



**HAL**  
open science

# Homogeneous strict polynomial functors as unstable modules

Nguyen The Cuong

► **To cite this version:**

Nguyen The Cuong. Homogeneous strict polynomial functors as unstable modules. 2014. hal-01235915

**HAL Id: hal-01235915**

**<https://hal.science/hal-01235915>**

Preprint submitted on 1 Dec 2015

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Homogeneous strict polynomial functors as unstable modules

NGUYỄN Thế Cường<sup>\*†</sup>

21st July 2014

## Abstract

A relation between Schur algebras and the Steenrod algebra is shown in [Hai10] where to each strict polynomial functor the author naturally associates an unstable module. We show that the restriction of Hai's functor to a sub-category of strict polynomial functors of a given degree is fully faithful.

## 1 Introduction

The search for a relation between Schur algebras and the Steenrod algebra has been a source of common interest between representation theorists and algebraic topologists for over thirty years. Functorial points of view, on unstable modules [HLS93], and on modules over Schur algebras [FS97], have given an efficient setting for studying such relation. For the Steenrod algebra side, [HLS93] uses Lannes' theory to construct a functor  $f$ , from the category  $\mathcal{U}$  of unstable modules, to the category  $\mathcal{F}$  of functors from finite dimensional  $\mathbb{F}_p$ -vector spaces to  $\mathbb{F}_p$ -vector spaces. This functor  $f$  induces an equivalence between the quotient category  $\mathcal{U}/\text{Nil}$  of  $\mathcal{U}$  by the Serre class of nilpotent modules, and the full sub-category  $\mathcal{F}_\omega$  of analytic functors. The interpretation of modules over Schur algebras given by [FS97] uses an algebraic version of the category of functors, the category  $\mathcal{P}$  of strict polynomial functors. The category  $\mathcal{P}$  decomposes as a direct sum  $\bigoplus_{d \geq 0} \mathcal{P}_d$  of its sub-categories of homogeneous functors of degree  $d$ . The category  $\mathcal{P}_d$  is equivalent to the category of modules over the Schur algebra  $S(n, d)$  for  $n \geq d$  [FS97, Theorem 3.2]. The presentation of  $\mathcal{P}_d$  as a category of functors with an extra structure comes with a functor  $\mathcal{P}_d \rightarrow \mathcal{F}$ . Nguyen D. H. Hai showed [Hai10] that this functor  $\mathcal{P}_d \rightarrow \mathcal{F}$  factors through the category  $\mathcal{U}$  by a functor  $\bar{m}_d : \mathcal{P}_d \rightarrow \mathcal{U}$ . These functors  $\bar{m}_d$  induce a functor  $\bar{m} : \mathcal{P} \rightarrow \mathcal{U}$ . The functor  $\bar{m}$  has remarkably interesting properties. In particular, it is exact and it commutes with tensor products and with the Frobenius twist. The relevance of this last property to computation will soon be apparent.

We observe that Hom-groups between unstable modules coming from strict polynomial functors via Hai's functor are computable: they are isomorphic to the Hom-groups of the corresponding strict polynomial functors in many interesting cases. The primary goal of this paper is to generalize these results to the whole category  $\mathcal{P}_d$ . The main theorem of the present work goes as follows:

**Theorem 1.1.** *The functor  $\bar{m}_d : \mathcal{P}_d \rightarrow \mathcal{U}$  is fully faithful.*

The theorem is proved by comparing corresponding Hom-groups in the two categories. We discuss an example of interest. Let  $n$  be a non-negative integer and  $V$  be an  $\mathbb{F}_2$ -vector space. The symmetric group

---

*Key words and phrases.* Steenrod algebra, strict polynomial functors, unstable modules.

\* I would like to thank Professor Lionel SCHWARTZ for urging me to prepare this paper. It is also a great pleasure for me to express my sincere thanks to all members of the VIASM for their hospitality during my two-month visit in Hanoi where the paper was baptised. I would like to take this opportunity to thank NGUYEN Dang Ho Hai and PHAM Van Tuan for having offered many significative discussions. This work would not be presented properly without precious suggestions from Vincent Franjou.

†Partially supported by the program ARCUS Vietnam MAE, Région IDF.

$\mathfrak{S}_n$  acts on  $V^{\otimes n}$  by permutations. Denote by  $\Gamma^n(V)$  the group of invariants  $(V^{\otimes n})^{\mathfrak{S}_n}$  and by  $S^n$  the group of co-invariants  $(V^{\otimes n})_{\mathfrak{S}_n}$ . Fix  $p = 2$ , let  $\mathcal{A}_2$  be the Steenrod algebra. The free unstable module generated by an element  $u$  of degree 1 is denoted by  $F(1)$ . It has an  $\mathbb{F}_2$ -basis consisting of  $u^{2^k}$  with  $k \geq 0$ .

Denote by  $\text{Sq}_0$  the operation, which associates to a homogeneous element  $x \in M^n$  of  $M \in \mathcal{U}$ , the element  $\text{Sq}_0^n x$ . An unstable module  $M$  is nilpotent if for every  $x \in M^n$  there exist an integer  $N_x$  such that  $\text{Sq}_0^{N_x} x = 0$ . An unstable module  $M$  is reduced if  $\text{Hom}_{\mathcal{U}}(N, M)$  is trivial for every nilpotent module  $N$ . It is called *Nil*-closed if  $\text{Ext}_{\mathcal{U}}^i(N, M)$ ,  $i = 0, 1$ , are trivial for every nilpotent module  $N$ .

Let  $G$  be  $\Gamma^2 \otimes \Gamma^1$  or  $S^3$ . To  $G$ , Hai's functor  $\bar{m}$  associates the unstable module  $G(F(1))$ . We now show that

$$\text{Hom}_{\mathcal{P}_3}(\Gamma^2 \otimes \Gamma^1, S^3) \cong \text{Hom}_{\mathcal{U}}(\Gamma^2(F(1)) \otimes \Gamma^1(F(1)), S^3(F(1))).$$

The readers of [HLS93] might expect the latter Hom-group to be isomorphic to  $\text{Hom}_{\mathcal{F}}(\Gamma^2 \otimes \Gamma^1, S^3)$ . However  $S^3(F(1))$  is not *Nil*-closed<sup>1</sup> then such an expectation fails. By classical functor techniques [FFSS99, Theorem 1.7], if  $\mathcal{C}$  is  $\mathcal{P}$  or  $\mathcal{F}$  then:

$$\text{Hom}_{\mathcal{C}}(\Gamma^2 \otimes \Gamma^1, S^3) \cong \bigoplus_{i=0}^3 \text{Hom}_{\mathcal{C}}(\Gamma^2, S^i) \otimes \text{Hom}_{\mathcal{C}}(\Gamma^1, S^{3-i}).$$

It follows that:

$$\begin{aligned} \text{Hom}_{\mathcal{F}}(\Gamma^2 \otimes \Gamma^1, S^3) &\cong \mathbb{F}_2^{\oplus 2}, \\ \text{Hom}_{\mathcal{P}_3}(\Gamma^2 \otimes \Gamma^1, S^3) &\cong \mathbb{F}_2. \end{aligned}$$

The module  $S^3(F(1))$  is not *Nil*-closed but it is reduced. On the other hand, the quotient of the module  $\Gamma^2(F(1)) \otimes \Gamma^1(F(1))$  by its sub-module generated by  $u \otimes u \otimes u^4$  is nilpotent. Therefore:

$$\text{Hom}_{\mathcal{U}}(\Gamma^2(F(1)) \otimes \Gamma^1(F(1)), S^3(F(1))) \cong \text{Hom}_{\mathcal{U}}(\mathcal{A}_2 \langle u \otimes u \otimes u^4 \rangle, S^3(F(1))) \cong \mathbb{F}_2.$$

This example is the key to the proof of Theorem 1.1.

We end the introduction by giving some further remarks on the result and stating the organization of the paper.

Theorem 1.1 implies that the category  $\mathcal{P}_d$  is a full sub-category of the category  $\mathcal{U}$ . Unfortunately the category  $\mathcal{P}$  itself cannot be embedded into  $\mathcal{U}$ . Fix  $p = 2$ , let  $F(2)$  be the free unstable module satisfying  $\text{Hom}_{\mathcal{U}}(F(2), M) \cong M^2$ , then there is no non-trivial morphism in the category  $\mathcal{P}$  from  $\Gamma^2$  to  $\Gamma^1$  but:

$$\text{Hom}_{\mathcal{U}}(\bar{m}(\Gamma^2), \bar{m}(\Gamma^1)) \cong \text{Hom}_{\mathcal{U}}(F(2), F(1)) \cong \mathbb{F}_2.$$

The category  $\mathcal{P}_d$  is not a thick sub-category of  $\mathcal{U}$ . Fix  $p = 2$ , let  $I^{(1)}$  denote the Frobenius twist in  $\mathcal{P}_2$ , that is the base change along the Frobenius. It is proved [FS97] that

$$\text{Ext}_{\mathcal{P}_2}^i(I^{(1)}, I^{(1)}) \cong \begin{cases} \mathbb{F}_2 & \text{if } i = 0, 2, \\ 0 & \text{otherwise.} \end{cases}$$

The corresponding Ext-group in the category  $\mathcal{U}$  is  $\text{Ext}_{\mathcal{U}}^i(\Phi F(1), \Phi F(1))$ . As in [Cuo14]:

$$\text{Ext}_{\mathcal{U}}^i(\Phi F(1), \Phi F(1)) \cong \begin{cases} \mathbb{F}_2 & \text{if } i = 2^n - 2, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,  $\text{Ext}_{\mathcal{P}_2}^i(I^{(1)}, I^{(1)})$  is not isomorphic to  $\text{Ext}_{\mathcal{U}}^i(\bar{m}_d(I^{(1)}), \bar{m}_d(I^{(1)}))$  for  $i = 2^n - 2$ ,  $n \geq 3$ .

<sup>1</sup> The sub-module of  $S^3(F(1))$  generated by  $u.u.u^4$  is concentrated in even degrees but this element does not has a square root.

## Organization of the article

In Section 2 we recall basic facts on the even Steenrod algebra and unstable modules following the presentation in [Hai10]. When  $p > 2$ , this so-called Steenrod algebra is slightly different from that of [Ste62] since we do not consider the Bockstein operation. We also recall Milnor's co-action on unstable modules and how it is used to determine the Steenrod action on certain type of elements.

The next section recalls strict polynomial functors. Main properties of Hai's functor are introduced and an easy observation on the existence of its adjoint functors is also given.

The structure of  $\Gamma^\lambda(F(1)) := \Gamma^{n_1}(F(1)) \otimes \Gamma^{n_2}(F(1)) \otimes \cdots \otimes \Gamma^{n_k}(F(1))$  is treated in Section 4. We show that there exists a monogeneous sub-module of  $\Gamma^\lambda(F(1))$  such that the quotient of  $\Gamma^\lambda(F(1))$  by this sub-module is nilpotent.

The last section deals with Theorem 1.1. The proof of this theorem is based on a combinatorial process followed by some Steenrod algebra techniques.

## 2 The even Steenrod algebra and unstable modules

In this section, we follow the simple presentation in [Hai10, Section 3] to define the *even* Steenrod algebra and unstable modules.

The letter  $p$  denotes a prime number. Let  $[-]$  be the integral part of a number. We denote by  $\mathcal{A}$  the quotient of the free associative unital graded  $\mathbb{F}_p$ -algebra generated by the  $\mathcal{P}^k, k \geq 0$ , of degree  $k(p-1)$  subject to the Adem relations

$$\mathcal{P}^i \mathcal{P}^j = \sum_{t=0}^{\lfloor \frac{i}{p} \rfloor} \binom{(p-1)(j-t)-1}{i-pt} \mathcal{P}^{i+j-t} \mathcal{P}^t$$

for every  $i \leq pj$  and  $\mathcal{P}^0 = 1$  [Hai10, Section 3].

An  $\mathcal{A}$ -module  $M$  is called unstable if for every homogeneous element  $x \in M^n$ ,  $\mathcal{P}^k x$  is trivial as soon as  $k$  is strictly greater than  $n$ . We denote by  $\mathcal{U}$  the category of unstable modules.

Let  $\mathcal{A}_p$  be the Steenrod algebra [Ste62, Sch94]. If  $p = 2$  then there is an isomorphism of algebras  $\mathcal{A} \rightarrow \mathcal{A}_2$ , obtained by identifying the  $\mathcal{P}^k$  with the Steenrod squares  $Sq^k$ . The category  $\mathcal{U}$  is equivalent to the category  $\mathcal{U}$  of unstable modules in [Sch94]. If  $p > 2$ ,  $\mathcal{A}$  is isomorphic, up to a grading scale, to the sub-algebra of  $\mathcal{A}_p$  generated by the reduced Steenrod powers  $P^k, k \geq 0$ , of degree  $2k(p-1)$ . The category  $\mathcal{U}$  is equivalent to the sub-category  $\mathcal{U}'$  of unstable  $\mathcal{A}_p$ -modules concentrating in even degrees [Sch94, Section 1.6].

We call  $\mathcal{A}$  the even Steenrod algebra and  $\mathcal{P}^k$  the  $k$ -th reduced Steenrod power.

Serre and Cartan [Ser53, Car55] introduced the notions of *admissible* and *excess*. A monomial

$$\mathcal{P}^{i_1} \mathcal{P}^{i_2} \dots \mathcal{P}^{i_k}$$

is called admissible if  $i_j \geq pi_{j+1}$  for every  $k-1 \geq j \geq 1$  and  $i_k \geq 1$ . The excess of this operation, denoted by  $e(\mathcal{P}^{i_1} \mathcal{P}^{i_2} \dots \mathcal{P}^{i_k})$ , is defined by

$$e(\mathcal{P}^{i_1} \mathcal{P}^{i_2} \dots \mathcal{P}^{i_k}) = pi_1 - (p-1) \left( \sum_{j=1}^k i_j \right).$$

The set of admissible monomials and  $\mathcal{P}^0$  is an additive basis of  $\mathcal{A}$ .

Let  $|x|$  be the degree of a homogeneous element. Denote by  $P_0$  the operation, which associates to a homogeneous element  $x \in M^n$  of  $M \in \mathcal{U}$ , the element  $\mathcal{P}^{|x|}x$ . An unstable module  $M$  is nilpotent if for every  $x \in M^n$  there exist an integer  $N_x$  such that  $P_0^{N_x}x = 0$ . Denote by *Nil* the class of all nilpotent modules.

An unstable module  $M$  is reduced if  $\text{Hom}_{\mathcal{A}}(N, M)$  is trivial for every nilpotent module  $N$ . It is called *Nil*-closed if  $\text{Ext}_{\mathcal{A}}^i(N, M)$ ,  $i = 0, 1$ , are trivial for every nilpotent module  $N$ .

Let  $n$  be a non-negative integer. We denote by  $F(n)$  the free unstable module generated by a generator  $\iota_n$  of degree  $n$ . These  $F(n)$  are projective satisfying  $\text{Hom}_{\mathcal{A}}(F(n), M) \cong M^n$ . When  $n = 1$  such a generator is denoted by  $u$  rather than  $\iota_1$ . As an  $\mathbb{F}_p$ -vector space,  $F(1)$  is generated by  $u^{p^i}$ ,  $i \geq 0$ . The action of the reduced Steenrod power  $\mathcal{P}^k$ ,  $k \geq 0$  is defined by:

$$\mathcal{P}^k(u^{p^i}) = \begin{cases} u^{p^i} & \text{if } k = 0, \\ u^{p^{i+1}} & \text{if } k = p^i, \\ 0 & \text{otherwise.} \end{cases}$$

In our setting, there is an isomorphism of unstable modules  $F(n) \cong (F(1)^{\otimes n})^{\mathfrak{S}_n}$  where the symmetric group  $\mathfrak{S}_n$  acts by permutations. Then we can identify  $F(n)$  with the sub-module of  $F(1)^{\otimes n}$  generated by  $u^{\otimes n}$  [LZ86, Sch94, Section 1.6].

Milnor [Mil58] established that  $\mathcal{A}$  has a natural co-product which makes it into a Hopf algebra and incorporates Thom's involution as the conjugation. The dual  $\mathcal{A}^*$  of  $\mathcal{A}$  is isomorphic to the polynomial algebra

$$\mathbb{F}_p[\xi_0, \xi_1, \dots, \xi_k, \dots], \quad |\xi_i| = p^i - 1, \quad \xi_0 = 1.$$

Let  $R = (r_1, r_2, \dots, r_k, \dots)$  be a sequence of non-negative integers with only finitely many non-trivial ones. Denote by  $\xi^R$  the product  $\xi_1^{r_1} \xi_2^{r_2} \dots \xi_k^{r_k} \dots$ . These monomials form a basis for  $\mathcal{A}^*$ .

**Definition 2.1** (Milnor's operations). *Let  $\mathbf{M}_r^n \in \mathcal{A}$  denote the dual of  $\xi_r^n$  with respect to the monomial basis  $\{\xi^R\}$  of  $\mathcal{A}^*$ .*

If  $M$  is an unstable module, the completed tensor product  $M \hat{\otimes} \mathcal{A}^*$  is the  $\mathbb{F}_p$ -graded vector space defined by:

$$\left( M \hat{\otimes} \mathcal{A}^* \right)^n = \prod_{l-k=n} M^l \otimes (\mathcal{A}^*)^k.$$

We recall how to use Milnor's co-action to determine the Steenrod action. There is Milnor's co-action  $\lambda : M \rightarrow M \hat{\otimes} \mathcal{A}^*$  for an unstable module  $M$ . We write  $\lambda(x)$  as a formal sum  $\sum_R x_R \otimes \xi^R$ . Let  $\theta$  be a Steenrod operation then

$$\theta x = \sum_R \xi^R(\theta) x_R.$$

Milnor's co-action on a tensor product is determined as follows:

$$\lambda(x \otimes y) = \sum_R \sum_{I+J=R} (x_I \otimes y_J) \otimes \xi^R.$$

Milnor's co-action on  $F(1)$  is defined by:

$$\begin{aligned} \lambda(u) &= \sum_{i \geq 0} u^{p^i} \otimes \xi_i, \\ \lambda(u^{p^s}) &= \sum_{i \geq 0} u^{p^{s+i}} \otimes \xi_i^{p^s}. \end{aligned}$$

The following observation on the action of Milnor's operations is easy and is left to the reader.

**Lemma 2.2.** *Let  $n$  be a non-negative integer then  $\mathbf{M}_r^n(u^{\otimes n}) = (u^{p^r})^{\otimes n}$ . Let  $l_1, \dots, l_q$  be a sequence of non-negative integers. If  $k_1, \dots, k_m$  is a sequence of non-negative integers such that  $p^{k_j} > n$  for every  $m \geq j \geq 1$  then:*

$$\mathbf{M}_r^n \left( \left( \bigotimes_{i=1}^q u^{p^{l_i}} \right) \otimes \left( \bigotimes_{i=1}^m u^{p^{k_i}} \right) \right) = \left( \mathbf{M}_r^n \left( \bigotimes_{i=1}^q u^{p^{l_i}} \right) \right) \otimes \left( \bigotimes_{i=1}^m u^{p^{k_i}} \right).$$

The following proposition is a strengthening of Lemma 2.2:

**Proposition 2.3.** *Let  $M$  and  $N$  be two connected unstable modules. For all homogeneous element  $x \in M$  of degree  $n$  then  $\mathbf{M}_r^n(x) = \mathbf{P}_0^r(x)$ . For all homogeneous elements  $y \in M$  and  $z \in N$  and all non-negative integer  $k$  such that  $p^k > n$ , then:*

$$\mathbf{M}_r^n(y \otimes \mathbf{P}_0^k(z)) = \mathbf{M}_r^n(y) \otimes \mathbf{P}_0^k(z).$$

*Proof.* Consider the morphism  $\varphi : F(n) \rightarrow M$ , defined by  $\varphi(u^{\otimes n}) = x$ . Lemma 2.2 yields:

$$\begin{aligned} \mathbf{M}_r^n(x) &= \mathbf{M}_r^n(\varphi(u^{\otimes n})) \\ &= \varphi(\mathbf{M}_r^n(u^{\otimes n})) \\ &= \varphi(\mathbf{P}_0^r(u^{\otimes n})) \\ &= \mathbf{P}_0^r(\varphi(u^{\otimes n})) \\ &= \mathbf{P}_0^r(x). \end{aligned}$$

Let  $\psi : F(|y|) \rightarrow M$  be the morphism defined by  $\psi(u^{\otimes |y|}) = y$ . Similarly, by considering the morphism  $\psi_1 : \Phi^k(F(|z|)) \rightarrow N$ , defined by

$$\psi(\mathcal{P}^{p^k} \iota_{|z|}) = \mathcal{P}^{p^k} z,$$

together with the product  $\psi \otimes \psi_1$ , we obtain the second equality.  $\square$

### 3 Strict polynomial functors and Hai's functor

The main goal of this section is to recall Hai's functor and give an easy observation on the existence of its adjoint functors.

Following the simple presentation introduced in [Pir03], we first recall the category of strict polynomial functors. Fix a prime number  $p$ , denote by  $\mathcal{V}$  the category of  $\mathbb{F}_p$ -vector spaces and by  $\mathcal{V}^f$  its full subcategory of spaces of finite dimension. Let  $n$  be a non-negative integer. Denote by  $\Gamma^n(V)$  the group of invariants  $(V^{\otimes n})^{\mathfrak{S}_n}$ . The category  $\Gamma^d \mathcal{V}^f$  is defined by:

$$\begin{aligned} \text{Ob}(\Gamma^d \mathcal{V}^f) &= \text{Ob}(\mathcal{V}^f), \\ \text{Hom}_{\Gamma^d \mathcal{V}^f}(V, W) &= \Gamma^d(\text{Hom}_{\mathcal{V}^f}(V, W)). \end{aligned}$$

A homogeneous strict polynomial functor of degree  $d$  is an  $\mathbb{F}_p$ -linear functor from  $\Gamma^d \mathcal{V}^f$  to  $\mathcal{V}^f$ . We denote by  $\mathcal{P}_d$  the category of all these functors. The notation  $\mathcal{P}$  stands for the direct sum  $\bigoplus_{d \geq 0} \mathcal{P}_d$ . A strict polynomial functor is an object of the category  $\mathcal{P}$ .

We now recall the parameterized version of  $\Gamma^d$  and  $S^d$ . For each  $W \in \mathcal{V}^f$ , let  $\Gamma^{d,W}$  be the functor which associates to an  $\mathbb{F}_p$ -vector space  $V$  the  $\mathbb{F}_p$ -vector space  $\Gamma^d(\text{Hom}_{\mathcal{V}^f}(W, V))$ , and let  $S^{d,W}$  be the functor which associates to an  $\mathbb{F}_p$ -vector space  $V$  the  $\mathbb{F}_p$ -vector space  $S^d(W^\sharp \otimes V)$ . Here,  $W^\sharp$  stands for the linear dual of  $W$ . The  $\Gamma^{d,W}$  are projective satisfying  $\text{Hom}_{\mathcal{P}_d}(\Gamma^{d,W}, F) \cong F(W)$  and the  $S^{d,W}$  are injective satisfying  $\text{Hom}_{\mathcal{P}_d}(F, S^{d,W}) \cong F(W)^\sharp$ .

Let  $\mathcal{F}$  denote the category of functors from finite dimensional  $\mathbb{F}_p$ -vector spaces to  $\mathbb{F}_p$ -vector spaces. Hai shows [Hai10] that the forgetful functor  $\mathcal{O} : \mathcal{P}_d \rightarrow \mathcal{F}$  factors through  $\mathcal{U}$  via a certain functor  $\bar{m}_d : \mathcal{P}_d \rightarrow \mathcal{U}$ . Let  $\bar{m}$  denote the induced functor from  $\mathcal{P}$  to  $\mathcal{U}$ . The functor  $\bar{m}$  has nice properties. It is exact and it commutes with tensor products and with Frobenius twists [Hai10, see Sections 3 and 4]. Moreover:

**Proposition 3.1.** *We have  $\bar{m}_d(\Gamma^d) = F(d)$ .*

The following observation is easy and is left to the reader:

**Proposition 3.2** ([Hai10]). *The functor  $\bar{m}_d$  admits a left adjoint and a right adjoint.*

## 4 The key lemma

As explained in the introduction, we prove Theorem 1.1 by comparing corresponding Hom-groups in the two categories  $\mathcal{P}_d$  and  $\mathcal{U}$ . This computation can be reduced to a smaller class of strict polynomial functors. This class is described in the following proposition. Before formulating this proposition, we fix:

**Notation 4.1.** Let  $\lambda = (\lambda_1, \dots, \lambda_k)$  be a sequence of non-negative integers. We denote:

$$\Gamma^\lambda := \Gamma^{\lambda_1} \otimes \dots \otimes \Gamma^{\lambda_k}.$$

**Proposition 4.2** ([FS97]). *If  $\dim_{\mathbb{F}_p} W \geq d$  then  $S^{d,W}$  is an injective generator of  $\mathcal{P}_d$ . The functors  $\Gamma^\lambda$ , where  $\lambda$  runs through the set of all sequences of non-negative integers whose sum is  $d$ , form a system of projective generators of  $\mathcal{P}_d$ .*

By abuse of notation, we denote by  $|\lambda|$  the sum of a sequence of integers. Theorem 1.1 is equivalent to the following lemma.

**Lemma 4.3.** *There are isomorphisms:*

$$\mathrm{Hom}_{\mathcal{U}}(\Gamma^\lambda(F(1)), S^{d,V}(F(1))) \cong S^{\lambda_1}(V^\sharp) \otimes \dots \otimes S^{\lambda_k}(V^\sharp)$$

for every  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ ,  $|\lambda| = d$ .

In this section we show that there exists a monogeneous sub-module of  $\Gamma^\lambda(F(1))$  such that the quotient  $\Gamma^\lambda(F(1))$  by this module is nilpotent. The following lemma is a consequence of [Lan92, Lemma 2.2.5.3], [FS90, Lemma 1.2.6].

**Lemma 4.4.** *Let  $M$  be a connected monogeneous unstable module. If  $n, q$  are non-negative integers such that  $p^q > n$  then  $F(n) \otimes \Phi^q M$  is monogeneous.*

By a simple induction, we obtain the following lemma:

**Lemma 4.5.** *Let  $\lambda$  be a sequence  $(\lambda_1, \lambda_2, \dots, \lambda_k)$  of non-negative integers. If the numbers  $q_1, q_2, \dots, q_{k-1}$  satisfy  $p^{q_i} > \lambda_i$  for all  $1 \leq i \leq k-1$ , then*

$$F(\lambda_1) \otimes \Phi^{q_1} F(\lambda_2) \otimes \Phi^{q_1+q_2} F(\lambda_3) \otimes \dots \otimes \Phi^{q_1+q_2+\dots+q_{k-1}} F(\lambda_k)$$

is monogeneous.

Denote by  $\mathcal{Q}$  the sequence  $(0, q_1, q_1 + q_2, \dots, q_1 + \dots + q_{k-1})$  and by  $\omega_{\lambda, \mathcal{Q}}$  the element

$$\iota_{\lambda_1} \otimes \mathbf{P}_0^{q_1} \iota_{\lambda_2} \otimes \mathbf{P}_0^{q_1+q_2} \iota_{\lambda_3} \otimes \dots \otimes \mathbf{P}_0^{q_1+q_2+\dots+q_{k-1}} \iota_{\lambda_k}.$$

Because  $M/\Phi^n M$ ,  $n \geq 1$ , is nilpotent for every unstable module  $M$ , an elementary induction yields:

**Corollary 4.6.** *Let  $\lambda$  be a sequence  $(\lambda_1, \lambda_2, \dots, \lambda_k)$  of non-negative integers. If the numbers  $q_1, q_2, \dots, q_{k-1}$  satisfy  $p^{q_i} > \lambda_i$  for all  $1 \leq i \leq k-1$ , then the quotient  $\Gamma^\lambda(F(1))/\mathcal{A}(\omega_{\lambda, \mathcal{Q}})$  is nilpotent.*

Because  $S^{d,V}(F(1))$  is reduced, hence  $\mathrm{Hom}_{\mathcal{U}}(\Gamma^\lambda(F(1))/\mathcal{A}(\omega_{\lambda, \mathcal{Q}}), S^{d,V}(F(1)))$  is trivial and then:

**Proposition 4.7.** *The inclusion  $\mathcal{A}(\omega_{\lambda, \mathcal{Q}}) \hookrightarrow \Gamma^\lambda(F(1))$  induces an injection:*

$$\mathrm{Hom}_{\mathcal{U}}(\Gamma^\lambda(F(1)), S^{d,V}(F(1))) \hookrightarrow \mathrm{Hom}_{\mathcal{U}}(\mathcal{A}(\omega_{\lambda, \mathcal{Q}}), S^{d,V}(F(1))).$$

We are thus led to the problem of determining the subspace of  $S^{d,V}(F(1))$  consisting of all possible images of  $\omega_{\lambda,\alpha}$ . Lemma 4.8 presents the desired determination. Before formulating this lemma, let us present an example of interest. There is an isomorphism of  $\mathbb{F}_p$ -vector spaces:

$$\mathrm{Hom}_{\mathcal{U}}(F(d), S^{d,V}(F(1))) \cong (S^{d,V}(F(1)))^d \cong S^d(V^\sharp).$$

Therefore if  $\varphi$  is a morphism from  $F(d)$  to  $S^{d,V}(F(1))$  then the image  $\varphi(\iota_d)$  is a sum of elements of the type  $\prod_{i=1}^d s_i \otimes u$  where  $s_i \in V^\sharp$ . The natural transformation  $\bigotimes_{j=1}^k S^{\lambda_j,V} \rightarrow S^{d,V}, \sum_{j=1}^k \lambda_j = d$ , induces a morphism

$$\rho : \bigotimes_{j=1}^k \mathrm{Hom}_{\mathcal{U}}(F(\lambda_j), S^{\lambda_j,V}(F(1))) \rightarrow \mathrm{Hom}_{\mathcal{U}}(F(\lambda), S^{d,V}(F(1))).$$

For  $1 \leq i \leq k$ , let  $f_i$  be a morphism from  $F(\lambda_i)$  to  $S^{\lambda_i,V}(F(1))$ . Then the image of  $\omega_{\lambda,\alpha}$  under  $\rho\left(\bigotimes_{i=1}^k f_i\right)$  is a sum of elements of the type

$$\prod_{j=1}^k \left( \prod_{i=1}^{\lambda_j} s_{i,j} \otimes u^{p^{q_1+q_2+\dots+q_{j-1}}} \right)$$

with  $s_{i,j} \in V^\sharp$ . We show that every morphism in  $\mathrm{Hom}_{\mathcal{U}}(F(\lambda), S^{d,V}(F(1)))$  is of this simple form.

**Lemma 4.8.** *Let  $\lambda = (\lambda_1, \dots, \lambda_k)$  be a sequence of non-negative integers whose sum is  $d$ . Let  $m$  be a number such that  $p^m > d^2$ . Denote by  $\alpha$  the sequence  $(0, m, 2m, \dots, (k-1)m)$  and by  $\omega_{\lambda,\alpha}$  the element  $\omega_{\lambda,\alpha}$ . If  $\varphi$  is a morphism from  $F(\lambda)$  to  $S^{d,V}(F(1))$  then  $\varphi(\omega_{\lambda,\alpha})$  is a sum of elements of the type*

$$\prod_{t=0}^{k-1} \left( \prod_{i=1}^{\lambda_{t+1}} v_{i,t} \otimes u^{p^{tm}} \right)$$

where  $v_{i,t} \in V^\sharp$ .

**Remark 4.9.** *Lemma 4.3 is a corollary of Lemma 4.8. Indeed since*

$$\bigotimes_{j=1}^k \mathrm{Hom}_{\mathcal{U}}(F(\lambda_j), S^{\lambda_j}(V^\sharp \otimes F(1))) \cong \bigotimes_{j=1}^k S^{\lambda_j}(V^\sharp),$$

it suffices to check that the morphism  $\rho$  is a bijection. The morphism  $\rho$  is clearly injective. Following Lemma 4.8, the morphism  $\rho$  is surjective as well and hence it is bijective. Lemma 4.3 have been proved and we are left with Lemma 4.8.

## 5 Proof of the key lemma

As discussed in the previous section, we are left with Lemma 4.8. The proof of this lemma now goes as follows.

*Proof of Lemma 4.8.* The image of  $\omega_{\lambda,\alpha}$  is a sum of elements of the type  $\prod_{i=1}^d v_i \otimes u^{p^{l_i}}$  in  $S^d(V^\sharp \otimes F(1))$ . Therefore, the following equality holds:

$$\sum_{i=0}^{k-1} p^{im} \lambda_{i+1} = \sum_{j=1}^d p^{l_j}. \quad (5.1)$$

We now show that in  $\prod_{i=1}^d v_i \otimes u^{p^{l_i}}$ , the element  $u^{p^{j_m}}, 0 \leq j \leq k-1$ , appears  $\lambda_{j+1}$  times. Without loss of generality, we suppose that  $l_1 \leq l_2 \leq \dots \leq l_d$ . We prove by induction on  $1 \leq i \leq k$  that there exist  $j_1, j_2, \dots, j_k$  such that for  $0 \leq i \leq k-1$ ,

$$p^{l_{j_{i+1}}} \leq \lambda_{i+1} p^{im} < p^{l_1+j_{i+1}},$$



$$\begin{aligned}
im &\leq l_t, 1 + j_i \leq t \leq j_{i+1}, \\
\lambda_{i+1} &= \sum_{s=1+j_i}^{j_{i+1}} p^{l_s - im}.
\end{aligned} \tag{5.2}$$

Let  $j_1$  be the index such that  $p^{l_{j_1}} \leq \lambda_1 < p^{l_{j_1+1}}$ . Denote by  $\mathcal{S}$  the set  $\{l_1, l_2, \dots, l_{j_1}\}$  and by  $p^{\mathcal{S}}$  the sum  $\sum_{i \in \mathcal{S}} p^i$ . We show that

$$p^{\mathcal{S}} = \lambda_1.$$

Since both sides of (5.1) and  $\lambda_1$  are congruent modulo  $p^{l_{j_1+1}}$ , it is enough to prove that  $p^{\mathcal{S}} \leq \lambda_1$ . Suppose that this inequality does not hold. Then

$$\lambda_1 < p^{\mathcal{S}} \leq \lambda_1 j_1 \leq d^2 < p^m.$$

Denote by  $r$  the sum  $\sum_{i=1}^d p^{l_i}$ . It follows from Proposition 2.3 that:

$$\mathbf{M}_r^{p^{\mathcal{S}}}(\omega_\alpha) = \mathbf{M}_r^{p^{\mathcal{S}}}(u^{\otimes \lambda_1}) \otimes \bigotimes_{i=1}^{k-1} (u^{p^{im}})^{\otimes \lambda_{i+1}} = 0.$$

On the other hand, we prove that the action of  $\mathbf{M}_r^{p^{\mathcal{S}}}$  on the image of  $\omega_\alpha$  is not trivial. In fact, the element

$$x := \mathbf{M}_r^{p^{\mathcal{S}}}\left(\prod_{i=1}^d v_i \otimes u^{p^{l_i}}\right)$$

is a sum of elements of the type

$$\prod_{i=1}^d v_i \otimes u^{p^{l_i + r\epsilon_i}}$$

where  $(\epsilon_1, \epsilon_2, \dots, \epsilon_d)$  is a sequence of 0 and 1 such that

$$\sum_{i=1}^d \epsilon_i p^{l_i} = \sum_{i=1}^{j_1} p^{l_i}.$$

Among these elements,

$$\left(\prod_{i=1}^{j_1} v_i \otimes u^{p^{l_i + r}}\right) \left(\prod_{i=1+j_1}^d v_i \otimes u^{p^{l_i}}\right)$$

is unique hence  $x$  is non-trivial. Suppose that there exist  $k_1 \leq k_2 \leq \dots \leq k_d$  such that

$$\begin{aligned}
\sum_{i=0}^{k-1} p^{im} \lambda_{i+1} &= \sum_{j=1}^d p^{k_j}, \\
\prod_{i=1}^d v_i \otimes u^{p^{l_i + r\epsilon_i}} &= \prod_{i=1}^d w_i \otimes u^{p^{k_i + r\tau_i}},
\end{aligned}$$

where  $(\tau_1, \tau_2, \dots, \tau_d)$  is a sequence of 0 and 1 such that

$$\sum_{i=1}^d \tau_i p^{k_i} = \sum_{i=1}^{j_1} p^{l_i}.$$

Since

$$r > \max\{l_i, k_i, 1 \leq i \leq d\},$$

then

$$\{v_i \otimes u^{p^{l_i}}, 1 \leq i \leq d\} = \{w_j \otimes u^{p^{k_j}}, 1 \leq j \leq d\}.$$

Therefore the action of  $\mathbf{M}_r^p$  on the image of  $\omega_\alpha$  is non-trivial in this case. This contradiction implies:

$$\sum_{i=1}^{j_1} p^{l_i} = \lambda_1$$

Suppose that there are  $j_1, j_2, \dots, j_t$  such that for  $0 \leq i \leq t-1$ ,

$$\begin{aligned} p^{l_{j_{i+1}}} &\leq \lambda_{i+1} p^{im} < p^{l_1 + j_{i+1}}, \\ im &\leq l_n, 1 + j_i \leq n \leq j_{i+1}, \\ \lambda_{i+1} &= \sum_{s=1+j_i}^{j_{i+1}} p^{l_s - im}. \end{aligned}$$

It follows from Proposition 2.3 that

$$\begin{aligned} \mathbf{M}_{km}^{\lambda_1 + p^m \lambda_2 + \dots + p^{(t-1)m} \lambda_t} (\omega_{(0, m, 2m, \dots, (k-1)m)}) &= \omega_{(km, (k+1)m, \dots, (k+t-1)m, tm, (t+1)m, \dots, (k-1)m)} \\ &= P_0^{tm} \omega_{((k-t)m, \dots, (k-1)m, 0, m, \dots, (k-t-1)m)}, \\ \mathbf{M}_{km}^{\lambda_1 + p^m \lambda_2 + \dots + p^{(t-1)m} \lambda_t} \left( \prod_{i=1}^d v_i \otimes u^{p^{l_i}} \right) &= P_0^{km} \left( \prod_{i=1}^{j_t} v_i \otimes u^{p^{l_i}} \right) \cdot \prod_{i=1+j_t}^d v_i \otimes u^{p^{l_i}}. \end{aligned} \quad (5.3)$$

Because every morphism in  $\text{Hom}_{\mathcal{A}}(F(\lambda), S^{d,V}(F(1)))$  is  $\mathcal{A}$ -linear, it follows from (5.3) that  $l_i \geq tm$  for all  $i \geq 1 + j_t$ . As  $S^{d,V}(F(1))$  is reduced, the image of  $\omega_{((k-t)m, \dots, (k-1)m, 0, m, \dots, (k-t-1)m)}$  is a sum of

$$P_0^{(k-t)m} \left( \prod_{i=1}^{j_t} v_i \otimes u^{p^{l_i}} \right) \cdot \prod_{i=1+j_t}^d v_i \otimes u^{p^{l_i - tm}}$$

with other elements. By the same manner for the case of  $\lambda_1$ , there exist  $j_{t+1}$  such that

$$\begin{aligned} p^{l_{j_{t+1}}} &\leq \lambda_{t+1} p^{tm} < p^{l_1 + j_{t+1}}, \\ \lambda_{t+1} &= \sum_{s=1+j_t}^{j_{t+1}} p^{l_s - tm}. \end{aligned}$$

The induction is then achieved. The lemma is now deduced from the equalities  $\sum_{i=1}^k \lambda_i = d$  and (5.2).  $\square$

## References

- [Car55] Henri Cartan. Sur l'itération des opérations de Steenrod. *Comment. Math. Helv.*, 29:40–58, 1955.
- [Cuo14] Nguyen The Cuong. *Some homological algebra computations in the category of unstable modules*. Theses, Université Paris 13, July 2014.
- [FFSS99] Vincent Franjou, Eric M. Friedlander, Alexander Scorichenko, and Andrei Suslin. General linear and functor cohomology over finite fields. *Ann. of Math. (2)*, 150(2):663–728, 1999.
- [FS90] V. Franjou and L. Schwartz. Reduced unstable  $A$ -modules and the modular representation theory of the symmetric groups. *Ann. Sci. École Norm. Sup. (4)*, 23(4):593–624, 1990.
- [FS97] Eric M. Friedlander and Andrei Suslin. Cohomology of finite group schemes over a field. *Invent. Math.*, 127(2):209–270, 1997.
- [Hai10] Nguyen Dang Ho Hai. Foncteurs polynomiaux stricts et modules instables sur l'algèbre de Steenrod. *J. Algebra*, 324(4):860–874, 2010.

- [HLS93] Hans-Werner Henn, Jean Lannes, and Lionel Schwartz. The categories of unstable modules and unstable algebras over the Steenrod algebra modulo nilpotent objects. *Amer. J. Math.*, 115(5):1053–1106, 1993.
- [Lan92] Jean Lannes. Sur les espaces fonctionnels dont la source est le classifiant d'un  $p$ -groupe abélien élémentaire. *Inst. Hautes Études Sci. Publ. Math.*, (75):135–244, 1992. With an appendix by Michel Zisman.
- [LZ86] Jean Lannes and Saïd Zarati. Sur les  $\mathcal{U}$ -injectifs. *Ann. Sci. École Norm. Sup. (4)*, 19(2):303–333, 1986.
- [Mil58] John Milnor. The Steenrod algebra and its dual. *Ann. of Math. (2)*, 67:150–171, 1958.
- [Pir03] Teimuraz Pirashvili. Introduction to functor homology. In *Rational representations, the Steenrod algebra and functor homology*, volume 16 of *Panor. Synthèses*, pages 1–26. Soc. Math. France, Paris, 2003.
- [Sch94] Lionel Schwartz. *Unstable modules over the Steenrod algebra and Sullivan's fixed point set conjecture*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1994.
- [Ser53] Jean-Pierre Serre. Cohomologie modulo 2 des complexes d'Eilenberg-MacLane. *Comment. Math. Helv.*, 27:198–232, 1953.
- [Ste62] N. E. Steenrod. *Cohomology operations*. Lectures by N. E. Steenrod written and revised by D. B. A. Epstein. Annals of Mathematics Studies, No. 50. Princeton University Press, Princeton, N.J., 1962.

DÉPARTAMENT DE MATHÉMATIQUES  
LAGA - UNIVERSITÉ PARIS 13  
99 AVENUE JEAN-BAPTISTE CLÉMENT  
93430 VILLETANEUSE

*E-mail address*, T. C. NGUYEN: [tdntcuong@gmail.com](mailto:tdntcuong@gmail.com) or [nguyentc@math.univ-paris13.fr](mailto:nguyentc@math.univ-paris13.fr)

LIAFV - CNRS  
FORMATH VIETNAM