New Hybrid FE-FV Method for Computing Current Distribution in 2-D Superconductors: Application to an HTS Cylinder in Transverse Magnetic Field

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Abstract— **This paper presents a new numerical method based on finite elements - finite volumes (FE-FV) for solving 2-D diffusion problems in high temperature superconductors (HTS). The approach does not involve directly the resistivity term (ρ), generally used to model the $E(J)$ characteristic as a power law, i.e $E(J) = \rho(J)J$, with $\rho(J) \propto J^{n-1}$. Instead, we use a $J(E)$ constitutive law $J \propto E^{\frac{1}{n}}$, with $\overrightarrow{E} = E\overrightarrow{e}_z$ (a single component), which leads to a scalar non-linear differential equation. After presenting in details the developments, the method is tested in the case of a superconducting cylinder submitted to a transverse magnetic field. The current density obtained is compared to another numerical technique (the semi-analytical method) in order to validate the results. Although not fully optimized yet, it appears that the proposed method is very stable, especially for large n-values (greater than 100).**

Index Terms— **Diffusion processes, electromagnetic analysis, finite-element methods (FEMs), high-temperature superconductors, integral equations, numerical analysis.**

I. INTRODUCTION

THE numerical models characterizing superconducting materials generally intend to solve Maxwell's equations, complemented by the non-linear constitutive equation $E(J) = E_c (J/J_c)^n$ (power law characteristic) [1]. The use of a resistivity term ρ, such as $E = \rho(J)J$, is the most common approach to treat the non-linearity. The resulting equations are then "semi-linear", and their resolutions depends on the calculation of $\rho(J)$. For instance, this is the underlying approach used by commercial finite element packages such as COMSOL Multiphysics or Flux-2D. However, in the case of large n values, this approach is prone to numerical oscillations and convergence problems. Others schemes using integral formulas in a semi-analytical form were proposed as an alternative to finite elements, but the same limitations occur for large n values [2], [3].

In this paper, we present a model combining Maxwell's equations directly to a $J(E)$ power law $J(E) = J_c (E/E_c)^{\frac{1}{n}}$. We assume the low frequency regime, as well as linear permittivity and permeability, such that $\nabla \cdot \overrightarrow{E} = 0$ and $\overrightarrow{B} = \mu_0 \overrightarrow{H}$. Under these assumptions, the electric field is the solution of a non-linear parabolic diffusion equation:

$$\nabla^2 u = c\frac{\partial \beta(u)}{\partial t} \tag{1}$$

with, $u = E/E_c$, $\beta(u) = u^\alpha = J/J_c$, $\alpha = 1/n$ and $c = \mu_0 J_c/E_c$.

This type of equation is often encountered in fluid mechanics and porous media. The numerical recipes developed in applied mathematics for these non-linear parabolic equations have been the subject of many works in the past. Among them, the finite volume (FV) method have been proposed and thoroughly justified and validated [4].

When applied to our case, there is no need to consider explicitly a resistivity term, and the discrete problem can remain fully non-linear. The fixed point theorem proves the existence of a discrete solution. The unknowns would correspond to the mean value of u (the normalized electric field) on each cell of the finite volume mesh. Nevertheless, it remains difficult to obtain the gradient of a mean value on a finite volume mesh, although this is required since $\nabla^2 u = \nabla \cdot (\nabla u)$. The discretization of the convective term (i.e. the divergence) containing a gradient therefore requires a specific treatment [5].

Footbridges between finite volumes (FV) and finite elements (FE) have been developed for convection-diffusion problems. They are used to connect the solutions FE and FV discretizations [6]. In the case of dual meshing, where the cells of finite volume mesh are constructed around the triangular mesh, it is possible to replace the FV discrete form of the convective term by a FE discrete form of the diffusive term (i.e. the Laplacian). These mixed schemes combine the advantages of both methods and are known to be robust, stable and efficient in the case of non-linear parabolic equations [7].

II. MODELING

A. The differential problem

The development presented below is valid for a single 2-D conducting domain (infinite length along z axis). No transport current is considered for now, but an uniform applied field can be applied along an arbitrary axis in the $x - y$ plane. We also assume parameters E_c, J_c and n as being constants. When the external magnetic field depends on time according to $\overrightarrow{B}(t) = B_x(t)\overrightarrow{e_x} + B_y(t)\overrightarrow{e_y}$, the induced electric field $\overrightarrow{E} = E(x, y, t)\overrightarrow{e_z}$ in the superconductor is the solution of the non-linear diffusion equation (1).

The Faraday's law and the continuity of the magnetic field lead to a boundary condition on ∇u:

$$\nabla \times \overrightarrow{E} = \mathcal{A}\nabla E(x, y, t) = -\frac{\partial \overrightarrow{B}}{\partial t}, \tag{2}$$

$$E_c \nabla u \cdot \overrightarrow{\nu} = -\mathcal{A}^{-1} \frac{\partial \overrightarrow{B}}{\partial t} \cdot \overrightarrow{\nu}, \tag{3}$$

where ν is the exterior normal to the domain boundary, and $\mathcal{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Our goal is therefore to solve the following differential system in accordance with the discretization rules presented in the next section :

$$(S) \begin{cases} c \dfrac{\partial \beta(u)}{\partial t} - \nabla^2 u = 0 \\ \nabla u \cdot \overrightarrow{\nu} = C_b(t) \\ u(t=0) = 0 \end{cases} \tag{4}$$

with $C_b(t) = -E_c^{-1} \mathcal{A}^{-1} \dfrac{\partial \overrightarrow{B}}{\partial t} \cdot \overrightarrow{\nu}$

B. Mathematical formalism and weak formulation

The problem domain is the space-time space $Q_T = \Omega \times [0,T]$. The FE and FV discretization methods consist in solving a weak formulation of the differential system (4), which is expressed as an integral equation, written in the appropriate spaces:

* ⋆ The Hilbert space on Ω, $L^2(\Omega)$ and the scalar product $(u,v)_{L^2} = \displaystyle\int_\Omega uv d\Omega$.
* ⋆ The sub-space $H^1(\Omega) = \{u \in L^2(\Omega), \nabla u \in L^2(\Omega)\}$
* ⋆ The space $L^\infty(0,T; L^2(\Omega))$: $\beta(u)$ is bounded on Q_T and at any given time t, $\beta(u) \in L^2(\Omega)$

The weak formulation is the projection on a distribution space of the differential system, when the later is approximated by a set of basis functions φ_i. The final equation (weak form) of our problem is obtained by multiplying the diffusion equation (4) by the test function φ and integrating it on Q_T, i.e.

$$\int_0^T \int_\Omega c \frac{\partial \beta(u)}{\partial t} \varphi d\Omega dt - \int_0^T \int_\Omega \nabla^2 u \varphi d\Omega dt = 0. \tag{5}$$

III. DISCRETIZATION

A. Temporal discretization

We chose to work with a prescribed partitioning of the time interval $[0,T]$ in $p+1$ points, which results in a series of discrete times $(t_p)_{0 \le p \le P}$. The time step is defined by $\Delta t_p = t_{p+1} - t_p$, and is small enough to use:

* ⋆ The rectangular method for the integration on the interval $[t_p, t_{p+1}]$,
* ⋆ Finite differences for the calculation of the derivatives.

Let's use the notation $u(x,y,t_p) = u^p$ for describing the solution at time t_p. On each interval $[t_p, t_{p+1}]$, we have the following integral equation on Ω:

$$c \int_\Omega \frac{\beta(u^{p+1}) - \beta(u^p)}{\Delta t_p} \varphi d\Omega - \int_\Omega \nabla^2 u^{p+1} \varphi d\Omega = 0. \tag{6}$$

The discrete problem is obtained by projecting this integral equation on the basis functions φ_i, defined on a finite dimension approximation subspace. These subspaces are built from the decomposition of the domain in triangles (FE) or control volumes (FV), as shown in Fig. 1.
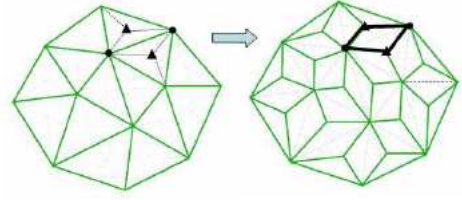

Fig. 1: FE triangular mesh (left) and FV control volumes (right): to build a FV cell from the FE mesh, we link the centers of gravity of two contiguous triangles through the vertex of their common edge.

B. Finite element discretization

For a set of nodes defining the triangular mesh, we denote $\mathcal{X}$ the approximation space determined by the basis functions φ_i, chosen as piecewise linear. Finding a FE discrete problem for (6) depends on the existence of the projection of $\beta(u)$ on $\mathcal{X}$. In our case, the existence is impossible because $\beta'(0) \hookrightarrow +\infty$. This difficulty is avoided in Finite volume discretization.

But, the discretization of the Laplace operator is easy and well-known [8]. It involves a diffusion matrix S, whose elements $S_{i,j}$ are the diffusion coefficients between the nodes N_i and N_j, corresponding to the unknowns u_i and u_j. For $u = \sum_j u_j \varphi_j$,

$$\int_\Omega \nabla^2 u \varphi_i d\Omega = \underbrace{-\int_\Omega \nabla u \cdot \nabla \varphi_i d\Omega}_{Diffusion} + \underbrace{\int_\Gamma \nabla u \cdot \varphi_i \overrightarrow{\nu} d\Gamma}_{Boundary} \tag{7}$$

The diffusive term $\displaystyle\int_\Omega \nabla u \cdot \nabla \varphi_i d\Omega = \sum_j S_{i,j} u_j$ is a function of

$$S_{i,j} = \sum_{K \in \mathcal{I}_h} \int_K \nabla \varphi_i \cdot \nabla \varphi_j dK.$$

C. Finite volume discretization

The basis functions of the FV approximation space, denoted $\mathcal{V}$, are the indicatrix functions 1_D of the cells D of the FV mesh ($1_D(x) = 1$ if $x \in D$, and $1_D(x) = 0$ elsewhere). The unknowns are the mean value of the solution (normalized electric field) on each cell. The definition of mean value is assumed by the continuity property of the function. Remembering that $\beta(u)$ is a non-linear but otherwise continuous function, the non-linearity is easily treated with the FV approach and the first integral in (6) is proportionnal of the mean value of β on D.

Let's define $\beta_D^p = \langle \beta(u^p) \rangle_D$ be the mean values of $\beta(u)$ on cell D at time $t = t_p$. We obtain:

$$\int_D \frac{\beta(u^{p+1}) - \beta(u^p)}{\Delta t_p} 1_D dD = \frac{\beta_D^{p+1} - \beta_D^p}{\Delta t_p} |D|, \tag{8}$$

where $|D|$ is the area of the FV element.

The FV method is based on the strict imposition of the flux conservation through the boundaries of each contiguous FV cells. In practice, it uses the divergence theorem to evaluate the convection through boundary integrals. The total outward flux Φ from a given cell D and time step p is given by:

$$\Phi = \int_D \nabla \cdot (\nabla u^p) dD = \int_{\partial D} \nabla u^p \cdot \overrightarrow{v} d(\partial D), \tag{9}$$

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where $\overrightarrow{v}$ is the outward-pointing normal vector.

In terms of the discrete problem, the total outward flux from cell D_i is the sum of the fluxes riding through each edge of the cell. The flux between two contiguous cells D_i and D_j therefore corresponds to $\Phi_{i,j}$. The FV formulation depends on the choice of the flux functions $\Phi_{i,j}$, which are chosen such that $\Phi_{i,j} = -\Phi_{j,i}$. The discrete equation on each cell can therefore be written as:

$$\int_{\partial D} \nabla u^p \cdot \overrightarrow{v} d(\partial D) = \sum_{D_j \in \mathcal{N}(D_i)} |D_i \cap D_j| \Phi_{i,j}(u_i, u_j), \quad (10)$$

where $\mathcal{N}(D_i)$ is the set of the contiguous cells of D_i. Unfortunately, approximating the gradient of the mean value is difficult. To solve this problem, we use footbridges between the FE and FV approximation spaces. These footbridges will allow expressing the fluxes $\Phi_{i,j}$ as a function of the $S_{i,j}$ terms.

D. The mixed scheme

When dual meshing is used to generate the FV cells, the footbridges beetween the FE and the FV approximation spaces are well described in [6]. Let u^{FE} and u^{FV} be the respective solutions of the FE and FV approaches. The footbridges operators π_{EV} and π_{VE} are projections defined as:

★ FV $(\mathcal{V}) \to$ FE $(\mathcal{X})$ $\pi_{EV}(u^{FV}) = u^{FE}$ is the solution of the FE discretization and satisfies:

$$\int_\Omega |\pi_{EV} u - u|^2 d\Omega = \min_{w \in \mathcal{X}} \int_\Omega |w - u|^2 d\Omega \quad (11)$$

★ FE $(\mathcal{X}) \to$ FV $(\mathcal{V})$ $\pi_{VE}(u^{FE}) = u^{FV}$ is the solution of the FV discretization and satisfies, for all $\varphi \in \mathcal{X}$:

$$\int_\Omega \pi_{VE} u \varphi d\Omega = \int_\Omega u \varphi d\Omega \quad (12)$$

These footbridges can be used to show that mixing the two discretizations (FE and FV) is possible only if $u^{FE} = u^{FV}$. The existence of the footbridge operators π_{EV} and π_{VE} defining in (11) and (12) ensures the required equality and have been proved in previous papers, e.g. [6].

For the kind of dual meshing considered here, each cell D_i of the FV mesh can be associated with one node N_i of the FE mesh, as shown in Fig. 2.

(a) The nodes N_i correspond to the middle of the edges

(b) The nodes N_i corresponds to the center of the cells D_i
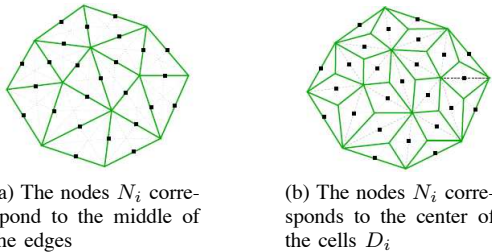
Fig. 2: Association of each node of the FE mesh to the cells of the FV mesh.

As a result, it is possible to calculate $\Phi_{i,j}$ as a function of $S_{i,j}$. On the internal FV cells (cells with 4 edges), we can consider that the FV convection and the FE diffusion are equal, i.e.

$$\int_D \nabla \cdot (\nabla u) dD = \int_\Omega \nabla u \cdot \nabla \varphi_i d\Omega \quad (13)$$

$$\sum_{D_j \in \mathcal{N}(D_i)} |D_i \cap D_j| \Phi_{i,j}(u_i, u_j) = -\sum_j S_{i,j} u_j \quad (14)$$

It is important that these expressions keep the local conservation property of the FV formulation ($\Phi_{i,j} = -\Phi_{j,i}$). This is the case if we define $S_{i,i} = -\sum_{j \neq i} S_{i,j}$, where $S_{i,j}$ are positive coefficients. Under these conditions, we obtain:

$$|D_i \cap D_j| \Phi_{i,j}(u_i, u_j) = S_{i,j}(u_j - u_i) \quad (15)$$

The discrete problem (hybrid FE and FV numerical schemes) consists in calculating, at each time step $p+1$, a set of unknowns defined by $\left(u_i^{p+1}\right)_{1 \leq i \leq N}$, N being the number of nodes in the FE mesh:

★ For the internal cells (4 edges), one has:

$$c \frac{\beta(u_i^{p+1}) - \beta(u_i^p)}{\triangle t_p} |D_i| - \sum_j S_{i,j} u_j^{p+1} = 0 \quad (16)$$

★ For the boundary cells (one edge belongs to the boundary Γ) and considering the boundary condition

$$\int_\Gamma \overrightarrow{\nabla} u^{p+1} \cdot \varphi_i \overrightarrow{\nu} d\Gamma = \int_{\Gamma_i} C_b^{p+1} \varphi_i d\Gamma = |\Gamma_i| C_b^{p+1} \quad (17)$$

one has:

$$c \frac{\beta(u_i^{p+1}) - \beta(u_i^p)}{\triangle t_p} |D_i| - \sum_j S_{i,j} u_j^{p+1} - |\Gamma_i| C_b^{p+1} = 0 \quad (18)$$

The existence of the discrete solution is proved by the fixed-point theorem. The numerical computation uses the unknowns $v_i^{p+1} = \beta(u_i^{p+1})$. Since β^{-1} is continuous and differentiable, we can use a Newton-Raphson algorithm, i.e.

$$u_i^{p,k} = \beta^{-1}(v_i^{p,k-1}) + \left(\beta^{-1}\right)' \Big|_{v_i^{p,k-1}} \left(v_i^{p,k} - v_i^{p,k-1}\right) \quad (19)$$

The convergence of this numerical scheme for each time step is obtained when the relative difference between two successive iterations $\epsilon = \sum_i \left[\frac{\left(v_i^{p,k} - v_i^{p,k-1}\right)}{v_i^{p,k}}\right]^2 < 10^{-9}$.

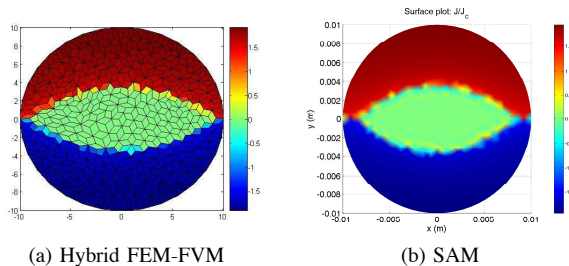## IV. NUMERICAL APPLICATION: LONG CYLINDER IN A TRANSVERSE MAGNETIC FIELD

Let's consider a cylinder whose parameters are $R = 10$ mm (radius), $J_c = 10^8$ A/cm^2, $E_c = 10^{-4}$ V/m^{-1} and $n = 15$, and immersed in a transverse external field defined as

$$\overrightarrow{B} = B_{max} sin(2\pi f t) \overrightarrow{e}_x. \quad (20)$$

In the cases presented below, $f = 50$ Hz, and we define $T_0 = 1/f = 20$ ms. Based on the Bean model, the field required for full penetration is $B_p = 2\mu_0 J_c R/\pi = 0.8T$. Setting $B_{max} = B_p$ implies that the full penetration will occur at $t = T_0/4$, where the applied field reaches its maximum value [9]. In the case of the power law characteristic with $n = 15$, however, the full penetration is not obtained, as will be shown below.

In Fig. 3, we present a comparative result between numerical solutions of the normalized current density (J/J_c) obtained with our hybrid method (HM) and with the semi-analytical method

(SAM, [3]) at time $t = T_0/4$. As can be seen, we find exactly the same solutions, to within numerical errors and mesh differences. As the SAM method was itself well validated against more classical finite element formulations, this proves the the formulation proposed is correct. Fig. 3 also reveals, as expected, that the full penetration is not reached because of the presence of the power law characteristic.
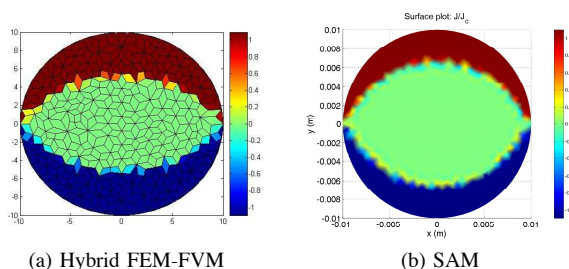
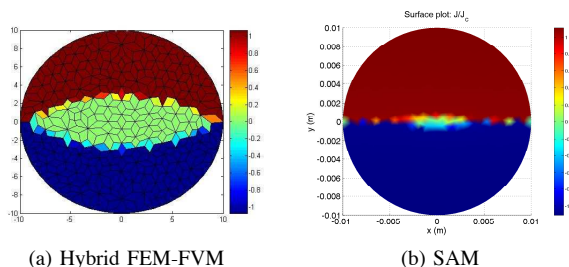in this case, and neither did another finite element method code we had access to. In this case, we reached the Bean penetration limit at $t = T_0/4$, which seems reasonable as $n = 200$ is a very good approximation of the Bean model. This suggests that our new hybrid method is very robust against strong non-linearities, thanks to the finite volume properties and the change of variable chosen (we solve for $\beta(u)$ instead of u directly).

(a) Hybrid FEM-FVM (b) SAM

Fig. 3: Comparison for $n = 15$, at $t = T_0/4$

(a) Hybrid FEM-FVM, $\frac{T_0}{8}$ (b) Hybrid FEM-FVM, $\frac{T_0}{4}$

Fig. 6: Results of hybrid FEM-FVM, for $n = 200$

As a further validation, we compared the results of both methods for $n = 100$, as shown in Fig. 4 (for $t = T_0/8$) and Fig. 5 (for $t = T_0/4$). In this case, however, we observe different penetration rates, which is surprising a priori, given the good correspondence found above. Even after a thorough investigation of both codes, this discrepancy could not be explained. However, as both codes solve for different variables, it is possible that tiny numerical drifts of the solution in the case of very non-linear cases such as this one are responsible of the differences.

V. CONCLUSION

In this paper, we presented the development of an hybrid FE-FV formulation to solve the non-linear electric field diffusion in 2-D superconducting materials. This method takes advantage of both the FE and FV methods, namely:

- ⋆ <u>FEM</u>: Easy to deal with irregular mesh and to process the Laplacian operator
- ⋆ <u>FVM</u>: Easy to treat the non-linear term $\beta(u)$

Interestingly, the accuracy of the results seems to be independent of mesh refinements. The hybrid method was validated by comparing the results against another validated method in the case of moderate non-linearity (power law exponent of $n = 15$), and proved to be very robust for n values up to 200. However, for strong non-linearities, it was impossible to obtain identical results between all numerical methods available to us. A full understanding of these discrepancies will require further investigation.

(a) Hybrid FEM-FVM (b) SAM

Fig. 4: Comparison for $n = 100$, at $t = T_0/8$

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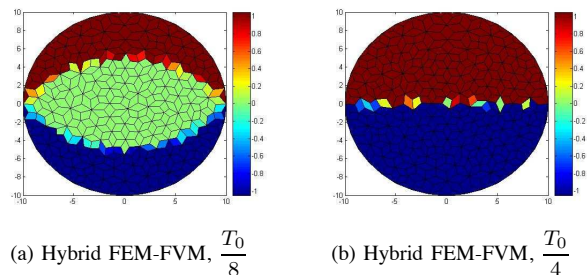
(a) Hybrid FEM-FVM (b) SAM

Fig. 5: Comparison for $n = 100$, at $t = T_0/4$

In order to further explore this aspect, we performed a simulation with $n = 200$. The result for the HM is presented in Fig. 6 at times $t = T_0/8$ and $t = T_0/4$. The SAM did not converge