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A MODEL OF MISCIBLE LIQUIDS IN POROUS MEDIA

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Abstract. In this article we study the interaction of two miscible liquids in porous media. The model consists of hydrodynamic equations with the Korteweg stress terms coupled with the reaction-diffusion equation for the concentration. We assume that the fluid is incompressible and its motion is described by the Darcy law. The global existence and uniqueness of solutions is established for some optimal conditions on the reaction source term and external force functions. Numerical simulations are performed to show the behavior of two miscible liquids subjected to the Korteweg stress.

1. Introduction

There exists transient interfacial phenomena between two miscible liquids similar to interfacial tension [7]. However they are rather weak and they decay in time because of the mixing of the two liquids because of the molecular diffusion [7, 12]. Investigation of such phenomena is motivated by enhanced oil recovery, hydrology, frontal polymerization, groundwater pollution and filtration [2, 6, 11, 15, 16].

In 1901 Korteweg [8] introduced additional stress terms in the Navier-Stokes equations to describe the influence of the composition gradients on fluid motion. In 1949, Zeldovich [18] studied the existence of a transitional interfacial tension and described it with the expression

$$\sigma = k\frac{[C]^2}{\delta},$$

where $[C]$ is the variation of mass fraction through the transition zone, and δ is the width of this zone. This relationship was generalized by Rousar and Nauman [14] to systems far from equilibrium, for linear concentration gradients. In 1958, Cahn and Hilliard [5] introduced the free energy density for a non-homogeneous fluid,

$$e = e_0 + k|\nabla\rho|^2,$$

where e_0 is the energy density of a homogeneous fluid and ρ denotes the density of the fluid.

A miscible liquid model with fully incompressible Navier-Stokes equations is studied in [9]. Modelling and experiments of miscible liquids in relation with micro-gravity experiments were carried out in [2, 3, 4, 13]. The existence and uniqueness of solutions for miscible liquids model in porous media is studied in [1].

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In this article, we continue the studies of miscible liquids in porous media. We consider a three-dimensional formulation and introduce the source terms in the equation of motion and in the equation for the concentration. The paper is organized as follows. The next section is devoted to the model presentation, while Section 3 deals with the existence of solutions. We establish the uniqueness of solutions in Section 4 followed in Section 5 by numerical simulations.

2. Model presentation

The model describing the interaction of two miscible liquids is written as follows:

$$\frac{\partial C}{\partial t} + u \cdot \nabla C = d\Delta C - Cg, \tag{2.1}$$

$$\frac{\partial u}{\partial t} + \frac{\mu}{K} u = -\nabla p + \nabla \cdot T(C) + f, \tag{2.2}$$

$$\operatorname{div}(u) = 0. \tag{2.3}$$

We consider the boundary conditions:

$$\frac{\partial C}{\partial n} = 0, \quad u.n = 0, \quad \text{on } \Gamma, \tag{2.4}$$

and the initial conditions:

$$C(x,0) = C_0(x), \quad u(x,0) = u_0(x), \quad x \in \Omega. \tag{2.5}$$

Here u is the velocity, p is the pressure, C is the concentration, d is the coefficient of mass diffusion, μ is the viscosity, K is the permeability of the medium, Γ is a Lipschitz continuous boundary of the open bounded domain Ω, n is the unit outward normal vector to Γ, f is the function describing the external forces such as gravity and buoyancy while the term, g stands for the reaction source term. The stress tensor terms are given by the relations:

$$T_{11} = k\Big(\frac{\partial C}{\partial x_2}\Big)^2, \quad T_{12} = T_{21} = -k\frac{\partial C}{\partial x_1}\frac{\partial C}{\partial x_2}, \quad T_{13} = T_{31} = -k\frac{\partial C}{\partial x_1}\frac{\partial C}{\partial x_3},$$
$$T_{23} = T_{32} = -k\frac{\partial C}{\partial x_2}\frac{\partial C}{\partial x_3}, \quad T_{22} = k\Big(\frac{\partial C}{\partial x_1}\Big)^2, \ T_{33} = k\Big(\frac{\partial C}{\partial x_3}\Big)^2,$$

where k is nonnegative constant. We set

$$\nabla \cdot T(C) = \begin{pmatrix} \frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} + \frac{\partial T_{13}}{\partial x_3} \\ \frac{\partial T_{21}}{\partial x_1} + \frac{\partial T_{22}}{\partial x_2} + \frac{\partial T_{23}}{\partial x_3} \\ \frac{\partial T_{31}}{\partial x_1} + \frac{\partial T_{32}}{\partial x_2} + \frac{\partial T_{33}}{\partial x_3} \end{pmatrix}. \tag{2.6}$$

To state the problem in a variational form we need to introduce function spaces:

$$S_u = \{u \in H(\operatorname{div};\Omega); \operatorname{div}(u) = 0, u.n = 0 \text{ on } \Gamma\},$$
$$S_C = \{C \in H^2(\Omega); \frac{\partial C}{\partial n} = 0 \text{ on } \Gamma\}.$$

The variational form of the problem is to find C, u such that for all B, v the following equalities hold:

$$(\frac{\partial C}{\partial t}, B) + d(\nabla C, \nabla B) + (u.\nabla C, B) + (gC, B) = 0, \tag{2.7}$$

$$(\frac{\partial u}{\partial t}, v) + \mu_p(u,v) - (div\ T(C), v) - (f,v) = 0. \tag{2.8}$$

Here $\mu_p = \mu/K$. The functions $f(x,t)$ and $g(x)$ are assumed to be a sufficiently regular in Ω such that the first is bounded in $L^\infty(0,t;L^2(\Omega))$, the second is bounded in $L^\infty(\Omega)$ and both of them are positive.

3. Existence of global solutions

We begin the proof of existence of solutions with the following lemmas.

Lemma 3.1. *The concentration C is bounded in the $L^\infty(0,t;L^2)$ space.*

Proof. Choosing C as test function in (2.7) and taking into account that g is a positive function, we get the inequality:

$$\frac{1}{2}\frac{\partial}{\partial t}(C,C) + d(\nabla C,\nabla C) + (u.\nabla C, C) \le 0.$$

Since $u \in S_u$, the last term vanishes. The second term is positive, so integrating by time we obtain:

$$\|C(t=s)\|_{L^2}^2 \le \|C_0\|_{L^2}^2.$$

From this inequality, it follows that C is bounded in $L^\infty(0,t;L^2)$. □

Lemma 3.2. *The concentration C is bounded in $L^\infty(0,t;H^1)$ and the velocity u is bounded in $L^\infty(0,t;L^2)$.*

Proof. Choosing $-k\Delta C$ as test function in equation (2.7), we have

$$(\frac{\partial C}{\partial t}, -k\Delta C) + (u.\nabla C, -k\Delta C) = d(\Delta C, -k\Delta C) + (gC, k\Delta C).$$

Next, since the reaction source term g is bounded, we get from the previous estimate:

$$\frac{k}{2}\frac{\partial}{\partial t}(\nabla C,\nabla C) + dk(\Delta C,\Delta C) - k(u.\nabla C,\Delta C) \le kg_0(\nabla C,\nabla C).$$

Then

$$\frac{1}{2}\frac{\partial}{\partial t}(\nabla C,\nabla C) + d(\Delta C,\Delta C) \le (u.\nabla C,\Delta C) + g_0(\nabla C,\nabla C). \tag{3.1}$$

Also, by choosing in (2.8), u as test function we obtain

$$\frac{1}{2}\frac{\partial}{\partial t}(u,u) + \mu_p(u,u) - (\nabla.T(C),u) = (f,u). \tag{3.2}$$

To have an explicit expression of $\nabla.T(C)$, we calculate its first component:

$$\begin{aligned}\frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} + \frac{\partial T_{13}}{\partial x_3} &= 2k\frac{\partial C}{\partial x_2}\frac{\partial^2 C}{\partial x_1 \partial x_2} - k\frac{\partial^2 C}{\partial x_1 \partial x_2}\frac{\partial C}{\partial x_2} - k\frac{\partial C}{\partial x_1}\frac{\partial^2 C}{\partial x_2^2}\\ &\quad - k\frac{\partial^2 C}{\partial x_1\partial x_3}\frac{\partial C}{\partial x_2} - k\frac{\partial C}{\partial x_1}\frac{\partial^2 C}{\partial x_3^2}.\end{aligned} \tag{3.3}$$

Hence

$$\frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} + \frac{\partial T_{13}}{\partial x_3} = k\frac{\partial C}{\partial x_1}\frac{\partial^2 C}{\partial x_1 \partial x_2} + k\frac{\partial C}{\partial x_1}\frac{\partial^2 C}{\partial x_1\partial x_3} + k\frac{\partial C}{\partial x_1}\frac{\partial^2 C}{\partial x_1^2} - k\frac{\partial C}{\partial x_1}\Delta C.$$

Then

$$\frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} + \frac{\partial T_{13}}{\partial x_3} = \frac{k}{2}\frac{\partial}{\partial x_1}(\nabla C)^2 - k\frac{\partial C}{\partial x_1}\Delta C.$$

Following the same steps for the second component, we conclude:

$$\nabla \cdot T = \frac{k}{2}\nabla(\nabla C)^2 - k\Delta C\nabla C\,.$$

Replacing this last equality in (3.2) and since $u \in S_u$, we have

$$\frac{1}{2}\frac{\partial}{\partial t}(u,u) + \mu_p(u,u) - k(\Delta C \nabla C, u) = (f,u). \tag{3.4}$$

Adding (3.4) to the inequality (3.1), and with the fact $u \in S_u$ and $C \in S_c$, we have

$$\frac{1}{2}\frac{\partial}{\partial t}\left((u,u) + k(\nabla C, \nabla C)\right) + \mu_p(u,u) + dk(\Delta C, \Delta C) \le (f,u) + g_0(\nabla C, \nabla C).$$

$$\begin{aligned}&\frac{1}{2}\frac{\partial}{\partial t}\left((u,u) + k(\nabla C, \nabla C)\right) + \mu_p(u,u) + dk(\Delta C, \Delta C)\\ &\le \frac{1}{2}(f,f) + \frac{1}{2}(u,u) + g_0(\nabla C, \nabla C).\end{aligned}$$

Since the third and the fourth terms in the left hand side of this inequality are positive, we have

$$\frac{\partial}{\partial t}\left((u,u) + k(\nabla C, \nabla C)\right) \le (f,f) + (u,u) + 2g_0(\nabla C, \nabla C).$$

Therefore,

$$\frac{\partial}{\partial t}\left((u,u) + (\nabla C, \nabla C)\right) \le \frac{(f,f)}{\min(1;k)} + \frac{\max(1;2g_0)}{\min(1;k)}\left((u,u) + (\nabla C, \nabla C)\right).$$

By integrating over time, and since f is bound in $L^\infty(0,t;L^2)$, we have

$$\begin{aligned}&\|u(t=s)\|_{L^2} + \|\nabla C(t=s)\|_{L^2}\\ &\le \|u_0\|_{L^2} + \|\nabla C_0\|_{L^2} + \frac{f_0}{\min(1;k)} + \frac{\max(1;2g_0)}{\min(1;k)}\int_0^t \left(\|u(s)\|_{L^2} + \|\nabla C(s)\|_{L^2}\right) ds.\end{aligned}$$

From the Gronwall's Lemma, it follows that

$$\begin{aligned}&\|u(t=s)\|_{L^2} + \|\nabla C(t=s)\|_{L^2}\\ &\le (\|u_0\|_{L^2} + \|\nabla C_0\|_{L^2} + \frac{f_0}{\min(1;k)})\exp\big(\frac{\max(1;2g_0)t}{\min(1;k)}\big).\end{aligned}$$

We conclude that C is bounded in $L^\infty(0,t;H^1)$ and u is bounded in $L^\infty(0,t;L^2)$ for $t \in [0;T]$. □

Lemma 3.3. *The time derivative of the concentration $\frac{\partial C}{\partial t}$ is bounded in $L^2(0,t;L^2)$.*

Proof. From (2.7), since g is a positive function and by the triangle inequality, we have

$$\|\frac{\partial C}{\partial t}\|_{L^2} \le d\|\Delta C\|_{L^2} + \|u \cdot \nabla C\|_{L^2}.$$

Using Hölder inequality, we obtain

$$\|\frac{\partial C}{\partial t}\|_{L^2} \le d\|\Delta C\|_{L^2} + \|u\|_{L^4}\|\nabla C\|_{L^4},$$

and from the Gagliardo-Nirenberg inequality, it follows that $\exists N > 0$ such that

$$\|\frac{\partial C}{\partial t}\|_{L^2} \le d\|\Delta C\|_{L^2} + N\|u\|_{L^2}^{1/2}\|\nabla u\|_{L^2}^{1/2}\|\nabla C\|_{L^2}^{1/2}\|\nabla C\|_{H^1}^{1/2}.$$

We conclude that $\frac{\partial C}{\partial t}$ is bounded in $L^2(0,t;L^2)$. □

Lemma 3.4. *The time derivative of the velocity $\frac{\partial u}{\partial t}$ is bounded in $L^2(0,t;L^2)$.*

Proof. To prove this lemma, it is sufficient to remark that $\nabla \cdot T(C)$ is a sum of the expressions of the form $\lambda D_i(D_jCD_lC)$, where $D_i = \frac{\partial}{\partial x_i}$, $i = 1, 2, 3$ and λ depending on i, j and l (see for example (3.3)). We have

$$\begin{aligned}\|D_i(D_jCD_lC)\|_{S'_C} &\le \|D_jCD_lC\|_{L^2(\Omega)}\\ &\le \|D_jC\|_{L^4(\Omega)}\|D_lC\|_{L^4(\Omega)}\\ &\le \mathcal{M}\|D_jC\|^{1/2}_{L^2(\Omega)}\|D_lC\|^{1/2}_{L^2(\Omega)}\|D_jC\|^{1/2}_{H^1(\Omega)}\|D_lC\|^{1/2}_{H^1(\Omega)}.\end{aligned}$$

We notice that f is a bounded function. Using the the same reasoning as for the previous lemmas, we prove that $\frac{\partial u}{\partial t}$ is bounded in $L^2(0,t;L^2)$. □

We can now formulate the main result of this section.

Theorem 3.5. *Problem* (2.1)-(2.5) *admits a global solution.*

Proof. It is easy to see that the problem admits a finite-dimensional solutions C_m and u_m defined on the interval of time $[0;T_m[$. From the previous Lemmas applied to C_m and u_m we deduce the global existence of those solutions.

Furthermore, the previous Lemmas provide the existence of subsequences, still denoted by C_m and u_m, such that

$$\begin{gathered}C_m \to C \quad \text{weakly in } L^2(0,T;S_C),\\ C_m \to C \quad \text{weak-star in } L^\infty(0,T;H^1),\\ C'_m \to C' \quad \text{weakly in } L^2(0,T;S'_C),\\ u_m \to u \quad \text{weakly in } L^2(0,T;S_u),\\ u_m \to u \quad \text{weak-star in } L^\infty(0,T;H_{\rm div}),\\ u'_m \to u' \quad \text{weakly in } L^2(0,T;S'_u)\,.\end{gathered}$$

By classical compactness theorems (see for example [10, 17]), we also obtain the strong convergence of $(C_m;u_m)$ and by passing to the limit we obtain the existence of solutions. □

4. Uniqueness of solution

To prove uniqueness of solution, we assume that problem (2.1)-(2.5) has two solutions (C_1,u_1) and (C_2,u_2). From (2.1), we have

$$\frac{\partial}{\partial t}(C_1-C_2) - d\Delta(C_1-C_2) + u_1\nabla C_1 - u_2\nabla C_2 + g(C_1-C_2) = 0, \tag{4.1}$$

and from (2.2), we also have

$$\begin{aligned}&\frac{\partial}{\partial t}(u_1-u_2) + \mu_p(u_1-u_2) + \nabla(p_1-p_2)\\ &= \frac{k}{2}\nabla\left((\nabla C_1)^2 - (\nabla C_2)^2\right) - k(\Delta C_1\nabla C_1 - \Delta C_2\nabla C_2).\end{aligned}\tag{4.2}$$

Multiplying (4.1) by $-k\Delta(C_1-C_2)$ and integrating, we obtain

$$\begin{aligned}&(\frac{\partial}{\partial t}(C_1-C_2), -k\Delta(C_1-C_2)) + dk(\Delta(C_1-C_2),\Delta(C_1-C_2))\\ &+ (u_1\nabla C_1, -k\Delta(C_1-C_2)) + (u_2\nabla C_2, k\Delta(C_1-C_2))\\ &+ (g(C_1-C_2), -k\Delta(C_1-C_2)) = 0.\end{aligned}$$

Similarly, multiplying (4.2) by $u_1 - u_2$ and integrating, we have

$$
\begin{aligned}
&(\frac{\partial}{\partial t}(u_1-u_2), u_1-u_2) + \mu_p(u_1-u_2, u_1-u_2) \\
&= \frac{k}{2}(\nabla\left((\nabla C_1)^2 - (\nabla C_2)^2\right), u_1-u_2) - k(\Delta C_1\nabla C_1 - \Delta C_2 \nabla C_2, u_1-u_2).
\end{aligned}
$$

Adding the two last equalities, using Green's formula and the fact that $u_i \in S_u$, we conclude that

$$
\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}(\|u_1-u_2\|_{L^2}^2 + k\|\nabla C_1 - \nabla C_2\|_{L^2}^2) + \mu_p\|u_1-u_2\|_{L^2}^2 + kd\|\Delta(C_1-C_2)\|_{L^2}^2 \\
&= k(u_1\nabla(C_1-C_2), \Delta(C_1-C_2)) + k((u_1-u_2)\nabla C_2, \Delta(C_1-C_2)) \\
&\quad - k(\Delta C_1 \nabla(C_1-C_2), u_1-u_2) + k(-\Delta C_1\nabla C_2 + \Delta C_2\nabla C_2, u_1-u_2) \\
&\quad + k(g(C_1-C_2), \Delta(C_1-C_2)).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}(\|u_1-u_2\|_{L^2}^2 + k\|\nabla C_1 - \nabla C_2\|_{L^2}^2) + \mu_p\|u_1-u_2\|_{L^2}^2 + kd\|\Delta(C_1-C_2)\|_{L^2}^2 \\
&= k(u_1\nabla(C_1-C_2), \Delta(C_1-C_2)) - k(\Delta C_1\nabla(C_1-C_2), u_1-u_2) \\
&\quad + k(g(C_1-C_2), \Delta(C_1-C_2)).
\end{aligned} \tag{4.3}
$$

We now estimate the right-hand side of this equality. We put $C = C_1 - C_2$ and $u = u_1 - u_2$. From the Hölder inequality it follows that

$$
|(\Delta C_1 \nabla C, u)| \leq \|\Delta C_1\|_{L^2}\|\nabla C.u\|_{L^2} \leq \|\Delta C_1\|_{L^2}\|\nabla C\|_{L^4}\|u\|_{(L^4)^2}.
$$

Also, from the Gagliardo-Nirenberg inequality we obtain

$$
|(\Delta C_1\nabla C, u)| \leq N_1\|\Delta C_1\|_{L^2}\|\nabla C\|_{L^2}^{1/2}\|\Delta C\|_{L^2}^{1/2}\|u\|_{L^2}^{1/2}\|\nabla u\|_{L^2}^{1/2}.
$$

Next, applying the Young's inequality, we obtain

$$
|(\Delta C_1\nabla C, u)| \leq \frac{N_1}{4}\|\Delta C\|_{L^2}^2 + \frac{3N_1}{4}\|\Delta C_1\|_{L^2}^{4/3}\|\nabla C\|_{L^2}^{2/3}\|u\|_{L^2}^{2/3}\|\nabla u\|_{L^2}^{2/3}.
$$

Using that same technics, we obtain the inequality

$$
|(u_1\nabla C, \Delta C)| \leq \|\Delta C\|_{L^2}\|\nabla C.u_1\|_{L^2} \leq \|\Delta C\|_{L^2}\|\nabla C\|_{L^4}\|u_1\|_{(L^4)^2}.
$$

Therefore,

$$
|(u_1\nabla C, \Delta C)| \leq N_2\|\Delta C\|_{L^2}^{3/2}\|\nabla C\|_{L^2}^{1/2}\|u_1\|_{L^2}^{1/2}\|\nabla u_1\|_{L^2}^{1/2}.
$$

Finally,

$$
|(u_1\nabla C, \Delta C)| \leq \frac{3N_2}{4}\|\Delta C\|_{L^2}^2 + \frac{N_2}{4}\|\nabla C\|_{L^2}^2\|u_1\|_{L^2}^2\|\nabla u_1\|_{L^2}^2.
$$

From (4.3) and assuming that $N_1 + 3N_2 \leq 4d$, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}(\|u\|_{L^2}^2 + k\|\nabla C\|_{L^2}^2) \\
&\leq \frac{3N_1 k}{2}\|\Delta C_1\|_{L^2}^{4/3}\|\nabla C\|_{L^2}^{2/3}\|u\|_{L^2}^{2/3}\|\nabla u\|_{L^2}^{2/3} \\
&\quad + \frac{N_2 k}{2}\|\nabla C\|_{L^2}^2\|u_1\|_{L^2}^2\|\nabla u_1\|_{L^2}^2 + kg_0\|\nabla C\|_{L^2}^2 \\
&\leq (\|u\|_{L^2}^2 + k\|\nabla C\|_{L^2}^2)\Big(\frac{N_2}{2}\|\nabla C\|_{L^2}^2\|u_1\|_{L^2}^2\|\nabla u_1\|_{L^2}^2
\end{aligned}
$$

$$+\frac{3N_1k}{2}\|\Delta C_1\|_{L^2}^{4/3}\|\nabla C\|_{L^2}^{2/3}\|u\|_{L^2}^{-4/3}\|\nabla u\|_{L^2}^{2/3}+kg_0\Big).$$

Denote

$$\phi(t)=\|\nabla C\|_{L^2}^2\|u_1\|_{L^2}^2\|\nabla u_1\|_{L^2}^2+\|\Delta C_1\|_{L^2}^{4/3}\|\nabla C\|_{L^2}^{2/3}\|u\|_{L^2}^{-4/3}\|\nabla u\|_{L^2}^{2/3}+1,$$

$$M=\max\big(\frac{N_2}{2},\frac{3N_1k}{2},kg_0\big).$$

Then we have the estimate

$$\frac{d}{dt}(\exp\Big(M\int_0^t\phi(s)ds\Big)(\|u\|_{L^2}^2+k\|\nabla C\|_{L^2}^2))\leq 0.$$

for all $t\geq 0$. From this we deduce that

$$\exp\Big(M\int_0^t\phi(s)ds\Big)(\|u\|_{L^2}^2+k\|\nabla C\|_{L^2}^2)\leq\|u(0)\|_{L^2}^2+k\|\nabla C(0)\|_{L^2}^2.$$

Since $u(0)=C(0)=0$, we conclude the uniqueness of solution. We can now state the theorem on the uniqueness of solution.

Theorem 4.1. *Problem* (2.1)-(2.5) *admits a unique solution.*

5. Numerical simulations

For numerical simulations, we will consider the 2D problem without reaction term and external forces. We will introduce the stream function defined by the equalities

$$u_1=\frac{\partial\psi}{\partial x_2},\quad u_2=-\frac{\partial\psi}{\partial x_1},$$

and the vorticity $\omega=rot(u)$. The problem becomes

$$\frac{\partial\omega}{\partial t}+\mu_p\omega=\frac{\partial}{\partial x_1}(\frac{\partial T_{21}}{\partial x_1}+\frac{\partial T_{22}}{\partial x_2})-\frac{\partial}{\partial x_2}(\frac{\partial T_{11}}{\partial x_1}+\frac{\partial T_{12}}{\partial x_2}),\tag{5.1}$$

$$\frac{\partial C}{\partial t}+(\frac{\partial\psi}{\partial x_2},-\frac{\partial\psi}{\partial x_1}).\nabla C=d\Delta C,\tag{5.2}$$

$$\omega=-\Delta\psi\,.\tag{5.3}$$

Numerical method. We begin with equation (5.2). It is solved by the alternative direction implicit finite difference method with Thomas algorithm:

$$\begin{aligned}
&\frac{C_{i,j}^{n+\frac12}-C_{i,j}^n}{h_t/2}\\
&=d\Big(\frac{C_{i-1,j}^{n+\frac12}-2C_{i,j}^{n+\frac12}+C_{i+1,j}^{n+\frac12}}{h_x^2}+\frac{C_{i,j-1}^n-2C_{i,j}^n+C_{i,j+1}^n}{h_y^2}\Big)\\
&\quad-\frac{\psi_{i,j+1}^n-\psi_{i,j-1}^n}{2h_y}\frac{C_{i+1,j}^{n+\frac12}-C_{i-1,j}^{n+\frac12}}{2h_x}+\frac{\psi_{i+1,j}^n-\psi_{i-1,j}^n}{2h_x}\frac{C_{i,j+1}^n-C_{i,j-1}^n}{2h_y}\,,
\end{aligned}$$

$$\begin{aligned}
&\frac{C_{i,j}^{n+1}-C_{i,j}^{n+\frac12}}{h_t/2}\\
&=d\Big(\frac{C_{i-1,j}^{n+\frac12}-2C_{i,j}^{n+\frac12}+C_{i+1,j}^{n+\frac12}}{h_x^2}+\frac{C_{i,j-1}^{n+1}-2C_{i,j}^{n+1}+C_{i,j+1}^{n+1}}{h_y^2}\Big)
\end{aligned}$$

$$
- \frac{\psi^n_{i,j+1} - \psi^n_{i,j-1}}{2h_y} \frac{C^{n+\frac{1}{2}}_{i+1,j} - C^{n+\frac{1}{2}}_{i-1,j}}{2h_x} + \frac{\psi^n_{i+1,j} - \psi^n_{i-1,j}}{2h_x} \frac{C^{n+1}_{i,j+1} - C^{n+1}_{i,j-1}}{2h_y} .
$$

In (5.1) we replace T_{ij} by their expressions through the concentrations

$$
\frac{\partial \omega}{\partial t} + \mu_p \omega = k\Big(\frac{\partial C}{\partial x_1}\Big(\frac{\partial^3 C}{\partial x_2^3} + \frac{\partial^3 C}{\partial x_1^2 \partial x_2}\Big) - \frac{\partial C}{\partial x_2}\Big(\frac{\partial^3 C}{\partial x_1^3} + \frac{\partial^3 C}{\partial x_1 \partial x_2^2}\Big)\Big) \tag{5.4}
$$

We use the finite difference scheme

$$
\begin{aligned}
\omega^{n+1}_{i,j} = {}& \frac{1}{1+h_t\mu_p}\omega^n_{i,j} + k\frac{h_t}{1+h_t\mu_p}\Big(\frac{C^{n+1}_{i+1,j} - C^{n+1}_{i-1,j}}{2h_x} \\
& \times \Big(\frac{C^{n+1}_{i,j+2} - 2C^{n+1}_{i,j+1} + 2C^{n+1}_{i,j-1} - C^{n+1}_{i,j-2}}{2h_y^3} \\
& + \frac{(C^{n+1}_{i+1,j+1} - C^{n+1}_{i+1,j-1}) - 2(C^{n+1}_{i,j+1} - C^{n+1}_{i,j-1}) + (C^{n+1}_{i-1,j+1} - C^{n+1}_{i-1,j-1})}{2h_x^2 h_y}\Big) \\
& - \frac{C^{n+1}_{i,j+1} - C^{n+1}_{i,j-1}}{2h_y}\Big(\frac{C^{n+1}_{i+2,j} - 2C^{n+1}_{i+1,j} + 2C^{n+1}_{i-1,j} - C^{n+1}_{i-2,j}}{2h_x^3} \\
& + \Big((C^{n+1}_{i+1,j+1} - C^{n+1}_{i-1,j+1}) - 2(C^{n+1}_{i+1,j} - C^{n+1}_{i-1,j}) + (C^{n+1}_{i+1,j-1} \\
& - C^{n+1}_{i-1,j-1})\Big) / \big(2h_y^2 h_x\big)\Big)\Big) .
\end{aligned}
$$

Equation (5.3) is solved by the fast Fourier transform method.

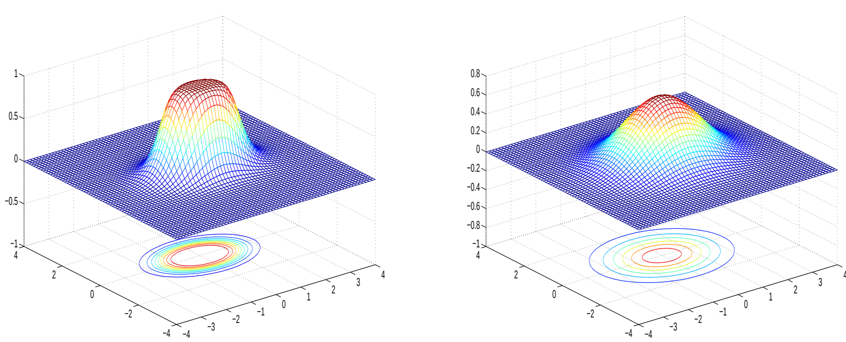

Figure 1. Evolution of the concentration during 100 seconds for $d = 3 \times 10^{-3}$, $k = 10^{-7}$ and $\mu_p = 100$.

Numerical results. An example of numerical simulations is shown in Figures 1 and 2. Figure 1 shows the evolution of the miscible drop in time. The transient interfacial tension affects the geometry of the drop and its shape becomes more spherical. At the same time, the maximum of the concentration decreases due to diffusion (Figure 2, left). The stream lines are shown in Figure 2 (right). Though transient interfacial phenomena are sufficiently weak, they provoke the motion of fluid which is initially quiescent.

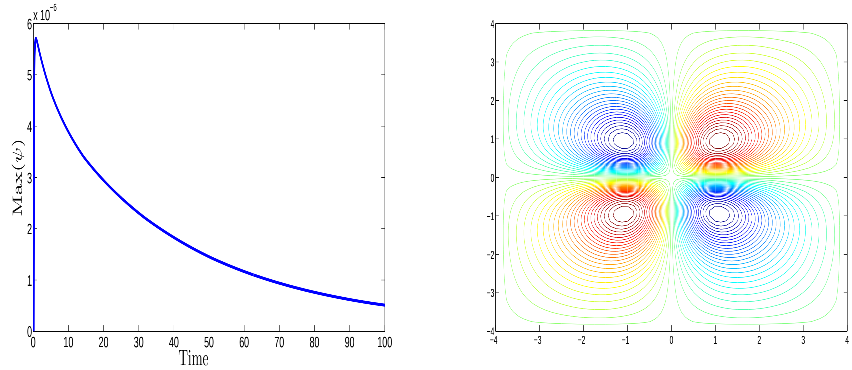

FIGURE 2. The maximum of stream function as function of time for $d = 3 \times 10^{-3}$, $k = 10^{-7}$ and $\mu_p = 10$ (left). The stream lines for the same parameters and after 100 s(right)

References

[1] K. Allali; *Existence and uniqueness of solutions for miscible liquids model in porous media. Electronic Journal of Differential Equations*, 2013, vol. 2013, no 254, p. 1-7.

[2] N. Bessonov, V. A. Volpert, J. A. Pojman, B. D. Zoltowski; *Numerical simulations of convection induced by Korteweg stresses in miscible polymermonomer systems*, Microgravity-Science and Technology **17** (2005) 8–12.

[3] N. Bessonov, J. Pojman, G. Viner, V. Volpert, B. Zoltowski Instabilities of diffuse interfaces. Math. Model. Nat. Phenom., 3 (2008), No. 1, 108-125.

[4] N. Bessonov, J. Pojman, V. Volpert; *Modelling of diffuse interfaces with temperature gradients.* J. Engrg. Math. 49 (2004), no. 4, 321–338.

[5] J. W. Cahn, J. E. Hilliard; *Free Energy of a Nonuniform System. I. Interfacial Free Energy,* J. Chem. Phys. 28 (1958), 258-267.

[6] C. A. Hutchinson, P. H. Braun; *Phase relations of miscible displacement in oil recovery*, AIChE Journal **7** (1961), 64–72.

[7] D. D. Joseph; *Fluid dynamics of two miscible liquids with diffusion and gradient stresses*, European journal of mechanics. B, Fluids **9** (1990) 565–596.

[8] D. Korteweg; *Sur la forme que prennent les quations du mouvement des fluides si l'on tient compte des forces capillaires causes par des variations de densit*, Arch. Nerl Sci. Exa. Nat., Series II. **6** (1901) 1–24.

[9] I. Kostin, M. Marion, R. Texier-Picard, V. Volpert; *Modelling of miscible liquids with the Korteweg stress*, ESAIM: Mathematical Modelling and Numerical Analysis **37**(2003) 741–753.

[10] J. L. Lions; *Quelques mthodes de rsolution des problmes aux limites non linaires*, Gauthier-Villars, Paris (1969).

[11] J. Offeringa, C. Van der Poel; *Displacement of oil from porous media by miscible liquids*, Journal of Petroleum Technology **5** (1954), 37–43.

[12] P. Petitjeans, T. Maxworthy; *Miscible displacements in capillary tubes. Part 1. Experiments*, Journal of Fluid Mechanics **326** (1996) 37–56.

[13] J. Pojman, N. Bessonov, V. Volpert, M. S. Paley; *Miscible Fluids in Microgravity (Mfmg): A Zero-Upmass Investigation on the International Space Station*, Microgravity Sci. Tech. 2007, XIX, Volume 19, issue 1, pages 33-41.

[14] I. Rousar, E. B. Nauman; *A Continuum Analysis of Surface Tension in Nonequilibrium Systems,* Chem. Eng. Comm. 129 (1994), 19-28.

[15] F. Schwille; *Groundwater pollution in porous media by fluids immiscible with water*, Science of the Total Environment **21** (1981), 173–185.

[16] F. Schwille; *Migration of organic fluids immiscible with water in the unsaturated zone*, Pollutants in porous media. Springer Berlin Heidelberg, (1984), 27–48.

[17] R. Temam; *Navier-Stokes equations. Theory and numerical analysis,* North-Holland Publishing Co., Amsterdam, New York, Stud. Math. Appl. 2 (1979).
[18] Y. B. Zeldovich; *About Surface Tension of a Boundary between two Mutually Soluble Liquids*, Zhur. Fiz. Khim. 23 (in Russian) (1949), 931-935.

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