Filtering problem for general modeling of the drift and application to portfolio optimization problems.

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Abstract

We study the filtering problem and the maximization problem of expected utility from terminal wealth in a partial information context. The special features is that the only information available to the investor is the vector of sock prices. The mean rate of return processes are not directly observed and supposed to be driven by a process μ_t modeled by a stochastic differential equations. The main result in this paper is to show under which assumptions on the coefficients of the model, we can estimate the unobserved market price of risks. Using the innovation approach, we show that under globally Lipschitz conditions on the coefficients of μ_t, the filters estimate of the risks satisfy a measure-valued Kushner-Stratonovich equations. On the other hand, using the pathwise density approach, we show that under a nondegenerate assumption and some regularity assumptions on the coefficients of μ_t, the density of the conditional distribution of μ_t given the observation data, can be expressed in terms of the solution to a linear parabolic partial differential equation parameterized by the observation path. Also, we can obtain an explicit formulae for the optimal wealth, the optimal portfolio and the value function for the cases of logarithmic and power utility function.

Keywords 0.1. *Partial information, filtering problem, Kushner-stratonovich equation, pathwise density approach, martingale duality method, utility maximization.*

1 Introduction

In financial market models, we do not have in general a complete knowledge of all parameters, which may be driven by unobserved random factors. This situation of partial information framework appears when investors only observe the vector of stock prices and cannot disentangle the drift term from the other sources of uncertainty. Investors observe changes in returns but cannot perfectly distinguish their dynamics.

Portfolio optimization problems under partial information are becoming more and more popular, also because of their practical interest. These problems have been studied widely via the filtering theory and using both portfolio optimization methodologies, namely the dynamic programming approach and the martingale approach. Models with incomplete information have been investigated by Dothan and Feldman [9] and Lakner [16], [17] have solved the partial optimization out the special case of the linear Gaussian filtering problem via respectively the dynamic programming methods and the martingale approach. Also, Karatzas and Xue [13] consider optimal consumption in an incomplete market setting the optimal terminal wealth is

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derived and the optimal strategy is determined for the linear Gaussian dynamics of the drift. Karatzas and Zhao [14] solve the problem if the drift is a fixed random variable with a known distribution (Bayesian case). Also, portfolio optimization problems under partial information for stochastic volatility models have been studied by Pham and Quenez [22], Ibrahim and Abergel [11].

In this paper, we are interested by studying the filtering problem and portfolio optimization problem in the context of partial information for general dynamics of the drift. We consider a market model where the mean rate of return processes are unobservable and follow a stochastic differential equations. Our main result focus on the following question: under which assumptions on the coefficients of the model, the drift of the stock and the coefficients of the drift dynamics, we can solve our filtering problem and therefore our portfolio optimization problem. The filtering problem consists in estimating the unobserved quantities (here they are the market price of risks) based on the nonlinear filtering theory. Using the change of measure techniques, the partial observation context can be transformed into a full information context such that coefficients depend only on past history of observed prices. We study two cases for the market price of risk, the boundeness case and the linear growth case. Therefore, for each case, we can make some assumptions on the coefficients of the drift dynamics in order to deduce the filter estimate using one of the following two approaches: the innovation approach, where we show that the filters estimate can be deduced as the solution to a measure-valued stochastic differential equations, or the density approach where we show that the density q_t of the unnormalized conditional distribution of the drift given the observation, is a solution to a linear stochastic differential equation. Therefore by Kallianpur-Striebel formula, we can deduce the equation satisfied by the density p_t of the conditional distribution, which is a nonlinear stochastic differential equation. Then, it is more easily to study the existence and uniqueness for q_t, and therefore deduce that for p_t. For that, we need to use the extended variational method to stochastic differential equation of Ito type, see Krylov-Rosovskii [15] and Pardoux [19]. With this variational method, we are limited to the case where the market price of risk is bounded. But we can make a suitable pathwise transformation, with which we prove that the density q_t respectively p_t can be expressed in terms of the solution to a linear partial differential equation parameterized by the observation process. Therefore classical partial differential equation methods can be applied in analyzing existence and uniqueness for q_t respectively p_t, for both cases of the market price of risk: the boundeness and the linear growth case.

After replacing the original partial information context by a full information one and computing the filters estimate, it is then possible to use the classical theory for stochastic control problem. Here we will be interested by the martingale approach to solve our utility optimization problem. We study the logarithmic and power utility functions, where we show that, due to the pathwise approach, the optimal wealth and the optimal portfolio can be computed explicitly in terms of the solution to a linear parabolic partial differential equation. In fact, in the literature, an explicit formulae for the optimal portfolio policies, in partial information context , have been obtained for usual cases for the modeling of the unobservable mean rate of return as: Gaussian, finite state Markov chain or Bayesian. But in our case, we have obtained these formulae for general modeling of the mean rate of return.

The structure of the present paper is as follows: In section 2, we describe the model and formulate the optimization problem. In section 3, we use results from filtering theory to reduce the partial information model to a model where the coefficients are adapted to the observation process. We present in section 4 two approaches in order to study our filtering problem. We study two cases for the market price of risk: the case where it is bounded and the case where it has linear growth. In both cases, we make the following assumptions on the coefficients of the drift dynamics in order to deduce the filters estimate. Finally in section 5, we use the martingale duality approach for the utility maximization problem. We obtain explicit formulae for the optimal portfolio, optimal wealth and value function of the utility maximization problem for the cases of logarithmic and power utility function.

2 Formulation of the problem

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space equipped with a filtration $\mathbb{F} = \{\mathcal{F}_t, 0 \leq t \leq T\}$ satisfying the usual conditions, where $T > 0$ is a fixed time horizon. The financial market consists of one risk-free-asset, whose price process is assumed for simplicity equal to 1, and n stocks of positive process $S = (S^1, ..., S^n)$ governed by:

$$dS_t = diag(S_t)\left(f(\mu_t)dt + \sigma dW_t\right), \tag{2.1}$$

$$d\mu_t = \zeta(\mu_t)dt + \vartheta(\mu_t)dW_t^1. \tag{2.2}$$

where W is a n-dimensional Brownian motion. Here $diag(S)$ denotes the diagonal $n \times n$ matrix with components $S^i, i = 1, ...n$. The mean rate of return $f(\mu_t)$, valued in $\mathbb{R}^n$, is not observable and is driven by some process μ_t, valued in $\mathbb{R}^n$, which is modeled by a stochastic differential equation. f is a measurable function from $\mathbb{R}^n$ into $\mathbb{R}^n$ and the matrix volatility σ, valued in $\mathbb{R}^{n\times n}$, is assumed to be a constant. Also W^1 is an n-dimensional Brownian motion, independent of W, ζ is a measurable function from $\mathbb{R}^n$ into $\mathbb{R}^n$ and ϑ is a measurable function from $\mathbb{R}^n$ into $\mathbb{R}^{n\times n}$.

We denote by $\lambda_t = \lambda(\mu_t) := \sigma^{-1}f(\mu_t)$, the unobservable market price of risk which is $\mathbb{F}$-adapted process.
Also we denote by $\mathbb{F}^S = \{F_t^S, 0 \leq t \leq T\}$ the filtration generated by the price process S.

2.1 The optimization problem

Consider an agent who invests at any time $t \in [0, T]$ a proportion π_t^i of his wealth in the $i-th$-risky asset $S^i, i = 1, ...n$. With $\pi_t = (\pi_t^1, ..., \pi_t^n)^*$ chosen, the proportion of wealth invested in the bond is $1 - \pi_t^* e_n$. Where e_n is the vector of one in $\mathbb{R}^n$.

Where the sign $*$ denotes the transposition operator.

We assume that the trading strategy is self-financing, then the wealth process corresponding to a portfolio π is defined by $R_0^\pi = x$ and evolves according to:

$$dR_t^\pi = R_t^\pi\left[\pi_t^* f(\mu_t)dt + \pi_t^* \sigma dW_t\right] \tag{2.3}$$

Given an utility function $U : \mathbb{R}^+ \to \mathbb{R}$, increasing, concave, the objective of the investor is to maximize the expected utility from terminal wealth. The value function of the agent is then:

$$J(x) = \sup_{\pi\in\mathcal{A}} \mathbb{E}[U(R_T^\pi)], \qquad x > 0. \tag{2.4}$$

Where $\mathcal{A}$ denotes the set of the admissible controls $(\pi_t)_{0\leq t\leq T}$ which are $\mathbb{F}^S$-adapted, and satisfies the integrability condition:

$$\int_0^T ||\pi_s||^2 ds < \infty.$$

We are in a context when an investor wants to maximize the expected utility from terminal wealth, where the only information available to the investor is the one generated by the asset prices, therefore leading to a utility maximization problem in partially observed incomplete model. In order to solve it, we aim to reduce it to a maximization problem with full information. For that, it becomes important to exploit all the information coming from the market itself in order to continuously update the knowledge of the not fully known quantities and this is where stochastic filtering problem becomes useful.

3 Reduction to a full observation model

We introduce the positive martingale defined by $L_0 = 1$ and $dL_t = -L_t\lambda_t^* dW_t$. It is explicitly given by

$$L_t = \exp\left(-\int_0^t \lambda_s^* dW_s - \frac{1}{2}\int_0^t ||\lambda_s||^2 ds\right). \tag{3.1}$$

We assume that L is a martingale, so that it defines a probability measure $\tilde{\mathbb{P}}$ equivalent to $\mathbb{P}$ on $(\Omega, \mathbb{F})$ characterized by:

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}|\mathcal{F}_t = L_t, \qquad 0 \leq t \leq T. \tag{3.2}$$

Then Girsanov's transformation ensures that

$$\tilde{W}_t = W_t + \int_0^t \lambda_s ds \quad , \qquad 0 \leq t \leq T,$$

is a $(\tilde{\mathbb{P}}, \mathbb{F})$-Brownian motion under $\tilde{\mathbb{P}}$, and the dynamics of S under $\tilde{\mathbb{P}}$ is

$$dS_t = diag(S_t)\sigma d\tilde{W}_t.$$

Remark 3.1. *An important property of the probability of reference $\tilde{\mathbb{P}}$ is that the filtration $\mathbb{F}^S$ is the augmented natural filtration of $\tilde{W}$.*

Now we assume that for all t, $\mathbb{E}|\lambda_t| < \infty$. Then under this assumption, we can introduce the conditional law, i.e, the filter estimate of the risk λ_t defined as follows:

$$\overline{\lambda}_t = \overline{\lambda}(\mu_t) := \mathbb{E}[\lambda(\mu_t)|\mathcal{F}_t^S].$$

Let us introduce $\overline{f}_t = \overline{f}(\mu_t) := \mathbb{E}[f(\mu_t)|\mathcal{F}_t^S]$. Since σ is $\mathbb{F}^S$-adapted, then we have that $\overline{f}_t = \sigma\overline{\lambda}_t$.

Remark 3.2. *Notice that the notation $\overline{f}(\mu_t)$ does not mean that the filter estimate $\overline{f}_t$ is a function of μ_t but it is just a notation to say that $\overline{f}_t$ is the conditional expectation of $f(\mu)$ given the observation data $\mathbb{F}^S$. So, in the sequel, when we say $\overline{\phi}(\mu_t)$ it is just to say that the filter estimate $\overline{\phi}_t$ is the conditional expectation of $\phi(\mu_t)$ given the observation data $\mathbb{F}^S$.*

Let us now consider the process

$$\overline{W}_t = \tilde{W}_t - \int_0^t \overline{\lambda}_s ds = W_t + \int_0^t \sigma^{-1}\left[f(\mu_s) - \overline{f}(\mu_s)\right] ds,$$

This is the so-called innovation process and by classical results in filtering theory (see for example proposition 2.30 in [1] and proposition 2.2.7 in [21]), we have that $\overline{W}$ is a $(\mathbb{P}, \mathbb{F}^S)$ Brownian motion.

Then, by means of the innovation processes, we can describe the dynamics of S within a framework of full observation model as follows:

$$dS_t = diag(S_t) \left(\overline{f}(\mu_t)dt + \sigma d\overline{W}_t \right). \tag{3.3}$$

We have showed that conditioning arguments can be used to replace the initial partial information problem by a full information problem one which depends only on the past history of observed prices. But the reduction procedure involves the filter estimate $\overline{f}(\mu_t)$, which will be the main interest of our article.

4 Filtering problem

Our filtering problem can be summarized as follows: the signal process is μ_t (the unobservable process) and we have from remark 3.1 that $\tilde{W}$ corresponds to the observation process, that is, we have the following signal-observation system:

$$d\mu_t = \zeta(\mu_t)dt + \vartheta(\mu_t)dW_t^1, \tag{4.1}$$

$$d\tilde{W}_t = \lambda(\mu_t)dt + dW_t. \tag{4.2}$$

where W^1 is independent of W.

Our aim is to characterize the filter estimate $\overline{f}(\mu_t)$; which is defined as the conditional expectation of $f(\mu_t)$ given the observation data $\mathbb{F}^S = \mathbb{F}^{\tilde{W}}$. For that, we need to use the nonlinear filtering theory. There are essentially two different approaches: the first is based on the important idea of innovation processes, where we show that the conditional distribution of the unobservable process having the observation, is the solution of a evolution stochastic equation often called the Kushner-Stratonovich equation. The second approach is focused on the existence of the conditional density. We will show for each approach, which conditions we need to impose on the coefficients f, ζ and ϑ in order to compute the filter estimate $\overline{f}(\mu_t)$.

Remark 4.1. *In what follows, since σ is a constant matrix, notice that when we say that the market price risk λ is bounded (resp.has linear growth) it's exactly equivalent to say that f is bounded (resp.has linear growth). That is, when we make an assumption on f it's exactly the same assumption on the market price risk λ.*

4.1 Innovation processes approach

We assume that both $\zeta = (\zeta_i)_{1 \leq i \leq n} : \mathbb{R}^n \to \mathbb{R}^n$ and $\vartheta = (\vartheta_{ij})_{1 \leq i,j \leq n} : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ are globally Lipschitz, that is, there exists a positive constant k such that:

$$||\zeta(x) - \zeta(y)|| \leq k||x - y|| \quad \text{and} \quad ||\vartheta(x) - \vartheta(y)|| \leq k||x - y||. \tag{4.3}$$

On the other hand, concerning the observation process (4.2), we will distinguish two cases for λ: the case where λ is bounded and the case where λ has linear growth.

It is well known that the innovation processes approach is based on the change of measure $\tilde{\mathbb{P}}$ given in (3.2):

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} | \mathcal{F}_t = L_t := \exp\left[-\int_0^t \lambda^*(\mu_s)dW_s - \frac{1}{2}|^2 \int_0^t ||\lambda(\mu_s)||^2 ds \right], \tag{4.4}$$

where we shall make the usual standing assumption on the filtering theory.

Assumption 1. *The process L is a martingale; that is, $\mathbb{E}[L_T] = 1$*

Let us now denote by Λ the $(\tilde{\mathbb{P}}, \mathbb{F})$-martingale given by $\Lambda_t = \dfrac{1}{L_t}$. Therefore the computation of the conditional distribution of μ_t given $\mathbb{F}^S$ can be obtained from the following Kallianpur-Striebel formula (4.5): for every $\phi \in \mathbb{B}(\mathbb{R}^n)$, we have the following representation:

$$\overline{\phi}_t = \overline{\phi}(\mu_t) := \mathbb{E}\left[\phi(\mu_t)|\mathcal{F}_t^S\right] = \frac{\tilde{\mathbb{E}}\left[\phi(\mu_t)\Lambda_t|\mathcal{F}_t^S\right]}{\tilde{\mathbb{E}}\left[\Lambda_t|\mathcal{F}_t^S\right]} := \frac{\psi_t(\phi)}{\psi_t(1)}, \tag{4.5}$$

with $\psi_t(\phi) := \tilde{\mathbb{E}}[\phi(\mu_t)\Lambda_t|\mathcal{F}_t^S]$ is the unnormalized conditional distributionof μ_t given the observation, $\psi_t(1)$ can be viewed as the normalising factor and $\mathbb{B}(\mathbb{R}^n)$ is the space of bounded measurable functions $\mathbb{R}^n \to \mathbb{R}$.

Remark 4.2. *Notice that the filter estimates $\overline{f}_t$ can be computed as a vector of n-filter estimates, that is, $\overline{f}_t = (\overline{f_t^i})_{1\le i\le n}$, where $f^i \in \mathbb{B}(\mathbb{R}^n)$.*

We shall also make the following assumption on the unnormalized conditional distribution:

Assumption 2. *For all $t \geq 0$,*

$$\tilde{\mathbb{P}}\left[\int_0^t [\psi_s(||\lambda||)]^2 ds < \infty\right] = 1 \tag{4.6}$$

.

Let us now introduce the following second order differential operator $\mathcal{A}$ as follows:

$$\mathcal{A}\phi = \frac{1}{2}\sum_{i,j=1}^n K_{ij}\partial^2_{x_i x_j}\phi + \sum_{i=1}^n \zeta_i \partial_{x_i}\phi, \qquad \text{for } \phi \in \mathbb{B}(\mathbb{R}^n). \tag{4.7}$$

and its adjoint $\mathcal{A}^*$ is given by:

$$\mathcal{A}^*\phi = \frac{1}{2}\sum_{i,j=1}^n \partial^2_{x_i x_j}(K_{ij}\phi) - \sum_{i=1}^n \partial_{x_i}(\zeta_i\phi), \qquad \text{for } \phi \in \mathbb{B}(\mathbb{R}). \tag{4.8}$$

with $K = \vartheta\vartheta^*$.

Now we present the following two results due to Bain and Crisan [1, Chap.3] and Pardoux [21, Chap.2]. Then, we will show which conditions can be imposed on the coefficients f, ζ and ϑ such that these two results can be applied. We show that depending on the condition taken on f (f is bounded or has linear growth), we can impose the necessary conditions on the coefficients ζ and ϑ in order to compute the filter estimate $\overline{f}(\mu_t)$. We show that this filter is the solution of a measure-valued stochastic differential equation.

Proposition 4.3. *If assumptions (1) and (2) are satisfied then the unnormalized conditional distribution ψ_t satisfies the following Zakai equation:*

$$d\psi_t(\phi) = \psi_t(\mathcal{A}\phi)dt + \sum_{j=1}^n \psi_t\left(\phi\lambda^j\right) d\tilde{W}_t^j, \tag{4.9}$$

for any $\phi \in B(\mathbb{R}^n)$.

We need now to impose the following fundamental condition in order to derive the Kushner-Stratonovich equation:

$$\mathbb{P}\left[\int_0^t ||\overline{\lambda}_s||^2 ds < \infty\right] = 1, \qquad \text{for all } t \geq 0. \tag{4.10}$$

Therefore from the above Zakai equation for $\psi_t(\phi)$ and $\psi_t(1)$ and from the Kallianpur-Striebel formula (4.5), we have the following evolution equation for the conditional distribution $\overline{\phi}$:

Corollary 4.4. *If assumptions (1) and (2) are satisfied then the conditional distribution $\overline{\phi}(\mu_t)$ satisfies the following Kushner-Stratonovich equation:*

$$d\overline{\phi}_t = \overline{(\mathcal{A}\phi)}_t dt + \sum_{j=1}^n \left(\overline{(\phi\lambda^j)}_t - \overline{\lambda_t^j}\ \overline{\phi}_t\right) d\overline{W}_t^j, \tag{4.11}$$

for any $\phi \in B(\mathbb{R}^n)$.

Remark 4.5. *The above Zakai and Kushner-Stratonovich equations hold true for any Borel measurable ϕ, not necessarily bounded. In fact, we can proceed by cutting of ϕ at a fixed level which we let tend to infinity. For this, let us introduce the functions $(\psi^k)_{k>0}$ defined as*

$$\psi^k(x) = \psi(x/k), \qquad x \text{ in } \mathbb{R}^n,$$

where

$$\psi(x) = \begin{cases} 1 & \text{if } ||x|| \leq 1 \\ \exp\left(\frac{||x||^2-1}{||x||^2-4}\right) & \text{if } 1 < ||x|| < 2 \\ 2 & \text{if } ||x|| \geq 2. \end{cases}$$

Let us introduce the following relations given in[1, P.151]:

$$\lim_{k\to\infty} \phi\psi^k(x) = \phi(x), \qquad |\phi(x)\psi^k(x)| \leq |\phi(x)|,$$
$$\lim_{k\to\infty} \mathcal{A}_s(\phi\psi^k)(x) = A_s\phi(x).$$

Then by replacing in equation (4.11) ϕ by $\phi\psi^k$ and from dominated convergence theorem, we may pass to the limit as $k \to \infty$ and then we deduce that $\overline{\phi}_t$ satisfies equation (4.11).

Now, we will be interested by studying both cases for f: the case where f is bounded and the case where it has linear growth. In each case, we make the necessary assumptions on the coefficients ζ and ϑ in order to compute the filter estimate $\overline{f}_t$.

4.1.1 The function f is bounded

In this case the risk λ is bounded, then we can deduce easily the evolution equation (4.12) satisfied by the filter estimate $\overline{f}$.

Proposition 4.6. *If condition (4.3) is satisfied and f is bounded, then the filter estimate $\overline{f}_t = (\overline{f_t^i})_{1\leq i\leq n}$ satisfy the following measure-valued Kushner-Stratonovich equation:*

$$d\overline{f_t^i} = \overline{(\mathcal{A}f^i)}_t dt + \sum_{j=1}^n \left(\sum_{k=1}^n \sigma_{jk}^{-1}\overline{(f^i f^k)}_t - \sum_{k=1}^n \sigma_{jk}^{-1}\overline{f_t^i}\ \overline{f_t^k}\right) d\overline{W}_t^j, \quad \textit{for } i = 1, ..., n. \tag{4.12}$$

Proof. Since f is bounded, hence λ also. Then assumptions 1 and 2 are satisfied. Therefore from corollary 4.4, we can deduce the above evolution equation for $\overline{f}_t$. ☐

4.1.2 The function f has linear growth

As the innovation processes approach is based on the change of measure, so the first important assumption was that L is martingale. Normally Novikov's condition is quite difficult to verify directly, so we need to use an alternative conditions under which the process L is a martingale.

From lemma 3.9 in Bain and Crisan [1], we have:

Lemma 4.7. *L is a martingale if the following conditions are satisfied:*

$$\mathbb{E}\left[\int_0^t ||\lambda_s||^2 ds\right] < \infty, \qquad \mathbb{E}\left[\int_0^t L_s||\lambda_s||^2 ds\right] < \infty \qquad \forall t > 0. \tag{4.13}$$

Now, we need to impose the following assumptions on the moments of μ_0:

- $A1)$ μ_0 has finite second moment.
- $A2)$ μ_0 has finite third moment.

Lemma 4.8. *Assume that condition (4.3) is satisfied. If λ has linear growth and $A1)$ is satisfied, then (4.13) is satisfied. Moreover, if $A2)$ is satisfied, then (4.6) is satisfied.*

Proof. The proof of the first part is given by lemma 4.1.1 in Bensoussan [3]. To prove (4.6), we need to use lemma 4.1.5 and part (b) in the proof of theorem 4.1.1 in Bensoussan [3]. □

Notice that condition (4.10) is a consequence of the first stronger condition (4.13) imposed on λ. Therefore, we can deduce the filter estimate $\overline{f}_t = (\overline{f_t^i})_{1\leq i\leq n}$, in the case where f has linear growth, as follows:

Proposition 4.9. *Assume that condition (4.3) is satisfied. If f has linear growth and assumptions $A1)$ and $A2)$ are satisfied, then the filter estimate $\overline{f}_t$ satisfy the following measure-valued Kushner-Stratonovich equation:*

$$d\overline{f_t^i} = \overline{(\mathcal{A}f^i)}_t dt + \sum_{j=1}^n \left(\sum_{k=1}^n \sigma_{jk}^{-1}\overline{(f^i f^k)}_t - \sum_{k=1}^n \sigma_{jk}^{-1}\overline{f_t^i}\ \overline{f_t^k}\right) d\overline{W}_t^j, \qquad \textit{for } i = 1, ..., n. \tag{4.14}$$

Proof. Since f has linear growth, hence λ also. Moreover, as assumptions $A1)$ and $A2)$ are satisfied, then from lemma 4.8 and lemma 1, we can deduce that assumptions 1 and 2 are satisfied. Therefore remark 4.5 and corollary 4.4 end the proof. □

4.1.3 Existence and uniqueness of the solution $\overline{f}_t$

Let us define the space of measure-valued stochastic processes within which we prove uniqueness of the solution to equations (4.12) and (4.14).

Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be the function $\psi(x) = 1 + ||x||$, for any $x \in \mathbb{R}^n$ and define $C^l(\mathbb{R}^n)$ to be the space of continuous functions ϕ such that $\phi/\psi \in C_b(\mathbb{R}^n)$(the space of bounded continuous functions).
Let us denote by $\mathbb{M}^l(\mathbb{R}^n)$ the space of finite measure $\mathcal{M}$ such that $\mathcal{M}(\psi) < \infty$. In particular, this implies that $\mathcal{M}(\phi) < \infty$ for all $\phi \in C^l(\mathbb{R}^n)$. Moreover, we endow $\mathbb{M}^l(\mathbb{R}^n)$ wit the corresponding weak topology: A sequence $(\mathcal{M}_k)$ of measures in $\mathbb{M}^l(\mathbb{R}^n)$ converges to $\mathcal{M} \in \mathbb{M}^l(\mathbb{R}^n)$ if and only if $\lim_{k\to\infty} \mathcal{M}_k(\phi) = \mathcal{M}(\phi)$, for all $\phi \in C^l(\mathbb{R}^n)$.

Definition 4.10. • *The Class $\mathbb{U}$ is the space of all Y_t-adapted $\mathbb{M}^l(\mathbb{R}^n)$-valued stochastic process $(\mathcal{M}_t)_{t\geqslant 0}$ with càdlàg paths such that, for all $t \geqslant 0$, we have*

$$\tilde{\mathbb{E}}\left[\int_0^t (\mathcal{M}_s(\psi))^2 ds\right] < \infty.$$

- *The Class $\tilde{\mathbb{U}}$ is the space of all Y_t-adapted $\mathbb{M}^l(\mathbb{R})$-valued stochastic process $(\mu)_{t\geqslant 0}$ with càdlàg paths such that the process $m_t\mathcal{M}$ belongs to the class $\mathbb{U}$, where the process $(m_t)_{t\geq 0}$ is defined as:*

$$m_t = \exp\left(\int_0^t \mathcal{M}_s(\lambda^*)\tilde{W}_s - \frac{1}{2}\int_0^t \mathcal{M}_s(\lambda^*)\mathcal{M}_s(\lambda)ds\right).$$

Now we state the uniqueness result of the solution to equations (4.12) and equation(4.14), see theorem 4.19 in Bain and Crisan [1, chap.4]

Proposition 4.11. *Suppose in addition to the above assumptions on ζ, ϑ and f imposed in proposition 4.6 (resp.proposition 4.9), that these coefficients have twice continuously differentiable components and all their derivatives of first and second order are bounded. Then equation (4.12)(resp. equation(4.14)) has a unique solution in the class $\tilde{\mathbb{U}}$.*

Remark 4.12. *The filter equations (4.12) and (4.14) describe an infinite dimensional stochastic differential equations driven by the innovations process. For that, we need to use some numerical methods adapting to infinite dimensional filtering problem. For example, the so-called particular Monte Carlo method which is based on a particle approximation of the conditional distribution. It has recently given raise to extensive studies, see for instance [5],[6]. Also we can use the Wiener chaos decomposition method of the Zakai equation developed and studied by Lototsky et al [18].*

4.2 Density approach

This approach is based on the following:
We assume that the law of μ_t given $\mathcal{F}_t^S$ admits a density $p_t(x)$ relative to some dominating measure $\eta(dx)$.

$$\mathbb{E}[\phi(\mu_t)|\mathcal{F}_t^S] = \int_{\mathbb{R}^n} \phi(x)p_t(x)\eta(dx), \tag{4.15}$$

Formally integrating by parts in the Kushner-Stratonovich equation (4.11) and using Fubini theorem, we can obtain as in Pardoux [21] and Bain and Crisan[1] that p_t satisfies the following stochastic partial differentiel equation (SPDE):

$$dp_t(x) = A^*p_t(x)dt + p_t(x)\sum_{j=1}^n(\lambda^j(x) - \overline{\lambda_t^j})d\overline{W}_t^j. \tag{4.16}$$

where the operator $\mathcal{A}^*$ is given by (4.8) and $\overline{\lambda_t^j} = \displaystyle\int_{\mathbb{R}^n} \lambda^j(x)p_t(x)\eta(dx)$ with $\lambda^j = \sum_{k=1}^n \sigma_{jk}^{-1}f^k$.

The equation (4.16) is a nonlinear stochastic partial differential equation. It is not easy to make mathematical sense of this equation, for example, how should the equation even be interpreted, and do such equations have solutions. So if we wish to work with the density approach, it usually makes more sense to work with the density q_t of the unnormalized conditional distribution of μ_t, which is formally the solution of a linear stochastic partial differential equation as follows:

$$dq_t(x) = A^*q_t(x)dt + q_t(x)\sum_{j=1}^{n}\lambda^j(x)d\tilde{W}_t^j, \tag{4.17}$$

where we assume that $\psi_t(\phi) := \tilde{\mathbb{E}}[\phi(\mu_t)\Lambda_t|\mathcal{F}_t^S] = \int_{\mathbb{R}^n}\phi(x)q_t(x)dx$.

We will be interested by the following questions: under which conditions on the coefficients f, ζ and ϑ, we can show that:

- the conditional distribution of μ_t given $\mathcal{F}_t^S$ has a density with respect to a reference measure, in particular with respect to Lebesgue measure.
- the linear stochastic partial differential equation (4.17) has a solution and this solution is the density of the unnormalized conditional distribution of μ_t given $\mathcal{F}_t^S$.

4.2.1 Study the existence of a regular density p_t and the solution of (4.16).

As in the innovation approach we will distinguish two cases for f: the case where f is bounded and the case where f has linear growth. Then depending on the condition taken on f, we can impose the necessary conditions on the coefficients ζ and ϑ in order to show that the conditional distribution of μ_t given the observation has a density with respect to Lebesgue measure and its density is the unique solution of the stochastic PDE (4.16).

Notations 1. *Let us introduce the Hilbert spaces*

$$\mathbb{H} = L^2(\mathbb{R}^n), \qquad \mathbb{K} = \mathbb{H}^1 := \left\{h \in L^2(\mathbb{R}^n)|\frac{\partial h}{\partial x_i} \in L^2(\mathbb{R}^n)\right\}$$

equipped with the inner products

$$(h_1, h_2) = \int_{\mathbb{R}^n} h_1(x)h_2(x)dx, \tag{4.18}$$

$$\tag{4.19}$$

We notice that $\mathbb{H}^1$ is dense in $\mathbb{H}$ with continuous injection. The topological dual space of $\mathbb{H}^1$ will be denoted by $(\mathbb{H}^1)'$. We denote by $< .,. >$ the duality between $\mathbb{H}^1$ and $(\mathbb{H}^1)'$.

Let now $\mathbb{X}$ and $\mathbb{Y}$ be two Hilbert spaces equipped with the norm $||.||_{\mathbb{X}}$ and $||.||_{\mathbb{Y}}$ respectively. We denote by $\mathcal{L}(\mathbb{X}, \mathbb{Y})$ the Banach space of continuous linear operators from $\mathbb{X}$ to $\mathbb{Y}$.

We denote by $\mathbb{M}^0([0,T];\mathbb{X})$ the set of all $\mathbb{X}$-valued measurable processes, and by $\mathbb{M}^2([0,T];\mathbb{X})$ (a subset of $L^2([0,T]\times\Omega;\mathbb{X})$) the space of $\mathbb{X}$-valued process Y such that $Y \in \mathbb{M}^0([0,T];\mathbb{X})$ and $\mathbb{E}\left[\int_0^T ||Y||_{\mathbb{X}}^2 ds\right] < \infty$.

In the following, we need to make the following assumption on the initial condition μ_0 of the signal process μ_t:

Assumption 3. *The initial condition μ_0 is a random variable with density $p_0 \in \mathbb{H}$.*

Here we need to make the following assumptions on the coefficients ϑ, ζ and f :

Assumption 4.
- *A.i) $\vartheta : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ is measurable and bounded.*
- *A.i) $\exists \alpha > 0$, such that $K(x) = \vartheta\vartheta^*(x) \geq \alpha I, \quad \forall x \in \mathcal{R}^n$.*

- *A.iii)* $\partial_{x_i}\vartheta_{ij} \in L^\infty(\mathbb{R}^n), \quad i,j = 1, ...n.$
- *A.iv)* ζ *and* f *are measurable and bounded functions from* $\mathbb{R}^n \to \mathbb{R}^n$.

Firstly, the following lemma about the existence of a density with respect to Lebesgue measure can be found in Bain and Crisan [1, P.173]:

Lemma 4.13. *If assumption 3 is satisfied and f is bounded, then almost surely the conditional distribution $\overline{\phi}(\mu_t)$ has a density with respect to Lebesque measure and this density is square integrable.*

Now we study the existence of a solution q_t to the stochastic PDE(4.17). Then, we show that q_t is the unnormalized conditional density of μ_t given the observation $\mathcal{F}_t^S$. By consequence, using Kallianpur-Striebel formula (4.5), we can deduce the conditional density p_t.

Using the extended variational method to stochastic equations of Ito type, see Krylov-Rosovskii [15] and Pardoux [19], we can deduce the following:

Proposition 4.14. *We suppose that assumptions 3 and 4 hold. Then equation (4.17) has a unique solution q_t belongs to $L^2(\Omega \times [0,T]; \mathbb{H}^1) \cap L^2(\Omega; C([0,T]; \mathbb{H}))$. Moreover, $q_t(x)$ is the density of the unnormalized conditional distribution of μ_t given the observation $\mathcal{F}_t^S$.*

Proof. All the notations used in this proof can be found in the above paragraph notations 1.

Firstly, in order to use the results of Krylov-Rosovskii [15] and Pardoux [19], we need to rewritte the operator $\mathcal{A}^*$ in its divergence form as a linear operator from an Hilbert space to its dual and to redefine the stochastic integral as an Hilbert space operator. In fact, due to assumption 4, the operator $\mathcal{A}^*$ can be rewritten as follows:

$$A^*\phi(x) = \frac{1}{2}\sum_{i,j=1}^n \partial_{x_i}(K_{ij}(x)\partial_{x_j}\phi(x)) - \sum_{i=1}^n \partial_{x_i}\left[\zeta_i(x)\phi(x) - \frac{1}{2}\sum_{j=1}^n \partial_{x_j}K_{ij}(x)\phi(x)\right], \qquad \text{for } x \in \mathbb{R}^n.$$

Secondly, we consider the operator $\mathcal{B} \in \mathcal{L}(\mathbb{H}^1, (L^2(\mathbb{R}^n))^n)$ as follows:

$$\mathcal{B}_j\phi(x) = \lambda^j\phi(x) = \sum_{k=1}^n \sigma_{jk}^{-1} f^k(x)\phi(x), \qquad j = 1, ...n.$$

Consequently, equation (4.17) can be rewritten as follows:

$$dq_t(x) = \mathcal{A}^* q_t(x)dt + \mathcal{B}q_t.dY_t, \qquad q_0 = p_0. \tag{4.20}$$

where . denotes the scalar product.

When we write $< \mathcal{A}^*\phi, \psi >$, then A^* is understood to be an operator from $\mathbb{H}^1$ to $(\mathbb{H}^1)'$, by using formally Green's formula, in the following way:

$$< \mathcal{A}^*\phi, \psi >= -\frac{1}{2}\sum_{i,j=1}^n (k_{ij}\partial_{x_j}\phi, \partial_{x_i}\psi) + \sum_{i=1}^n((\zeta_i - \partial_{x_j}k_{ij})\phi, \partial_{x_i}\psi).$$

where $(.,.)$ is the inner product given in (4.18).

Now, after rewritten equation (4.17) as in form (4.20) and in order to apply the results of pardoux [19] about the existence of a solution to (4.20), we need to show that the pair operators $(\mathcal{A}^*, \mathcal{B})$ satisfy the coercivity condition, that is, $\exists\alpha_1 > 0$ and α_2 such that: $\forall u \in \mathbb{H}^1$, we have:

$$2 < \mathcal{A}^*u, u > +\alpha_2|u|^2_{L^2(\mathbb{R}^n)} \geq \alpha_1||u||^2_{\mathbb{H}^1} + |\mathcal{B}u|^2_{(L^2(\mathbb{R}^n))^n}. \tag{4.21}$$

where the norm in $(L^2(\mathbb{R}^n))^n$ is defined as follows:

$$|u|_{(L^2(\mathbb{R}^n))^n} = \left(\sum_{i=1}^{n} |u_i|^2_{L^2(\mathbb{R}^n)}\right)^{1/2}.$$

In fact the condition 4.21 is satisfied from assumption 4, especially from the positive definite condition on K and the fact that f is bounded. Therefore from (4.21), assumption 4 and assumption 3, the first part of the proof ends by applying theorem 2.1 in pardoux [19].

Using the same arguments for the proof of corollary 3.2 in pardoux [19], more precisely with the help of unique solvability theorem for Backward kolomogrov equation, we can deduce that $q_t(x)$ is the density of the unnormalized conditional distribution of μ_t given $\mathbb{F}^S$. □

Proposition 4.15. *Under assumption 3 and 4, the filter estimate $\overline{f}_t = (\overline{f}^i_t)_{1\le i\le n}$ can be computed as follows :*

$$\overline{f^i_t} = \int_{\mathbb{R}^n} f^i(x)p_t(x)dx, \qquad \textit{where } p_t(x) = \frac{q_t(x)}{\displaystyle\int_{\mathbb{R}^n} q_t(x)dx}, \tag{4.22}$$

$p_t(x)$ is the density of the conditional distribution of μ_t given the observations $\mathbb{F}^S$ and $q_t(x)$ is solution of (4.17). Moreover, $p_t(x)$ satisfies (4.16).

Proof. First of all, we need to verify that $\int_{\mathbb{R}^n} q_t(x)dx$ does not reach 0 nor infinity. This can be deduced from the fact $\int_{\mathbb{R}^n} q_t(x)dx = \psi_t(1) =: \tilde{\mathbb{E}}[\Lambda_t|\mathcal{F}^S_t]$ verifies

$$\int_{\mathbb{R}^n} q_t(x)dx = 1 + \int_0^t \int_{\mathbb{R}^n} q_s(x)\lambda(x)dxd\tilde{W}_s$$

From Kallianpur-Striebel formula (4.5) and proposition 4.14, we can deduce (4.22). Finally, applying Itô's formula in the Hilbert space case, we can deduce that $p_t(x)$ verifies (4.16). □

Remark 4.16. *It is possible to recast the stochastic partial differential equation (4.17) into a form in which there are no stochastic integral terms. This reduction form can be obtained using a suitable pathwise transformation, as in proposition 4.17. This form can be analyzed and establishing the existence and uniqueness of a fundamental solution to (4.17). Also, the importance of this form, is that we can prove the existence and uniqueness solution q_t to (4.17) without requiring the boundeness assumptions on f which is an necessary assumption (assumption 4, A.iv)) in proposition 4.14 to study the existence and uniqueness for q_t thus for p_t. Also, we see in section 5 the advantage of this pathwise technique to obtain an explicit formulas for the optimal portfolio, optimal wealth and value function up to the solution of a linear partial differential equation.*

4.2.2 The pathwise density approach

The pathwise theory of filtering is concerned with casting the filtering equations in a form in which the filtered estimates can be computed separately for each sample path of the observation process. The ideas of this approach are due to Clark [4], Davis [7] [8],.... The essential idea of this approach comes from the definition of the conditional expectation, that is, as $\mathbb{F}^S_t$ is the filtration generated by the observation process $\tilde{W}$, then by definition of the conditional expectation, the filtered estimates can be seen as a function of $\tilde{W}_t$ a.s. For more details about this approach and the technical tools used, see Fleming-Pardoux [10] and Pardoux [19].

We show in proposition (4.17) that for a given simple path $\tilde{W}_t$, q_t can be expressed in terms of a solution of a linear nondegenerate parabolic partial differential equation, and therefore classical partial differential equation can be applied in analyzing existence, uniqueness and regularity of solutions.

We study both cases for f and for each case, we make the necessary assumptions on the coefficients ζ and ϑ.:

Assumption 5. *(f is bounded)*

- *B.i) assumptions 3 and 4 hold.*
- *B.ii)* $\partial_{x_i} f$ *and* $\partial^2_{x_i,x_j} f \in L^\infty(\mathbb{R}^n, \mathbb{R}^n), \quad i,j = 1,...n.$

Assumption 6. *(f has linear growth)*

- *C.i)* $\vartheta : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ *is measurable and bounded.*
- *C.ii)* $\exists \alpha > 0$, *such that* $K(x) = \vartheta\vartheta^*(x) \geq \alpha I, \quad \forall x \in \mathbb{R}^n.$
- *C.iii)* $\partial_{x_i} \vartheta_{ij} \in L^\infty(\mathbb{R}^n), \quad i,j = 1, ...n.$
- *C.iv)* ζ *and* f *are measurable from* $\mathbb{R}^n \to \mathbb{R}^n$ *and have linear growth, that is,* $\exists c$ *such that:*

$$|\zeta(x)|, |f(x)| \leq c(1 + |x|), \qquad \forall x \in \mathbb{R}^n.$$

- *C.v)* $\partial_{x_i} f$ $(i = 1, ...n)$ *and* $\sum_{i,j=1}^n K_{ij} \partial^2_{x_i,x_j} f \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$
- *C.vi)* $div(\zeta)$ *and* $\sum_{i,j=1}^n \partial^2_{x_i,x_j} K_{ij} \in L^\infty(\mathbb{R}^n).$
- *C.vii) assumption 3 holds.*

Proposition 4.17. *Under one of the assumptions 5 or 6, the density* $q_t(x)$ *of the unnormalized conditional distribution of* μ_t *given* $\mathcal{F}_t^S$ *is given by:*

$$q_t(x) = \exp(\tilde{W}_t^* \lambda(x)) \nu_t^{\tilde{W}_t}(x),$$

where $\nu_t^{\tilde{W}_t}$ *is the solution of the following linear parabolic partial differential equation parametrized by the observation path* $\tilde{W}_t$*:*

$$\partial_t \nu^{\tilde{W}} = \frac{1}{2} Tr(K(x) D_x^2 \nu^{\tilde{W}}) + (\Gamma^{\tilde{W}}(t,x))^* D_x \nu^{\tilde{W}} + F^{\tilde{W}}(t,x) \nu^{\tilde{W}}, \qquad \nu_0^{\tilde{W}}(x) = p_0. \tag{4.23}$$

The functions $\Gamma^{\tilde{W}}(t,x)$ *and* $F^{\tilde{W}}(t,x)$ *are given respectively by:*

$$\begin{aligned}
\Gamma^{\tilde{W}}(t,x) &= -\zeta(x) + K(x) D_x(\tilde{W}_t^* \lambda) + \chi(x), \\
F^{\tilde{W}}(t,x) &= \frac{1}{2} Tr(K(x) D_x^2(\tilde{W}_t^* \lambda)) + \zeta(x) D_x(\tilde{W}_t^* \lambda) + \frac{1}{2} D_x(\tilde{W}_t^* \lambda) K D_x^*(\tilde{W}_t^* \lambda) \\
&- div\left(\zeta(x) - K D_x(\tilde{W}_t^* \lambda)\right) + \frac{1}{2} \sum_{i,j=1}^n \partial^2_{x_i,x_j} K_{ij}(x) - \frac{1}{2} \lambda^*(x) \lambda(x).
\end{aligned}$$

Where $K = \vartheta\vartheta^*$ *is a* $n \times n$ *matrix-valued function,* $\lambda(x) = \sigma^{-1} f(x)$ *is a* $n \times 1$ *matrix-valued function,* $\chi(x)$ *is a* $n \times 1$ *matrix-valued function where* $\chi_i(x) = \sum_{j=1}^n \partial_{x_j} K_{ij}(x), \quad i = 1, ...n.$ *Here* D_x *and* D_x^2 *denote the gradient and the Hessian operators with respect to the variable* x.

Proof. The proof for the case of assumption 5 can be found in Fleming-Pardoux [10] (see equation (7.1) and theorem 7.1) and [19] (see equation (4.10) and theorem 4.2). Using the smoothness assumptions *A.iii)* in assumption 4 and *B.ii)*, we have done an integration by part in order to write the equation satisfied by $\nu_t^{\tilde{W}}$ as that in (4.23).

On the other hand, for the case of assumption 6, the proof can be found in Pardoux [20, P.212-2015] . $\square$

Due to the classical results of Bensoussan-Lions [2], we have, for fixed $\tilde{W}_t$, a unique solution of $\nu_t^{\tilde{W}} \in L^2([0,T];\mathbb{H}^1) \cap C([0,T];L^2(\mathbb{R}^n))$, thus for q_t.

Therefore, we can deduce that p_t given by 4.24 is the density of the conditional distribution of μ_t given the observation, and therefore the filter $\overline{f}_t$ can be computed as follows:

Corollary 4.18. *Under on the assumptions 5 or 6, the filter estimate* $\overline{f}_t = (\overline{f_t^i})_{1\leq i\leq n}$ *can be computed as follows :*

$$\overline{f_t^i} = \int_{\mathbb{R}^n} f^i(x)p_t(x)dx, \qquad \textit{where } p_t(x) = \frac{\nu_t^{\tilde{W}_t}(x)\exp(\tilde{W}_t^*\lambda(x))}{\displaystyle\int_{\mathbb{R}^n} \nu_t^{\tilde{W}_t}(x)\exp(\tilde{W}_t^*\lambda(x))dx}. \tag{4.24}$$

$p_t(x)$ *is the density of the conditional distribution of* μ_t *given the observations* $\mathbb{F}^S$ *and* $\nu_t^{\tilde{W}}(x)$ *is solution of (4.23).*

Proof. The proof can be deduced from the definition of $\overline{f}_t$ and theorem 5.3 in Pardoux [20]. $\square$

5 Application to portfolio optimization

Recall that the trader's objective is to solve the following optimization problem

$$J(x) = \sup_{\pi\in\mathcal{A}} \mathbb{E}[U(R_T^\pi)], \qquad x > 0. \tag{5.1}$$

where from (3.3), the dynamics of the wealth process R_t^π in the full information context is given by:

$$dR_t^\pi = R_t^\pi \pi_t^* \left[\overline{f}(\mu_t)dt + \sigma d\overline{W}_t\right].$$

We are now faced an optimization problem in full observation context. Then, one may apply the classical stochastic optimization approach like the martingale approach or the PDE approach. Here we will use the martingale approach.

Before presenting our results, let Z_t be the optimal projection of the $\mathbb{P}$-martingale L to $\mathbb{F}^S$, so $Z_t := \mathbb{E}[L_t|\mathbb{F}_t^S]$, where L is given by (3.1).
By applying Kallainpur-Striebel formula to L_t, we obtain that $Z_t = \dfrac{1}{\tilde{\Lambda}_t}$, where $\tilde{\Lambda}_t = \tilde{\mathbb{E}}[\Lambda_t|\mathcal{F}_t^S]$ and $\Lambda_t = \dfrac{1}{L_t}$.

Then, from (3.2), we define the measure transformation on $\mathbb{F}^S$ as follows:

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}|\mathbb{F}_t^S = Z_t. \tag{5.2}$$

As in Lakner [**?**], we have the following result for the representation of Z_t and $\tilde{\Lambda}$.

Proposition 5.1. *Under assumptions 1 and* $\mathbb{E}|\lambda_t| < \infty$ *, we have*

$$Z_t = \exp\left(- \int_0^t \overline{\lambda}_s^* d\overline{W}_s - \frac{1}{2} \int_0^t ||\overline{\lambda}_s||^2 ds \right), \tag{5.3}$$

$$\tilde{\Lambda}_t = \exp\left(\int_0^t \overline{\lambda}_s^* d\tilde{W}_s - \frac{1}{2} \int_0^t ||\overline{\lambda}_s||^2 ds \right). \tag{5.4}$$

5.1 Martingale approach

Let us impose the standard Inada conditions on the utility function: U is C^1 on $(0, \infty)$, and satisfies $U^{'}(0) = \infty$ and $U^{'}(\infty) = 0$. We denote by $I = (U^{'})^{-1}$ the inverse of the derivatives of U, which is decreasing function from $(0, \infty)$ into $(0, \infty)$.

Using the martingale approach, see for example Karatzas's [12], we have the following result:

Theorem 5.2. *The optimal wealth for the utility maximization problem (5.1) is given by*

$$\tilde{R}_t = \mathbb{E}\left[\frac{Z_T}{Z_t} I(z_x Z_T) | \mathcal{F}_t^S \right] \tag{5.5}$$

where z_x *is the Lagrange multiplier such that* $\mathbb{E}[Z_T I(z_x Z_T)] = x$ *and* x *is the initial wealth. Also the optimal portfolio* $\tilde{\pi}$ *is implicity determined by the equation*

$$d\tilde{R}_t = \tilde{R}_t \tilde{\pi}_t^* d\tilde{W}_t. \tag{5.6}$$

We now give some examples of applications of the martingale approach combined with the filtering problem. We solve our optimization problem with the logarithmic and power utility functions.

5.1.1 Logarithmic utility function

We consider an utility function $U(x) = \ln(x)$. In this case $I(x) = \dfrac{1}{x}$ and the Lagrange multiplier $z_x = \dfrac{1}{x}$. Therefore from (5.5), the optimal wealth is given by

$$\tilde{R}_t = x\tilde{\Lambda}_t. \tag{5.7}$$

By applying Itô's formula to (5.7) and from (5.4), we have that $d\tilde{R}_t = \tilde{R}_t \overline{\lambda}_t^* d\tilde{W}_t$. Hence, comparing this dynamic for $\tilde{R}_t$ with (5.6), we obtain that the optimal portfolio $\tilde{\pi}$ is given by

$$\tilde{\pi}_t = \overline{\lambda}_t := \sigma^{-1} \overline{f}_t.$$

Finally, the value function is given by $J(x) = \ln(x) + \dfrac{1}{2}\mathbb{E}[\int_0^T ||\overline{\lambda}_s||^2 ds]$.

5.1.2 Power utility function

Here, we consider an utility function $U(x) = \dfrac{x^p}{p}$, $0 < p < 1$

Proposition 5.3. *the optimal wealth is given by:*

$$\tilde{R}_t = x \frac{M_t}{M_0} (\tilde{\Lambda}_t)^{1/1-p}.$$

where $M_0 = \tilde{\mathbb{E}}[(\tilde{\Lambda}_T)^{1/1-p}]$ *and* $M_t = M(t, \tilde{W}_t)$ *is solution of the following linear PDE:*

$$\partial_t M + \frac{1}{2}Tr(D_w^2 M) + \frac{1}{1-p}(\sigma^{-1}\overline{f}(t,w))^* D_w M + \frac{p}{2(1-p)^2}||\sigma^{-1}\overline{f}(t,w)||^2 M = 0.$$

$\overline{f}(t,w) = (\overline{f^i}(t,w))_{1\le i\le n}$ *is given by:*

$$\overline{f^i}(t,w) = \int_{\mathbb{R}^n} f^i(x)p_t(x)dx, \qquad \text{where } \; p_t(x) = \frac{\nu_t^w(x)\exp(w^*\lambda(x))}{\displaystyle\int_{\mathbb{R}^n}\nu_t^w(x)\exp(w^*\lambda(x))dx}.$$

and $\nu_t^w(x)$ *is solution of (4.23). The associated optimal portfolio* $\tilde{\pi}_t$ *is given by:*

$$\tilde{\pi}_t = \frac{1}{1-p}\sigma^{-1}\overline{f}_t + \frac{D_w M(t,\tilde{W}_t)}{M(t,\tilde{W}_t)}. \tag{5.8}$$

Moreover, the value function is given by:

$$J(x) = \frac{x^p}{p}(M(0,\tilde{W}_0))^{1-p}. \tag{5.9}$$

.

Proof. In this case $I(x) = x^{1/(p-1)}$ and the Lagrange multiplier z_x is given by

$$z_x = \frac{(x)^{p-1}}{\mathbb{E}[(Z_T)^{p/p-1}]} = \frac{(x)^{p-1}}{\tilde{\mathbb{E}}[(\tilde{\Lambda}_T)^{1/1-p}]}.$$

Therefore from theorem 5.2, the optimal wealth process is given by:

$$\tilde{R}_t = x\frac{M_t}{M_0}(\tilde{\Lambda}_t)^{1/1-p}. \tag{5.10}$$

with

$$M_t = \tilde{\mathbb{E}}\left[\left(\frac{\tilde{\Lambda}_T}{\tilde{\Lambda}_t}\right)^{1/1-p}\Big|\mathcal{F}_t^S\right].$$

We notice that from (5.4), M_t can be rewritten as follows:

$$M_t = \tilde{\mathbb{E}}\left[\exp\left(\int_t^T \frac{1}{1-p}\overline{\lambda}_s^* d\tilde{W}_s - \frac{1}{2}(\frac{1}{1-p})^2\int_t^T ||\overline{\lambda}_s||^2 ds\right)\exp\left(\frac{1}{2}\int_t^T \frac{p}{(1-p)^2}||\overline{\lambda}_s||^2 ds\right)\Big|\mathcal{F}_t^S\right].$$

Let us now consider the process $\tilde{M}_t = \exp\left(\int_t^T \frac{1}{1-p}\overline{\lambda}_s^* d\tilde{W}_s - \frac{1}{2}(\frac{1}{1-p})^2\int_t^T ||\overline{\lambda}_s||^2 ds\right)$.

We assume that $\tilde{M}_t$ is a martingale, then we can define a change of probability measure as follows:

$$\frac{d\mathbb{P}^M}{d\tilde{\mathbb{P}}}|\mathcal{F}_t^S = \tilde{M}_t.$$

Therefore

$$M_t = \mathbb{E}^M\left[\exp\left(\frac{1}{2}\int_t^T \frac{p}{(1-p)^2}||\overline{\lambda}_s||^2 ds\right)\Big|\mathcal{F}_t^S\right],$$

$$= \mathbb{E}^M\left[\exp\left(\frac{1}{2}\int_t^T \frac{p}{(1-p)^2}||\sigma^{-1}\overline{f}_s||^2 ds\right)\Big|\mathcal{F}_t^S\right], \tag{5.11}$$

$$= M(t,\tilde{W}_t). \tag{5.12}$$

Where the second equality comes from the fact that $\overline{\lambda}_t = \sigma^{-1}\overline{f}_t$ and the last equality, which say that $M_t = M(t, \tilde{W}_t)$ is a function of $(t, \tilde{W}_t)$, is deduced from corollary 4.18. Moreover, the process $\tilde{W}_t$ has the following dynamics under $\mathbb{P}^M$:

$$d\tilde{W}_t = dW_t^M + \frac{1}{1-p}\overline{\lambda}_t dt.$$

Now, by Feynman-Kac representation, the function $M(t, w)$ for $(t, w) \in [0, T] \times \mathbb{R}^n$ satisfies the following linear PDE:

$$\partial_t M + \frac{1}{2}Tr(D_w^2 M) + \frac{1}{1-p}(\sigma^{-1}\overline{f}(t, w))^* D_w M + \frac{p}{2(1-p)^2}||\sigma^{-1}\overline{f}(t, w)||^2 M = 0, \tag{5.13}$$

with terminal condition $M(T, w) = 1$. We write $\overline{f}_t$ as $\overline{f}(t, w)$ in order to indicate the dependence on the observation path $\tilde{W}_t = w$. From corollary 4.18, $\overline{f}(t, w) = (\overline{f^i}(t, w))_{1 \le i \le n}$ is given by:

$$\overline{f^i}(t, w) = \int_{\mathbb{R}^n} f^i(x) p_t(x) dx,$$

where $p_t(x) = \dfrac{\nu_t^w(x)\exp(w^*\lambda(x))}{\displaystyle\int_{\mathbb{R}^n} \nu_t^w(x)\exp(w^*\lambda(x))dx}$.

It remains to show that the optimal portfolio $\tilde{\pi}_t$ is given by (5.8). From theorem 5.2, we have that the optimal portfolio can be determined from (5.6). Then, from (5.4) and the fact that M_t is solution to (5.13), if we apply Itô's formula on (5.10) and comparing it to (5.6), we can deduce that the optimal poftrolio is given by (5.8).

Finally, from (5.10) and (5.2), we can deduce that the value function $J(x)$ is given by (5.9). □

Notice that the advantage to use the pathwise approach is appeared, for the power utility case, in the step when we have expressed M_t in terms of $\tilde{W}_t$ as in (5.12). In fact, if the filter estimate $\overline{f}_t$ is given as in (4.22), then in this case M_t will be written as a function of p_t, that is, $M_t = M(t, p_t)$, where the dynamcis of p_t is given by (4.16). Therefore, from Feynman-Kac representation, we have that M will be the solution of a partial differential equation with infinite dimensional state variable p, where the partial derivatives terms with respect to p are Frechet derivatives.

References

[1] Alan Bain and Dan Crisan. *Fundamentals of stochastic filtering*, volume 60 of *Stochastic Modelling and Applied Probability*. Springer, New York, 2009.

[2] A. Bensoussan and J.-L. Lions. *Applications des inéquations variationnelles en contrôle stochastique*. Dunod, Paris, 1978. Méthodes Mathématiques de l'Informatique, No. 6.

[3] Alain Bensoussan. *Stochastic control of partially observable systems*. Cambridge University Press, Cambridge, 1992.

[4] J.M.C Clark. The design of robust approximations to the stochastic differential equations of nonlinear filtering, communication systems and random process theory. *Sijthoff and Noordhoff Alphen 1978. NATO Advanced Study Inst. Ser.*, pages 721–734.

[5] D. Crisan and T. Lyons. A particle approximation of the solution of the Kushner-Stratonovitch equation. *Probab. Theory Related Fields*, 115(4):549–578, 1999.

[6] Dan Crişan, Pierre Del Moral, and Terry J Lyons. Interacting particle systems approximations of the kushner-stratonovitch equation. *Advances in Applied Probability*, pages 819–838, 1999.

[7] M. H. A. Davis. A pathwise solution of the equations of nonlinear filtering. *Theory of Probability and its Applications*, 27(1):167–175, 1982.

[8] M.H.A. Davis. *On a Multiplicative Functional Transformation Arising in Nonlinear Filtering Theory*. Publication - Dept of Computing & Control Imperial College of Science & Technology. Imperial College of Science & Technology, 1987.

[9] Michael U. Dothan and David Feldman. Equilibrium interest rates and multiperiod bonds in a partially observable economy. *The Journal of Finance*, 41(2):369–382, 1986.

[10] Wendell H. Fleming. Exit probabilities and optimal stochastic control. *Appl. Math. Optim.*, 4(4):329–346, 1977/78.

[11] Dalia Ibrahim and Frédéric Abergel. Non-linear filtering and optimal investment under partial information for stochastic volatility models. *https://hal.archives-ouvertes.fr/hal-01018869v4/document*, 2015.

[12] Ioannis Karatzas. *Lectures on the mathematics of finance*, volume 8 of *CRM Monograph Series*. American Mathematical Society, Providence, RI, 1997.

[13] Ioannis Karatzas and Xlng-Xlong Xue. A note on utility maximization under partial observations. *Mathematical Finance*, 1(2):57–70, 1991.

[14] Ioannis Karatzas and Xiaoliang Zhao. Bayesian adaptive portfolio optimization. In *Option pricing, interest rates and risk management*, Handb. Math. Finance, pages 632–669. Cambridge Univ. Press, Cambridge, 2001.

[15] N.V Krylov and B.L Rozovskii. On conditional distributions of diffusion processes. *Mathematics of the USSR-Izvestiya*, 12(2):336, 1978.

[16] Peter Lakner. Utility maximization with partial information. *Stochastic Process. Appl.*, 56(2):247–273, 1995.

[17] Peter Lakner. Optimal trading strategy for an investor: the case of partial information. *Stochastic Process. Appl.*, 76(1):77–97, 1998.

[18] Sergey Lototsky, Remigijus Mikulevicius, and Boris L. Rozovskii. Nonlinear filtering revisited: a spectral approach. *SIAM J. Control Optim.*, 35(2):435–461, 1997.

[19] E. Pardoux. Stochastic partial differential equations and filtering of diffusion processes. *Stochastics*, 3(2):127–167, 1979.

[20] E. Pardoux. Équations du filtrage non linéaire, de la prédiction et du lissage. *Stochastics*, 6(3-4):193–231, 1981/82.

[21] Étienne Pardoux. Filtrage non linéaire et équations aux dérivées partielles stochastiques associées. In *École d'Été de Probabilités de Saint-Flour XIX—1989*, volume 1464 of *Lecture Notes in Math.*, pages 67–163. Springer, Berlin, 1991.

[22] Huyên Pham and Marie-Claire Quenez. Optimal portfolio in partially observed stochastic volatility models. *Ann. Appl. Probab.*, 11(1):210–238, 2001.