f \rangle_{L^2(m)} \lesssim -\|\langle v \rangle^{\frac{\gamma}{2}} \tilde{\nabla}_v f\|_{L^2(m)}^2 - \|\langle v \rangle^{\frac{\gamma}{2}} \tilde{\nabla}_v(mf)\|_{L^2}^2 - \|\langle v \rangle^{\frac{\gamma+\sigma}{2}} f\|_{L^2(m)}^2.$$

- If $\mu^{-1/2} \leq m \prec \mu^{-1}$, there holds

$$(2.15) \quad \langle \mathcal{B}f, f \rangle_{L^2(m)} \lesssim -\|\langle v \rangle^{\frac{\gamma}{2}} \tilde{\nabla}_v(mf)\|_{L^2}^2 - \|\langle v \rangle^{\frac{\gamma+\sigma}{2}} f\|_{L^2(m)}^2.$$

Proof. From the definition (2.7)-(2.8) of \mathcal{B} , we have

$$\begin{aligned}\int (\mathcal{B}f) f m^2 &= \int \bar{a}_{ij} \partial_{ij} f f m^2 - \int \bar{c} f^2 m^2 - \int M \chi_R f^2 m^2 \\ &=: T_1 + T_2 + T_3.\end{aligned}$$

Let us compute the term T_1 . Writing $g = mf$ and thus $\partial_{ij} f f m^2 = \partial_{ij}(m^{-1}g)gm$, an integration by parts yields

$$T_1 = - \int \{\bar{b}_j gm + \bar{a}_{ij} \partial_i gm + \bar{a}_{ij} g \partial_i m\} \partial_j(m^{-1}g).$$

Using that $\partial_j(m^{-1}g) = m^{-1}\partial_j g - m^{-2}\partial_j m g$ in the last equation, we first get

$$T_1 = - \int \bar{a}_{ij} \partial_i g \partial_j g + \int \left\{ \bar{a}_{ij} \frac{\partial_i m}{m} \frac{\partial_j m}{m} + \bar{b}_j \frac{\partial_j m}{m} \right\} g^2 - \int \bar{b}_j g \partial_j g,$$

and thanks to another integration by parts for the last term, we finally obtain

$$\int (\mathcal{B}f) f m^2 = - \int \bar{a}_{ij} \partial_i (mf) \partial_j (mf) + \int \{\tilde{\zeta}_m - M\chi_R\} f^2 m^2.$$

In a similar (and even simpler) way, we can also obtain

$$\int (\mathcal{B}f) f m^2 = - \int \bar{a}_{ij} \partial_i f \partial_j f m^2 + \int \{\zeta_m - M\chi_R\} f^2 m^2.$$

Thanks to Lemma 2.2, we may choose $M, R > 0$ large enough such that

$$\zeta_m(v) - M\chi_R(v) \lesssim -\langle v \rangle^{\gamma+\sigma}, \quad \tilde{\zeta}_m(v) - M\chi_R(v) \lesssim -\langle v \rangle^{\gamma+\sigma}, \quad \text{if } m \prec \mu^{-1/2},$$

and

$$\tilde{\zeta}_m(v) - M\chi_R(v) \lesssim -\langle v \rangle^{\gamma+\sigma}, \quad \text{if } \mu^{-1/2} \preceq m \prec \mu^{-1},$$

and we then conclude using the coercivity of \bar{a}_{ij} from Lemma 2.1. \square

For any admissible weight function m , we define the operator $\mathcal{B}_m g = m\mathcal{B}(m^{-1}g)$, which writes

$$(2.16) \quad \begin{aligned} \mathcal{B}_m g &= \bar{a}_{ij} \partial_{ij} g - 2\bar{a}_{ij} \frac{\partial_i m}{m} \partial_j g + \left\{ 2\bar{a}_{ij} \frac{\partial_i m}{m} \frac{\partial_j m}{m} - \bar{a}_{ij} \frac{\partial_{ij} m}{m} - \bar{c} - M\chi_R \right\} g \\ &=: \bar{a}_{ij} \partial_{ij} g + \beta_j \partial_j g + (\delta - M\chi_R)g. \end{aligned}$$

We then define its formal adjoint operator \mathcal{B}_m^* that verifies

$$(2.17) \quad \mathcal{B}_m^* \phi = \bar{a}_{ij} \partial_{ij} \phi + 2 \left\{ \bar{a}_{ij} \frac{\partial_i m}{m} + \bar{b}_j \right\} \partial_j \phi + \left\{ \bar{a}_{ij} \frac{\partial_{ij} m}{m} + 2\bar{b}_i \frac{\partial_i m}{m} - M\chi_R \right\} \phi.$$

Observe that if f satisfies the equation $\partial_t f = \mathcal{B}f$ then $g = mf$ satisfies $\partial_t g = \mathcal{B}_m g$, and also that $\langle \mathcal{B}f, f \rangle_{L^2(m)} = \langle \mathcal{B}_m g, g \rangle_{L^2}$. Moreover there holds by duality

$$\forall t \geq 0, \quad \langle S_{\mathcal{B}_m}(t)g, \phi \rangle_{L^2} = \langle g, S_{\mathcal{B}_m^*}(t)\phi \rangle_{L^2},$$

where we recall that $S_{\mathcal{B}_m}(t)$ is the semigroup generated by \mathcal{B}_m and $S_{\mathcal{B}_m^*}(t)$ the semigroup generated by \mathcal{B}_m^* .

We now prove weakly dissipative properties of the adjoint \mathcal{B}_m^* .

Lemma 2.4. *Let m be an admissible weight functions such that $m \succ \langle v \rangle^{(\gamma+3)/2}$. We can choose $M, R > 0$, large enough, such that \mathcal{B}_m^* is weakly dissipative in L^2 in the following sense:*

$$(2.18) \quad \langle \mathcal{B}_m^* \phi, \phi \rangle_{L^2} \lesssim -\|\phi\|_{L^2(\langle v \rangle^{(\gamma+\sigma)/2})}^2 - \|\tilde{\nabla}_v \phi\|_{L^2(\langle v \rangle^{\gamma/2})}^2.$$

Proof. We compute

$$\begin{aligned} \int (\mathcal{B}_m^* \phi) \phi &= \int \left(\bar{a}_{ij} \frac{\partial_{ij} m}{m} + 2\bar{b}_j \frac{\partial_j m}{m} - M\chi_R \right) \phi^2 + \int \left(\bar{a}_{ij} \frac{\partial_j m}{m} + \bar{b}_i \right) \partial_i (\phi^2) + \int \bar{a}_{ij} \partial_{ij} \phi \phi \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

Performing one integration by parts, we obtain

$$I_2 = - \int \partial_i \left(\bar{a}_{ij} \frac{\partial_j m}{m} + \bar{b}_i \right) \phi^2 = \int \left\{ -\bar{a}_{ij} \frac{\partial_{ij} m}{m} + \bar{a}_{ij} \frac{\partial_i m}{m} \frac{\partial_j m}{m} - \bar{b}_j \frac{\partial_j m}{m} - \bar{c} \right\} \phi^2.$$

Using that $\partial_{ij} \phi \phi = \frac{1}{2} \partial_{ij} (\phi^2) - \partial_i \phi \partial_j \phi$, it follows

$$I_3 = - \int \bar{a}_{ij} \partial_i \phi \partial_j \phi + \frac{1}{2} \int \partial_{ij} (\bar{a}_{ij}) \phi^2 = - \int \bar{a}_{ij} \partial_i \phi \partial_j \phi + \frac{1}{2} \int \bar{c} \phi^2.$$

Finally, we get

$$\begin{aligned} \int (\mathcal{B}_m^* \phi) \phi &= - \int \bar{a}_{ij} \partial_i \phi \partial_j \phi + \int \{\tilde{\zeta}_m - M\chi_R\} \phi^2 \\ &\lesssim - \|\langle v \rangle^{\frac{\gamma}{2}} \tilde{\nabla}_v \phi\|_{L^2}^2 - \|\langle v \rangle^{\frac{\gamma+\sigma}{2}} \phi\|_{L^2}^2, \end{aligned}$$

by using the coercivity of \bar{a}_{ij} from Lemma 2.1 and choosing $M, R > 0$ large enough such that $\tilde{\zeta}_m(v) - M\chi_R(v) \lesssim -\langle v \rangle^{\gamma+\sigma}$ thanks to Lemma 2.2. That completes the proof. \square

2.5. Estimates on the operator \mathcal{A} . We prove boundedness properties for the operator \mathcal{A} .

Lemma 2.5. *For any $\theta \in (0, 1)$, $\ell = 0, 1$ and $p \in [1, \infty]$, there holds $\mathcal{A} \in \mathcal{B}(W^{\ell,p}, W^{\ell,p}(\mu^{-\theta}))$.*

Proof. We only prove the case $\ell = 0$, the case $\ell = 1$ being similar. We only investigate \mathcal{A}_0 since $\mathcal{A} = \mathcal{A}_0 + M\chi_R$, and we recall that $\mathcal{A}_0 g = (a_{ij} * g) \partial_{ij} \mu + (c * g) \mu$. We decompose a and c into a bounded part and a singular part. More precisely, we split $a_{ij}(z) = a_{ij}(z) \mathbf{1}_{|z|>1} + a_{ij}(z) \mathbf{1}_{|z|\leq 1} =: a_{ij}^+(z) + a_{ij}^-(z)$, and similarly for $c(z)$.

Assume first $\gamma \in (-3, -2)$. For the bounded parts a^+ and c^+ , we easily have

$$|(a_{ij}^+ * g)(v)| + |(c^+ * g)(v)| \lesssim \|g\|_{L^1},$$

and therefore

$$\|(a_{ij}^+ * g) \mu\|_{L^p(\mu^{-\theta})} + \|(c^+ * g) \mu\|_{L^p(\mu^{-\theta})} \lesssim \|g\|_{L^1}.$$

We now turn to the singular terms. We first have

$$\|(a_{ij}^- * g) \mu\|_{L^1(\mu^{-\theta})} \lesssim \int_{v_*} |g(v_*)| \left(\int_v |v - v_*|^{(\gamma+2)} \mathbf{1}_{|v-v_*|\leq 1} \mu^{1-\theta}(v) \right) \lesssim \|g\|_{L^1}$$

and similarly,

$$\|(c^- * g) \mu\|_{L^1(\mu^{-\theta})} \lesssim \int_{v_*} |g(v_*)| \left(\int_v |v - v_*|^\gamma \mathbf{1}_{|v-v_*|\leq 1} \mu^{1-\theta}(v) \right) \lesssim \|g\|_{L^1}.$$

As a consequence, we already obtain that \mathcal{A} is a bounded operator from $L^1 \rightarrow L^1(\mu^{-\theta})$. Moreover, we can estimate

$$|(a_{ij}^- * g)(v)| \lesssim \|g\|_{L^\infty} \left(\int |v - v_*|^{(\gamma+2)} \mathbf{1}_{|v-v_*|\leq 1} dv_* \right) \lesssim \|g\|_{L^\infty}$$

and in a similar way

$$|(c^- * g)(v)| \lesssim \|g\|_{L^\infty} \left(\int |v - v_*|^\gamma \mathbf{1}_{|v-v_*|\leq 1} dv_* \right) \lesssim \|g\|_{L^\infty},$$

which imply

$$\|(a_{ij}^- * g) \mu\|_{L^\infty(\mu^{-\theta})} \lesssim \|g\|_{L^\infty}, \quad \|(c^- * g) \mu\|_{L^\infty(\mu^{-\theta})} \lesssim \|g\|_{L^\infty}.$$

These estimates prove that \mathcal{A} is bounded from $L^\infty \rightarrow L^\infty(\mu^{-\theta})$. We can then conclude to the boundedness of \mathcal{A} for any $p \in [1, \infty]$ by Riesz-Thorin interpolation theorem.

Assume now $\gamma = -3$. In that case the term $(a_{ij} * g) \mu$ can be treated exactly in the same way as above, but now we have $c = -\delta_0$ and then $c * g = -g$. Therefore,

$$\|(c * g) \mu\|_{L^p(\mu^{-\theta})} = \|g \mu^{1-\theta}\|_{L^p} \lesssim \|g\|_{L^p},$$

which completes the proof. \square

3. SEMIGROUP DECAY

This section is devoted to the proof of decay and regularity estimates for the linearized semigroup $S_{\mathcal{L}}$. Given two admissible weight functions $m_0 \prec m_1$, we define

$$\Theta_{m_1, m_0}(t) = \frac{(\log t)^{2(k_1 - k_0)/|\gamma|}}{\langle t \rangle^{(k_1 - k_0)/|\gamma|}}, \quad \text{if } m_1 = \langle v \rangle^{k_1} \text{ and } m_0 = \langle v \rangle^{k_0}$$

and

$$\Theta_{m_1, m_0}(t) = e^{-\lambda t^{\frac{s}{|\gamma|}}}, \quad \text{for some } \lambda > 0, \quad \text{if } m_1 = e^{\kappa \langle v \rangle^s}.$$

Observe that when $m \succ \langle v \rangle^{(3-\gamma)/2}$, we can choose m_0 such that $m \succ m_0 \succ \langle v \rangle^{(\gamma+3)/2}$, and then $\Theta_{m, m_0} \in L^1(\mathbb{R}_+)$.

Here and below, we define the time convolution product $S_1 * S_2$ of two functions S_i defined on the half real line \mathbb{R}_+ by

$$(S_1 * S_2)(t) = \int_0^t S_1(t-s)S_2(s) ds,$$

and we also define $S^0 = I$ and $S^{(*n)} = S * S^{(*n-1)}$ for any $n \geq 1$.

3.1. Decay estimates for $S_{\mathcal{B}}$. We first prove decay estimates for the semigroup $S_{\mathcal{B}}$.

For any admissible weight function m , we define the space $H_*^1(m)$ associated to the norm

$$(3.1) \quad \|f\|_{H_*^1(m)}^2 := \|f\|_{L^2(m \langle v \rangle^{(\gamma+\sigma)/2})}^2 + \|\tilde{\nabla}_v(mf)\|_{L^2(\langle v \rangle^{\gamma/2})}^2,$$

and we easily observe that $H^1(m) \subset H_*^1(m) \subset H^1(\langle v \rangle^{\gamma/2}m)$. We then define the negative Sobolev space $H_*^{-1}(m)$ in duality with $H_*^1(m)$ with respect to the duality product on $L^2(m)$, more precisely

$$(3.2) \quad \|f\|_{H_*^{-1}(m)} := \sup_{\|\phi\|_{H_*^1(m)} \leq 1} \langle f, \phi \rangle_{L^2(m)} = \sup_{\|\phi\|_{H_*^1(m)} \leq 1} \langle mf, m\phi \rangle_{L^2},$$

and observe that $\|f\|_{H_*^{-1}(m)} = \|mf\|_{H_*^{-1}}$.

Lemma 3.1. *Let m_0, m_1 be two admissible weight functions such that $m_1 \succ m_0 \succ \langle v \rangle^{(\gamma+3)/2}$. For any $t \geq 0$, there holds*

$$(3.3) \quad \|S_{\mathcal{B}}(t)\|_{L^2(m_1) \rightarrow L^2(m_0)} \lesssim \Theta_{m_1, m_0}(t).$$

Moreover, for any admissible polynomial weight function $m \succ \langle v \rangle^{(\gamma+3)/2}$, the following regularity estimate holds

$$(3.4) \quad \int_0^\infty \|S_{\mathcal{B}}(t)\|_{H_*^{-1}(m) \rightarrow L^2(m)}^2 dt < \infty.$$

Proof. We denote $X(m) = L^2(m)$. We observe that for $\tilde{m}_0 := m_0 \langle v \rangle^{(\gamma+\sigma)/2} \prec m_0 \prec m_1$ (where we recall that $\sigma = 0$ if m_0 is a polynomial function and $\sigma = s$ if m_0 is an exponential function) and for any $R \in (0, \infty)$, we have

$$\frac{\tilde{m}_0^2}{m_0^2}(R) \|f\|_{X(m_0)}^2 \leq \|f\|_{X(\tilde{m}_0)}^2 + \frac{\tilde{m}_0^2}{m_1^2}(R) \|f\|_{X(m_1)}^2.$$

We write that estimate as

$$(3.5) \quad \epsilon_R \|f\|_{X(m_0)}^2 \leq \|f\|_{X(\tilde{m}_0)}^2 + \theta_R \|f\|_{X(m_1)}^2,$$

with

$$\epsilon_R := \frac{\tilde{m}_0^2}{m_0^2}(R), \quad \theta_R := \frac{\tilde{m}_0^2}{m_1^2}(R), \quad \epsilon_R, \frac{\theta_R}{\epsilon_R} \rightarrow 0 \quad \text{as } R \rightarrow \infty.$$

Let us denote $f_{\mathcal{B}}(t) = S_{\mathcal{B}}(t)f_0$ for any $t \geq 0$. Thanks to (2.14) for the weight m_1 , we have

$$\|f_{\mathcal{B}}(t)\|_{X(m_1)} \leq \|f_0\|_{X(m_1)}, \quad \forall t \geq 0.$$

Writing now (2.14) for m_0 , using the interpolation (3.5) and the above estimate, for any $R > 0$, we get

$$\begin{aligned} \frac{d}{dt} \|f_{\mathcal{B}}\|_{X(m_0)}^2 &\lesssim -\|f_{\mathcal{B}}\|_{X(m_0 \langle v \rangle^{(\gamma+\sigma)/2})}^2 \\ &\lesssim -\epsilon_R \|f_{\mathcal{B}}\|_{X(m_0)}^2 + \theta_R \|f_{\mathcal{B}}\|_{X(m_1)}^2 \\ &\lesssim -\epsilon_R \|f_{\mathcal{B}}\|_{X(m_0)}^2 + \theta_R \|f_0\|_{X(m_1)}^2, \end{aligned}$$

with $\epsilon_R = \langle R \rangle^{\gamma+\sigma}$ and $\theta_R/\epsilon_R = m_0^2(R)/m_1^2(R)$. Integrating that last differential inequality, we obtain

$$\begin{aligned} \|f_{\mathcal{B}}(t)\|_{X(m_0)}^2 &\lesssim e^{-\epsilon_R t} \|f_0\|_{X(m_0)}^2 + \frac{\theta_R}{\epsilon_R} \|f_0\|_{X(m_1)}^2 \\ &\lesssim \Gamma_{m_1, m_0}^2(t) \|f_0\|_{X(m_1)}^2, \end{aligned}$$

with

$$\Gamma_{m_1, m_0}^2(t) := \inf_{R>0} \left(e^{-\epsilon_R t} + \frac{\theta_R}{\epsilon_R} \right).$$

We can complete the proof of (3.3) by establishing $\Gamma_{m_1, m_0}(t) \lesssim \Theta_{m_1, m_0}(t)$ for the different choices of weights $m_0 \prec m_1$.

Case 1: $m_0 = \langle v \rangle^{k_0}$ and $m_1 = \langle v \rangle^{k_1}$ with $k_0 < k_1$. We have

$$\Gamma_{m_1, m_0}^2(t) = \inf_{R>0} \left(e^{-\langle R \rangle^{\gamma} t} + \langle R \rangle^{2(k_0 - k_1)} \right).$$

We take $\langle R \rangle = (\langle t \rangle \theta(t))^{1/|\gamma|}$ with $\theta(t) := [\log(1+t)]^{-2}$ and we get

$$\Gamma_{m_1, m_0}^2(t) \leq e^{-\theta(t)} + [\log(1+t)]^{4(k_1 - k_0)/|\gamma|} \langle t \rangle^{-2(k_1 - k_0)/|\gamma|},$$

which behaves just as $\Theta_{m_1, m_0}^2(t)$ as $t \rightarrow \infty$.

Case 2: $m_0 = e^{\kappa_0 \langle v \rangle^s}$ and $m_1 = e^{\kappa_1 \langle v \rangle^s}$ with $\kappa_0 < \kappa_1$. We have

$$\Gamma_{m_1, m_0}^2(t) = \inf_{R>0} \left(e^{-\langle R \rangle^{\gamma+s} t} + e^{2(\kappa_0 - \kappa_1) \langle R \rangle^s} \right).$$

We take $\langle R \rangle = t^{1/|\gamma|}$ and we get

$$\Gamma_{m_1, m_0}^2(t) \leq e^{-t^{s/|\gamma|}} + e^{-2(\kappa_1 - \kappa_0) t^{s/|\gamma|}},$$

which is nothing but $\Theta_{m_1, m_0}^2(t)$. The general case $m_1 \succ m_0$ follows from that estimate.

We turn now to the proof of the regularity estimate (3.4) for a fixed admissible weight function $m \succ \langle v \rangle^{(\gamma+3)/2}$. Thanks to (2.18) and denoting $\phi_{\mathcal{B}_m^*}(t) = S_{\mathcal{B}_m^*}(t)\phi_0$, we have

$$\frac{d}{dt} \|\phi_{\mathcal{B}_m^*}\|_{L^2}^2 \lesssim -\|\phi_{\mathcal{B}_m^*}\|_{H_*^1}^2,$$

which readily implies

$$\int_0^\infty \|S_{\mathcal{B}_m^*}(t)\|_{L^2 \rightarrow H_*^1}^2 dt < \infty.$$

We then deduce (3.4) by duality. \square

3.2. Decay estimates for $S_{\mathcal{L}}$. We first prove decay estimates in a family of small reference spaces included in $L^2(\mu^{-1/2})$.

Proposition 3.2. *For any admissible weight ω such that $\mu^{-1/2} \prec \omega \prec \mu^{-1}$, there holds*

$$\forall t \geq 0, \quad \|S_{\mathcal{L}}(t)\Pi\|_{L^2(\omega) \rightarrow L^2(\mu^{-1/2})} \lesssim e^{-\lambda t^{\frac{2}{|\gamma|}}}.$$

Proof. Let us denote for simplicity $E = L^2(\mu^{-1/2}) \supset E_1 = L^2(\omega)$. We already know from (2.6) and (2.14) that

$$t \mapsto \|S_{\mathcal{L}}(t)\Pi\|_{E \rightarrow E}, \|S_{\mathcal{B}}(t)\|_{E_1 \rightarrow E_1} \in L_t^\infty(\mathbb{R}_+).$$

We then write, thanks to Duhamel's formula,

$$S_{\mathcal{L}}\Pi = S_{\mathcal{B}}\Pi + S_{\mathcal{B}}\mathcal{A} * S_{\mathcal{L}}\Pi,$$

and using Lemma 2.5 and Lemma 3.1, we obtain that $\|S_{\mathcal{B}}\mathcal{A}(t)\|_{E \rightarrow E_1} \in L_t^1(\mathbb{R}_+)$, whence

$$(3.6) \quad \|S_{\mathcal{L}}(t)\Pi\|_{E_1 \rightarrow E_1} \lesssim \|S_{\mathcal{B}}(t)\|_{E_1 \rightarrow E_1} + \|S_{\mathcal{B}}\mathcal{A}(t)\|_{E \rightarrow E_1} * \|S_{\mathcal{L}}(t)\Pi\|_{E_1 \rightarrow E \rightarrow E} \in L_t^\infty(\mathbb{R}_+).$$

Finally, we conclude the proof by writing (2.6) and using an interpolation argument (as in the proof of Lemma 3.1) together with (3.6). \square

As an immediate consequence, we prove uniform in time bounds for the semigroup $S_{\mathcal{L}}$ in large spaces.

Lemma 3.3. *For any admissible weight function $m \succ \langle v \rangle^{\frac{\gamma+3}{2}}$, there holds*

$$t \mapsto \|S_{\mathcal{L}}(t)\Pi\|_{L^2(m) \rightarrow L^2(m)} \in L_t^\infty(\mathbb{R}_+).$$

Proof. Let us denote $E = L^2(\mu^{-1/2})$, $E_1 = L^2(\omega)$ and $X = L^2(m)$, with $\mu^{-1/2} \prec \omega \prec \mu^{-1}$. We only need to treat the case $\langle v \rangle^{\frac{\gamma+3}{2}} \prec m \prec \mu^{-1/2}$ so that $E \subset X$ (the other cases have already been treated in (3.6)). We first write

$$S_{\mathcal{L}}\Pi = \Pi S_{\mathcal{B}} + S_{\mathcal{L}}\Pi * \mathcal{A}S_{\mathcal{B}},$$

and observe that $\|S_{\mathcal{B}}(t)\|_{X \rightarrow X} \in L_t^\infty(\mathbb{R}_+)$ from (2.14) and $\|S_{\mathcal{L}}(t)\Pi\|_{E_1 \rightarrow E} \in L_t^1(\mathbb{R}_+)$ from Proposition 3.2. Moreover, Lemma 2.5 and Lemma 3.1 yield $\|\mathcal{A}S_{\mathcal{B}}(t)\|_{X \rightarrow E_1} \in L_t^\infty(\mathbb{R}_+)$, so that

$$\|S_{\mathcal{L}}(t)\Pi\|_{X \rightarrow X} \lesssim \|S_{\mathcal{B}}(t)\|_{X \rightarrow X} + \|S_{\mathcal{L}}(t)\Pi\|_{E_1 \rightarrow E \rightarrow X} * \|\mathcal{A}S_{\mathcal{B}}(t)\|_{X \rightarrow E_1} \in L_t^\infty(\mathbb{R}_+),$$

and the proof is complete. \square

We can now prove that $S_{\mathcal{L}}$ inherits the decay and regularity estimates already established for the semigroup $S_{\mathcal{B}}$.

Theorem 3.4. *Let m_0, m_1 be two admissible weight functions such that $\langle v \rangle^{(\gamma+3)/2} \prec m_0 \prec m_1$ and $m_0 \preceq \mu^{-1/2}$. There hold*

$$(3.7) \quad \|S_{\mathcal{L}}(t)\Pi\|_{L^2(m_1) \rightarrow L^2(m_0)} \lesssim \Theta_{m_1, m_0}(t), \quad \forall t \geq 0.$$

and

$$(3.8) \quad \int_0^\infty \|S_{\mathcal{L}}(t)\Pi\|_{H_*^{-1}(m_0) \rightarrow L^2(m_0)}^2 dt < \infty.$$

Proof. We fix an admissible weight function ω such that $\omega \succ \mu^{-1/2}$ and $\omega \succeq m_1$, and we split the proof into two steps.

Step 1. We denote $X_0 = L^2(m_0)$, $X_1 = L^2(m_1)$, $E_0 = L^2(\mu^{-1/2})$ and $E_1 = L^2(\omega)$. We write the factorization identity

$$S_{\mathcal{L}}\Pi = \Pi S_{\mathcal{B}} + S_{\mathcal{L}}\Pi * \mathcal{A}S_{\mathcal{B}}.$$

Thanks to Lemma 3.1, Proposition 3.2 and Lemma 2.5, we have

$$\begin{aligned} \|S_{\mathcal{L}}(t)\Pi\|_{X_1 \rightarrow X_0} &\leq \|S_{\mathcal{B}}(t)\Pi\|_{X_1 \rightarrow X_0} + \|\Pi S_{\mathcal{L}}(t)\|_{E_1 \rightarrow E_0} * \|\mathcal{A}S_{\mathcal{B}}(t)\|_{X_1 \rightarrow X_0 \rightarrow E_1} \\ &\lesssim \Theta_{m_1, m_0}(t) + (e^{-\lambda t^{2/|\gamma|}} * \Theta_{m_1, m_0})(t), \end{aligned}$$

which concludes the proof of (3.7).

Step 2. Denote $Z_0 = H_*^{-1}(m_0)$. Writing the factorization identity as in step 1, we have

$$\|S_{\mathcal{L}}(t)\Pi\|_{Z_0 \rightarrow X_0} \lesssim \|S_{\mathcal{B}}(t)\|_{Z_0 \rightarrow X_0} + \|S_{\mathcal{L}}(t)\Pi\|_{E_1 \rightarrow X_0} * \|\mathcal{A}S_{\mathcal{B}}(t)\|_{Z_0 \rightarrow E_1}.$$

Thanks to Lemma 2.5, (3.4) in Lemma 3.1, and Proposition 3.2, we deduce

$$\|S_{\mathcal{B}}(t)\|_{Z_0 \rightarrow X_0} \in L_t^2(\mathbb{R}_+), \quad \|S_{\mathcal{L}}(t)\Pi\|_{E_1 \rightarrow E_0 \rightarrow X_0} \in L_t^1(\mathbb{R}_+), \quad \|\mathcal{A}S_{\mathcal{B}}(t)\|_{Z_0 \rightarrow E_1} \in L_t^2(\mathbb{R}_+),$$

which implies (3.8). \square

3.3. Weak dissipativity of \mathcal{L} . As a final step, we establish that \mathcal{L} is weakly dissipative in some appropriate spaces. In order to do that, we define the spaces

$$(3.9) \quad X := L^2(m), \quad Y := H_*^1(m), \quad Z := H_*^{-1}(m), \quad X_0 := L^2,$$

where we recall that $H_*^1(m)$ and $H_*^{-1}(m)$ have been introduced in (3.1) and (3.2). For any $\eta > 0$, we also define the norm $\|\cdot\|_X$ by

$$(3.10) \quad \|f\|_X^2 := \eta \|f\|_X^2 + \int_0^\infty \|S_{\mathcal{L}}(\tau)f\|_{X_0}^2 d\tau,$$

and we denote by $\langle \cdot, \cdot \rangle_X$ the associated duality product.

Proposition 3.5. *Let m be an admissible weight function such that $m \succ \langle v \rangle^{\frac{3}{2}}$. The norm $\|\cdot\|_X$ is equivalent to $\|\cdot\|_X$, and, moreover, there exists $\eta > 0$ small enough such that*

$$(3.11) \quad \frac{d}{dt} \|S_{\mathcal{L}}(t)f\|_X^2 \lesssim -\|S_{\mathcal{L}}(t)f\|_Y^2, \quad \forall f \in \Pi X.$$

As a consequence, the following regularity estimate holds

$$(3.12) \quad \|S_{\mathcal{L}}(t)\Pi\|_{X \rightarrow Y} \in L_t^2(\mathbb{R}_+).$$

Proof. We easily observe that, thanks to Theorem 3.4,

$$\int_0^\infty \|S_{\mathcal{L}}(\tau)f\|_{X_0}^2 d\tau \lesssim \|f\|_X^2 \int_0^\infty \Theta^2(\tau) d\tau,$$

for some decay function $\Theta \in L^2(\mathbb{R}_+)$ under the condition $m \succ \langle v \rangle^{3/2}$, thus $\|\cdot\|_X$ is equivalent to $\|\cdot\|_X$. Now denote $f_{\mathcal{L}}(t) = S_{\mathcal{L}}(t)f_0$, $f_0 \in \Pi X$, so that $f_{\mathcal{L}}(t) \in \Pi X$ for any $t \geq 0$, recall that $\mathcal{L} = \mathcal{A} + \mathcal{B}$ and write

$$\frac{1}{2} \frac{d}{dt} \|f_{\mathcal{L}}(t)\|_X^2 = \eta \langle \mathcal{B}f_{\mathcal{L}}(t), f_{\mathcal{L}}(t) \rangle_X + \eta \langle \mathcal{A}f_{\mathcal{L}}(t), f_{\mathcal{L}}(t) \rangle_X + \frac{1}{2} \int_0^\infty \frac{d}{d\tau} \|S_{\mathcal{L}}(\tau)f_{\mathcal{L}}(t)\|_{X_0}^2 d\tau.$$

Thanks to Lemma 2.3 and Lemma 2.5, we have

$$\eta \langle \mathcal{B}f_{\mathcal{L}}(t), f_{\mathcal{L}}(t) \rangle_X \leq -\eta K' \|f_{\mathcal{L}}(t)\|_Y^2, \quad \eta \langle \mathcal{A}f_{\mathcal{L}}(t), f_{\mathcal{L}}(t) \rangle_X \leq \eta C \|f_{\mathcal{L}}(t)\|_{X_0}^2.$$

Moreover, for the last term, we have

$$\int_0^\infty \frac{d}{d\tau} \|S_{\mathcal{L}}(\tau)f_{\mathcal{L}}(t)\|_{X_0}^2 d\tau = \lim_{\tau \rightarrow \infty} \|S_{\mathcal{L}}(\tau)f_{\mathcal{L}}(t)\|_{X_0}^2 - \|f_{\mathcal{L}}(t)\|_{X_0}^2 = -\|f_{\mathcal{L}}(t)\|_{X_0}^2,$$

where we have used

$$\forall t \geq 0, \quad \|S_{\mathcal{L}}(\tau)f_{\mathcal{L}}(t)\|_{X_0} \leq C \Theta_m(\tau) \|f_0\|_X \quad \text{with} \quad \lim_{\tau \rightarrow \infty} \Theta_m(\tau) = 0,$$

thanks to Lemma 3.3 and Theorem 3.4. We conclude the proof of (3.11) gathering previous estimates and taking $\eta > 0$ small enough. We finally obtain (3.12) by integrating (3.11) in the time variable. \square

3.4. Summarizing the decay and dissipativity estimates. We summarize the set of information we have established in this section and that we will use in order to get our main existence, uniqueness and stability result for the nonlinear equation in Section 5 (in the spatially homogeneous case).

Corollary 3.6. *Consider an admissible weight function m such that $m \succ \langle v \rangle^{2+3/2}$. With the above assumptions and notation, there exists $\eta > 0$ such that the norm $\|\cdot\|_X$ defined in (3.10) is equivalent to the initial norm on X and*

$$(3.13) \quad \langle \mathcal{L}\Pi f, \Pi f \rangle_X \lesssim -\|\Pi f\|_Y^2, \quad \forall f \in X_1^{\mathcal{L}},$$

$$(3.14) \quad \|S_{\mathcal{L}}(t)\Pi\|_{Y \rightarrow X_0} \in L_t^2(\mathbb{R}_+),$$

$$(3.15) \quad \|S_{\mathcal{L}}(t)\Pi\|_{Z \rightarrow X_0} \in L_t^2(\mathbb{R}_+).$$

Proof. Estimate (3.13) is just a reformulation of (3.11) in Proposition 3.5. We now prove estimate (3.14). We fix an admissible polynomial weight m_1 such that $\langle v \rangle^{3/2} \prec m_1 \prec \langle v \rangle^{\gamma/2} m$. Thanks to the embeddings $H_*^1(m_1) \subset X_0$ and $Y \subset L^2(m_1)$, and to the estimate (3.12) in Theorem 3.4, we deduce

$$\|S_{\mathcal{L}}(t)\Pi\|_{Y \rightarrow X_0} \lesssim \|S_{\mathcal{L}}(t)\Pi\|_{L^2(m_1) \rightarrow H_*^1(m_1)} \in L_t^2(\mathbb{R}_+).$$

Finally, estimate (3.15) follows directly from (3.8) in Theorem 3.4 and the embedding $X \subset X_0$. \square

4. NONLINEAR ESTIMATES

In this section, we present some estimates on the nonlinear Landau operator Q . We start with two auxiliary results.

Lemma 4.1. ([14, Lemma 3.2]) *Let $-3 < \alpha < 0$ and $\theta > 3$. Then*

$$A_{\alpha}(v) := \int_{\mathbb{R}^3} |v - v_*|^{\alpha} \langle v_* \rangle^{-\theta} dv_* \lesssim \langle v \rangle^{\alpha}.$$

Lemma 4.2. *There holds*

(i) *For any $3/(3 + \gamma + 2) < p \leq \infty$ and $\theta > 2 + 3(1 - 1/p)$*

$$|(a_{ij} * f)(v) v_i v_j| + |(a_{ij} * f)(v) v_i| + |(a_{ij} * f)(v)| \lesssim \langle v \rangle^{\gamma+2} \|f\|_{L^p(\langle v \rangle^{\theta})}.$$

(ii) *For any $3/(3 + \gamma + 1) < p \leq \infty$ and any $\theta' > 3(1 - 1/p)$*

$$|(b_j * f)(v)| \lesssim \langle v \rangle^{\gamma+1} \|f\|_{L^p(\langle v \rangle^{\theta'})}.$$

Proof. (i) Recall that 0 is an eigenvalue of the matrix $a_{ij}(z)$ so that $a_{ij}(v - v_*)v_i = a_{ij}(v - v_*)v_{*i}$ and $a_{ij}(v - v_*)v_i v_j = a_{ij}(v - v_*)v_{*i}v_{*j}$. Thanks to Holder's inequality and using Lemma 4.1, we obtain for any $3/(3 + \gamma + 2) < p \leq \infty$ and any $\theta > 3(1 - 1/p)$,

$$\begin{aligned} |(a_{ij} * f)(v) v_i v_j| &= \left| \int_{v_*} a_{ij}(v - v_*) v_{*i} v_{*j} f_* \right| \\ &\lesssim \int_{v_*} |v - v_*|^{\gamma+2} \langle v_* \rangle^{-\bar{\theta}} \langle v_* \rangle^{\bar{\theta}+2} |f_*| \\ &\lesssim \left(\int_{v_*} |v - v_*|^{(\gamma+2)\frac{p}{p-1}} \langle v_* \rangle^{-\bar{\theta}\frac{p}{p-1}} \right)^{(p-1)/p} \|f\|_{L^p(\langle v \rangle^{\bar{\theta}+2})} \\ &\lesssim \langle v \rangle^{\gamma+2} \|f\|_{L^p(\langle v \rangle^{\bar{\theta}+2})}. \end{aligned}$$

We can get the estimates for $(a_{ij} * f)(v) v_i$ and $(a_{ij} * f)(v)$ in a similar way. Remark that we can choose $p = 2$ since $\gamma \in [-3, -2)$.

(ii) For the term $(b * f)$ we recall that $b_i(z) = -2|z|^\gamma z_i$. Thanks to Holder's inequality and Lemma 4.1, we obtain for any $3/(3 + \gamma + 1) < p \leq \infty$ and any $\theta' > 3(1 - 1/p)$,

$$\begin{aligned} |(b_i * f)(v)| &\lesssim \int_{v_*} |v - v_*|^{\gamma+1} \langle v_* \rangle^{-\theta'} \langle v_* \rangle^{\theta'} |f_*| \\ &\lesssim \left(\int_{v_*} |v - v_*|^{(\gamma+1)\frac{p}{p-1}} \langle v_* \rangle^{-\theta' \frac{p}{p-1}} \right)^{(p-1)/p} \|f\|_{L^p(\langle v \rangle^{\theta'})} \\ &\lesssim \langle v \rangle^{\gamma+1} \|f\|_{L^p(\langle v \rangle^{\theta'})}. \end{aligned}$$

Remark now that we have $3/(3 + \gamma + 1) \in (3/2, 3]$, thus we can choose $p = 4$ for any $\gamma \in [-3, -2)$. \square

We establish our main estimate on the Landau collision operator.

Lemma 4.3. *Consider any admissible weight function $m \succeq 1$. Then, for any $\theta > 2 + 3/2$ and $\theta' > 9/4$, there holds*

$$(4.1) \quad \langle Q(f, g), h \rangle_{L^2(m)} \lesssim \left(\|f\|_{L^2(\langle v \rangle^\theta)} \|g\|_{H^1_+(m)} + \|f\|_{H^1(\langle v \rangle^{\theta'})} \|g\|_{L^2(m)} \right) \|h\|_{H^1_+(m)}.$$

Proof. Let us denote $G = mg$ and $H = mh$. We write

$$\begin{aligned} \langle Q(f, g), h \rangle_{L^2(m)} &= \int \partial_j \{ (a_{ij} * f) \partial_i g - (b_j * f) g \} h m^2 \\ &= \int \partial_j \{ (a_{ij} * f) \partial_i (m^{-1} G) \} H m - \int \partial_j \{ (b_j * f) m^{-1} G \} H m =: A + B. \end{aligned}$$

Performing an integration by parts and developing terms, we easily get $A = A_1 + A_2 + A_3 + A_4$ and $B = B_1 + B_2$, with

$$\begin{aligned} A_1 &:= - \int (a_{ij} * f) \partial_i G \partial_j H, & A_2 &:= - \int (a_{ij} * f) \frac{\partial_j m}{m} \partial_i G H, \\ A_3 &:= \int (a_{ij} * f) \frac{\partial_i m}{m} G \partial_j H, & A_4 &:= \int (a_{ij} * f) \frac{\partial_i m}{m} \frac{\partial_j m}{m} G H, \\ B_1 &:= \int (b_j * f) G \partial_j H, & B_2 &:= \int (b_j * f) \frac{\partial_j m}{m} G H. \end{aligned}$$

We then estimate each term separately.

Step 1. Term A_1 . We only consider the case $|v| > 1$, since the estimate for $|v| \leq 1$ is evident. We decompose $\partial_i G = P_v \partial_i G + (I - P_v) \partial_i G =: \partial_i^\parallel G + \partial_i^\perp G$, and similarly for $\partial_j H = \partial_j^\parallel H + \partial_j^\perp H$. We write

$$\begin{aligned} A_1 &= \int_{|v|>1} (a_{ij} * f) \{ \partial_i^\parallel G \partial_j^\parallel H + \partial_i^\parallel G \partial_j^\perp H + \partial_i^\perp G \partial_j^\parallel H + \partial_i^\perp G \partial_j^\perp H \} \\ &=: T_1 + T_2 + T_3 + T_4. \end{aligned}$$

Using Lemma 4.2-(i) with $p = 2$, for any $\theta > 2 + 3/2$, we have

$$\begin{aligned} T_1 &= \int_{|v|>1} (a_{ij} * f) v_i v_j \frac{(v \cdot \nabla_v G)}{|v|^2} \frac{(v \cdot \nabla_v H)}{|v|^2} \\ &\lesssim \|f\|_{L^2(\langle v \rangle^\theta)} \int_{|v|>1} \langle v \rangle^{\gamma+2} |v|^{-2} |\nabla_v G| |\nabla_v H| \\ &\lesssim \|f\|_{L^2(\langle v \rangle^\theta)} \|\langle v \rangle^{\frac{\gamma}{2}} \nabla_v (mg)\|_{L^2} \|\langle v \rangle^{\frac{\gamma}{2}} \nabla_v (mh)\|_{L^2}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned}
T_2 &= \int_{|v|>1} (a_{ij} * f) v_i \frac{(v \cdot \nabla_v G)}{|v|^2} \partial_j^\perp h \\
&\lesssim \|f\|_{L^2(\langle v \rangle^\theta)} \int_{|v|>1} \langle v \rangle^{\gamma+2} |v|^{-1} |\nabla_v G| |\nabla_v^\perp H| \\
&\lesssim \|f\|_{L^2(\langle v \rangle^\theta)} \|\langle v \rangle^{\frac{\gamma}{2}} \nabla_v(mg)\|_{L^2} \|\langle v \rangle^{\frac{\gamma+2}{2}} \nabla_v^\perp(mh)\|_{L^2},
\end{aligned}$$

and similarly

$$T_3 \lesssim \|f\|_{L^2(\langle v \rangle^\theta)} \|\langle v \rangle^{\frac{\gamma+2}{2}} \nabla_v^\perp(mg)\|_{L^2} \|\langle v \rangle^{\frac{\gamma}{2}} \nabla_v(mh)\|_{L^2}.$$

For the term T_4 , we have

$$\begin{aligned}
T_4 &\lesssim \|f\|_{L^2(\langle v \rangle^\theta)} \int \langle v \rangle^{\gamma+2} |\nabla_v^\perp G| |\nabla_v^\perp H| \\
&\lesssim \|f\|_{L^2(\langle v \rangle^\theta)} \|\langle v \rangle^{\frac{\gamma+2}{2}} \nabla_v^\perp(mg)\|_{L^2} \|\langle v \rangle^{\frac{\gamma+2}{2}} \nabla_v^\perp(mh)\|_{L^2}.
\end{aligned}$$

All in all, we obtain

$$A_1 \lesssim \|f\|_{L^2(\langle v \rangle^\theta)} \|g\|_{H_*^1(m)} \|h\|_{H_*^1(m)}.$$

Step 2. Term A_2 . Recall that $\partial_j m^2 = C v_j \langle v \rangle^{\sigma-2} m^2$. The case $|v| \leq 1$ is evident so we only consider $|v| > 1$. The same argument as for A_1 gives us

$$\begin{aligned}
A_2 &= C \int_{|v|>1} (a_{ij} * f) v_j \langle v \rangle^{\sigma-2} \{\partial_i^\parallel G + \partial_i^\perp G\} H \\
&\lesssim \|f\|_{L^2(\langle v \rangle^\theta)} \int \left\{ \langle v \rangle^{\gamma+\sigma-1} |\nabla_v G| + \langle v \rangle^{\gamma+\sigma} |\nabla_v^\perp G| \right\} |H| \\
&\lesssim \|f\|_{L^2(\langle v \rangle^\theta)} \left\{ \|\langle v \rangle^{\frac{\gamma}{2}} \nabla_v(mg)\|_{L^2} + \|\langle v \rangle^{\frac{\gamma+2}{2}} \nabla_v^\perp(mg)\|_{L^2} \right\} \|\langle v \rangle^{\frac{\gamma+2\sigma-2}{2}} h\|_{L^2(m)} \\
&\lesssim \|f\|_{L^2(\langle v \rangle^\theta)} \|g\|_{H_*^1(m)} \|h\|_{H_*^1(m)}.
\end{aligned}$$

Step 3. Term A_3 . In a similar way as for the term A_2 , we also have

$$A_3 \lesssim \|f\|_{L^2(\langle v \rangle^\theta)} \|\langle v \rangle^{\frac{\gamma+\sigma}{2}} g\|_{L^2(m)} \|h\|_{H_*^1(m)} \lesssim \|f\|_{L^2(\langle v \rangle^\theta)} \|g\|_{H_*^1(m)} \|h\|_{H_*^1(m)}.$$

Step 4. Term A_4 . Arguing as before, we easily get

$$\begin{aligned}
A_4 &= C \int (a_{ij} * f) v_i v_j \langle v \rangle^{2\sigma-4} G H \\
&\lesssim \|f\|_{L^2(\langle v \rangle^\theta)} \int \langle v \rangle^{\gamma+2\sigma-2} |G| |H| \\
&\lesssim \|f\|_{L^2(\langle v \rangle^\theta)} \|\langle v \rangle^{\frac{\gamma+2\sigma-2}{2}} g\|_{L^2(m)} \|\langle v \rangle^{\frac{\gamma+2\sigma-2}{2}} h\|_{L^2(m)} \\
&\lesssim \|f\|_{L^2(\langle v \rangle^\theta)} \|g\|_{H_*^1(m)} \|h\|_{H_*^1(m)}.
\end{aligned}$$

Step 5. Term B_1 . Thanks to Lemma 4.2-(ii) with $p = 4$, for any $\theta' > 9/4$, it follows

$$\begin{aligned}
B_1 &\lesssim \|f\|_{L^4(\langle v \rangle^{\theta'})} \int \langle v \rangle^{\gamma+1} |G| |\nabla_v H| \\
&\lesssim \|f\|_{H^1(\langle v \rangle^{\theta'})} \|\langle v \rangle^{\frac{\gamma+2}{2}} g\|_{L^2(m)} \|\langle v \rangle^{\frac{\gamma}{2}} \nabla_v(mh)\|_{L^2} \\
&\lesssim \|f\|_{H^1(\langle v \rangle^{\theta'})} \|\langle v \rangle^{\frac{\gamma+2}{2}} g\|_{L^2(m)} \|h\|_{H_*^1(m)},
\end{aligned}$$

where we have used the embedding $H^1(\langle v \rangle^{\theta'}) \subset L^4(\langle v \rangle^{\theta'})$.

Step 6. Term B_2 . Using $\partial_j m = Cv_j \langle v \rangle^{\sigma-2} m$, we have

$$\begin{aligned} B_2 &\lesssim \|f\|_{L^4(\langle v \rangle^{\theta'})} \int \langle v \rangle^{\gamma+\sigma} |G| |H| \\ &\lesssim \|f\|_{H^1(\langle v \rangle^{\theta'})} \|\langle v \rangle^{\frac{\gamma+\sigma}{2}} g\|_{L^2(m)} \|\langle v \rangle^{\frac{\gamma+\sigma}{2}} h\|_{L^2(m)} \\ &\lesssim \|f\|_{H^1(\langle v \rangle^{\theta'})} \|\langle v \rangle^{\frac{\gamma+\sigma}{2}} g\|_{L^2(m)} \|h\|_{H_*^1(m)}. \end{aligned}$$

Step 7. Conclusion. Gathering previous estimates and using that $\|\langle v \rangle^{\frac{\gamma+\sigma}{2}} g\|_{L^2(m)}$ and $\|\langle v \rangle^{\frac{\gamma+2}{2}} g\|_{L^2(m)}$ can be controlled by $\|g\|_{L^2(m)}$, we obtain, for any $\theta > 2 + 3/2$ and $\theta' > 9/4$,

$$\langle Q(f, g), h \rangle_{L^2(m)} \lesssim \|f\|_{L^2(\langle v \rangle^\theta)} \|g\|_{H_*^1(m)} \|h\|_{H_*^1(m)} + \|f\|_{H^1(\langle v \rangle^{\theta'})} \|g\|_{L^2(m)} \|h\|_{H_*^1(m)},$$

which concludes the proof of (4.1). \square

Corollary 4.4. *Consider an admissible weight function m such that $m \succ \langle v \rangle^{2+3/2}$. With the notation (3.9), there holds*

$$(4.2) \quad \langle Q(f, g), h \rangle_X \lesssim (\|f\|_X \|g\|_Y + \|f\|_Y \|g\|_X) \|h\|_Y,$$

and in particular

$$(4.3) \quad \|Q(f, g)\|_Z \lesssim \|f\|_X \|g\|_Y + \|f\|_Y \|g\|_X.$$

Proof. The proof of (4.2) easily follows from (4.1) observing that $H_*^1(m) \hookrightarrow H^1(\langle v \rangle^{\theta'})$ (see (3.1)). The proof of (4.3) is then straightforward by the definition of $Z = H_*^{-1}(m)$ (see (3.2)). \square

5. NONLINEAR STABILITY

This section is devoted to the proof of the spatially homogeneous version of Theorem 1.1.

Consider a solution F to the homogeneous Landau equation (1.2) and define the variation $f = F - \mu$, which satisfies,

$$(5.1) \quad \begin{cases} \partial_t f = \mathcal{L}f + Q(f, f) \\ f|_{t=0} = f_0 = F_0 - \mu. \end{cases}$$

We observe that, $\Pi_0 f_0 = 0$ and therefore, thanks to the conservation laws,

$$\Pi_0 f(t) = \Pi_0 Q(f(t), f(t)) = 0 \quad \text{for any } t > 0.$$

Hereafter in this section, we fix an admissible weight function m satisfying $m \succ \langle v \rangle^{2+3/2}$ and consider the spaces X, Y, Z and X_0 defined in (3.9). We also recall the norm $\|\cdot\|_X$ defined in (3.10), which is equivalent to $\|\cdot\|_X$.

We first prove a stability estimate.

Proposition 5.1. *There exist some constants $C, K \in (0, \infty)$ such that any solution f to (5.1) satisfies, at least formally, the following differential inequality*

$$\frac{d}{dt} \|f\|_X^2 \leq (C \|f\|_X - K) \|f\|_Y^2.$$

Proof. We write

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|f\|_X^2 &= \eta \langle \mathcal{L}f, f \rangle_X + \eta \langle Q(f, f), f \rangle_X + \int_0^\infty \langle S_{\mathcal{L}}(\tau) Q(f, f), S_{\mathcal{L}}(\tau) g \rangle_{X_0} d\tau \\ &=: T_1 + T_2 + T_3. \end{aligned}$$

On the one hand, thanks to (3.13) in Corollary 3.6 and to Corollary 4.4, there exist $K, C' > 0$ such that

$$T_1 + T_2 \leq -K\|f\|_Y^2 + C'\|f\|_X \|f\|_Y^2.$$

On the other hand, we have

$$\begin{aligned} & \int_0^\infty \langle S_{\mathcal{L}}(\tau)Q(f, f), S_{\mathcal{L}}(\tau)f \rangle_{X_0} d\tau \\ & \leq \int_0^\infty \|S_{\mathcal{L}}(\tau)Q(f, f)\|_{X_0} \|S_{\mathcal{L}}(\tau)f\|_{X_0} d\tau \\ & \leq \left(\int_0^\infty \|S_{\mathcal{L}}(\tau)Q(f, f)\|_{X_0}^2 d\tau \right)^{1/2} \left(\int_0^\infty \|S_{\mathcal{L}}(\tau)f\|_{X_0}^2 d\tau \right)^{1/2} \\ & \lesssim \|Q(f, f)\|_Z \|f\|_Y \lesssim \|f\|_X \|f\|_Y^2, \end{aligned}$$

where we have used (3.14) and (3.15) in Corollary 3.6 as well as Corollary 4.4 again in the last line. We conclude the proof by gathering these two estimates. \square

A consequence of the stability estimate in Proposition 5.1 we obtain the spatially homogeneous version of Theorem 1.1.

Proof of Theorem 1.1. The spatially homogeneous case. We split the proof into three steps.

Step 1. Existence. At least formally, if $\epsilon_0 > 0$ is small enough, we easily obtain (1.12) thanks to Proposition 5.1. The rigorous proof of existence of solutions such that (1.12) holds follows a classical argument based on an iterative scheme that approximates (5.1), see e.g. [20, Proof of Theorem 5.3].

Step 2. Uniqueness. Consider two solutions f_1 and f_2 to (5.1) with same initial data $\|f_0\|_X \leq \epsilon_0$ and such that

$$\|f_i\|_{L^\infty([0, \infty); X)} + \|g_i\|_{L^2([0, \infty); Y)} \leq \epsilon_1.$$

The difference $f_1 - f_2$ satisfies

$$\partial_t(f_1 - f_2) = \mathcal{L}(f_1 - f_2) + Q(f_1, f_1 - f_2) + Q(f_1 - f_2, f_2),$$

hence

$$\begin{aligned} \frac{d}{dt} \|f_1 - f_2\|_X^2 &= \langle \mathcal{A}(f_1 - f_2), f_1 - f_2 \rangle_X + \langle \mathcal{B}(f_1 - f_2), f_1 - f_2 \rangle_X \\ &\quad + \langle Q(f_1, f_1 - f_2), f_1 - f_2 \rangle_X + \langle Q(f_1 - f_2, f_2), f_1 - f_2 \rangle_X. \end{aligned}$$

Thanks to Lemmas 2.3, 2.5 and 4.3, we easily obtain

$$\begin{aligned} \frac{d}{dt} \|f_1 - f_2\|_X^2 + \|f_1 - f_2\|_Y^2 &\lesssim \|f_1 - f_2\|_X^2 + (\|f_1\|_X + \|f_2\|_X) \|f_1 - f_2\|_Y^2 \\ &\quad + (\|f_1\|_Y + \|f_2\|_Y) \|f_1 - f_2\|_X \|f_1 - f_2\|_Y \\ &\leq C [1 + \eta^{-1}(\|f_1\|_Y^2 + \|f_2\|_Y^2)] \|f_1 - f_2\|_X^2 \\ &\quad + C [\|f_1\|_X + \|f_2\|_X + \eta] \|f_1 - f_2\|_Y^2, \end{aligned}$$

for any $\eta > 0$. Choosing $\epsilon_1 > 0$ small enough and next $\eta > 0$ small enough, we deduce that

$$\frac{d}{dt} \|f_1 - f_2\|_X^2 \lesssim [1 + \eta^{-1}(\|f_1\|_Y^2 + \|f_2\|_Y^2)] \|f_1 - f_2\|_X^2,$$

and we conclude to the uniqueness thanks to Grönwall's lemma.

Step 3. Decay. Let \tilde{m} be an admissible weight function such that $\langle v \rangle^{2+3/2} \prec \tilde{m} \prec m$, and denote $\tilde{X} = L^2(\tilde{m})$ and $\tilde{Y} = H_*^1(\tilde{m})$. Thanks to Proposition 5.1 in both spaces X and \tilde{X} , it follows

$$\frac{d}{dt} \|f\|_X^2 \leq (C\epsilon_0 - K)\|f\|_Y^2 \leq 0, \quad \frac{d}{dt} \|f\|_{\tilde{X}}^2 \leq (C\epsilon_0 - K)\|f\|_{\tilde{Y}}^2 \lesssim -\|f\|_{\tilde{Y}}^2.$$

These two estimates together imply (see the proof of Lemma 3.1) the decay

$$\|f(t)\|_{\bar{X}} \lesssim \Theta_{m, \bar{m}}(t) \|f_0\|_X.$$

We hence obtain

$$\|f(t)\|_{X_0} \lesssim \Theta_m(t) \|f_0\|_X,$$

where we recall that Θ_m is defined in (1.9), and that completes the proof. \square

We conclude the section by presenting a proof of our improvement of the speed of convergence to the equilibrium for solutions to the spatially homogenous Landau equation in a non perturbative framework.

Proof of Corollary 1.3. We claim that for some time $t_0 > 0$ (smaller than some explicit constant $T > 0$) we have

$$(5.2) \quad \|f(t_0)\|_{L_v^2(m_1)} \leq \epsilon_0,$$

where we denote $m_1 = m^{1/2} \langle v \rangle^{-9/2}$ and $\epsilon_0 > 0$ is given in Theorem 1.1. Indeed, thanks to [16] there holds

$$\forall t, T > 0, \quad \int_t^{t+T} \|f(\tau)\|_{L_v^3(\langle v \rangle^{-3})} d\tau \lesssim 1 + T,$$

and from [13, Theorem 2] we have the convergence

$$\|f(t)\|_{L^1(m)} \lesssim \theta(t), \quad \theta(t) \simeq e^{-\lambda t^{\frac{s}{s+|\gamma|}} (\log(1+t))^{-\frac{|\gamma|}{s+|\gamma|}}}.$$

Thanks to the interpolation inequality

$$\|f\|_{L_v^2(m_1)} \leq \|f\|_{L_v^1(m)}^{1/4} \|f\|_{L_v^3(\langle v \rangle^{-3})}^{3/4},$$

we obtain, for any $t > 0$,

$$\theta(t)^{-1/4} \int_t^{t+1} \|f(\tau)\|_{L_v^2(m_1)} d\tau \lesssim \int_t^{t+1} \theta^{-1}(\tau) \|f(\tau)\|_{L_v^1(m)} d\tau + \int_t^{t+1} \|f(\tau)\|_{L_v^3(\langle v \rangle^{-3})} d\tau \lesssim 1,$$

which proves (5.2). Therefore we can apply Theorem 1.1 starting from $t_0 > 0$ and we deduce the convergence

$$\|f(t)\|_{L_v^2(m_1)} \lesssim \Theta_{m_1}(t).$$

The proof is then complete by remarking that $\Theta_{m_1}(t) \simeq \Theta_m(t)$, since m is an exponential weight. \square

6. THE SPATIALLY INHOMOGENEOUS CASE

In this section, we explain how we may adapt to the spatially inhomogeneous case the arguments presented in the previous sections. The novelties come from the facts that:

(1) We establish a first weak hypocoercivity estimate in the (small) space $\mathcal{H}_{x,v}^1(\mu^{-1})$ (see (6.3) below);

(2) We prove a set of weak dissipativity estimates on an appropriate operator $\bar{\mathcal{B}}$ and of regularization results on the time functions $(\mathcal{A}S_{\bar{\mathcal{B}}})^{(*n)}$ and $(S_{\bar{\mathcal{B}}}\mathcal{A})^{(*\ell)}$ in order to transfer the above information to the space $H_x^2 L_v^2(m)$, which is suitable for establishing our existence, uniqueness and stability results.

6.1. The linearized inhomogeneous operator. We denote by $\bar{\mathcal{L}}$ the inhomogeneous linearized Landau operator given by

$$(6.1) \quad \bar{\mathcal{L}} := \mathcal{L} - v \cdot \nabla_x,$$

where we recall that \mathcal{L} is defined in (2.3). We have

$$\ker(\bar{\mathcal{L}}) = \text{span}\{\mu, v_1\mu, v_2\mu, v_3\mu, |v|^2\mu\}$$

and the projection $\bar{\Pi}_0$ onto $\ker(\bar{\mathcal{L}})$ is given by

$$\bar{\Pi}_0(f) = \left(\int f \, dx \, dv \right) \mu + \sum_{j=1}^3 \left(\int v_j f \, dx \, dv \right) v_j \mu + \left(\int \frac{|v|^2 - 3}{6} f \, dx \, dv \right) \frac{|v|^2 - 3}{6} \mu.$$

Hereafter we denote $\bar{\Pi} := I - \bar{\Pi}_0$ the projection onto the orthogonal of $\ker(\bar{\mathcal{L}})$. Recall the factorization for the homogeneous operator $\mathcal{L} = \mathcal{A} + \mathcal{B}$ in (2.8), then we write

$$\bar{\mathcal{L}} = \mathcal{A} + \bar{\mathcal{B}}, \quad \bar{\mathcal{B}} := \mathcal{B} - v \cdot \nabla_x.$$

6.2. Functional spaces. We denote by $L_{x,v}^2 = L_{x,v}^2(\mathbb{T}_x^3 \times \mathbb{R}_v^3)$ the standard Lebesgue space on $\mathbb{T}_x^3 \times \mathbb{R}_v^3$. For a velocity weight function $m = m(v) : \mathbb{R}_v^3 \rightarrow \mathbb{R}_+$, we then define the weighted Lebesgue spaces $L_{x,v}^2(m)$ and weighted Sobolev spaces $H_x^n L_v^2(m)$, $n \in \mathbb{N}$, associated to the norm

$$\|f\|_{L_{x,v}^2(m)} = \|mf\|_{L_{x,v}^2}, \quad \|f\|_{H_x^n L_v^2(m)}^2 := \sum_{0 \leq j \leq n} \|\nabla_x^j(mf)\|_{L_{x,v}^2}^2.$$

We similarly define the weighted Sobolev space $H_{x,v}^n(m)$, $n \in \mathbb{N}$, through the norm

$$(6.2) \quad \|f\|_{H_{x,v}^n(m)} := \|mf\|_{H_{x,v}^n},$$

where $H_{x,v}^n = H_{x,v}^n(\mathbb{T}_x^3 \times \mathbb{R}_v^3)$ denotes the usual Sobolev space on $\mathbb{T}_x^3 \times \mathbb{R}_v^3$. We also define the space $\mathcal{H}_{x,v}^1(m)$ associated to the norm defined by

$$(6.3) \quad \|f\|_{\mathcal{H}_{x,v}^1(m)}^2 := \|mf\|_{L_{x,v}^2}^2 + \|\nabla_x(mf)\|_{L_{x,v}^2}^2 + \|\langle v \rangle^\alpha \nabla_v(mf)\|_{L_{x,v}^2}^2,$$

with

$$(6.4) \quad \alpha := \alpha(m) := \max \left\{ \gamma + \sigma, \frac{\gamma}{2} + \frac{\sigma}{4} \right\} < 0.$$

We easily observe that

$$(6.5) \quad H_{x,v}^1(m) \subset \mathcal{H}_{x,v}^1(m) \subset H_{x,v}^1(\langle v \rangle^\alpha m),$$

and also that, for any $\gamma \in [-3, -2)$,

$$\alpha = \frac{\gamma}{2} + \frac{\sigma}{4} \text{ if } \sigma \in [0, 4/3], \quad \alpha = \gamma + 2 \text{ if } \sigma = 2.$$

We remark that we shall use the spaces $\mathcal{H}_{x,v}^1(m)$ (instead of $H_{x,v}^1(m)$) in order to obtain weakly dissipative estimates for $\bar{\mathcal{B}}$, and the reason for that will be explained in Lemma 6.4.

Recall the space $H_{v,*}^1(m)$ defined in (3.1), then we define the space $H_x^2(H_{v,*}^1(m))$ associated to the norm

$$(6.6) \quad \|f\|_{H_x^2(H_{v,*}^1(m))}^2 := \sum_{0 \leq j \leq 2} \|\nabla_x^j f\|_{L_x^2(H_{v,*}^1(m))}^2 := \sum_{0 \leq j \leq 2} \int_{\mathbb{T}_x^3} \|\nabla_x^j f\|_{H_{v,*}^1(m)}^2.$$

We also define the negative weighted Sobolev space $H_x^2(H_{v,*}^{-1}(m))$ in duality with $H_x^2(H_{v,*}^1(m))$ with respect to the $H_x^n L_v^2(m)$ duality product, more precisely

$$\begin{aligned} \|f\|_{H_x^2(H_{v,*}^{-1}(m))} &:= \sup_{\|\phi\|_{H_x^2(H_{v,*}^1(m))} \leq 1} \langle f, \phi \rangle_{H_x^2 L_v^2(m)} \\ &:= \sup_{\|\phi\|_{H_x^2(H_{v,*}^1(m))} \leq 1} \sum_{0 \leq j \leq 2} \langle \nabla_x^j(mf), \nabla_x^j(m\phi) \rangle_{L_{x,v}^2}, \end{aligned}$$

and observe that $\|f\|_{H_x^2(H_{v,*}^{-1}(m))} = \|mf\|_{H_x^2(H_{v,*}^{-1})}$.

6.3. Weak coercivity estimate of $\bar{\mathcal{L}}$. Starting from the weak coercivity estimate (2.6) for the homogeneous linearized operator \mathcal{L} in $L_v^2(\mu^{-1/2})$, we can exhibit an equivalent norm to the usual norm in $\mathcal{H}_{x,v}^1(\mu^{-1/2})$ such that $\bar{\mathcal{L}}$ is weakly coercive related to that norm. Our method of proof follows the method developed in [34] for proving (strong) coercivity estimate and then spectral gap estimate in the case of the linearized Landau equation for harder potentials. We also refer to [21, 43] where related arguments have been introduced.

Lemma 6.1. *There exists a Hilbert norm $\|\cdot\|_{\tilde{\mathcal{H}}_{x,v}^1(\mu^{-1/2})}$ (which associated scalar product is denoted by $\langle \cdot, \cdot \rangle_{\tilde{\mathcal{H}}_{x,v}^1(\mu^{-1/2})}$) equivalent to $\|\cdot\|_{\mathcal{H}_{x,v}^1(\mu^{-1/2})}$ such that, for any $f \in \mathcal{H}_{x,v}^1(\mu^{-1/2})$, there holds*

$$(6.7) \quad \langle \bar{\mathcal{L}}f, f \rangle_{\tilde{\mathcal{H}}_{x,v}^1(\mu^{-1/2})} \lesssim -\|\bar{\Pi}f\|_{\tilde{\mathcal{H}}_{x,v}^1(\langle v \rangle^{(\gamma+2)/2}\mu^{-1/2})}^2.$$

Proof. We only sketch the proof presenting the main steps, and we refer to [34] for more details. We define

$$Lh = \mu^{-1/2} \mathcal{L}(\mu^{1/2}h).$$

Observe that $f = \mu^{1/2}h$ satisfies $Lh = \mu^{-1/2}\mathcal{L}f$ and $\langle Lh, h \rangle_{L_v^2} = \langle \mathcal{L}f, f \rangle_{L_v^2(\mu^{-1/2})}$. Following [21, Section 2] we can decompose $L = A + K$ such that the following properties holds:

(i) Generalized coercivity estimate (see (2.6))

$$\langle Lh, h \rangle_{L_v^2} \lesssim -\|h - \Pi_L h\|_{H_{v,**}^1}^2,$$

where Π_L is the projection onto $\ker(L)$ in L_v^2 , and we denote

$$\|h\|_{H_{v,**}^1(\omega)}^2 := \|\langle v \rangle^{\frac{\gamma+2}{2}} h\|_{L_v^2(\omega)}^2 + \|\langle v \rangle^{\frac{\gamma}{2}} \tilde{\nabla}_v h\|_{L_v^2(\omega)}^2.$$

(ii) [21, Lemma 5]: For $\theta \in \mathbb{R}$ and $\delta > 0$, there holds

$$\langle \langle v \rangle^{2\theta} Kh, h \rangle_{L_v^2} \lesssim \delta \|h\|_{H_{v,**}^1(\langle v \rangle^\theta)}^2 + C(\delta) \|g\|_{L_v^2}^2,$$

and also

$$\langle \langle v \rangle^{2\theta} Lh_1, h_2 \rangle_{L_v^2} \lesssim \|h_1\|_{H_{v,**}^1(\langle v \rangle^\theta)} \|h_2\|_{H_{v,**}^1(\langle v \rangle^\theta)}.$$

(iii) [21, Lemma 6]: For $\theta \in \mathbb{R}$ and $\delta > 0$, there holds

$$\langle \langle v \rangle^{2\theta} \nabla_v(Ah), \nabla_v h \rangle_{L_v^2} \lesssim -\|\nabla_v h\|_{H_{v,**}^1(\langle v \rangle^\theta)}^2 + \eta \|h\|_{H_{v,**}^1(\langle v \rangle^\theta)}^2 + \eta^{-1} \|\mu h\|_{L_v^2}^2,$$

and also

$$\langle \langle v \rangle^{2\theta} \nabla_v(Kh), \nabla_v h \rangle_{L_v^2} \lesssim \left(\eta \|h\|_{H_{v,**}^1(\langle v \rangle^\theta)} + \eta \|\nabla_v h\|_{H_{v,**}^1(\langle v \rangle^\theta)} + \eta^{-1} \|\mu h\|_{L_v^2} \right) \|\nabla_v h\|_{H_{v,**}^1(\langle v \rangle^\theta)}.$$

We now consider the inhomogeneous operator $\bar{L} := L - v \cdot \nabla_x$, we denote $\Pi_{\bar{L}}$ the projection onto $\ker(\bar{L})$ in $L_{x,v}^2$ and we consider a solution h to the evolution equation $\partial_t h = \bar{L}h$ with initial datum $h(0) = h_0 \in \ker(\bar{L})^\perp$. Thanks to (i) and the fact that ∇_x commutes with \bar{L} , we immediately have

$$\frac{1}{2} \frac{d}{dt} \left(\|h\|_{L_{x,v}^2}^2 + \|\nabla_x h\|_{L_{x,v}^2}^2 \right) \lesssim -\|h - \Pi_{\bar{L}} h\|_{L_{x,v}^2(H_{v,**}^1)}^2 - \|\nabla_x h - \Pi_{\bar{L}}(\nabla_x h)\|_{L_{x,v}^2(H_{v,**}^1)}^2.$$

We next look to the v -derivative. For $\eta > 0$ small enough, we may similarly obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\langle v \rangle^{\gamma+2} \nabla_v h\|_{L_{x,v}^2}^2 &\lesssim -\|\nabla_v h\|_{L_{x,v}^2(H_{v,**}^1(\langle v \rangle^{\gamma+2}))}^2 + \eta \|h\|_{L_{x,v}^2(\langle v \rangle^{3\gamma/2+3})}^2 \\ &\quad + \eta^{-1} \|\mu h\|_{L_{x,v}^2}^2 + \|\nabla_x h\|_{L_{x,v}^2(\langle v \rangle^{\gamma/2+1})}^2, \end{aligned}$$

where we mainly use (iii) and the Cauchy-Schwarz inequality. We also compute the evolution of the mixed term

$$\frac{1}{2} \frac{d}{dt} \langle \langle v \rangle^{\gamma+2} \nabla_x h, \nabla_v h \rangle_{L^2_{x,v}} = - \|\langle v \rangle^{\frac{\gamma+2}{2}} \nabla_x h\|_{L^2_{x,v}}^2 + \langle \langle v \rangle^{\gamma+2} \nabla_x Lh, \nabla_v h \rangle_{L^2_{x,v}},$$

Thanks to (i) and $\nabla_x Lh = L(\nabla_x h - \Pi_L(\nabla_x h))$, for any $\eta > 0$, it follows that

$$\langle \langle v \rangle^{\gamma+2} \nabla_x Lh, \nabla_v h \rangle_{L^2_{x,v}} \lesssim \eta^{-1} \|\nabla_x h - \Pi_L(\nabla_x h)\|_{L^2_x(H^1_{v,**}(\langle v \rangle^{(\gamma+2)/2}))}^2 + \eta \|\nabla_v h\|_{L^2_x(H^1_{v,**}(\langle v \rangle^{(\gamma+2)/2}))}^2.$$

We finally introduce the norm

$$\|h\|^2 := \|h\|_{L^2_{x,v}}^2 + \alpha_1 \|\nabla_x h\|_{L^2_{x,v}}^2 + \alpha_2 \|\langle v \rangle^{\gamma+2} \nabla_v h\|_{L^2_{x,v}}^2 + \alpha_3 \|\langle v \rangle^{\gamma+2} \nabla_x h, \nabla_v h\|_{L^2_{x,v}},$$

for positive constants α_i with $\alpha_3 < 2\sqrt{\alpha_1 \alpha_2}$, so that $\|h\|^2$ is equivalent to

$$\|h\|_{L^2_{x,v}}^2 + \|\nabla_x h\|_{L^2_{x,v}}^2 + \|\langle v \rangle^{\gamma+2} \nabla_v h\|_{L^2_{x,v}}^2.$$

Gathering the previous estimates, we get

$$\begin{aligned} \frac{d}{dt} \|h\|^2 &\lesssim -\|h - \Pi_L h\|_{L^2_x H^1_{v,**}}^2 - \alpha_1 \|\nabla_x h - \Pi_L(\nabla_x h)\|_{L^2_x(H^1_{v,**})}^2 \\ &\quad - \alpha_2 \|\nabla_v h\|_{L^2_x(H^1_{v,**}(\langle v \rangle^{\gamma+2}))}^2 + \alpha_2 \eta^{-1} \|h - \Pi_L h\|_{L^2_{x,v}(\langle v \rangle^{3\gamma/2+3})}^2 \\ &\quad + \alpha_2 \eta^{-1} \|\nabla_x h\|_{L^2_{x,v}(\langle v \rangle^{3\gamma/2+3})}^2 + \alpha_2 \|\nabla_x h\|_{L^2_{x,v}(\langle v \rangle^{(\gamma+2)/2})}^2 - \alpha_3 \|\nabla_x h\|_{L^2_{x,v}(\langle v \rangle^{(\gamma+2)/2})}^2 \\ &\quad + \alpha_3 \eta^{-1} \|\nabla_x h - \Pi_L(\nabla_x h)\|_{L^2_x(H^1_{v,**}(\langle v \rangle^{(\gamma+2)/2}))}^2 + \alpha_3 \eta \|\nabla_v h\|_{L^2_x(H^1_{v,**}(\langle v \rangle^{(\gamma+2)/2}))}^2. \end{aligned}$$

We choose the constants $\alpha_i, \eta > 0$ small enough, and we get

$$\begin{aligned} \frac{d}{dt} \|h\|^2 &\lesssim -\|h - \Pi_L h\|_{L^2_x(H^1_{v,**})}^2 - \alpha_1 \|\nabla_x h - \Pi_L(\nabla_x h)\|_{L^2_x(H^1_{v,**})}^2 \\ &\quad - \alpha_3 \|\nabla_x h\|_{L^2_{x,v}(\langle v \rangle^{(\gamma+2)/2})}^2 - \alpha_2 \|\nabla_v h\|_{L^2_x(H^1_{v,**}(\langle v \rangle^{\gamma+2}))}^2. \end{aligned}$$

Because $\Pi_{\bar{L}} h = 0$, the function $\Pi_L h$ has zero mean on the torus \mathbb{T}_x^3 and Poincaré's inequality implies

$$\|h\|_{L^2_{x,v}(\langle v \rangle^{(\gamma+2)/2})}^2 \lesssim \|h - \Pi_L h\|_{L^2_{x,v}(\langle v \rangle^{(\gamma+2)/2})}^2 + \frac{\alpha_3}{2} \|\nabla_x h\|_{L^2_{x,v}(\langle v \rangle^{(\gamma+2)/2})}^2.$$

We put together the two last estimates and we get

$$\begin{aligned} \frac{d}{dt} \|h\|^2 &\lesssim -\|h\|_{L^2_{x,v}(\langle v \rangle^{(\gamma+2)/2})}^2 - \|\nabla_x h\|_{L^2_{x,v}(\langle v \rangle^{(\gamma+2)/2})}^2 - \|\nabla_v h\|_{L^2_{x,v}(\langle v \rangle^{3(\gamma+2)/2})}^2 \\ &\lesssim -\|\langle v \rangle^{(\gamma+2)/2} h\|^2. \end{aligned}$$

Coming back to the function $f = \mu^{1/2} h$ and defining $\|f\|_{\tilde{\mathcal{H}}^1_{x,v}(\mu^{-1/2})} := \|\mu^{-1/2} f\|$, we have $\partial_t f = \bar{\mathcal{L}} f$ and

$$\langle \bar{\mathcal{L}} f, f \rangle_{\tilde{\mathcal{H}}^1_{x,v}(\mu^{-1/2})} = \frac{d}{dt} \|f\|_{\tilde{\mathcal{H}}^1_{x,v}(\mu^{-1/2})}^2 \lesssim -\|f\|_{\tilde{\mathcal{H}}^1_{x,v}(\langle v \rangle^{(\gamma+2)/2} \mu^{-1/2})}^2,$$

from which (6.7) immediately follows. \square

6.4. Weak dissipativity properties on $\bar{\mathcal{B}}$. We prove in this section weak dissipativity properties of $\bar{\mathcal{B}}$ using the analogous results already proven in Lemmas 2.3 and 2.4 for the homogeneous operator \mathcal{B} .

Lemma 6.2. *Let m be an admissible weight function such that $m \succ \langle v \rangle^{(\gamma+3)/2}$ and $n \in \mathbb{N}$. There exist $M, R > 0$ large enough such that $\bar{\mathcal{B}}$ is weakly dissipative in $H_x^n L_v^2(m)$ in the following sense:*

- If $m \prec \mu^{-1/2}$, there holds

$$(6.8) \quad \langle \bar{\mathcal{B}} f, f \rangle_{H_x^n L_v^2(m)} \lesssim -\|\langle v \rangle^{\frac{\gamma}{2}} \tilde{\nabla}_v f\|_{H_x^n L_v^2(m)}^2 - \|\langle v \rangle^{\frac{\gamma}{2}} \tilde{\nabla}_v(mf)\|_{H_x^n L_v^2}^2 - \|\langle v \rangle^{\frac{\gamma+\sigma}{2}} f\|_{H_x^n L_v^2(m)}^2.$$

- If $\mu^{-1/2} \preceq m \prec \mu^{-1}$, there holds

$$(6.9) \quad \langle \bar{\mathcal{B}}f, f \rangle_{H_x^n L_v^2(m)} \lesssim -\|\langle v \rangle^{\frac{\gamma}{2}} \tilde{\nabla}_v(mf)\|_{H_x^n L_v^2}^2 - \|\langle v \rangle^{\frac{\gamma+\sigma}{2}} f\|_{H_x^n L_v^2(m)}^2.$$

Proof. Since the operator $\bar{\mathcal{B}}$ commutes with ∇_x we only need to treat the case $n = 0$. The proof follows the same argument as for the homogeneous case in Lemma 2.3 thanks to the divergence structure of the transport operator. \square

We define the operator

$$(6.10) \quad \bar{\mathcal{B}}_m g = m\bar{\mathcal{B}}(m^{-1}g) = \mathcal{B}_m g - v \cdot \nabla_x g,$$

where we recall that \mathcal{B}_m is defined in (2.16), as well as its formal adjoint operator $\bar{\mathcal{B}}_m^*$ that verifies

$$(6.11) \quad \bar{\mathcal{B}}_m^* \phi = \mathcal{B}_m^* \phi + v \cdot \nabla_x \phi,$$

with \mathcal{B}_m^* defined in (2.17). Observe that if f satisfies $\partial_t f = \bar{\mathcal{B}}f$, then $g = mf$ satisfies $\partial_t g = \bar{\mathcal{B}}_m g$ and $\langle \bar{\mathcal{B}}f, f \rangle_{\mathcal{H}_{x,v}^1(m)} = \langle \bar{\mathcal{B}}_m g, g \rangle_{\mathcal{H}_{x,v}^1}$. Moreover, we have by duality

$$\forall t \geq 0, \quad \langle S_{\bar{\mathcal{B}}_m}(t)g, \phi \rangle_{H_x^n L_v^2} = \langle g, S_{\bar{\mathcal{B}}_m^*}(t)\phi \rangle_{H_x^n L_v^2}.$$

Lemma 6.3. *Let m be an admissible weight function such that $m \succ \langle v \rangle^{(\gamma+3)/2}$ and $n \in \mathbb{N}$. We can choose M, R large enough such that $\bar{\mathcal{B}}_m^*$ is weakly dissipative in $H_x^n L_v^2$ in the sense*

$$\langle \bar{\mathcal{B}}_m^* \phi, \phi \rangle_{H_x^n L_v^2} \lesssim -\|\langle v \rangle^{\frac{\gamma}{2}} \tilde{\nabla}_v \phi\|_{H_x^n L_v^2}^2 - \|\langle v \rangle^{\frac{\gamma+\sigma}{2}} \phi\|_{H_x^n L_v^2}^2.$$

Proof. The proof follows the same arguments as in the proof of Lemma 2.4, thanks to the divergence structure of the transport operator and since ∇_x commutes with $\bar{\mathcal{B}}_m^*$. \square

We turn now to weakly dissipative properties of $\bar{\mathcal{B}}$ in the spaces $\mathcal{H}_{x,v}^1(m)$ defined in (6.3).

Lemma 6.4. *Let m be an admissible weight function such that $m \succ \langle v \rangle^{(\gamma+3)/2}$. For any $\eta > 0$, we define the norm*

$$\|f\|_{\mathcal{H}_{x,v}^1(m)}^2 := \|mf\|_{L_{x,v}^2}^2 + \|\nabla_x(mf)\|_{L_{x,v}^2}^2 + \eta\|\langle v \rangle^\alpha \nabla_v(mf)\|_{L_{x,v}^2}^2,$$

and its associated scalar product $\langle \cdot, \cdot \rangle_{\mathcal{H}_{x,v}^1(m)}$, which is equivalent to the standard $\mathcal{H}_{x,v}^1(m)$ -norm defined in (6.3). There exist $M, R, \eta > 0$ such that $\bar{\mathcal{B}}$ is weakly dissipative in $\mathcal{H}_{x,v}^1(m)$ in the sense

$$\begin{aligned} \langle \bar{\mathcal{B}}f, f \rangle_{\mathcal{H}_{x,v}^1(m)} &\lesssim -\|f\|_{\mathcal{H}_{x,v}^1(m\langle v \rangle^{(\gamma+\sigma)/2})}^2 - \|\langle v \rangle^{\frac{\gamma}{2}} \tilde{\nabla}_v(mf)\|_{L_{x,v}^2}^2 \\ &\quad - \|\langle v \rangle^{\frac{\gamma}{2}} \tilde{\nabla}_v(\nabla_x(mf))\|_{L_{x,v}^2}^2 - \eta\|\langle v \rangle^{\frac{\gamma}{2}+\alpha} \tilde{\nabla}_v(\nabla_v(mf))\|_{L_{x,v}^2}^2. \end{aligned}$$

Proof. We remark that we have introduced the spaces (6.3), in which the term $\nabla_v(mf)$ has a weight $\langle v \rangle^\alpha$ with $\alpha < 0$, in order to treat the terms coming from the derivative in the v -variable of the transport operator.

For the sake of simplicity, we shall equivalently prove that

$$\begin{aligned} &\frac{d}{dt} \left(\|g_{\bar{\mathcal{B}}_m}\|_{L_{x,v}^2}^2 + \|\nabla_x g_{\bar{\mathcal{B}}_m}\|_{L_{x,v}^2}^2 + \eta\|\langle v \rangle^\alpha \nabla_v g_{\bar{\mathcal{B}}_m}\|_{L_{x,v}^2}^2 \right) \\ &\lesssim - \left(\|\langle v \rangle^{\frac{\gamma+\sigma}{2}} g_{\bar{\mathcal{B}}_m}\|_{L_{x,v}^2}^2 + \|\langle v \rangle^{\frac{\gamma+\sigma}{2}} \nabla_x g_{\bar{\mathcal{B}}_m}\|_{L_{x,v}^2}^2 + \eta\|\langle v \rangle^{\frac{\gamma+\sigma}{2}+\alpha} \nabla_v g_{\bar{\mathcal{B}}_m}\|_{L_{x,v}^2}^2 \right) \\ &\quad - \|\langle v \rangle^{\frac{\gamma}{2}} \tilde{\nabla}_v g_{\bar{\mathcal{B}}_m}\|_{L_{x,v}^2}^2 - \|\langle v \rangle^{\frac{\gamma}{2}} \tilde{\nabla}_v(\nabla_x g_{\bar{\mathcal{B}}_m})\|_{L_{x,v}^2}^2 - \eta\|\langle v \rangle^{\frac{\gamma}{2}+\alpha} \tilde{\nabla}_v(\nabla_v g_{\bar{\mathcal{B}}_m})\|_{L_{x,v}^2}^2, \end{aligned}$$

for any solution $g_{\bar{\mathcal{B}}_m}$ to the equation $\partial_t g_{\bar{\mathcal{B}}_m} = \bar{\mathcal{B}}_m g_{\bar{\mathcal{B}}_m}$, so that, with $g_{\bar{\mathcal{B}}} = mf_{\bar{\mathcal{B}}}$, $f_{\bar{\mathcal{B}}}$ is a solution to $\partial_t f_{\bar{\mathcal{B}}} = \bar{\mathcal{B}}f_{\bar{\mathcal{B}}}$. We now use the shorthand $g = g_{\bar{\mathcal{B}}_m}$ and split the proof into three steps.

Step 1. We first obtain from Lemma 6.2 (for $M, R > 0$ large enough)

$$(6.12) \quad \frac{d}{dt} \|g\|_{L^2_{x,v}}^2 \lesssim -\|\langle v \rangle^{\frac{\gamma}{2}} \tilde{\nabla}_v g\|_{L^2_{x,v}}^2 - \|\langle v \rangle^{\frac{\gamma+\sigma}{2}} g\|_{L^2_{x,v}}^2$$

and

$$(6.13) \quad \frac{d}{dt} \|\nabla_x g\|_{L^2_{x,v}}^2 \lesssim -\|\langle v \rangle^{\frac{\gamma}{2}} \tilde{\nabla}_v (\nabla_x g)\|_{L^2_{x,v}}^2 - \|\langle v \rangle^{\frac{\gamma+\sigma}{2}} \nabla_x g\|_{L^2_{x,v}}^2.$$

Step 2. We write

$$\frac{1}{2} \frac{d}{dt} \|\langle v \rangle^\alpha \nabla_v g\|_{L^2_{x,v}}^2 = \int_{x,v} \nabla_v (\mathcal{B}_m g) \cdot \nabla_v g \langle v \rangle^{2\alpha} - \int_{x,v} \nabla_x g \cdot \nabla_v g \langle v \rangle^{2\alpha}.$$

where the equation satisfied by $\nabla_v g$ is

$$\partial_t \nabla_v g = \mathcal{B}_m (\nabla_v g) + (\nabla_v \bar{a}_{ij}) \partial_{ij} g + (\nabla_v \beta_j) \partial_j g + (\nabla_v \delta - M \nabla_v \chi_R) g - \nabla_x g.$$

We first compute

$$\int_{x,v} \nabla_v (\mathcal{B}_m g) \cdot \nabla_v g \langle v \rangle^{2\alpha} =: T_1 + T_2 + T_3 + T_4,$$

where

$$\begin{aligned} T_1 &= \int (\mathcal{B}_m \nabla_v g) \cdot \nabla_v g \langle v \rangle^{2\alpha}, & T_2 &= \int (\nabla_v \bar{a}_{ij}) \partial_{ij} g \nabla_v g \langle v \rangle^{2\alpha}, \\ T_3 &= \int (\nabla_v \beta_j) \partial_j g \nabla_v g \langle v \rangle^{2\alpha}, & T_4 &= \int (\nabla_v \delta - M \nabla_v \chi_R) g \nabla_v g \langle v \rangle^{2\alpha}. \end{aligned}$$

From Lemma 6.2, we have

$$T_1 \leq -K \|\langle v \rangle^{\frac{\gamma}{2} + \alpha} \tilde{\nabla}_v (\nabla_v g)\|_{L^2}^2 + \int \{\tilde{\zeta}_m(v) - M \chi_R(v)\} |\nabla_v g|^2 \langle v \rangle^{2\alpha}.$$

Terms T_3 and T_4 are easy to estimate. As in the proof of Lemma 2.2, we can compute explicitly $\beta_j(v)$ and $\delta(v)$, thus we easily deduce

$$|\nabla_v \beta_j(v)| + |\nabla_v \delta(v)| \lesssim \langle v \rangle^{\gamma + \sigma - 1}.$$

Therefore

$$T_3 + T_4 \lesssim \int \{\langle v \rangle^{\gamma + \sigma - 1} + \frac{M}{R} \mathbf{1}_{R \leq |v| \leq 2R}\} |\nabla_v g|^2 \langle v \rangle^{2\alpha} + \int \{\langle v \rangle^{\gamma + \sigma - 1} + \frac{M}{R} \mathbf{1}_{R \leq |v| \leq 2R}\} g^2 \langle v \rangle^{2\alpha}.$$

Thanks to Lemma 2.2, for $M, R > 0$ large enough, we have

$$T_1 + T_3 + T_4 \lesssim -\|\langle v \rangle^{\frac{\gamma}{2} + \alpha} \tilde{\nabla}_v (\nabla_v g)\|_{L^2_{x,v}}^2 - \|\langle v \rangle^{\frac{\gamma+\sigma}{2} + \alpha} \nabla_v g\|_{L^2_{x,v}}^2 + \|\langle v \rangle^{\frac{\gamma+\sigma-1}{2} + \alpha} g\|_{L^2_{x,v}}^2.$$

Thanks to an integration by parts, we first obtain

$$\begin{aligned} T_2 &= - \int (\nabla_v \bar{b}_j) \partial_j g \nabla_v g \langle v \rangle^{2\alpha} - \int (\nabla_v \bar{a}_{ij}) \partial_j g \partial_i \nabla_v g \langle v \rangle^{2\alpha} - \int (\nabla_v \bar{a}_{ij}) \partial_j g \nabla_v g \partial_i \langle v \rangle^{2\alpha} \\ &=: U + V + W. \end{aligned}$$

Thanks to Lemma 2.1, we easily have

$$U + W \lesssim \|\langle v \rangle^{\frac{\gamma}{2} + \alpha} \nabla_v g\|_{L^2_{x,v}}^2.$$

We make another integration by parts for V (now with respect to ∇_v), we get

$$V = \int (\Delta_v \bar{a}_{ij}) \partial_i g \partial_j g \langle v \rangle^{2\alpha} + \int (\nabla_v \bar{a}_{ij}) \partial_i g \partial_j \nabla_v g \langle v \rangle^{2\alpha} + \int (\nabla_v \bar{a}_{ij}) \partial_i g \partial_j g \nabla_v \langle v \rangle^{2\alpha},$$

and we recognize that the middle term is equal to $-V$, so that

$$V = \frac{1}{2} \int (\Delta_v \bar{a}_{ij}) \partial_i g \partial_j g \langle v \rangle^{2\alpha} + \frac{1}{2} \int (\nabla_v \bar{a}_{ij}) \partial_i g \partial_j g \nabla_v \langle v \rangle^{2\alpha} \lesssim \|\langle v \rangle^{\frac{\gamma}{2} + \alpha} \nabla_v g\|_{L^2_{x,v}}^2.$$

We finally obtain (for $M, R > 0$ large enough)

$$(6.14) \quad \int_{x,v} \nabla_v (\mathcal{B}_m g) \cdot \nabla_v g \langle v \rangle^{2\alpha} \lesssim -\|\langle v \rangle^{\frac{\gamma}{2}+\alpha} \tilde{\nabla}_v (\nabla_v g)\|_{L_{x,v}^2}^2 - \|\langle v \rangle^{\frac{\gamma+\sigma}{2}+\alpha} \nabla_v g\|_{L_{x,v}^2}^2 \\ + \|\langle v \rangle^{\frac{\gamma+\sigma-1}{2}+\alpha} g\|_{L_{x,v}^2}^2 + \|\langle v \rangle^{\frac{\gamma}{2}+\alpha} \tilde{\nabla}_v g\|_{L_{x,v}^2}^2.$$

By Cauchy-Schwarz inequality, we also get

$$(6.15) \quad \int_{x,v} \nabla_x g \cdot \nabla_v g \langle v \rangle^{2\alpha} \lesssim \eta^{-1/2} \|\langle v \rangle^{\frac{\gamma+\sigma}{2}} \nabla_x g\|_{L_{x,v}^2}^2 + \eta^{1/2} \|\langle v \rangle^{2\alpha-\frac{\gamma+\sigma}{2}} \nabla_v g\|_{L_{x,v}^2}^2.$$

Remark that the first term in the right-hand side of (6.15) can be controlled by the second term in the right-hand side of (6.13), as well as

$$2\alpha - \frac{\gamma + \sigma}{2} = \frac{\gamma}{2} \quad \text{if} \quad \frac{\gamma}{2} + \frac{\sigma}{4} \geq \gamma + \sigma, \\ 2\alpha - \frac{\gamma + \sigma}{2} = \frac{\gamma + \sigma}{2} + \alpha = \frac{3}{2}(\gamma + \sigma) \quad \text{if} \quad \frac{\gamma}{2} + \frac{\sigma}{4} < \gamma + \sigma.$$

As a consequence, the last term in (6.15) can be controlled by the first term in the right-hand side of (6.12) or by the second term in the right-hand-side of (6.14).

Step 3. Putting together previous estimates, it follows, for any $\eta > 0$,

$$\begin{aligned} \frac{d}{dt} \|g\|_{\tilde{\mathcal{H}}_{x,v}^1}^2 &\lesssim -\|\langle v \rangle^{\frac{\gamma+\sigma}{2}} g\|_{L_{x,v}^2}^2 + \eta \|\langle v \rangle^{\frac{\gamma+\sigma-1}{2}+\alpha} g\|_{L_{x,v}^2}^2 \\ &\quad - \|\langle v \rangle^{\frac{\gamma}{2}} \tilde{\nabla}_v g\|_{L_{x,v}^2}^2 + \eta \|\langle v \rangle^{\frac{\gamma}{2}+\alpha} \tilde{\nabla}_v g\|_{L_{x,v}^2}^2 \\ &\quad - \|\langle v \rangle^{\frac{\gamma+\sigma}{2}} \nabla_x g\|_{L_{x,v}^2}^2 + \eta^{1/2} \|\langle v \rangle^{\frac{\gamma+\sigma}{2}} \nabla_x g\|_{L_{x,v}^2}^2 \\ &\quad - \|\langle v \rangle^{\frac{\gamma}{2}} \tilde{\nabla}_v g\|_{L_{x,v}^2}^2 - \eta \|\langle v \rangle^{\frac{\gamma+\sigma}{2}+\alpha} \nabla_v g\|_{L_{x,v}^2}^2 + \eta^{3/2} \|\langle v \rangle^{2\alpha-\frac{\gamma+\sigma}{2}} \nabla_v g\|_{L_{x,v}^2}^2 \\ &\quad - \|\langle v \rangle^{\frac{\gamma}{2}} \tilde{\nabla}_v (\nabla_x g)\|_{L_{x,v}^2}^2 - \eta \|\langle v \rangle^{\frac{\gamma}{2}+\alpha} \tilde{\nabla}_v (\nabla_v g)\|_{L_{x,v}^2}^2, \end{aligned}$$

and we conclude the proof by taking $\eta > 0$ small enough. \square

Corollary 6.5. *Let m_0, m_1 be admissible weight functions such that $m_1 \succ m_0 \succ \langle v \rangle^{(\gamma+3)/2}$. There hold*

$$(6.16) \quad \|S_{\mathcal{B}}(t)\|_{H_x^2 L_v^2(m_1) \rightarrow H_x^2 L_v^2(m_0)} \lesssim \Theta_{m_1, m_0}(t), \quad \forall t \geq 0,$$

$$(6.17) \quad \|S_{\mathcal{B}}(t)\|_{\mathcal{H}_{x,v}^1(m_1) \rightarrow \mathcal{H}_{x,v}^1(m_0)} \lesssim \Theta_{m_1, m_0}(t), \quad \forall t \geq 0.$$

Moreover, the following regularity estimate holds

$$(6.18) \quad \int_0^\infty \|S_{\mathcal{B}}(t)\|_{H_x^2(H_{v,*}^{-1}(m_0)) \rightarrow H_x^2 L_v^2(m_0)}^2 dt < \infty.$$

Proof. The proof follows the same arguments of Lemma 3.1, using the weakly dissipative estimates of Lemmas 6.2, 6.3 and 6.4. \square

6.5. Regularisation properties of $S_{\mathcal{B}}$ and $(\mathcal{A}S_{\mathcal{B}})^{(*n)}$. We start proving regularisation properties of the semigroup $S_{\mathcal{B}}$ in some large weighted Lebesgue and Sobolev spaces in the spirit of Hérau's quantitative version [22] of the Hörmander hypoellipticity property of the kinetic Fokker-Planck equation.

Lemma 6.6. *Let m, m_1 be admissible polynomial weight functions such that $\langle v \rangle^{3/2} \prec m_1 \prec m$ with $m_1 \prec \mu^{-1/2}$. For any $n \in \mathbb{N}^*$, there holds*

$$(6.19) \quad \forall t \geq 0, \quad \|S_{\mathcal{B}}(t)\|_{L_{x,v}^2(m) \rightarrow H_{x,v}^n(m_1 \langle v \rangle^{\gamma/2})} \lesssim \frac{\Theta_{m, m_1}(t)}{t^{3n/2} \wedge 1}.$$

Proof. We prove (6.19) in the case $n = 1$. Let us denote $m_0 := m_1 \langle v \rangle^{\gamma/2}$ and $f_t = \mathcal{S}_{\bar{\mathcal{B}}}(t)f$. Define $g_t^0 = m_0 f_t$ and $g_t^1 = m_1 f_t$, which verify $g_t^0 = \mathcal{S}_{\bar{\mathcal{B}}_{m_0}}(t)g^0$ and $g_t^1 = \mathcal{S}_{\bar{\mathcal{B}}_{m_1}}(t)g^1$. We define the functional

$$\mathcal{F}(t) := \|g_t^1\|_{L_{x,v}^2}^2 + \alpha_1 t \|\nabla_v g_t^0\|_{L_{x,v}^2}^2 + \alpha_2 t^2 \langle \nabla_x g_t^0, \nabla_v g_t^0 \rangle_{L_{x,v}^2} + \alpha_3 t^3 \|\nabla_x g_t^0\|_{L_{x,v}^2}^2,$$

and choose α_i , $i = 1, 2, 3$ such that $0 < \alpha_3 \leq \alpha_2 \leq \alpha_1 \leq 1$ and $\alpha_2^2 \leq 2\alpha_1\alpha_3$. Then, there holds

$$\forall t \in [0, 1], \quad \mathcal{F}(t) \gtrsim \alpha_3 t^3 \|\nabla_{x,v} g_t^0\|_{L_{x,v}^2}^2.$$

We have moreover

$$\begin{aligned} \frac{d}{dt} \mathcal{F}(t) &= \frac{d}{dt} \|g_t^1\|_{L_{x,v}^2}^2 + \alpha_1 \|\nabla_v g_t^0\|_{L_{x,v}^2}^2 + \alpha_1 t \frac{d}{dt} \|\nabla_v g_t^0\|_{L_{x,v}^2}^2 \\ &\quad + 2\alpha_2 t \langle \nabla_x g_t^0, \nabla_v g_t^0 \rangle_{L_{x,v}^2} + \alpha_2 t^2 \frac{d}{dt} \langle \nabla_x g_t^0, \nabla_v g_t^0 \rangle_{L_{x,v}^2} \\ &\quad + 3\alpha_3 t^2 \|\nabla_x g_t^0\|_{L_{x,v}^2}^2 + \alpha_3 t^3 \frac{d}{dt} \|\nabla_x g_t^0\|_{L_{x,v}^2}^2. \end{aligned}$$

Recall that $\bar{\mathcal{B}}_m$ is defined in (6.10), so that we compute

$$\begin{aligned} \frac{d}{dt} \langle \nabla_x g^0, \nabla_v g^0 \rangle_{L_{x,v}^2} &= \int \bar{a}_{ij} \partial_{ij} (\nabla_x g^0) \nabla_v g^0 + \beta_j \partial_j (\nabla_x g^0) \nabla_v g^0 + (\delta - M\chi_R) \nabla_x g^0 \nabla_v g^0 \\ &\quad + \int \bar{a}_{ij} \partial_{ij} (\nabla_v g^0) \nabla_x g^0 + \beta_j \partial_j (\nabla_v g^0) \nabla_x g^0 + (\delta - M\chi_R) \nabla_v g^0 \nabla_x g^0 \\ &\quad + \int (\nabla_v \bar{a}_{ij}) \partial_{ij} g^0 \nabla_x g^0 + (\nabla_v \beta_j) \partial_j g^0 \nabla_x g^0 + \nabla_v (\delta - M\chi_R) g^0 \nabla_x g^0. \end{aligned}$$

Integrating by parts in last expression, we obtain (with the same type of arguments as in Lemma 6.4)

$$\begin{aligned} \frac{d}{dt} \langle \nabla_x g^0, \nabla_v g^0 \rangle_{L_{x,v}^2} &= -2 \int \bar{a}_{ij} \partial_i (\nabla_x g^0) \partial_j (\nabla_v g^0) + \int \{\partial_j \beta_j + \bar{c} + 2\delta - 2M\chi_R\} \nabla_x g^0 \nabla_v g^0 \\ &\quad + \int \nabla_v (\beta_j - \bar{b}_j) \partial_j g^0 \nabla_x g^0 + \frac{1}{2} \int (\Delta_v \bar{a}_{ij}) \partial_i g^0 \partial_j g^0 - \|\nabla_x g^0\|_{L_{x,v}^2}^2. \end{aligned}$$

From that equation, we deduce

$$\begin{aligned} \frac{d}{dt} \langle \nabla_x g_t^0, \nabla_v g_t^0 \rangle_{L_{x,v}^2} &\lesssim \|\langle v \rangle^{\frac{\gamma}{2}} \tilde{\nabla}_v (\nabla_x g_t^0)\|_{L_{x,v}^2} \|\langle v \rangle^{\frac{\gamma}{2}} \tilde{\nabla}_v (\nabla_v g_t^0)\|_{L_{x,v}^2} \\ (6.20) \quad &\quad + \|\langle v \rangle^{\frac{\gamma+\sigma_0}{2}} \nabla_x g_t^0\|_{L_{x,v}^2} \|\langle v \rangle^{\frac{\gamma+\sigma_0}{2}} \nabla_v g_t^0\|_{L_{x,v}^2} \\ &\quad + \|\langle v \rangle^{\frac{\gamma}{2}} \nabla_v g_t^0\|_{L_{x,v}^2}^2 - \|\nabla_x g_t^0\|_{L_{x,v}^2}^2. \end{aligned}$$

Recall that from Lemma 6.2, we already have

$$(6.21) \quad \frac{d}{dt} \|g_t^1\|_{L_{x,v}^2}^2 \lesssim -\|\langle v \rangle^{\frac{\gamma}{2}} \tilde{\nabla}_v g_t^1\|_{L_{x,v}^2}^2 - \|\langle v \rangle^{\frac{\gamma}{2}} m_1 \tilde{\nabla}_v f_t\|_{L_{x,v}^2}^2 - \|\langle v \rangle^{\frac{\gamma+\sigma_1}{2}} g_t^1\|_{L_{x,v}^2}^2.$$

Moreover, thanks to the proof of Lemma 6.4, we get

$$\begin{aligned} \frac{d}{dt} \|\nabla_v g_t^0\|_{L_{x,v}^2}^2 &\lesssim -\|\langle v \rangle^{\frac{\gamma}{2}} \tilde{\nabla}_v (\nabla_v g_t^0)\|_{L_{x,v}^2}^2 - \|\langle v \rangle^{\frac{\gamma+\sigma_0}{2}} \nabla_v g_t^0\|_{L_{x,v}^2}^2 \\ (6.22) \quad &\quad + \|\langle v \rangle^{\frac{\gamma+\sigma_0-1}{2}} g_t^0\|_{L_{x,v}^2}^2 + \|\langle v \rangle^{\frac{\gamma}{2}} \tilde{\nabla}_v g_t^0\|_{L_{x,v}^2}^2 \\ &\quad + \|\nabla_x g_t^0\|_{L_{x,v}^2} \|\nabla_v g_t^0\|_{L_{x,v}^2}. \end{aligned}$$

Using Lemma 6.2 and the fact that ∇_x commutes with $\bar{\mathcal{B}}$, we also have

$$(6.23) \quad \frac{d}{dt} \|\nabla_x g_t^0\|_{L_{x,v}^2}^2 \lesssim -\|\langle v \rangle^{\frac{\gamma}{2}} \tilde{\nabla}_v (\nabla_x g_t^0)\|_{L_{x,v}^2}^2 - \|\langle v \rangle^{\frac{\gamma+\sigma_0}{2}} \nabla_x g_t^0\|_{L_{x,v}^2}^2.$$

Let us denote by D_1 the absolute value of the dissipative terms in (6.21), by D_2 the absolute value of the dissipative terms in (6.22), by D_3 the absolute value of the dissipative terms in (6.20), and finally by D_4 the absolute value of the dissipative terms in (6.23). Observe that

$$\|\nabla_v g_t^0\|_{L_{x,v}^2}^2 + \|\langle v \rangle^{\frac{\gamma}{2}} \tilde{\nabla}_v g_t^0\|_{L_{x,v}^2}^2 \lesssim D_1.$$

Gathering estimates (6.20), (6.21), (6.22) and (6.23), we obtain, for any $t \in (0, 1]$,

$$\begin{aligned} \frac{d}{dt} \mathcal{F}(t) &\lesssim (-1 + \alpha_1 + \alpha_1 t + \alpha_2 t^2) D_1 + (\alpha_1 t + \alpha_2 t + \alpha_2 t^2) D_1^{1/2} D_3^{1/2} \\ &\quad - \alpha_1 t D_2 - \alpha_2 t^2 D_3 + \alpha_2 t^2 D_2^{1/2} D_4^{1/2} + \alpha_3 t^2 D_3 - \alpha_3 t^3 D_4 \\ &\lesssim (-1 + \alpha_1 + \alpha_2) D_1 + (\alpha_1 + \alpha_2) t D_1^{1/2} D_3^{1/2} \\ &\quad - \alpha_1 t D_2 - \alpha_2 t^2 D_3 + \alpha_2 t^2 D_2^{1/2} D_4^{1/2} + \alpha_3 t^2 D_3 - \alpha_3 t^3 D_4. \end{aligned}$$

Using Cauchy-Schwarz inequality it follows, for $t \in (0, 1]$,

$$\frac{d}{dt} \mathcal{F}(t) \lesssim (-1 + \alpha_1 + \alpha_2 + \frac{\alpha_1^2}{\alpha_3} + \frac{\alpha_2^2}{\alpha_3}) D_1 + t(-\alpha_1 + \frac{\alpha_2^2}{\alpha_3^{5/4}}) D_2 + t^2(-\alpha_2 + \alpha_3) D_3 + t^3(-\alpha_3 + \alpha_3^{5/4}) D_4,$$

and choosing $\alpha_1 = \epsilon$, $\alpha_2 = \epsilon^{5/4}$, and $\alpha_3 = \epsilon^{4/3}$, we easily conclude to, for $\epsilon > 0$ small enough,

$$\forall t \in (0, 1], \quad \frac{d}{dt} \mathcal{F}(t) \leq 0.$$

This implies, coming back to the function $f_t = S_{\bar{\mathcal{B}}}(t)f$,

$$\forall t \in (0, 1], \quad t^3 \|\nabla_{x,v}(m_0 f_t)\|_{L_{x,v}^2}^2 \lesssim \mathcal{F}(t) \leq \mathcal{F}(0) = \|f_0\|_{L_{x,v}^2(m_1)}^2.$$

From that inequality together with (6.16) from Corollary 6.5, we deduce

$$\forall t \in (0, 1], \quad \|S_{\bar{\mathcal{B}}}(t)f\|_{H_{x,v}^1(m_1 \langle v \rangle^{\gamma/2})} \lesssim t^{-3/2} \|f\|_{L_{x,v}^2(m_1)},$$

which already gives (6.19) for small times $t \in (0, 1]$. For $t > 1$ and $m \succ m_1$ (recall that $m_1 \langle v \rangle^{\gamma/2} \succ \langle v \rangle^{(\gamma+3)/2}$) we write, using again (6.16),

$$\begin{aligned} \|S_{\bar{\mathcal{B}}}(t)f\|_{H_{x,v}^1(m_1 \langle v \rangle^{\gamma/2})} &= \|S_{\bar{\mathcal{B}}}(1)(S_{\bar{\mathcal{B}}}(t-1)f)\|_{H_{x,v}^1(m_1 \langle v \rangle^{\gamma/2})} \\ &\lesssim \|S_{\bar{\mathcal{B}}}(t-1)f\|_{L_{x,v}^2(m_1)} \\ &\lesssim \Theta_{m,m_1}(t) \|f\|_{L_{x,v}^2(m)}, \end{aligned}$$

which completes the proof of (6.19) when $n = 1$. The proof of the general case (6.19) follows by differentiating the equation and applying the same arguments as above. \square

As a consequence of Lemma 2.5, we also obtain an analogous result for high-order Sobolev spaces.

Corollary 6.7. *For any $\theta \in (0, 1)$ and $n \in \mathbb{N}$, there hold $\mathcal{A} \in \mathcal{B}(H_x^n L_v^2, H_x^n L_v^2(\mu^{-\theta}))$ and $\mathcal{A} \in \mathcal{B}(H_{x,v}^n, H_{x,v}^n(\mu^{-\theta}))$.*

We finally obtain the following regularity properties, as a consequence of Corollary 6.5, Lemma 6.6 and Corollary 6.7.

Corollary 6.8. *Let m, ω be admissible weight functions such that $\langle v \rangle^{3/2} \prec m \prec \omega$ and $\mu^{-1/2} \prec \omega \prec \mu^{-1}$. There hold*

$$(6.24) \quad \|(\mathcal{A}S_{\bar{\mathcal{B}}})^{(*2)}(t)\|_{L_{x,v}^2(\omega) \rightarrow \mathcal{H}_{x,v}^1(\omega)} \in L_t^1(\mathbb{R}_+).$$

and

$$(6.25) \quad \|(S_{\bar{\mathcal{B}}}\mathcal{A})^{(*4)}(t)\|_{L_{x,v}^2(m) \rightarrow H_{x,v}^2(m)} \in L_t^1(\mathbb{R}_+).$$

Proof. Observe that with the choice of the weight ω we have $\Theta_{\omega,m}(t) \simeq e^{-\lambda t^{2/|\gamma|}} \in L_t^1(\mathbb{R}_+)$.

Step 1. Thanks to Corollary 6.5 and Corollary 6.7, we already have,

$$\|\mathcal{A}S_{\bar{\mathcal{B}}}(t)\|_{L_{x,v}^2(\omega) \rightarrow L_{x,v}^2(\omega)} \lesssim \|\mathcal{A}\|_{L_{x,v}^2(m) \rightarrow L_{x,v}^2(\omega)} \|S_{\bar{\mathcal{B}}}(t)\|_{L_{x,v}^2(\omega) \rightarrow L_{x,v}^2(m)} \lesssim \Theta_{\omega,m}(t)$$

and

$$\|S_{\bar{\mathcal{B}}}\mathcal{A}(t)\|_{L_{x,v}^2(m) \rightarrow L_{x,v}^2(m)} \lesssim \|S_{\bar{\mathcal{B}}}(t)\|_{L_{x,v}^2(\omega) \rightarrow L_{x,v}^2(m)} \|\mathcal{A}\|_{L_{x,v}^2(m) \rightarrow L_{x,v}^2(\omega)} \lesssim \Theta_{\omega,m}(t)$$

so that, for any $j \in \mathbb{N}^*$,

$$\|(\mathcal{A}S_{\bar{\mathcal{B}}})^{(*j)}(t)\|_{L_{x,v}^2(\omega) \rightarrow L_{x,v}^2(\omega)}, \|(S_{\bar{\mathcal{B}}}\mathcal{A})^{(*j)}(t)\|_{L_{x,v}^2(m) \rightarrow L_{x,v}^2(m)} \lesssim \Theta_{\omega,m}(t),$$

and similarly

$$\|(\mathcal{A}S_{\bar{\mathcal{B}}})^{(*j)}(t)\|_{\mathcal{H}_{x,v}^1(\omega) \rightarrow \mathcal{H}_{x,v}^1(\omega)}, \|(S_{\bar{\mathcal{B}}}\mathcal{A})^{(*j)}(t)\|_{\mathcal{H}_{x,v}^1(m) \rightarrow \mathcal{H}_{x,v}^1(m)} \lesssim \Theta_{\omega,m}(t).$$

Step 2. We prove (6.24). We first write

$$\begin{aligned} & \|(\mathcal{A}S_{\bar{\mathcal{B}}})^{(*2)}(t)\|_{L_{x,v}^2(\omega) \rightarrow \mathcal{H}_{x,v}^1(\omega)} \\ & \lesssim \int_0^{t/2} \|\mathcal{A}S_{\bar{\mathcal{B}}}(t-s)\mathcal{A}S_{\bar{\mathcal{B}}}(s)\|_{L_{x,v}^2(\omega) \rightarrow \mathcal{H}_{x,v}^1(\omega)} ds \\ & \quad + \int_{t/2}^t \|\mathcal{A}S_{\bar{\mathcal{B}}}(t-s)\mathcal{A}S_{\bar{\mathcal{B}}}(s)\|_{L_{x,v}^2(\omega) \rightarrow \mathcal{H}_{x,v}^1(\omega)} ds =: I_1 + I_2. \end{aligned}$$

Using Corollary 6.5, (6.19) of Lemma 6.6, Corollary 6.7 and Step 1, we have

$$\begin{aligned} I_1 & \lesssim \int_0^{t/2} \|\mathcal{A}\|_{H_{x,v}^1(m\langle v \rangle^{\gamma/2}) \rightarrow \mathcal{H}_{x,v}^1(\omega)} \|S_{\bar{\mathcal{B}}}(t-s)\|_{L_{x,v}^2(\omega) \rightarrow H_{x,v}^1(m\langle v \rangle^{\gamma/2})} \|\mathcal{A}S_{\bar{\mathcal{B}}}(s)\|_{L_{x,v}^2(\omega) \rightarrow L_{x,v}^2(\omega)} ds \\ & \lesssim \int_0^{t/2} \frac{\Theta_{\omega,m}(t-s)}{(t-s)^{3/2} \wedge 1} \Theta_{\omega,m}(s) ds \lesssim \Theta_{\omega,m}(t/2) \int_0^{t/2} \frac{\Theta_{\omega,m}(s)}{(t-s)^{3/2} \wedge 1} ds \lesssim \frac{\Theta_{\omega,m}(t)}{t^{1/2} \wedge 1} \in L_t^1(\mathbb{R}_+), \end{aligned}$$

where in the last line the decay function $\Theta_{\omega,m_1}(t)$ is defined with another positive constant $\lambda' < \lambda$, i.e. $\Theta_{\omega,m_1}(t) \simeq e^{-\lambda' t^{2/|\gamma|}}$. For the other term I_2 , we use again Corollary 6.5, (6.19) of Lemma 6.6, Corollary 6.7 and Step 1, but in a different order, to obtain

$$\begin{aligned} I_2 & \lesssim \int_{t/2}^t \|\mathcal{A}S_{\bar{\mathcal{B}}}(t-s)\|_{\mathcal{H}_{x,v}^1(\omega) \rightarrow \mathcal{H}_{x,v}^1(\omega)} \|\mathcal{A}\|_{H_{x,v}^1(m\langle v \rangle^{\gamma/2}) \rightarrow \mathcal{H}_{x,v}^1(\omega)} \|S_{\bar{\mathcal{B}}}(s)\|_{L_{x,v}^2(\omega) \rightarrow H_{x,v}^1(m\langle v \rangle^{\gamma/2})} ds \\ & \lesssim \int_{t/2}^t \Theta_{\omega,m}(t-s) \frac{\Theta_{\omega,m}(s)}{s^{3/2} \wedge 1} ds \lesssim \Theta_{\omega,m}(t/2) \int_{t/2}^t \frac{\Theta_{\omega,m}(t-s)}{s^{3/2} \wedge 1} ds \lesssim \frac{\Theta_{\omega,m}(t)}{t^{1/2} \wedge 1} \in L_t^1(\mathbb{R}_+), \end{aligned}$$

where again $\Theta_{\omega,m}(t)$ in the last line is defined with a different choice of constants, and the proof of (6.24) is complete.

Step 3. We now turn to the proof of (6.25). We claim that, for any $j \in \mathbb{N}$, there holds

$$(6.26) \quad \|(S_{\bar{\mathcal{B}}}\mathcal{A})^{*(j+1)}(t)\|_{L_{x,v}^2(m) \rightarrow H_{x,v}^n(m)} \lesssim \frac{\Theta_{\omega,m}(t)}{t^{3n/2-j} \wedge 1},$$

so that we can conclude to (6.25) by choosing $j = 3$ when $n = 2$.

The case $j = 0$ follows directly from Lemma 6.6 and Corollary 6.7, and we prove the claim by induction. Suppose that (6.26) holds for some j then we compute, splitting again the integral

into two parts,

$$\begin{aligned} \|(S_{\bar{\mathcal{B}}}\mathcal{A})^{*(j+2)}(t)\|_{L_{x,v}^2(m) \rightarrow H_x^n L_v^2(m)} &\lesssim \int_0^{t/2} \|(S_{\bar{\mathcal{B}}}\mathcal{A})^{*(j+1)}(t-s)S_{\bar{\mathcal{B}}}\mathcal{A}(s)\|_{L_{x,v}^2(m) \rightarrow H_x^n L_v^2(m)} ds \\ &\quad + \int_{t/2}^t \|S_{\bar{\mathcal{B}}}\mathcal{A}(t-s)(S_{\bar{\mathcal{B}}}\mathcal{A})^{*(j+1)}(s)\|_{L_{x,v}^2(m) \rightarrow H_x^n L_v^2(m)} ds \\ &=: T_1 + T_2. \end{aligned}$$

In a similar way as in Step 2, using Corollary 6.5, (6.19) of Lemma 6.6, Corollary 6.7 and Step 1, we obtain

$$\begin{aligned} T_1 &\lesssim \int_0^{t/2} \|(S_{\bar{\mathcal{B}}}\mathcal{A})^{*(j+1)}(t-s)\|_{L_{x,v}^2(m) \rightarrow H_{x,v}^n(m)} \|S_{\bar{\mathcal{B}}}\mathcal{A}(s)\|_{L_{x,v}^2(m) \rightarrow L_{x,v}^2(m)} ds \\ &\lesssim \int_0^{t/2} \frac{\Theta_{\omega,m}(t-s)}{(t-s)^{3n/2-j} \wedge 1} \Theta_{\omega,m}(s) ds \lesssim \frac{\Theta_{\omega,m}(t)}{t^{3n/2-(j+1)} \wedge 1}. \end{aligned}$$

Moreover,

$$\begin{aligned} T_2 &\lesssim \int_{t/2}^t \|S_{\bar{\mathcal{B}}}\mathcal{A}(t-s)\|_{L_{x,v}^2(m) \rightarrow L_{x,v}^2(m)} \|(S_{\bar{\mathcal{B}}}\mathcal{A})^{*(j+1)}(s)\|_{L_{x,v}^2(m) \rightarrow H_{x,v}^n(m)} ds \\ &\lesssim \int_{t/2}^t \Theta_{\omega,m}(t-s) \frac{\Theta_{\omega,m}(s)}{s^{3n/2-j} \wedge 1} ds \lesssim \frac{\Theta_{\omega,m}(t)}{t^{3n/2-(j+1)} \wedge 1}, \end{aligned}$$

which completes the proof. \square

6.6. Decay of the semigroup $S_{\bar{\mathcal{L}}}$. With the results above we obtain the decay of the semigroup $S_{\bar{\mathcal{L}}}\bar{\Pi}$ in large spaces as considered in the statement of Theorem 1.1.

We first write a semigroup factorization. Recall that $\bar{\mathcal{L}} = \mathcal{A} + \bar{\mathcal{B}}$ and that $\bar{\Pi}$ commutes with $\bar{\mathcal{L}}$. For any $\ell, n \in \mathbb{N}^*$, we can write the iterated Duhamel formulas

$$\begin{aligned} \bar{\Pi}S_{\bar{\mathcal{L}}} &= \sum_{0 \leq j \leq \ell-1} \bar{\Pi}S_{\bar{\mathcal{B}}} * (\mathcal{A}S_{\bar{\mathcal{B}}})^{*(j)} + \bar{\Pi}S_{\bar{\mathcal{L}}} * (\mathcal{A}S_{\bar{\mathcal{B}}})^{*(\ell)} \\ S_{\Lambda}\bar{\Pi} &= \sum_{0 \leq i \leq n-1} (S_{\bar{\mathcal{B}}}\mathcal{A})^{*(i)} * S_{\bar{\mathcal{B}}}\bar{\Pi} + (S_{\bar{\mathcal{B}}}\mathcal{A})^{*(n)} * S_{\bar{\mathcal{L}}}\bar{\Pi}, \end{aligned}$$

and then deduce

$$(6.27) \quad \begin{aligned} S_{\bar{\mathcal{L}}}\bar{\Pi} &= \sum_{0 \leq j \leq \ell-1} \bar{\Pi}S_{\bar{\mathcal{B}}} * (\mathcal{A}S_{\bar{\mathcal{B}}})^{*(j)} + \sum_{0 \leq i \leq n-1} (S_{\bar{\mathcal{B}}}\mathcal{A})^{*(i)} * S_{\bar{\mathcal{B}}}\bar{\Pi} * (\mathcal{A}S_{\bar{\mathcal{B}}})^{*(\ell)} \\ &\quad + (S_{\bar{\mathcal{B}}}\mathcal{A})^{*(n)} * S_{\bar{\mathcal{L}}}\bar{\Pi} * (\mathcal{A}S_{\bar{\mathcal{B}}})^{*(\ell)}. \end{aligned}$$

Theorem 6.9. *Let m_0, m_1 be two admissible weight functions such that $\langle v \rangle^{3/2} \prec m_0 \prec m_1$ and $m_0 \leq \mu^{-1/2}$. Then we have the uniform in time bound*

$$(6.28) \quad \|S_{\bar{\mathcal{L}}}(t)\bar{\Pi}\|_{H_x^2 L_v^2(m_1) \rightarrow H_x^2 L_v^2(m_1)} \in L_t^\infty(\mathbb{R}_+),$$

as well as the decay estimate

$$(6.29) \quad \|S_{\bar{\mathcal{L}}}(t)\bar{\Pi}\|_{H_x^2 L_v^2(m_1) \rightarrow H_x^2 L_v^2(m_0)} \lesssim \Theta_{m_1, m_0}(t) \quad \forall t \geq 0.$$

Moreover, for any admissible weight function $m_1 \succ \langle v \rangle^{3/2}$, there holds

$$(6.30) \quad \int_0^\infty \|S_{\bar{\mathcal{L}}}(t)\bar{\Pi}\|_{H_x^2(H_{v,*}^{-1}(m_1)) \rightarrow H_x^2 L_v^2(m_1)}^2 dt < \infty.$$

Proof. We split the proof into five steps.

Step 1. Decay in the small functions space. Let us denote $E_0 = \mathcal{H}_{x,v}^1(\mu^{-1/2})$ and $E_1 = \mathcal{H}_{x,v}^1(\omega)$, for some admissible weight $\mu^{-1/2} \prec \omega \prec \mu^{-1}$. Arguing exactly as in Proposition 3.2, using Lemma 6.1 and Lemma 6.4 we obtain

$$\forall t \geq 0, \quad \|S_{\bar{\mathcal{L}}}(t)\bar{\Pi}\|_{E_1 \rightarrow E_0} \lesssim e^{-\lambda t \frac{2}{|\gamma|}} \in L_t^1(\mathbb{R}_+).$$

Step 2. Factorization and bounds. Let us denote $X_0 = H_x^2 L_v^2(m_0)$, $X_1 = H_x^2 L_v^2(m_1)$, $X_2 = H_x^2 L_v^2(\omega)$ and $Z_1 = H_x^2(H_{v,*}^{-1}(m_1))$. We first write the factorization identity, thanks to (6.27),

$$\begin{aligned} (6.31) \quad S_{\bar{\mathcal{L}}}\bar{\Pi} &= \sum_{0 \leq j \leq 2} \bar{\Pi} S_{\bar{\mathcal{B}}} * (\mathcal{A} S_{\bar{\mathcal{B}}})^{(*j)} + \sum_{0 \leq i \leq 3} (S_{\bar{\mathcal{B}}}\mathcal{A})^{(*i)} * S_{\bar{\mathcal{B}}}\bar{\Pi} * (\mathcal{A} S_{\bar{\mathcal{B}}})^{(*3)} \\ &\quad + (S_{\bar{\mathcal{B}}}\mathcal{A})^{(*4)} * S_{\bar{\mathcal{L}}}\bar{\Pi} * (\mathcal{A} S_{\bar{\mathcal{B}}})^{(*3)} \\ &=: \sum_{0 \leq j \leq 2} S_1^j + \sum_{0 \leq i \leq 3} S_2^i + S_3. \end{aligned}$$

Let us now gather several bounds that are immediate consequences of previous results for \mathcal{A} and $S_{\bar{\mathcal{B}}}$. Thanks to Corollary 6.8, we already have

$$(6.32) \quad \|(\mathcal{A} S_{\bar{\mathcal{B}}})^{(*2)}(t)\|_{X_2 \rightarrow E_1} \in L_t^1(\mathbb{R}_+) \quad \text{and} \quad \|(S_{\bar{\mathcal{B}}}\mathcal{A})^{(*4)}(t)\|_{E_0 \rightarrow X_1} \in L_t^1(\mathbb{R}_+).$$

From Corollary 6.5 and Corollary 6.7, it also holds, for any $i, j \geq 1$,

$$(6.33) \quad \|S_{\bar{\mathcal{B}}}(t)\|_{X_2 \rightarrow X_1}, \|(\mathcal{A} S_{\bar{\mathcal{B}}})^{(*j)}(t)\|_{X_2 \rightarrow X_2}, \|(S_{\bar{\mathcal{B}}}\mathcal{A})^{(*i)}(t)\|_{X_1 \rightarrow X_1} \in L_t^1(\mathbb{R}_+),$$

moreover

$$(6.34) \quad \|S_{\bar{\mathcal{B}}}(t)\|_{X_1 \rightarrow X_1}, \|\mathcal{A} S_{\bar{\mathcal{B}}}\|_{X_1 \rightarrow X_2} \in L_t^\infty(\mathbb{R}_+),$$

and also

$$(6.35) \quad \|S_{\bar{\mathcal{B}}}(t)\|_{X_1 \rightarrow X_0} \lesssim \Theta_{m_1, m_0}(t), \quad \|\mathcal{A} S_{\bar{\mathcal{B}}}\|_{X_1 \rightarrow X_2} \lesssim \Theta_{m_1, m_0}(t),$$

Finally, thanks to (6.18) in Corollary 6.5 and Corollary 6.7, it follows

$$(6.36) \quad \|S_{\bar{\mathcal{B}}}(t)\|_{Z_1 \rightarrow X_1}, \|\mathcal{A} S_{\bar{\mathcal{B}}}(t)\|_{Z_1 \rightarrow X_2} \in L_t^2(\mathbb{R}_+).$$

Step 3. Proof of (6.28). Using the factorization (6.31) together with the bounds (6.32), (6.33) and (6.34), we first get

$$\|S_1^0(t)\|_{X_1 \rightarrow X_1} \in L_t^\infty(\mathbb{R}_+),$$

Moreover, for $1 \leq j \leq 2$ and $0 \leq i \leq 3$, we also have

$$\begin{aligned} \|S_1^j(t)\|_{X_1 \rightarrow X_1} &\lesssim \|S_{\bar{\mathcal{B}}}(t)\|_{X_2 \rightarrow X_1} * \|(\mathcal{A} S_{\bar{\mathcal{B}}})^{*(j-1)}(t)\|_{X_2 \rightarrow X_2} * \|\mathcal{A} S_{\bar{\mathcal{B}}}(t)\|_{X_1 \rightarrow X_2} \\ &\in L_t^1(\mathbb{R}_+) * L_t^1(\mathbb{R}_+) * L_t^\infty(\mathbb{R}_+) \subset L_t^\infty(\mathbb{R}_+), \end{aligned}$$

and

$$\begin{aligned} \|S_2^i(t)\|_{X_1 \rightarrow X_1} &\lesssim \|(S_{\bar{\mathcal{B}}}\mathcal{A})^{(*i)}(t)\|_{X_1 \rightarrow X_1} * \|S_{\bar{\mathcal{B}}}(t)\|_{X_2 \rightarrow X_1} * \|(\mathcal{A} S_{\bar{\mathcal{B}}})^{(*2)}(t)\|_{X_2 \rightarrow X_2} * \|\mathcal{A} S_{\bar{\mathcal{B}}}(t)\|_{X_1 \rightarrow X_2} \\ &\in L_t^1(\mathbb{R}_+) * L_t^1(\mathbb{R}_+) * L_t^1(\mathbb{R}_+) * L_t^\infty(\mathbb{R}_+) \subset L_t^\infty(\mathbb{R}_+). \end{aligned}$$

Finally, using Step 1, it follows

$$\begin{aligned} \|S_3(t)\|_{X_1 \rightarrow X_1} &\lesssim \|(S_{\bar{\mathcal{B}}}\mathcal{A})^{(*4)}\|_{E_0 \rightarrow X_1} * \|S_{\bar{\mathcal{L}}}(t)\bar{\Pi}\|_{E_1 \rightarrow E_0} * \|(\mathcal{A} S_{\bar{\mathcal{B}}})^{(*2)}(t)\|_{X_2 \rightarrow E_1} * \|\mathcal{A} S_{\bar{\mathcal{B}}}(t)\|_{X_1 \rightarrow X_2} \\ &\in L_t^1(\mathbb{R}_+) * L_t^1(\mathbb{R}_+) * L_t^1(\mathbb{R}_+) * L_t^\infty(\mathbb{R}_+) \subset L_t^\infty(\mathbb{R}_+), \end{aligned}$$

which completes the proof of (6.28).

Step 4. Proof of (6.29). The proof of (6.29) can be obtained using the factorization (6.31), the bounds (6.32), (6.33) and (6.35), and arguing as in Step 3.

Step 5. Proof of (6.30). The proof of (6.30) follows using the factorization (6.31), the bounds (6.32), (6.33) and (6.36), and arguing as in Step 3. \square

6.7. Summary of the decay and dissipativity results for $\bar{\mathcal{L}}$. We introduce the appropriate functional spaces and we summarize the decay and dissipativity properties of the semigroup $S_{\bar{\mathcal{L}}}$ which will be useful in the next section.

From now on, for a given admissible weight function m such that $m \succ \langle v \rangle^{2+3/2}$, we define

$$(6.37) \quad \mathcal{X} := H_x^2 L_v^2(m), \quad \mathcal{Y} := H_x^2 H_{v,*}^1(m), \quad \mathcal{Z} := H_x^2 H_{v,*}^{-1}(m), \quad \mathcal{X}_0 := H_x^2 L_v^2.$$

We also define the norm $\|\cdot\|_{\mathcal{X}}$ on \mathcal{X} , and its associated scalar product $\langle\langle \cdot, \cdot \rangle\rangle_{\mathcal{X}}$, given by

$$(6.38) \quad \|g\|_{\mathcal{X}}^2 := \eta \|g\|_{\mathcal{X}}^2 + \int_0^\infty \|S_{\bar{\mathcal{L}}}(\tau)g\|_{\mathcal{X}_0}^2 d\tau,$$

for $\eta > 0$ small enough.

Theorem 6.10. *Consider an admissible weight function m such that $m \succ \langle v \rangle^{2+3/2}$. With the above assumptions and notations, the norm $\|\cdot\|_{\mathcal{X}}$ is equivalent to the initial norm $\|\cdot\|_{\mathcal{X}}$ on \mathcal{X} , and moreover, there exists $\eta > 0$ small enough such that*

$$(6.39) \quad \langle\langle \bar{\mathcal{L}}\bar{\Pi}f, \bar{\Pi}f \rangle\rangle_{\mathcal{X}} \lesssim -\|\bar{\Pi}f\|_{\mathcal{Y}}^2, \quad \forall f \in \mathcal{X}_1^{\bar{\mathcal{L}}},$$

$$(6.40) \quad \|S_{\bar{\mathcal{L}}}(t)\bar{\Pi}\|_{\mathcal{Y} \rightarrow \mathcal{X}_0} \in L_t^2(\mathbb{R}_+),$$

$$(6.41) \quad \|S_{\bar{\mathcal{L}}}(t)\bar{\Pi}\|_{\mathcal{Z} \rightarrow \mathcal{X}_0} \in L_t^2(\mathbb{R}_+).$$

Proof. The proof follows exactly the same arguments as in Proposition 3.5 and Corollary 3.6. First of all, the equivalence of the norms follows as in Proposition 3.5 since $m \succ \langle v \rangle^{3/2}$. Moreover, for any admissible weight $m_1 \succ \langle v \rangle^{3/2}$, inequality (6.39) can be proven as in the proof of estimate (3.11) in Proposition 3.5, by using Lemma 6.2, Corollary 6.7 and Theorem 6.9. As a consequence, arguing as in the proof of (3.12) in Proposition 3.5, we obtain the regularity estimate

$$(6.42) \quad \|S_{\bar{\mathcal{L}}}(t)\bar{\Pi}\|_{H_x^n L_v^2(m_1) \rightarrow H_x^n(H_{v,*}^1(m_1))} \in L_t^2(\mathbb{R}_+).$$

Then, fixing a polynomial weight function m_1 so that $\langle v \rangle^{3/2} \prec m_1 \preceq \langle v \rangle^{\gamma/2} m$, estimate (6.40) follows from (6.42) and the embeddings $H_x^n(H_{v,*}^1(m_1)) \subset \mathcal{X}_0$ and $\mathcal{Y} \subset H_x^n L_v^2(m_1)$. Finally, estimate (6.41) is an easy consequence of (6.30) in Theorem 6.9 together with $\mathcal{X} \subset \mathcal{X}_0$. \square

6.8. Nonlinear estimate. From the nonlinear estimate for the homogeneous case established in Lemma 4.3 and Corollary 4.4, we deduce the following estimate.

Lemma 6.11. *Let m be an admissible weight function such that $m \succ \langle v \rangle^{2+3/2}$. Then*

$$(6.43) \quad \langle Q(f, g), h \rangle_{\mathcal{X}} \lesssim (\|f\|_{\mathcal{X}} \|g\|_{\mathcal{Y}} + \|f\|_{\mathcal{Y}} \|g\|_{\mathcal{X}}) \|h\|_{\mathcal{Y}}.$$

As a consequence

$$(6.44) \quad \|Q(f, g)\|_{\mathcal{Z}} \lesssim \|f\|_{\mathcal{X}} \|g\|_{\mathcal{Y}} + \|f\|_{\mathcal{Y}} \|g\|_{\mathcal{X}}.$$

Proof. We proceed similarly as in [14, Lemma 3.5] and thus only sketch the proof. For the most difficult term, we have thanks to Lemma 4.3,

$$\begin{aligned} \langle \nabla_x^2 Q(f, g), \nabla_x^2 h \rangle_{L_{x,v}^2(m)} &= \langle Q(\nabla_x^2 f, g) + 2Q(\nabla_x f, \nabla_x g) + Q(f, \nabla_x^2 g), \nabla_x^2 h \rangle_{L_{x,v}^2(m)} \\ &\lesssim \int_{\mathbb{T}^3} \left(\|\nabla_x^2 f\|_{L^2(m)} \|g\|_{H_*^1(m)} + \|\nabla_x^2 f\|_{H_*^1(m)} \|g\|_{L^2(m)} \right. \\ &\quad + \|\nabla_x f\|_{L^2(m)} \|\nabla_x g\|_{H_*^1(m)} + \|\nabla_x f\|_{H_*^1(m)} \|\nabla_x g\|_{L^2(m)} \\ &\quad \left. + \|f\|_{L^2(m)} \|\nabla_x^2 g\|_{H_*^1(m)} + \|f\|_{H_*^1(m)} \|\nabla_x^2 g\|_{L^2(m)} \right) \|\nabla_x^2 h\|_{H_*^1(m)} dx. \end{aligned}$$

Using first the Cauchy-Schwarz inequality in the x variable and next the two Sobolev embeddings $L_x^\infty \subset H_x^2$ and $L_x^4 \subset H_x^1$, we straightforwardly obtain that the above RHS term is bounded by the RHS term in (6.43). The proof of (6.44) is then straightforward. \square

6.9. Proof of the main result. For a solution F to the inhomogeneous Landau equation (1.1), we consider the perturbation $f = F - \mu$ that verifies

$$(6.45) \quad \begin{cases} \partial_t f = \bar{\mathcal{L}}f + Q(f, f) \\ f_0 = F_0 - \mu. \end{cases}$$

Observe that, thanks to the conservation laws, there holds $\bar{\Pi}_0 f(t) = \bar{\Pi}_0 f_0 = 0$ and also that $\bar{\Pi}_0 Q(f(t), f(t)) = 0$ for any $t \geq 0$.

Proof of Theorem 1.1. Consider the spaces and norms defined in (6.37) and (6.38). The proof then follows the same arguments as in the proof of the spatially homogeneous version of Theorem 1.1 presented in Section 5, by using the dissipative, decay and regularity estimates of Theorem 6.10 and the nonlinear estimates in Lemma 6.11. \square

We conclude the section by presenting a proof of our improvement of the speed of convergence to the equilibrium for solutions to the spatially inhomogeneous Landau equation in a non perturbative framework.

Proof of Corollary 1.4. Under the assumptions (1.15) and (1.16), [17, Theorem 2 & Section I.5] implies that

$$\|f(t)\|_{L_{x,v}^1} \lesssim \langle t \rangle^{-\theta},$$

for some explicit constant $\theta > 0$. We then write the interpolation inequality

$$\|f\|_{H_{x,v}^2(m^\alpha)} \lesssim \|f\|_{H_{x,v}^3}^{\beta_1} \|f\|_{L_{x,v}^1}^{\beta_2} \|f\|_{L_{x,v}^1(m)}^{1-\beta_1-\beta_2},$$

where $\alpha, \beta_1, \beta_2 \in (0, 1)$ are explicit constants. We conclude taking $t_0 > 0$ large enough so that $\|f(t_0)\|_{H_x^2 L_v^2(m^\alpha)} \leq \epsilon_0$, applying Theorem 1.1 and observing that $\Theta_{m^\alpha}(t) \simeq \Theta_m(t)$. \square

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