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CONTINUITY OF THE TIME AND ISOPERIMETRIC CONSTANTS IN SUPERCRITICAL PERCOLATION

OLIVIER GARET, RÉGINE MARCHAND, EVIATAR B. PROCACCIA,
AND MARIE THÉRET

ABSTRACT. We consider two different objects on super-critical Bernoulli percolation on \mathbb{Z}^d : the time constant for i.i.d. first-passage percolation (for $d \geq 2$) and the isoperimetric constant (for $d = 2$). We prove that both objects are continuous with respect to the law of the environment. More precisely we prove that the isoperimetric constant of supercritical percolation in \mathbb{Z}^2 is continuous in the percolation parameter. As a corollary we prove that normalized sets achieving the isoperimetric constant are continuous with respect to the Hausdorff metric. Concerning first-passage percolation, equivalently we consider the model of i.i.d. first-passage percolation on \mathbb{Z}^d with possibly infinite passage times: we associate with each edge e of the graph a passage time $t(e)$ taking values in $[0, +\infty]$, such that $\mathbb{P}[t(e) < +\infty] > p_c(d)$. We prove the continuity of the time constant with respect to the law of the passage times. This extends the continuity property previously proved by Cox and Kesten [8, 10, 18] for first passage percolation with finite passage times.

1. INTRODUCTION

We first introduce briefly the studied objects, then state the corresponding results, and finally say a few words about the proofs. We consider super-critical bond percolation on \mathbb{Z}^d , with parameter $p > p_c(d)$, the critical parameter for this percolation. The a.s. existence of a unique infinite cluster \mathcal{C}_∞ is well known in this model. We study two different objects defined on \mathcal{C}_∞ : the time constant for an independent first passage percolation and the isoperimetric (or Cheeger) constant.

1.1. First passage percolation. First-passage percolation on \mathbb{Z}^d was introduced by Hammersley and Welsh [16] as a model for the spread of a fluid in a porous medium. To each edge of the \mathbb{Z}^d lattice is attached a nonnegative random variable $t(e)$ which corresponds to the travel time needed by the fluid to cross the edge. When the passage times are independent identically distributed variables with common distribution F , Cox and Durrett [9] showed that, under some moment conditions, the time that is needed to travel from 0 to nx is $n\mu_F(x) + o(n)$, where μ_F is a semi-norm associated to F .

A natural extension is to replace the \mathbb{Z}^d lattice by a random environment given by the infinite cluster \mathcal{C}_∞ of a super-critical Bernoulli percolation model. This is equivalent to allow $t(e)$ to be equal to $+\infty$.

The existence of a time constant in first-passage percolation on the \mathbb{Z}^d lattice was first proved by Garet and Marchand in [12], in the case where $(t(e)\mathbb{1}_{t(e) < +\infty})$ is a

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stationary integrable ergodic field. Recently, Cerf and Th eret [6] focused of the case where $(t(e)\mathbb{1}_{t(e)<+\infty})$ is an independent field, and managed to prove the existence of an appropriate time constant $\mu(x)$, without any assumption of integrability.

In the following, we adopt the settings of Cerf and Th eret: the passage times are independent random variables with a common distribution F taking its values in $[0, +\infty]$.

1.2. Isoperimetric constant. Isoperimetry in super critical percolation is also a well studied subject. For a finite graph $\mathfrak{J} = (V(\mathfrak{J}), E(\mathfrak{J}))$, the isoperimetric constant is defined as

$$\varphi_{\mathfrak{J}} = \min \left\{ \frac{|\partial A|}{|A|} : A \subset V(\mathfrak{J}), 0 < |A| \leq \frac{|V(\mathfrak{J})|}{2} \right\},$$

where ∂A is the edge boundary of A , $\partial A = \{e = (x, y) \in E(\mathfrak{J}) : x \in A, y \notin A, \text{ or } x \notin A, y \in A\}$.

We consider the isoperimetric constant $\varphi_n(p)$ of $\mathcal{C}_{\infty} \cap [-n, n]^d$, the intersection of the infinite component of super-critical percolation of parameter p with the box $[-n, n]^d$. In several papers (e.g. [2],[20], [21],[3]) it was shown that there exist constants $c, C > 0$ such that $c < n\varphi_n(p) < C$, with probability tending rapidly to 1. This led Benjamini to conjecture the existence of $\lim_{n \rightarrow \infty} n\varphi_n(p)$. In [23], Rosenthal and Procaccia prove that the variance of $n\varphi_n(p)$ is smaller than Cn^{2-d} , which implies $n\varphi_n(p)$ is concentrated around it's mean for $d \geq 3$. In [4], Biskup, Louidor, Procaccia and Rosenthal prove the existence of $\lim_{n \rightarrow \infty} n\varphi_n(p)$ for $d = 2$. In addition a shape theorem is obtained: any set yielding the isoperimetric constant converges in the Hausdorff metric to the normalized Wulff shape \widehat{W}_p , with respect to a specific norm given in an implicit form. For additional background and a wider introduction on Wulff construction in this context the reader is referred to [4].

1.3. Main results. The goal of this paper is to prove that the objects we have just defined are continuous with respect to the law of the environment. Concerning the isoperimetric constant, we restrict ourselves to the case $d = 2$, and we prove the continuity of the Wulff shape.

Theorem 1.1. *For $d = 2$,*

$$p \in (p_c(2), 1] \mapsto \lim_{n \rightarrow \infty} n\varphi_n(p)$$

is continuous. Moreover

$$p \in (p_c(2), 1] \mapsto \widehat{W}_p$$

is continuous for the Hausdorff distance between non-empty compact sets of \mathbb{R}^d .

Concerning first passage percolation, we consider a general dimension $d \geq 2$ and we prove the continuity of the map $F \mapsto \mu_F$. More precisely, let $(G_n)_{n \in \mathbb{N}}$ and G be probability measures on $[0, +\infty]$. We say that G_n converges weakly towards G when n goes to infinity, and we write $G_n \xrightarrow{d} G$, if for any continuous bounded function $f : [0, +\infty] \mapsto [0, +\infty)$ we have

$$\lim_{n \rightarrow \infty} \int_{[0, +\infty]} f dG_n = \int_{[0, +\infty]} f dG.$$

It is well known that $G_n \xrightarrow{d} G$ if and only if $\lim_{n \rightarrow \infty} G_n([t, +\infty]) = G([t, +\infty])$ for all $t \in [0, +\infty)$ such that G is continuous at t . Our main result concerning first-passage percolation is the continuity of the time constants μ_G with respect

to the law G of the passage time of an edge, uniformly in the direction. Let $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : \|x\|_2 = 1\}$.

Theorem 1.2. *Let $G, (G_n)_{n \in \mathbb{N}}$ be probability measures on $[0, +\infty]$ such that for every $n \in \mathbb{N}$, $G_n([0, +\infty)) > p_c(d)$ and $G([0, +\infty)) > p_c(d)$. If $G_n \xrightarrow{d} G$, then*

$$\lim_{n \rightarrow \infty} \sup_{x \in \mathbb{S}^{d-1}} |\mu_{G_n}(x) - \mu_G(x)| = 0.$$

This result extends the continuity of the time constant in classical first-passage percolation proved by Cox and Kesten [8, 10, 18] to first-passage percolation with possibly infinite passage times. We can quite easily deduce from Theorem 1.2 the following continuity of the asymptotic shapes when they exist, i.e., when μ_G is a norm:

Corollary 1.3. *Let $G, (G_n)_{n \in \mathbb{N}}$ be probability measures on $[0, +\infty]$ such that for every $n \in \mathbb{N}$, $G_n([0, +\infty)) > p_c(d)$, $G([0, +\infty)) > p_c(d)$ and $G(\{0\}) < p_c(d)$. If $G_n \xrightarrow{d} G$, then*

$$\lim_{n \rightarrow \infty} d_H(\mathcal{B}_{\mu_{G_n}}, \mathcal{B}_{\mu_G}) = 0,$$

where \mathcal{B}_μ is the unit ball for the norm μ and d_H is the Hausdorff distance between non-empty compact sets of \mathbb{R}^d .

Particularly, when $G_p = p\delta_1 + (1-p)\delta_{+\infty}$, the functional μ_{G_p} governs the asymptotic distance in the infinite cluster of a supercritical Bernoulli percolation (see [12, 13, 14]). We get the following corollary:

Corollary 1.4. *For $p > p_c(d)$, let us denote by \mathcal{B}_p the unit ball for the norm that is associated to the cheminal distance in supercritical bond percolation. Then,*

$$p \in (p_c(d), 1] \mapsto \mathcal{B}_p$$

is continuous for the Hausdorff distance between non-empty compact sets of \mathbb{R}^d .

As a key step of the proof of Theorem 1.2, we study the effect of truncations of the passage time on the time constant. Let G be a probability measure on $[0, +\infty]$ such that $G([0, +\infty)) > p_c(d)$. For every $K > 0$, we set

$$G^K = 1_{[0, K)}G + G([K, +\infty))\delta_K,$$

i.e., G^K is the law of the truncated passage time $t_G^K(e) = \min(t_G(e), K)$. We have the following control on the effect of these truncations on the time constants:

Theorem 1.5. *Let G be a probability measure on $[0, +\infty]$ such that $G([0, +\infty)) > p_c(d)$. Then*

$$\forall x \in \mathbb{Z}^d \quad \lim_{K \rightarrow \infty} \mu_{G^K}(x) = \mu_G(x).$$

We even have the finer uniform control:

Theorem 1.6. *Let G_0 be a probability measure on $[0, +\infty]$ and $p_0 \in [0, 1]$ such that $G_0(\{0\}) < p_c(d) < p_0$. Then there exists a function Ψ , that depends on (G_0, p_0, M_0) , such that $\lim_{+\infty} \Psi = 0$ and for every $K \geq M_0$ and every probability measure G with $G \succeq G_0$ and $G([0, M_0]) \geq p_0$ we have*

$$\forall x \in \mathbb{Z}^d \quad 0 \leq \mu_{G^K}(x) - \mu_G(x) \leq (\mu_G(x) + \|x\|_1) \Psi(K).$$

As a consequence of these results, we can approximate the time constants for the chemical distance in supercritical percolation on \mathbb{Z}^d by the time constants for some finite passage times:

Corollary 1.7. *Let $p > p_c(d)$, and consider $G = p\delta_1 + (1-p)\delta_{+\infty}$. Then $G^K = p\delta_1 + (1-p)\delta_K$ for all $K \geq 1$ and*

$$\forall x \in \mathbb{Z}^d \quad \lim_{K \rightarrow \infty} \mu_{G^K}(x) = \mu_G(x).$$

1.4. Idea of the proofs. Obviously, the two main theorems of the paper, Theorems 1.1 and 1.2, state results of the same nature. Beyond this similarity, they are linked by their proofs, that share a common structure and a common renormalisation step. The idea of the delicate part of both proofs is inspired by Cox and Kesten's method in [10]. Consider that some edges of \mathbb{Z}^d are "good" (i.e. open, or of passage time smaller than some constant), and the others are bad, for a given law of the environment (a parameter p for the percolation, or a given law G of passage times), and look at a path of good edges in this setting. Then change a little bit your environment : decrease p to $p - \varepsilon$, or increase the passage times of the edges. Some edges of the chosen path become bad. To recover a path of good edges, you have to bypass these edges. The most intuitive idea is to consider the cluster of bad edges around each one of them, and to bypass the edge by a short path along the boundary of this cluster. This idea works successfully in Cox and Kesten's paper. Unfortunately in our setting the control we have on these boundaries, or on the number of new bad edges we create, is not good enough. This is the reason why we cannot perform our construction of a modified good path at the scale of the edges. Thus we need to use a coarse graining argument to construct the bypasses at the scale of good blocks.

In section 2, we give more precise definitions of the studied objects and state some preliminary results. In Section 3, we present the renormalization process and the construction of modified paths that will be useful to study both the time constant and the isoperimetric constant. Sections 4 and 5 are devoted to the study of first-passage percolation. In Section 4, we use the renormalization argument to study the effect of truncating the passage times on the time constant. We then use it in Section 5 to prove the continuity of the time constant. Finally Section 6 is devoted to the proof of the continuity of the isoperimetric constant, using again the renormalization argument.

2. DEFINITIONS AND PRELIMINARY RESULTS

In this section we give a formal definition of the objects we briefly presented in the introduction. We also present the coupling that will be useful in the rest of the paper, and prove the monotonicity of the time constant.

2.1. Lattice and passage times. Let $d \geq 2$. We consider the graph whose vertices are the points of \mathbb{Z}^d , and we put an edge between two vertices x and y if and only if the Euclidean distance between x and y is equal to 1. We denote this set of edges by \mathbb{E}^d . We denote by 0 the origin of the graph. For $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, we define $\|x\|_1 = \sum_{i=1}^d |x_i|$, $\|x\|_2 = \sqrt{\sum_{i=1}^d x_i^2}$ and $\|x\|_\infty = \max\{|x_i| : i \in \{1, \dots, d\}\}$.

Let $(t(e), e \in \mathbb{E}^d)$ be a family of i.i.d. random variables taking values in $[0, +\infty]$ with common distribution G . We emphasize that $+\infty$ is a possible value for the

passage times, on the contrary to what is assumed in classical first-passage percolation. The random variable $t(e)$ is called the *passage time* of e , i.e., it is the time needed to cross the edge e . If x, y are vertices in \mathbb{Z}^d , a path from x to y is a sequence $r = (v_0, e_1, \dots, e_n, v_n)$ of vertices $(v_i)_{i=0, \dots, n}$ and edges $(e_i)_{i=1, \dots, n}$ for some $n \in \mathbb{N}$ such that $v_0 = x$, $v_n = y$ and for all $i \in \{1, \dots, n\}$, e_i is the edge of endpoints v_{i-1} and v_i . We define the length $|r|$ of a path r as its number of edges and we define the passage time of r by $T(r) = \sum_{e \in r} t(e)$. We obtain a random pseudo-metric T on \mathbb{Z}^d in the following way (the only possibly missing property is the separation of distinct points):

$$\forall x, y \in \mathbb{Z}^d, \quad T(x, y) = \inf\{T(r) : r \text{ is a path from } x \text{ to } y\} \in [0, +\infty].$$

Since different laws appear in this article, we put subscript G on our notations to emphasize the dependance with respect to the probability measure $G : t_G(e), T_G(r)$ and $T_G(x, y)$.

As we are interested in the asymptotic behavior of the pseudo-metric T_G , we will only consider laws G on $[0, +\infty]$ such that $G([0, +\infty)) > p_c(d)$. Here and in the following, $p_c(d)$ denotes the critical parameter for bond Bernoulli percolation on $(\mathbb{Z}^d, \mathbb{E}^d)$. Thus there a.s. exists a unique infinite cluster $\mathcal{C}_{G, \infty}$ in the super-critical percolation $(1_{\{t_G(e) < \infty\}}, e \in \mathbb{E}^d)$ that only keeps edges with finite passage times. Our generalized first-passage percolation model with time distribution G is then equivalent to standard i.i.d. first-passage percolation (where the passage time of an edge e is the law of $t_G(e)$ conditioned to be finite) on a super-critical Bernoulli percolation performed independently (where the parameter for an edge to be closed is $G(\{+\infty\})$).

For instance, if we take $G = p\delta_1 + (1-p)\delta_{+\infty}$ with $p > p_c(d)$, the pseudo-distance T_G is the chemical distance in supercritical bond percolation with parameter p .

2.2. Definition of the time constant. As announced in the introduction, we follow the approach by Cerf and Th  ret in [6], which requires no integrability condition on the restriction of G to $[0, +\infty)$. We collect in this subsection the definition and properties of the time constants obtained in their paper.

To get round the fact that the times T_G can take infinite values, we introduce some regularized times $\tilde{T}_G^{\mathcal{C}}$, for well chosen sets \mathcal{C} . These regularized passage times have better integrability properties.

Let \mathcal{C} be a subgraph of $(\mathbb{Z}^d, \mathbb{E}^d)$. Typically, \mathcal{C} will be the infinite cluster of an embedded supercritical Bernoulli bond percolation. For every $x \in \mathbb{Z}^d$, we define the random vertex $\tilde{x}^{\mathcal{C}}$ as the vertex of \mathcal{C} which minimizes $\|x - \tilde{x}^{\mathcal{C}}\|_1$, with a deterministic rule to break ties. We then define the passages times $\tilde{T}_G^{\mathcal{C}}$ by

$$\forall x, y \in \mathbb{Z}^d, \quad \tilde{T}_G^{\mathcal{C}}(x, y) = T_G(\tilde{x}^{\mathcal{C}}, \tilde{y}^{\mathcal{C}}).$$

Let G be a probability measure on $[0, +\infty]$ such that $G([0, +\infty)) > p_c(d)$, and let $M > 0$ be such that $G([0, M]) > p_c(d)$. We denote by $\mathcal{C}_{G, M}$ the a.s. unique infinite cluster of the percolation $(1_{\{t_G(e) \leq M\}}, e \in \mathbb{E}^d)$, i.e. the percolation obtained by keeping only edges with passage times less than M . For any $x, y \in \mathbb{Z}^d$, the (level M) regularized passage time $\tilde{T}_G^{\mathcal{C}_{G, M}}(x, y)$ is then

$$\tilde{T}_G^{\mathcal{C}_{G, M}}(x, y) = T_G(\tilde{x}^{\mathcal{C}_{G, M}}, \tilde{y}^{\mathcal{C}_{G, M}}).$$

The parameter M only plays a role in the choice of $\tilde{x}^{\mathcal{C}_{G, M}}$ and $\tilde{y}^{\mathcal{C}_{G, M}}$. Once these points are chosen, the optimization in $\tilde{T}_G^{\mathcal{C}_{G, M}}(x, y)$ is on all paths between $\tilde{x}^{\mathcal{C}_{G, M}}$

and $\tilde{y}^{\mathcal{C}_{G,M}}$, paths using edges with passage time larger than M included. But as $\tilde{x}^{\mathcal{C}_{G,M}} \in \mathcal{C}_{G,M}$ and $\tilde{y}^{\mathcal{C}_{G,M}} \in \mathcal{C}_{G,M}$, we know that exists a path using only edges with passage time less than M between these two points. To be more precise, we denote by $D^{\mathcal{C}}(x, y)$ the chemical distance (or graph distance) between two vertices x and y on \mathcal{C} :

$$\forall x, y \in \mathbb{Z}^d, \quad D^{\mathcal{C}}(x, y) = \inf\{|r| : r \text{ is a path from } x \text{ to } y, r \subset \mathcal{C}\},$$

where $\inf \emptyset = +\infty$. The event that the vertices x and y are connected in \mathcal{C} is denoted by $\{x \xrightarrow{\mathcal{C}} y\}$. Then, for any $x, y \in \mathbb{Z}^d$,

$$\tilde{T}_G^{\mathcal{C}_{G,M}}(x, y) \leq MD^{\mathcal{C}_{G,M}}(\tilde{x}^{\mathcal{C}_{G,M}}, \tilde{y}^{\mathcal{C}_{G,M}}).$$

The regularized passage time $\tilde{T}_G^{\mathcal{C}_{G,M}}$ enjoys then the same good integrability properties as the chemical distance on a supercritical percolation cluster (see [1]):

Proposition 2.1 (Moments of \tilde{T} , [6]). *Let G be a probability measure on $[0, +\infty]$ such that $G([0, +\infty)) > p_c(d)$. For every $M \in [0, +\infty)$ such that $G([0, M]) > p_c(d)$, there exist positive constants C_1, C_2 and C_3 such that*

$$\forall x \in \mathbb{Z}^d, \forall l \geq C_3 \|x\|_1, \quad \mathbb{P}\left[\tilde{T}_G^{\mathcal{C}_{G,M}}(0, x) > l\right] \leq C_1 e^{-C_2 l}.$$

This result implies in particular that the times $\tilde{T}_G^{\mathcal{C}_{G,M}}(0, x)$ are integrable. A classical application of a subadditive ergodic theorem gives the existence of a time constant:

Proposition 2.2 (Convergence to the time constant, [6]). *Let G be a probability measure on $[0, +\infty]$ such that $G([0, +\infty)) > p_c(d)$. There exists a deterministic function $\mu_G : \mathbb{Z}^d \rightarrow [0, +\infty)$ such that for every $M \in [0, +\infty)$ satisfying $G([0, M]) > p_c(d)$, we have the following properties:*

- (1) $\forall x \in \mathbb{Z}^d \quad \mu_G(x) = \inf_{n \in \mathbb{N}^*} \frac{\mathbb{E}\left[\tilde{T}_G^{\mathcal{C}_{G,M}}(0, nx)\right]}{n} = \lim_{n \rightarrow +\infty} \frac{\mathbb{E}\left[\tilde{T}_G^{\mathcal{C}_{G,M}}(0, nx)\right]}{n},$
- (2) $\forall x \in \mathbb{Z}^d \quad \lim_{n \rightarrow \infty} \frac{\tilde{T}_G^{\mathcal{C}_{G,M}}(0, nx)}{n} = \mu_G(x) \quad \text{a.s. and in } L^1,$
- (3) $\forall x \in \mathbb{Z}^d \quad \lim_{n \rightarrow \infty} \frac{\tilde{T}_G^{\mathcal{C}_{G,\infty}}(0, nx)}{n} = \mu_G(x) \quad \text{in probability,}$
- (4) $\forall x \in \mathbb{Z}^d \quad \lim_{n \rightarrow \infty} \frac{T_G(0, nx)}{n} = \theta_G^2 \delta_{\mu_G(x)} + (1 - \theta_G^2) \delta_{+\infty} \quad \text{in distribution,}$

where $\theta_G = \mathbb{P}[0 \in \mathcal{C}_{G,\infty}]$.

Note that even if the definition (1) of the time constants $\mu_G(x)$ requires to introduce a parameter M in the definition of the regularized passage times $\tilde{T}_G^{\mathcal{C}_{G,M}}(0, nx)$, these time constants $\mu_G(x)$ do not depend on M . Note also that if instead of taking the $\tilde{x}^{\mathcal{C}_{G,M}}$ in the infinite cluster $\mathcal{C}_{G,M}$ of edges with passage time less than M , we take the $\tilde{x}^{\mathcal{C}_{G,\infty}}$ in the infinite cluster $\mathcal{C}_{G,\infty}$ of edges with finite passage time, the almost sure convergence is weakened into the convergence in probability (3). Without any regularization, the convergence in (4) is only in law.

As in the classical first-passage percolation model, the function μ_G can be extended, by homogeneity, into a pseudo-norm on \mathbb{R}^d (the only possibly missing property of μ_G is the strict positivity):

Proposition 2.3 (Properties of μ_G , [6]). *Let G be a probability measure on $[0, +\infty)$ such that $G([0, +\infty)) > p_c(d)$. Then either μ_G is identically equal to 0 or $\mu_G(x) > 0$ for all $x \neq 0$, and we know that*

$$\mu_G = 0 \iff G(\{0\}) \geq p_c(d).$$

Proposition 2.1 gives strong enough integrability properties of $\tilde{T}_G^{\mathcal{C}_{G,M}}(0, x)$ to ensure that the convergence to the time constants is uniform in the direction:

Proposition 2.4 (Uniform convergence, [6]). *Let G be a probability measure on $[0, +\infty)$ such that $G([0, +\infty)) > p_c(d)$. Then for every $M \in [0, +\infty)$ such that $G([0, M]) > p_c(d)$, we have*

$$\lim_{n \rightarrow \infty} \sup_{x \in \mathbb{Z}^d, \|x\|_1 \geq n} \left| \frac{\tilde{T}_G^{\mathcal{C}_{G,M}}(0, x) - \mu_G(x)}{\|x\|_1} \right| = 0 \quad a.s.$$

When $\mu_G > 0$, this uniform convergence is equivalent to the so called *shape theorem*, that we briefly present now. We define $B_{G,t}$ (resp. $\tilde{B}_{G,t}^{\mathcal{C}_{G,M}}, \tilde{B}_{G,t}^{\mathcal{C}_{G,\infty}}$) as the set of all points reached from the origin within a time t :

$$B_{G,t} = \{z \in \mathbb{Z}^d : T_G(0, z) \leq t\},$$

(resp. $\tilde{T}_G^{\mathcal{C}_{G,M}}, \tilde{T}_G^{\mathcal{C}_{G,\infty}}$), and when μ_G is a norm we denote by \mathcal{B}_{μ_G} its closed unit ball. Roughly speaking, the shape theorem states that the rescaled set $B_{G,t}/t$ (respectively $\tilde{B}_{G,t}^{\mathcal{C}_{G,M}}/t, \tilde{B}_{G,t}^{\mathcal{C}_{G,\infty}}/t$) converges towards \mathcal{B}_{μ_G} . The convergence holds in a sense that depends on the regularity of times considered (see [6] for more precise results).

2.3. Definition of the Cheeger constant. We collect in this subsection the definitions and properties of the Cheeger constant obtained in [4]. The connection in this paper between the time constant and the Cheeger constant is made possible due to the representation of the Cheeger as the solution of a continuous isoperimetric problem with respect to some norm. To define this norm we first require some definitions. For uniformity of notations we remark that Bernoulli percolation can be viewed as a special case of FPP, with law $G_p = p\delta_1 + (1-p)\delta_\infty$. We denote by \mathcal{C}_p the infinite cluster $\mathcal{C}_{G_p,1}$. For a path $r = (v_0, e_1, \dots, e_n, v_n)$, and $i \in \{2, \dots, n-1\}$, an edge $e = (x_i, z)$ is said to be a right-boundary edge if z is a neighbor of x_i between x_{i+1} and x_{i-1} in the clockwise direction. The right boundary $\partial^+ r$ of r is the set of right-boundary edges. A path is called right-most if it uses every edge at most once in every orientation and it doesn't contain right-boundary edges. See Figure 1; the solid lines represent the path, dashed lines represent the right-boundary edges, and the curly line is a path in the medial graph which shows the orientation (See [4] for a thorough discussion). For $x, y \in \mathbb{Z}^2$, let $\mathcal{R}(x, y)$ be the set of right-most paths from x to y . For a path $r \in \mathcal{R}(x, y)$, define $\mathbf{b}(r) = |\{e \in \partial^+ r : e \text{ is open}\}|$. For $x, y \in \mathcal{C}_p$ we define the right boundary distance, $b(x, y) = \inf\{\mathbf{b}(r) : r \in \mathcal{R}(x, y), \text{ open}\}$. The next result yields uniform convergence of the right boundary distance to a norm on \mathbb{R}^2 .

Theorem 2.5 ([4] Theorem 2.1). *For any $p > p_c(2)$ and any $x \in \mathbb{R}^2$, the limit*

$$\beta_p(x) := \lim_{n \rightarrow \infty} \frac{b(\tilde{0}^{\mathcal{C}_p}, \tilde{n}x^{\mathcal{C}_p})}{n}$$

The set \widehat{W}_p is a minimizer for the isoperimetric problem associated with the norm β_p . By Theorem 2.7 and (5) both parts of Theorem 1.1 follow the continuity of $p \mapsto \beta_p$. Denote by $\mathcal{U}(n)$ be the set of minimizers of $\varphi_n(p)$.

Theorem 2.8 ([4] Theorem 1.7). *For every $p > p_c(2)$,*

$$\max_{U \in \mathcal{U}(n)} \inf_{\substack{\xi \in \mathbb{R}^2: \\ \xi + \widehat{W}_p / \sqrt{2} \subseteq B(1)}} d_{\mathbb{H}}(n^{-1}U, \xi + \widehat{W}_p / \sqrt{2}) \xrightarrow{n \rightarrow \infty} 0,$$

and

$$\max_{U \in \mathcal{U}(n)} \left| \frac{|U|}{(\theta_p |B(n)|/2)} - 1 \right| \xrightarrow{n \rightarrow \infty} 0,$$

hold for \mathbb{P}_p almost every realization of ω , where $\theta_p = \mathbb{P}_p(0 \in \mathcal{C}_p)$.

2.4. Coupling. To understand how μ_G depends on G , it is useful to consider passage times $(t_G(e))$ with common distribution G , that also have good coupling properties. For any probability measure G on $[0, +\infty]$, we denote by \mathfrak{G} the function

$$\begin{aligned} \mathfrak{G} : [0, +\infty) &\rightarrow [0, 1] \\ t &\mapsto G([t, +\infty)), \end{aligned}$$

which characterizes G . For two probability measures G_1, G_2 on $[0, +\infty]$, we define the following stochastic domination relation:

$$G_1 \succeq G_2 \quad \Leftrightarrow \quad \forall t \in [0, +\infty) \quad \mathfrak{G}_1(t) \geq \mathfrak{G}_2(t).$$

This is to have this equivalence that we choose to characterize a probability measure G by \mathfrak{G} instead of the more standard distribution function $t \mapsto G([0, t])$.

Given a probability measure G on $[0, +\infty]$, we define the two following pseudo-inverse functions for \mathfrak{G} :

$$\begin{aligned} \forall t \in [0, 1], \hat{\mathfrak{G}}(t) &= \sup\{s \in [0, +\infty) : \mathfrak{G}(s) \geq 1 - t\} \text{ and} \\ \tilde{\mathfrak{G}}(t) &= \sup\{s \in [0, +\infty) : \mathfrak{G}(s) > 1 - t\}. \end{aligned}$$

These pseudo-inverse functions can be used to simulate random variable with distribution G :

Lemma 2.9. *Let U be a random variable with uniform law on $(0, 1)$. If G is a probability measure on $[0, +\infty]$, then $\hat{\mathfrak{G}}(U)$ and $\tilde{\mathfrak{G}}(U)$ are random variables taking values in $[0, +\infty]$ with distribution G , and $\tilde{\mathfrak{G}}(U) = \hat{\mathfrak{G}}(U)$ a.s.*

Proof. The function $\hat{\mathfrak{G}}$ has the following property

$$(6) \quad \forall t \in [0, 1], \forall s \in [0, +\infty), \quad \hat{\mathfrak{G}}(t) \geq s \iff \mathfrak{G}(s) \geq 1 - t.$$

Then for all $s \in [0, +\infty)$, we have $\mathbb{P}[\hat{\mathfrak{G}}(U) \geq s] = \mathbb{P}[U \geq 1 - \mathfrak{G}(s)] = \mathfrak{G}(s)$, thus $\hat{\mathfrak{G}}(U)$ has distribution G . The function $\tilde{\mathfrak{G}}$ does not satisfy the property (6). However, $\hat{\mathfrak{G}}(t) \neq \tilde{\mathfrak{G}}(t)$ only for $t \in [0, 1]$ such that $\mathfrak{G}^{-1}(\{1 - t\})$ contains an open interval, thus the set $\{t \in [0, 1] : \hat{\mathfrak{G}}(t) \neq \tilde{\mathfrak{G}}(t)\}$ is at most countable. This implies that $\hat{\mathfrak{G}}(U) = \tilde{\mathfrak{G}}(U)$ a.s., thus $\hat{\mathfrak{G}}(U)$ has the same law as $\tilde{\mathfrak{G}}(U)$. \square

In the following, we will always build the passage times of the edges with this lemma. Let then $(u(e), e \in \mathbb{E}^d)$ be a family of i.i.d. random variables with uniform

law on $(0, 1)$. For any given probability measure G on $[0, +\infty]$, the family of i.i.d passage times with distribution G will always be

$$(7) \quad \forall e \in \mathbb{E}^d, \quad t_G(e) = \hat{\mathfrak{G}}(u(e)).$$

Of course the main interest of this construction is to obtain couplings between laws. In particular, if G_1 and G_2 are probability measures on $[0, +\infty]$,

$$G_1 \preceq G_2 \quad \Rightarrow \quad t_{G_1}(e) \leq t_{G_2}(e) \text{ for all edges } e.$$

In particular in the case of Bernoulli percolation, if $p \leq q$, $G_q = q\delta_1 + (1 - q)\delta_\infty \preceq G_p = p\delta_1 + (1 - p)\delta_\infty$ thus $\mathcal{C}_p \subset \mathcal{C}_q$. Moreover, we have the following pleasant property:

Lemma 2.10. *Let $G, (G_n)_{n \in \mathbb{N}}$ be probability measures on $[0, +\infty]$. We define the passage times $t_G(e)$ and $t_{G_n}(e)$ as in equation (7). If $G_n \xrightarrow{d} G$, then*

$$\text{a.s.}, \forall e \in \mathbb{E}^d, \quad \lim_{n \rightarrow \infty} t_{G_n}(e) = t_G(e).$$

Proof.

(i) Let us prove that if $G_n \succeq G$ for all n , then

$$(8) \quad \forall t \in [0, 1] \quad \lim_{n \rightarrow \infty} \hat{\mathfrak{G}}_n(t) = \hat{\mathfrak{G}}(t).$$

Consider $t \in [0, 1]$, let $x = \hat{\mathfrak{G}}(t)$ and $x_n = \hat{\mathfrak{G}}_n(t)$. Since $G_n \succeq G$, we have $\mathfrak{G}_n \geq \mathfrak{G}$ thus $x_n \geq x$. Suppose that $\overline{\lim}_{n \rightarrow +\infty} x_n := \bar{x} > x$. Up to extraction, we suppose that $\lim_{n \rightarrow +\infty} x_n = \bar{x}$. Choose $\beta \in (x, \bar{x})$ such that \mathfrak{G} is continuous at β , thus $\lim_{n \rightarrow \infty} \mathfrak{G}_n(\beta) = \mathfrak{G}(\beta)$. On one hand, by the definition of $\hat{\mathfrak{G}}$ and the monotonicity of \mathfrak{G} , we have $\mathfrak{G}(\beta) < 1 - t$. On the other hand, $\beta < x_n$ for all n large enough, thus $\mathfrak{G}_n(\beta) \geq 1 - t$ for all n large enough, and we conclude that $\mathfrak{G}(\beta) = \lim_{n \rightarrow \infty} \mathfrak{G}_n(\beta) \geq 1 - t$, which is a contradiction, and (8) is proved.

(ii) Similarly, if $G_n \preceq G$ for all n , then $\forall t \in [0, 1] \quad \lim_{n \rightarrow \infty} \hat{\mathfrak{G}}_n(t) = \tilde{\mathfrak{G}}(t)$.

(iii) We define $\underline{\mathfrak{G}}_n = \min\{\mathfrak{G}, \mathfrak{G}_n\}$ (resp. $\overline{\mathfrak{G}}_n = \max\{\mathfrak{G}, \mathfrak{G}_n\}$), and we denote by \underline{G}_n (resp. \overline{G}_n) the corresponding probability measure on $[0, +\infty]$. Notice that $\overline{G}_n \xrightarrow{d} G$ and $\underline{G}_n \xrightarrow{d} G$. Fix an edge e . Then $\overline{G}_n \succeq G$ for all n , and (i) implies that

$$\text{a.s.} \quad \lim_{n \rightarrow \infty} t_{\overline{G}_n}(e) = t_G(e).$$

As $\underline{G}_n \preceq G$ for all n and $t_G(e) = \tilde{\mathfrak{G}}(u(e))$ almost surely, (ii) implies that

$$\text{a.s.} \quad \lim_{n \rightarrow \infty} t_{\underline{G}_n}(e) = t_G(e).$$

Finally, as $\underline{G}_n \preceq G_n \preceq \overline{G}_n$ for all n , we know by coupling that $t_{\underline{G}_n}(e) \leq t_{G_n}(e) \leq t_{\overline{G}_n}(e)$, which gives the desired convergence. \square

2.5. Stabilization of the point \tilde{x} . We need to establish the monotonicity of the time constant in first-passage percolation. Since we work with different probability measures, the fact that, in the regularization process, the point $\tilde{x}^{\mathcal{C}_{G,M}}$ depends on G may be disturbing. We get round this problem by considering an alternative probability measure H :

Lemma 2.11. *Let G and H be probability measures on $[0, +\infty]$ such that $G \preceq H$. For all $M \in [0, +\infty)$ satisfying $H([0, M]) > p_c(d)$, we have*

$$\mu_G(x) = \inf_{n \in \mathbb{N}^*} \frac{\mathbb{E} \left[\tilde{T}_G^{\mathcal{C}_{H,M}}(0, nx) \right]}{n} = \lim_{n \rightarrow \infty} \frac{\tilde{T}_G^{\mathcal{C}_{H,M}}(0, nx)}{n} \text{ a.s. and in } L^1.$$

Proof. Since $G \preceq H$ we get by coupling that $t_G(e) \leq t_H(e)$ for all $e \in \mathbb{E}^d$. Let $M \in [0, +\infty)$ satisfying $H([0, M]) > p_c(d)$, then $G([0, M]) > p_c(d)$ and $\mathcal{C}_{H,M} \subset \mathcal{C}_{G,M}$. The proof of the convergence of $\tilde{T}_G^{\mathcal{C}_{H,M}}(0, nx)/n$ is a straightforward adaptation of the proof of the convergence of $\tilde{T}_G^{\mathcal{C}_{G,M}}(0, nx)/n$: by the subadditive ergodic theorem, there exists a function $\mu'_{G,H} : \mathbb{R}^d \rightarrow [0, +\infty)$ such that for all $x \in \mathbb{Z}^d$ we have

$$\mu'_{G,H}(x) = \inf_{n \in \mathbb{N}^*} \frac{\mathbb{E} \left[\tilde{T}_G^{\mathcal{C}_{H,M}}(0, nx) \right]}{n} = \lim_{n \rightarrow \infty} \frac{\tilde{T}_G^{\mathcal{C}_{H,M}}(0, nx)}{n} \text{ a.s. and in } L^1.$$

It remains to prove that $\mu'_{G,H} = \mu_G$. For any $x \in \mathbb{Z}^d$, for any $\varepsilon > 0$, we have

$$\begin{aligned} & \mathbb{P} \left[\left| \tilde{T}_G^{\mathcal{C}_{H,M}}(0, nx) - \tilde{T}_G^{\mathcal{C}_{G,M}}(0, nx) \right| > n\varepsilon \right] \\ & \leq \mathbb{P} \left[T_G(\tilde{0}^{\mathcal{C}_{G,M}}, \tilde{0}^{\mathcal{C}_{H,M}}) + T_G(\tilde{nx}^{\mathcal{C}_{G,M}}, \tilde{nx}^{\mathcal{C}_{H,M}}) > n\varepsilon \right] \\ (9) \quad & \leq 2 \mathbb{P} \left[T_G(\tilde{0}^{\mathcal{C}_{G,M}}, \tilde{0}^{\mathcal{C}_{H,M}}) > n\varepsilon/2 \right]. \end{aligned}$$

Since $\tilde{0}^{\mathcal{C}_{G,M}} \in \mathcal{C}_{G,M} \subset \mathcal{C}_{G,\infty}$ and $\tilde{0}^{\mathcal{C}_{H,M}} \in \mathcal{C}_{H,M} \subset \mathcal{C}_{G,M} \subset \mathcal{C}_{G,\infty}$, the time $T_G(\tilde{0}^{\mathcal{C}_{G,M}}, \tilde{0}^{\mathcal{C}_{H,M}})$ is finite a.s. thus the right hand side of inequality (9) goes to 0 as n goes to infinity. This concludes the proof of Lemma 2.11. \square

2.6. Monotonicity of the time constant in first-passage percolation. As a simple consequence of the coupling built in section 2.4 and the stabilization of the points \tilde{x} proposed in section 2.5, we obtain the monotonicity of the function $G \mapsto \mu_G$.

Lemma 2.12. *Let G, H be probability measures on $[0, +\infty]$. we have*

$$G \preceq H \implies \mu_G \leq \mu_H.$$

Proof. By construction of μ_G and μ_H , it suffices to prove that $\mu_G(x) \leq \mu_H(x)$ for all $x \in \mathbb{Z}^d$. By coupling, since $G \preceq H$, we have $t_G(e) \leq t_H(e)$ for every edge e . Using Lemma 2.11 the conclusion is immediate, since we have a.s.

$$\mu_G(x) = \lim_{n \rightarrow \infty} \frac{\tilde{T}_G^{\mathcal{C}_{H,M}}(0, nx)}{n} \leq \lim_{n \rightarrow \infty} \frac{\tilde{T}_H^{\mathcal{C}_{H,M}}(0, nx)}{n} = \mu_H(x).$$

\square

3. RENORMALIZATION

In this section we present the renormalization process and the construction of modified paths that will be useful to study both the time constant and the isoperimetric constant.

3.1. Definition of the renormalization process. Let G be a probability measure on $[0, +\infty]$, and fix M_0 such that $G([0, M_0]) > p_c(d)$. Denote as before by \mathcal{C}_{G, M_0} the a.s. unique infinite cluster of the supercritical Bernoulli field $\{1_{t_G(e) \leq M_0} : e \in \mathbb{E}^d\}$. We call this field the M_0 -percolation and its clusters the M_0 -clusters.

We use a renormalization process in the spirit of the work of Antal and Pisztora [1]. For a large integer N , that will be appropriately chosen later, we chop \mathbb{Z}^d into disjoint N -boxes as follows: we set B_N to be the box $[-N, N]^d \cap \mathbb{Z}^d$ and define the family of N -boxes by setting, for $\mathbf{i} \in \mathbb{Z}^d$,

$$B_N(\mathbf{i}) = \tau_{\mathbf{i}(2N+1)}(B_N),$$

where τ_b stands for the shift in \mathbb{Z}^d with vector $b \in \mathbb{Z}^d$. We will also refer to the box $B_N(\mathbf{i})$ as the N -box with coordinate \mathbf{i} . The coordinates of N -boxes will be denoted in bold and considered as macroscopic sites, to distinguish them from the microscopic sites in the initial graph \mathbb{Z}^d . We also introduce larger boxes: for $\mathbf{i} \in \mathbb{Z}^d$,

$$B'_N(\mathbf{i}) = \tau_{\mathbf{i}(2N+1)}(B_{3N}).$$

As in [1], we say that a connected cluster C is a crossing cluster for a box B , if for all d directions there is an open path contained in $C \cap B$ joining the two opposite faces of the box B .

Let $\mathcal{C}'_{G, M_0} = (\mathbb{Z}^d, \{e \in \mathbb{E}^d : t_g(e) \leq M_0\})$ be the graph whose edges are opened for the Bernoulli percolation $(1_{t_G(e) \leq M_0}, e \in \mathbb{E}^d)$. We recall that \mathcal{C}_{G, M_0} is the infinite cluster of \mathcal{C}'_{G, M_0} , and we have $D^{\mathcal{C}_{G, M_0}}(x, y) = D^{\mathcal{C}'_{G, M_0}}(x, y)$ for every vertices x and y in \mathcal{C}_{G, M_0} , and $D^{\mathcal{C}_{G, M_0}}(x, y) = +\infty$ if x or y are not in \mathcal{C}_{G, M_0} . Let us recall the following result, obtained by Antal and Pisztora [1, Theorem 1.1], that says that the chemical distance $D^{\mathcal{C}'_{G, M_0}}$ can't be too large when compared to $\|\cdot\|_1$ or $\|\cdot\|_\infty$ (or any other equivalent norm): there exist positive constants \hat{A}, \hat{B}, β such that

$$(10) \quad \forall x \in \mathbb{Z}^d \quad \mathbb{P}(\beta\|x\|_1 \leq D^{\mathcal{C}'_{G, M_0}}(0, x) < +\infty) \leq \hat{A} \exp(-\hat{B}\|x\|_1),$$

and

$$(11) \quad \forall x \in \mathbb{Z}^d \quad \mathbb{P}(\beta\|x\|_\infty \leq D^{\mathcal{C}'_{G, M_0}}(0, x) < +\infty) \leq \hat{A} \exp(-\hat{B}\|x\|_\infty).$$

In fact Antal and Pisztora proved (10), but different norms being equivalent in \mathbb{R}^d , we can obtain (11) by changing the constants.

Definition 3.1. *We say that the macroscopic site \mathbf{i} is good (or that the box $B_N(\mathbf{i})$ is good) if the following events occur:*

- (i) *There exists a unique M_0 -cluster \mathcal{C} in $B'_N(\mathbf{i})$ which has more than N vertices;*
- (ii) *This M_0 -cluster \mathcal{C} is crossing for each of the 3^d N -boxes included in $B'_N(\mathbf{i})$;*
- (iii) *For all $x, y \in B'_N(\mathbf{i})$, if $\|x - y\|_\infty \geq N$ and x and y belong to this M_0 -cluster \mathcal{C} , then $D^{\mathcal{C}'_{G, M_0}}(x, y) \leq 3\beta N$.*

We call this cluster \mathcal{C} the crossing M_0 -cluster of the good box $B_N(\mathbf{i})$.

Otherwise, $B_N(\mathbf{i})$ is said to be *bad*.

On the macroscopic grid \mathbb{Z}^d , we consider the same standard nearest neighbour graph structure as on the microscopic initial grid \mathbb{Z}^d . Moreover we say that two macroscopic sites \mathbf{i} and \mathbf{j} are $*$ -neighbors if and only if $\|\mathbf{i} - \mathbf{j}\|_\infty = 1$. If C is a connected set of macroscopic sites, we define its exterior vertex boundary

$$\partial_v C = \{\mathbf{i} \in \mathbb{Z}^d \setminus C : \mathbf{i} \text{ has a neighbour in } C\}.$$

For a bad macroscopic site \mathbf{i} , denote by $C(\mathbf{i})$ the connected cluster of bad macroscopic sites containing \mathbf{i} : the set $\partial_v C(\mathbf{i})$ is then a $*$ -connected set of good macroscopic sites. For a good macroscopic site \mathbf{i} , we define $\partial_v C(\mathbf{i})$ to be $\{\mathbf{i}\}$.

3.2. Modification of a path. Let $K \geq M_0$ and N be fixed. Let now γ be a path in \mathbb{Z}^d , starting from the origin. To γ , we associate the connected set $\Gamma \subset \mathbb{Z}^d$ of N -boxes it visits: this is a lattice animal containing the origin, *i.e.* a connected finite set of \mathbb{Z}^d containing the origin. We decompose γ into two parts, namely $\gamma_a = \{e \in \gamma : t_G(e) < K\}$ and $\gamma_b = \{e \in \gamma : t_G(e) \geq K\}$. We denote by Bad the (random) set of bad connected components of the macroscopic percolation given by the states of the N -boxes. The idea is to remove from γ the edges with passage time larger than K , and to look for bypasses for these edges using only edges with passage time less than M_0 .

Lemma 3.2. *Assume that $0 \in C_{G,M_0}$, that $x \in C_{G,M_0}$, that the N -boxes containing 0 and x are good and let γ be a path between 0 and x . Then there exists in C_{G,M_0} a path γ' between 0 and x that has the following properties :*

- (1) $\gamma' \setminus \gamma$ is a collection of disjoint self avoiding paths that intersect $\gamma' \cap \gamma$ only at their endpoints;
- (2) $|\gamma' \setminus \gamma| \leq \rho_d \left(N \sum_{C \in \text{Bad}: C \cap \Gamma \neq \emptyset} |C| + N^d |\gamma_b| \right)$, where ρ_d is a constant depending only on the dimension d .

See Figure 2.

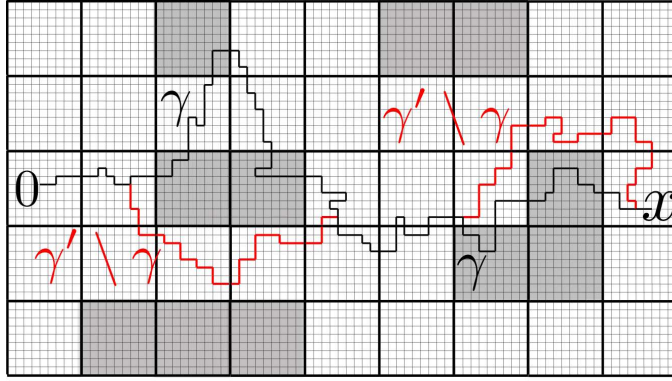


FIGURE 2. The paths γ and γ' (bad blocks are colored in dark gray).

Before proving Lemma 3.2, we need a simpler estimate on the cardinality of a path inside a set of good blocks.

Lemma 3.3. *There exists a constant $\hat{\rho}_d$, depending only on d , such that for every fixed N , for every $n \in \mathbb{N}^*$, if $(B_N(\mathbf{i}))_{\mathbf{i} \in \mathcal{I}}$ is a $*$ -connected set of n good N -blocks, if $x \in \mathcal{B}_N(\mathbf{j})$ for $\mathbf{j} \in \mathcal{I}$ and x is in the crossing M_0 -cluster of $\mathcal{B}_N(\mathbf{j})$, if $y \in \mathcal{B}_N(\mathbf{k})$ for $\mathbf{k} \in \mathcal{I}$ and y is in the crossing M_0 -cluster of $\mathcal{B}_N(\mathbf{k})$, then there exists a path from x to y in C'_{G,M_0} of length at most equal to $\hat{\rho}_d(Nn + N^d)$.*

Proof of Lemma 3.3. Since $(B_N(\mathbf{i}))_{\mathbf{i} \in \mathcal{I}}$ is a $*$ -connected set of good blocks, the definition of good boxes ensures that there exists a M_0 -cluster \mathcal{C} in $\mathcal{C}'_{G, M_0} \cap \cup_{\mathbf{i} \in \mathcal{I}} \{e \in B'_N(\mathbf{i})\}$ which is crossing for every N -box included in $\cup_{\mathbf{i} \in \mathcal{I}} B'_N(\mathbf{i})$ (see Proposition 2.1 in Antal and Pisztora [1]). Since x and y are in \mathcal{C} , there exists a path $\gamma = (\gamma_1, \dots, \gamma_p)$ from x to y in $\mathcal{C}'_{G, M_0} \cap \cup_{\mathbf{i} \in \mathcal{I}} \{e \in B'_N(\mathbf{i})\}$. Let $(\varphi_i)_{1 \leq i \leq r}$ be the path of macroscopic sites corresponding to the path of good blocks visited by γ (φ may not be injective). Notice that $r \leq 3^d n$. We now extract a sequence of points along γ . Let $\Psi(1) = 1$ and $j(1) = 1$. If $\Psi(1), \dots, \Psi(k)$ and $j(1), \dots, j(k)$ are defined, if the set $\{i \geq \Psi(k) : \|\varphi_i - \varphi_{\Psi(k)}\|_\infty \geq 2\}$ is non empty we define $\Psi(k+1) = \inf\{i \geq \Psi(k) : \|\varphi_i - \varphi_{\Psi(k)}\|_\infty \geq 2\}$ and we choose $j(k+1) \geq j(k)$ such that $\gamma_{j(k+1)} \in B_N(\varphi_{\Psi(k+1)})$; if the set $\{i \geq \Psi(k) : \|\varphi_i - \varphi_{\Psi(k)}\|_\infty \geq 2\}$ is empty we stop the process. We obtain a sequence $(\gamma_{j(k)}, k = 1, \dots, r')$ of points, with $r' \leq r$. By construction, for all $k \in \{1, \dots, r' - 1\}$, we have $\|\gamma_{j(k+1)} - \gamma_{j(k)}\|_\infty \geq N$ and

$$\|\varphi_{\Psi(k+1)} - \varphi_{\Psi(k+1)-1}\|_\infty = \|\varphi_{\Psi(k+1)-1} - \varphi_{\Psi(k)}\|_\infty = 1,$$

thus $\gamma_{j(k)} \in B'_N(\varphi_{\Psi(k+1)-1})$ and $\gamma_{j(k+1)} \in B'_N(\varphi_{\Psi(k+1)-1})$. For all $k \in \{1, \dots, r' - 1\}$, $B_N(\varphi_{\Psi(k+1)-1})$ is a good box, and $\gamma_{j(k)}$ and $\gamma_{j(k+1)}$ belong to the crossing M_0 -cluster of $B_N(\varphi_{\Psi(k+1)-1})$, thus there exists a path from $\gamma_{j(k)}$ to $\gamma_{j(k+1)}$ in \mathcal{C}'_{G, M_0} of length at most $3\beta N$. By glueing these paths, we obtain a path from $x = \gamma_{j(1)}$ to $\gamma_{j(r')}$ in \mathcal{C}'_{G, M_0} of length at most $3\beta N r' \leq 3^{d+1} \beta N n$. Finally, since y and $\gamma_{j(r')}$ belong to the crossing M_0 -cluster of $B'_N(\varphi_{\Psi(r')})$, there exists a path from $\gamma_{j(r')}$ to y in \mathcal{C}'_{G, M_0} of length at most $|\{e \in B'_N(\varphi_{\Psi(r')})\}| \leq 2d 3^d N^d$. \square

Proof of Lemma 3.2. To the path γ , we associate the sequence $\varphi_0 = (\varphi_0(j))_{1 \leq j \leq r_0}$ of N -boxes it visits. Note that φ is not necessarily injective, and that the previously defined lattice animal Γ is equal to $\varphi_0(\{1, \dots, r_0\})$.

From the sequence φ_0 , we extract the subsequence $(\varphi_1(j))_{1 \leq j \leq r_1}$, with $r_1 \leq r_0$, of N -boxes B such that $\gamma \cap B$ contains at least one edge with passage time larger than K (more precisely, we keep the indices of the boxes B that contain the smallest extremity, for the lexicographic order, of an edge of γ with passage time larger than K). Notice that $r_1 \leq |\gamma_b|$. The idea is the following:

- (1) If $\varphi_1(j)$ is good, we add to γ all the edges in B with passage time less than M_0 : there will be enough such edges in the good N -box to find a by-pass for the edge of γ with too large passage time.
- (2) If $\varphi_1(j)$ is bad, we will look for such a by-pass in the exterior vertex boundary $\partial_v C(\varphi_1(j))$ of the connected component of bad boxes of $\varphi_1(j)$.

In both cases, we use Lemma 3.3. We recall that if \mathbf{i} is good, then $\partial_v C(\mathbf{i}) = \{\mathbf{i}\}$. Note that some $\partial_v C(\varphi_1(j))$ may coincide or be nested one in another or overlap. In order to define properly the modification of our path, we need thus to extract a subsequence once again. We first consider the $*$ -connected components $(S_{\varphi_2(j)})_{1 \leq j \leq r_2}$, with $r_2 \leq r_1$, of the union of the $(\partial_v C(\varphi_1(j)))_{1 \leq j \leq r_1}$, by keeping only the smallest index for each connected component. Next, in case of nesting, we only keep the largest connected component. We denote by $(S_{\varphi_3(j)})_{1 \leq j \leq r_3}$, with $r_3 \leq r_2$, the remaining hypersurfaces of good N -boxes. Finally it may happen that γ visits several times the same $S_{\varphi_3(j)}$ for some j : in this situation we can and must remove the loops that γ makes between its different visits in $S_{\varphi_3(j)}$. Thus by a last extraction we obtain $(S_{\varphi_4(j)})_{1 \leq j \leq r_4}$, where $S_{\varphi_4(1)} = S_{\varphi_3(1)}$ and for all $k \geq 1$,

$\varphi_4(k+1)$ is the infimum of the indices $(\varphi_3(j))_{1 \leq j \leq r_3}$ such that γ visits $S_{\varphi_3(j)}$ after it exits $S_{\varphi_4(k)}$ for the last time (if such a j exists).

Note that the path γ must visit each $(S_{\varphi_4(j)})_{1 \leq j \leq r_4}$. We now cut γ in several pieces. Let $\Psi_{in}(1) = \min\{k \geq 1 : \gamma_k \in \cup_{\mathbf{i} \in S_{\varphi_4(1)}} B_N(\mathbf{i})\}$ and $\Psi_{out}(1) = \max\{k \geq \Psi_{in}(1) : \gamma_k \in \cup_{\mathbf{i} \in S_{\varphi_4(1)}} B_N(\mathbf{i})\}$. By recurrence, for all $2 \leq j \leq r_4$, we define $\Psi_{in}(j) = \min\{k \geq \Psi_{out}(j-1) : \gamma_k \in \cup_{\mathbf{i} \in S_{\varphi_4(j)}} B_N(\mathbf{i})\}$ and $\Psi_{out}(j) = \max\{k \geq \Psi_{in}(j) : \gamma_k \in \cup_{\mathbf{i} \in S_{\varphi_4(j)}} B_N(\mathbf{i})\}$. For all $1 \leq j \leq r_4 - 1$, let γ_j be the part of γ from $\gamma_{\Psi_{out}(j)}$ to $\gamma_{\Psi_{in}(j+1)}$. By construction γ_j contains no edge with passage time larger than K , and has at least N vertices in $B'_N(\mathbf{i})$ for some $\mathbf{i} \in S_{\varphi_4(j)}$ (resp. in $B'_N(\mathbf{k})$ for some $\mathbf{k} \in S_{\varphi_4(j+1)}$), thus $\gamma_j \cap B'_N(\mathbf{i})$ (resp. $\gamma_j \cap B'_N(\mathbf{k})$) is included in the crossing M_0 -cluster of $B'_N(\mathbf{i})$ (resp. $B'_N(\mathbf{k})$). This implies that for all $2 \leq j \leq r_4 - 1$, we can apply Lemma 3.3 to state that there exists a path γ'_j from $\gamma_{\Psi_{in}(j)}$ to $\gamma_{\Psi_{out}(j)}$ in \mathcal{C}'_{G, M_0} of length at most $\hat{\rho}_d(N^d + N|S_{\varphi_4(j)}|)$. To do the same in $S_{\varphi_4(1)}$, we have to study more carefully the beginning of the path γ . If $\mathbf{0}$ is in $S_{\varphi_4(1)}$, then $\Psi_{in}(1) = 1$ and $\gamma_{\Psi_{in}(1)} = 0$. As $0 \in \mathcal{C}_{G, M_0}$ and $B_N(\mathbf{0})$ is good, 0 is in the crossing M_0 -cluster of $B_N(\mathbf{0})$, and applying Lemma 3.3 we obtain the existence of a path γ'_1 from $x = \gamma_{\Psi_{in}(1)}$ to $\gamma_{\Psi_{out}(1)}$ in \mathcal{C}'_{G, M_0} of length at most $\hat{\rho}_d(N^d + N|S_{\varphi_4(1)}|)$. In this case we define $\gamma_0 = \emptyset$. If $\mathbf{0}$ is not in $S_{\varphi_4(1)}$, denote by γ_0 the portion of γ from 0 to $\gamma_{\Psi_{in}(1)}$. The same reasoning as before ensures that $\gamma_{\Psi_{in}(1)}$ belongs to the crossing M_0 -cluster of $B'_N(\mathbf{i})$ for some $\mathbf{i} \in S_{\varphi_4(1)}$, and we can again apply Lemma 3.3 to obtain the existence of a path γ'_1 from $\gamma_{\Psi_{in}(1)}$ to $\gamma_{\Psi_{out}(1)}$ in \mathcal{C}'_{G, M_0} of length at most $\hat{\rho}_d(N^d + N|S_{\varphi_4(1)}|)$. This reasoning can be readily transposed for the end of the path γ , thus we obtain the existence of a path γ'_{r_4} from $\gamma_{\Psi_{in}(r_4)}$ to $\gamma_{\Psi_{out}(r_4)}$ in \mathcal{C}'_{G, M_0} of length at most $\hat{\rho}_d(N^d + N|S_{\varphi_4(r_4)}|)$, and we denote by γ_{r_4} the portion of γ from $\gamma_{\Psi_{out}(r_4)}$ to x ($\gamma_{r_4} = \emptyset$ if $\gamma_{\Psi_{out}(r_4)} = x$).

We can glue together the paths $\gamma_0, \gamma'_1, \gamma_1, \gamma'_2, \dots, \gamma'_{r_4}, \gamma_{r_4}$ in this order to obtain a path γ' from 0 to x in \mathcal{C}'_{G, M_0} . Up to cutting parts of these paths, we can suppose that each γ'_i is a self-avoiding path, that the γ'_i are disjoint and that each γ'_i intersects only γ_{i-1} and γ_i , and only with its endpoints.

Finally we need an estimate on $|\gamma' \setminus \gamma|$. Obviously $\gamma' \setminus \gamma \subset \cup_{i=1}^{r_4} \gamma'_i$, thus

$$\begin{aligned} |\gamma' \setminus \gamma| &\leq \sum_{i=1}^{r_4} \hat{\rho}_d(N^d + N|S_{\varphi_4(j)}|) \\ &\leq r_4 \hat{\rho}_d N^d + \hat{\rho}_d N \sum_{i=1}^{r_4} |S_{\varphi_4(j)}| \\ &\leq 2\hat{\rho}_d N^d |\gamma_b| + \hat{\rho}_d N \sum_{C \in \text{Bad}: C \cap \Gamma \neq \emptyset} |\partial_v C|. \end{aligned}$$

To conclude, we just have to remark that $|\partial_v C| \leq 2d|C|$. □

4. TRUNCATED PASSAGE TIMES

This section is devoted to the proof of Theorems 1.5 and 1.6.

Let us denote by $\mathcal{M}(G_0, p_0, M_0)$ the set of probability measure on $[0, +\infty]$ such that $G \geq G_0$ and $G([0, M_0]) \geq p_0$ for a fixed $p_0 > p_c(d)$. For $G \in \mathcal{M}(G_0, p_0, M_0)$, denote as before by \mathcal{C}_{G, M_0} the a.s. unique infinite cluster of the supercritical

Bernoulli field $\{1_{t_G(e) \leq M_0} : e \in \mathbb{E}^d\}$. We call this field the M_0 -percolation and its clusters the M_0 -clusters.

4.1. Estimation for the passage time of the modified path.

Lemma 4.1. *Let G be in $\mathcal{M}(G_0, p_0, M_0)$. There exists a positive constant ρ'_d (depending only on d and M_0) such that the following holds: Assume that $0 \in \mathcal{C}_{G, M_0}$, that $x \in \mathcal{C}_{G, M_0}$ and that the N -boxes containing 0 and x are good, then*

$$T_G(0, x) \leq T_{G^\kappa}(0, x) \left(1 + \frac{\rho'_d N^d}{K}\right) + \rho'_d N \sum_{C \in \text{Bad}: C \cap \Gamma \neq \emptyset} |C|,$$

where Γ is the lattice animal of N -boxes visited by an optimal path between 0 and x for the passage times with distribution G^K .

Proof. As $0 \in \mathcal{C}_{G, M_0}$ and $x \in \mathcal{C}_{G, M_0}$, the quantities $T_G(0, x)$ and $T_{G^\kappa}(0, x)$ are bounded by M_0 times the chemical distance in \mathcal{C}_{G, M_0} between 0 and x , and are thus finite. Let γ be an optimal path between 0 and x for $T_{G^\kappa}(0, x)$, and consider the modification γ' given by Lemma 3.2. Since γ' is a path between 0 and x , we have

$$\begin{aligned} T_G(0, x) &\leq \sum_{e \in \gamma'} t_G(e) = \sum_{e \in \gamma \cap \gamma'} t_G(e) + \sum_{e \in \gamma' \setminus \gamma} t_G(e) \\ &\leq \sum_{e \in \gamma_a} t_G(e) + M_0 |\gamma' \setminus \gamma|. \end{aligned}$$

On one hand, since γ is an optimal path between 0 and x for $T_{G^\kappa}(0, x)$, we have

$$\sum_{e \in \gamma_a} t_G(e) = \sum_{e \in \gamma_a} t_{G^\kappa}(e) \leq \sum_{e \in \gamma} t_{G^\kappa}(e) = T_{G^\kappa}(0, x).$$

On the other hand, using the estimate on the cardinality of $\gamma' \setminus \gamma$ given in Lemma 3.2, and noticing that the number of edges in γ_b is less than $T_{G^\kappa}(\gamma)/K$, we obtain

$$|\gamma' \setminus \gamma| \leq \rho_d \left(\frac{N^d T_{G^\kappa}(\gamma)}{K} + N \sum_{C \in \text{Bad}: C \cap \Gamma \neq \emptyset} |C| \right).$$

□

Lemma 4.2. *For every $p < 1$, there exists an integer $N(p_0)$ such that for each $G \in \mathcal{M}(G_0, p_0, M_0)$, the field $(\mathbf{1}_{\{B_N(\mathbf{i}) \text{ good}\}})_{\mathbf{i} \in \mathbb{Z}^d}$ stochastically dominates a family of independent Bernoulli random variables with parameter p .*

Proof. Obviously, the states of $(B_N(\mathbf{i}))_{\mathbf{i} \in \mathbb{Z}^d}$ have a finite range of dependance and are identically distributed. Then, by the Liggett–Schonmann–Stacey Theorem [19], it is sufficient to check that $\lim_{N \rightarrow +\infty} \mathbb{P}(B_N \text{ good}) = 1$.

Consider first the properties (i) and (ii) of the Definition 3.1. When $d \geq 3$, the fact that $\lim_{N \rightarrow +\infty} \mathbb{P}(B_N \text{ satisfies (i) and (ii)}) = 1$ follows from the Pisztora coarse graining argument (see Pisztora [22] or the coarse graining section in Cerf [5]). When $d = 2$, see Couronné and Messikh [7]. It remains to study the property (iii) in the Definition 3.1. Using Antal and Pisztora's estimate (10), we have for all N

and \mathbf{i}

$\mathbb{P}[B_N(\mathbf{i}) \text{ does not satisfy (iii)}]$

$$\begin{aligned} &\leq \sum_{x \in B'_N(\mathbf{i})} \sum_{y \in B'_N(\mathbf{i})} 1_{\|x-y\|_\infty \geq N} \mathbb{P} \left[x \xleftrightarrow{C'_{G, M_0}} y, D^{C'_{G, M_0}}(x, y) \geq 3\beta N \right] \\ &\leq \sum_{x \in B'_N(\mathbf{i})} \sum_{y \in B'_N(\mathbf{i})} 1_{\|x-y\|_\infty \geq N} \mathbb{P} \left[x \xleftrightarrow{C'_{G, M_0}} y, D^{C'_{G, M_0}}(x, y) \geq \beta \|x-y\|_\infty \right] \\ &\leq \sum_{x \in B'_N(\mathbf{i})} \sum_{y \in B'_N(\mathbf{i})} 1_{\|x-y\|_\infty \geq N} \hat{A} e^{-\hat{B} \|x-y\|_\infty} \\ &\leq (3N)^d \cdot (3N)^d \hat{A} e^{-\hat{B} N} \end{aligned}$$

that goes to 0 when N goes to infinity. \square

Lemma 4.3. *There exists a constant \tilde{C}_d , depending only on d , such that for every path γ of \mathbb{Z}^d , for every $N \in \mathbb{N}^*$, if Γ is the animal of N -blocks that γ visits, then*

$$|\Gamma| \leq \tilde{C}_d \left(1 + \frac{|\gamma| + 1}{N} \right) - 1.$$

Proof. Let $\gamma = (\gamma_i)_{i=1, \dots, n}$ be a path of \mathbb{Z}^d for a $n \in \mathbb{N}^*$ (γ_i is the i -th vertex of γ , $n = |\gamma| + 1$), and fix $N \in \mathbb{N}^*$. Let Γ be the animal of N -blocks that γ visits. We will include Γ in a bigger set of blocks whose size can be controlled. Let $p(1) = 1$ and \mathbf{i}_1 be the macroscopic site such that $\gamma_1 \in B_N(\mathbf{i}_1)$. If $p(1), \dots, p(k)$ and $\mathbf{i}_1, \dots, \mathbf{i}_k$ are constructed, define $p(k+1) = \inf\{j \in \{p(k), \dots, n\} : \gamma_j \notin B'_N(\mathbf{i}_k)\}$ if this set is not empty and let \mathbf{i}_{k+1} be the macroscopic site such that $\gamma_{p(k+1)} \in B_N(\mathbf{i}_{k+1})$, and stop the process if for every $j \in \{p(k), \dots, n\}$, $\gamma_j \in B'_N(\mathbf{i}_k)$. We obtain two finite sequences $(p(1), \dots, p(r))$ and $(\mathbf{i}_1, \dots, \mathbf{i}_r)$. First notice that

$$\Gamma \subset \bigcup_{k=1}^r B'_N(\mathbf{i}_k)$$

by construction, thus $|\Gamma| \leq 3^d r - 1$. Moreover for every $k \in \{1, \dots, r-1\}$, $\|\gamma_{p(k+1)} - \gamma_{p(k)}\|_1 \geq N$, thus $p(k+1) - p(k) \geq N$. This implies that $N(r-1) \leq p(r) - p(1) \leq n$, and we conclude that

$$|\Gamma| \leq 3^d \left(1 + \frac{n}{N} \right) - 1.$$

\square

Lemma 4.4. *Suppose that $G_0(\{0\}) < p_c(d)$. For every $\varepsilon > 0$ there exists $p_1(\varepsilon) > 0$ and $A(\varepsilon) > 0$ such that for every $K \geq M_0$, for each $G \in \mathcal{M}(G_0, p_0, M_0)$, for all x large enough,*

$$\mathbb{P} \left(\tilde{T}_G^{C_{G, M_0}}(0, x) \leq \tilde{T}_{G^K}^{C_{G, M_0}}(0, x) \left(1 + \frac{A(\varepsilon)}{K} \right) + \varepsilon \|x\|_1 \right) \geq p_1(\varepsilon).$$

Proof. Let $\varepsilon > 0$ be fixed. Let $p_c^{\text{site}}(d)$ be the critical parameter for independent Bernoulli site percolation on \mathbb{Z}^d . Choose $\alpha = \alpha(\varepsilon) > 0$ and then $p = p(\varepsilon) \in$

$(p_c^{\text{site}}(d), 1)$, such that

$$(12) \quad 7^d \exp(-\alpha\varepsilon) \leq \frac{1}{3},$$

$$(13) \quad p + \frac{e^\alpha 7^d (1-p)}{1 - e^\alpha 7^d (1-p)} \leq \frac{3}{2}.$$

Let finally $N = N(\varepsilon)$ be large enough to have the stochastic comparison of Lemma 4.2 with this parameter p .

Since $G_0^K(\{0\}) = G_0(\{0\}) < p_c(d)$, there exist positive constants A', B', β' such that for all $k \in \mathbb{N}^*$ (see Proposition 5.8 in Kesten [18]):

$$(14) \quad \mathbb{P} \left[\exists r \text{ s.a. path starting at } 0 \text{ s.t. } |r| \geq k \text{ and } T_{G_0^K}(r) \leq \beta'k \right] \leq A' \exp(-B'k).$$

Obviously, for each $G \in \mathcal{M}(G_0, p_0, M_0)$, we also have

$$(15) \quad \mathbb{P} \left[\exists r \text{ s.a. path starting at } 0 \text{ s.t. } |r| \geq k \text{ and } T_{G^K}(r) \leq \beta'k \right] \leq A' \exp(-B'k).$$

Let $K \geq M_0$. Fix a large x . Let E_x be the following good event: $0 \in \mathcal{C}_{G, M_0}$ and $x \in \mathcal{C}_{G, M_0}$ and $D^{\mathcal{C}_{G, M_0}}(0, x) \leq \beta \|x\|_1$ and the N -boxes containing 0 and x are good. Note that the FKG inequality and Antal and Pisztor's estimate (10) ensure that

$$(16) \quad P_1(x) \stackrel{\text{def}}{=} \mathbb{P}(E_x) \geq \mathbb{P}(0 \in \mathcal{C}_{G, M_0})^2 (1 - \hat{A} \exp(-\hat{B} \|x\|_1))^2.$$

From now on, we only work on the event E_x . As $0 \in \mathcal{C}_{G, M_0}$ and $x \in \mathcal{C}_{G, M_0}$, we have $\tilde{T}_G^{\mathcal{C}_{G, M_0}}(0, x) = T_G(0, x)$ and $\tilde{T}_{G^K}^{\mathcal{C}_{G, M_0}}(0, x) = T_{G^K}(0, x) \leq M_0 D^{\mathcal{C}_{G, M_0}}(0, x)$. Equation (15) implies the existence of a geodesic γ_x for $T_{G^K}(0, x)$. Note that

$$\begin{aligned} \left\{ E_x, |\gamma_x| \geq \frac{\beta M_0}{\beta'} \|x\|_1 \right\} &\subset \{ |\gamma_x| \geq \frac{\beta M_0}{\beta'} \|x\|_1, T_{G^K}(\gamma_x) \leq \beta M_0 \|x\|_1 \} \\ &\cup \{ E_x, T_{G^K}(\gamma_x) > \beta M_0 \|x\|_1 \} \\ &\subset \{ |\gamma_x| \geq \frac{\beta M_0}{\beta'} \|x\|_1, T_{G^K}(\gamma_x) \leq \beta M_0 \|x\|_1 \} \\ &\cup \{ \beta \|x\|_1 \leq D^{\mathcal{C}_{G, M_0}}(0, x) < +\infty \} \end{aligned}$$

Then, (10) and (15) imply the existence of positive constants A'', B'' such that all $x \in \mathbb{Z}^d$:

$$\mathbb{P} \left(E_x, |\gamma_x| \geq \frac{\beta M_0}{\beta'} \|x\|_1 \right) \leq A'' \exp(-B'' \|x\|_1).$$

Thus we set $\beta'' = \frac{\beta M_0}{\beta'} > 0$ and if E'_x denotes the event $E_x \cap \{ |\gamma_x| \leq \beta'' \|x\|_1 \}$, then

$$(17) \quad P_2(x) \stackrel{\text{def}}{=} \mathbb{P}(E'_x) \geq \mathbb{P}(E_x) - A'' \exp(-B'' \|x\|_1).$$

Denote by Γ_x the lattice animal of the N -boxes visited by this geodesic. Note that by construction, on the event E_x , we have $|\Gamma_x| \geq \|x\|_1/N$. On the other hand Lemma 4.3 implies that $|\Gamma_x| \leq \tilde{C}_d(1 + (|\gamma_x| + 1)/N) - 1 \leq 2\tilde{C}_d |\gamma_x|/N$ at least for x large enough (remember that $|\gamma_x| \geq \|x\|_1$). Denote by $\mathcal{Animals}$ the set of lattice animals containing 0, and $\mathcal{Animals}_n$ the subset of those having size n . Using

Lemma 4.1, we have for x large enough

$$\begin{aligned}
P_3(x) &\stackrel{\text{def}}{=} \mathbb{P} \left(E'_x, \tilde{T}_G^{\mathcal{C}G, M_0}(0, x) \geq \tilde{T}_{GK}^{\mathcal{C}G, M_0}(0, x) \left(1 + \frac{\rho'_d N(\varepsilon)^d}{K} \right) + 2\varepsilon \tilde{C}_d \rho'_d \beta'' \|x\|_1 \right) \\
&\leq \mathbb{P} \left(E'_x, \sum_{C \in \text{Bad}: C \cap \Gamma_x \neq \emptyset} |C| \geq \varepsilon |\Gamma_x| \right) \\
(18) \quad &\leq \mathbb{P} \left(\exists \Gamma \in \text{Animals}, |\Gamma| \geq \frac{\|x\|_1}{N}, \sum_{C \in \text{Bad}: C \cap \Gamma \neq \emptyset} |C| \geq \varepsilon |\Gamma| \right) \stackrel{\text{def}}{=} P_4(\|x\|_1) \\
&\leq \sum_{n \geq \frac{\|x\|_1}{N}} \sum_{\Gamma \in \text{Animals}_n} \mathbb{P} \left(\sum_{C \in \text{Bad}: C \cap \Gamma \neq \emptyset} |C| \geq \varepsilon |\Gamma| \right) \\
&\leq \sum_{n \geq \frac{\|x\|_1}{N}} \sum_{\Gamma \in \text{Animals}_n} \mathbb{P}_p \left(\sum_{C \in \text{Bad}: C \cap \Gamma \neq \emptyset} |C| \geq \varepsilon |\Gamma| \right).
\end{aligned}$$

For the last inequality, we use the coupling Lemma 4.2 to replace the locally dependent states of our N -boxes by an independent Bernoulli site percolation with parameter p chosen in (13). From now on, we work with this Bernoulli site percolation with parameter p . Denote by $C(0)$ the connected component of closed sites containing 0 (with the convention that if 0 is open, then $C(0) = \emptyset$). Let $(\tilde{C}(i))_{i \in \mathbb{Z}^d}$ be independent and identically distributed random sets of \mathbb{Z}^d with the same law as $C(0)$. Fix a set $\Gamma = (\Gamma(i))_{1 \leq i \leq n}$ of sites; we first prove that, for the independent Bernoulli site percolation, the following stochastic comparison holds:

$$(19) \quad \sum_{C \in \text{Bad}: C \cap \Gamma \neq \emptyset} |C| \preceq \sum_{i=1}^n |\tilde{C}(i)|.$$

The idea is to build algorithmically the real clusters from the sequence of pre-clusters $(\tilde{C}(i))_{i \in \mathbb{Z}^d}$, as in the work of Fontes and Newman [11], proof of Theorem 4. Note however that in our sum (19), each visited cluster is only counted once, while they count each cluster the number of times it is visited, which explains the difference between our stochastic domination and their one. We proceed by induction on $j \in \{1, \dots, n\}$ to build a new family $(\bar{C}(i))_{1 \leq i \leq n}$ such that

$$A_j \stackrel{\text{def}}{=} \bigcup_{C \in \text{Bad}: C \cap \{\Gamma(i): 1 \leq i \leq j\} \neq \emptyset} C \stackrel{\text{law}}{\subset} \bigcup_{i=1}^j \bar{C}(i) \subset \bigcup_{i=1}^j (\Gamma(i) + \tilde{C}(i)).$$

Set $\bar{C}(1) = \Gamma(1) + \tilde{C}(1)$. Assume now that $(\bar{C}(i))_{1 \leq i \leq j}$ are built for some $j < n$:

- if $\Gamma(j+1) \in A_j$, then $A_{j+1} = A_j$, so we set $\bar{C}(j+1) = \emptyset$;
- if $\Gamma(j+1) \in \partial_v A_j$ (the exterior vertex boundary of A_j), then it is a good site, so we set $\bar{C}(j+1) = \emptyset$;
- otherwise, the conditional distribution of the bad cluster C containing the site $\Gamma(j+1)$, given A_j , is that of the percolation cluster of $\Gamma(j+1)$ in a site percolation model where \mathbb{Z}^d is replaced by $\mathbb{Z}^d \setminus (A_j \cup \partial_v A_j)$; thus, it has the same law as the connected component of $\Gamma(j+1)$ in

$$\bar{C}(j+1) = (\Gamma(j+1) + \tilde{C}(j+1)) \setminus (A_j \cup \partial_v A_j),$$

which ends the construction and proves (19). As the number of lattice animals containing 0 with size n is bounded from above by $(7^d)^n$ (see Kesten [17], p 82. or Grimmett [15], p.85), we have, by the Markov inequality,

$$P_4(\|x\|_1) \leq \sum_{n \geq \frac{\|x\|_1}{N}} (7^d)^n \exp(-\alpha \varepsilon n) (\mathbb{E}_p(\exp(\alpha |C(0)|)))^n.$$

But

$$\begin{aligned} \mathbb{E}_p(\exp(\alpha |C(0)|)) &= p + \sum_{k \geq 1} \exp(\alpha k) \mathbb{P}(|C(0)| = k) \leq p + \sum_{k \geq 1} \exp(\alpha k) \mathbb{P}_p(|C(0)| \geq k) \\ &\leq p + \sum_{k \geq 1} \exp(\alpha k) (7^d)^k (1-p)^k = p + \frac{e^\alpha 7^d (1-p)}{1 - e^\alpha 7^d (1-p)}. \end{aligned}$$

With the choices (12) and (13) we made for α and p , this ensures that

$$(20) \quad P_4(\|x\|_1) \leq \sum_{n \geq \frac{\|x\|_1}{N}} 2^{-n} \leq 2^{-\frac{\|x\|_1}{N} + 1}.$$

To conclude, note that with (16), (17), (18) and (20), we get

$$\begin{aligned} \mathbb{P} \left(\tilde{T}_G^{\mathcal{C}_{G, M_0}}(0, x) \geq \tilde{T}_{G^K}^{\mathcal{C}_{G, M_0}}(0, x) \left(1 + \frac{\rho'_d N(\varepsilon)^d}{K} \right) + 2\varepsilon \tilde{C}_d \rho'_d \beta'' \|x\|_1 \right) \\ \leq 1 - P_2(x) + P_3(x) \leq 1 - P_2(x) + P_4(\|x\|_1) \leq 1 - p_1(\varepsilon), \end{aligned}$$

for a well-chosen $p_1(\varepsilon) > 0$ and every x large enough. \square

4.2. Proof of Theorem 1.5 and Theorem 1.6. Suppose that $G_0(\{0\}) < p_c(d)$. For any $\varepsilon > 0$, consider $p_1(\varepsilon)$ and $A(\varepsilon)$ as given by Lemma 4.4, and define, for $K \geq M_0$, $\Psi(K) = \inf_{\varepsilon > 0} \frac{A(\varepsilon)}{K} + \varepsilon$. It is easy to see that $\lim_{K \rightarrow +\infty} \Psi(K) = 0$. Fix $\varepsilon > 0$, $\delta > 0$, $K \geq M_0$ and $x \in \mathbb{Z}^d$. Now consider $G \in \mathcal{M}(G_0, p_0, M_0)$.

With the convergence (2) in Proposition 2.2 and Lemma 4.4, we can choose n large enough such that

$$\begin{aligned} \mathbb{P} \left(\mu_G(x) - \delta \leq \frac{\tilde{T}_G^{\mathcal{C}_{G, M_0}}(0, nx)}{n} \right) &\geq 1 - \frac{p_1(\varepsilon)}{3}, \\ \mathbb{P} \left(\frac{\tilde{T}_{G^K}^{\mathcal{C}_{G, M_0}}(0, nx)}{n} \leq \mu_{G^K}(x) + \delta \right) &\geq 1 - \frac{p_1(\varepsilon)}{3}, \\ \mathbb{P} \left(\tilde{T}_G^{\mathcal{C}_{G, M_0}}(0, nx) \leq \tilde{T}_{G^K}^{\mathcal{C}_{G, M_0}}(0, nx) \left(1 + \frac{A(\varepsilon)}{K} \right) + \varepsilon n \|x\|_1 \right) &\geq p_1(\varepsilon). \end{aligned}$$

For every $\varepsilon > 0$, for every $\delta > 0$, on the intersection of these 3 events, that has positive probability, we obtain

$$\begin{aligned} \forall G \in \mathcal{M}(G_0, p_0, M_0), K \geq M_0, x \in \mathbb{Z}^d \\ \mu_G(x) - \delta \leq (\mu_{G^K}(x) + \delta) \left(1 + \frac{A(\varepsilon)}{K} \right) + \varepsilon \|x\|_1, \end{aligned}$$

and by letting δ going to 0 we get

$$\forall \varepsilon > 0, G \in \mathcal{M}(G_0, p_0, M_0), K \geq M_0, x \in \mathbb{Z}^d \quad \mu_G(x) \leq \mu_{G^K}(x) \left(1 + \frac{A(\varepsilon)}{K} \right) + \varepsilon \|x\|_1.$$

It follows that for every $\varepsilon > 0$,

$$\begin{aligned} 0 \leq \mu_{G^K}(x) - \mu_G(x) &\leq \mu_{G^K} \frac{A(\varepsilon)}{K} + \varepsilon \|x\|_1 \\ &\leq (\mu_G(x) + \|x\|_1) \left(\frac{A(\varepsilon)}{K} + \varepsilon \right), \end{aligned}$$

thus, by optimizing ε ,

$$0 \leq \mu_{G^K}(x) - \mu_G(x) \leq (\mu_G(x) + \|x\|_1) \Psi(K),$$

so Theorem 1.6 is proved.

Let us prove Theorem 1.5. If $G(\{0\}) \geq p_c(d)$, then $\mu_{G^K}(x) = \mu_G(x) = 0$, so there is nothing to prove. Otherwise, take $G = G_0$ and simply consider M_0 large enough so that $G([0, M_0]) > p_c(d)$ and take $p_0 = G([0, M_0])$. Now, since $G \in \mathcal{M}(G_0, p_0, M_0)$, we can apply Theorem 1.5 and use the fact that $\lim_{K \rightarrow +\infty} \Psi(K) = 0$.

5. CONTINUITY OF THE TIME CONSTANT, THEOREM 1.2

We state now two properties that will be proved in the next sections.

Lemma 5.1. *Suppose that $G, (G_n)_{n \in \mathbb{N}}$ are probability measures on $[0, +\infty]$ such that $G([0, +\infty)) > p_c(d)$ and $G_n([0, +\infty)) > p_c(d)$ for all $n \in \mathbb{N}$. If $G_n \xrightarrow{d} G$ and $G_n \succeq G$ for all n , then*

$$\forall x \in \mathbb{Z}^d, \quad \overline{\lim}_{n \rightarrow +\infty} \mu_{G_n}(x) \leq \mu_G(x).$$

Lemma 5.2. *Suppose that $G, (G_n)_{n \in \mathbb{N}}$ are probability measures on $[0, R]$ for some common and finite $R \in [0, +\infty)$. If $G_n \xrightarrow{d} G$, then*

$$\forall x \in \mathbb{Z}^d, \quad \lim_{n \rightarrow \infty} \mu_{G_n}(x) = \mu_G(x).$$

5.1. Proof of Theorem 1.2. We follow the general structure of Cox and Kesten's proof of the continuity of the time constant in first-passage percolation with finite passage times in [10]. We first deduce Theorem 1.2 from Theorem 1.5 and Lemmas 5.1 and 5.2.

Proof of Theorem 1.2. Let $G, (G_n)_{n \in \mathbb{N}}$ be probability measures on $[0, +\infty]$. We first prove that for all fixed $x \in \mathbb{Z}^d$, we have

$$(21) \quad \lim_{n \rightarrow \infty} \mu_{G_n}(x) = \mu_G(x).$$

We define $\underline{\mathfrak{G}}_n = \min\{\mathfrak{G}, \mathfrak{G}_n\}$ (resp. $\overline{\mathfrak{G}}_n = \max\{\mathfrak{G}, \mathfrak{G}_n\}$), and we denote by \underline{G}_n (resp. \overline{G}_n) the corresponding probability measure on $[0, +\infty]$. Then $\underline{\mathfrak{G}}_n \leq \mathfrak{G} \leq \overline{\mathfrak{G}}_n$ (resp. $\underline{G}_n \leq G_n \leq \overline{G}_n$), thus by Lemma 2.12 we have $\mu_{\underline{G}_n}(x) \leq \mu_G(x) \leq \mu_{\overline{G}_n}(x)$. To conclude that (21) holds, it is sufficient to prove that

$$(i) \quad \underline{\lim}_{n \rightarrow \infty} \mu_{\underline{G}_n}(x) \geq \mu_G(x) \quad \text{and} \quad (ii) \quad \overline{\lim}_{n \rightarrow +\infty} \mu_{\overline{G}_n}(x) \leq \mu_G(x).$$

Notice that $\overline{G}_n \xrightarrow{d} G$ and $\underline{G}_n \xrightarrow{d} G$. Inequality (ii) is obtained by a straightforward application of Lemma 5.1. For any $K \in [0, +\infty)$, we define $G^K = 1_{[0, K]}G + G([K, +\infty])\delta_K$ (resp. $\underline{G}_n^K = 1_{[0, K]}\underline{G}_n + \underline{G}_n([K, +\infty])\delta_K$), the distribution of

$t_G^K(e) = \min(t_G(e), K)$ (resp. $t_{\underline{G}_n}^K(e) = \min(t_{\underline{G}_n}(e), K)$). Using Lemmas 2.12 and 5.2, since $\underline{G}_n^K \xrightarrow{d} G^K$, we obtain for all K

$$\varliminf_{n \rightarrow \infty} \mu_{\underline{G}_n}(x) \geq \lim_{n \rightarrow \infty} \mu_{\underline{G}_n^K}(x) = \mu_{G^K}(x),$$

and by Theorem 1.5 we have $\lim_{K \rightarrow \infty} \mu_{G^K}(x) = \mu_G(x)$. This concludes the proof of (i), and of (21).

By homogeneity, (21) also holds for all $x \in \mathbb{Q}^d$. We know that $|\mu_{G_n}(x) - \mu_{G_n}(y)| \leq \mu_{G_n}(e_1) \|x - y\|_1$, where $e_1 = (1, 0, \dots, 0)$. Moreover $\lim_{n \rightarrow \infty} \mu_{G_n}(e_1) = \mu_G(e_1)$, thus for all $n \geq n_0$ large enough we have $|\mu_{G_n}(x) - \mu_{G_n}(y)| \leq 2\mu_G(e_1) \|x - y\|_1$ for all $x, y \in \mathbb{R}^d$. This implies that for any fixed $\varepsilon > 0$, there exists $\eta > 0$ such that for all $x, y \in \mathbb{R}^d$ such that $\|x - y\|_1 \leq \eta$, we have

$$\sup\{|\mu_G(x) - \mu_G(y)|, |\mu_{G_n}(x) - \mu_{G_n}(y)|, n \geq n_0\} \leq \varepsilon.$$

There exists a finite set (y_1, \dots, y_m) of rational points of \mathbb{R}^d such that

$$\mathbb{S}^{d-1} \subset \bigcup_{i=1}^m \{x \in \mathbb{R}^d : \|y_i - x\|_1 \leq \eta\}.$$

Thus $\varlimsup_{n \rightarrow +\infty} \sup_{x \in \mathbb{S}^{d-1}} |\mu_{G_n}(x) - \mu_G(x)| \leq 2\varepsilon + \lim_{n \rightarrow +\infty} \max_{i=1, \dots, m} |\mu_{G_n}(y_i) - \mu_G(y_i)| = 2\varepsilon$.

Since ε was arbitrary, Theorem 1.2 is proved. \square

We prove Lemmas 5.1 and 5.2 in the next sections, using a bound on sequences of probability measures.

5.2. Bound on sequences of probability measures.

Lemma 5.3. *Suppose that G and $(G_n)_{n \in \mathbb{N}}$ are probability measures on $[0, +\infty]$ such that $G_n \xrightarrow{d} G$.*

- (i) *If $G([0, +\infty)) > p_c(d)$ and $G_n([0, +\infty)) > p_c(d)$ for all $n \in \mathbb{N}$, then there exists a probability measure H^+ on $[0, +\infty]$ such that $G_n \preceq H^+$ for all n and $H^+([0, +\infty)) > p_c(d)$.*
- (ii) *If $G(\{0\}) < p_c(d)$ and $G_n(\{0\}) < p_c(d)$ for all $n \in \mathbb{N}$, then there exists a probability measure H^- on $[0, +\infty]$ such that $G_n \succeq H^-$ for all n and $H^-(\{0\}) < p_c(d)$.*

Proof. (i) We define $\hat{\mathfrak{H}}^+ = \sup_{n \in \mathbb{N}} \mathfrak{G}_n$, and $\mathfrak{H}^+(x) = \inf\{\hat{\mathfrak{H}}^+(y) : y < x\}$ for all $x \in [0, +\infty)$. Then $\hat{\mathfrak{H}}^+$ and \mathfrak{H}^+ are non-increasing functions defined on $[0, +\infty)$ and they take values in $[0, 1]$. By construction \mathfrak{H}^+ is left continuous and $\mathfrak{H}^+ \geq \mathfrak{G}_n$, for all $n \in \mathbb{N}$. Moreover we have $\hat{\mathfrak{H}}^+(x) = \mathfrak{H}^+(x) = 1$ for all $x \leq 0$. Thus there exists a probability measure H^+ on $[0, +\infty]$ such that $\mathfrak{H}^+(t) = H^+([t, +\infty])$ for all $t \in [0, +\infty)$. It remains to prove that $H^+([0, +\infty)) > p_c(d)$. Since $G([0, +\infty)) > p_c(d)$, i.e. $\lim_{+\infty} \mathfrak{G} < 1 - p_c(d)$, there exist $A \in [0, +\infty)$ and $\varepsilon > 0$ such that \mathfrak{G} is continuous at A and $\mathfrak{G}(A) \leq 1 - p_c(d) - 2\varepsilon$. Moreover $G_n \xrightarrow{d} G$ and \mathfrak{G} is continuous at A , thus there exists n_0 such that for all $n \geq n_0$ we have $\mathfrak{G}_n(A) \leq \mathfrak{G}(A) + \varepsilon \leq 1 - p_c(d) - \varepsilon$. For any $i \in \{1, \dots, n_0 - 1\}$, $G_i([0, +\infty)) > p_c(d)$ thus there exists $A_i < +\infty$ such that $\mathfrak{G}_i(A_i) < 1 - p_c(d)$. Fix $A' = \max(A, A_0, \dots, A_{n_0-1}) < +\infty$.

We conclude that

$$\begin{aligned}\hat{\mathfrak{H}}^+(A') &= \max\left(\mathfrak{G}_0(A'), \dots, \mathfrak{G}_{n_0-1}(A'), \sup_{n \geq n_0} \mathfrak{G}_n(A')\right) \\ &\leq \max\left(\mathfrak{G}_0(A_0), \dots, \mathfrak{G}_{n_0-1}(A_{n_0-1}), \sup_{n \geq n_0} \mathfrak{G}_n(A)\right) < 1 - p_c(d),\end{aligned}$$

thus $H^+([0, +\infty)) = 1 - \lim_{+\infty} \mathfrak{H}^+ > p_c(d)$.

(ii) We define $\mathfrak{H}^- = \inf_{n \in \mathbb{N}} \mathfrak{G}_n$. Then \mathfrak{H}^- is non-increasing, defined on $[0, +\infty)$ and it takes values in $[0, 1]$. Fix $t_0 \in [0, +\infty)$. Let us prove that \mathfrak{H}^- is left continuous at t_0 . By definition of \mathfrak{H}^- , for any $\varepsilon > 0$, there exists n_0 such that $\mathfrak{H}^-(t_0) \geq \mathfrak{G}_{n_0}(t_0) - \varepsilon$. Since \mathfrak{G}_{n_0} is left continuous, there exists $\eta > 0$ such that for all $t \in (t_0 - \eta, t_0]$ we have $\mathfrak{G}_{n_0}(t) \leq \mathfrak{G}_{n_0}(t_0) + \varepsilon$. Thus for all $t \in (t_0 - \eta, t_0]$, we obtain

$$\mathfrak{H}^-(t) \leq \mathfrak{G}_{n_0}(t) \leq \mathfrak{G}_{n_0}(t_0) + \varepsilon \leq \mathfrak{H}^-(t_0) + 2\varepsilon,$$

thus \mathfrak{H}^- is right continuous. By construction $\mathfrak{H}^- \leq \mathfrak{G}_n$, for all $n \in \mathbb{N}$. Moreover $\mathfrak{H}^-(t) = 1$ for all $t \leq 0$. Thus there exists a probability measure H^- on $[0, +\infty)$ such that $\mathfrak{H}^-(t) = H^-([t, +\infty))$ for all $t \in [0, +\infty)$. It remains to prove that $H^-({0}) < p_c(d)$. Since $G({0}) < p_c(d)$, there exists $\eta > 0$ such that $G([0, \eta)) < p_c(d)$, i.e., $\mathfrak{G}(\eta) > 1 - p_c(d)$. Let $\varepsilon > 0$ such that $\mathfrak{G}(\eta) \geq 1 - p_c(d) + 2\varepsilon$. There exists $\delta \in [0, \eta)$ such that \mathfrak{G} is continuous at δ . Then $\lim_{n \rightarrow \infty} \mathfrak{G}_n(\delta) = \mathfrak{G}(\delta)$, thus there exists n_0 such that for all $n \geq n_0$, $\mathfrak{G}_n(\delta) \geq \mathfrak{G}(\delta) - \varepsilon \geq 1 - p_c(d) + \varepsilon$. For any $i \in \{1, \dots, n_0 - 1\}$, there exists $\delta_i > 0$ such that $\mathfrak{G}_i(\delta_i) > 1 - p_c(d)$. Fix $\delta' = \min(\delta, \delta_0, \dots, \delta_{n_0-1}) > 0$. We conclude that

$$\begin{aligned}\mathfrak{H}^-(\delta') &= \min\left(\mathfrak{G}_0(\delta'), \dots, \mathfrak{G}_{n_0-1}(\delta'), \inf_{n \geq n_0} \mathfrak{G}_n(\delta')\right) \\ &\geq \min\left(\mathfrak{G}_0(\delta_0), \dots, \mathfrak{G}_{n_0-1}(\delta_{n_0-1}), \inf_{n \geq n_0} \mathfrak{G}_n(\delta)\right) > 1 - p_c(d),\end{aligned}$$

and

$$H^-({0}) = 1 - \lim_{t \rightarrow 0, t > 0} \mathfrak{H}^-(t) \leq 1 - \mathfrak{H}^-(\delta') < p_c(d).$$

□

5.3. Upper bound on $\lim_{n \rightarrow \infty} \mu_{G_n}$. This section is devoted to the proof of Lemma 5.1. We follow the structure of Cox and Kesten's proof of Lemma 1 in [10].

Proof of Lemma 5.1. We take H^+ as given in Lemma 5.3 (i), and we fix $M \in [0, +\infty)$ such that $H^+([0, M]) > p_c(d)$. We work with the stabilized points $\tilde{x}^{\mathcal{C}_{H^+, M}}$. We consider a point $x \in \mathbb{Z}^d$, and $k \in \mathbb{N}^*$. For any path r from $\tilde{0}^{\mathcal{C}_{H^+, M}}$ to $kx^{\mathcal{C}_{H^+, M}}$, using Lemma 2.10 we have a.s.

$$T_G(r) = \sum_{e \in r} t_G(e) = \lim_{n \rightarrow +\infty} \sum_{e \in r} t_{G_n}(e) \geq \overline{\lim}_{n \rightarrow +\infty} \tilde{T}_{G_n}^{\mathcal{C}_{H^+, M}}(0, kx).$$

Taking the infimum over any such path r , we obtain

$$\tilde{T}_G^{\mathcal{C}_{H^+, M}}(0, kx) \geq \overline{\lim}_{n \rightarrow +\infty} \tilde{T}_{G_n}^{\mathcal{C}_{H^+, M}}(0, kx).$$

Conversely, since $G \preceq G_n$, thanks to the coupling of the laws we get $\tilde{T}_G^{\mathcal{C}_{H^+,M}}(0, kx) \leq \tilde{T}_{G_n}^{\mathcal{C}_{H^+,M}}(0, kx)$ for all n , thus

$$\forall k \in \mathbb{N}^*, \text{ a.s.}, \quad \lim_{n \rightarrow \infty} \tilde{T}_{G_n}^{\mathcal{C}_{H^+,M}}(0, kx) = \tilde{T}_G^{\mathcal{C}_{H^+,M}}(0, kx).$$

Since for all n we have $\tilde{T}_{G_n}^{\mathcal{C}_{H^+,M}}(0, kx) \leq \tilde{T}_{H^+}^{\mathcal{C}_{H^+,M}}(0, kx)$ that is integrable by Proposition 2.1, the dominated convergence theorem implies that, for all $k \in \mathbb{N}^*$,

$$(22) \quad \lim_{n \rightarrow \infty} \mathbb{E} \left[\tilde{T}_{G_n}^{\mathcal{C}_{H^+,M}}(0, kx) \right] = \mathbb{E} \left[\tilde{T}_G^{\mathcal{C}_{H^+,M}}(0, kx) \right].$$

By Lemma 2.11, we know that $\mu_G(x) = \inf_{k \in \mathbb{N}^*} \mathbb{E} \left[\tilde{T}_G^{\mathcal{C}_{H^+,M}}(0, kx) \right] / k$. For any $\varepsilon > 0$, there exists $K(G, \varepsilon)$ such that

$$(23) \quad \mu_G(x) \geq \frac{\mathbb{E} \left[\tilde{T}_G^{\mathcal{C}_{H^+,M}}(0, Kx) \right]}{K} - \varepsilon,$$

and using (22) we know that there exists $n_0(\varepsilon, K)$ such that for all $n \geq n_0$ we have

$$(24) \quad \frac{\mathbb{E} \left[\tilde{T}_{G_n}^{\mathcal{C}_{H^+,M}}(0, Kx) \right]}{K} \geq \frac{\mathbb{E} \left[\tilde{T}_G^{\mathcal{C}_{H^+,M}}(0, Kx) \right]}{K} - \varepsilon.$$

Since $\mu_{G_n}(x) = \inf_{k \in \mathbb{N}^*} \mathbb{E} \left[\tilde{T}_{G_n}^{\mathcal{C}_{H^+,M}}(0, kx) \right] / k$, combining equations (23) and (24), we obtain that for any $\varepsilon > 0$, for all n large enough,

$$\mu_G(x) \geq \mu_{G_n}(x) - 2\varepsilon.$$

This concludes the proof of Lemma 5.1. \square

Remark 5.4. *The domination we use to prove (22) is free, since whatever the probability measure H^+ on $[0, +\infty]$ we consider, the regularized times $\tilde{T}_{H^+}^{\mathcal{C}_{H^+,M}}(0, x)$ are always integrable. In [8], Cox considered the (non regularized) times $T_{G_n}(0, x)$ for probability measures G_n on $[0, +\infty)$. By Lemma 5.3 it is easy to obtain $T_{G_n}(0, x) \leq T_H(0, x)$ for some probability measure H on $[0, +\infty)$. However, without further assumption, $T_H(0, x)$ may not be integrable. This is the reason why Cox supposed that the family $(G_n, n \in \mathbb{N})$ was uniformly integrable. In [9], Cox and Kesten circumvent this problem by considering some regularized passage times that are always integrable. There is no straightforward generalization of their regularized passage times to the case of possibly infinite passage times, but the \tilde{T} introduced in [6] plays the same role.*

5.4. Compact case. This section is devoted to the proof of Lemma 5.2. Of course, Lemma 5.2 can be seen as a particular case of the continuity result by Cox and Kesten. But, as noted by Kesten in his Saint-Flour course [18], the Cox–Kesten way makes use of former results by Cox in [9] and is not the shortest path to a proof in the compact case. In [18] Kesten also gave a sketch of a shorter proof in the compact case. We thought the reader would be pleased to have a self-contained proof, so we present a short but full proof of Lemma 5.2, quite inspired by Kesten [18].

Proof of Lemma 5.2. Let $G, (G_n)_{n \in \mathbb{N}}$ be probability measures on $[0, R]$, and consider $x \in \mathbb{Z}^d$. As in the proof of Theorem 1.2, we have $G_n \preceq \overline{G}_n$, where $\overline{\mathfrak{G}}_n =$

$\max(\mathfrak{G}, \mathfrak{G}_n)$, thus $\mu_{G_n} \leq \mu_{\overline{G}_n}$. Applying Lemma 5.1, we know that

$$\overline{\lim}_{n \rightarrow +\infty} \mu_{\overline{G}_n}(x) \leq \mu_G(x).$$

If $\mu_G(x) = 0$, then $\lim_{n \rightarrow \infty} \mu_{G_n}(x) = \lim_{n \rightarrow \infty} \mu_{\overline{G}_n}(x) = \mu_G(x) = 0$ and the proof is complete. We suppose from now on that $\mu_G(x) > 0$, thus $x \neq 0$. Since the passage times $t_G(e)$ are finite, it is well known that $\mu_G(x) > 0$ for $x \neq 0$ if and only if $G(\{0\}) < p_c(d)$ (see Theorem 6.1 in [18], or Proposition 2.3 in a more general setting). We want to prove that $\underline{\lim}_{n \rightarrow \infty} \mu_{\underline{G}_n}(x) \geq \mu_G(x)$, where $\underline{G}_n = \min(\mathfrak{G}, \mathfrak{G}_n)$. Notice that $\tilde{x}^{\mathcal{C}_{\underline{G}_n, M}} = \tilde{x}^{\mathcal{C}_{G, M}} = x$ for any $M \geq R$, thus we do not need to introduce regularized times \tilde{T} . In what follows we note *s.a.* for *self avoiding*. Since $\underline{G}_n \xrightarrow{d} G$, we have $\lim_{n \rightarrow \infty} \underline{G}_n(\{0\}) \leq G(\{0\}) < p_c(d)$, thus we consider only n large enough so that $\underline{G}_n(\{0\}) < p_c(d)$. Applying Lemma 5.3 (ii) to the sequence of functions \underline{G}_n , we obtain the existence of a probability measure H^- on $[0, +\infty]$ (in fact on $[0, R]$) such that $H^- \preceq \underline{G}_n$ for all n and $H^-(\{0\}) < p_c(d)$. Thanks to the coupling, we know that $t_{H^-}(e) \leq t_{\underline{G}_n}(e) \leq t_G(e)$ for every edge e , thus we obtain that for all $A \in \mathbb{N}^*$, for all $C \in [0, +\infty)$,

$$\begin{aligned} & \mathbb{P}[T_{\underline{G}_n}(0, kx) \leq T_G(0, kx) - \varepsilon k] \\ & \leq \mathbb{P}[\exists r \text{ s.a. path starting at } 0 \text{ s.t. } |r| \geq Ak \text{ and } T_{\underline{G}_n}(r) \leq ACk] \\ & \quad + \mathbb{P}[T_{\underline{G}_n}(0, kx) > ACk] + \sum_{\substack{r \text{ s.a. path from } 0 \\ \text{s.t. } |r| \leq Ak}} \mathbb{P}\left[\sum_{e \in r} t_G(e) - t_{\underline{G}_n}(e) \geq \varepsilon k\right] \\ & \leq \mathbb{P}[\exists r \text{ s.a. path starting at } 0 \text{ s.t. } |r| \geq Ak \text{ and } T_{H^-}(r) \leq ACk] \\ & \quad + \mathbb{P}[T_G(0, kx) > ACk] + (2d)^{Ak} \mathbb{P}\left[\sum_{i=1}^{Ak} t_G(e_i) - t_{\underline{G}_n}(e_i) \geq \varepsilon k\right], \end{aligned}$$

where $(e_i, i = 1, \dots, Ak)$ is a collection of distinct edges. Since $H^-(\{0\}) < p_c(d)$, we know that we can choose $C \in (0, +\infty)$ (depending on d and H) such that there exist finite and positive constants D, E (depending also on d and H) satisfying, for all $k \in \mathbb{N}^*$,

$$\mathbb{P}[\exists r \text{ s.a. path starting at } 0 \text{ s.t. } |r| \geq k \text{ and } T_{H^-}(r) \leq Ck] \leq De^{-Ek}$$

(see Proposition 5.8 in [18]). Since the support of G is included in $[0, R]$ for some finite R , we know that $T_G(0, kx) \leq Rk\|x\|_1$, thus we choose A large enough (depending on F, d and C) so that

$$\mathbb{P}[T_G(0, kx) > ACk] = 0.$$

If we prove that there exists $n_0(G, (\underline{G}_n), \varepsilon)$ such that for all $n \geq n_0$,

$$(25) \quad \sum_{k>0} (2d)^{Ak} \mathbb{P}\left[\sum_{i=1}^{Ak} t_G(e_i) - t_{\underline{G}_n}(e_i) \geq \varepsilon k\right] < +\infty,$$

then for all $n \geq n_0$ we have $\sum_k \mathbb{P}[T_{\underline{G}_n}(0, kx) \leq T_G(0, kx) - \varepsilon k] < +\infty$. By Borel-Cantelli's lemma we obtain that for all $n \geq n_0$, a.s., for all $k \geq k_0(n)$ large enough,

$$T_{\underline{G}_n}(0, kx) > T_G(0, kx) - \varepsilon k,$$

thus for all $n \geq n_0$ we get

$$\mu_{\underline{G}_n}(x) \geq \mu_G(x) - \varepsilon.$$

We conclude that $\liminf_{n \rightarrow \infty} \mu_{\underline{G}_n}(x) \geq \mu_G(x)$. It remains to prove (25). For any $\alpha > 0$, by Markov's inequality we have

$$\begin{aligned} (2d)^{Ak} \mathbb{P} \left[\sum_{i=1}^{Ak} t_G(e_i) - t_{\underline{G}_n}(e_i) \geq \varepsilon k \right] \\ \leq \left(2d \exp \left(\frac{-\alpha \varepsilon}{A} \right) \mathbb{E} \left[\exp(\alpha(t_G(e) - t_{\underline{G}_n}(e))) \right] \right)^{Ak}. \end{aligned}$$

By Lemma 2.10 we have $\lim_{n \rightarrow \infty} t_{\underline{G}_n}(e) = t_G(e)$ a.s. Since $t_{\underline{G}_n}(e), t_G(e) \leq R$ we obtain by a dominated convergence theorem that

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\exp(\alpha(t_G(e) - t_{\underline{G}_n}(e))) \right] = 1.$$

We choose $\alpha(\varepsilon)$ large enough so that

$$2d \leq \exp \left(\frac{\alpha \varepsilon}{4A} \right),$$

and then $n_0(G, (\underline{G}_n), \varepsilon)$ large enough so that for all $n \geq n_0$, we have

$$\mathbb{E} \left[\exp(\alpha(t_G(e) - t_{\underline{G}_n}(e))) \right] \leq \exp \left(\frac{\alpha \varepsilon}{4A} \right).$$

Thus for all $n \geq n_0$, we have

$$(2d)^{Ak} \mathbb{P} \left[\sum_{i=1}^{Ak} t_G(e_i) - t_{\underline{G}_n}(e_i) \geq \varepsilon k \right] \leq \exp \left(-\frac{\alpha \varepsilon}{2A} \right),$$

so (25) is proved. \square

6. ISOPERIMETRY OF PLANAR PERCOLATION

This section is devoted to the proof of Theorem 1.1. The definition of the objects used in this section are given in section 2.3. The main step in the proof of Theorem 1.1 is the following lemma.

Lemma 6.1. *For every $p > p_c(2)$,*

$$\limsup_{p' \rightarrow p} \sup_{x \in \mathbb{S}^1} |\beta_{p'}(x) - \beta_p(x)| = 0.$$

Proof. Right continuity:

First we couple \mathbb{P}_p and $\mathbb{P}_{p+\delta}$ by sprinkling i.e. first sample \mathbb{P}_p , then add edges with an independent probability $\delta/(1-p)$, abbreviate the coupling $Q_{p,\delta}$. We extend the definition of $\tilde{\gamma}^C$ to any $y \in \mathbb{R}^d$. By Lemma 2.6 there exist $C, C', \alpha > 0$ such that $\forall x \in \mathbb{S}^1$, if $t > \alpha n$, then

$$(26) \quad \mathbb{P}_p[\exists \gamma \in \mathcal{R}(\tilde{0}^{C_p}, \tilde{n}x^{C_p}) : \text{open}, |\gamma| > t] \leq C e^{-C't}.$$

For any path γ open in \mathbb{P}_p , γ is also open in $\mathbb{P}_{p+\delta}$. However some additional right-boundary edges may be open. To bound the difference, note that if $|\gamma| < \alpha n$ by [4, Lemma 2.5] $|\partial^+ \gamma| < 3\alpha n$. For a path $r \in \mathcal{R}(x, y)$, let us define $\mathbf{b}_p(r) = |\{e \in \partial^+ r : e \text{ is } \mathbb{P}_p\text{-open}\}|$. By the independence of the extra edges in the coupling $Q_{p,\delta}$, we

can bound $\mathbf{b}_{p+\delta}(\gamma) - \mathbf{b}_p(\gamma)$ by Cramér's theorem. For every fixed path γ such that $|\gamma| < \alpha n$, for every $\varepsilon > 0$,

$$(27) \quad Q_{p,\delta} [\mathbf{b}_{p+\delta}(\gamma) - \mathbf{b}_p(\gamma) > 3\varepsilon\alpha n] \leq e^{-3\alpha n(\varepsilon \log \frac{\varepsilon}{\delta} + (1-\varepsilon) \log \frac{1-\varepsilon}{1-\delta})}.$$

Since there are at most $4^{\alpha n}$ paths of length smaller than αn containing 0, by choosing δ small enough we obtain for every $x \in \mathbb{S}^1$

$$(28) \quad \begin{aligned} & Q_{p,\delta} [\exists \gamma \in \mathcal{R}(\tilde{0}^{C_p}, \tilde{n}\tilde{x}^{C_p}) : \mathbb{P}_p - \text{open}, \mathbf{b}_{p+\delta}(\gamma) - \mathbf{b}_p(\gamma) > 3\varepsilon\alpha n] \\ & \leq Q_{p,\delta} [\exists \gamma \in \mathcal{R}(\tilde{0}^{C_p}, \tilde{n}\tilde{x}^{C_p}) : \mathbb{P}_p - \text{open}, |\gamma| \leq \alpha n, \mathbf{b}_{p+\delta}(\gamma) - \mathbf{b}_p(\gamma) > 3\varepsilon\alpha n] \\ & \quad + \mathbb{P}_p[\exists \gamma \in \mathcal{R}(\tilde{0}^{C_p}, \tilde{n}\tilde{x}^{C_p}) : \text{open}, |\gamma| > \alpha n] \\ & \leq \sum_{y \in \mathbb{Z}^d : \|0-y\|_2 \leq n} Q_{p,\delta} [\exists \gamma : \gamma \text{ starts at } y, |\gamma| \leq \alpha n, \mathbf{b}_{p+\delta}(\gamma) - \mathbf{b}_p(\gamma) > 3\varepsilon\alpha n] \\ & \quad + \mathbb{P}_p[\|\tilde{0}^{C_p} - 0\|_2 > n] + Ce^{-C'\alpha n} \\ & \leq Kn^d 4^{\alpha n} e^{-3\alpha n(\varepsilon \log \frac{\varepsilon}{\delta} + (1-\varepsilon) \log \frac{1-\varepsilon}{1-\delta})} + \hat{C}e^{\hat{C}'n} + Ce^{-C'\alpha n} \\ & \leq \tilde{C}e^{-\tilde{C}'n} \end{aligned}$$

where K is a constant depending only on the dimension and $\hat{C}, \hat{C}', \tilde{C}, \tilde{C}'$ are constants depending also on p . Thus for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every $\delta \leq \delta_0$, for every $x \in \mathbb{S}^1$,

$$\beta_{p+\delta}(x) < \beta_p(x) + 3\alpha\varepsilon.$$

Left continuity: We couple \mathbb{P}_p and $\mathbb{P}_{p-\delta}$ by reverse sprinkling i.e. first sample \mathbb{P}_p and then remove each open edge independently with probability $\frac{\delta}{p}$, abbreviate the coupling $Q'_{p,\delta}$. Given a \mathbb{P}_p -open path $\gamma \in \mathcal{R}(\tilde{0}^{C_{p-\delta}}, \tilde{n}\tilde{x}^{C_{p-\delta}})$, γ may not be open in $\mathbb{P}_{p-\delta}$. Thus we use the results of Section 3 to modify the path to an open path in $\mathbb{P}_{p-\delta}$ which does not gain too many extra right-boundary edges. We have to change a little bit our point of view, since we are not dealing with passage times and truncated passage times anymore, but with edges that are open for \mathbb{P}_p or $\mathbb{P}_{p-\delta}$ respectively. The definition of good and bad boxes (see Definition 3.1) is now associated to the $\mathbb{P}_{p-\delta}$ -open clusters, i.e., we say that a macroscopic site \mathbf{i} is *good* (or that the box $B_N(\mathbf{i})$ is good) if the following events occur:

- There exists a unique $\mathbb{P}_{p-\delta}$ -open cluster in $B'_N(\mathbf{i})$ which has more than N vertices;
- This $\mathbb{P}_{p-\delta}$ -open cluster is crossing for each of the 3^d N -boxes included in $B'_N(\mathbf{i})$;
- For all $x, y \in B'_N(\mathbf{i})$, if $\|x-y\|_\infty \geq N$ and x and y belong to this $\mathbb{P}_{p-\delta}$ -open, then $D^{C'_{p-\delta}}(x, y) \leq 3\beta N$,

where $C'_{p-\delta}$ is the subgraph of \mathbb{Z}^d whose edges are $\mathbb{P}_{p-\delta}$ -open. Otherwise, $B_N(\mathbf{i})$ is said to be *bad*. In order to use the notations of Lemma 3.2, define $G_{p-\delta} = (p-\delta)\delta_1 + (1-p+\delta)\delta_\infty$ and choose any $1 < K = M_0 < \infty$. Let $x \in \mathbb{S}^1$, we denote by $[nx]$ the point y of \mathbb{Z}^d which minimizes $\|nx - y\|_1$ (with a deterministic rule to break ties). For every path γ from 0 to $[nx]$, let $\Gamma = \Gamma(\gamma)$ be the lattice animal of

N -boxes it visits. Thus we define

$$(29) \quad \begin{aligned} \gamma_a &= \{e \in \gamma : t_{G_{p-\delta}}(e) < \infty\} = \{e \in \gamma : e \text{ is open in } \mathbb{P}_{p-\delta}\} \\ \gamma_b &= \{e \in \gamma : t_{G_{p-\delta}}(e) = \infty\} = \{e \in \gamma : e \text{ is closed in } \mathbb{P}_{p-\delta}\}. \end{aligned}$$

We now mimic the proof of Lemma 4.4. Fix $\varepsilon > 0$, and choose $N = N(\varepsilon)$ as in this proof (see below (13)). Let E be the following good event: $0 \in \mathcal{C}_{p-\delta}$, $[nx] \in \mathcal{C}_{p-\delta}$ and the N -boxes containing 0 and $[nx]$ are good. Let $\gamma \in \mathcal{R}(0, [nx])$ such that γ is \mathbb{P}_p -open. By Lemma 3.2, on the event E , there exists a path γ' with the following properties:

- (1) γ' is a path from 0 to $[nx]$ which is $\mathbb{P}_{p-\delta}$ -open;
- (2) $\gamma' \setminus \gamma$ is a collection of simple paths (and also right-most) that intersect $\gamma \cap \gamma'$ only at their endpoints thus γ' is a right-most path (see [4, Lemma 2.6]);
- (3) $|\gamma' \setminus \gamma| \leq \rho_d \left(N^d |\gamma_b| + N \sum_{C \in \text{Bad}: C \cap \Gamma \neq \emptyset} |C| \right)$.

Note that

$$(30) \quad \mathbf{b}_{p-\delta}(\gamma') \leq \mathbf{b}_p(\gamma') \leq \mathbf{b}_p(\gamma) + 3|\gamma' \setminus \gamma|.$$

For $x, y \in \mathcal{C}_p$ (resp. $x, y \in \mathcal{C}_{p-\delta}$), we define $b_p(x, y) = \inf\{\mathbf{b}_p(r) : r \in \mathcal{R}(x, y), \mathbb{P}_p\text{-open}\}$ (resp. $b_{p-\delta}(x, y) = \inf\{\mathbf{b}_{p-\delta}(r) : r \in \mathcal{R}(x, y), \mathbb{P}_{p-\delta}\text{-open}\}$). For a fixed α (to be chosen later), abbreviate

$$\begin{aligned} A_\varepsilon &= E \cap \{\exists \gamma \in \mathcal{R}(0, [nx]) : \mathbb{P}_p\text{-open}, \mathbf{b}_{p-\delta}(\gamma') > \mathbf{b}_p(\gamma) + 12\alpha\tilde{C}_d\rho_d\varepsilon n\}, \\ S_\varepsilon &= \left\{ \forall \gamma \in \mathcal{R}(\tilde{0}^{\mathcal{C}_{p-\delta}}, \tilde{n}x^{\mathcal{C}_{p-\delta}}) : \mathbb{P}_p\text{-open}, |\gamma| \leq \alpha n \right\}, \\ B_\varepsilon &= \left\{ \forall \gamma \in \mathcal{R}(\tilde{0}^{\mathcal{C}_{p-\delta}}, \tilde{n}x^{\mathcal{C}_{p-\delta}}) : \mathbb{P}_p\text{-open}, |\gamma_b| \leq \frac{2\tilde{C}_d\alpha\varepsilon n}{N^d} \right\}, \\ C_\varepsilon &= \left\{ \exists \gamma \in \mathcal{R}(\tilde{0}^{\mathcal{C}_{p-\delta}}, \tilde{n}x^{\mathcal{C}_{p-\delta}}) : \mathbb{P}_p\text{-open}, 2d \sum_{C \in \text{Bad}: C \cap \Gamma(\gamma) \neq \emptyset} |C| > \frac{2\tilde{C}_d\alpha\varepsilon n}{N} \right\}. \end{aligned}$$

We have

$$(31) \quad \begin{aligned} & Q'_{p,\delta} \left(|b_p(\tilde{0}^{\mathcal{C}_p}, \tilde{n}x^{\mathcal{C}_p}) - b_{p-\delta}(\tilde{0}^{\mathcal{C}_{p-\delta}}, \tilde{n}x^{\mathcal{C}_{p-\delta}})| > 12\alpha\tilde{C}_d\rho_d\varepsilon n \right) \\ & \leq \mathbb{P}_{p-\delta}(E^c) + Q'_{p,\delta}(A_\varepsilon) \\ & \leq \mathbb{P}_{p-\delta}(E^c) + Q'_{p,\delta}(E \cap S_\varepsilon^c) + Q'_{p,\delta}(B_\varepsilon^c \cap S_\varepsilon \cap E) + Q'_{p,\delta}(B_\varepsilon \cap S_\varepsilon \cap A_\varepsilon) \\ & \leq \mathbb{P}_{p-\delta}(E^c) + Q'_{p,\delta}(E \cap S_\varepsilon^c) + Q'_{p,\delta}(B_\varepsilon^c \cap S_\varepsilon \cap E) + Q'_{p,\delta}(E \cap B_\varepsilon \cap S_\varepsilon \cap C_\varepsilon) \end{aligned}$$

where the last inequality is due to (30). By the FKG inequality, we have

$$\mathbb{P}_{p-\delta}(E^c) \leq 1 - \theta_{p-\delta}^2 p(\varepsilon)^2,$$

where $p(\varepsilon)$ is the probability that a box is good (as chosen in (13)). When ε goes to 0, $p(\varepsilon)$ goes to one, so for ε small enough we can suppose that $p(\varepsilon) \geq 1/2$, and we get

$$(32) \quad \mathbb{P}_{p-\delta}(E^c) \leq 1 - \theta_{p-\delta}^2/4 < 1.$$

By Lemma 2.6 there exist $C, C', \alpha > 0$ (we choose this α in the definition of the previous events), depending on p but not on δ , such that for every $x \in \mathbb{S}^1$ if $t > \alpha n$,

then

$$Q'_{p-\delta}[E \cap \{\exists \gamma \in \mathcal{R}(\tilde{0}^{C_p}, \tilde{n}x^{C_p}) : \mathbb{P}_p - \text{open}, |\gamma| > t\}] \leq Ce^{-C't}.$$

Thus

$$(33) \quad \begin{aligned} Q'_{p,\delta}(E \cap S_\varepsilon^c) &= Q'_{p,\delta}[E \cap \{\exists \gamma \in \mathcal{R}(\tilde{0}^{C_p}, \tilde{n}x^{C_p}) : \mathbb{P}_{p-\delta} - \text{open}, |\gamma| > t\}] \\ &< Ce^{-C'\alpha n}. \end{aligned}$$

If γ is a path from 0 to $\lfloor nx \rfloor$, then $|\Gamma| \geq n/N$ and by Lemma 4.3 we have $|\Gamma| \leq \tilde{C}_d(1 + (|\gamma| + 1)/N) - 1 \leq 2\tilde{C}_d|\gamma|/N$, at least for n large enough, thus by (20) we have

$$(34) \quad \begin{aligned} &Q'_{p,\delta}(E \cap B_\varepsilon \cap S_\varepsilon \cap C_\varepsilon) \\ &\leq Q'_{p,\delta} \left(\exists \Gamma \in \mathcal{A}\text{imals}, |\Gamma| \geq \frac{n}{N}, \sum_{C \in \text{Bad}: C \cap \Gamma \neq \emptyset} |C| \geq \varepsilon |\Gamma| \right) \stackrel{\text{def}}{=} P_4(n) \\ &\leq 2^{-\frac{n}{N}+1}. \end{aligned}$$

Finally, by the independence of the inverse sprinkling and again Cramér's theorem (see (27)), for every $\varepsilon > 0$,

$$(35) \quad \begin{aligned} &Q'_{p,\delta}[B_\varepsilon^c \cap S_\varepsilon \cap E] \\ &\leq Q'_{p,\delta} \left(\exists \gamma : \mathbb{P}_p - \text{open}, \gamma \text{ starts at } 0, |\gamma| \leq \alpha n, |\gamma_b| > \frac{2\tilde{C}_d\alpha\varepsilon n}{N^d} \right) \\ &\leq 4^{\alpha n} e^{-3\alpha n \left(\frac{2\tilde{C}_d\varepsilon}{N^d} \log \frac{2\tilde{C}_d\varepsilon}{\delta N^d} + (1 - \frac{2\tilde{C}_d\varepsilon}{N^d}) \log \frac{1-2\tilde{C}_d\varepsilon/N^d}{1-\delta} \right)}. \end{aligned}$$

For a fixed ε , and thus a fixed large $N(\varepsilon)$, we can choose $\delta \leq \delta_0$ small enough to control this term. Combining (31), (32), (33), (35) and (34), we deduce that for every $\varepsilon > 0$ there exists $\delta_0(\varepsilon) > 0$ and $p_2(\varepsilon) > 0$ such that for every $\delta \leq \delta_0$, for every $x \in \mathbb{S}^1$, for every n large enough,

$$Q'_{p,\delta} \left(|b_p(\tilde{0}^{C_p}, \tilde{n}x^{C_p}) - b_{p-\delta}(\tilde{0}^{C_{p-\delta}}, \tilde{n}x^{C_{p-\delta}})| > 12\alpha\tilde{C}_d\rho_d\varepsilon n \right) \leq 1 - p_2(\varepsilon),$$

thus for every $\varepsilon > 0$ there exists $\delta_0(\varepsilon) > 0$ such that for every $\delta \leq \delta_0$, for every $x \in \mathbb{S}^1$,

$$\beta_{p-\delta}(x) < \beta_p(x) + 12\alpha\tilde{C}_d\rho_d\varepsilon.$$

This ends the proof of Lemma 6.1. \square

We can now prove the main theorem of this section

Proof of Theorem 1.1. First consider the limit $\lim_{n \rightarrow \infty} n\varphi_n(p)$. Let $p > p_c(2)$. For any rectifiable Jordan curve λ , with $\text{Leb}(\text{int}(\lambda)) = 1$,

$$\text{len}_{\beta_p}(\lambda) = \sup_{N \geq 1} \sup_{0 \leq t_0 < \dots < t_N \leq 1} \sum_{i=1}^N \beta_p(\lambda(t_i) - \lambda(t_{i-1})),$$

and

$$\beta_p(\lambda(t_i) - \lambda(t_{i-1})) = \beta_p \left(\frac{\lambda(t_i) - \lambda(t_{i-1})}{\|\lambda(t_i) - \lambda(t_{i-1})\|_2} \right) \|\lambda(t_i) - \lambda(t_{i-1})\|_2.$$

By Lemma 6.1 for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every $q > p_c(2)$ satisfying $|p - q| < \delta$ we have $\sup_{x \in \mathbb{S}^1} |\beta_q(x) - \beta_p(x)| < \varepsilon$, thus

$$(36) \quad |\text{len}_{\beta_p}(\lambda) - \text{len}_{\beta_q}(\lambda)| \leq \varepsilon \text{len}_{\|\cdot\|_2}(\lambda).$$

The infimum in Theorem 2.7 is achieved (by compactness of the set of Lipschitz curves), so let us denote by λ_p (resp. λ_q) a Jordan curve such that $\text{Leb}(\text{int}(\lambda_p)) = 1$ and $\text{len}_{\beta_p}(\lambda_p) = \sqrt{2}\theta_p \lim_{n \rightarrow \infty} n\varphi_n(p)$ (resp. $\text{Leb}(\text{int}(\lambda_q)) = 1$ and $\text{len}_{\beta_q}(\lambda_q) = \sqrt{2}\theta_q \lim_{n \rightarrow \infty} n\varphi_n(q)$). All norms in \mathbb{R}^2 are equivalent thus we know that $\text{len}_{\|\cdot\|_2}(\lambda_p) < \infty$ and $\text{len}_{\|\cdot\|_2}(\lambda_q) < \infty$. From (36) we deduce that for every $\varepsilon > 0$ there exists an $\delta > 0$ such that if $|p - q| < \delta$ then

$$\begin{aligned} \sqrt{2}\theta_p \lim_{n \rightarrow \infty} n\varphi_n(p) &= \text{len}_{\beta_p}(\lambda_p) \\ &\geq \text{len}_{\beta_q}(\lambda_p) - \varepsilon \text{len}_{\|\cdot\|_2}(\lambda_p) \\ (37) \quad &\geq \sqrt{2}\theta_q \lim_{n \rightarrow \infty} n\varphi_n(q) - \varepsilon \text{len}_{\|\cdot\|_2}(\lambda_p) \end{aligned}$$

and

$$\begin{aligned} \sqrt{2}\theta_p \lim_{n \rightarrow \infty} n\varphi_n(p) &\leq \text{len}_{\beta_p}(\lambda_q) \\ &\leq \text{len}_{\beta_q}(\lambda_q) + \varepsilon \text{len}_{\|\cdot\|_2}(\lambda_q) \\ (38) \quad &\leq \sqrt{2}\theta_q \lim_{n \rightarrow \infty} n\varphi_n(q) + \varepsilon \text{len}_{\|\cdot\|_2}(\lambda_q). \end{aligned}$$

Let $\beta_q^{\min} = \inf_{x \in \mathbb{S}^1} \beta_q(x)$, for all q . By Lemma 6.1 again we know that for every q satisfying $|p - q| < \delta$ we have $\beta_q^{\min} \geq \beta_p^{\min} - \varepsilon$, which is positive for ε small enough (β_p^{\min} is not zero since β_p is a norm), thus

$$\text{len}_{\|\cdot\|_2}(\lambda_q) \leq \frac{\text{len}_{\beta_q}(\lambda_q)}{\beta_q^{\min}} \leq \frac{\text{len}_{\beta_q}(\lambda_q)}{\beta_p^{\min} - \varepsilon}.$$

Thanks to Equation (38) we obtain

$$(39) \quad \sqrt{2}\theta_p \lim_{n \rightarrow \infty} n\varphi_n(p) \leq \sqrt{2}\theta_q \lim_{n \rightarrow \infty} n\varphi_n(q) \left(1 + \frac{\varepsilon}{\beta_p^{\min} - \varepsilon}\right).$$

Combining (37) and (39) we obtain that

$$\lim_{q \rightarrow p} \sqrt{2}\theta_q \lim_{n \rightarrow \infty} n\varphi_n(q) = \sqrt{2}\theta_p \lim_{n \rightarrow \infty} n\varphi_n(p).$$

Since $p \mapsto \theta_p$ is continuous on $(p_c(2), 1]$, this concludes the first part of the proof.

Next we prove that $p \mapsto \widehat{W}_p$ is continuous for the Hausdorff distance. Fix $\eta > 0$ and $p > p_c(2)$ and let $\varepsilon = \varepsilon(\eta, p) > 0$ be small enough such that

$$(40) \quad \varepsilon \leq \frac{\beta_p^{\min}}{2} \min(\eta, 1).$$

As previously let $\delta > 0$ satisfy $\sup_{x \in \mathbb{S}^1} |\beta_q(x) - \beta_p(x)| < \varepsilon$ for all $q > p_c(2)$ such that $|p - q| < \delta$. For every $x \in W_q$ we have by definition of W_q that for every $\hat{n} \in \mathbb{S}^1$, $\hat{n} \cdot x \leq \beta_q(\hat{n})$. Thus for all $q > p_c(2)$ such that $|p - q| < \delta$,

$$\hat{n} \cdot x \leq \beta_q(\hat{n}) \leq \beta_p(\hat{n}) + \varepsilon \leq (1 + \eta)\beta_p(\hat{n}),$$

where the last inequality comes from (40), thus $x \in (1 + \eta)W_p$. We obtain that for all $p > p_c(2)$, for all $\eta > 0$, there exists $\delta > 0$ such that for every $q > p_c(2)$ satisfying $|p - q| < \delta$,

$$(41) \quad W_q \subset (1 + \eta)W_p.$$

For every $q > p_c(2)$ satisfying $|p - q| < \delta$, we also have $\beta_q^{\min} \geq \beta_p^{\min} - \varepsilon \geq \beta_p^{\min}/2 \geq \varepsilon/\eta$ by (40), thus by the same method we obtain that for every $x \in W_p$, for every $\hat{n} \in \mathbb{S}^1$,

$$\hat{n} \cdot x \leq \beta_p(\hat{n}) \leq \beta_q(\hat{n}) + \varepsilon \leq (1 + \eta)\beta_q(\hat{n}),$$

thus

$$(42) \quad W_p \subset (1 + \eta)W_q.$$

For every $x \in W_p$, $\|x\|_2 = x \cdot x / \|x\|_2 \leq \beta_p(x) \leq \beta_p^{\max}$, where $\beta_p^{\max} = \sup_{x \in \mathbb{S}^1} \beta_p(x) < \infty$, thus $\|(1 + \eta)x - x\|_2 \leq \eta\beta_p^{\max}$. Similarly, for all $q > p_c(2)$ satisfying $|p - q| < \delta$, $\|x\|_2 \leq \beta_q^{\max} \leq 2\beta_p^{\max}$ and $\|(1 + \eta)x - x\|_2 \leq 2\eta\beta_p^{\max}$. With (41) and (42), we conclude that for every $p > p_c(2)$, for every $\eta > 0$, there exists $\delta > 0$ such that for every $q > p_c(2)$ satisfying $|p - q| < \delta$,

$$d_H(W_p, W_q) \leq 2\eta\beta_p^{\max},$$

thus $\lim_{q \rightarrow p} d_H(W_p, W_q) = 0$. This implies that $\lim_{q \rightarrow p} \text{Leb}(W_q) = \text{Leb}(W_p)$, and since $\widehat{W}_p = \frac{W_p}{\sqrt{\text{Leb}(W_p)}}$ we deduce from (41) and (42) by a similar argument that $\lim_{q \rightarrow p} d_H(\widehat{W}_p, \widehat{W}_q) = 0$. This concludes the proof of Theorem 1.1. \square

Remark 6.2. *To deduce the continuity of the Wulff crystal from Lemma 6.1, we can also consider a more general setting. Consider β_p^* the dual norm of β_p , defined by*

$$\forall x \in \mathbb{R}^d, \quad \beta_p^*(x) = \sup\{x \cdot y : \beta_p(y) \leq 1\}.$$

Then β_p^ is a norm, and what we did is equivalent to deduce from Lemma 6.1 the same result concerning β_p^* :*

$$(43) \quad \lim_{q \rightarrow p} \sup_{x \in \mathbb{S}^1} |\beta_q^*(x) - \beta_p^*(x)| = 0.$$

Notice that W_p , the Wulff crystal associated to β_b , is in fact the unit ball associated to β_b^ , then (43) implies the continuity of $p \mapsto W_p$ according to the Hausdorff distance.*

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UNIVERSITÉ DE LORRAINE, INSTITUT ÉLIE CARTAN DE LORRAINE, UMR 7502, VANDOEUVRE-LÈS-NANCY, F-54506, FRANCE, AND, CNRS, INSTITUT ÉLIE CARTAN DE LORRAINE, UMR 7502, VANDOEUVRE-LÈS-NANCY, F-54506, FRANCE,
E-mail address: `Olivier.Garet@univ-lorraine.fr`

UNIVERSITÉ DE LORRAINE, INSTITUT ÉLIE CARTAN DE LORRAINE, UMR 7502, VANDOEUVRE-LÈS-NANCY, F-54506, FRANCE, AND, CNRS, INSTITUT ÉLIE CARTAN DE LORRAINE, UMR 7502, VANDOEUVRE-LÈS-NANCY, F-54506, FRANCE,
E-mail address: `Regine.Marchand@univ-lorraine.fr`

TEXAS A&M UNIVERSITY, MAILSTOP 3368, COLLEGE STATION TX 77843, USA¹
E-mail address: `procaccia@math.tamu.edu`

LPMA, UNIVERSITÉ PARIS DIDEROT, 5 RUE THOMAS MANN, 75205 PARIS CEDEX 13, FRANCE
E-mail address: `marie.theret@univ-paris-diderot.fr`

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